r-Bell numbers for Graphs

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Contents

Abstract

The r-Bell number $\hat{B}_r(G)$ of a simple labeled graph G is the number of partitions of its vertex set whose blocks are independent sets of G where the first r-vertices are in different blocks. The number of these partitions with k blocks is the (graphical) r- Stirling number $\hat{S}_r(G,k)$ of G. On this particular paper we will investigate the integer sequence of r-Bell numbers for different kinds of graphs.

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Chapter 1

Stirling numbers of the second kind

1.1 Set Partition

Definition 1. *A Partition of a set A is defined as a set of non empty pair wise disjoint subsets of A whose union is A.*

To show this,Let $A_1, A_2, ..., A_n$ be a subset of A. $A_1, A_2, ..., A_n$ Is said to be a partition of A, if and only if,

1. $A_i \neq \emptyset \quad \forall i \in \{1, 2, ..., n\}$
2. $A_i \cap A_j = \emptyset \quad i \neq j \quad i, j \in \{1, 2, ..., n\}$
3. $\cup_{i=1}^{n} A_i = A$.

The subsets $A_1, A_2, ..., A_n$ are also called blocks of the partition.

Example 1. *Let $A = \{1, 2, 3, 4\}$, then A can be partitioned into 4 subsets or blocks. The first one is the block which contains all element of A, which is {1,2,3,4}*

Into two classes or blocks are:-
{{1},{2,3,4} }, {{2}{1,3,4} }, {{3}{1,2,4} }, {{4}{1,2,3} }, {{1,2}{3,4} }, {{1,3}{2,4} }, {{1,4}{2,3} }

Into three blocks are:-
{{1}{2}{3,4} }, {{1}{3}{2,4} }, {{1}{4}{2,3} }, {{2}{3}{1,4} }, {{2}{4}{1,3} }, {{3}{4}{1,2} }

Into four blocks are: -
{1}, {2}, {3}, {4}.

Definition 2. *Let $n, k \in \mathbb{Z}^+$ such that $n \geq k$, the number of ways of partitioning of a set with n elements into k nonempty blocks or classes is called **Stirling numbers of the second kind**. It is denoted by S(n,k) or $\left\{ n \atop k \right\}$.*

Stirling number of the second kind also counts the number of ways of distributing n distinguishable balls into k (indistinguishable)identical boxes such that no box is empty. On example 1, we can count the Stirling numbers of the second kind. These are S(4,1)=1; this is {1,2,3,4}
S(4,2)=7; these are,
{{1},{2,3,4} }, {{2}{1,3,4} }, {{3}{1,2,4} }, {{4}{1,2,3} }, {{1,2}{3,4} }, {{1,3}{2,4} }, {{1,4}{2,3} }
S(4,3)=6; these are,
{{1}{2}{3,4} }, {{1}{3}{2,4} }, {{1}{4}{2,3} }, {{2}{3}{1,4} }, {{2}{4}{1,3} }, {{3}{4}{1,2} } and
S(4,4)=1; {1}, {2}, {3}, {4}.

1.2 Properties of Stirling numbers of the second kind

For any nonnegative integers n and k $n \geq k$

1. S(0,0)=1
2. S(n,0)= S(0,k)=0 $\forall k, n \in \mathbb{Z}^+$
3. S(n,k)=0 if $k > n \geq 1$
4. S(n,1)=1 for $n \geq 1$
5. S(n,n)=1 for $n \geq 1$
6. S(n,n-1)=$\binom{n}{2}$
7. S(n,2)=$2^{(n-1)} - 1 \; for \; n \geq 1$
8. S(n,3)=$\frac{1}{2}(3^{(n-1)} + 1) - 2^{(n-1)}$

Proof.

1. If there is no object and no sets then there is one way.
2. Given n objects, we cannot put the object into no sets. This means that there is no way that distributing n balls into no boxes. And if there is no ball and k boxes we cannot distribute.
3. We cannot distribute n balls into k boxes with no box is left empty if $n \leq k$.
4. Distributing n balls into 1 box means there is only one set.
5. We can divide n objects into n-nonempty disjoint subsets only in one way; each subset contains only one object.
6. To put n objects into n-1 parts where no box is left empty, one of the parts contains two objects and each of the others contains one object. We can choose the two elements in $\binom{n}{2}$ ways. We prove the others later.

□

1.3 Recurrence formula about Stirling numbers of the second kind

The recurrence formula for Stirling numbers of the second kind is written in the following form.

Theorem 1. *S(n,k)= kS(n-1,k)+S(n-1,k-1)*

Proof. If we add the singleton part {n} to any partition of {1,2,. . .,n-1} into k-1 parts then we obtain a partition of {1,2,. . .,n} into k parts. Also given any partition of {1,2,. . .,n-1} into k parts we may, by inserting the element n into one of these k parts gives rise to k partitions of {1,2,. . .,n} into k parts. Each of the partitions of {1,2,. . .,n} into k parts will eventually arise by way of the above processes. Therefore, S(n,k)= kS(n-1,k)+S(n-1,k-1).

□

Stirling numbers of the second kind can also be defined as a coefficient of$(x)_k$.

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k$$

Where $\quad (x)_k = x(x-1)(x-2)(x-k+1)$

Proof. The number of surjective mappings from an n-set to a k-set is k!S(n,k) or a block of the partition is the inverse image of an element of the k-set.

Example 2. *Let n=3, k=2, k!S(n,k)= 2!S(3,2)= 2.3 = 6. The number of surjective mappings from {1,2,3} to {1,2}*$f_i = 1,2,31,2, i = 1,2,,6$ *is 6.*
These are:-

$f_1(1) = 1$	$f_1(2) = 2$	$f_1(3) = 2$
$f_2(1) = 1$	$f_2(2) = 1$	$f_2(3) = 2$
$f_3(1) = 2$	$f_3(2) = 1$	$f_3(3) = 1$
$f_4(1) = 2$	$f_4(2) = 2$	$f_4(3) = 1$
$f_5(1) = 1$	$f_5(2) = 2$	$f_5(3) = 1$
$f_6(1) = 2$	$f_6(2) = 1$	$f_6(3) = 2$

Let x be an integer, there are x^n mappings from the n-set, N={1,2,,n} to the x-set {1,2,,x}. Means each element of N have x posibilities. For any k-subset Y of {1,2,,x}, there are k!S(n,k) surjections from N to Y. So we can find

$$x^n = \sum_{k=0}^{n} \binom{x}{k} k! S(n,k) = \sum_{k=0}^{n} S(n,k)(x)_k$$

□

1.4 Generating function of Stirling Numbers of the Second Kind

We can find the ordinary generating function for Stirling numbers of the second kind from the recurrence formula, S(n,k)= kS(n-1,k)+S(n-1,k-1)

Let $B_k(x) = \sum_{n=0} S(n,k)\, x^n$

$$= \sum S(n-1,k-1)\, x^n + \sum kS(n-1,k)\, x^n$$

$$= x\sum S(n-1,k-1)\, x^{n-1} + xk\sum S(n-1,k)\, x^{n-1}$$

$$B_k(x) = xB_{k-1}(x) + kxB_k(x)$$

$(1-kx)\, B_k(x) = xB_{k-1}(x)$

$$B_k(x) = \frac{x}{1-kx} B_{k-1}(x), k = 1, \quad B_0(x) = 1$$

$$B_k(x) = \frac{x}{1-kx} B_{k-1}(x)$$

$$= \frac{x}{1-kx} * \frac{x}{1-(k-1)x} B_{k-2}(x)$$

$$= \prod_{i=0}^{k-1} \frac{x}{1-(k-i)x} B_{k-(i+1)}(x)$$

$$= \frac{x}{1-kx} * \frac{x}{1-(k-1)x} * \cdots * \frac{x}{1-2x} * \frac{x}{1-x}$$

$$= \frac{x^k}{(1-x)(1-2x)\dots(1-kx)}$$

So that $\sum_{n=0} S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\dots(1-kx)}$

By partial fraction, let $a_k \in R,\ k \in Z^+$

$$B_k(x) = \frac{a_1}{1-x} + \frac{a_2}{1-2x} + \cdots + \frac{a_k}{1-kx}$$

To find a's, fix r for $1 = r = k$

$$\frac{x^k}{(1-x)(1-2x)\dots(1-rx)(1-(r+1)x)\dots(1-kx)}$$

$$= \frac{a_1}{1-x} + \frac{a_2}{1-2x} + \cdots + \frac{a_r}{1-rx} + \frac{a_{r+1}}{1-(1+r)x} + \cdots + \frac{a_k}{1-kx}$$

Multiply both side by $1-rx$ and let $x = \frac{1}{r}$

$$\frac{x^k(1-rx)}{(1-x)(1-2x)\dots(1-rx)(1-(r+1)x)\dots(1-kx)}$$

$$= \frac{x^k}{(1-x)(1-2x)\dots(1-(r-1)x)(1-(r+1)x)\dots(1-kx)}$$

$$= \frac{a_1(1-rx)}{1-x} + \frac{a_2(1-rx)}{1-2x} + \dots + \frac{a_{r-1}(1-rx)}{1-(r-1)x} + a_r + \frac{a_{r+1}(1-rx)}{1-(1+r)x} + \dots + \frac{a_k(1-rx)}{1-kx}$$

Since$x = \frac{1}{r}$

$$\frac{\left(\frac{1}{r}\right)^k}{\left(1-\left(\frac{1}{r}\right)\right)\left(1-2\left(\frac{1}{r}\right)\right)\dots\left(1-(r-1)\left(\frac{1}{r}\right)\right)\left(1-(r+1)\left(\frac{1}{r}\right)\right)\dots\left(1-k\left(\frac{1}{r}\right)\right)} = a_r$$

$$= \frac{r^{-k}}{\left(\frac{r-1}{r}\right)\left(\frac{r-2}{r}\right)\dots\left(\frac{r-r+1}{r}\right)\left(\frac{r-r-1}{r}\right)\dots\left(\frac{r-k}{r}\right)}$$

$$= \frac{r^{-k}r^{k-1}}{(r-1)(r-2)\dots(1)(-1)(-2)\dots(r-k)}$$

$$= \frac{r^{-1}}{(r-1)!(-1)^{k-r}(k-r)!}$$

$$= \frac{(-1)^{k-r}}{(r)!(k-r)!} = a_r$$

Therefore

$$S(n,k) = [x^n]\frac{x^k}{(1-x)(1-2x)\dots(1-kx)}$$

$$= [x^n]\sum_{r=1}^{k}\frac{a_r}{1-rx}$$

$$= \sum_{r=1}^{k} a_r [x^n]\frac{1}{1-rx}$$

$$= \sum_{r=1}^{k} a_r r^n$$

$$S(n,k) = \sum_{r=1}^{k}\frac{(-1)^{k-r}r^n}{(r)!(k-r)!} \quad (1)$$

This is the same as$S(n,k) = \frac{1}{k!}\sum_{r=1}^{k}(-1)^{k-r}r^n\binom{k}{r}$ (2)

From equation (1) we can prove the following.

7. $S(n,2) = \sum_{r=1}^{2} \frac{(-1)^{2-r} r^n}{(r)!\,(2-r)!}$

$$= \frac{-1}{1} + \frac{2^n}{2!}$$

$$= \frac{2^n - 2}{2} = 2^{n-1} - 1$$

8. $S(n,3) = \sum_{r=1}^{3} \frac{(-1)^{3-r} r^n}{(r)!\,(3-r)!}$

$$= \frac{1}{2!} + \frac{-1\,(2^n)}{2!} + \frac{3^n}{3!}$$

$$= \frac{3^{n-1} - 2^n + 1}{2}$$

$$= \frac{1}{2}\left(3^{n-1} + 1\right) - 2^{n-1}$$

The following table-1 is the triangle of Stirling numbers of the second kind.

k / n	1	2	3	4	5	6	7	8
1								
2	1	1						
3	1	3	1					
4	1	7	6	1				
5	1	15	25	10	1			
6	1	31	90	65	15	1		
7	1	63	301	350	140	21	1	
8	1	127	966	1701	1050	266	28	1

Table1.1 is the triangle of Stirling numbers of the second kind

1.5 Bell Numbers

Definition 3. *The Bell number is the number of partitions of a set of n elements .*

Let B_n denote the n^{th} Bell number, defined as the number of partitions of a set of n elements. The Bell numbers are just the sum of the entries in a row of the triangle of Stirling numbers of the second kind. This is

$$B_n = \sum_{k=0}^{n} S(n,k) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$$

We can also formulate the bell numbers as follows.

$$B_n = \sum_{k=1}^{n} \frac{1}{k!} \sum_{r=1}^{k} (-1)^{k-r} r^n \binom{k}{r} \qquad from\ (2)$$

1.5.1 Dobinskis formula for Bell Numbers

Earlier in this chapter we have seen an explicit formula (1) for $\left\{ {n \atop k} \right\}$. If we sum that formula from k=1 to n we will have an explicit formula forB_n. However, there is one important thing that is reasonable to observe. The formula (1) is valid for all positive integer values of n and k. In particular, it is valid if $k > n$.But $\left\{ {n \atop k} \right\} = 0$ for $k > n$. This means the formula need not to be told that $\left\{ {5 \atop 7} \right\} = 0$; it knows it. That means if we insert n=5,k=7 into (1) and we get 0. Thus, to calculate the Bell numbers, we can sum the last member of (1) from k=1 to P, where P is any number you please that is greater or equal to n. Lets do it.

$$\begin{aligned} B_n &= \sum_{k=1}^{P} \sum_{r=1}^{k} (-1)^{k-r} \frac{r^{n-1}}{(r-1)!\,(k-r)!} \\ &= \sum_{r=1}^{P} \frac{r^{n-1}}{(r-1)!} \sum_{k=r}^{P} \frac{(-1)^{k-r}}{(k-r)!} \\ &= \sum_{r=1}^{P} \frac{r^{n-1}}{(r-1)!} \left\{ \sum_{i=0}^{P-r} \frac{(-1)^i}{i!} \right\} \end{aligned}$$

But now the number P is arbitrary, except that $P \geq n$. Let's keep n and r fixed, and let $P \to \infty$. This gives the following remarkable formula for the Bell numbers $B_n = \frac{1}{e} \sum_{r \geq 0} \frac{r^n}{r!}$, $\qquad (n \geq 0)$
This formula is called Dobinski's formula for the Bell numbers.

1.5.2 Triangle scheme for calculating Bell numbers

The Bell numbers can easily be calculated by creating the so-called Bell triangle, also called Aitken's array.

1. Start with the number one. Put this on a row by itself.
2. Start a new row with the rightmost element from the previous row as the leftmost number.
3. Determine the number not on the left column by taking the sum of the numbers to the left and the number above to the left(the number diagonally up and left of the number we are calculating)
4. Repeat step3 until there is a new row with one more number for that row than the previous row.
5. The first number on the given row is the Bell number for that row.

For example, if we notice the first row it is made by placing one by itself. The next (second) row is made by taking the rightmost number from the previous row, and placing it on a new row. We have a structure like this:

1	
1	x

The value of x here is determined by adding the number to the left of x and the number above it.

1	
1	2
y	

The value of y is determined by copying over the rightmost number from the previous row. Since this rightmost number has a value of 2, y is given a value of two.

Here are the first few rows of this triangle:

1									
1	2								
2	3	5							
5	7	10	15						
15	20	27	37	52					
52	67	87	114	151	203				
203	255	322	409	523	674	877			
877	1080	1335	1657	2066	2589	3263	4140		
4140	5017	6097	7432	9089	11155	13744	17007	21147	
21147	25287	30304	36401	43833	52922	64077	77821	94828	115975

Table1.2 the Bell triangle with ten rows

1.5.3 Generating Function for Bell numbers

The exponential generating function of Bell numbers is

$$\sum_{n=1}^{8} B_n \frac{x^n}{n!} = e^{e^x - 1} \tag{3}$$

Proof. Let

$$B(x) = \sum_{n=1}^{8} B_n \frac{x^n}{n!}$$

$$B_n = \sum_{k=1}^{n} S(n,k)$$

$$= \sum_{k=1}^{n} \sum_{r=1}^{k} \frac{(-1)^{k-r} r^n}{(r)!\,(k-r)!}$$

$$= \sum_{r=1}^{n} \frac{r^n}{r!} \sum_{r=1}^{k} \frac{(-1)^{k-r}}{(k-r)!}$$

$$= \frac{1}{e} \sum_{r=1}^{n} \frac{r^n}{r!} \qquad from\ (1)$$

$$B(x) = \sum_{n=1}^{8} \frac{1}{e} \sum_{r=1}^{n} \frac{r^n}{r!} \frac{x^n}{n!}$$

$$= \frac{1}{e} \sum_{n=1}^{8} \sum_{r=1}^{n} \frac{r^n}{r!} \frac{x^n}{n!}$$

$$= \frac{1}{e} \sum_{r=1}^{8} \frac{1}{r!} \sum_{n=0} \frac{(rx)^n}{n!}$$

$$= \frac{1}{e} \sum_{r=1}^{8} \frac{e^{xr}}{r!}$$

$$= \frac{1}{e} \sum_{r=1}^{8} \frac{(e^x)^r}{r!}$$

$$B(x) = \frac{1}{e} e^{e^x} = e^{e^x - 1}$$

□

Theorem 2. *The Bell numbers satisfy this recursion formula:*

$$B_{n+1} = \sum_{r=0}^{n} \binom{n}{r} B_r \tag{4}$$

Proof. Let

$$B(x) = \sum_{n=0}^{8} B_n \frac{x^n}{n!} \qquad \text{then,}$$

$$B'(x) = \sum_{n=1}^{8} B_n n \frac{x^{n-1}}{n!}$$

$$= \sum_{n=1}^{8} B_n \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{8} B_{n+1} \frac{x^n}{n!}$$

And $$B'(x) = \left(e^{e^x - 1}\right)(e^x) \qquad \textit{from } (3)$$

$$= e^x B(x)$$

$$= \sum_{n=0} \frac{x^n}{n!} \sum_{r=0} B_r \frac{x^r}{r!}$$

$$= \sum_{n=0}^{8} \sum_{r=0}^{n} \binom{n}{r} B_r \frac{x^n}{n!}$$

$$= \sum_{n=0}^{8} B_{n+1} \frac{x^n}{n!}$$

Therefore $B_{n+1} = \sum_{r=1}^{n} \binom{n}{r} B_r$

□

Chapter 2

The r-Stirling number of the second kind

The r-Stirling numbers represent a certain generalization of the regular Stirling numbers, the r-Stirling numbers count certain restricted partitions and are defined, for all positive r as follows.

Definition 4. *Let n, k and r be non-negative integers. Then the r- Stirling numbers of the second kind which is denoted by $S_r\,(n, k)$ enumerates the k-partitions of n elements such that the first r elements are in distinct subsets.*

The alternative definition follows from the following terminology each non-empty subset, in a permutation of an ordered set has a minimal element; a partition of the set $\{1,\ 2,\ \ ...,\ \ n\}$ into k non-empty subsets has k associated minimal elements.

Definition 5. *The r- Stirling numbers of the second kind which is denoted by $\left\{ {n \atop k} \right\}_r$ enumerates thenumber of ways to partition the set $\{1,\ 2,\ .\ .\ .,\ n\}$ in to k non-empty disjoint subset such that the numbers 1, 2, . . ., r are all minimal elements.*

The regular Stirling numbers of the second kind can be expressed as

$$\left\{ {n \atop k} \right\}_0 = \left\{ {n \atop k} \right\} = \left\{ {n \atop k} \right\}_1$$

2.1 Recurrence formula about r-Stirling numbers of the second kind

Theorem 3. *The r-Stirling numbers of the second kind comply with the following recurrence relation.*

$$\left\{ {n \atop k} \right\}_r = 0, \qquad n < r \; OR \; (n = r \cap k \neq r)$$

$$\left\{ {n \atop k} \right\}_r = 1 \qquad n = r = k$$

$$\left\{ {n \atop k} \right\}_r = k \left\{ {n-1 \atop k} \right\}_r + \left\{ {n-1 \atop k-1} \right\}_r \qquad n > r$$

Proof. Given a set [n] to be partitioned into k nonempty parts, we either put the last object , "n" into a class by itself or we put it together with some nonempty subset of the first n -1 objects.A partition of the set [n-1] into k block is enumerated by $\left\{ {n-1 \atop k} \right\}$ and we can add the element "n" in to any of the k blocks, this can be done in k ways; So a partition of the set [n] into k blocks where the element "n" is in a block with other elements is given by $k \left\{ {n-1 \atop k} \right\}$. And if we put the last object in to a block by itself we can partition the set [n-1] into $k-1$ blocks in $\left\{ {n-1 \atop k-1} \right\}$ ways. As we can see, for $n > r$ this process does not affect the distribution of the numbers $1, \; 2, \; \; . \; .., \; \; r$ into different blocks. Hence $\left\{ {n \atop k} \right\}_r = k \left\{ {n-1 \atop k} \right\}_r + \left\{ {n-1 \atop k-1} \right\}_r \quad n > r$ □

The following special values can be easily calculated.

$$\left\{ {n \atop n} \right\}_r = 1, \qquad n \geq r$$

$$\left\{ {n \atop k} \right\}_r = 0, \qquad r \geq k$$

$$\left\{ {n \atop r} \right\}_r = r^{n-r}, \qquad n \geq r$$

For future reference the following tables were computed using the recurrence in Theorem2.1.

$\left\{ {n \atop k} \right\}_r$	k=1	k=2	k=3	k=4	k=5	k=6
n=1	1					
n=2	1	1				
n=3	1	3	1			
n=4	1	7	6	1		
n=5	1	15	20	10	1	
n=6	1	31	90	65	15	1

Table 2.1 The first few r-*Stirling numbers of the second kind with* r=1

$\left\{ {n \atop k} \right\}_r$	k=2	k=3	k=4	k=5	k=6	k=7
n=2	1					
n=3	2	1				
n=4	4	5	1			
n=5	8	19	9	1		
n=6	16	65	55	14	1	
n=7	32	211	285	125	20	1

Table 2.2 The first few r-*Stirling numbers of the second kind with* r=2

$\left\{ {n \atop k} \right\}_r$	k=3	k=4	k=5	k=6	k=7	k=8
n=3	1					
n=4	3	1				
n=5	9	7	1			
n=6	27	37	12	1		
n=7	81	175	97	18	1	
n=8	243	781	660	205	25	1

Table 2.3 The first few r-*Stirling numbers of the second kind with* r=3

Theorem 4. *The r-stirling numbers of the second kind satisfy the recurrence*

$$\left\{ {n \atop k} \right\}_r = \left\{ {n \atop k} \right\}_{r-1} - (r-1) \left\{ {n-1 \atop k} \right\}_{r-1} 1 \leq r \leq n$$

Proof. The above recurrence can be written as

$$(r-1) \left\{ {n-1 \atop k} \right\}_{r-1} = \left\{ {n \atop k} \right\}_{r-1} - \left\{ {n \atop k} \right\}_r$$

The right side of this equation counts the number of partitions of the set [n] into k non-empty subsets such that $\{1,\ 2,\ \ldots,\ r-1\}$ are minimal elements but r is not. But this number is equal to $(r-1)\left\{ {n-1 \atop k} \right\}_{r-1}$ because such partitions can be obtained in $r-1$ ways from partitions of $\{1,\ 2,\ \ldots,\ n\} - \{r\}$ into k non-empty subsets such that $\{1,\ 2,\ \ldots,\ r-1\}$ are minimal, by including r in any of the $r-1$ subsets containing a smaller element. □

2.2 Ordinary Generating Functions (ogf)

Theorem 5. *The r-Stirling number of the second kind have the "vertical" generating function:*

$$\sum_n \left\{ {n \atop k} \right\}_r z^n = \begin{cases} \frac{z^k}{(1-rz)(1-(r+1)z)\cdots(1-kz)}\ , & m \geq r \geq 0 \\ 0 & \textit{otherwise}, \end{cases}$$

Corollary 1.

$$\left\{ {n \atop k} \right\}_r = \sum_{p=r}^{k} \frac{(-1)^{k-p} p^{n-r}}{(p-r)!\,(k-p)!}$$

Theorem 6.

$$\left\{ {n \atop k} \right\}_r = \sum_i \binom{n-r}{i} \left\{ {n-p-i \atop k-p} \right\}_{r-p} p^i, \quad 0 < p \leq r$$

Proof. To form a partition with k subsets such that1, 2, $\ldots$, r are subset leaders first choose i numbers to be in the subsets led by 1, 2, $\ldots$, p and construct these subsets; this can be done in $\binom{n-r}{i}$ ways. The remaining $n-p-i$ numbers must form $k-p$ subsets such that $p+1,\ \ldots,\ r$ are subset leaders which can be clone in $\left\{ {n-p-i \atop k-p} \right\}_{r-p}$ ways; using the identity $\left\{ {n \atop m} \right\}_r = r^{n-r}\ ,\ n > r$ and summing for all i completes the proof. □

Corollary 2.

$$\left\{ {n \atop k} \right\}_r = \sum_i \binom{n-r}{i} \left\{ {n-1-i \atop k-1} \right\}_{r-1}$$

Theorem 7.

$$\left\{ {n \atop k} \right\}_r = \sum_i \binom{n-r}{i} \left\{ {i \atop k-r} \right\} r^{n-r-i}$$

Proof. $\left\{ {n \atop k} \right\}_r$ enumerates the k-partitions of n elements such that the first r elements are in distinct subsets. The number of such partitions can be enumerated in the following way. We split $1,\ 2,\ \ \ldots,\ \ r$ into singletons, and we create $k-r$ additional blocks to have k blocks. To fill the $k-r$ blocks, we choose i elements from $\{r+1,\ r+2,\ \ \ldots,\ \ n\}$ into them. This can be done in $\binom{n-r}{i}$ ways. And we can make $\left\{ {i \atop k-r} \right\}$ different $(k-r)$-partitions from these elements. The remaining $(n-r-i)$ elements that are left unchosen among $\{r+1,\ r+2,\ \ \ldots,\ \ n\}$ distributed in to the blocks containing the first r elements, this can be done in $r^{(n-r-i)}$ different ways since each of $(n-r-i)$ element have r places to go. Finally summing on i we get the intended result. □

2.3 Exponential Generating Function (egf)

Theorem 8. *The r-Stirling number of the second kind have the following exponential generating function*

$$\sum_k \left\{ {k+r \atop m+r} \right\}_r \frac{z^k}{k!} = \begin{cases} \frac{1}{m!} e^{rz} (e^z-1)^m & \\ 0, & \textit{otherwise} \end{cases}, \quad m \geq 0$$

Proof. Using the expansions

$$e^{rz} = \sum_k r^k \frac{z^k}{k!}$$

$$\frac{1}{m!}(e^z-1)^m = \sum_k \left\{ {k \atop m} \right\} \frac{z^k}{k!}$$

And *theorem 7* will give us the desired result. □

Theorem 9. *The r-stirling number of the second kind satisfy*

$$\left\{ {n \atop k} \right\}_r = \sum_i \left\{ {p+i \atop p} \right\}_r \left\{ {n-i \atop k} \right\}_{p+1}, \quad r \leq p < n$$

Proof. From *theorem5*

$$\sum_i \left\{ {n+i \atop i} \right\}_r z^i = \frac{1}{(1-rz)\cdots(1-pz)\,(1-(p+1)\,z)\cdots(1-nz)}$$

Expressing this result as a convolution we obtain

$$\left\{ {n+k \atop n} \right\}_r = \sum_i \left\{ {p+i \atop p} \right\}_r \left\{ {n+k-i \atop n} \right\}_{p+1}$$

And the theorem follows by suitable change of variable. □

Chapter 3

r-Bell Numbers and the r-Bell Polynomial

The Bell number B_n counts the partitions of a set with n elements; B_n can be given by the sum

$$\mathrm{B}_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$$

Where $\left\{ {n \atop k} \right\}$ denotes Stirling partition numbers (the Stirling numbers of the second kind).

The n-th Bell polynomial is

$$\mathrm{B}_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k$$

.

These numbers and polynomials have many interesting properties and appear in several combinatorial identities. A more general notion the r-Stirling number of the second kind, $\left\{ {n \atop k} \right\}_r$, with parameters $r \le k \le n$ that enumerates the partitions of a set of n elements into k nonempty, disjoint subsets such that the first r elements are in distinct subsets, was introduced in the previous chapter. Now is the time to talk about the r- Bell numbers.

Definition 6. *The r-Bell number with parameters $r \le n$ counts the number of the partitions of a set with n elements such that the first r elements $1, 2, ..., r$ are in distinct blocks. It is denoted by $\mathrm{B}_{n,r}$ or B_r^n. B_r^n is given by the sum*

$$B_r^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r, n \ge r \ge 0$$

It is observable that, ($\mathrm{B}_n = B_0^n = B_1^n$), the standard Bell number, B_n, is a special case ($r = 0$ *and* $r = 1$) of the r-*Bell* number B_r^n .The r-th Bell polynomial is

$$B_r^n\left(x\right)=\sum_{k=0}^{n}\left\{ n \atop k \right\}_r x^k, n\geq r\geq 0$$

As a result $B_r^n=B_r^n(1)$ and $B_0^n\left(x\right)=B_n(x)$ is the standard Bell polynomial.

3.1 Some elementary facts about the r-Bell polynomials

Theorem 10. *The r-Bell numbers are polynomials in r and can be generate from the the r-th Bell polynomial using the following equality*

$$B_r^{n+r}\left(x\right)=\sum_{k=0}^{n+r}\left(\sum_{i=0}^{n}\binom{n}{i}\left\{ i \atop k \right\} r^{n-i}\right)x^k,\quad n,r\geq 0$$

Proof. B_r^{n+r} is the sum of coefficients of $B_r^{n+r}\left(x\right)$ and by definition $B_r^{n+r}\left(x\right)=\sum_{k=0}^{n+r}\left\{ n+r \atop k \right\}_r x^k$
Applying *theorem7* on the above equality gives us $B_r^{n+r}\left(x\right)=\sum_{k=0}^{n+r}\left(\sum_{i=0}^{n}\binom{n}{i}\left\{ i \atop k \right\} r^{n-i}\right)x^k$
Now set $x=1$ then,

$$\begin{aligned}B_r^{n+r} &= B_r^{n+r}\left(1\right)\\ &=\sum_{k=0}^{n}\left(\sum_{i=0}^{n}\binom{n}{i}\left\{ i \atop k \right\} r^{n-i}\right)\end{aligned}$$

Since $\left\{ i \atop k \right\}=0\ k>i\geq 0$. □

For instance let n=3 then,
$B_r^{3+r}=\sum_{k=0}^{3}\left(\sum_{i=0}^{3}\binom{3}{i}\left\{ i \atop k \right\} r^{3-i}\right)$

$$=r^3+3r^2+3r+1+3r+3+1$$

$$=r^3+3r^2+6r+5.$$

A consequence is that, the r-Bell numbers can be expressed in terms of the standard Bell numbers.

Corollary 3. $B_r^n(x)=\sum_{i=0}^{n-r} r^{n-r-i}\binom{n-r}{i}B_i \qquad n\geq r\geq 0$

The identity in this corollary can be derived by changing the order of the summation in theorem 10.

3.2 Examples and Tables over the r-Bell polynomials and r-Bell numbers

Here the following example will clarify more the meaning of the r- Bell numbers. By definition$B_3^5 = \left\{ {5 \atop 3} \right\}_3 + \left\{ {5 \atop 4} \right\}_3 + \left\{ {5 \atop 5} \right\}_3$

$\left\{ {5 \atop 3} \right\}_3$ Enumerates the partitions of [5] in to three blocks such that the first three elements, 1, 2, and 3 are in distinct blocks.

$\{1,4,5\}\{2\}\{3\}$; $\{1,4\}\{2,5\}\{3\}$; $\{1,4\}\{2\}\{3,5\}$;

$\{1,5\}\{2,4\}\{3\}$; $\{1,5\}\{2\}\{3,4\}$; $\{1\}\{2,4,5\}\{3\}$;

$\{1\}\{2,4\}\{3,5\}$; $\{1\}\{2,5\}\{3,4\}$; $\{1\}\{2\}\{3,4,5\}$

$\left\{ {5 \atop 4} \right\}_3$ Counts the number of partitions of [5] in to four blocks such that the first three elements 1, 2, and 3, belongs to distinct blocks.

$\{1,5\}\{2\}\{3\}\{4\}$; $\{1\}\{2,5\}\{3\}\{4\}$;

$\{1\}\{2\}\{3,5\}\{4\}$; $\{1\}\{2\}\{3\}\{4,5\}$;

$\{1,4\}\{2\}\{3\}\{5\}$; $\{1\}\{2,4\}\{3\}\{5\}$;

$\{1\}\{2\}\{3,4\}\{5\}$

$\left\{ {5 \atop 5} \right\}_3$ Also counts the number of partitions of [5] in to five blocks such that the first three elements 1, 2, and 3, are in different blocks.

$\{1\}\{3\}\{3\}\{4\}\{5\}$

As a result $B_3^5 = \left\{ {5 \atop 3} \right\}_3 + \left\{ {5 \atop 4} \right\}_3 + \left\{ {5 \atop 5} \right\}_3 = 9 + 7 + 1 = 17$
is the total number of ways to partition the set [5] such that the first three elements are in distinct blocks.

r n	0	1	2	3	4	5	6
0	1						
1	1	1					
2	2	2	1				
3	5	5	3	1			
4	15	15	10	4	1		
5	52	52	37	17	5	1	
6	203	203	151	77	26	6	1

Table 3.1 The first few r- Bell numbers

$B_r^r(x) = 1$
$B_r^{r+1}(x) = x + r$
$B_r^{r+2}(x) = x^2 + (1+2r)\,x + r^2$
$B_r^{r+3}(x) = x^3 + (3+3r)\,x^2 + (1+3r+3r^2)x + r^3$
$B_r^{r+4}(x) = x^4 + (6+4r)\,x^3 + (7+12r+6r^2)x^2 + (1+4r+6r^2+4r^3)x + r^4$
$B_r^{r+5}(x) = x^5 + (10+5r)\,x^4 + (25+30r+10r^2)x^3 + (15+35r+30r^2+10r^3)x^2$
$\quad + (1+5r+10r^2+10r^3+5r^4)x + r^5$

Figure 3.1 The first few r- Bell polynomials

3.3 Some New Properties of r-Bell numbers

Theorem 11. *The $r-$Bell numbers B_r^n is given by the formula*

$$B_r^n = \sum_{i=0}^{n}\sum_{k=0}^{n-r}\binom{n-r}{k}\left\{{n-r-k \atop i}\right\}r^k$$

Proof. $\left\{{n \atop k}\right\}_r$ enumerates the k-partitions of the set [n] such that the first r elements are in distinct subsets. The number of such partitions can be enumerated in the following way. First we split the first r elements into singletons, and we can create $(i-r)$ additional blocks to have i blocks. Now choose k elements from the remaining $n-r$ elements this can be done in $\binom{n-r}{k}$ways and these k elements are distributed into any of the r singletons in r^k ways; after doing this we are left with $n-r-k$ elements in hand and we can distribute these elements in$\left\{{n-r-k \atop i}\right\}$ ways into i blocks. Finally, summing over k and i appropriately we get the desired result. □

Corollary 4. *The r- Bell numbers can also be calculated using the following formulas:*

$$B_r^n = \sum_{i=r}^{n}\sum_{k=0}^{n-r}\binom{n-r}{k}\left\{{n-r-k \atop i-r}\right\}r^k$$

OR

$$B_r^n = \sum_{i=0}^{n}\sum_{k=0}^{n-r}\binom{n-r}{k}\left\{{k \atop i}\right\}r^{n-r-k}$$

Theorem 12. *If $n-r=2$, then B_r^n is the sum of the first r odd numbers plus 1.*

$$B_r^n = \left(\sum_{i=0}^{r}(2i+1)\right) + 1$$

Lemma 1. *If $n-r=2$, then*

$$B_r^n = nr + 2$$

Proof. Given that n-r=2, we want to show that$B_r^n = nr + 2$ .From *theorem 11* we know that

$$B_r^n = \sum_{i=0}^{n} \sum_{k=0}^{n-r} \binom{n-r}{k} \left\{ {n-r-k \atop i} \right\} r^k$$

But $n-r=2$ then

$$\begin{aligned} B_r^n &= \sum_{i=0}^{n} \sum_{k=0}^{2} \binom{2}{k} \left\{ {2-k \atop i} \right\} r^k \\ &= r^2 + 2r + 2 \\ &= r(r+2) + 2 \\ &= nr + 2 \end{aligned}$$

□

Proof. (*Theorem12*)
Given that $n-r=2$, we want to show that

$$B_r^n = \left(\sum_{i=0}^{r} (2i+1) \right) + 1$$

Now consider only the summation part: $B_r^n = \sum_{i=0}^{r} (2i+1)$

$$\begin{aligned} B_r^n = \sum_{i=0}^{r} (2i+1) &= 2\sum_{i=0}^{r} i + \sum_{i=0}^{r} 1 + 1 \\ &= 2 \cdot \frac{r(r+1)}{2} + r + 1 \\ &= r^2 + 2r + 1 \\ &= r(r+2) + 1 \\ &= nr + 1 \end{aligned}$$

Adding 1 to the last equation above and using *Lemma1* we get,

$$B_r^n = \left(\sum_{i=0}^{r} (2i+1) \right) + 1$$

□

Theorem 13. *If $r = 2$ or $r = 3$, then B_r^n is given by*

$$B_2^n = \sum k = 0^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} - \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}$$

OR

$$B_3^n = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} - 3 \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + 2 \left\{ \begin{matrix} n-2 \\ k \end{matrix} \right\}$$

respectively.

Proof.

Case 1: ($r = 2$). The standard Bell number B_0^n enumerates the total number of partitions of the set [n] subtracting the number of partitions where the first two elements are in the same subset leaves us with the value for B_2^n is $B_2^n = \sum k = 0^n \left\{ {n \atop k} \right\} - \left\{ {n-1 \atop k} \right\}$

Case 2: (r=3). Using similar combinatorial argument we can show that $B_3^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} - 3 \left\{ {n-1 \atop k} \right\} + 2 \left\{ {n-2 \atop k} \right\}$

□

3.4 Dobinsk's formula for r-Bell numbers

In the first chapter we have seen that the Bell numbers are involved in Dobnisk's formula:

$$B_n = \frac{1}{e} \sum_{s \geq 0} \frac{s^n}{s!} , \qquad (n \geq 0)$$

Our aim is to generalize this identity to our case

Theorem 14. *(Dobinsk's formula) the r-Bell numbers satisfy the identity*

$$B_r^n = \frac{1}{e} \sum_{s \geq r} \frac{s^{n-r}}{(s-r)!} , \qquad (n \geq r \geq 0)$$

Consequently, the r-Bell polynomials are given by

$$B_r^n(x) = \frac{1}{e} \sum_{s \geq r} \frac{s^{n-r}}{(s-r)!} \, x^{s-r}, \qquad (n \geq r \geq 0)$$

Proof. The r-Stirling number for a fixed n and r have the explicit formula

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r = \sum_{p=r}^{k} \frac{(-1)^{k-p} p^{n-r}}{(p-r)! \, (k-p)!}$$

The formula is valid for all positive integer values of n, r and k in particular for $k > n$ and $r > k$. But if $k > n$ and/or $r > k$ then $\left\{ {n \atop k} \right\}_r = 0$. This means the above formula need not tobe told that $\left\{ {7 \atop 9} \right\}_r = 0$ or $\left\{ {9 \atop 5} \right\}_7 = 0$. If we wonder and insert $n = 9$, $r = 7$ and $k = 5$ into the above machine it will work it out, we obtain 0.
If we sum that formula from $k = 1$ *to* M, where M is any number that is greater or equal to n we will have an explicit formula for B_r^n. Lets work it out.

$$B_r^n = \sum_{k=r}^{M} \left\{ {n \atop k} \right\}_r = \sum_{k=r}^{M} \sum_{p=r}^{k} \frac{(-1)^{k-p} p^{n-r}}{(p-r)!\,(k-p)!}$$

$$= \sum_{p=r}^{M} \frac{p^{n-r}}{(p-r)!} \sum_{k=r}^{M} \frac{(-1)^{k-p}}{(k-p)!}$$

$$= \sum_{p=r}^{M} \frac{p^{n-r}}{(p-r)!} \left\{ \sum_{s=0}^{M-p} \frac{(-1)^s}{s!} \right\}$$

$$= \frac{1}{e} \sum_{p=r}^{M} \frac{p^{n-r}}{(p-r)!}$$

But now the number M is arbitrary, except that $M \geq n$. Since the partial sum of the exponential series in the curly braces above is so tempting, let's keep n and r fixed, and let $M \to \infty$. This gives the following notable formula for the r-Bell numbers.

$$B_r^n = \frac{1}{e} \sum_{p=r}^{\infty} \frac{p^{n-r}}{(p-r)!}, \qquad (n \geq r \geq 0)$$

□

This formula for the r-Bell numbers, although it has a certain charm, doesn't lend itself to computation. From it, however, we can derive a generating function for the Bell numbers that is unexpectedly simple and elegant. We will look for the generating function in the form

$$B_r^n(x) = \sum_{n=0}^{\infty} \frac{B_r^n\, x^n}{n!}$$

3.5 Exponential Generating Function

Theorem 15. *The exponential generating function for the r-Bell numbers is*

$$\sum_{n=0}^{\infty} B_r^n \frac{x^n}{n!} = e^{(e^x - 1) + rx}$$

Proof. Consider the Dobinski's formula for $B_r^n = \frac{1}{e}\sum_{p=r}^{\infty} \frac{(p)^{n-r}}{(p-r)!}, \quad n \geq r$

$$\sum_{n=r}^{\infty} B_r^n \frac{x^n}{n!} = \sum_{n=r}^{\infty} \frac{1}{e} \sum_{p=r}^{\infty} \frac{(p)^{n-r}}{(p-r)!} \frac{x^{n-r}}{(n-r)!}$$

$$= \sum_{n=r}^{\infty} \frac{1}{e} \frac{1}{(n-r)!} \sum_{p=r}^{\infty} \frac{(n)^{p-r} x^{p-r}}{(p-r)!}$$

$$= \frac{1}{e} \sum_{n=r}^{\infty} \frac{1}{(n-r)!} \sum_{p=r}^{\infty} \frac{(nx)^{p-r}}{(p-r)!}$$

$$= \frac{1}{e} \sum_{n=r}^{\infty} \frac{1}{(n-r)!} e^{nx}$$

$$= \frac{e^{rx}}{e} \sum_{n=r}^{\infty} \frac{e^{(n-r)x}}{(n-r)!}$$

$$= \frac{e^{rx}}{e} e^{e^x} = e^{e^x + rx - 1}$$

□

3.6 Triangle scheme for calculating r-Bell numbers

In the first chapter, we have discussed about triangular scheme for standard Bell number. Now, it would be appropriate to generalize such nice scheme to the r-Bell numbers. The rules are given as follow:

Step 1: Start with the 0th column copy and place the standard Bell numbers from the Bell triangle down the column.

Step 2: Start a new column copy the standard Bell number from the Bell triangle, replace the 0th Bell number(B_0) by zero, and place them down the first column.

Step 3: Start a new column put 0 in the first k positions down, then put 1 in the (k+1)th position. Where k is the corresponding column number.

Step 4: The next position (downward) in column k is (k+2). We fill this position by taking difference of the number to the left of it and k (the same k in step 3) times the number (diagonally up and to the left)of the number we are calculating. The next postition is filled in similar manner.

Step 5: Repeat step 3 and 4.

For example, the 0^{th} column is made by copying and placing the standard Bell numbers from the Bell triangle down the column. Similarly first column is made according to step2. The next column is the second column to construct this column start by putting zero in the first two position down the column. We now have a structure like this:

r n	0	1	2
0	1	0	0
1	1	1	0
2	2	2	1
3	5	5	x
4	15	15	
$\vdots$	$\vdots$	$\vdots$	

The value of x here is determind by subtracting the number (diagonaly up and to the left of x), 2, multiplied by 1 from the number to the left of x (**??**). So, $5 - 2 \cdot 1$ =3, that means the value of x is 3.

r / n	0	1	2
0	1	0	0
1	1	1	0
2	2	2	1
3	5	5	3
4	15	15	y
$\vdots$	$\vdots$	$\vdots$	

Similarly we can find the value of y, doing so we get that $y = 10$.
Here is a list of the first few r-Bell numbers:

r / n	0	1	2	3	4	5	6
0	1						
1	1	1					
2	2	2	1				
3	5	5	3	1			
4	15	15	10	4	1		
5	52	52	37	17	5	1	
6	203	203	151	77	26	6	1

Table 3.2 The first few r- Bell numbers

As we see from the table what makes it different from the standard Bell table is that, here every entry of the table represents a specific r-Bell number for a particular value of r while in the bell triangle the Bell numbers are found only on the zero-th column. But here the Bell numbers are also found in the first column other than the zero-th column.

3.7 Hankel Matrix

Hankel matrices of integer sequences and their determinants have been studied in several papers by Ehrenborg [16] and Peart and Woan [13].
The Hankel matrix H of the integer sequence $\{a_0,\ a_1, a_2, a_3, \cdots\}$ is the infinite matrix

$$H = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

With elements $h_{i,j} = a_{i+j-1}$. The Hankel martrix H_n of order n of A is the upper-left $(n+1) \times (n+1)$ Submartix of H, and h_n , the *Hankel determinant of order n* of A, is the determinant of the corresponding Hankel matrix of order n, h_n= det(H_n).

For example, the Hankel matrix of order 3 of the Bell sequence $1,\ 1,\ 2,\ 5,\ 15,\ 52, \cdots$ is

$$H_3 = \begin{pmatrix} 1 & 1 & 2 & 5 \\ 1 & 2 & 5 & 15 \\ 2 & 5 & 15 & 52 \\ 5 & 15 & 52 & 203 \end{pmatrix}$$

with 3^{rd} order Hankel determinant $h_n = 12$.
Given an integer sequence $A = \{a_0, a_1, a_2, a_3, \cdots\}$ the sequence $\{h_n\} = \{h_1, h_2, h_3, \cdots\}$ of Hankel determinants of A is called the *Hankel transform of A* [9], a term first introduced by the author in sequence A055878 of the On-Line Encyclopedia of Integer Sequences (EIS) [15].
For example, the Hankel matrix of order 3 of the sequence of Bell numbers $\{1,\ 1,\ 2,\ 5,\ 15,\ 52, \cdots\}$ is

$$H_3 = \begin{pmatrix} 1 & 1 & 2 & 5 \\ 1 & 2 & 5 & 15 \\ 2 & 5 & 15 & 52 \\ 5 & 15 & 52 & 203 \end{pmatrix}$$

and the determinants of orders 0 through 4 give the Hankel transform $\{1,\ 1,\ 2,\ 12, 288, \cdots\}$
Aigner [12] showed that the Hankel transform of the Bell numbers is (1!, 1!2!, 1!2!3!, . . .), that is, for any fixed n,

$$\begin{vmatrix} B_1 & B_2 & B_3 & \cdots & B_n \\ B_2 & B_3 & B_4 & \cdots & B_{n+1} \\ B_3 & B_4 & B_5 & \cdots & B_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \\ B_n & B_{n+1} & B_{n+2} & \cdots & B_{2n} \end{vmatrix} = \prod_{i=0}^{n} i!$$

We can determine the Hankel transform of r-Bell numbers easily for $r \leq n$.

Theorem 16. *The r-Bell number have the Hankel transform*

$$\begin{vmatrix} B_r^0 & B_r^1 & B_r^2 & \cdots & B_r^n \\ B_r^1 & B_r^2 & B_r^3 & \cdots & B_r^{n+1} \\ B_r^2 & B_{3,r} & B_r^4 & \cdots & B_r^{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_r^n & B_r^{n+1} & B_r^{n+2} & \cdots & B_r^{2n} \end{vmatrix} = \begin{cases} 0 & if\ r > n \\ 1 & if\ r = n\ \cap\ r\,(mod4) = 0\ OR\ 3 \\ -1 & if\ r = n\ \cap r\,(mod4) = 1\ OR\ 2 \end{cases}$$

Proof. If $r > n$, then we have a row and a column of all zero entries in the Hankel matrix. Calculating the determinant using this row or column we get the desired result that is the determinant is zero. Now suppose $r = n$, then we have a matrix of size (r+1) which is lower triangular with respect to the minor diagonal with all the minor diagonal entries one .

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & B_r^{n+1} \\ 0 & 0 & 0 & 1 & B_r^{n+1} & B_r^{n+2} \\ 0 & 0 & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & B_r^{n+1} & \iddots & \iddots & B_r^{2n-1} \\ 1 & B_r^{n+1} & B_r^{n+2} & \cdots & \cdots & B_r^{2n} \end{pmatrix}$$

Let us do the determinant for such matrices. The first row has only one nonzero entry; using this row to calculate the determinant makes computation simple. And doing the determinant we get

$$(-1)(-1)^4(-1)^5\cdots(-1)^{r+2}(-1)^{r+1}(-1)^{r+2} = -\prod_{i=4}^{r+2}(-1)^i$$

Now we know that $(-1)^2 = 1$ and dividing by 1 does not change the result. Thus dividing each of the factor in the product by $(-1)^2$ we end up with

$$(-1)(-1)^2\cdots(-1)^{r-2}(-1)^{r-1}(-1)^r = \prod_{i=1}^{r}(-1)^i$$

This gives us 1 if the exponents sum up to an even number otherwise we get(-1).
Case 1: suppose $r = n$ and $r\,(mod 4) = 0$, then $r = 4k$ for some $k > 0$. And we know that

$$\begin{aligned} 1+2+\cdots+(r-1)+r &= \frac{r\,(r+1)}{2} \quad then \\ &= \frac{4k\,(4k+1)}{2} \\ &= 2\left(4k^2+k\right) \end{aligned}$$

Which is an even number, so we conclude that $\prod_{i=1}^{r}(-1)^i = 1$.
Case 2: suppose $r = n$ and $r\,(mod 4) = 3$, then $r = 4k+3$ for some $k > 0$. And we know that

$$\begin{aligned} 1+2+\cdots+(r-1)+r &= \frac{r\,(r+1)}{2} \quad , \ then \ , \\ &= \frac{(4k+3)\,(4k+4)}{2} \\ &= 2(k+1)\,(4k+3) \end{aligned}$$

Which is also an even number, so, we conclude that $\prod_{i=1}^{r}(-1)^i = 1$.
Case 3: suppose $r = n$ and $r\,(mod 4) = 1$, then $r = 4k+1$ for some $k > 0$. And we know that

$$1+2+\cdots+(r-1)+r = \frac{r\,(r+1)}{2} \qquad then$$

$$= \frac{(4k+1)(4k+2)}{2}$$
$$= (2k+1)(4k+1)$$

Which is an odd number, so, we conclude that $\prod_{i=1}^{r} (-1)^{i} = -1$.
Case 4: suppose $r = n$ and $r\,(mod 4) = 2$, then $r = 4k + 2$ for some $k > 0$.
And we know that

$$1 + 2 + \cdots + (r-1) + r = \frac{r(r+1)}{2} \qquad then$$
$$= \frac{(4k+2)(4k+3)}{2}$$
$$= (2k+1)(4k+3)$$

Which is again an odd number, so, we conclude that $\prod_{i=1}^{r} (-1)^{i} = -1$. $\square$

Chapter 4

Graphical r-Bell Numbers

4.1 Labeled Graph

Definition 7. *A graph labeling is the assignment of labels, usually represented by integers, to the edges or vertices, or both, of a graph.*

Formally, given a graph G, a vertex labeling is a function of $V(G)$ to a set of *labels*. A graph with such a function defined is called a vertex-labeled graph. Likewise, an edge labeling is a function mapping edges of G to a set of "labels". In this case, G is called an edge-labeled graph. When used without qualification, the term labeled graph generally refers to a vertex-labeled graph with all labels distinct. Such a graph may equivalently be labeled by the consecutive integers $\{1, ..., n\}$, where n is the number of vertices in the graph. For the sake of this paper, the vertices in any graph will be given labels by consecutive integers. To make life easier let us see an example of such labeling using a graph with five vertices using the figure below as we can see there are five vertices and they are labeled $\{1, ..., 5\}$.

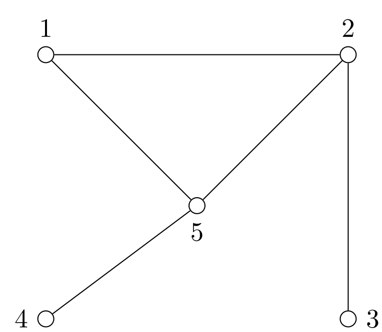

Figure 2. A graph with five vertices

Definition 8. *A split graph is a graph in which the vertices can be partitioned into a clique and an independent set. And a complete split graph G is a split graph in which every possible edge that could connect vertices in different parts is part of the graph.*

Note that, from the definition, split graph is closed under complementation.

Example 3. *Consider the following graph which has five vertices*

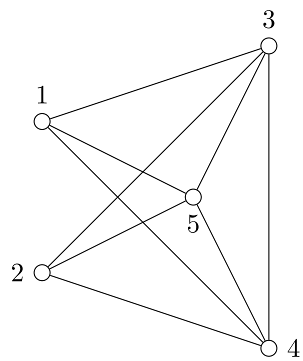

Figure 3. A complete split graph(KS(2,3))

This graph is a complete split graph (KS(2,3)) over five vertices, the first two vertices (according to the label) makes the empty graph over two vertices(E_2) where as the rest three vertices, 3,4, and 5 makes the complete graph(K_3).

4.2 Bell and Stirling Numbers for Graphs

Definition 9. *For a simple graph G = (V, E), a partition of the full vertex set V of G is called stable if each of its blocks is an independent set of G. The (graphical) Bell number $\hat{B}(G)$ of a simple graph G is the number of such stable vertex partitions [17].*

This numbers are first introduced by Bryce Duncan and Rhodes Peele [17]. The number of these partitions with k blocks is the (graphical) Stirling number $\hat{S}$(G, k) of G. As a running example, and to fix ideas, suppose G is the graph with V = {1, 2, 3, 4, 5} and E = 12, 23, 34, 45, 51, 25. Then $\hat{B}\,(G) = 8$, the list of stable partitions being:
{{1, 3}, {2, 4}, {5} }, { {1, 4}, {2}, {3, 5} }, {{1}, {2, 4}, {3, 5} }
{{1, 3}, {2}, {4}, {5}}, {{1, 4}, {2}, {3}, {5}}, {{1}, {2, 4}, {3}, {5}},

{{1}, {2}, {3, 5}, {4}}, {{1}, {2}, {3}, {4}, {5}}

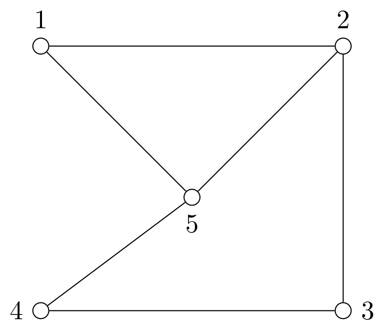

Figure 4. Graph of G

As suggested by our running example above, the stable partitions of G can be enumerated by listing them in groups according to the number of blocks they contain. Accordingly, for any k in the range $c(G) \leq k \leq |V|$, we define the (graphical) *Stirling number* $\hat{S}(G, k)$ to be the number of stable partitions of G consisting of exactly k blocks. (Here c(G) is the chromatic number of G). For instance if we let k=3 for the above simple graph G, we obtain $\hat{S}(G, k) = 3$.

Definition 10. *The graphical r-Bell number* $\hat{B}_r(G^\pi)$ *of a simple labeled graph G of order n with parameters* $r \leq n$*,where '*π*' is a labeling, counts the number of stable partitions of the full vertex set such that the first r vertices* $1, 2, ..., r$ *are in distinct blocks. If we take the sum over all labeling, '*π*', we get the general r-Bell number* $\hat{B}_r(G)$ *for G(unlabeled).*

$$\hat{B}_r(G) = \sum_{\pi \ a \ labeling} \hat{B}_r(G^\pi)$$

Here 'n' is the cardinality of the full vertex set $V(G)$. similarly for any 'r' and 'k' in the range $c(G) \leq r \leq k \leq n = |V|$,we can define the graphical *r-Stirling number* $\hat{S}_r(G, k)$ to be the number of stable partitions of G consisting of exactly k blocks and the first r vertices $1, 2, ..., r$ are in distinct blocks.
The work done by Bryce Duncan and Rhodes Peele [17] focused on $\hat{B}(G)$ for various graphs G. In particular, they considered trees, cycles, stars, and complements of paths. Here our focus will be on $\hat{B}_r(G)$ for complete split and complete n-partite graph. Where a **complete n-partite graph** is a graph whose vertices can be partitioned into k subsets, $V_1, V_2, V_3, \ldots, V_n$,(n being positive integer), such that no edge has both endpoints in the same subset and every possible edge that could connect vertices in different subsets is part of the graph. A complete n-partite graph with partitions of size $|V_j| = m_j$, is denoted by $K_{m_1, m_2, \ldots, m_n}$.

4.3 $\widehat{\mathrm{B}}_{\mathrm{r}}(\mathrm{G})$ for Particular Graph G

Let $n \in \mathbb{N}$. Denote by K_n the complete graph on n vertices, and $\overline{K}_n$ its complement, the graph with n vertices and no edges. Then it is immediate that

$$\hat{B}\left(K_n\right) = 1 \textit{ and } \hat{B}\left(\overline{K}_n\right) = B_n.$$

4.3.1 $\widehat{\mathrm{B}}_{\mathrm{r}}(\mathrm{G})$ for Complete Split Graph G

Theorem 17. *Let $n, m \in \mathbb{N}$ where, n is the size of the independent set and m is the size of the clique. Denote by $(KS)_{n,m}$ the complete split graph on $(n+m)$ vertices, and $\left(\overline{KS}\right)_{n,m}$ its complement, the graph with $(n+m)$ vertices and edges that are not in $(KS)_{n,m}$. Then*

$$\hat{B}\left((KS)_{n,\ m}\right) = B_n \quad and \quad \hat{B}\left(\left(\overline{KS}\right)_{n,m}\right) = B_n^{m+n}$$

Proof. Let the vertices of $(\mathrm{KS})_{\mathrm{n,\ m}}$ be labeled $\{1,\ 2,\ 3,\ \ldots, \mathrm{n},\ \mathrm{n}+1,\ \ldots,\ \mathrm{n}+\mathrm{m}\}$. Then consider the vertices in the independent set are labeled by $\{1,\ 2,\ 3,\ \ldots, \mathrm{n}\}$ and the vertices that make the clique are labeled accordingly with $\{\ \mathrm{n}+1,\ \ \mathrm{n}+2, \ldots,\ \mathrm{n}+\mathrm{m}\}$. Here we have an edgeless graph $\mathrm{E_n}= \mathrm{G}[\{1,\ 2,\ 3,\ \ldots, \mathrm{n}\}\ ]$ that is induced by the independent set and we knew that $\widehat{\mathrm{B}}\left(\overline{\mathrm{K}}_{\mathrm{n}}\right) = \mathrm{B_n}$. So $\widehat{\mathrm{B}}(\mathrm{E_n}) = \mathrm{B_n}$. After doing this we are now left with the clique and by its nature it induces a complete graph $\mathrm{K_m}$ where $\mathrm{m} = |\ \{\ \mathrm{n}+1,\ \ \mathrm{n}+2, \ldots,\ \mathrm{n}+\mathrm{m}\}|$ which leads to the trivial case, $\widehat{\mathrm{B}}\left(\mathrm{K_m}\right) = 1$. Finally, multiplying $\widehat{\mathrm{B}}\left(\mathrm{K_m}\right)$ by $\widehat{\mathrm{B}}\left(\overline{\mathrm{K}}_{\mathrm{n}}\right)$ gives us $\widehat{\mathrm{B}}\left((\mathrm{KS})_{\mathrm{n,\ m}}\right) = \mathrm{B_n}$. The complement case the new clique is the previous independent set and the independent set is the previous clique the graph we obtain is still split but not complete. First distribute the vertices in the new clique into n singletons. Then from the m vertices of the new independent set choose i vertices, this can be done in $\binom{m}{i}$ ways. Now we can put these i vertices into any of the n singletons in n^i ways, partitition the remaining $(\mathrm{m}-\mathrm{i})$ vertices into additional k parts, this can be done in $\left\{ {m-i \atop k} \right\}$ ways. Finally summing up over i then over k and using theorem2.5 we get that

$$B\ {}_{n}^{n+m} = \sum_{k=0}^{m}\sum_{i=0}^{m}\binom{m}{i}\left\{ {m-i \atop k} \right\} n^i$$

□

Question 1. *Earlier in this chapter we have defined a labeled graph, now let us consider a labeled complete split graph. If we use $[n+m]$, $n, m \in \mathbb{N}$ as set of labels, then in how many ways can we label $(\mathrm{KS})_{\mathrm{n,\ m}}$?*

Generally speaking there are $(n+m)!$ possible ways of labeling. But, this machinery over counts (counts redundantly) what is really out there. We have equal number of vertices and labels (n+m). Recall that n is the size of the independent set and m is the size of the clique in $(\mathrm{KS})_{\mathrm{n,\ m}}$. Now we can choose n labels in $\binom{n+m}{n}$ ways out of the given label set, which will be used for labeling the n vertices in the independent set. The chosen n labels will be assigned to the n vertices in only one way since each vertex is adjacent with every vertex in the clique. There are m labels left and m vertices which are all in the clique that need to be labeled. There is only one to assign the m labels left to the m vertices, the reason is that every pair of vertices, whether both vertices are from the clique or one from the clique and the other from the independent set are always adjacent.

So, the honest and right answer for the above question is $\binom{n+m}{n}$.

Question 2. *Consider a randomly labeled complete split graph* $(\mathrm{KS})_{\mathrm{n,\ m}}$ *, where the labels come from* $[n+m]$*. Suppose we want the first 'r' vertices (vertices labeled with members of* $[r]$ *,* $r \leq n \in \mathbb{N}$*) to be in distinct blocks(partitions), in how many ways can we accomplish this task? Will it be the same as* $\widehat{\mathrm{B}}\left((\mathrm{KS})_{\mathrm{n,\ m}}\right)$*?*

Theorem 18. *Let* $S \subseteq [r]$ *,* $(s = |S|) \leq r \leq n, m \in \mathbb{N}$ *and consider* $U, V \subseteq [n+m]$ *as a label set for the independent set and the clique of the labeled complete split graph* $G = (KS)_{n,m}$ *respectively. Then* $\hat{B}_r(G^\pi)$*, (where* π *is labeling randomly chosen from* $\binom{n+m}{n}$ *different kinds of labeling) is given by*

$$\hat{B}_r\left((\mathrm{KS})^\pi_{\mathrm{n,m}}\right) = B^n_{r-s} \quad , \quad if \ \ ([r]-S) \subseteq U \ and \ S \subseteq V$$

Proof. We will do the proof by considering three cases.
<u>Case 1:</u> Assume that $[r] \subseteq V$. Then the r vertices we are interested on are part of a clique, we can partition the element of the clique in only one possible way since every pair vertices are adjacent. Hence $\hat{B}_r\left((\mathrm{KS})_{\mathrm{n,\ m}}\right) = \hat{B}\left((\mathrm{KS})_{\mathrm{n,\ m}}\right) = \mathrm{B}_{\mathrm{n}}$.
<u>Case2:</u> Suppose$[r] \subseteq U$ and relabeling by $[n = |U|]$ leaving the vertices labeled by $[r]$ intact. Then whole thing is reduced to finding r-Bell number of the set $[n]$ which is given by B^n_r .
<u>Case3:</u> Here in this case, the element we are interested are in both the independent set and the clique. Those vertices in the clique are already put in different blocks by definition. Letting $r = r - s$ and using case2 we get $\hat{B}_r\left((\mathrm{KS})_{\mathrm{n,\ m}}\right) = B^n_{r-s}$. □

4.3.2 $\widehat{\mathbf{B}}_{\mathbf{r}}(\mathbf{G})$ for Complete n-Partite Graph G

Theorem 19. *Let* G *be a bipartite graph and* $n, m \in \mathbb{N}$*, where each number represents the size of the each parts respectively. Denote by* $K_{n,m}$ *the complete bipartite graph on* $(n+m)$ *vertices, and* $\overline{K}_{n,m}$ *its complement, the graph with* $(n+m)$ *vertices and edges that are not in* $K_{n,m}$*. Then*

$$\hat{B}\left(K_{n,\ m}\right) = B_n B_m \quad And$$

$$\hat{B}(\overline{K}_{n,\ m}) = \begin{cases} \sum_{i=0}^{n} \left(\binom{n}{i} \prod_{k=(m-n+i+1)}^{m} k \right) & if \quad n \le m \\ \sum_{i=0}^{m} \left(\binom{m}{i} \prod_{k=(n-m+i+1)}^{n} k \right) & if \quad n \ge m \end{cases}$$

Proof. Let the vertices of $K_{n,\ m}$ be labeled $\{1,\ 2,\ 3,\ \ldots, \mathrm{n},\ \mathrm{n}+1,\ \ldots,\ \mathrm{n}+\mathrm{m}\}$. Then this label set it is divided in to two subsets of size n, m (recall that n and m are the size of the each parts in $K_{n,\ m}$ respectively) we can partition each subset in B_n and B_m ways according to their size, this together with multiplication principle gives us what we want. Hence $\hat{B}\left(K_{n,\ m}\right) = B_n B_m$ □

In sub-section 4.3.1, we have showed that there are $\binom{\mathrm{n+m}}{\mathrm{n}}$ ways of labeling $(\mathrm{KS})_{\mathrm{n,\ m}}$. The same is true for $K_{n,\ m}$ since they have equal number of vertices and share important properties.

Theorem 20. *Let $S \subseteq [r]$, $(s = |S|) \le r \le n, m \in \mathbb{N}$ and consider $U, V \subseteq [n+m]$ as a label set for the parts $V_1$and V_2 of the labeled complete bipartite graph $G = K_{n,m}$ respectively. Where $|V_1| = n, |V_2| = m$. Then $\hat{B}_r(G^{\pi})$,(where π is a random labeling) is given by*

$$\hat{B}_r\left(\mathrm{K}_{\mathrm{n,\ m}}^{\pi}\right) = B_{r-s}^{n} \quad B_s^{m} \quad , \quad if \quad ([r] - S) \subseteq U \ and \ S \subseteq V$$

Proof.
<u>**Case1**:</u> Suppose$[r] \subseteq U$. Then vertices we want to work on all belong to V_1 we can partition V_1 as we want in $B_r^{n=|V_1|}$ ways and V_2 can be partitioned in b_m ways multiplying the two gives us $\hat{B}_r\left(\mathrm{K}_{\mathrm{n,\ m}}^{\pi}\right) = B_r^n \mathrm{B_m}$.
<u>**Case2**:</u> Suppose$[r] \subseteq V$. Then vertices we want to work on all belong to V_2 we can partition V_2 as we want in $B_r^{m=|V_2|}$ ways and V_1 can be partitioned in B_n ways multiplying the two gives us $\hat{B}_r\left(\mathrm{K}_{\mathrm{n,\ m}}^{\pi}\right) = B_r^m \mathrm{B_n}$.
<u>**Case3:**</u> Here assume that $([r] - S) \subseteq U$, this means some of the element that we want to manipulate are inV_1 ,so, we can partition V_1 in $B_{r-s}^{n=|V_1|}$ ways. And the rest of the element we want to manipulate belongs to V_2 and we can also partition V_2 in $B_s^{m=|V_2|}$ ways. Thus by multiplying the two we end up with the desired result $\hat{B}_r\left(\mathrm{K}_{\mathrm{n,\ m}}^{\pi}\right) = B_{r-s}^{n} \quad B_s^{m}$, which concludes the proof. □

Example 4. *Let* $\mathrm{G} = \mathrm{K}_{4,4}$ *then find* $\hat{B}_2(\mathrm{G})$ *where G is unlabeled.*

Solution: Here $n = m = 4$ *and* $r = 2$, now

1. If both of the vertices we are working belong to a single part, then $\hat{B}_2(\mathrm{G}) = \mathrm{B}_4 \mathrm{B}_2^4 = 15 * 10 = 150$.

2. If they belongs to different parts, then $\hat{B}_2(\mathrm{G}) = \mathrm{B}_1^4 \mathrm{B}_1^4 = 15 * 15 = 225$.

There are $\frac{\binom{4+4}{4}}{2} = \frac{70}{2} = 35$, different ways of labeling G, here we divide by 2 because $n = m$ and order is not important. Out of these 15 are of the case both vertices 1and 2 are in the same partition multiplying this by 150 gives us 2250, and the rest 20 labeling take vertex 1 and 2 in different parts multiply this by 225 to get 4500 . Finally taking the sum of the two (2250, 4500) we get

$$\hat{B}_2\left(\mathrm{G}\right) = 2250 + 4500 = 6750$$

Here what we need to know is that we merely considered bipartite case just to make the idea touch the ground and have a firm grasp of it. But, it is generally true for n-partite graphs we will see this in the next theorem.

Theorem 21. *Let $G = K_{m_1,\ m_2,\ \dots,m_k}$ be a complete k-partite graph and $S_j \subseteq [r]\,,\ (s_j = |S_j| \leq r \leq m_j)$ where m_j is the number of vertices in the j^{th} part and consider $U_j \subseteq [n]\,, |U_j| = m_j$ as a label set for the j^{th} part of G. Then $\hat{B}_r\left(\mathrm{G}\right)$ is given by*

$$\hat{B}_r\left(\mathrm{G}\right) = \prod_{\mathrm{i}=1}^{\mathrm{k}} \mathrm{B}_{\mathrm{s_i}}^{\mathrm{n_i}}\ ,\quad \text{if } \mathrm{S_i} \subseteq U_j$$

Proof. We will do the proof by dividing in to two cases, when all of the r elements (vertices) are in one part and when they are spread in two and more parts.
Case1: Suppose $[r] \subseteq U_j$ for some $j \in \{1,\ 2,\ \cdots,\ k\}$, then all of the vertices we want to control belongs to a single part. We can partition U_j as we want in $\mathrm{B}_{\mathrm{r}}^{\mathrm{n_j}}$ possible ways. And we can partition $U_{i\neq j}$ in $b_{\mathrm{n_i}}$ possible ways. Finally multiplying out over all possible values i with $\mathrm{B}_{\mathrm{r}}^{\mathrm{n_j}}$ we end up getting what we want.
Case2: Consider the collection of subsets $\left\{S_1,\ S_{2,\ \dots,}\ S_k\right\}$ of $[r]$, such that $\bigcup_{j=1}^{k} S_j = [r]$, where $|S_j| = s_j$ and $S_i \cap S_j = \emptyset, \forall i, j \in \{1,\ 2,\ \cdots,\ k\}$. And suppose $S_j \subseteq U_j$, this means that the vertices we want to manipulate are spread among different parts. If we take a particular label set U_j it contains s_j elements out of the r elements we are manipulating and we know that U_j partitioned in $\mathrm{B}_{s_j}^{\mathrm{n_j}}$ ways taking a product over all i we get $\hat{B}_r\left(\mathrm{G}\right) = \prod_{\mathrm{i}=1}^{\mathrm{k}} \mathrm{B}_{\mathrm{s_i}}^{\mathrm{n_i}}$.

□

Chapter 5

Conclusion

In our day to day activity we often face problems such as an arrangement of objects in certain way, the distribution of items according to a certain specification, partition of things under a certain condition and so on. In order to solve the problem regarding to partition of a set, the r-Bell number takes the key position in enumerating the number of partitions of a given set under restricted condition. For this reason and other studying their nature and properties will add up a handy tool to our toolbox. As the same time if we look back at the r-Bell number of graphs, a lot can be said about them, in relation with the following types of graphs, path graphs, cycles, general bipartite graph as whole and their compliment and any random graphs. Hopefully much will be said in the future regarding this topic.

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