College of Natural and Computational Sciences (CNCS)

Department of Mathematics

Further More on the Theory of BH-Lattices

By

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Declaration

I, Mekonnen Mamo Elema, with student ID. number $GSR/4953/12$, hereby declare that this dissertation is my own work and that it has not been previously submitted for assessment or completion of any post-graduate qualification to another university or for another qualification.

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Certificate

I hereby certify that I have read this dissertation prepared by Mekonnen Mamo Elema under my supervision and recommend thatit should be accepted as fulfilling the dissertation requirement.

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Abstract

In this dissertation, we study further properties of BH-lattices, which is a subclass of BH-monoids. We furnish certain examples of BH-monoids that are not BH-lattices. We give a characterization of BH-lattice in terms of bounded BH-lattice and commutative l-group. Also, we prove that every BH-lattice is a direct product of Heyting algebra and commutative l-group under certain conditions. Further, we obtain the decomposition theorem in terms of Boolean algebra and a commutative l-group.

Moreover, we introduce the concept of filters in BH-lattices and furnish certain examples. We obtain certain basic properties of BH-lattices. Also, we characterize the filter generated by a given subset of a BH-lattice. Besides these, we prove that the set of all filters with set inclusion forms a Heyting algebra

Furthermore, we define the congruence relation on BH-lattices and obtain a one-to-one correspondence between the set of congruences and the filter of BH-lattices, which gives more insight for constructing quotient algebra. Also, we prove that the quotient algebra is a BH-lattice.

Finally, we introduce different types of filters in BH-lattices, furnish examples, and prove certain properties of each type of filter, their interrelation, and state some open problems for further study in the area.

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Mekonnen Mamo Elema

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List of Symbols and Abbreviations

Symbol	Explanation	Symbol	Explanation
$\in$	- an element of	a'	- complement of element a
$\notin$	- not an element of	a^*	- pseudo-complement of element a
$\wedge$	- meet	x^-	$e \to x$
$\vee$	- join	Po-group	- partially ordered group
$\leq$	- partial order	l-group	- lattice-ordered groups
$<$	- strict partial order	BH	- Brouwer-Heyting
$\subseteq$	- subset	Drl	- Dually residuated lattice ordered
$\subset$	- proper subset	P(X)	- power set of set X
$\cap$	- Intersection (set theoretic)	$\mathbb{F}$	- the set of all filters of L
$\cup$	- union (set theoretic)	$\mathbb{F}_P(L)$	- the set of all prime filters of L
$\mathbb{R}$	- the set of real numbers		
$\mathbb{Q}$	- the set of rational numbers		
$\mathbb{Z}$	- the set of integers		
$\mathbb{N}$	- the set of natural numbers		
$[a)$	$= \{x : a \leq x\}$		
$(a]$	$= \{x : x \leq a\}$		
$[b, c]$	$= \{x : b \leq x \leq c\}$		

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Introduction

The ring theoretic generalizations of Boolean rings and lattice theoretic generalizations of Boolean algebras have been studied by many authors. Some of the lattice theoretic generalizations of Boolean algebras are Distributive lattices, Brouwerain algebras, Heyting algebras, Semi Heyting algebras and residuated lattices.

It is Ward and Dilworth [31] who initiated the study of residuated lattices and, as a result, a study on lattice-ordered semigroups with residuation as an operation has been introduced by Swamy [27] under the name Drl-semigroups. He did extensive study in this class [28, 29]. Drl-semigroups are a common abstraction of Brouwerian algebras (of [6]) and commutative lattice ordered groups in the sense that every Drl-semi group can be represented as a direct product of Browerain algebra and a commutative l-group.

Different scholars introduced and investigated certain properties of filters in various classes of algebraic systems (both in fuzzy and crisp algebraic systems). Michiro K and Wieslaw A. D [16] investigated fundamental properties of some types of filters (Boolean, positive implicative, implicative and fantastic filters) in BL-algebra - which were introduced by M. Haveshki et al. [15]. Abdullah M. Al. Roq et al. [2] introduced the notion of normal uni-soft filters in R_o-algebra and investigated related properties. A. Badawy [1] introduced and characterized De Morgan filters of decomposable MS-algebra. M. Sambasiva Rao [17, 18] introduced the notion of β-filters and e-filters of MS-algebra and A. E. Badawy [3] studied the concept of filters of p-algebras with respect to a closure operator.

After Zadeh L. A [33] introduced the theory of fuzzy sets and fuzzy relations, various scholars developed the concept of fuzzy filters in various fuzzy algebraic systems.

Berhanu Assaye Alaba and Teferi Getachew Alemayehu, [4, 5], introduced and characterized β- fuzzy filters and e- fuzzy filters of MS-algebra and their characterizations.

Yiquan Zhu and Yang Xu [32] developed the theory of general residuated lattices, and they also introduced the concept of regular filters and fuzzy regular filters in general residuated lattices and derived some of their characterizations.

Two algebras, Browerian and Heyting, are generalizations to Boolean algebra (lattice) and dual to each other. There is confusion in the literature regarding the nomenclature of these two algebras. Brouwerian algebra defined in [19] by Nordhaus, EA and Leolapidus is called as Heyting algebra by Birkhoff G [6].

To clear the confusion between the two algebras, recently Swamy [26] introduced the notion of Brower-Heyting monoids (for short, BH-monoids) as a general class containing both Browerian algebra and its dual Heyting algebra. He also observed that commutative po-groups and dual Drl-semigroups are examples of BH-monoids. Further, he obtained decomposition theorems for both BH-Monoids and for BH-lattices.

Motivated by the above-mentioned ideas, here we study further properties of BH-lattices, which are a subclass of BH-monoids. We furnish certain examples of BH-monoids that are not BH-lattices. Moreover, we introduce the concept of filters in BH-lattices and furnish certain examples. Characterize the filter generated by a given subset of a BH-lattice. Furthermore, we define congruence relations on BH-lattices and prove the one - to - one correspondence theorem between the set of congruence (Con L) and the st of all filters of BH-lattice L ($\mathbb{F}$).

Finally, we introduce different types of filters in BH-lattices, furnish examples, and prove certain properties of each type of filter, their interrelation, and state some open problems for further study in the area.

This thesis is divided into five chapters. The first chapter is preliminaries. To avoid the confusion mentioned above, we recall the two definitions in the preliminaries and also we recall all the existing literature on BH-monoids and BH-lattices. The second chapter is further properties of BH-lattices. In this chapter, we obtain further properties on

BH-lattices which we use in the sequel. The third chapter is devoted to decomposition theorems. In this we prove every BH-lattice can be represented as a direct product of Heyting algebra and a commutative l-group with certain conditions. The fourth chapter is on filters and congruence relations of BH-lattices. In this chapter we introduced the concept of filters and congruence relations, give examples and prove certain properties of them. The last chapter is about some types of filters on BH-lattices. Here we introduce prime and maximal filters, deductive filters, implicative and positive implicative filters and Boolean filters. We also prove certain properties of each filter and also their relations.

Chapter 1

Preliminaries

This chapter is devoted to some basic definitions and results concerning certain special types of lattices. For the basic notation and results, we refer to different standard sources, which are cited at each point where required.

1.1 Basic Concepts of Lattice

Definition 1.1.1. *[6],[8],[11],[13],[24] A set P equipped with a binary relation $\leq$ is called a partially ordered set (also called a poset) if it satisfying the following three properties $\forall x, y, z \in P$.*

1. *$x \leq x$ (reflexive property).*
2. *If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetric property).*
3. *If $x \leq y$ and $y \leq z$,then $x \leq z$ (transitive property).*

Definition 1.1.2. *[6],[8],[11],[13],[24] A partial order $\leq$ on P is a total order, if for $x, y \in P$, $x \leq y$ or $y \leq x$. If the order on a poset is total order, then it is called a totally ordered set, or a linearly ordered set, or simply a chain.*

Notation: If the set P together with binary relation $\leq$ is a Poset, then it is simply denoted by $(P, \leq)$.

Note: In a Poset P, for $x, y \in P$ we use the expression $x < y$ to mean $x \leq y$ but $x \neq y$.

Example 1.1.3. *For a set X, $(P(X), \subseteq)$ is a poset, where $\subseteq$ is the set theoretic subset. This is not a total order.*

Example 1.1.4. *For the set of all real numbers $\mathbb{R}$, the usual ordering $\leq$ is a partial order. It is also a total order on $\mathbb{R}$.*

Definition 1.1.5. *[6],[24] Let $(P, \leq)$ be a poset. The dual poset $(P^d, \leq)$ is the poset with the same underlying set but whose order relation is the opposite of $\leq$, that is, $x \leq y \Leftrightarrow x \geq y$.*

Definition 1.1.6. *[6], [11],[24] Suppose that A is a subset of a poset P and $p, q \in P$. Then P is called an upper bound for A if $a \leq p, \forall a \in A$. And it is called the least upper bound of A (or simply lub of A), or supremum of A (Sup A) if p is an upper bound of A, and $a \leq b$ for every a in A, then $p \leq b$.*
Analogously q is called a lower bound for A if $q \leq a, \forall a \in A$. And q is the greatest lower bound of A (or simply glb of A), or infimum of A (inf A) if q is a lower bound of A, and $b \leq a$ for every a in A, then $b \leq q$.

Definition 1.1.7. *[6], [8], [11] A poset L is a lattice if and only if for every x, y in L both $sup\{x, y\}$ and $inf\{x, y\}$ exist (in L).*

An alternative definition of a lattice is given under

Definition 1.1.8. *[6], [8], [11] Suppose L is a non-empty set and $\vee$ and $\wedge$ are two binary operations (read "join" and "meet" respectively) on L. Then the system $(L, \vee, \wedge)$ is called a lattice if it satisfies the following identities:*

1. *commutative laws*

   (a) $x \vee y = y \vee x$ (b) $x \wedge y = y \wedge x$

2. *associative laws*

   (a) $x \vee (y \vee z) = (x \vee y) \vee z$ (b) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

3. *absorption laws*

   (a) $x = x \vee (x \wedge y)$ *(b)* $x = x \wedge (x \vee y)$

Note

1. Observe that the idempotent laws

   (a) $x \vee x = x$ (b) $x \wedge x = x$, follows from the absorption laws, as $x \vee x = x \vee [x \wedge (x \vee x)] = x$. Dually, one can prove $x \wedge x = x$.

2. The remaining axioms in the definition of lattice are independent. i.e, no one of them can be proved from the others.

Remark 1.1.9. *Here after we write L for the lattice $(L, \vee, \wedge)$.*

Definition 1.1.10. *[6], [8], [11], [21] Let L be a lattice with 0 and 1. The two special elements $0, 1 \in L$ are called bound elements if $0 \leq a \leq 1$ for all $a \in L$. In this case L is called a bounded lattice.*
The element 1 is called the upper bound, or top of L and the element 0 is called the lower bound or bottom of L.

Definition 1.1.11. *[6], [8], [11] A poset P is called complete if both sup A and inf A exist ($\in P$), $\forall A \subseteq P$.*

Remark 1.1.12. *A lattice L which is complete as a poset is a complete lattice, and all complete posets are lattices.*

Definition 1.1.13. *[10], [11], [12] If L is a bounded lattice, then we say that $y \in L$ is a complement of $x \in L$ if $x \wedge y = 0$ and $x \vee y = 1$.*
If every element has a complement, then we say L is complemented lattice.

Remark 1.1.14. *A complement of an element is not necessarily unique, it may not exist.*

For instance, in M_5 (see Figure 1.1), all elements other than zero and the unit have two complements each.

Definition 1.1.15. *[10], [11], [12] Let the interval $I = [b, c]$ is contained in a lattice*

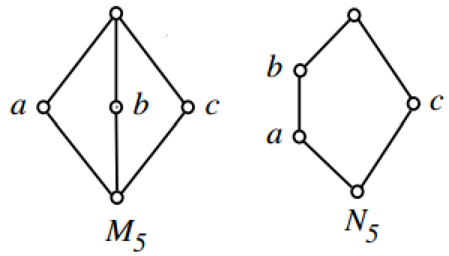


Figure 1.1: The structure of M_5 and N_5 lattice

L and $a \in I$. Then $x \in L$ is called a relative complement of a in [b, c] if $a \vee x = c$ and $a \wedge x = b$

An element $a \in L$ is called relatively complemented if for every interval I in L with $a \in I$, it has a complement relative to I. The lattice L itself is called a relatively complemented lattice if every element of L is relatively complemented. Equivalently, L is relatively complemented iff each of its interval is a complemented lattice.

Remark 1.1.16. *In a lattice L,*

1. *If y is a complement of x, then x is a complement of y.*
2. *For a lattice L, if 0 and 1 exists, then they are complements of each other.*
3. *A relatively complemented lattice is complemented if it is bounded.*
4. *Complemented lattices are not necessarily relatively complemented.*

Example 1.1.17. *Consider the lattice given in fig 1.2 which is a complemented lattice . But it is not relatively complemented as c has no complement in $[b, 1]$.*

Definition 1.1.18. *[6],[10], [11], [12], [21] Let L be a lattice with zero. An element $a^* \in L$ is a pseudo-complement of a ($\in L$) if*

1 $a \wedge a^* = 0$

2 $a \wedge x = 0 \Rightarrow x \leq a^*$

Definition 1.1.19. *[6],[10], [11], [12], [21] A pseudo-complemented lattice is one in which every element has a pseudo-complement.*

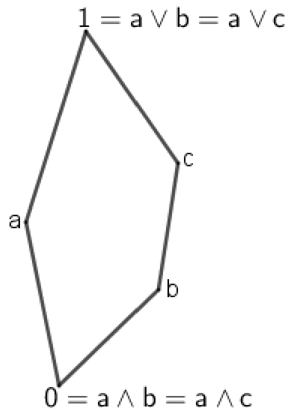


Figure 1.2: Example of complemented but not relatively complemented lattice

Remark 1.1.20. *Complemented lattices are not necessarily pseudo-complemented.*

Example 1.1.21. *Consider the lattice M_5. Both the elements a, b and c have no pseudo-complement. So it is not pseudo-complemented.*

Note

1. An element can have at most one pseudo-complement.

2. Let L is a lattice, $a \in L$ and a^* is a pseudo-complement of a. Then:
   $a \wedge x = 0 \Leftrightarrow a^* \wedge x = x \Leftrightarrow x \leq a^*$.

Remark 1.1.22. *Every pseudo-complemented lattice is bounded. Indeed, since $0 \wedge y = 0$ for every $y \in L$, we validate that $y \leq 0^*$ for every $y \in L$. Hence 0^* is the upper bound of L.*

Example 1.1.23. *For any non-empty set X, $(P(X), \cap, \cup)$ is pseudo-complemented lattice, where for each $A \in P(X), A^* = A'$.*

Theorem 1.1.24. *[6], [10], [11], [12] In any lattice L, for $x, y, z \in L$,*
$x \leq y \Rightarrow x \wedge z \leq y \wedge z$ and $x \vee z \leq y \vee z$.

Theorem 1.1.25. *[6],[10], [11], [12] In any lattice L, for $x, y, z \in L$, (distributive inequalities holds)*
a. $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$
b. $(x \vee y) \wedge (x \vee z) \geq x \vee (y \wedge z)$

Theorem 1.1.26. *[6],[10], [11], [12], [21] In any lattice L the following identities are equivalent, $\forall x, y, z \in L$.*
a. $(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)$
b. $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$

Definition 1.1.27. *[6],[10], [11], [12], [21] A lattice L is called a distributive lattice if it satisfies either of the distributive laws, $\forall x, y, z \in L$.*
$D_1 : (x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)$
$D_2 : (x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$

Example 1.1.28. *For any non empty set A, $(P(A), \cup, \cap)$ is a distributive lattice.*

Theorem 1.1.29. *([21], Theorem 9.3) A lattice L is a distributive lattice iff $x \wedge c = y \wedge c$ and $x \vee c = y \vee c$ implies $x = y$, for every elements $c, x, y \in L$.*

Corollary 1.1.30. *Complements and relative complements that exist are unique in a distributive lattice.*

Theorem 1.1.31. *([11] Corollary 103, [21] Theorem 11.2) A lattice L is distributive iff each element has at most one relative complement in any interval.*

Definition 1.1.32. *[6] A sub-lattice of a lattice L is a subset L' of L in which for $x, y \in L'$, both $x \wedge y, x \vee y \in L'$.*

Definition 1.1.33. *[6] In a lattice L, the subset J of L is called convex if $x, y \in J$ and $c \in L$ such that $x \leq c \leq y$, then $c \in J$.*

Example 1.1.34. *For any lattice L, $[x, y]$ is convex, $\forall x, y \in L$.*

Definition 1.1.35. *[8] A closure operator on set X is a mapping $C : P(X) \longrightarrow P(X)$ such that for $A, B \subseteq X$, it satisfies:*

1. $A \subseteq C(A)$ *(extensive)*
2. $C^2(A) = C(A)$ *(idempotent)*
3. $A \subseteq B \Rightarrow C(A) \subseteq C(B)$ *(isotone).*

If A is subset of X and C(A) = A, then A is called a closed subset of X. L_c denote the poset of closed subsets of X with set inclusion as the partial ordering.

Theorem 1.1.36. *([8], Theorem 5.2) For a closure operator C on a set A, L_c is a complete lattice with*

$$\bigwedge_{i\in I} C(A_i) = \bigcap_{i\in I} C(A_i)$$

and

$$\bigvee_{i\in I} C(A_i) = C(\bigcup_{i\in I} A_i)$$

1.2 Certain Special Classes of Lattices (Algebras)

In this section we collect certain definitions, examples and results concerning certain types of lattices (algebras). We refer from [6], [9], [10], [13], [14], [19], [21], [23], [25], [27],[28] and [29].

1.2.1 Boolean Algebra

Definition 1.2.1. *[6],[23] [24] A Boolean algebra is a non-empty set B, together with two binary operations $\wedge$ and $\vee$, a unary operation $'$, and two distinguished elements 0 and 1, satisfying the following axioms:*
$B_1 : (B, \vee, \wedge)$ *is a distributive lattice*
$B_2 : x \wedge 0 = 0; x \vee 1 = 1$
$B_3 : x \wedge x' = 0; x \vee x' = 1.$
In other words a Boolean algebra is a complemented distributive lattice.

Example 1.2.2. *Let $P(X)$ be the class of all subsets of X (the power set of X). The class of all subsets of an arbitrary set X is an example of a Boolean algebra, under the natural set-theoretic operations on $P(X)$ (the binary operations of union and intersection, and the unary operation of complementation).*

Lemma 1.2.3. *[23] If p, q are elements of Boolean algebra B, then $p \wedge q = p$ if and only if $p \vee q = q$*

Proof. Suppose B is any Boolean algebra, $p, q \in B$ and $p \wedge q = p$

$\Rightarrow p \vee q = (p \wedge q) \vee q = q$ (by law of absorption in the definition of lattice).

Conversely let $p \vee q = q$. $\Rightarrow p \wedge q = p \wedge (p \vee q) = p$ (by law of absorption in the definition of lattice) □

Lemma 1.2.4. *If p, q, r, s are any elements of a Boolean algebra B, then $p \leq q$ and $r \leq s \Rightarrow p \wedge r \leq q \wedge s$ and $p \vee r \leq q \vee s$*

Proof. Let B be a Boolean algebra, $p, q, r, s \in B$, $p \leq q$ and $r \leq s$. $\Rightarrow p \wedge r \leq q \wedge r$ and $q \wedge r \leq q \wedge s$ (using Theorem 1.1.24)

$\Rightarrow p \wedge r \leq q \wedge s$ (transitive property of the partial order).

Again from the given inequities

$p \vee r \leq q \vee r$ and $q \vee r \leq q \vee s$ (using Theorem 1.1.24)

$\Rightarrow p \vee r \leq q \vee s$ (transitive property of the partial order). □

Theorem 1.2.5. *In a Boolean algebra B, every element a has a unique complement and hence B is relatively complemented.*

Proof. The uniqueness of complement follows from Corollary 1.1.30.

Let B is a Boolean algebra, $[x, y] \subset B$ and $a \in [x, y]$

Then using the distributive property

$(a' \vee x) \wedge y = (a' \wedge y) \vee (x \wedge y) = (a' \wedge y) \vee x.$

Thus using distributive and associative property

$a \vee [(a' \vee x) \wedge y] = [a \vee (a' \vee x)] \wedge (a \vee y) = 1 \wedge (a \vee y) = (a \vee y) = y$

$a \wedge [(a' \wedge y) \vee x] = [a \wedge (a' \wedge y)] \vee (a \wedge x) = 0 \vee (a \wedge x) = x$

Hence $(a' \vee x) \wedge y$ is the relative complement of a in $[x, y]$. Since both a and $[x, y]$ are arbitrary, B is relatively complemented. □

Theorem 1.2.6. *[6], [24] In a Boolean algebra B, for $a, b \in B$*

1. *$(a')' = a$ (Involution)*
2. *$(a \wedge b)' = a' \vee b'$ and $(a \vee b)' = a' \wedge b'$ (De Morgan's Law)*

Proof. Let B be a Boolean algebra and $a, b \in B$. Since $a \wedge a' = 0$ and $a \vee a' = 1$, the

unique complement (by Theorem 1.2.5) of a' is a. Hence $(a')' = a$. Hence (1) holds.
Using the commutative, associative and distributive property
$(a \wedge b) \vee (a' \vee b') = (a \vee (a' \vee b')) \wedge (b \vee (a' \vee b')) = 1 \wedge 1 = 1$
$(a \wedge b) \wedge (a' \vee b') = ((a \wedge b) \wedge a') \vee ((a \wedge b) \wedge b') = 0 \vee 0 = 0.$
Hence the unique complement (by Theorem 1.2.5) of $a \wedge b$ is $a' \vee b'$. Hence $(a \wedge b)' = a' \vee b'$.
The other law is established similarly. Thus (2) holds. □

Theorem 1.2.7. *[6], [24] In a Boolean algebra B,* $\forall a, b, c \in B$

1 $a \leq b \Leftrightarrow b' \leq a'$ *(Order-reversing))*

2 $a \leq b \Leftrightarrow a \wedge b' = 0 \Leftrightarrow a' \vee b = 1$ *(Order and complements)*

3. $a \wedge b \leq c \Leftrightarrow a \leq c \vee b'$*;*

4. $a \vee b \geq c \Leftrightarrow a \geq c \wedge b'$*.*

Proof. Suppose that B is a Boolean algebra and $a, b, c \in B$. $a \leq b \Leftrightarrow a \wedge b = a$ (by the definition of $\leq$)
$\Leftrightarrow (a \wedge b)' = a' \vee b' = a'$ (by 2 of Theorem 1.2.6)
$\Leftrightarrow b' \leq a'$ (by Lemma 1.2.4 and definition of $\leq$). Hence (1) holds.
Let $a \leq b$, then $a \vee b = b$
$a \wedge b' = a \wedge (a \vee b)'$
$= a \wedge (a' \wedge b')$ (De Morgan's Law)
$= (a \wedge a') \wedge b' = 0 \wedge b' = 0$ (associativety)
Hence $a \leq b \Rightarrow a \wedge b' = 0$ (i)
Let $a \wedge b' = 0$. Then
$a' \vee b = a' \vee (b')'$ (Involution)
$= (a \wedge b')' = 0' = 1$ (De Morgan's Law)
Hence
$a \wedge b' = 0 \Rightarrow a' \vee b = 1$ (ii)
Finally let $a' \vee b = 1$. Then
$a = a \wedge 1 = a \wedge (a' \vee b) = (a \wedge a') \vee (a \wedge b)$ (Distributive)
$= 0 \vee (a \wedge b) = (a \wedge b) \leq b.$

$\Rightarrow a \leq b$.

Hence $a' \vee b = 1 \Rightarrow a \leq b$ (iii)

Thus from (i), (ii) and (iii) it follows that $a \leq b \Leftrightarrow a \wedge b^{'} = 0 \Leftrightarrow a^{'} \vee b = 1$. Thus (2) holds.

Let $a \wedge b \leq c$.

$\Rightarrow (a \wedge b) \vee b' \leq c \vee b'$ (by Theorem 1.1.24)

$\Rightarrow (a \vee b') \wedge (b \vee b') \leq c \vee b'$ (distributive law)

$\Rightarrow (a \vee b') \wedge 1 \leq c \vee b'$

$\Rightarrow a \leq a \vee b' \leq c \vee b'$.

Conversely, let $a \leq c \vee b'$.

$\Rightarrow a \wedge b \leq (c \vee b') \wedge b$ (by Theorem 1.1.24).

$= (c \wedge b) \vee (b' \wedge b) = c \wedge b \leq c$ (distributive law and definitions of complement and $\wedge$).

Hence (3) holds.

Let $(a \vee b) \geq c$.

$\Rightarrow (a \vee b) \wedge b' \geq c \wedge b'$ (by Theorem 1.1.24)

$\Rightarrow (a \wedge b') \vee (b \wedge b') = (a \wedge b') \geq c \wedge b'$ (by definition of complement and distributive property).

$\Rightarrow a \geq a \wedge b' \geq c \wedge b'$.

$\Rightarrow a \geq c \wedge b'$.

Conversely, let $a \geq c \wedge b'$

$\Rightarrow a \vee b \geq (c \wedge b') \vee b$

$\Rightarrow a \vee b \geq (c \vee b) \wedge (b' \vee b') = c \vee b \geq c$ (by definition of complement and distributive property). Thus (4) holds. □

1.2.2 Brouwerian Algebra

The following results are referred to from [19]

Definition 1.2.8. *A lattice $(L, \vee, \wedge)$ is called Brouwerian algebra if it is bounded and for each $a, b \in L$ there is a least x such that $x \vee b \geq a$. The least element is denoted by $a - b$.*

Example 1.2.9. *Any finite chain is a Brouwerian algebra.*

Example 1.2.10. *Every Boolean lattice is Brouwerian algebra. To show this consider the following. For a Boolean algebra B define $a - b = b' \wedge a$. Then*

1. $(a \wedge b') \vee b = (b \vee b') \wedge (a \vee b)$ *(Boolean algebra is distributive.)*
   $= 1 \wedge (a \vee b) = (a \vee b) \geq a$

2. *Let $x \in B$ such that $x \vee b \geq a$.*
   $\Rightarrow ((x \vee b) \vee a') = 1$ *(by Theorem 1.2.7)*
   Now consider the following
   $(a \wedge b')' \vee x = (a' \vee b'') \vee x = (x \vee b) \vee a' = 1$ *(by Theorem 1.2.6, commutative and associative property).*
   $\Rightarrow (b' \wedge a) \leq x$ *(by Theorem 1.2.7). Hence $b' \wedge a$ is the least element of B such that $x \vee b \geq a$.*

Hence B is a Brouwerian algebra.

Remark 1.2.11. *The converse is not true in general. For instance consider the lattice given in Fig 1.3. Since it is not complemented , it is not Boolean algebra. Clearly it is a Brouwerian algebra.*

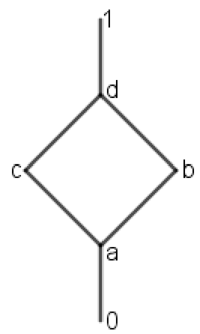


Figure 1.3: Example of Brouwerian lattice which is not a Boolean lattice

Theorem 1.2.12. *In a Brouwerian algebra L, the following holds, $\forall x, y, z \in L$.*

1. $y - x \leq y, \qquad 0 - x = 0$

2. $x \leq y \Leftrightarrow x - y = 0$ *and* $x - x = 0$

3. $x \vee y = y \vee (x - y)$ *and* $x - 0 = x$

4. $x \leq y \Rightarrow z - y \leq z - x.$

5. $(x - y) - y = x - y$

Proof. Suppose that L is a Brouwerian algebra and $x, y, z \in L$.

1. As $x \vee y \geq y$, Clearly $y - x \leq y$ and consequently $0 - x = 0$.

2. Let $x \leq y$. $\Rightarrow x \leq x \vee z \leq y \vee z, \forall z \in L$ (by Theorem 1.1.24).
   $\Rightarrow x - y \leq z, \forall z \in L$
   $\Rightarrow x - y = 0$.
   Conversely let $x - y = 0$. $\Rightarrow x - y \leq 0$
   $\Rightarrow x \leq 0 \vee y = y$. Hence the first equality holds. And trivially $x - x = 0$ as a consequence of the first equality. Hence (2) holds.

3. $x - y \leq x$ (by 1)
   $\Rightarrow y \vee (x - y) \leq x \vee y$ (by Theorem 1.1.24).
   Again by definition $x \leq y \vee (x - y)$. Since $y \leq y \vee (x - y)$, it follows that $x \vee y \leq y \vee (x - y)$. Hence the first equality $x \vee y = y \vee (x - y)$ holds. For the second equality let $y = 0$ in the first equality. So (3) holds.

4. Since by definition $x - y$ is the least z such that $z \vee y \geq x$, so $x \leq y \vee (x - y)$.
   Let $x \leq y$.
   $\Rightarrow z \leq x \vee (z - x) \leq y \vee (z - x)$ (by Theorem 1.1.24).
   $\Rightarrow z - y \leq z - x$ (by definition of least). Hence the inequality in (4) holds.

5. $(x - y) - y \leq x - y$ (by Theorem 1.2.12(1)) (i)
   Again $((x - y) - y) \vee y = (x - y) \vee y = x \vee y \geq x$ (by Theorem 1.2.12(3))
   $\Rightarrow x - y \leq (x - y) - y$ (by definition) (ii)
   Hence from (i) and (ii), we obtain that $(x - y) - y = x - y$

□

Theorem 1.2.13. *In a Brouwerian algebra L, the following holds, $\forall x, y, z \in L$.*

1. $x \leq y \Rightarrow x - z \leq y - z$

2. $(x \vee y) - z = (x - z) \vee (y - z)$

3. *L is a bounded distributive lattice.*

4. $(x - y) \vee (x \wedge y) = x$

Proof. Let L be a Brouwerian algebra and $x, y, z \in L$

1. Let $x \leq y \Rightarrow x \leq y \leq z \vee y = z \vee (y - z)$ (by Theorem 1.2.12)
   $\Rightarrow z \vee (y - z) \geq x$ (by transitivity)
   $\Rightarrow y - z \geq x - z$

2. $x \leq x \vee y$ and $y \leq x \vee y$
   $\Rightarrow x - z \leq (x \vee y) - z$ and $y - z \leq (x \vee y) - z$ (by 1)
   $\Rightarrow (x - z) \vee (y - z) \leq (x \vee y) - z$ (i)
   Again by definition it is true that
   $(x - z) \vee z \geq x$ and $(y - z) \vee z \geq y$
   $\Rightarrow (x - z) \vee (y - z) \vee z \geq (x \vee y)$
   $\Rightarrow (x - z) \vee (y - z) \geq (x \vee y) - z$ (ii)
   From (i) and (ii), $(x \vee y) - z = (x - z) \vee (y - z)$

3. Obviously, L is bounded. By the theorem of distributive inequalities, it is enough to show that $x \vee (y \wedge z) \geq (x \vee y) \wedge (x \vee z)$. To show this, consider the following:
   $x \vee z \geq (x \vee y) \wedge (x \vee z)$ and $x \vee y \geq (x \vee y) \wedge (x \vee z)$
   $\Rightarrow z \geq [(x \vee y) \wedge (x \vee z)] - x$ and $y \geq [(x \vee y) \wedge (x \vee z)] - x$
   $\Rightarrow y \wedge z \geq [(x \vee y) \wedge (x \vee z)] - x$
   $\Rightarrow x \vee (y \wedge z) \geq (x \vee y) \wedge (x \vee z)$. Hence it is distributive.

4. $(x - y) \vee (x \wedge y) = [(x - y) \vee x] \wedge [(x - y) \vee y]$ (distributive)
   But $x - y \leq x$ (by Theorem 1.2.12). So $(x - y) \vee x = x$ and by Theorem 1.2.12(3), $(x - y) \vee y = x \vee y$. Hence $(x - y) \vee (x \wedge y) = [(x - y) \vee x] \wedge [(x - y) \vee y] = x \wedge (x \vee y) = x$

□

Definition 1.2.14. *Suppose that* $(L, \vee, \wedge)$ *is a Brouwerian algebra and* $a \in L$ *be any element. The Brouwerian complement of* a *is* $1 - a$*, which is denoted by* $\neg a$*. Similarly* $\neg\neg a = 1 - \neg a$

Theorem 1.2.15. *A Brouwerian algebra L is a Boolean algebra* $\Leftrightarrow a \wedge (1-a) = 0, \forall a \in L$

Proof. ($\Rightarrow$) Let a Brouwerian algebra L is a Boolean algebra. Hence every element $a \in L$ has a complement a'.
$\Rightarrow a \wedge a' = 0$ and $a \vee a' = 1$.
From $a \vee a' = 1$, it follows that $a \vee a' \geq 1$ and consequently from the definition of a Brouwerian algebra we have $1 - a \leq a'$
$\Rightarrow (1-a) \vee a' = a'$ (i)
Since for any $a, b \in L, a \vee b = (a - b) \vee b \geq a$ (by Theorem 1.2.12), it follows that $(1-a) \vee a = a \vee 1 = 1$ (ii)
Again from $a \wedge a' = 0$, we have
$(1 - a) \vee (a \wedge a') = (1 - a) \vee 0 = 1 - a$
$\Rightarrow ((1 - a) \vee a) \wedge ((1 - a) \vee a') = 1 - a$ (distributive)
$\Rightarrow 1 \wedge ((1 - a) \vee a') = 1 - a$ (by (ii))
$\Rightarrow (1-a) \vee a' = 1-a$ (iii)
From (i) and (iii), it follows that $a' = 1 - a$. Hence $a \wedge (1 - a) = a \wedge a' = 0$
($\Leftarrow$) Since $(1 - a) \vee a = a \vee 1 = 1$, if $a \wedge (1 - a) = 0$, then $a' = 1 - a$. Hence L is Boolean algebra. □

1.2.3 Heyting Algebra

The following results are referred to from [22]

Definition 1.2.16. *A lattice* $(L, \vee, \wedge)$ *is called a Heyting algebra if it is bounded and for any given elements a and b in L, there is a greatest* x *such that* $x \wedge a \leq b$

Remark 1.2.17. *The greatest element* x *is denoted by* $a \to b$*. Clearly* $a \to b$ *is unique.*

Example 1.2.18. *Any chain that contain a least element 0 and a greatest element 1* *is a Heyting algebra. In such case* $a \to b = 1$ *when* $a \leq b$*, and* b *otherwise.*

Example 1.2.19. *Every Boolean algebra is Heyting algebra. To show this consider the following. Let* $(B, \vee, \wedge, ', 0, 1)$ *be a Boolean algebra and for* $a, b \in B$*; define* $a \to b$ *to be* $a' \vee b$*. Then*

1. $(a' \vee b) \wedge a = (a \wedge a') \vee (a \wedge b)$ *(Boolean algebra is distributive)*
   $= 0 \vee (a \wedge b) = (a \wedge b) \leq b$

2. *Let* $x \wedge a \leq b$
   $\Rightarrow (x \wedge a) \wedge b' = 0$ *(by Theorem* 1.2.7(2)*)*
   $\Rightarrow x \wedge (a \wedge b') = 0$ *(associative)*
   $\Rightarrow x \wedge (a' \vee b)' = 0$ *(by Theorem* 1.2.7*)*
   $\Rightarrow x \leq a' \vee b$ *(by Theorem* 1.2.7(2)*)*

Hence $(B, \vee, \wedge, \to, 0, 1)$ *is a Heyting algebra. That is, Every Boolean algebra is a Heyting algebra, with* $a \to b$ *given by* $a' \vee b$*.*

Remark 1.2.20. *The converse fails.i.e; Heyting algebra is not a Boolean algebra. For instance consider the example given at the end of this section.*

Remark 1.2.21. *Both Brouwerian and Heyting algebras are generalizations to Boolean algebras and are dual to each other.*

Theorem 1.2.22. *The following hold in a Heyting algebra* L*,* $\forall a, b, c \in L$*.*

1. $b \leq a \to b$

2. $1 = a \to 1$

3. $a \leq b \Leftrightarrow a \to b = 1$*. Hence* $0 \to b = 1$ *for any* $b \in L$*.*

4. $a \wedge (a \to b) = a \wedge b$

5. $b \wedge (a \to b) = b$ *and* $((a \wedge b) \to a) \wedge c = c$

6. $b = 1 \to b$

7. *If* $a \leq b$*, then* $(c \to a) \leq (c \to b)$

Proof. Suppose that L is a Heyting algebra and $a, b, c \in L$.

1. Since $b \wedge a \leq b$, the inequality holds by the definition.

2. This is simply a consequence of 1

3. Let $a \leq b$. Since $1 \wedge a = a \leq b$. Hence $a \to b = 1$. Conversely let $a \to b = 1$, then trivially $a \leq b$ holds.

4. By (1), $b \leq a \to b$, so $a \wedge b \leq a \wedge (a \to b)$. Alternatively, by definition of Heyting algebra, $a \wedge (a \to b) \leq b$. Further, from the definition of $\wedge$ we have $a \wedge (a \to b) \leq a$. So $a \wedge (a \to b) \leq a \wedge b$.

5. Follows from 3 and 4.

6. This is consequence of 4 (Simply replace a by 1 in 4)

7. using (4), $c \wedge (c \to a) = c \wedge a \leq a \leq b$
$\Rightarrow c \to a \leq c \to b$ (by definition)

□

Theorem 1.2.23. *In a Heyting algebra L the following hold,* $\forall a, b, c \in L$.

1. $a \leq b \Rightarrow (b \to c) \leq (a \to c)$

2. $a \to (b \to c) = (a \wedge b) \to c = (a \to b) \to (a \to c)$

3. $(a \to b) \wedge (a \to c) = a \to (b \wedge c)$

4. $(a \to b) \wedge c = ((c \wedge a) \to (c \wedge b)) \wedge c$

5. *L is a bounded distributive lattice.*

Proof. Assume that L is a Heyting algebra and $a, b, c \in L$.

1. Let $a \leq b \Rightarrow a \wedge (b \to c) \leq b \wedge (b \to c) = b \wedge c \leq c$(By Theorem 1.1.24 and 1.2.22

)
$\Rightarrow a \wedge (b \to c) \leq c$
$\Rightarrow b \to c \leq a \to c$ (by definition).

2. We shall use (4) of Theorem 1.2.22 above a number of times, and the fact that $x = y$ iff $x \leq y$, $y \leq x$. First equality:
$(a \to (b \to c)) \wedge (a \wedge b) = (a \wedge (b \to c)) \wedge b$
$= (b \wedge (b \to c)) \wedge a$
$= (b \wedge c) \wedge a \leq c.$
So $a \to (b \to c) \leq (a \wedge b) \to c.$ (i)
Then again, $((a \wedge b) \to c) \wedge a \wedge b = a \wedge b \wedge c \leq c$
so $((a \wedge b) \to c) \wedge a \leq b \to c$, and consequently
$(a \wedge b) \to c \leq a \to (b \to c).$ (ii)
From (i) and (ii) the first equality holds.
Second equality:
Using Theorem 1.2.22(4), $((a \wedge b) \to c) \wedge (a \to b) \wedge a = ((a \wedge b) \to c) \wedge (a \wedge b) = (a \wedge b) \wedge c \leq c$, so $((a \wedge b) \to c) \wedge (a \to b) \leq a \to c$
and consequently $(a \wedge b) \to c \leq (a \to b) \to (a \to c).$ (iii)
On the other hand,
$((a \to b) \to (a \to c)) \wedge (a \wedge b) = ((a \to b) \to (a \to c)) \wedge (a \wedge (a \to b)) = ((a \to b) \wedge (a \to c)) \wedge a = (a \wedge b) \wedge (a \to c)$
$= b \wedge (a \wedge c) \leq c,$
so $(a \to b) \to (a \to c) \leq (a \wedge b) \to c.$ (iv)
From (iii) and (iv) the second equality holds.

3. Since $b \wedge c \leq b$ and $b \wedge c \leq c$, using 7 of Theorem 1.2.22 it follows that $a \to (b \wedge c) \leq a \to b$ and $a \to (b \wedge c) \leq a \to c$. Hence $a \to (b \wedge c) \leq (a \to b) \wedge (a \to c)$. Again by definition it is true that, $a \wedge (a \to b) \leq b$ and $a \wedge (a \to c) \leq c$. so $a \wedge (a \to c) \wedge (a \to b) \leq (b \wedge c)$. Which implies that $(a \to c) \wedge (a \to b) \leq a \to (b \wedge c)$. Hence $(a \to c) \wedge (a \to b) = a \to (b \wedge c)$

4. $(a \to b) \wedge (c \wedge a) = a \wedge (a \to b) \wedge c$ (commutativity, associativity of $\wedge$)
$= a \wedge b \wedge c \leq b \wedge c$ (Using 4 of Theorem 1.2.22)

$\Rightarrow a \to b \leq (c \wedge a) \to (b \wedge c)$ (by definition of Heyting algebra).

$\Rightarrow (a \to b) \wedge c \leq [(c \wedge a) \to (b \wedge c)] \wedge c.$ (i)

On the other hand using 2 and 4 of Theorem 1.2.22

$(a \to b) \wedge c = c \wedge (c \to (a \to b)) = c \wedge ((a \wedge c) \to b).$

But using 1 of Theorem 1.2.22, $b \leq (a \wedge c) \to b.$

$\Rightarrow b \wedge c \leq ((a \wedge c) \to b) \wedge c \leq (a \wedge c) \to b.$

$\Rightarrow (a \wedge c) \to (b \wedge c) \leq (a \wedge c) \to [(a \wedge c) \to b]$ (Using 7 of Theorem 1.2.22)

$= (a \wedge c) \to b$(Using 2).

$\Rightarrow [(a \wedge c) \to (b \wedge c)] \wedge c \leq [(a \wedge c) \to b] \wedge c.$ (ii)

From (i) and (ii), $[(a \wedge c) \to (b \wedge c)] \wedge c = [(a \wedge c) \to b] \wedge c.$

5. Obviously L is bounded. By the theorem of distributive inequalities, it is enough to illustrate that $a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$. To show this, consider the following: $a \wedge b \leq (a \wedge b) \vee (a \wedge c)$, so $b \leq a \to ((a \wedge b) \vee (a \wedge c)).$
Similarly, $c \leq a \to ((a \wedge b) \vee (a \wedge c)).$
Hence $b \vee c \leq a \to ((a \wedge b) \vee (a \wedge c)),$
$\Rightarrow a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c).$

□

The alternative definition of Heyting algebra is given under.

Definition 1.2.24. *A non empty set L with three binary operations $\wedge$, $\vee$ and $\to$ and two distinguished elements 0 and 1 is a Heyting Algebra if the following conditions hold:*

(H_1) $(L, \vee, \wedge, 0, 1)$ *is a lattice with 0,1*

(H_2) $x \wedge (x \to y) = x \wedge y$

(H_3) $x \wedge (y \to z) = x \wedge [(x \wedge y) \to (x \wedge z)]$

(H_4) $(x \wedge y) \to x = 1$

Theorem 1.2.25. *The definitions given in* 1.2.16 *and* 1.2.24 *are equivalent.*

Proof. Assume 1.2.16. Then 1.2.24 follows from the Theorem 1.2.22 and 1.2.23. Suppose 1.2.24 holds. Let $a, b \in H$. By $(H_2), a \wedge (a \to b) = a \wedge b \leq b$, i.e., $a \wedge (a \to$

$b) \leq b$. If $a \wedge c \leq b$ for some $c \in L$, then by (H_3),
$c \wedge (a \to b) = c \wedge ((c \wedge a) \to (c \wedge b)) = c \wedge ((a \wedge b \wedge c) \to (b \wedge c))$.
Applying (H_4) yields
$c \wedge ((a \wedge b \wedge c) \to (b \wedge c)) = c$.
So $c \wedge (a \to b) = c$, i.e.,$c \leq a \to b$. Thus,$a \to b$ is the greatest of the elements c satisfying $a \wedge c \leq b$. □

Remark 1.2.26. *Heyting algebras are pseudocomplemented (with $a^* = a \to 0$ as the pseudocomplement of a). To show this consider the following. Suppose that L is a Heyting algebra and $a \in L$. Then*

1. $a \wedge a^* = a \wedge (a \to 0) = a \wedge 0 = 0$*(by H_2)*

2. *Assume $a \wedge x = 0$. $x \to (a \to 0) = (x \wedge a) \to 0 = 0 \to 0 = 1$(by Theorem* 1.2.23(2) *and $a \to a = 1, \forall a \in L$).*
   $\Rightarrow x \leq (a \to 0) = a^*$ *(by Theorem* 1.2.22(3) *). Hence $a^* = a \to 0$. Since a is arbitrary, L is pseudocomplemented.*

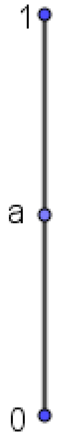
For $a^* = a \to 0$, it is not in general true that $a \vee a^* = 1$, thus a^* is only a pseudo-complement, not a true complement, as would be the case in a Boolean algebra.

Remark 1.2.27. *Any Heyting algebra L is a Boolean algebra $\Leftrightarrow$ for each $a \in L, a \vee \neg a = 1$, where $\neg a = a \to 0$.*

To show this consider the following. Let Heyting algebra L is Boolean algebra. Hence every element $a \in L$ has a complement a'. Hence $a \wedge a' = 0$.
$\Rightarrow a' \leq \neg a$ (by definition of pseudo-complement)
$\Rightarrow \neg a \wedge a' = a'$ (i)
Again, $a \vee a' = 1$
$\Rightarrow \neg a \wedge (a \vee a') = \neg a \wedge 1 = \neg a$
$\Rightarrow (\neg a \wedge a) \vee (\neg a \wedge a') = \neg a \wedge a' = \neg a$ (ii)
From (i) and (ii) it follows that $a' = \neg a$. Hence $a \vee \neg a = a \vee a' = 1$.
The converse follows trivially from the definition of Boolean algebra.

Example 1.2.28. *(Heyting algebra is not necessarily a Boolean algebra)*

*For instance consider the three element chain whose $\rightarrow$ operation is defined in the table given in fig.*1.4. *In this example,* $a \vee \neg a = a \vee (a \rightarrow 0) = a \vee 0 = a \neq 1$. *falsifies that*



$\rightarrow$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

Figure 1.4: Example of Heyting algebra which is not a Boolean algebra

it is Boolean algebra.

1.2.4 Po-Group

The following results are referred to from [7], [13].

Definition 1.2.29. *[7], [13] A non-empty set G with binary operation . and binary relation $\leq$ is called a partially ordered group(po-group) if the following axioms are satisfied.*

1. $(G, .)$ *is a group*
2. $(G, \leq)$ *is a poset*
3. $x \leq y \Rightarrow xz \leq yz,$
4. $x \leq y \Rightarrow zx \leq zy \forall x, y, z \in G.$

Note

The above two conditions can be equivalently stated as, for
$a, b, x, y \in G, x \leq y \Rightarrow axb \leq ayb$ (called translation-invariant).

Example 1.2.30. *The additive group of real numbers with the usual order relation;*

the group $F(X,\mathbb{R})$ of functions from an arbitrary non-empty set X into $\mathbb{R}$, with the operation $(f+g)(x)=f(x)+g(x)$ and order relation $f\leq g$ iff $f(x)\leq g(x), \forall x\in X$ are examples of po-group.

Definition 1.2.31. *A po-group is called an l-group if $(G,\leq)$ is a lattice.*

Example 1.2.32. *The additive groups $\mathbb{Z}$ - integers, $\mathbb{Q}$ -rationals, $\mathbb{R}$ - reals are the simplest examples of l- groups with respect to the usual ordering.*

Example 1.2.33. *Let X be a non empty set, $G=F(X,\mathbb{R})$ be a set of all functions $f:X\longrightarrow\mathbb{R}$ with the coordinate order (i.e $f\leq g, f,g\in G\Leftrightarrow$ for $f(x)\leq g(x)$ for all x in X). Then G is an l-group under the partial order $\leq$ and the addition $(f+g)(x)=f(x)+g(x)$.*

Remark 1.2.34. *Every subgroup H of a partially ordered group G is a partially ordered group itself, where H inherits the partial order from G. Note that a subgroup of an l-group is not necessarily an l-group.*

Lemma 1.2.35. *[13] The following relations hold in a po-group G, $\forall x,y,z,t\in G$.*
(1) $x\leq y\Rightarrow z^{-1}xz\leq z^{-1}yz$;
(2) $x\leq y\Leftrightarrow y^{-1}\leq x^{-1}$;
(3) $x\leq y$ and $z\leq t\Rightarrow xz\leq yt$.

Proof. Directly follows from the definition by a routine verification. □

Theorem 1.2.36. *The following proprties hold in a po-group G, $\forall g,h\in G$.*

1. *If $x\vee y$ exists, then so does $x^{-1}\wedge y^{-1}$. Furthermore, $x^{-1}\wedge y^{-1}=(x\vee y)^{-1}$.*

2. *If $x\wedge y$ exists, then so does $x^{-1}\vee y^{-1}$. Furthermore, $x^{-1}\vee y^{-1}=(x\wedge y)^{-1}$.*

Proof. Since $x\leq x\vee y$, using Theorem 1.2.35 it follows that $(x\vee y)^{-1}\leq x^{-1}$. Similarly,$(x\vee y)^{-1}\leq y^{-1}$. If $z\leq x^{-1},y^{-1}$, then $x,y\leq z^{-1}$. Then $x\vee y\leq z^{-1}$. Therefore, $z\leq(x\vee y)^{-1}$. Thus by definition, $x^{-1}\wedge y^{-1}=(x\vee y)^{-1}$. Hence 1 hold. Using duality, we could prove 2 by interchanging $\vee$ and $\wedge$. □

Theorem 1.2.37. *[7] A po-group G is lattice-ordered iff sup $\{x, e\}$ exists for $\forall x \in G$, where e is the identity element of the group G.*

Proof. The forward is clear.

Conversely, assume $a, b \in G$ and suppose that $x = (ab^{-1} \vee e)b \in G$. Obviously , $x \geq ab^{-1}b = a$ and $x \geq eb = b$ (since $ab^{-1} \leq ab^{-1} \vee e, e \leq ab^{-1} \vee e$).

Hence x is upper bound of $\{a, b\}$. Now let $y \in G$ and $y \geq a$ and $y \geq b$. Then $yb^{-1} \geq ab^{-1}$ and $yb^{-1} \geq bb^{-1} = e$

Hence $yb^{-1} \geq ab^{-1} \vee e$ and accordingly $y \geq (ab^{-1} \vee e)b$. Hence $a \vee b$ exists in G and is $(ab^{-1} \vee e)b$.

Again consider the element $t = (a^{-1} \vee b^{-1})^{-1}$.

$a^{-1} \leq a^{-1} \vee b^{-1}$ (property of $\vee$)

$\Rightarrow (a^{-1} \vee b^{-1})^{-1} \leq (a^{-1})^{-1} = a$ (by Theorem 1.2.35(2))

Similarly $(a^{-1} \vee b^{-1})^{-1} \leq (b^{-1})^{-1} = b$. And hence $t = (a^{-1} \vee b^{-1})^{-1}$ is a lower bound for $\{a, b\}$.

Let $z \in G$ be such that $z \leq a, z \leq b$.

$\Rightarrow a^{-1} \leq z^{-1}, b^{-1} \leq z^{-1}$ (by Theorem 1.2.35(2))

$\Rightarrow a^{-1} \vee b^{-1} \leq z^{-1}$ (by $\vee$ property)

$\Rightarrow z \leq (a^{-1} \vee b^{-1})^{-1} = t$ (by Theorem 1.2.35(2))

Thus $a \wedge b$ exists in G and is $(a^{-1} \vee b^{-1})^{-1} = t$. Hence G is a lattice by definition of a lattice. □

Theorem 1.2.38. *[13] In any l-group G*

1 $(x \vee y)^{-1} = x^{-1} \wedge y^{-1}$, $\qquad (x \wedge y)^{-1} = x^{-1} \vee y^{-1}$

2 $x(x \wedge y)^{-1}y = x \vee y$, $\qquad x(x \vee y)^{-1}y = x \wedge y$.

Proof. Let $x, y \in G$. Since $x \vee y \geq x, y$, using 2 of Theorem 1.2.35, $(x \vee y)^{-1} \leq x^{-1}, y^{-1}$.

$\Rightarrow (x \vee y)^{-1} \leq x^{-1} \wedge y^{-1}$ (i)

Again as $x^{-1} \wedge y^{-1} \leq x^{-1}, y^{-1}$, $(x^{-1} \wedge y^{-1})^{-1} \geq x, y$ (using Theorem 1.2.35(2)).

$\Rightarrow (x^{-1} \wedge y^{-1})^{-1} \geq x \vee y$.

$\Rightarrow x^{-1} \wedge y^{-1} \leq (x \vee y)^{-1}$ (using Theorem 1.2.35(2)). (ii)

Hence from (i) and (ii) $(x \vee y)^{-1} = x^{-1} \wedge y^{-1}$. The dual identity can be proved analogously. Hence Property (1) holds.
From (1) and the definition $x(x \wedge y)^{-1}y = x(x^{-1} \vee y^{-1})y$. Now observe the following.
$x^{-1} \leq x^{-1} \vee y^{-1}$ and $y^{-1} \leq x^{-1} \vee y^{-1}$
$\Rightarrow y = xx^{-1}y \leq x(x^{-1} \vee y^{-1})y$ and $x = xy^{-1}y \leq x(x^{-1} \vee y^{-1})y$
$\Rightarrow x \vee y \leq x(x^{-1} \vee y^{-1})y = x(x \wedge y)^{-1}y$ (iii)
Again $x \wedge y \leq x, y$
$\Rightarrow (x \wedge y)^{-1} \geq x^{-1}, y^{-1}$ (by Theorem 1.2.35(2)).
$\Rightarrow x(x \wedge y)^{-1}y \geq xx^{-1}y = y, xy^{-1}y = x$
$\Rightarrow x(x \wedge y)^{-1}y \geq y \vee x$ (iv)
Hence from (iii) and (iv) $x \vee y = x(x \wedge y)^{-1}y$. Here also, the dual identity can be proved analogously. Hence property (2) holds. □

Theorem 1.2.39. *[13] In l-group G, $(G, \vee, \wedge)$ is distributive.*

Proof. If $x \vee y = x \vee z$ and $x \wedge y = x \wedge z$ for $x, y, z \in G$, then
$y = (x \wedge y)x^{-1}x(x \wedge y)^{-1}y = (x \wedge y)x^{-1}(x \vee y)$ (by Theorem 1.2.38)
$= (x \wedge z)x^{-1}(x \vee z)$ (by assumption)
$= (x \wedge z)x^{-1}x(x \wedge z)^{-1}z$ (by Theorem 1.2.38)
$= Z$. Hence by Theorem 1.1.29, G is a distributive lattice. □

Theorem 1.2.40. *([7], Theorem 9.8) If $(G, \leq)$ is an ordered group such that $(G, \leq)$ is a semi-lattice, then it is a lattice.*

1.2.5 Drl-monoid

Dually residuated lattice ordered semigroups (briefly Drl-semigroups) were introduced and studied by K.L.N. Swamy as a common generalization of commutative lattice ordered groups (l-groups) and Brouwerian (and hence also Boolean) algebras.

Definition 1.2.41. *A system $A = (A, +, 0, \vee, \wedge, -)$ is called a Drl-monoid if the following conditions are satisfied, $(\forall x, y, z \in A)$*

1. *$(A, +, 0)$ is an abelian monoid;*

2. $(A, \vee, \wedge)$ *is a lattice;*

3. $(A, +, \vee, \wedge, 0)$ *is an l-monoid;*

4. *If* $\leq$ *denotes the order on* A *induced by the lattice* $(A, \vee, \wedge)$*, then* $x - y$ *is the smallest* $z \in A$ *such that* $y + z \geq x$*;*

5. $((x - y) \vee 0) + y \leq x \vee y$.

Example 1.2.42. *Let A be the multiplicative semigroup of the set of non negative integers ordered by the divisibility relation. Then A is a Drl semigroup with least element 1 and greatest element 0.*

Example 1.2.43. *Let A be the additive semigroup of the non negative integers with usual ordering. Then A is a Drl semigroup with least element 0.*

1.3 Brouwer-Heyting Monoids

Swamy [27] initiated the study of dually residualed lattice ordered semigroups (Drl. semigroups for short) in the context of obtaining a common abstraction of Brouwerian algebras (of [6]) and lattice ordered groups. They are called dually residuated because the residuation on is dual to that of ward and Dilworth.
In this section we study the concept of Brouwer-Heyting (BH- for short) Monoids and results concerning BH-Monoids.

1.3.1 Definition and Examples of BH- Monoids

We begin with the following.

Definition 1.3.1. *[26] A system* $(G, \circ, e, \rho, \rightarrow)$ *is a BH- monoid where*

1 $(G.\circ, e)$ *is a commutative semigroup with identity "e"*

2 (G, ρ) *is a partially ordered set and* $\rightarrow$ *is binary operation on G such that for all* x, a, b *in G,* $(x \circ b)\rho a \Leftrightarrow x\rho(a \rightarrow b)$.

We now give some examples.

Example 1.3.2. *[26] Suppose that $(G, \circ, e, \leq)$ is a commutative po-group. Define $a \to b = aob^{-1}$. Then G is a BH-Monoid.*

To show that this system is a BH- monoid, we have to show that condition (ii) holds.
Let (G, o, e, ρ) is a commutative po-group and $x, a, b \in G$
Clearly $\to$ is a binary operation on the set G.
Let $(xob) \leq a \Leftrightarrow (xob)ob^{-1} \leq aob^{-1}$ (by the translation invariant property in the definition of a po-group)
$\Leftrightarrow x \leq aob^{-1}$ (using the group property of the po-group)
Hence it is a BH- monoid.

Example 1.3.3. *[26] Suppose that $(B, \vee, \wedge, 0, 1)$ is a Boolean algebra. Let $\circ = \wedge$, $\rho = \leq$ and define $a\rho b$ if $a \wedge b = a$, $e = 1$, $a \to b = a \vee b'$.*

To show that this system is a BH- monoid consider the following.

1. Clearly from the definition of a Boolean algebra, $(B, \wedge, 1)$ is a commutative semi group with identity 1

2. Again from the order relation in Boolean algebra, (B, ρ) is a partially ordered set and

3. Since $\vee$ is a binary relation on the Boolean algebra B, it follows that $\to$ is binary operation on B.
   To show that $(xob)\rho a \Leftrightarrow x\rho(a \to b)$, $\forall x, a, b \in B$, consider the following. This is equivalent to showing $(x \wedge b) \leq a \Leftrightarrow x \leq (a \vee b')$, $\forall x, a, b \in B$.
   $(\Leftarrow)$ Let $x \leq (a \vee b')$
   $\Rightarrow x \wedge (a \vee b') = x$ (by the definition of $\leq$)
   $\Rightarrow (x \wedge a) \vee (x \wedge b') = x$ (distributive property of $\wedge$ over $\vee$ in Boolean algebra)
   $\Rightarrow [(x \wedge a) \vee (x \wedge b')] \wedge b = x \wedge b$ (by the Property of $\wedge$)
   $\Rightarrow [(x \wedge a) \wedge b] \vee [(x \wedge b') \wedge b] = x \wedge b$ (distributive property)
   $\Rightarrow [(x \wedge a) \wedge b] \vee 0 = x \wedge b$ (since $[(x \wedge b') \wedge b] = 0$)
   $\Rightarrow (x \wedge b) \wedge a = x \wedge b$ (by commutative and associative property of $\wedge$)
   $\Rightarrow (x \wedge b) \leq a$ (by definition of $\leq$ in the Boolean algebra)

($\Rightarrow$) Let $(x \wedge b) \leq a$

$\Rightarrow (x \wedge b) \wedge a = x \wedge b$ (by the definition of $\leq$ in the Boolean algebra)

$\Rightarrow [(x \wedge b) \wedge a] \vee b' = [x \wedge b] \vee b'$ (by property of $\vee$)

$\Rightarrow [(x \vee b')] \wedge [b \vee b'] \wedge [a \vee b'] = [x \vee b'] \wedge [b \vee b']$ (by distributivity)

$\Rightarrow [x \vee b'] \wedge [a \vee b'] = [x \vee b']$ (as $b \vee b' = 1$ and $x \wedge 1 = x$)

$\Rightarrow x \wedge [(x \vee b') \wedge (a \vee b')] = [x \wedge (x \vee b')]$ (by property of $\wedge$)

$\Rightarrow [x \wedge (x \vee b')] \wedge (a \vee b') = [x \wedge (x \vee b')]$ (by associativity of $\wedge$)

$\Rightarrow x \wedge (a \vee b') = x$ (by absorption law)

$\Rightarrow x \leq (a \vee b')$ (by the definition of $\leq$ in the Boolean algebra)

Hence it is a BH-monoid.

Example 1.3.4. *[26] Suppose that $(G, \vee, \wedge, 0, 1)$ is a Heyting algebra. Let $\circ = \wedge$, $e = 1$, ρ is the lattice order and $a \to b$ is the largest x such that $x \wedge b \leq a$.*

To show that this system is a BH- monoid consider the following.(Notice that the implication here and the one in the Heyting algebra are reverse of each other.)

1. Clearly from the definition of a Heyting algebra, $(G, \wedge, 1)$ is a commutative semi-group with identity 1

2. Again from the order relation in Heyting algebra (G, ρ) is a partially ordered set and

3. Also from the definition of Heyting algebra, it follows that $\to$ is binary operation on G (by remark 1.2.17).

   To show that for all $x, a, b \in G$, $(xob)\rho a \Leftrightarrow x\rho(a \to b)$. i.e, $x, a, b \in G$, $(x \wedge b) \leq a \Leftrightarrow x \leq (a \to b)$.

   This follows from the definition of Heyting algebra and Theorem 1.2.22(3). Hence it is a BH-monoid.

Example 1.3.5. *[26] Let $(L, \vee, \wedge, 0, 1)$ be Brouwerian lattice. Let $\rho =$ (the dual of $\leq$ (the lattice order)) $\geq$, $o = \vee$ monoid operation, the least $e = 0$ as identity element and $a \to b$ be the smallest x (denoted by $a - b$) such that $x \vee b \geq a$ i.e .$a \leq x \vee b \Leftrightarrow a - b \leq x$. Then $(L, \vee, 0, \geq, -)$ is a BH- monoid.*

To show that this system is a BH- monoid consider the following.

1. Clearly from the definition of Brouwerian lattice, $(L, \vee, 0)$ is a commutative semi group with identity 0

2. Again from the order relation in lattice, $(L, \geq)$ is a partially ordered set (as the dual of partial ordering is a partial ordering) and

3. From the definition of $-$, it is a binary operation on the Brouwerian lattice L. Showing that $(xob)\rho a \Leftrightarrow x\rho(a \to b), \forall x, a, b \in L$ is equivalent to showing $(x \vee b) \geq a \Leftrightarrow x \geq (a - b), \forall x, a, b \in B$. And this follows directly from the definition of Brouwerian lattice.

Example 1.3.6. *[26] Let $(G, +, \leq, -, 0)$ be a Drl- monoid. We have $x + b \geq a \Leftrightarrow a - b \leq x$(By the definition of Drl monoid). Thus $(G, +, \geq, -, 0)$ is a BH-monoid (the dual of Drl-monoid is a BH- monoid).*

To show that this system is a BH- monoid consider the following.

1. Clearly from the definition of Drl- monoid, $(G, +, 0)$ is a commutative semi group with identity 0

2. Again from the order relation in lattice $(G, \leq)$ it follows that $(G, \geq)$ is a partially ordered set.

3. From the definition of Drl- monoid, $-$ is a binary operation.
   To show that for all $x, a, b \in G$, $(xob)\rho a \Leftrightarrow x\rho(a \to b)$. i.e, for $x, a, b \in G$, $(x + b) \geq a \Leftrightarrow x \geq (a - b)$. And this follows directly from the definition of Drl- monoid.

1.3.2 Some results on BH Monoids

Here after for the sake of convenience, we use $\leq$ instead of ρ. Let $(G, o, e, \leq, \to)$ be a BH- monoid and $a, b, c \in L$. Then:

Lemma 1.3.7. *[26] $b \leq c \Rightarrow a \circ b \leq a \circ c$*

Proof. Let $b \leq c, coa = aoc$ (o is commutative)
$\Rightarrow coa \leq aoc$ ($\leq$ is reflexive).
$\Rightarrow c \leq (aoc) \rightarrow a$ (by definition condition of $\rightarrow$).
$\Rightarrow b \leq c \leq (aoc) \rightarrow a$(by the given hypothesis).
$\Rightarrow b \leq (aoc) \rightarrow a$ (by the transitivity of $\leq$).
$\Rightarrow (boa) \leq (aoc)$ (by definition condition of $\rightarrow$).
$\Rightarrow (aob) \leq (aoc)$ (by the commutativity of o). □

Lemma 1.3.8. *[26] $a \leq (a \circ b) \rightarrow b$.*

Proof. $aob \leq aob$ (by reflexivity of $\leq$)
$\Rightarrow a \leq (aob) \rightarrow b$ (by definition condition of $\rightarrow$). □

Lemma 1.3.9. *[26] $(a \rightarrow b) \circ b \leq a$*

Proof. $(a \rightarrow b) \leq a \rightarrow b$ (by reflexivity of $\leq$)
$\Rightarrow bo(a \rightarrow b) \leq a$ (by definition condition of $\rightarrow$). Hence the result holds □

Lemma 1.3.10. *[26] $c \rightarrow (a \circ b) = (c \rightarrow b) \rightarrow a = (c \rightarrow a) \rightarrow b$.*

Proof. $(c \rightarrow (aob))o(aob) \leq c$ (by Lemma 1.3.9)
$\Rightarrow ((c \rightarrow (aob))oa)ob \leq c$ (o is associative)
$\Rightarrow ((c \rightarrow (aob))oa) \leq c \rightarrow b$ (by definition of $\rightarrow$)
$\Rightarrow (c \rightarrow (aob)) \leq (c \rightarrow b) \rightarrow a$ (by definition of $\rightarrow$) i
Moreover, $((c \rightarrow b) \rightarrow a)oa \leq c \rightarrow b$ (by Lemma 1.3.9)
$\Rightarrow ((c \rightarrow b) \rightarrow a)o(aob) \leq (c \rightarrow b)ob)$ (by Lemma 1.3.7, commutativity and associativity of o)
$\leq c$ (by Lemma 1.3.9). That is $((c \rightarrow b) \rightarrow a)o(aob) \leq c$
$\Rightarrow (c \rightarrow b) \rightarrow a \leq c \rightarrow (aob)$ (by definition of $\rightarrow$) ii
From (i) and (ii) we have, $c \rightarrow (aob) = (c \rightarrow b) \rightarrow a$ iii
Using the commutativity of o in (iii), $c \rightarrow (aob) = c \rightarrow (boa) = (c \rightarrow a) \rightarrow b$. So the result holds □

Lemma 1.3.11. *[26]* $e \to e = e$

Proof. $eoe = e$ (since e is the identity element)

$\Rightarrow eoe \leq e$ (by reflexivity of $\leq$)

$\Rightarrow e \leq e \to e$ (by the definition condition of $\to$) i

$\Rightarrow e = eoe \leq (e \to e)oe$ (by Lemma 1.3.7) $\leq e$(by Lemma 1.3.9)

That is $e \to e = (e \to e)oe \leq e$ (because e is an identity element) ii

Hence from (i) and (ii) we have $e \to e = e$ which is the required result. □

Lemma 1.3.12. *[26]* $a \to e = a$

Proof. $aoe = a$ (because e is an identity element).

$\Rightarrow aoe \leq a$ (because of reflexivity of $\leq$)

$\Rightarrow a \leq a \to e$ (by the definition condition of $\to$)

$\Rightarrow a \leq (a \to e)oe$ (because e is an identity element) $\leq a$ (by Lemma 1.3.9)

That is $a \leq a \to e \leq a$. Hence by antisymmetry property of $\leq, a = a \to e$ □

Lemma 1.3.13. *[26]* $e \leq a \to a$

Proof. $eoa = a \leq a$ (by reflexivity of $\leq$)

$\Rightarrow e \leq a \to a$ (by definition condition of $\to$) □

Lemma 1.3.14. *[26]* $a \leq b \Rightarrow a \to c \leq b \to c$

Proof. $co(a \to c) = (a \to c)oc$ (o is commutative.)

$\leq a$ (by Lemma 1.3.9) $\leq b$ (by the given hypothesis). That is $(a \to c)oc \leq b$.

$\Rightarrow a \to c \leq b \to c$ (using the definition condition of $\to$)

Hence $a \leq b \Rightarrow a \to c \leq b \to c$ which is the required result. □

Lemma 1.3.15. *[26]* $a \leq b \Rightarrow c \to b \leq c \to a$

Proof. Let $a \leq b$. $\Rightarrow (c \to b)oa \leq (c \to b)ob$ (by Lemma 1.3.7) $\leq c$ (by Lemma 1.3.9)

That is $(c \to b)oa \leq c$ (by transitivity of $\leq$).

$\Rightarrow c \to b \leq c \to a$ (using the definition condition of $\to$). □

Lemma 1.3.16. *[26]* $b \leq a \Leftrightarrow e \leq a \to b$

Proof. Let $b \leq a \Leftrightarrow eob \leq a$ (e is an identity element).

$\Leftrightarrow e \leq a \to b$ (by defining condition of $\to$) □

Lemma 1.3.17. *[26]* $(a \to b) \circ (b \to c) \leq a \to c$

Proof. $co(a \to b)o(b \to c) = co(b \to c)o(a \to b)$

$= (b \to c)oco(a \to b)$ (o is commutative.)

$\leq bo(a \to b)$ (by Lemma 1.3.9, $(b \to c)oc \leq b$, and Lemma 1.3.7)

$\leq a$ (by Lemma 1.3.9 and commutativity of o).

$\Rightarrow (a \to b)o(b \to c)oc \leq a$ (by transitivity of $\leq$ and commutativity of o)

$\Rightarrow (a \to b)o(b \to c) \leq a \to c$ (by definition condition of $\to$). Hence the result holds. □

Lemma 1.3.18. *[26] If* $a \vee b$ *exists, then* $(c \circ a) \vee (c \circ b)$ *exists for any* c *and* $c \circ (a \vee b) = (c \circ a) \vee (c \circ b)$

Proof. Let $t = a \vee b = lub\{a, b\}$ exists for any a, b.

$\Rightarrow a \leq t, b \leq t$ (by the definition of lub)

$\Rightarrow coa \leq cot, cob \leq cot$ (by Lemma 1.3.7)

$\Rightarrow cot = co(a \vee b)$ is an upper bound for coa and cob. (i)

To show the other containment consider the following. Let u be any upper bound for $\{coa, cob\}$. Then $coa \leq u, cob \leq u$(by the definition of an upper bound)

$\Rightarrow aoc \leq u, boc \leq u$ (o is commutative).

$\Rightarrow a \leq u \to c, b \leq u \to c$ (by the definition condition of $\to$)

$\Rightarrow a \vee b \leq u \to c$ (by the definition of lub)

$\Rightarrow co(a \vee b) \leq u$ (by the definition condition of $\to$).

That is $co(a \vee b) \leq u$ (ii)

From (i) and (ii) we have $co(a \vee b) = (coa) \vee (cob)$. □

Lemma 1.3.19. *[26] If $a \vee b$ exists, then $(c \to a) \wedge (c \to b)$ exist for any c and $c \to (a \vee b) = (c \to a) \wedge (c \to b)$*

Proof. Let $u = a \vee b$ exists for any a, b. since $a \leq a \vee b$, $b \leq a \vee b$ (by the definition of *lub*), we obtain that $c \to a \vee b \leq c \to a, c \to a \vee b \leq c \to b$ (by Lemma 1.3.15).
So $c \to a \vee b$ is a lower bound for $c \to a$ and $c \to b$. To show the other containment, consider the following. Let u be any lower bound for $c \to a$ and $c \to b$.
$\Rightarrow u \leq c \to a$ and $u \leq c \to b$ (by the definition of lower bound).
$\Rightarrow uoa \leq c, uob \leq c$ (by the definition condition of $\to$)
$\Rightarrow aou \leq c, bou \leq c$ (by the commutativity of o)
$\Rightarrow a \leq c \to u, b \leq c \to u$ (by the definition condition of $\to$)
$\Rightarrow a \vee b \leq c \to u$ (by the property of $\vee$)
$\Rightarrow c \to (c \to u) \leq c \to (a \vee b)$ (by Lemma 1.3.15) (i)
Since by Lemma 1.3.9, $(c \to u)ou \leq c$, it follows that, $uo(c \to u) \leq c$ (commutativity of o).
$u \leq c \to (c \to u)$ (by definition condition of $\to$) (ii)
From (i) and (ii) , $u \leq c \to (c \to u) \leq c \to (a \vee b)$
$\Rightarrow u \leq c \to (a \vee b)$ (by transitivity of $\leq$). So the result $c \to (a \vee b) = (c \to a) \wedge (c \to b)$ holds. □

Lemma 1.3.20. *[26] If $a \wedge b$ exists, then $(a \wedge b) \to c$ exists for any c and $(a \wedge b) \to c = (a \to c) \wedge (b \to c)$.*

Proof. Let $a \wedge b$ exists. Then $a \wedge b \leq a, a \wedge b \leq b$ (by definition of *glb*)
$\Rightarrow (a \wedge b) \to c \leq a \to c, (a \wedge b) \to c \leq b \to c$ (by Lemma 1.3.14). So $(a \wedge b) \to c$ is a lower bound of $a \to c$ and $b \to c$. Let u be any lower bound for $(a \to c)$ and $(b \to c)$. Then $u \leq a \to c, u \leq b \to c$ (by definition of lower bound).
$\Rightarrow uoc \leq a, uoc \leq b$ (by definition condition of $\to$). Hence uoc is lower bound for a and b. So $uoc \leq a \wedge b$ (by definition of *glb*).
$\Rightarrow u \leq (a \wedge b) \to c$ (by definition condition of $\to$). So for any lower bound u of $(a \to c)$ and $(b \to c)$, $u \leq (a \wedge b) \to c$. Accordingly, $(a \to c) \wedge (b \to c) = (a \wedge b) \to c$. □

1.3.3 BH-lattices

Definition 1.3.21. *[26] A BH- monoid system* $(L, \circ, e, \leq, \rightarrow)$ *is a BH- lattice if*

1. $(L, \leq)$ *is a lattice with glb and lub denoted* $\wedge$ *and* $\vee$ *respectively*
2. $x \circ (y \wedge z) = (x \circ y) \wedge (x \circ z), \forall x, y, z \in L$
3. $((y \rightarrow x) \wedge e) \circ x = x \wedge y, \forall x, y \in L$

Theorem 1.3.22. *[26] A lattice* $(L, \vee, \wedge, o, e, \rightarrow)$*, where* $(L, \circ, e)$ *is a commutative monoid and* $\rightarrow$ *be a binary operation on L,is a BH lattice* $\Leftrightarrow$

1. $(y \rightarrow x) \circ x \leq y$
2. $(x \wedge z) \rightarrow y \leq x \rightarrow y$
3. $x \leq x \circ y \rightarrow y, \forall x, y \in L$
4. $a \circ (b \wedge c) = (a \circ b) \wedge (a \circ c), \forall a, b, c \in L$
5. $((b \rightarrow a) \wedge e) \circ a = a \wedge b, \forall a, b, c \in L$

Proof. $(\Rightarrow)$ Let a lattice $(L, \vee, \wedge, o, e, \rightarrow)$, where (L, o, e) is a monoid and $\rightarrow$ be a binary operation on L,be a BH- lattice. Then

1. Follows from Lemma 1.3.9
2. $x \wedge z \leq x$ (by the definition of $\leq$ in the lattice $(L, \vee, \wedge, o, e, \rightarrow)$)
   $\Rightarrow (x \wedge z) \rightarrow y \leq x \rightarrow y$ (by Lemma 1.3.14)
3. Since $(L, \leq)$ is a partial ordering, $xoy \leq xoy$ (by reflexivity)
   $\Rightarrow x \leq xoy \rightarrow y$ (by definition condition of $\rightarrow$)
4. Follows from (2) in the definition of BH lattice.
5. Follows from (3) in the definition of BH lattice.

$(\Leftarrow)$ Let $(L, \vee, \wedge, o, e, \rightarrow)$ be a lattice, where (L, o, e) is a commutative monoid and $\rightarrow$

be a binary operation on L. And let the conditions (1) to (5) be satisfied. Then

1. (L, o, e) is a commutative monoid (given)

2. If we define $\leq$ on L by $a \leq b$ iff $a = a \wedge b$, then $(L, \leq)$ is a lattice with glb and lub denoted $\wedge$ and $\vee$ respectively.

3. $\rightarrow$ be a binary operation on L (Given). To show $(xob) \leq a \Leftrightarrow x \leq (a \rightarrow b)$, consider the following.
   Let $xob \leq a$. Then $a \wedge (xob) = xob$. (by the definition of $\leq$ in the lattice)
   Now $x \leq (xob) \rightarrow b$ (by 3 in the given condition)
   $= (a \wedge (xob)) \rightarrow b$ (by assumption condition $a \wedge (xob) = xob$)
   $\leq a \rightarrow b$ (by 2 in the given condition).
   Hence $xob \leq a \Rightarrow x \leq a \rightarrow b$ (i)
   Let $x \leq a \rightarrow b \Rightarrow x \wedge (a \rightarrow b) = x$ (by definition of $\leq$ in the lattice)
   $\Rightarrow bo[x \wedge (a \rightarrow b)] = xob$ (o is a commutative binary operation on L).
   $\Rightarrow (box) \wedge [bo(a \rightarrow b)] = xob$ (by 4 in the given condition)
   $\Rightarrow xob = (box) \wedge [bo(a \rightarrow b)] \leq (a \rightarrow b)ob \leq a$ (by 1 and property of $\wedge$). Hence $x \leq a \rightarrow b \Rightarrow xob \leq a$. (ii)
   So from (i) and (ii) it follows that $(xob) \leq a \Leftrightarrow x \leq (a \rightarrow b)$. Hence $(L, o, e, \leq, \rightarrow)$ is a BH- monoid.

4. $ao(b \wedge c) = (aob) \wedge (aoc)$ for all $a, b, c \in L$ (follows from 4 in the given condition).

5. $((b \rightarrow a) \wedge e)oa = a \wedge b$, $\forall a, b \in L$ (follows from 5 in the given condition). Hence it is a BH- lattice.

□

Remark 1.3.23. *[26] Theorem* 1.3.22 *shows that BH- lattices can be defined by means of identities alone.*

Here after, throughout this section L denote a BH- lattice.

Theorem 1.3.24. *[26] In BH-lattice $a \rightarrow a = e$, $\forall a \in L$.*

Proof. $e \leq a \to a$ (by Lemma 1.3.13) (i)

Let $d = a \to a$, then

$dod = (a \to a)o(a \to a) \leq a \to a = d$ (by Lemma 1.3.17) (ii)

Moreover, $e \leq a \to a$ (by Lemma 1.3.13)

$\Rightarrow eod \leq (a \to a)od$ (by Lemma 1.3.7)

$\Rightarrow d = a \to a \leq (a \to a)o(a \to a) = dod$ (iii)

Hence from (ii) and (iii) $dod = d$. By (3) in the definition of BH-lattice, for any x with $e \leq x$, $((e \to x) \wedge e)ox = e \wedge x = e$ (by definition of $\wedge$).

Now, for $x = d$ using (i), $((e \to d) \wedge e)od = e \wedge d = e$

$\Rightarrow d = eod = ((e \to d) \wedge e)odod$

But $(e \to d) \wedge e \leq e \to d$ (using the definition of $\wedge$)

$\Rightarrow d = ((e \to d) \wedge e)odod \leq (e \to d)odod$ (by Lemma 1.3.7)

$= (e \to d)od \leq e$ (as d o d = d, by Lemma 1.3.9)

Hence by transitivity $d \leq e$ (iv)

From (i) and (iv), $a \to a = e$ which is the required result. □

Theorem 1.3.25. *[26] Any BH-lattice L is distributive.*

Proof. Recall that, by Theorem 1.1.31 a lattice L is distributive iff any element has at most one relative complement in any interval. So it is enough to show that relative complements are uniquely determined. Let for $a, b, x, y, y' \in L, a \leq x \leq b$, such that $x \wedge y = x \wedge y' = a, x \vee y = x \vee y' = b$.

Now $y = y \wedge (y \vee x)$ (absorption law)

$= ((y \to (y \vee x)) \wedge e)o(y \vee x)$ (by (3) in the definition of BH-lattices)

$= ((y \to y) \wedge (y \to x) \wedge e)o(y \vee x)$ (using Lemma 1.3.19)

$= (e \wedge (y \to x) \wedge e)o(y \vee x)$ (by Theorem 1.3.24)

$= ((y \to x) \wedge e)o(y \vee x)$ (by the property of $\wedge$)

$= [(y \to x) \wedge (x \to x)]o(y \vee x)$(by Theorem 1.3.24)

$= ((y \wedge x) \to x)o(y \vee x)$ (by Lemma 1.3.20)

$= ((y' \wedge x) \to x)o(y' \vee x)$ (by the hypothesis)

$= [(y' \to x) \wedge (x \to x)]o(y' \vee x)$ (by Lemma 1.3.20)

$= ((y' \to x) \wedge e)o(y' \vee x)$ (by Theorem 1.3.24)

$= [((y' \to x) \wedge e) \wedge e]o(y' \vee x)$ (by the property of $\wedge$)
$= ((y' \to x) \wedge (y' \to y') \wedge e)o(y' \vee x)$ (by Theorem 1.3.24)
$= ((y' \to (y' \vee x) \wedge e)o(y' \vee x)$ (by Lemma 1.3.19)
$= y' \wedge (y' \vee x)$ (by (3) in the definition of BH-lattices)
$= y'$ (absorption law). Hence relative complements are uniquely determined and consequently it is a distributive lattice. □

Theorem 1.3.26. *In BH-lattice* L $(a \vee b) \circ (a \wedge b) = a \circ b$, $\forall a, b \in L$.

Proof. $[a \to (a \vee b)]ob = [(a \to a) \wedge (a \to b)]ob$ (by Lemma 1.3.19) $= [e \wedge (a \to b)]ob$ (by Theorem 1.3.24) $= [(a \to b) \wedge e]ob$ ($\wedge$ is commutative in a lattice) $= a \wedge b$ (by (3) in the definition of BH-lattices).
Hence $a \wedge b = [a \to (a \vee b)]ob$ (i)
Now $(a \wedge b)o(a \vee b) = [[a \to (a \vee b)]ob]o(a \vee b)$ (using (i))
$= [[a \to (a \vee b)]o(a \vee b)]ob$ (by commutativity and associativity of o) (ii)
$\leq aob$ (by Lemma 1.3.9 and 1.3.7)
Hence $(a \wedge b)o(a \vee b) = [[a \to (a \vee b)]ob]o(a \vee b) \leq aob$ (iii)
Again $[a \to (a \vee b)] \wedge e \leq [a \to (a \vee b)]$ (by the definition of $\wedge$) $\Rightarrow [[a \to (a \vee b)] \wedge e]o(a \vee b)ob \leq [a \to (a \vee b)]o(a \vee b)ob$ (by Lemma 1.3.7) $\Rightarrow [a \wedge (a \vee b)]ob = [[a \to (a \vee b)] \wedge e]o(a \vee b)ob \leq [a \to (a \vee b)]o(a \vee b)ob$ (by (3) in the definition of BH-lattices) $\Rightarrow [a \wedge (a \vee b)]ob \leq [a \to (a \vee b)]o(a \vee b)ob \Rightarrow aob = [a \wedge (a \vee b)]ob \leq [a \to (a \vee b)]o(a \vee b)ob$ (by absorption law)
Hence $aob \leq [a \to (a \vee b)]o(a \vee b)ob = (a \wedge b)o(a \vee b)$ (by ii) (iv)
Hence from (iii) and (iv), $(a \wedge b)o(a \vee b) = aob$ □

Corollary 1.3.27. *[26] In BH-lattice* L, $a = a \circ e = (a \vee e) \circ (a \wedge e)$, $\forall a \in L$.

Proof. Let $b = e$ in the above Theorem 1.3.26. □

Theorem 1.3.28. *[26] In BH-lattice* L*, any* $x \in L$ *with* $e \leq x$ *is invertible.*

Proof. let $e \leq x \Rightarrow e \to x \leq e \to e = e$ (by Lemma 1.3.15 and 1.3.11) $\Rightarrow (e \to x) \wedge e = e \to x, e \wedge x = e$ (by the definition of $\wedge$) (i)

Now by (3) in the definition of BH- lattice, for any x with $e \leq x$ we have $((e \to x) \wedge e)ox = e \wedge x \Rightarrow (e \to x)ox = e$ (by (i) above) and hence inverse of x is $e \to x$. □

Theorem 1.3.29. *[26] In BH- lattice L, for $x \in L$, $e \vee x$ is invertible.*

Since by Theorem 1.3.28, any x with $e \leq x$ is invertible and from the definition of $\vee, e \leq e \vee x$. So it is invertible.

Theorem 1.3.30. *[26] In BH-lattice L, for $x \in L$, if x is invertible, then $e \to x$ is inverse of x.*

Proof. Let x is invertible. $\Rightarrow \exists y$ such that $yox = e \Rightarrow yox \leq e$ (Reflexive.) $\Rightarrow y \leq e \to x$ (by definition condition of $\to$). $\Rightarrow yox \leq (e \to x)ox$ (by Lemma 1.3.7). $\Rightarrow e = yox \leq (e \to x)ox \leq e$ (by Lemma 1.3.9). $\Rightarrow (e \to x)ox = e$ (antisymmetric property of $\leq$) Hence $e \to x$ is the inverse of x (by uniqueness of an inverse in associative algebraic structure). □

Remark 1.3.31. *Theorem* 1.3.30 *holds also in BH- moniods.*

Theorem 1.3.32. *[26] In BH-lattice L, if b is invertible, then $a \to b = a \circ (e \to b)$, $\forall a, b \in L$.*

Proof. Let b is invertible. By Theorem 1.3.30, $e \to b$ is the inverse of b. So that $(e \to b)ob = e$. Since for any a, we have $aoe = a \Rightarrow ao(e \to b)ob = a$. Suppose that $x = a \to b$, then $xob = (a \to b)ob \leq a$ (by Lemma 1.3.9) $\Rightarrow xobo(e \to b) \leq ao(e \to b)$ (by Lemma 1.3.7) $\Rightarrow xoe \leq ao(e \to b)$ (since $bo(e \to b) = e$) $\Rightarrow x \leq ao(e \to b)$

That is $x = a \to b \leq ao(e \to b)$ (i)

But by Lemma 1.3.12, $a \to e = a$, $ao(e \to b) = (a \to e)o(e \to b) \leq a \to b$ (by Lemma 1.3.17) Hence $ao(e \to b) \leq a \to b$ (ii)

Thus from (i) and (ii), $ao(e \to b) = a \to b$ □

Theorem 1.3.33. *[26] If in BH-lattice L, a and b are invertible, then aob is invertible and $(e \to a)o(e \to b)$ is the inverse of aob, $\forall a, b \in L$.*

Proof. Let in BH-lattice L, $a, b \in L$ are invertible. It follows from Theorem 1.3.30 and associativity of the algebraic operation and uniqueness of an inverse. ☐

Theorem 1.3.34. *[26] In BH-lattice L, for $x \in L$, $e \to x$ is invertible.*

Proof. For any $x, xoe = x = (x \vee e)o(x \wedge e)$ (by Theorem 1.3.26) $\Rightarrow e \to x = ((e \to x) \vee e)o((e \to x) \wedge e)$, (replacing x by $e \to x$). Now $x \wedge e \leq e$ (by the definition of $\wedge$) $\Rightarrow e \to e \leq e \to (x \wedge e)$ (by Lemma 1.3.15) $\Rightarrow e \leq e \to (x \wedge e)$ (by Theorem 1.3.24). Hence by Theorem 1.3.28, $e \to (x \wedge e)$ is invertible.
Again by Theorem 1.3.29, $x \vee e$ is invertible.
Hence by Theorem 1.3.33(2), $(e \to (x \wedge e)) \to (x \vee e)$ is invertible.
Thus, $(e \to (x \wedge e)) \to (x \vee e) = (e \to [(x \wedge e)o(x \vee e)])$ (by Lemma 1.3.10)
$= e \to x$ (by Theorem 1.3.26) which is invertible. ☐

.

Theorem 1.3.35. *[26] In BH-lattice L, $G = \{x \in L : x$ is invertible element $\}$ is an l-group.*

Proof. since $eoe = e, e \in G$. By Theorem 1.3.33, $a, b \in G \Rightarrow aob \in G$. Hence (G, o) is a group (by Theorem 1.3.30, 1.3.34). And by Lemma 1.3.7, $a \leq b \Rightarrow coaod \leq cobod$. Consequently using the definition G is a po-group . For any $a \in G, a \vee e \in G$ (by Theorem 1.3.28). Hence by Theorem 1.2.37 G is an l-group. ☐

Theorem 1.3.36. *(Decomposition Theorem for BH- Monoids)*
[26] A BH- monoid L is the direct product of po- group and a BH- monoid with greatest element iff

1. $e \to a$ *is invertible,* $\forall a \in L$.
2. $a \to a = e$, $\forall a \in L$.

Proof. Let L is a BH- monoid and assume that (1) and (2)holds. Let $G = \{x \in L : x$ is invertible element $\}$ and $H = \{a \in L : e \to a = e\}$. Clearly $(G, \leq, o)$ is a po-group.

To show that H is a BH- monoid with greatest element, consider the following.

i. By (2) $e \in H$. So that H is non - empty.

ii. Let $a \in H \Rightarrow e \to a = e$ (by the definition of H)
$\Rightarrow e \leq e \to a$ (reflexive)
$\Rightarrow a = eoa \leq e$. (using the definition condition of $\to$ in L). Hence e is the greatest element in H

iii. Let $a, b \in H$, $e \to (aob) = (e \to a) \to b$ (by Lemma 1.3.10) $= e \to b = e$ (using the definition of H). Thus H is closed under o.

iv. Let $a, b \in H \Rightarrow a \leq e$ (e is the greatest element) $\Rightarrow a \to b \leq e \to b = e$ (by Lemma 1.3.14) $\Rightarrow e \to (a \to b) \geq e \to e = e$. (using Lemma 1.3.15).
Hence $e \to (a \to b) \geq e$ (i)
Now as $b \in H$, $b \leq e \Rightarrow a \to b \geq a \to e = a$ (by Lemma 1.3.15 and Lemma 1.3.12) $\Rightarrow e \to (a \to b) \leq e \to a = e$ (by Lemma 1.3.15).
Hence $e \to (a \to b) \leq e$ (ii)
Hence from (i) and (ii) it follows that $e \to (a \to b) = e$ and hence $a, b \in H \Rightarrow (a \to b) \in H$. Consequently $\to$ is a binary operation on H. So that H is a BH- monoid with greatest element e (the other axioms for BH- monoid holds in H, as $H \subset L$).

Now to show that L is direct product of G and H, consider the following. Now let $a \in L$. Put $x = ao(e \to a), y = e \to (e \to a)$. $e \to x = e \to [ao(e \to a)] = (e \to a) \to (e \to a) = e$(by Lemma 1.3.10 and (2) in the hypothesis given). So that x belongs to H and y is invertible by hypothesis (1) and so is in G. Now $xoy = [ao(e \to a)]o[e \to (e \to a)] = aoe = a$. (since $(e \to a)$ and $e \to (e \to a)$ are inverses of each other (by Theorem 1.3.30)). Now let $a = x'oy', x' \in H, y' \in G$. $e \to a = e \to x'oy' = (e \to x') \to y'$ (by Lemma 1.3.10) $= e \to y'$ (as $e \to x' = e$, being $x' \in H$). Also $e \to a = e \to xoy = e \to y$. Now $e \to y = e \to y'$. That is inverse of y and inverse of y' are equal. $\Rightarrow e \to (e \to y) = e \to (e \to y')$ (since $\to$ is a binary operation). $\Rightarrow y = y'$ (as y and y' are elements of a group). Now $a = xoy = x'oy' \Rightarrow xoy = x'oy$ (since $y = y'$) $\Rightarrow xoyo(e \to y) = x'oyo(e \to y)$

(since o is a binary operation) $\Rightarrow x = x'$. To show that $G \cap H = \{e\}$. Clearly $e \in G \cap H$. Let $a \in G \cap H$. $\Rightarrow a \in G$ and $a \in H \Rightarrow$ a is invertible and $e \to a = e$. $\Rightarrow ao(e \to a) = e \Rightarrow aoe = e \Rightarrow a = e$. Hence a BH- monoid L is the direct product of po- group and a BH- monoid with greatest element. □

Theorem 1.3.37. *(Decomposition Theorem for BH- Lattices)*
[26] Any BH- lattice L is the direct product of a commutative l-group and a BH- lattice with greatest element. The converse is obvious.

Proof. Suppose that $G = \{x \in L : x$ is invertible element $\}$ and $H = \{a \in L : e \to a = e\}$. By Theorem 1.3.36, L is already direct product of G and H, as BH- monoids. G is a commutative l-group by Theorem 1.3.35 and definition of BH- monoids. Now we show that H is a BH- lattice. For $a, b \in H$, clearly $a \vee b, a \wedge b \in L$. $e \to (a \vee b) = (e \to a) \wedge (e \to b) = e \wedge e = e$. (By Lemma 1.3.19). Hence $a \vee b \in H$. To show $e \to (a \wedge b) = e$. Using the proof of Theorem 1.3.36, $aob \in H$. $\Rightarrow e = e \to (aob) = e \to ((a \vee b)o(a \wedge b))$ (by Lemma 1.3.10 and Theorem 1.3.26) $= (e \to (a \vee b)) \to (a \wedge b)$ (by Lemma 1.3.10) $= e \to (a \wedge b)$ (as $e \to (a \vee b) = e$). Hence $a \wedge b \in H$. i.e, $a, b \in H$, $e \to (a \wedge b) \in H$ and $e \to (a \vee b) \in H$. So $(H, \leq)$ is a lattice with glb and lub denoted by $\wedge$ and $\vee$ respectively. The other properties for a BH-lattice holds in H as they holds in L. Hence L is the direct product of a commutative l-group and a BH-lattice with greatest element. □

Chapter 2

Further Properties of BH-lattices

In this part we furnish examples of BH-lattices and develop further properties of BH-lattices.

2.1 Examples of BH-lattices

Example 2.1.1. *Boolean algebra as defined in example 1.3.3 above are BH-lattices. To show this consider a Boolean algebra* $(B, \vee, \wedge)$.

1. *Clearly by example 1.3.3 above, it is a BH-monoid.*

2. *Clearly it is a lattice.*

3. $xo(y \wedge z) = x \wedge (y \wedge z) = (x \wedge x) \wedge (y \wedge z) = (x \wedge y) \wedge (x \wedge z)$.

4. $((x \to y) \wedge e)oy = ((x \vee y') \wedge e) \wedge y = (x \vee y') \wedge y = (x \wedge y) \vee (y \wedge y') = x \wedge y$

Hence a Boolean algebra is a BH-lattice as defined in example 1.3.3 above.

Example 2.1.2. *Heyting algebra as defined in example 1.3.4 above are BH-lattices. To show this consider a Heyting algebra* $(H, \vee, \wedge)$.

1. *Clearly by example 1.3.4 above, it is a BH-monoid.*

2. *Clearly it is a lattice.*

3. $xo(y \wedge z) = x \wedge (y \wedge z) = (x \wedge x) \wedge (y \wedge z) = (x \wedge y) \wedge (x \wedge z)$.

4. $((x \to y) \wedge e)oy = (x \to y) \wedge y = x \wedge y$ *(by (4) of theorem 1.2.22).*

Hence a Heyting algebra is a BH-lattice as defined in example 1.3.4 above.

Example 2.1.3. *Commutative l-groups are BH-lattices with* $\to$ *defined as in example* 1.3.2 *above.*
To show this consider a commutative l-group G.

1. *Clearly by example 1.3.2 above, it is a BH-monoid.*

2. *Clearly by definition it is a lattice.*

3. $xo(y \wedge z) = (x^{-1})^{-1}o(y \wedge z) = (y \wedge z) \to x^{-1}$ *(by definition of* $\to$ *and property of inverse in a group)* $= (y \to x^{-1}) \wedge (z \to x^{-1})$ *(by Lemma 1.3.20)* $= (yo(x^{-1})^{-1}) \wedge (zo(x^{-1})^{-1}) = (yox) \wedge (zox)$.

4. $((x \to y) \wedge e)oy = ((xoy^{-1}) \wedge e)oy = ((xoy^{-1})oy) \wedge (eoy)$ *(by 3 above)* $= xo(y^{-1}oy) \wedge y = x \wedge y$.

Hence a commutative l-group is a BH-lattice as defined in example 1.3.2 above.

Bounded commutative l-groups, Boolean algebra and Heyting algebra are examples of bounded BH-lattices while unbounded l-groups are examples of unbounded BH-lattices. Now we have the following examples which are BH-monoids but not BH-lattices.

Example 2.1.4. *In a commutative po-group* $(G, \circ, e, \rho)$ *given in example* 1.3.2 *above the system* $(G, \leq)$ *may not be a lattice. So it is not a BH-lattice.*

Example 2.1.5. *Consider the lattice given in Fig* 2.1*. Clearly it is a Brouwerian algebra and hence a BH-monoid. Since* $b = [(c \to b) \wedge e]ob = [(c \to b) \wedge 0]ob \neq c \wedge b = a$. *Hence condition 3 in the definition of BH-lattices is failed. Thus Brouwerian algebra as given in Example* 1.3.5 *is not a BH-lattice.*

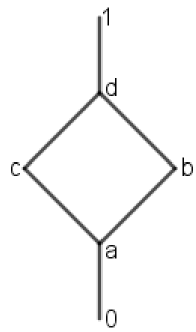


Figure 2.1: Example of BH-monoid which is not a BH-lattice

Example 2.1.6. *Consider the set A - the multiplicative semigroup of the set of non negative integers ordered by the divisibility relation. Then A is a Drl- semigroup with least element 1 and greatest element 0. By example 1.3.6, it is a BH-monoid. For* $x, y \in A$, $x \to y = \lfloor \frac{x}{y} \rfloor$, *where* $\lfloor . \rfloor$ *is the floor function for* $y \neq 0$, $y \wedge x = GCF(x, y)$ *and* $y \vee x = LCM(x, y)$. *Then for the BH-monoid induced from the above Drl-monoid,* $[(3 \to 2) \wedge e]o2 = (1 \wedge 1).2 = 1.2 = 2 \neq 3 \wedge 2 = 6$. *Hence condition 3 in the definition of BH-lattices is failed. Hence it is not a BH-lattice. Thus Drl-monoid is not a BH-lattice.*

Note: After now on $(L, o, e, \leq, \to)$ or simply L stands for a BH-lattice $(L, o, e, \leq, \to)$ and $x, y, z \in L$.

2.2 Further Properties of BH-lattices

Theorem 2.2.1. *In BH- lattice L, for* $x, y \in L$, *if x and y are invertible, then* $x \to y$ *is invertible and* $y \to x$ *is the inverse of* $x \to y$.

Proof. Suppose that L is a BH-lattice and $x, y \in L$ are invertible elements.
$\Rightarrow x \to y = xo(e \to y)$ and $y \to x = yo(e \to x)$ (By Theorem 1.3.32).
$\Rightarrow (x \to y)o(y \to x) = [xo(e \to y)]o[yo(e \to x)]$
$= xo[[(e \to y)oy]o(e \to x)]$(associative)
$= xo[eo(e \to x)] = e.$ (By Theorem 1.3.30)
Hence the result holds. □

Theorem 2.2.2. *For every BH- lattice L, the following properties holds* $\forall x, y, z \in L$.

1. $x \leq y \to (y \to x)$. *So* $x \wedge y \leq [y \to (y \to x)] \wedge [x \to (x \to y)]$.

2. $z \leq (xoz) \to (x \wedge y)$ *or equivalently* $(x \wedge y) \leq (xoz) \to z$

3. $x \wedge y \leq [(xoy) \to y] \wedge [(xoy) \to x] = xoy \to (x \vee y)$

Proof. Assume that L is a BH-lattice and $x, y, z \in L$.
$y \to x \leq y \to x$(reflexive).
$\Rightarrow (y \to x)ox = xo(y \to x) \leq y$ (commutative and the defining condition of $\to$).
$\Rightarrow x \leq y \to (y \to x)$ (the defining condition of $\to$).
Similarly one can show that $y \leq x \to (x \to y)$.
Hence $x \wedge y \leq [x \to (x \to y)] \wedge [y \to (y \to x)]$. Hence (1) holds.
$x \wedge y \leq x$ (from the definition of $\wedge$).
$\Rightarrow (x \wedge y)oz \leq xoz$ (by Lemma 1.3.7).
$\Rightarrow (x \wedge y) \leq (xoz) \to z$ and $z \leq (xoz) \to (x \wedge y)$ (by definition condition of $\to$).
Similarly $x \wedge y \leq (xoy) \to y$ and $x \wedge y \leq (xoy) \to x$. Hence $x \wedge y \leq [(xoy) \to x] \wedge [(xoy) \to y] = xoy \to (x \vee y)$ (by Lemma 1.3.19). Hence (2) and (3) holds. □

Remark 2.2.3. *In any BH-lattice L and* $a, b \in L$, $b \to a = max\{x \in L : aox \leq b\}$

Theorem 2.2.4. *Every BH-lattice L satisfies the following properties,* $\forall x, y, z \in L$.

1. $x \leq ((x \vee y)oz) \to z, x \leq ((x \vee y)ox) \to y$

2. $x \to y \leq (z \to y) \to (z \to x)$

3. $x \to y \leq (x \to z) \to (y \to z)$

Proof.

1. $xoz \leq (x \vee y)oz, xoy \leq (x \vee y)ox$ (by definition of $\vee$ and Lemma 1.3.7)
$\Rightarrow x \leq (x \vee y)oz \to z, x \leq (x \vee y)ox \to y$ (by defining condition of $\to$).

2. $x \leq z \to (z \to x)$ (by 1 of Theorem 2.2.2 above)
$\Rightarrow x \to y \leq [z \to (z \to x)] \to y$ (by Lemma 1.3.14).
$= z \to (yo(z \to x))$ (by Lemma 1.3.10).

$= (z \to y) \to (z \to x)$ (by Lemma 1.3.10).

Thus, $x \to y \leq (z \to y) \to (z \to x)$.

3. $(y \to z)oz \leq y$ (by Lemma 1.3.9)

$\Rightarrow x \to y \leq x \to [(y \to z)oz]$ (by Lemma 1.3.15))

$= (x \to z) \to (y \to z)$ (by Lemma 1.3.10). □

Corollary 2.2.5. *Let $x, y \in L$. Then $x \to y \leq e \to (y \to x)$*

Proof. Replace z by x in 3 of Theorem 2.2.4 □

Notation: For $x \in L$, $e \to x$ is denoted by the symbol x^-.

Theorem 2.2.6. *Every BH-lattice L satisfies the following properties, $\forall x, y \in L$.*

1. $(xoy)^- = x^- \to y = y^- \to x$
2. $x \leq y \Rightarrow y^- \leq x^-$
3. $x \leq x^{--}$ *and* $x^- = x^{---}$
4. $(x \vee y)^- = x^- \wedge y^-$ *and* $x^- \vee y^- \leq (x \wedge y)^-$
5. $x \to y \leq y^- \to x^-$
6. $x^{--} \to y^{--} = y^- \to x^-$

Proof. Let $x, y \in L$ and $x^- = e \to x$.

$(xoy)^- = e \to (xoy) = (e \to x) \to y = (e \to y) \to x$ (by Lemma 1.3.10)

$\Rightarrow (xoy)^- = x^- \to y = y^- \to x$. Hence (1) holds.

Let $x \leq y \Rightarrow e \to y \leq e \to x$ (by Lemma 1.3.15)

$\Rightarrow y^- \leq x^-$. Hence (2) holds

$x \leq e \to (e \to x) = x^{--}$ (by 1 of Theorem 2.2.2).

$\Rightarrow x \leq x^{--}$.

Now replacing x^- in the place of x in the above inequality we have $x^- \leq x^{---}$. But

by 2 in the inequality $x \leq x^{--}$, we obtain that $x^{---} = e \to (x^{--}) \leq (e \to x) = x^{-}$.
So $x^{---} = x^{-}$. Hence (3) holds
Moreover, $(x \vee y)^{-} = e \to (x \vee y) = (e \to x) \wedge (e \to y) = x^{-} \wedge y^{-}$ (by Lemma 1.3.19)
$x \wedge y \leq x, y$ (by definition of $\wedge$)
$\Rightarrow e \to x = x^{-} \leq e \to (x \wedge y) = (x \wedge y)^{-}$ (by Lemma 1.3.15)
Similarly $y^{-} \leq (x \wedge y)^{-}$. Hence $x^{-} \vee y^{-} \leq (x \wedge y)^{-}$. Hence (4) holds.
$x \to y \leq (e \to y) \to (e \to x) = y^{-} \to x^{-}$ (by 2 of Theorem 2.2.4). Hence (5) holds.
Furthermore, $y^{--} = e \to y^{-}$ is invertible (by Theorem 1.3.34)
$\Rightarrow x^{--} \to y^{--} = x^{--}o(e \to y^{--})$ (by Theorem 1.3.32)
$= x^{--}oy^{---}$ (as $e \to y^{--} = y^{---}$)
$= x^{--}oy^{-}$ (by 3 of Theorem 2.2.6)
$= y^{-}o(e \to x^{-})$
$= y^{-} \to x^{-}$ (by Theorem 1.3.32 and Theorem 1.3.34). Hence (6) holds. □

Theorem 2.2.7. *Every BH-lattice L satisfies the following properties,* $\forall x, y \in L$.

1. $(x \wedge y) \to x \leq e$
2. $e \leq x \to (x \wedge y)$

Proof. $x \wedge y \leq x$ (by definition of $\wedge$)
$\Rightarrow x \wedge y \to x \leq x \to x = e$ (by Lemma 1.3.14 and Theorem 1.3.24).
Further, $x \wedge y \leq x$ (by definition of $\wedge$)
$\Rightarrow e = x \to x \leq x \to x \wedge y$ (by Lemma 1.3.15 and Theorem 1.3.24). □

Theorem 2.2.8. *For* $x, y \in L, x \to (y \to x) = x \Rightarrow x \to (x \to y) = y \to (y \to x)$

Proof. Let $x, y \in L$ and $x \to (y \to x) = x$.
$x \to (x \to y) = (x \to (y \to x)) \to (x \to y)$
$= (x \to (x \to y)) \to (y \to x)$ (by Lemma 1.3.10).
$y \leq x \to (x \to y)$ (by 1 of Theorem 2.2.2)
$\Rightarrow y \to (y \to x) \leq (x \to (x \to y)) \to (y \to x)$ (by Lemma 1.3.14)

$= (x \to (y \to x)) \to (x \to y)$ (by Lemma 1.3.10)

$= x \to (x \to y)$ (by the given hypothesis).

$\Rightarrow y \to (y \to x) \leq x \to (x \to y)$. Analogously, $x \to (x \to y) \leq y \to (y \to x)$.

Hence $x \to (x \to y) = y \to (y \to x)$ □

Theorem 2.2.9. *Let $a, b, x, y \in L$ and $a \leq b$, $x \leq y$, then the following properties hold.*

1. $aox \leq boy$
2. $(a \to x) \vee (b \to y) \leq b \to x$
3. $a \to y \leq (a \to x) \wedge (b \to y)$
4. $a \to y \leq b \to x$

Proof. Suppose $a \leq b$ and $x \leq y$.

$\Rightarrow aox \leq box$ and $box \leq boy$ (by Lemma 1.3.7).

$\Rightarrow aox \leq boy$. Hence (1) holds.

Further, $a \to x \leq b \to x$ (by Lemma 1.3.14) and $b \to y \leq b \to x$. (by Lemma 1.3.15)

$\Rightarrow (a \to x) \vee (b \to y) \leq b \to x$. Hence (2) holds.

Moreover, $a \to y \leq b \to y$ (by Lemma 1.3.14) and $a \to y \leq a \to x$ (by Lemma 1.3.15).

$\Rightarrow a \to y \leq (a \to x) \wedge (b \to y)$. Hence (3)holds.

From (2) and (3), $a \to y \leq (a \to x) \wedge (b \to y) \leq (a \to x) \vee (b \to y) \leq b \to x$. Thus, by transtivity $a \to y \leq b \to x$. □

Theorem 2.2.10. *Every BH-lattice L satisfies the following properties, $\forall x, y, z, s, t \in L$.*

1. $x \to y \leq xoz \to yoz$
2. $(x \to y)o(s \to t) \leq (xos) \to (yot)$

Proof. $x \leq (xoz) \to z$ (by Lemma 1.3.8).

$\Rightarrow x \to y \leq [(xoz) \to z] \to y$ (by Lemma 1.3.14)

$= xoz \to yoz$ (Lemma 1.3.10). Hence the first inequality holds.

Now using the first inequality we have $x \to y \leq xos \to yos$ and $s \to t \leq yos \to toy$.

$\Rightarrow (x \to y)o(s \to t) \leq (xos \to yos)o(yos \to yot)$ (by Theorem 2.2.9)

$\leq xos \to yot$ (by Lemma 1.3.17). So the second inequality holds. □

Theorem 2.2.11. *Let $x, y \in L$. $(x \wedge y) \to (x \vee y) = (x \to y) \wedge (y \to x)$*

Proof. $x \wedge y \to x \vee y = [(x \wedge y) \to x] \wedge [(x \wedge y) \to y]$ (by Lemma 1.3.19)

$= [(x \to x) \wedge (y \to x)] \wedge [(x \to y) \wedge (y \to y)]$ (by Lemma 1.3.20)

$= e \wedge (y \to x) \wedge (x \to y)$ (by Theorem 1.3.24) (i)

Further,

$(x \vee y)o\{(x \to y) \wedge (y \to x)\} = [xo\{(x \to y) \wedge (y \to x)\}] \vee [yo\{(x \to y) \wedge (y \to x)\}]$ (by Lemma 1.3.18)

$\leq \{xo(y \to x)\} \vee \{yo(x \to y)\}$ (by Lemma 1.3.7 and property of $\vee$ and $\wedge$)

$\leq x \vee y$(by Lemma 1.3.9 and property of $\vee$ and $\wedge$)

i.e; $(x \vee y)o\{(x \to y) \wedge (y \to x)\} \leq x \vee y$

$\Rightarrow (x \to y) \wedge (y \to x) \leq (x \vee y) \to (x \vee y) = e$ (by definition condition of $\to$)

$\Rightarrow (x \to y) \wedge (y \to x) \wedge e = (x \to y) \wedge (y \to x)$ (ii)

Hence from (i) and (ii) we have $x \wedge y \to x \vee y = (x \to y) \wedge (y \to x)$ □

Theorem 2.2.12. *Let $x, y \in L$. $y \leq x \Rightarrow (y \to x)ox = y$*

Proof. Suppose $y \leq x \Rightarrow (y \to x) \leq e$ (by Lemma 1.3.14) and Theorem 1.3.24

$\Rightarrow (y \to x) \wedge e = (y \to x)$

$\Rightarrow ((y \to x) \wedge e)ox = (y \to x)ox$

$\Rightarrow x \wedge y = (y \to x)ox$ (by 3 in the definition of BH-lattice)

$\Rightarrow y = (y \to x)ox.$ □

Theorem 2.2.13. *In a BH-lattice L $((x \to y) \wedge e)o(x \vee y) = x, \forall x, y \in L.$*

Proof. $x = (x \to (x \vee y))o(x \vee y)$ (by Theorem 2.2.12)

$= ((x \to y) \wedge (x \to x))o(x \vee y)$ (by Lemma 1.3.19).

$= ((x \to y) \wedge e)o(x \vee y)$ (by Lemma 1.3.24). □

Theorem 2.2.14. *In a BH-lattice L* $[(x \to y) \wedge (y \to x)]o(x \vee y) = x \wedge y, \forall x, y \in L$.

Proof. $[(x \to y) \wedge (y \to x)]o(x \vee y) = [(x \wedge y) \to (x \vee y)]o(x \vee y)$ (by Theorem 2.2.11)

$= x \wedge y$ (by Theorem 2.2.12) □

Theorem 2.2.15. *In a BH-lattice L* $x \leq y \leq z \Rightarrow (x \to y)o(y \to z) = x \to z, \forall x, y, z \in L$.

Proof. Suppose that $x \leq y \leq z$.

$\Rightarrow (y \to z)oz = y$ (by Theorem 2.2.12)

$\Rightarrow x \to y = x \to \{(y \to z)oz\}$

$= (x \to z) \to (y \to z)$ (by Lemma 1.3.10).

$\Rightarrow (x \to y)o(y \to z) = ((x \to z) \to (y \to z))o(y \to z)$.

$x \leq y \Rightarrow x \to z \leq y \to z$ (by Lemma 1.3.14)

$\Rightarrow (x \to y)o(y \to z) = ((x \to z) \to (y \to z))o(y \to z)$

$= x \to z$ ((by Theorem 2.2.12)). □

Theorem 2.2.16. *In any BH-lattice L, the following properties hold,* $\forall x, y, z \in L$.

1. $(x \to y)oz \leq x \to (y \to z)$,

2. $(x \to z)oy \leq (xoy) \to z$

3. $(e \to x)o(e \to y) \leq e \to (xoy)$

Proof. $(y \to z)oz \leq y$. (by Lemma 1.3.9)

$\Rightarrow x \to y \leq x \to (y \to z)oz$ (by Lemma 1.3.15)

$= (x \to (y \to z)) \to z$ (by Lemma 1.3.10,)

$\Rightarrow (x \to y)oz \leq x \to (y \to z)$. Thus, the first inequality holds.

$x \leq (xoy) \to y$ (by Lemma 1.3.8).

$\Rightarrow x \to z \leq ((xoy) \to y) \to z$ (by Lemma 1.3.14)

$= ((xoy) \to z) \to y$ (by Lemma 1.3.10)

$\Rightarrow (x \to z)oy \leq (xoy) \to z$ (by defining condition of $\to$). Thus the second inequality holds.

Finally, $(e \to x)o(e \to y) \leq e \to (x \to (e \to y))$ and $x \to (e \to y) \geq (x \to e)oy$ (using the first inequality)

$= xoy$ (by Lemma 1.3.12).

$\Rightarrow e \to [x \to (e \to y)] \leq e \to (xoy)$ (by Lemma 1.3.15).

Hence $(e \to x)o(e \to y) \leq e \to (xoy)$. □

One can also prove (3) in Theorem 2.2.16 by using Theorem 2.2.10.

Theorem 2.2.17. *For $x, y \in L, (xoy)^{--} \leq (x^{-}oy^{-})^{-} = x^{--}oy^{--}$*

Proof. $(e \to x)o(e \to y) \leq e \to xoy$ (by 3 of Theorem 2.2.16).

$(xoy)^{--} \leq e \to [(e \to x)o(e \to y)]$ (by Lemma 1.3.15)

$= (e \to (e \to x)) \to (e \to y)$ (by Lemma 1.3.10)

$= x^{--} \to y^{-}$

$= (x^{-}oy^{-})^{-}$ (by 1 of Theorem 2.2.6).

$\Rightarrow (x^{-}oy^{-})^{-} = x^{--} \to y^{-} = x^{--}o(e \to y^{-})$ (by Theorem 1.3.32 and Theorem 1.3.34)

$= x^{--}oy^{--}$ □

Theorem 2.2.18. *For $x, y, z \in L$ the following properties hold.*

1. $x \wedge y = e = x \wedge z \Rightarrow x \wedge (yoz) = e$

2. $x \vee y = e = x \vee z \Rightarrow x \vee (yoz) = e$

Proof. Let $x, y, z \in L$ and $x \wedge y = e = x \wedge z$.

$\Rightarrow e \leq x, y, z$.

$\Rightarrow e \leq x \wedge y \wedge z$ and $e \leq yoz$ (by Theorem 2.2.9).

$\Rightarrow e \leq x \wedge (yoz)$. (i)

and $x = xoe \leq xo(x \wedge y \wedge z)$ (by Lemma 1.3.7)

$\Rightarrow x \wedge (yoz) \leq \{xo(x \wedge y \wedge z)\} \wedge (yoz) = (xox) \wedge (xoz) \wedge (yox) \wedge (yoz) = (x \wedge y)o(x \wedge z) = eoe = e$ (by property of $\wedge$ and distributive property)

i.e, $x \wedge (yoz) \leq e$. (ii)

So from (i) and (ii) it follows that $x \wedge (yoz) = e$.

Suppose that $x \vee y = e = x \vee z$.

$\Rightarrow x, y, z \leq e.$

$\Rightarrow x \vee y \vee z \leq e$ and $yoz \leq e$ (by Theorem 2.2.9).

$\Rightarrow x \vee (yoz) \leq e.$ (iii)

and $xo(x \vee y \vee z) \leq xoe = x$ (by Lemma 1.3.7 and property of $\vee$

$\Rightarrow \{xo(x \vee y \vee z)\} \vee (yoz) = (xox) \vee (xoz) \vee (yox) \vee (yoz)$ (by Lemma 1.3.18))

$= (x \vee y)o(x \vee z) = eoe = e$ (by Lemma 1.3.18))

and $\{xo(x \vee y \vee z)\} \vee (yoz) \leq x \vee (yoz)$ (by the property of $\vee$).

i.e, $e \leq x \vee (yoz)$. (iv)

Hence from (iii) and (iv) it follows that $x \vee (yoz) = e$ □

Notation: For any x and y in a BH-lattice L, $(x \to y) \wedge (y \to x)$ is denoted by $x * y$. That is $x * y = (x \to y) \wedge (y \to x)$.

Theorem 2.2.19. *For any $x, y, z \in L$, the following properties hold.*

1. $x * y \leq e$ *with equality iff* $x = y$.
2. $x * y = y * x$
3. $(x \vee y) * (x \wedge y) = x * y$
4. $(x * y)o(y * z) \leq x * z$
5. $x * y \leq (x \to y) * e$
6. $x \leq e \Rightarrow x * e = x$

Proof. $x * y = (x \to y) \wedge (y \to x)$

$= (x \wedge y) \to (x \vee y)$ (by Theorem 2.2.11)

$\leq e$ (by Lemma 1.3.14, Theorem 1.3.24 and the fact $(x \wedge y) \leq (x \vee y)$). If $x = y$, then by Theorem 1.3.24, $x * y = e$. Conversely, if $x * y = e$, then

$x * y = (x \wedge y) \to (x \vee y) = e$

$\Rightarrow e \leq (x \wedge y) \to (x \vee y).$

$\Rightarrow (x \vee y) \leq (x \wedge y)$ (by defining condition of $\to$)

$\Rightarrow x \vee y = x \wedge y$ and consequently we obtain that $x = y$. Thus, (1) holds.

Evidently (2) is trivial.

$(x \vee y) * (x \wedge y) = [(x \vee y) \wedge (x \wedge y)] \to [(x \vee y) \vee (x \wedge y)]$ (by Theorem 2.2.11)

$= (x \wedge y) \to (x \vee y)$

$= x * y$. Hence (3) holds.

$(x \to y) \wedge (y \to x) \leq x \to y, y \to x$ and $(y \to z) \wedge (z \to y) \leq y \to z, z \to y$.

$\Rightarrow (x * y)o(y * z) = [(x \to y) \wedge (y \to x)]o[(y \to z) \wedge (z \to y)]$

$\leq (x \to y)o(y \to z)$ (by 1 of Theorem 2.2.9)

$\leq x \to z$ (by Lemma 1.3.17).

That is $(x * y)o(y * z) \leq x \to z$. By a similar argument $(x * y)o(y * z) \leq z \to x$. So we obtain that $(x * y)o(y * z) \leq (x \to z) \wedge (z \to x) = x * z$. Hence (4) holds.

$y \to x \leq (x \to x) \to (x \to y)$ (by 2 of Theorem 2.2.4)

$= e \to (x \to y)$ (by Theorem 1.3.24)

$\Rightarrow x * y = (x \to y) \wedge (y \to x)$

$\leq (x \to y) \wedge (e \to (x \to y))$

$= (x \to y) * e$.

So $x * y \leq (x \to y) * e$. Hence (5) holds.

Finally, let $x \leq e$

$\Rightarrow e \to e = e \leq e \to x$ (by Theorem 1.3.15).

$\Rightarrow x = e \wedge x \leq x \wedge (e \to x) = x * e \leq x$ (by Theorem 1.1.24). Hence $x * e = x$ □

Definition 2.2.20. *For $n \in \mathbb{N}$ and $x \in L$, define $x^n = xoxo...ox$ (n times).*

Theorem 2.2.21. *In a BH-lattice L, if there exists an element $x \in L$ such that $e < x$, then the set L is an infinite and not bounded above.*

Proof. Suppose that $e < x$ and $x \neq e$. Consider the sequence $\{x^{2^n}\}_{n \in \mathbb{N}}$, where $\mathbb{N}$ is the set of non-negative integers. With Lemma 1.3.7, $x \leq x^2$. If $x^2 = x$, then by Lemma 1.3.8, $x \leq x^2 \to x = x \to x = e$. i.e; $x \leq e$. Since $e < x$, it follows that $x = e$, which is a contradiction. So $x^2 \neq x$. If $x^2 = e$, then $x = x \to e = x \to x^2 = (x \to x) \to x = e \to x$. Since $e < x$, by Lemma 1.3.15, $e \to x \leq e \to e = e$. This implies $x = e \to x \leq e$. Hence $x = e$. In both cases there is a contradiction. So $e < x < x^2$.

This implies that $x^2 \leq x^4$. If $x^2 = x^4$, then by Lemma 1.3.8, $x^2 \leq [x^2 o x^2] \to x^2 = e$. i.e; $x^2 \leq e$. Hence $x^2 = e$ which is a contradiction. Hence $e < x < x^2 < x^4$.
Suppose that $e < x < x^2 < x^4 < ... < x^{2^n}$ for some $n \in \mathbb{N}$. This implies that $x^{2^n} \leq x^{2^n} o x^{2^n}$. If $x^{2^n} = x^{2^n} o x^{2^n}$, then by Lemma1.3.8, $x^{2^n} \leq [x^{2^n} o x^{2^n}] \to x^{2^n} = e$. i.e; $x^{2^n} \leq e$. Hence $x^{2^n} = e$ which is a contradiction. Hence by the principle of mathematical induction, the chain $e < x < x^2 < x^4 < ... < x^{2^n}...$ does not terminate at some point. So, the sequence of elements x^{2^n} are all distinct. Hence the set L is infinite and unbounded above. □

Corollary 2.2.22. *If L is bounded above by the element t, then $t = e$.*

Proof. Directly follows from Theorem 2.2.21. □

Theorem 2.2.23. *In any BH-lattice L, the following properties hold, $\forall x, y \in L$.*

1. $x \leq e$ *and* $y \leq e \Leftrightarrow xoy \leq x \wedge y$
2. $e \leq x$ *and* $e \leq y \Rightarrow x \vee y \leq xoy$
3. $e \leq x$ *and* $y \leq e \Rightarrow y \leq xoy \leq x$

Proof. Suppose that $x, y \in L$, $x \leq e$ and $y \leq e$
$\Rightarrow xoy \leq y$ and $xoy \leq x$ (by Lemma 1.3.7).
$\Rightarrow xoy \leq x \wedge y$ (by the definition of $\wedge$).
Conversely, let $xoy \leq x \wedge y$.

$\Rightarrow xoy \leq x \wedge y \leq y$ and $xoy \leq x \wedge y \leq x$ (by definition of $\wedge$)

$\Rightarrow xoy \leq y$ and $xoy \leq x$ (transitive)

$\Rightarrow x \leq y \to y = e$ and $y \leq x \to x = e$ (by defining condition of $\to$ and Theorem 1.3.24). Hence (i) holds.

Suppose that $x, y \in L$, $e \leq x$ and $e \leq y$
$\Rightarrow y \leq xoy$ and $x \leq xoy$ (by Lemma 1.3.7).
$\Rightarrow x \vee y \leq xoy$ (by definition of $\vee$).
Suppose $e \leq x$ and $y \leq e$.
$\Rightarrow y \leq xoy \leq x$ (by Lemma 1.3.7). □

Theorem 2.2.24. *For $x \in L$ and any $n \in \mathbb{N}, x^n \leq e \Leftrightarrow x \leq e$.*

Proof. For $n = 2, (x \vee e)^2 = (x \vee e)ox \vee (x \vee e)oe$(by Lemma 1.3.18).

$= xox \vee eox \vee xoe \vee eoe$

$= x^2 \vee x \vee e.$

Suppose that for some $k \in \mathbb{N}, (x \vee e)^k = x^k \vee x^{k-1} \vee ... \vee x \vee e$. Then

$(x \vee e)^{k+1} = (x^k \vee x^{k-1} \vee ... \vee x \vee e) \vee (x \vee e)$

$= \{(x^k \vee x^{k-1} \vee ... \vee x \vee e)ox\} \vee \{(x^k \vee x^{k-1} \vee ... \vee x \vee e)oe\}$ (by Lemma 1.3.18)

$= (x^{k+1} \vee x^k \vee ... \vee x \vee e) \vee (x^k \vee x^{k-1} \vee ... \vee x \vee e)$ (by Lemma 1.3.18)

$= x^{k+1} \vee x^k \vee ... \vee x \vee e.$

Thus, by principle of mathematical induction we have

$(x \vee e)^n = x^n \vee x^{n-1} \vee ... \vee x \vee e$

$= (x^n \vee e) \vee (x^{n-1} \vee ... \vee x \vee e)$

$= x^{n-1} \vee ... \vee x \vee e$ (as $x^n \leq e$)

$= (x \vee e)^{n-1}$. That is $(x \vee e)o(x \vee e)^{n-1} = (x \vee e)^{n-1}$

$\Rightarrow (x \vee e) \leq [(x \vee e)o(x \vee e)^{n-1}] \to (x \vee e)^{n-1} = (x \vee e)^{n-1} \to (x \vee e)^{n-1} = e$ (by of Theorem 1.3.8).

$\Rightarrow x \vee e = e$

$\Rightarrow x \leq e$. Hence $x \leq e$.

The converse follows from Theorem 2.2.9 by induction. □

Corollary 2.2.25. *For $x \in L, x * e = x \Leftrightarrow x \leq e$.*

Proof. Suppose that, $x \wedge (e \to x) = x * e = x$.

$\Rightarrow x \leq e \to x$

$\Rightarrow x^2 \leq e$ (by defining condition of $\to$).

$\Rightarrow x \leq e$ (by Theorem 2.2.24). The converse follows from Theorem 2.2.19. □

Corollary 2.2.26. *In a BH-lattice, if $x, y \in L$ and y is invertible element, then for any $n \in \mathbb{N}$,*

$x^n \leq y^n \Leftrightarrow x \leq y$.

Proof. Let $x^n \leq y^n$.

$\Rightarrow (xo(e \rightarrow y))^n = x^n o(e \rightarrow y)^n \leq y^n o(e \rightarrow y)^n$ (by Theorem 1.3.7).

$= (yo(e \rightarrow y))^n = e$.

$\Rightarrow xo(e \rightarrow y) \leq e$ (by Theorem 2.2.24)

$\Rightarrow xo(e \rightarrow y)oy \leq eoy$ (by Lemma 1.3.7)

$\Rightarrow x \leq y$ (by Theorem 1.3.30 and y is invertible element).

The converse follows from Theorem 2.2.9 by induction. □

Theorem 2.2.27. *For $x \in L$ and any $n \in \mathbb{N}, x^n = e \Leftrightarrow x = e$.*

Proof. Let $x^n = e \Rightarrow x^n \leq e$.

$\Rightarrow x \leq e$ (by Theorem 2.2.24). Perceive that

$x = x \rightarrow e = x \rightarrow x^n = x \rightarrow (xox^{n-1}) = (x \rightarrow x) \rightarrow x^{n-1}$ (by Lemma 1.3.10)

$= e \rightarrow x^{n-1} = x^n \rightarrow x^{n-1}$(by Theorem 1.3.24).

Since $x \leq e, x \wedge e = x$. Thus, by the distributive property of o over $\wedge$, using repeatedly, we have $x^n = (x \wedge e)^n = x^n \wedge x^{n-1} \wedge ... \wedge x \wedge e = x^{n-1} \wedge ... \wedge x \wedge e = (x \wedge e)^{n-1} = x^{n-1}$. Hence by Theorem 1.3.24, $x = x^n \rightarrow x^{n-1} = e$. The converse is trivial. □

Corollary 2.2.28. *In an l-group, every element other than the identity element has an infinite order.*

Proof. Directly follows from Theorem 2.2.27. □

Definition 2.2.29. *An element $l \in L$ is called unity if and only if $xo(l \rightarrow x) = lol, \forall x \in L$.*

Example 2.2.30. *Every bounded chain is a BH-lattice with unity $l = 0$.*

To show this consider the following. Let H be a bounded chain. Hence by Example 1.2.18, H is a Heyting algebra. Hence it is a BH-lattice and for any $a \in H, 0 \rightarrow a = 0$. Hence, $ao(0 \rightarrow a) = a \wedge 0 = 0 = 0o0$. Thus, $l = 0$ is the unity element of H.

Lemma 2.2.31. *In L with unity l, the following properties hold.*

1. *$lol = l$*

2. $l \leq e$

3. $e = e \to l = (e \to l) \to l$

Proof. Let $x = e$ in definition 2.2.29. Then it is immediate that $lol = l$. So (1) holds. $l \leq (lol) \to l = e$ (by Lemma 1.3.8 , Theorem 1.3.24 and condition (1)). So (2) is satisfied.
Let $t = e \to l$. Then
$t = e \to (lol)$ (by (1))
$= (e \to l) \to l$ (by Lemma 1.3.10)
$= t \to l$ i.e $t = t \to l$.
$\Rightarrow e = t \to t = (t \to l) \to t = t \to (lot) = (t \to t) \to l = e \to l = t$ (by Lemma 1.3.10 and Theorem 1.3.25). Thus $e \to l = e$. Thus (3) holds. □

Theorem 2.2.32. *Unity element is unique, if it exists in L.*

Proof. Let l and $l^{'}$ be unities.
$\Rightarrow l \leq e$ and $e \to l = e = e \to l'$ (by Lemma 2.2.31).
$\Rightarrow l \to l' \leq e \to l' = e$ (by Lemma 1.3.14 and Lemma 2.2.31).
Hence by (3) in the definition of BH-lattices and definition of unity element,
$l \wedge l' = [(l \to l') \wedge e]ol' = (l \to l')ol' = l$. By a similar argument $l' \wedge l = l'$. Hence $l = l'$. □

Corollary 2.2.33. *L with unity is an l-group if and only if $l = e$.*

Proof. Suppose that L is a BH-lattice with unity. If L is an l-group, then every element of L is invertible. Hence by Theorem 1.3.30, $xo(e \to x) = e = eoe$. Thus by the definition of unity e is a unity element and hence by Theorem 2.2.32, $l = e$. Conversely, if $l = e$, then $xo(e \to x) = e = eoe$. Hence every element of L has an inverse. Thus, (L, o) is an group. Hence by the definition L is an l-group. □

Theorem 2.2.34. *If L with unity l contains a least element α, then $\alpha = l$*

Proof. Since α is least element of L, $\alpha \leq e$

$\Rightarrow (\alpha o \alpha) \leq e o \alpha = \alpha$ (by Theorem 1.3.7).

$\Rightarrow \alpha o \alpha = \alpha$ (as α is least). Thus,

$\Rightarrow l = (l \to \alpha) o \alpha = (l \to \alpha) o (\alpha o \alpha)$ (by the definition of unity)

$= \{(l \to \alpha) o \alpha\} o \alpha$ (associative)

$= l o \alpha$ (by the definition of unity and 1 of Theorem 2.2.31)

$\leq e o \alpha = \alpha$ (since $l \leq e$ and Lemma 1.3.7).

$\Rightarrow \alpha = l$ (since α is least). □

Theorem 2.2.35. *If a BH-lattice L with unity l, contains an element x such that $x < l$, then the set L is an infinite set and unbounded below.*

Proof. Assume that L is a BH-lattice with unity l and there exists an element $x \in L$ such that $x < l$ and $x \neq l$.

$\Rightarrow l \leq e$ (by Lemma 2.2.31)

$\Rightarrow x < l \leq e$

$\Rightarrow x^2 \leq l o x \leq x$ (by Theorem 1.3.7).

If $x^2 = x$, then $x = x^2 \leq l o x \leq x$

$\Rightarrow x = l o x$. Hence,

$e = e \to l$ (by Lemma 2.2.31,)

$= (x \to x) \to l$ (by Theorem 1.3.24)

$= x \to (x o l)$ (by Lemma 1.3.10)

$= x \to [x o (l \to x)] o x = x \to [x^2 o (l \to x)]$ (definition of unity element)

$= x \to x o (l \to x)$

$= x \to l$ (definition of unity element).

$\Rightarrow e \leq x \to l$

$\Rightarrow l < x$ (by defining condition of $\to$) which is a contradiction. Thus $x^2 \neq x$.

If $x^2 = l$, then

$x \to l = x \to x^2 = (x \to x) \to x = e \to x$ (by Lemma 1.3.10 and Theorem 1.3.24).

Since $x \leq e$, by Theorem 1.3.15, $e \leq e \to x$. So $e \leq e \to x = x \to l$

$\Rightarrow l \leq x$ which is a contradiction. Hence $x^2 \neq l$. In both cases there is a contradiction.

Hence $x^2 < x < l \leq e$.

$\Rightarrow x^2 o x^2 \leq x^2$ (by Lemma 1.3.7).

If $x^2 = x^4$, then

$e = e \to l$ (by Lemma 2.2.31,)

$= (x^2 \to x^2) \to l$ (by Theorem 1.3.24)

$= x^2 \to (x^2 o l)$ (by Lemma 1.3.10 and)

$= x^2 \to [x^2 o(l \to x^2)]o x^2$ (by definition of unity element)

$= x^2 \to x^4 o(l \to x^2) = x^2 \to x^2 o(l \to x^2)$ (by assumption $x^2 = x^4$)

$= x^2 \to l$ (by definition of unity element).

$\Rightarrow e \leq x^2 \to l$

$\Rightarrow l \leq x^2$ which is a contradiction. Hence $x^4 < x^2 < l \leq e$.

Suppose that $x^{2^k} < x^{2^{k-1}} < ... < x^4 < x^2 < l \leq e$.

$\Rightarrow x^{2^{k+1}} = x^{2^k} o x^{2^k} \leq e o x^{2^k}$ (by Lemma 1.3.7). If $x^{2^k} = x^{2^{k+1}}$, then

$e = e \to l$ (by Lemma 2.2.31,)

$= (x^{2^k} \to x^{2^k}) \to l$ (by Theorem 1.3.24)

$= x^{2^k} \to x^{2^k} o l)$ (by Lemma 1.3.10 and)

$= x^{2^k} \to [x^{2^k} o(l \to x^{2^k})]o x^{2^k}$ (by definition of unity element)

$= x^{2^k} \to x^{2^{k+1}} o(l \to x^{2^k}) = x^{2^k} \to x^{2^k} o(l \to x^{2^k})$ (by assumption $x^{2^k} = x^{2^{k+1}}$)

$= x^{2^k} \to l$ (by definition of unity element).

$\Rightarrow e \leq x^{2^k} \to l$

$\Rightarrow l \leq x^{2^k}$ which is a contradiction. Hence by principle of mathematical induction $... < x^{2^n} < x^{2^{n-1}} < ... < x^4 < x^2 < l \leq e$ and hence the sequence of elements x^{2^n} are all distinct. Hence the set L is infinite and unbounded below. □

Corollary 2.2.36. *If L with unity l is bounded below by the element t, then $t = l$.*

Proof. Directly follows from Theorem 2.2.35. □

Theorem 2.2.37. *Let L be a BH-lattice with unity l. Then $L_l = \{x \in L : x \to l = e\}$ is a BH-lattice with least and greatest element. And L_l is subset of the BH-lattice $L^e = \{x \in L : e \to x = e\}$.*

Proof. $l \to l = e$ and $e \to l = e$ (by Theorem 1.3.24 and Lemma 2.2.31). Hence both l and e belong to L_l. Let $x \in L_l$,

$\Rightarrow x \to l = e$ and $e \to x = (x \to l) \to x$

$= (x \to x) \to l = e \to l = e$ (by Lemma 1.3.10 and 2.2.31).

$\Rightarrow e \leq x \to l$ and $e \leq e \to x$

$\Rightarrow x \leq e, l \leq x$ (by defining condition of $\to$).

$\Rightarrow e$ is the greatest element and l is the least element in L_l.

Let $x, y \in L_l$.

$\Rightarrow x \leq e$ and $y \leq e$.

$\Rightarrow xoy \leq e$ (by 1 of Theorem 2.2.9).

$\Rightarrow (xoy) \to l \leq e \to l = e$ (by Lemma 1.3.14).

Furthermore, $(x \to l)oy \leq (xoy) \to l$ (by 2 of Theorem 2.2.16).

$\Rightarrow (x \to l)oy = eoy = y \leq (xoy) \to l$.

$\Rightarrow e = y \to l \leq \{(xoy) \to l\} \to l = (xoy) \to lol$

$= (xoy) \to l$ (by Lemma 1.3.10, 1.3.14 and 2.2.31).

Hence $(xoy) \to l = e$. Thus $xoy \in L_l$.

$(x \wedge y) \to l = (x \to l) \wedge (y \to l) = e \wedge e = e$ (by Lemma 1.3.20). Hence L_l is closed under both o and $\wedge$.

$x, y \leq x \vee y \leq e$ (e is greatest element)

$\Rightarrow e = (x \to l) \wedge (y \to l) \leq (x \vee y) \to l \leq e \to l = e$ (by Lemma 1.3.14 and 2.2.31).

Hence $x \vee y \in L_l$.

Moreover, by Lemma 1.3.10 and Lemma 2.2.31, $(x \to y) \to l = (x \to l) \to y = e \to y = (y \to l) \to y = (y \to y) \to l = e \to l = e$. So $x \to y \in L_l$. Hence L_l is a BH-lattice with l as least element and e as greatest element.

Let $x \in L_l$. Then

$e \to x = (x \to l) \to x$

$= (x \to x) \to l$ (by Lemma 1.3.10)

$= e \to l = e$ (by Theorem 1.3.24 and Lemma 2.2.31).

Thus $x \in L^e$ and hence $L_l \subseteq L^e$. Finally, by the proof of Theorem 1.3.36 (Decomposition Theorem for BH- Monoids), L^e is a BH-lattice with greatest element e. □

Theorem 2.2.38. *In a BH-lattice L,* (L, o) *is a group* $\Leftrightarrow x \to y = x \to z \Rightarrow y =$

$z, \forall x, y, z \in L$.

Proof. Suppose that (L, o) is a group and let for any $x, y, z \in L, x \to y = x \to z$. Then $(x \to y)^{-1} = y \to x = z \to x = (x \to z)^{-1}$ (by Theorem 2.2.1).
$\Rightarrow (y \to x) \to (e \to x) = (z \to x) \to (e \to x)$.
$\Rightarrow (y \to (xo(e \to x)) = (z \to (xo(e \to x))$ (by Lemma 1.3.10).
$\Rightarrow y = y \to e = z \to e = z$ (by Lemma 1.3.12 and Theorem 1.3.30).
Conversely, assume that for $x, y, z \in L, x \to y = x \to z \Rightarrow y = z$. Then by Lemma 1.3.10, $e \to \{yo(e \to y)\} = (e \to y) \to (e \to y) = e = e \to e$. So that $yo(e \to y) = e$. Hence y is invertible element. Since y is arbitrary, every element of L is invertible. Hence (L, o) is a group. □

Theorem 2.2.39. *L is an l-group if (L, o) is a group and further $a \to b$ is the solution of the equation $box = a$.*

Proof. If (L, o) is a group, then by the definition of BH-lattice and Theorem 1.3.7, $L = (L, o, \leq, \to)$ is an l-group. Further, by Theorem 1.3.30 and Theorem 1.3.32, $bo(a \to b) = bo[ao(e \to b)] = a$. Hence $a \to b$ is the solution of the equation $box = a$. □

Theorem 2.2.40. *BH-lattice L bounded below is Heyting algebra if $xoy = y \wedge x$, $\forall x, y \in L$. Also, in L, if $(L, \leq, \to)$ is a Heyting Algebra, then $xoy = y \wedge x, \forall x, y \in L$.*

Proof. Let BH-lattice L is bounded below by 0 and $xoy = x \wedge y, \forall x, y \in L$. Then

1. $x = eox = e \wedge x \Rightarrow x \leq e$. Hence $(L, \vee, \wedge, 0, e = 1)$ is a lattice with 0, 1

2. $x \wedge y = \{(x \to y) \wedge e\}oy = \{(x \to y)oe\}oy = (x \to y) \wedge y$

3. $y \to z \leq (yox) \to (zox)$ (by 1 of Theorem 2.2.10).
   $= (y \wedge x) \to (z \wedge x)$ (by hypothesis $o = \wedge$)
   $\Rightarrow (y \to z) \wedge x \leq [(y \wedge x) \to (z \wedge x)] \wedge x$. (i)
   Moreover, using 2 and Lemma 1.3.10

$(y \to z) \wedge x = x \wedge ((y \to z) \to x) = x \wedge (y \to zox)$

$= x \wedge (y \to z \wedge x)$ (by hypothesis $o = \wedge$). (ii)

Further, using Lemma 1.3.8, $y \leq (yox) \to x$.

$\Rightarrow yox = y \wedge x \leq ((yox) \to x) \wedge x$ (by hypothesis $o = \wedge$)

$\leq (y \wedge x) \to x \leq y \to x$ (by Lemma 1.3.14 and $y \wedge x \leq y$).

$\Rightarrow (y \wedge x) \to (z \wedge x) \leq [y \to x] \to (z \wedge x)$ (Using Theorem 1.3.14)

$= y \to xo((z \wedge x)$ (by Lemma 1.3.10)

$= y \to x \wedge (z \wedge x)$ (by hypothesis $o = \wedge$)

$= y \to z \wedge x$.

$\Rightarrow [(y \wedge x) \to (z \wedge x)] \wedge x \leq [y \to z \wedge x] \wedge x = [y \to z] \wedge x$ (by (ii)).

(iii) From (i) and (iii) it follows that $[(y \wedge x) \to (z \wedge x)] \wedge x = [y \to z] \wedge x$.

4. $x \wedge y \leq x \Rightarrow e \leq x \to x \wedge y$ (by Lemma 1.3.15).

   $\Rightarrow e = x \to x \wedge y$ (e is the greatest element).

For the second part let $(L, \leq, \to)$ be a Heyting algebra.

$\Rightarrow$ For $a, b \in L, a \to b$ is the largest x such that $x \wedge b \leq a$.

$a \wedge b \leq a \Rightarrow a \leq a \to b$

$\Rightarrow a \wedge b \leq (a \to b) \wedge b = a \wedge b$ (by 4 of Theorem 1.2.22, as L is Heyting algebra).

$\Rightarrow a \wedge (a \to a) = a \wedge a$

$\Rightarrow a \wedge e = a$

$\Rightarrow a \leq e$. Hence e is the largest element of the lattice.

Hence by Theorem 1.3.7, $aob \leq a, b \Rightarrow aob \leq a \wedge b$.

$\Rightarrow aob = (aob) \wedge (a \wedge b) = (a \wedge b) \wedge [aob \to a \wedge b]$ (by 4 of Theorem 1.2.22)

$= a \wedge b$ (by 3 of Theorem 1.2.22). □

Theorem 2.2.41. *If in a BH-lattice L with unity l $(L, \vee, \wedge)$ is a Boolean algebra, then $o = \wedge$ and $x\prime = l \to x$.*

Proof. By Theorem 2.2.34 and Corollary 2.2.22, l is the least element and e is the greatest element of the Boolean algebra. Let $x \in L$. Then there exists an element $x\prime \in L$ such that $x \vee x\prime = e$ and $x \wedge x\prime = l$.

$\Rightarrow xox\prime = x \wedge x\prime = l$ (by Theorem 2.2.40, $o = \wedge$).

$\Rightarrow x\prime \leq l \rightarrow x$ (by the defining condition of $\rightarrow$).

$\Rightarrow e = x \vee x\prime \leq x \vee (l \rightarrow x)$

$\Rightarrow x \vee (l \rightarrow x) = e$ (e is the greatest element).

Further, $x \wedge (l \rightarrow x) = xo(l \rightarrow x) = l$ (by the definition of unity element). So $l \rightarrow x$ is the complement of x in the Boolean algebra L. Hence by uniqueness of complement $x\prime = l \rightarrow x$ □

Chapter 3

Decomposition Theorems of BH-lattices

3.1 Decomposition Theorems

Definition 3.1.1. *An element x of a BH-lattice L is called an idempotent element if $x^2 = x$.*

In any BH-lattice L, e is an idempotent element.

Lemma 3.1.2. *If L is a BH-lattice which satisfies the property $x \to (zoz) \leq (x \to z)o(e \to z), \forall x, y, z \in L$, then the following are equivalent.*

1. $H = \{x \in L : xox \to x = e\}$
2. $H' = \{x \in L : xox = x\}$
3. $B = \{x \in L : e \to x = e\}$

Proof. Suppose that $x \to (zoz) \leq (x \to z)o(e \to z)$.
$(x \to z)o(e \to z) \leq (xoe) \to (zoz)$ (by Theorem 2.2.10)
Hence $(xoe) \to (zoz) = (x \to z)o(e \to z)$. α
Let $x \in H \Rightarrow xox \to x = e$

$\Rightarrow e \leq xox \rightarrow x$

$\Rightarrow x \leq xox$ (by the definition of $\rightarrow$ in BH-lattices).

Further, $x \leq xox \rightarrow x = e$ (by Lemma 1.3.8).

$\Rightarrow xox \leq x$(by Lemma 1.3.7). So that $xox = x$.

Also if $xox = x$, then clearly $xox \rightarrow x = x \rightarrow x = e$. Thus H is the set of all idempotent elements with respect to the operation o. That is, $H = H'$.

Let $x \in H \Rightarrow (xox) \rightarrow x = e$.

$\Rightarrow e \rightarrow x = \{(xox) \rightarrow x\} \rightarrow x = (xox) \rightarrow (xox) = e$ (by Lemma 1.3.10). Hence $x \in B$.

Furthermore, let $y \in B \Rightarrow e \rightarrow y = e$.

$\Rightarrow y = yoe = yo(e \rightarrow y) \leq e$ (by Lemma 1.3.9)

$\Rightarrow y^2 \leq y$ (by Lemma 1.3.7)

Moreover, $e = y^2 \rightarrow y^2 = (y^2 \rightarrow y)o(e \rightarrow y)$ (by equation α above)

$= y^2 \rightarrow y$

$\Rightarrow y \leq y^2$ (by defining condition of $\rightarrow$). Hence $y^2 = y$. Therefore $y \in H'$. Thus $y \in H$ and so H = B. □

Theorem 3.1.3. *A BH-lattice L is direct product of Heyting algebra and a commutative l-group if*

1. $x \rightarrow (zoz) \leq (x \rightarrow z)o(e \rightarrow z)$
2. *there exists an idempotent element* $0 \in L$ *such that* $0 \leq x$, *for any idempotent element* $x \in L$.

Furthermore, if L is the direct product of a Heyting algebra and a commutative l-group, then condition (1) holds.

Proof. Suppose that $x \rightarrow (zoz) \leq (x \rightarrow z)o(e \rightarrow z)$. Since by Theorem 2.2.10, $(xoe) \rightarrow (zoz) \geq (x \rightarrow z)o(e \rightarrow z)$ it follows that

$(xoe) \rightarrow (zoz) = (x \rightarrow z)o(e \rightarrow z)$. (1)

Consider the set $H = \{a \in L : (aoa) \rightarrow a = e\}$. Since $e \in H, H \neq \emptyset$. By Lemma 3.1.2, H is the set of all idempotent elements with respect to the operation o.

Further, consider the set $G = \{a \in L : (aoa) \to a = a\}$.

Let $a \in G$. Then $ao(e \to a) = \{(aoa) \to a\}o(e \to a) = (aoa) \to (aoa) = e$ (by (1) above and Theorem 1.3.24). Hence a is an invirtible element. Moreover, let $a \in L$ be an invirtible element. Then $ao(e \to a) = e$ (by Theorem 1.3.30)
$= (aoaoe) \to (aoa) = [(aoa) \to a]o(e \to a)$(by (1) above and Theorem 1.3.24)
$\Rightarrow a = (aoa) \to a$(by uniqueness of inverse). Hence G is the set of all invertible elements. Thus by Theorem 1.3.35, G is a commutative l-group.

Now let's show that H is a Heyting algebra. Since $e \in H$, H is non empty subset of the BH-lattice L. Let $x \in H$. Then $x^2 = xox = x$
$\Rightarrow e = e \to x$ (by Lemma 3.1.2)
$\Rightarrow eox = x \leq e$ (by the defining condition of $\to$). Hence H is bounded above by e.
Let $x, y \in H$. Then:
$(xoy)o(xoy) = (xox)o(yoy) = xoy$. Hence H is closed under o. Since H is bounded above by $e, x \leq e$.
$\Rightarrow x \to y \leq (e \to y) = e$ (by Lemma 1.3.14 and Lemma 3.1.2)
$\Rightarrow e = e \to e \leq e \to (x \to y)$ (by Lemma 1.3.15). Hence $e \leq e \to (x \to y)$.
Further, as $y \in H, y \leq e$.
$\Rightarrow x = x \to e \leq x \to y$ (by Lemma 1.3.12 and Lemma 1.3.15)
$\Rightarrow e \to (x \to y) \leq e \to x = e$ (by Lemma 1.3.15 and Lemma 3.1.2). Thus $e \to (x \to y) \leq e$ and hence $e \to (x \to y) = e$. So by Lemma 3.1.2, H is closed under $\to$. Furthermore, by condition (3) in the definition of BH-lattices $x \wedge y = [(x \to y) \wedge e]oy = (x \to y)oy \in H$ (as H is bounded above by e). Hence H is closed under $\wedge$ (as H is closed under o and $\to$). Now as $x, y \leq e$, using Lemma 1.3.7, $xoy \leq x, xoy \leq y$ and consequently it follows that $xoy \leq x \vee y$. Thus using Lemma 1.3.18, it follows that $(x \vee y)o(x \vee y) = x^2 \vee xoy \vee xoy \vee y^2 = (x \vee y) \vee (xoy) = x \vee y$. Hence H is closed under $\vee$. Hence H is a sub BH-lattice of L.

In addition to this, as $xoy \leq x$ and $xoy \leq y$, in that case $xoy \leq x \wedge y$. Again using Theorem 2.2.9, $x \wedge y \leq x, y$ implies that $x \wedge y = (x \wedge y)o(x \wedge y) \leq xoy$. Hence $x \wedge y = xoy$.

Finally by condition 2, H is bounded below. Hence by Theorem 2.2.40, H is a Heyting algebra.

Clearly $G \cap H = e$

For $a \in L$, let $t = ao(e \to a)$ and $s = e \to (e \to a)$.
Then $e \to t = e \to [ao(e \to a)] = (e \to a) \to (e \to a) = e$ (by Lemma 1.3.10 and Theorem 1.3.24). Hence by Lemma 3.1.2, $t \in H$. Using Theorem 1.3.34 s is invertible element. SO that $s \in G$.
Furthermore, Using Theorem 1.3.34, and Theorem 1.3.30 for any $a \in L, tos = [ao(e \to a)]o[e \to (e \to a)] = a$.
Now, suppose that $a = xoy, x \in H, y \in G$. Then by Lemma 1.3.10, $e \to a = e \to xoy = (e \to x) \to y = e \to y$ (as $e \to x = e$, being $x \in H$). Also by Lemma 1.3.10, $e \to a = e \to tos = (e \to t) \to s = e \to s$. Now $e \to s = e \to y$. That is inverse of y and inverse of s are equal.
$\Rightarrow e \to (e \to y) = e \to (e \to s)$ (since $\to$ is a binary operation).
$\Rightarrow y = s$ (as y and s are elements of a group).
Now $a = xoy = tos$
$\Rightarrow xoy = tos$ (since $y = s$)
$\Rightarrow xoyo(e \to y) = toyo(e \to y)$ (since o is a binary operation)
$\Rightarrow x = t$. Thus L is the direct product of H and G.
Furthermore, if L is the direct product of a Heyting algebra and a commutative l-group, then trivially condition (1) holds. □

Theorem 3.1.4. *BH-lattice L is direct product of a BH-lattice with least and greatest elements and a commutative l-group iff*

1. $(xoy) \to (zoz) \leq (x \to z)o(y \to z)$
2. *There exists an element* l *such that* $xo(l \to x) = lol$

Proof. Let the conditions given in (1) and (2) holds. From (2) for $l = x$, $lo(l \to l) = lol$. Implying that $lol = l$. And from (1) using Theorem 2.2.10 it follows that $(xoy) \to (zoz) = (x \to z)o(y \to z)$. (2)
$l^- = e \to (lol) = (e \to l) \to l = l^- \to l$ (by Lemma 1.3.10).
$\Rightarrow e = l^- \to l^- = (l^- \to l) \to l^- = (l^- \to l^-) \to l = e \to l = l^-$ (by Lemma 1.3.10 and Theorem 1.3.24).
Let $G = \{x \in L : x \to l = x\}$ and $B = \{x \in L : x \to l = e\}$. Since $e \in G$ and $l \in B$,

both G and B are non empty. Then

$(xoy) \to l = (xoy) \to (lol) = (x \to l)o(y \to l)$ (by (2) above) and

$(x \wedge y) \to l = (x \to l) \wedge (y \to l)$ (by Lemma 1.3.20). Hence both G and B are closed under o and $\wedge$.

Now assume that $x, y \in B$. Then $x \to l = e \Rightarrow l \leq x$. That is, l is the least element of H.

$e \to x = (x \to l) \to x = (x \to x) \to l = e \to l = e$ (by Lemma 1.3.10).

$\Rightarrow x \leq e$. Similarly $y \leq e$ and so, e is the greatest element of H.

$\Rightarrow x \vee y \leq e$

$\Rightarrow (x \vee y) \to l \leq e \to l$ (by Lemma 1.3.14)

Since $x, y \leq x \vee y$, by Lemma 1.3.14, $e = x \to l, e = y \to l \leq (x \vee y) \to l$

$\Rightarrow e = (x \to l) \vee (y \to l) \leq (x \vee y) \to l \leq e \to l = e$. Hence $x \vee y \in B$. Further, by Lemma 1.3.10 $(x \to y) \to l = (x \to l) \to y = e \to y = (y \to l) \to y = (y \to y) \to l = e \to l = e$. So $x \to y \in B$. Hence B is a BH-lattice with l as least element and e as greatest element.

Now let $x, y \in G$. Then by Lemma 1.3.10, $(x \to y) \to l = (x \to l) \to y = x \to y$ and consequently $x \to y \in G$. Furthermore,

$xo(e \to x) = (x \to l)o\{(l \to l) \to x\} = (x \to l)o\{(l \to x) \to l\}$ (by Lemma 1.3.10)

$= \{xo(l \to x)\} \to (lol)$ (by condition (2) above)

$= l \to l = e$ (by 2 in the hypothesis, Theorem 1.3.24 and Lemma 2.2.31). Hence x is invertible element. And by Lemma 1.3.10, $(e \to x) \to l = (e \to l) \to x = e \to x$.. Hence $e \to x \in G$, whenever $x \in G$.

Moreover, by Theorem 1.3.26, $(x \vee y)o(x \wedge y) = xoy$. So by Theorem 1.3.30, $x \vee y = (xoy)o(e \to (x \wedge y))$. Thus $x \vee y \in G$ and so G is closed under all the operation. i.e, G is a BH-lattice. And consequently using Theorem 1.3.35, it follows that G is a commutative l-group

Now for $a \in L$, let $t = a \to l$ and $s = a \to t$. Then by Lemma 1.3.10 and Theorem 1.3.24,

$t \to l = (a \to l) \to l = a \to (lol) = a \to l = t$ and $s \to l = (a \to (a \to l)) \to l = (a \to l) \to (a \to l) = e$. Thus $t \in G$ and $s \in B$. Beside these,

$l \leq e$ (by Lemma 2.2.31)

$\Rightarrow a = a \to e \leq a \to l$ (by Lemma 1.3.12 and Theorem 1.3.15).

$\Rightarrow tos = (a \to l)o(a \to (a \to l)) = a$ (by Theorem 2.2.12)

Now let $a = t'os'$, where $t' \in G$ and $s' \in B$. Then by (2) above, $t = a \to l = (t'os') \to lol = (t' \to l)o(s' \to l) = eo(t' \to l) = t'$. Hence $t = t'$ and consequently by Theorem 1.3.34, $a = tos = t'os' = tos' \Rightarrow (e \to t)(tos) = (e \to t)(tos') \Rightarrow s = s'$. Finally, as $e \to l = e$, clearly $\{e\} = G \cap B$. Thus L is the direct product of B and G.

Conversely, if L is the direct product of BH-lattice with least and greatest element B and commutative l-group G, then trivially conditions (1) and (2) in the Theorem hold. □

Theorem 3.1.5. *BH-lattice L is direct product of a Boolean algebra and a commutative l-group iff*

1. $x \to (yoy) \leq (x \to y)o(e \to y)$ *for all* $x, y \in L$

2. *there exists an element* l *in* L *such that* $(l \to x)ox = lol$ *and* $l \to (l \to x) = x, \forall x \in L$.

Proof. Suppose that the condition in (1) and (2) hold. Let G be the set of all invertible elements of L and H be the set of all idempotent elements of L. By the same argument as in the proof of Theorem 3.1.3, H is a BH-lattice with greatest element e, $o = \wedge$ and L is direct product of G and H.

For any $x \in H$, $x \leq e$ (is the greatest element)

$\Rightarrow e \leq e \to x \leq e$ (by Lemma 1.3.15, Theorem 1.3.24 and e is greatest element)

$\Rightarrow e = e \to x$.

By Lemma 2.2.31 $l \in H$, so it follows that $e = e \to (l \to x)$. So

$l \to (l \to x) = x$ (by the condition given in 2)

$\Rightarrow [l \to (l \to x)] \to l = x \to l$

$\Rightarrow e = e \to (l \to x) = [l \to (l \to x)] \to l = x \to l$ (by Lemma 1.3.10 and Theorem 1.3.24)

$\Rightarrow l \leq x$ (by defining condition of $\to$). Hence H is bounded below by the element l.

Thus by Theorem 2.2.40, H is a Heyting algebra with greatest element e , least element l and $o = \wedge$.

Now for any $x \in H$, by the given condition (2) and Lemma 2.2.31,

$x \wedge (l \to x) = xo(l \to x) = lol = l$.

Moreover, $l = [l \to \{x \vee (l \to x)\}]o[x \vee (l \to x)]$ (by condition (2))

$= [[l \to \{x \vee (l \to x)\}] \wedge x] \vee [[l \to \{x \vee (l \to x)\}] \wedge (l \to x)]$ (by Lemma 1.3.18, $o = \wedge$).

$\Rightarrow [l \to \{x \vee (l \to x)\}] \wedge x = l$ and $[l \to \{x \vee (l \to x)\}] \wedge (l \to x)] = l$ (as l is the least element of H).

$\Rightarrow [l \to \{x \vee (l \to x)\}]ox = l$ and $[l \to \{x \vee (l \to x)\}]o(l \to x)] = l$ (because $o = \wedge$).

$\Rightarrow l \to \{x \vee (l \to x)\} \leq l \to x$ and $l \to \{x \vee (l \to x)\} \leq l \to (l \to x) = x$ (by the defining condition of $\to$ and condition (2)).

$\Rightarrow l \to \{x \vee (l \to x)\} \leq (l \to x) \wedge x = (l \to x)ox = lol = l$ (by condition (2), $o = \wedge$ and Lemma 2.2.31).

$\Rightarrow l \to \{x \vee (l \to x)\} = l$ (because l is least element).

$\Rightarrow e = l \to l = l \to [l \to \{x \vee (l \to x)\}] = x \vee (l \to x)$ (by condition (2)).

Hence H is complemented with $x' = l \to x$ and consequently by definition H is a Boolean algebra. Thus L is the direct product of Boolean algebra and a commutative l-group.

Conversely if L is the direct product of a Boolean algebra H and a commutative l-group G, then trivially condition (1) and (2) hold. □

Definition 3.1.6. *A BH-lattice L is called idempotent if $x^2 = x, \forall x \in L$. That is, an idempotent BH-latice is a BH-lattice in which every element is idempotent element.*

Theorem 3.1.7. *An idempotent BH-lattice L with unity l is a direct product of Boolean algebra and commutative l-group iff $l \to (l \to x) = x, \forall x \in L$.*

Proof. Suppose that $l \to (l \to x) = x, \forall x \in L$. Let $H = \{x \in L : e \to x = e\}$ and G be the set of all invertible elements of L. The proof of H is a BH-lattice with greatest element e is analogous to the proof 2 of Theorem 1.3.36[26] and further L is the direct product of H and G can be obtained. Furthermore for $x, y \in H$, $x, y \leq e$.

$\Rightarrow xoy \leq x$ and $xoy \leq y$ (by Lemma 1.3.7).

$\Rightarrow xoy \leq x \wedge y \leq x, y$.

$\Rightarrow xoy \leq x \wedge y = (x \wedge y)o(x \wedge y) \leq xoy$ (by Theorem 2.2.9). Thus $xoy = x \wedge y$. Finally by the same argument as in proof of Theorem 3.1.5, $x' = l \to x, \forall x \in H$. Thus H is a Boolean algebra.

Conversely if L is the direct product of Boolean algebra and commutative l-group, then the condition $l \to (l \to x) = x, \forall x \in L$ is trivial. □

Chapter 4

Congruences and Filters of BH-Lattices

4.1 Filters on BH-lattices

In this section we introduce the notion of filters in BH-lattices , furnish examples and prove certain properties of filters.

Definition 4.1.1. *A nonempty subset F of a BH-lattice L is called a filter of L iff*

1. $x, y \in F \Rightarrow xoy \in F$
2. $x * e \leq y * e$ *and* $x \in F \Rightarrow y \in F$.

Theorem 4.1.2. *Every filter F of a BH-lattice L contains the identity element e.*

Proof. Since F is non empty subset of L, there exists some element, say $a \in F$. By Theorem 2.2.19, $a * e \leq e = e * e$. So, by definition of a filter $e \in F$. □

Theorem 4.1.3. *Let F be a filter of BH-lattice L. Then the following properties hold.*

1. *If* $x, y \in F$ *and* $x \wedge y \leq a \leq x \vee y$, *then* $a \in F$.
2. *F is closed under all the operations in L.*

Proof. Let $x \in F$. $\Rightarrow x * e \leq e$ (by Theorem 2.2.19)
$\Rightarrow e \to e = e \leq e \to (x * e)$ (By Theorem 1.3.15 and Theorem 1.3.24).
$\Rightarrow e \wedge (x * e) = x * e \leq (x * e) \wedge (e \to (x * e)) = (x * e) * e$. Thus $x * e \in F$. Hence, for $x, y \in F$, $x * e, y * e \in F$, and $(x * e)o(y * e) \in F$.
Let, $x, y, a \in L$, such that $x \wedge y \leq a \leq x \vee y$. Then
$a * e = a \wedge (e \to a)$
$\geq a \wedge (e \to (x \vee y))$ (by Lemma 1.3.15)
$\geq (x \wedge y) \wedge (e \to (x \vee y))$
$= (x \wedge y) \wedge (e \to x) \wedge (e \to y) = (x * e) \wedge (y * e)$ (by Lemma 1.3.19)
$\geq (x * e)o(y * e)$ (by Theorem 2.2.19 and Theorem 2.2.23)
$= [(x * e)o(y * e)] * e$ (by Theorem 2.2.9 and Theorem 2.2.19).
Hence $a \in F$. Hence replacing a by $x \wedge y$ and $x \vee y$, we obtain that $x \wedge y, x \vee y \in F$.
$[(x * e)o(y * e)] * e = (x * e)o(y * e)$ (by Theorem 2.2.9 and Theorem 2.2.19)
$\leq x * y = (x * y) * e$ (by Theorem 2.2.19). Thus,in that case $x * y \in F$.
Furthermore, $(x \to y)oy \leq x$ (by Lemma 1.3.9).
$\Rightarrow y \to x \leq y \to (x \to y)oy$(by Lemma 1.3.15)
$= e \to (x \to y)$ (by Lemma 1.3.10 and Theorem 1.3.24). So that $(x \to y) * e = (x \to y) \wedge (e \to (x \to y)) \geq (x \to y) \wedge (y \to x) = x * y = (x * y) * e$. Hence by definition $x \to y \in F$. Thus (2) holds. ☐

Corollary 4.1.4. *A filter F of BH-lattice L is sub BH-lattice of L.*

Proof. Directly follows from the above theorem by definition. ☐

Corollary 4.1.5. *If x is an element of filter F of BH-lattice L, then the interval $[x \wedge e, x \vee e] \subseteq F$.*

Proof. Directly follows from the above theorem. ☐

For a BH-lattice L, a sub BH-lattice H of L may not be a filter of L. For this consider the l-group $(\mathbb{R}, +)$ with the usual ordering and $H = (\mathbb{Z}, +)$. H is not a filter because $4 * e = -4 \leq \frac{1}{2} * e = -\frac{1}{2}$. But $4 \in F$ and $-\frac{1}{2} \notin \mathbb{Z}$.

Definition 4.1.6. *Every BH-lattice L is a filter of itself, called the improper filter; all other filters will be called proper.*

Theorem 4.1.7. *For any BH-lattice L, $E = \{e\}$ is filter of L.*

Proof. Clearly $e * e = e$. Let $x \in L$ such that $e * e \leq x * e$.
$\Rightarrow e \leq x * e \leq e$ (by Theorem 2.2.19).
$\Rightarrow x * e = x \wedge (e \rightarrow x) = e$
$\Rightarrow e \leq x$ and $e \leq e \rightarrow x$ (by definition of $\wedge$).
$\Rightarrow e \leq x$ and $x \leq e$ (by defining condition of $\rightarrow$).
Hence $x = e \in E$. Thus E is a filter. □

Definition 4.1.8. *The filter $E = \{e\}$ is called the trivial filter.*

Theorem 4.1.9. *$\mathbb{Z}$ is the only non trivial filter in the BH-lattice $(\mathbb{Z}, +, 0, \leq, -)$, where $\leq$ is the usual ordering.*

Proof. Let F be a filter of BH-lattice $(\mathbb{Z}, +, 0, \leq, -)$ and $0 \neq x_0 \in \mathbb{Z}$. If there exists an element $y \in F$ such that $y * 0 \leq x_0 * 0$, then $x_0 \in F$ and $\mathbb{Z} \subseteq F$ and the proof is over. Suppose not. This implies, for all $y \in F, x_0 * 0 \leq y * 0$ (because $\mathbb{Z}$ is a chain) .
$\Rightarrow x_0 \wedge (-x_0) \leq y \wedge (-y)$ (in l-group $x \rightarrow y = xoy^{-1}$).
$\Rightarrow -|x_0| \leq -|y| \Rightarrow |y| \leq |x_0|$.
Since F is filter and $y \in F, |y| \in F$ and further by Theorem 4.1.3, for all $n \in \mathbb{N}, n|y| \in F$. Hence $n|y| \leq |x_0|, \forall n \in \mathbb{N}$ which is impossible. Hence there exists an element $y \in F$ such that $y * 0 \leq x_0 * 0$. Thus $x_0 \in F$. □

Example 4.1.10. *Let $L = \{0, x, y, z, 1\}$ be the lattice given by the Hasse diagram in fig.* 4.1. *For $o = \wedge$ and $\rightarrow$ defined by the table* 4.1.

Then L is a BH-lattice, $F_1 = \{x, z, 1\}, F_2 = \{y, z, 1\}$ and $F_3 = \{1\}$ is a filter of L.

Example 4.1.11. *Let $B = \{1, 2, 3, 5, 6, 10, 15, 30\}$, positive factors of 30 is a Boolean algebra under divisibility. That is, for $x, y \in B, x \leq y$ means x divides y, $x \vee y = lcm\{x, y\}, x \wedge y = gcf\{x, y\}$ and $1' = 30, 2' = 15, 3' = 10, 5' = 6$. Then clearly*

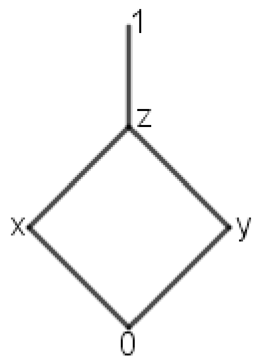


Figure 4.1: Hasse diagram of L $= \{0, x, y, z, 1\}$

$\to$	0	x	y	z	1
0	1	y	x	0	0
x	1	1	x	x	x
y	1	y	1	y	y
z	1	1	1	1	z
1	1	1	1	1	1

Table 4.1: definition of $\to$

$F_4 = \{2, 6, 10, 30\}$, $F_5 = \{3, 6, 15, 30\}$ *and* $F_6 = \{30\}$ *are filters of B.*

Theorem 4.1.12. *The set of filters of a BH-lattice are closed under arbitrary intersection.*

Proof. Let $F_i, i \in I$ be family of filters in L and

$$F = \bigcap_{i \in I} F_i.$$

$x, y \in F \Rightarrow x, y \in F_i, \forall i \in I$. Thus $xoy \in F_i, \forall i \in I$ and hence $xoy \in F$. Suppose $x \in F$ and $x * e \leq y * e$. Then $x \in F_i, \forall i \in I$ and $x * e \leq y * e$. Hence $y \in F_i, \forall i \in I$ and thus $y \in F$. Therefore F is filter of L. □

Consider the filters F_4 and F_5 in example 4.1.11 above. $F_4 \cup F_5 = \{2, 3, 6, 10, 15, 30\}$. Clearly, $10, 15 \in F_1 \cup F_2$ but $10o15 = 10 \wedge 15 = gcf(10, 15) = 5 \notin F_1 \cup F_2$. Hence $F_1 \cup F_2$ is not a filter. Hence we have the following remark.

Remark 4.1.13. *Union of two filters of BH-lattice L is not necessarily a filter.*

Theorem 4.1.14. *Let L be a BH-lattice with the special property $e \to (e \to x) = x, \forall x \in L$. Let F and G be filters of L. Then $F \cup G$ is a filter of L if and only if $G \subseteq F$ or $F \subseteq G$.*

Proof. Let $F \cup G$ be a filter of L. Suppose that $F \nsubseteq G$ and $G \nsubseteq F$. Thus there exists elements $x \in F - G$ and $y \in G - F$. Hence $xoy \in F \cup G$. And consequently $xoy \in F$ or $xoy \in G$. If $xoy \in F$, then by Lemma 1.3.10, Theorem 1.3.24 and Theorem 4.1.3, $x \to xoy = (x \to x) \to y = e \to y \in F$. Thus $e \to (e \to y) = y \in F$, which is a contradiction. If $xoy \in G$, then $y \to xoy = e \to x \in G$. Thus $e \to (e \to x) = x \in G$, which is also a contradiction. In both cases there is a contradiction. Hence $G \subseteq F$ or $F \subseteq G$. The converse is trivial. □

Corollary 4.1.15. *If a BH-lattice L is an l-group and F and G are filters of L, then $F \cup G$ is a filter of L iff $G \subseteq F$ or $F \subseteq G$.*

Proof. In l-group, every element is invertible and by Theorem 1.3.30 $e \to x$ is the inverse of x. Hence $e \to (e \to x) = x$. So it follows by the above theorem. □

Theorem 4.1.16. *For any filter F of a BH-lattice L, the following hold, $\forall x, y, z \in L$.*

1. $x * y \in F \Leftrightarrow x \to y, y \to x \in F$.

2. $x * y, y * z \in F \Rightarrow x * z \in F$.

3. $x, y \to x \in F$, *then* $x \wedge y \in F$.

Proof. $(x * y) * e = x * y \leq (x \to y) * e$ (by Theorem 2.2.19)
Hence $x * y \in F$ implies $x \to y \in F$. By a similar argument by interchanging the role of x and y, we obtain that $y \to x \in F$. Conversely, if $x \to y, y \to x \in F$, then by Theorem 4.1.3, $x * y = (x \to y) \wedge (y \to x) \in F$. Hence (1) holds.
Further, by Theorem 2.2.19, Theorem 2.2.23 and Theorem 4.1.3 if $(x * y), (y * z) \in F$, then $(x * y)o(y * z) \in F$. And $\{(x * y)o(y * z)\} * e = (x * y)o(y * z) \leq x * z = (x * z) * e$.

And hence $x * z \in F$. So (2) holds. Moreover, if $x, y \to x \in F$, by Theorem 4.1.2 and Theorem 4.1.3 we have $x \wedge y = ((y \to x) \wedge e)ox \in F$. Hence (3) holds. □

Theorem 4.1.17. *If F is a filter of a BH-lattice L and $x \in L$. Then*

1. $x \in F \Leftrightarrow (x * e) \in F$.

2. $x \in F \Leftrightarrow (x \vee e), (x \wedge e) \in F$.

Proof. Let $x \in F$. Then by Theorem 4.1.2 and Theorem 4.1.3, $x * e \in F$. If $x * e \in F$, then $(x * e) * e = x * e \leq x * e$. Hence $x \in F$. Thus (1) holds. By Theorem 4.1.2, $e \in F$. Hence by Theorem 4.1.3, $x \in F \Rightarrow x \vee e, x \wedge e \in F$. Conversely, let $(x \vee e), (x \wedge e) \in F$. Then by Theorem 4.1.3 and Theorem 2.2.19, $(x \vee e) * (x \wedge e) = x * e \in F$. Hence from (1), we acquire that $x \in F$. Hence (2) holds. □

Theorem 4.1.18. *If F is a filter of a BH-lattice L and $x, y \in L$, then*

1. $(x \vee y) * e \leq (x * e) \vee (y * e)$

2. *If* $x \vee y \in F$, *then* $(x * e) \vee (y * e) \in F$.

Proof. $(x \vee y) * e = (x \vee y) \wedge (e \to (x \vee y))$

$= (x \vee y) \wedge \{(e \to x) \wedge (e \to y)\}$ (by Lemma 1.3.19)

$= \{x \wedge ((e \to x) \wedge (e \to y))\} \vee \{y \wedge ((e \to x) \wedge (e \to y))\}$ (distributive)

$= \{(x * e) \wedge (e \to y)\} \vee \{(y * e) \wedge (e \to x)\}$

$\leq (x * e) \vee (y * e)$. Hence (1) holds. Now by Theorem 2.2.9 and Theorem 2.2.19 in (1), we obtain that $(x \vee y) * e \leq (x * e \vee y * e) * e$. Hence $x \vee y \in F$ implies that $(x * e) \vee (y * e) \in F$. Hence (2) holds. □

Theorem 4.1.19. *If F is a filter of a BH-lattice L bounded above and $x, y, z \in L$, then the following hold.*

1. $x \to y, y \to z \in F \Rightarrow x \to z \in F$.

2. $x \to y, yoz \in F \Rightarrow xoz \in F$.

3. $x, z \in F, x \leq y \to z \Rightarrow y \in F$.

4. $x \leq y \Rightarrow y \to x \in F$.

Proof. Suppose that F is a fiter of a BH-lattice L bounded above. By Corollary 2.2.22, L is bounded above by e. Hence by Theorem 2.2.19, for any $x \in L, x * e = x$.
Suppose that $x \to y, y \to z \in F$
$\Rightarrow (x \to y)o(y \to z) \leq x \to z$ (by Theorem 1.3.17) and $(x \to y)o(y \to z) \in F$ (by definition of filter). Hence by definition of filter $x \to z \in F$. Thus (1) holds.
Assume $x \to y, yoz \in F \Rightarrow (x \to y)o(yoz) = ((x \to y)oy)oz \in F$ (by definition of a filter and associative property). Since by Lemma 1.3.9 $((x \to y)oy) \leq x$, $((x \to y)oy)oz \leq xoz$ (by Lemma 1.3.7). Hence $xoz \in F$. Thus (2) holds.
Further, let $x, z \in F$ and $x \leq y \to z$
$\Rightarrow y \to z, z \in F. \Rightarrow (y \to z)oz \in F$ and by Lemma 1.3.9 $(y \to z)oz \leq y$. Hence $y \in F$.
Finally to prove (4), let $x \leq y \Rightarrow e \leq y \to x$ (by Lemma 1.3.14) and by Theorem 4.1.2 $e \in F$. Hence $y \to x \in F$. □

Theorem 4.1.20. *If F and G are filters of BH-lattice L which is bounded above, then $F \cup G$ is a filter of L if and only if $G \subseteq F$ or $F \subseteq G$.*

Proof. Let L is bounded above. Then by Theorem 2.2.19 and Corollary 2.2.22 $\forall x \in L, x * e = x$. Suppose that $F \cup G$ is a filter of L and $F \nsubseteq G$, $G \nsubseteq F$. This implies there exist elements $x \in F - G$ and $y \in G - F$. Hence by Theorem 4.1.3 $x * y = (x \to y) \wedge (y \to x) \in F \cup G$.
$\Rightarrow (x \to y) \wedge (y \to x) \in F$ or $(x \to y) \wedge (y \to x) \in G$.
If $(x \to y) \wedge (y \to x) \in F$, then as $(x \to y) \wedge (y \to x) \leq y \to x, x \to y \in F$. Thus by definition of a filter and Lemma 1.3.9 $(y \to x)ox \leq y$ and $(y \to x)ox \in F$. So, $y \in F$ which is a contradiction. If $x * y \in G$, then $x \to y \in G$. Thus by the same argument as above, $x \in G$, which is also a contradiction. In both cases there is a contradiction. Hence $G \subseteq F$ or $F \subseteq G$. The converse is trivial. □

Theorem 4.1.21. *If L is a BH-lattice which is bounded above, then $L^e = \{x \in L : e \to x = e\}$ is a filter of L.*

Proof. Let $x, y \in L^e$. Then by Lemma 1.3.10 $e \to xoy = (e \to x) \to y = e \to y = e$. Hence $xoy \in L^e$. Let $x \in L^e$ and $x \leq y$. Then $x \leq y \leq e$. Thus by Lemma 1.3.15 $e = e \to e \leq e \to y \leq e \to x = e$. Hence $y \in L^e$. □

Theorem 4.1.22. *If L is a BH-lattice which is bounded above with unity l, then* $L_l = \{x \in L : x \to l = e\}$ *is a filter of L.*

Proof. By Theorem 2.2.37, L_l is a sub BH-lattice of L. Hence, if $x, y \in L_l$, then $xoy \in L_l$. Let $x \in L_l$ and $x \leq y$. Then by Theorem 1.3.14, $e = x \to l \leq y \to l \leq e$. Thus $y \to l = e$ and hence $y \in L_l$. □

Definition 4.1.23. *A non-empty subset H of a BH-lattice L is called*

1. *multiplicatively closed if* $xoy \in H$*, whenever x and y belong to H.*
2. *implicatively closed if* $x \to y \in H$*, whenever x and y belong to H.*

Example 4.1.24. *For any BH-lattice L, by Theorem 1.3.36* $L^e = \{x \in L : e \to x = e\}$ *is both multiplicatively and implicatively closed set. Further, if L is with unity, then by Theorem 2.2.37* $L_l = \{x \in L : x \to l = e\}$ *is also both multiplicatively and implicatively closed set.*

Definition 4.1.25. *For a subset X of a given BH-lattice L we denote the least filter containing X by* $F(X)$ *or* $[X)$*. i.e., the intersection of all filters containing X, called the filter generated by X. The elements of X are called generators of the filter [X).*
If $X = \{x_1, ..., x_n\}$*, then the filter [X) is simply denoted by* $[x_1, x_2, ..., x_n)$ *and said to be finitely generated. We can write* $F(a)$ *instead of writing* $F(\{a\})$*, when* $X = \{a\}$ *and we call it the principal filter generated by a.*

Remark 4.1.26. *For a BH-lattice L and* $X \subseteq L$*, clearly by the definition F(X) is unique.*

Theorem 4.1.27. *For any non-empty subset X of a BH-lattice L, the filter generated by X is given by,* $F(X) = \{x \in L : (a_1 * e)o(a_2 * e)o...o(a_k * e) \leq x * e, a_i \in X, k \in \mathbb{N}\}$*,* a_i*'s are not necessarily distinct.*

Proof. Since for any $a \in X, a * e \leq a * e, X \subseteq F(X)$. Let $x, y \in F(X)$. Then there exist elements $a_1, ..., a_k, b_1, ..., b_t \in X$ such that $(a_1 * e)o(a_2 * e)o...o(a_k * e) \leq x * e$ and $(b_1 * e)o(b_2 * e)o...o(b_t * e) \leq y * e$.
$\Rightarrow \{(a_1 * e)o...o(a_k * e)\}o\{(b_1 * e)o...o(b_t * e)\} \leq (x * e)o(y * e)$ (by Theorem 2.2.9)
$= (x \wedge (e \to x))o(y \wedge (e \to y)) = \{(x \wedge (e \to x))oy\} \wedge \{x \wedge (e \to x))o(e \to y)\}$
$= xoy \wedge (yo(e \to x)) \wedge (xo(e \to y)) \wedge \{(e \to x)o(e \to y)\}$ (by distributive property of o over $\wedge$)
$\leq xoy \wedge \{(e \to x)o(e \to y)\}$.
$\leq (xoy) \wedge \{e \to xoy\} = (xoy) * e$ (by Theorem 2.2.10).
$\Rightarrow \{(a_1 * e)o...o(a_k * e)\}o\{(b_1 * e)o...o(b_t * e)\} \leq (xoy) \wedge (e \to (xoy)) = (xoy) * e$. Hence $xoy \in F(X)$. Let $x \in F(X), y \in L$ and $x * e \leq y * e$. Then there exist elements $a_1, ..., a_k \in X$ such that $(a_1 * e)o(a_2 * e)o...o(a_k * e) \leq x * e \leq y * e$. Hence clearly, $y \in F(X)$. Moreover, let F be any filter of L containing X. Let $x \in F(X)$. There exist elements $a_1, ..., a_k \in X$ such that $(a_1 * e)o(a_2 * e)o...o(a_k * e) \leq x * e$. And trivially by Theorem 2.2.9, Theorem 2.2.19, Theorem 4.1.3 and Theorem 4.1.17, $\{(a_1 * e)o(a_2 * e)o...o(a_k * e)\} * e = \{(a_1 * e)o(a_2 * e)o...o(a_k * e)\} \in F$. Hence $x \in F$ and consequently we obtain $F(X) \subseteq F$. Therefore $F(X)$ is the filter generated by X. □

Corollary 4.1.28. *If X is a non-empty implicatively closed subset of BH-lattice L and $e \in X$, then $F(X) = \{x \in L : (a_1 \wedge a_2)o(a_3 \wedge a_4)o...o(a_{k-1} \wedge a_k) \leq x * e, a_i \in X\}$, where a_i's are not necessarily distinct.*

Proof. Since for any $a \in X, a*e = a \wedge (e \to a) = a_1 \wedge a_2$, where $a_1 = a, a_2 = e \to a \in X$. In a similar manner by renaming and by Theorem 4.1.27 , the Corollary follows. □

Remark 4.1.29. *For BH-lattice L, the filter generated by empty set is the intersection of all filters of L. By Theorem 4.1.2 and Theorem 4.1.7 we have $F(\emptyset) = E = \{e\}$.*

Corollary 4.1.30. *If L is a BH-lattice and $a \in L$, then the filter generated by a is $F(a) = \{x \in L : (a * e)^n \leq x * e, n \in \mathbb{N}\}$.*

Proof. Directly follows from Theorem 4.1.27. □

Corollary 4.1.31. *If L is a BH-lattice and non-empty subset X of L is bounded above by e, then* $F(X) = \{x \in L : a_1 o a_2 o ... a_k \leq x * e, a_i \in X\}$.

Proof. Follows from Theorem 2.2.19 and Theorem 4.1.27. □

Corollary 4.1.32. *If T is filter of BH-lattice L, then* $F(T) = T$.

Proof. Follows from the definition. Or as a corollary we can prove as follows. Clearly $T \subseteq F(T)$. Let $x \in F(T)$. Then by Theorem 4.1.27 there exists $a_1, a_2, ..., a_k \in T$ such that $(a_1 * e)o(a_2 * e)o...o(a_k * e) \leq x * e$. By Theorem 2.2.9, Theorem 2.2.19 and Theorem 4.1.3, we have $a_1, a_2, ..., a_k \in T$ such that $(a_1 * e)o(a_2 * e)o...o(a_k * e) \in T$. So by definition it follows that $x \in T$. Thus $F(T) = T$. □

From the above results we observe that

1. if $a \in L, a \leq e$, then $F(a) = \{x \in L : a^n \leq x * e, n \in \mathbb{N}\}$.
2. if L is a BH-lattice bounded above, then for any $\emptyset \neq X \subseteq L$, $F(X) = \{x \in L : x_1 o x_2 o ... o x_n \leq x, x_i \in X, \forall i = 1, 2, 3, ..., n\}$ and $F(a) = \{x \in L : a^n \leq x, n \in \mathbb{N}\}$. Moreover, if X is multiplicatively closed subset of L, then $F(X)$ reduces to the form $F(X) = \{x \in L : a \leq x, a \in X\}$.

Corollary 4.1.33. *For a Heyting algebra H and* $a \in H$, $H(a) = \{x : a^n \leq x,$ *for some positive integer n* $\}$ *is the filter generated by a.*

Proof. Heyting algebra is bounded above by e and by Theorem 2.2.19, $a * e = x$. Hence the result follows from Corollary 4.1.31. □

Corollary 4.1.34. *In a commutative l-group G and* $a \in G$, $\{x : (a \wedge a^{-1})^n \leq x \wedge x^{-1}, n \in \mathbb{N}\}$ *is the filter generated by a. Particularly, if* $a \leq e$, *then the filter generated by a is simply given by* $\{x : a^n \leq x \wedge x^{-1}, n \in \mathbb{N}\}$

Proof. In a commutative l-group, $a * e = a \wedge (e \to a) = a \wedge a^{-1}$. And by Theorem 2.2.19, if $a \leq e, a * e = a$. Hence it holds. □

Example 4.1.35. *Consider the set of all positive real numbers $\mathbb{R}^+$ with the usual ordering and multiplication operation. Then it is a commutative l-group and so* $(\mathbb{R}^+, ., 1, \leq, \rightarrow)$ *is a BH-lattice. Thus:*
$F(1) = \{x : 1 \leq x \wedge x^{-1} = x \wedge \frac{1}{x}\} = \{1\}$.
$F(3) = \{x : (\frac{1}{3})^n \leq x \wedge x^{-1} = x \wedge \frac{1}{x}, n \in \mathbb{N}\} = [\frac{1}{3}, 3] \cup [\frac{1}{9}, 9] \cup [\frac{1}{27}, 27] \cup ... = \mathbb{R}^+$.

Corollary 4.1.36. *If L is a BH-lattice, $x, y \in L$ and $A, B \subseteq L$, then the following properties hold.*

1. *$F(e) = \{e\}$ and $F(L) = L$,*
2. *$A \subseteq B$ implies $F(A) \subseteq F(B)$,*
3. *$a \leq b \leq e$ implies $F(b) \subseteq F(a)$,*
4. *if A is a filter, then $F(A) = A$,*
5. *if A is a filter and $a \in A$, then $F(a) \subseteq A$.*

Proof. Directly follows from Theorem 2.2.9, Theorem 2.2.19, Theorem 4.1.27 and definition of filter. □

Recall that
- a subdirectly irreducible algebra is an algebra that cannot be factored as a subdirect product of "simpler" algebras [8].
- by Theorem 8.5 in [8]a subdirectly irreducible algebra is directly indecomposable.
- in l-group $(\mathbb{Z}, +, \leq)$, where $\leq$ is the usual ordering in the set of integers, the property $e \leq (x \rightarrow y) \vee (y \rightarrow x) = (x+y^{-1}) \vee (y+x^{-1}) = (x-y) \vee (-(x-y)), \forall x, y \in \mathbb{Z}$ hold. This show that, there exists a BH-lattice with the special property $e \leq (x \rightarrow y) \vee (y \rightarrow x)$.

Theorem 4.1.37. *Let L be a BH-lattice which satisfies the special property $e \leq (x \rightarrow y) \vee (y \rightarrow x)$. If L is subdirectly irreducible, then L is totally ordered.*

Proof. Suppose L is subdirectly irreducible and let L is not totally ordered. Hence there exist a pair of incomparable elements $x, y \in L$. By Lemma 1.3.16, this implies that $(x \rightarrow y) \wedge e \neq e$ and $(y \rightarrow x) \wedge e \neq e$.

Let $F_1 = F((x \to y) \wedge e)$ and $F_2 = F((y \to x) \wedge e)$. Let $a \in F_1 \cap F_2$. Then by Corollary 4.1.30 there exist positive integers m and n such that $\{(x \to y) \wedge e\}^m \leq a * e$ and $\{(y \to x) \wedge e\}^n \leq a * e$. Hence $[\{(x \to y) \wedge e\}^m] \vee [\{(y \to x) \wedge e\}^n] \leq a * e$.
Further more, by the hypothesis given and distributive property $[\{(x \to y) \wedge e\}] \vee [\{(y \to x) \wedge e\}] = \{(x \to y) \vee (y \to x)\} \wedge e = e$. For any $t, s \in L$ such that $t \vee s = e$, by Theorem 1.3.7, Theorem 2.2.9 and the properties of $\vee$, $s^2 \leq e, (t \vee s)ot \leq t$. Thus $t \vee s^2 \leq e$ and $(t \vee s)ot \vee sos = tot \vee sot \vee sot \vee sos = (t \vee s)o(t \vee s) = e$. And $e = to(t \vee s) \vee (sos) \leq t \vee s^2$. Therefore $t \vee s^2 = e$. Therefore by induction we have $t \vee s^n = e$. By a similar argument by interchanging the role of t and s in this result, we obtain $t^m \vee s^n = e$. Now, taking $t = \{(x \to y) \wedge e\}$ and $s = \{(y \to x) \wedge e\}$ and using Theorem 2.2.19 gives, $e = [\{(x \to y) \wedge e\}^m] \vee [\{(y \to x) \wedge e\}^n] \leq a * e \leq e$. Thus $a * e = e$ and hence $a = e$. Therefore L is subdirectly reducible. □

Corollary 4.1.38. *Any subdirectly irreducible Boolean algebra is a chain.*

Proof. In a Boolean algebra B, for $x, y \in B, (x \to y) \vee (y \to x) = (x \vee y') \vee (y \vee x') = (x \vee x') \vee (y \vee y') = e \vee e = e$. Hence the corollary follows from the theorem above. □

Corollary 4.1.39. *Any subdirectly irreducible commutative l-group is a chain.*

Proof. Suppose that L is a BH-lattice and (L, o) is a group. Then for any $x \in L$,
$e = (x \vee x^{-1}) \to (x \vee x^{-1})$ (by Theorem 1.3.24)
$= (x \vee x^{-1})o(x \vee x^{-1})^{-1}$ (by definition of $\to$ in l-group)
$= (x \vee x^{-1})o(x^{-1} \wedge x)$ (by Theorem 1.2.38)
$\leq (x \vee x^{-1})^2$ (by Theorem 1.3.10).
$\Rightarrow \{(x \vee x^{-1})^{-1}\}^2 \leq e$ (operating both sides by $\{(x \vee x^{-1})^{-1}\}^2$).
$\Rightarrow (x \vee x^{-1})^{-1} \leq e$ (by Theorem 2.2.24)
$\Rightarrow e \leq x \vee x^{-1}$ (operating both sides by $x \vee x^{-1}$). Therefore for any commutative l-group L, and $x, y \in L, e \leq (xoy^{-1}) \vee (xoy^{-1})^{-1} = (xoy^{-1}) \vee (x^{-1}oy) = (x \to y) \vee (y \to x)$. Hence by the above theorem any subdirectly irreducible commutative l-group is a chain. □

Theorem 4.1.40. *If L is a BH-lattice and $x, y, z \in L$ such that $x, y, z \leq e$, then*

1. $(x \wedge y)o(x \wedge z) \leq x \wedge (yoz)$.

2. $(x \vee y)o(x \vee z) \leq x \vee (yoz)$.

Proof. Suppose $x, y, z \leq e$. By Theorem 2.2.23, $x \leq e, y \leq e, z \leq e$ implies that $xox \leq x \wedge x, xoy \leq x \wedge y, xoz \leq x \wedge z$. Hence by distributive property of o over $\wedge$, $(x \wedge y)o(x \wedge z) = (xox) \wedge (yox) \wedge (xoz) \wedge (yoz) \leq (x \wedge x) \wedge (y \wedge x) \wedge (x \wedge z) \wedge (yoz) \leq x \wedge x \wedge x \wedge (yoz) = x \wedge (yoz)$. Further, by distributive property of o over $\vee$ $(x \vee y)o(x \vee z) = (xox) \vee (yox) \vee (xoz) \vee (yoz) \leq (x \wedge x) \vee (y \wedge x) \vee (x \wedge z) \vee (yoz) \leq x \vee x \vee x \vee (yoz) = x \vee (yoz)$. □

Theorem 4.1.41. *If L is any BH-lattice and F and T are filters of L, then $F \vee T = F(F \cup T) = \{x \in L : fot \leq x * e, f \in F, t \in T, f, t \leq e\}$ is the smallest filter containing $F \cup T$ (under set inclusion).*

Proof. By Theorem 4.1.27, $F(F \cup T) = \{x \in L : (a_1 * e)o...o(a_k * e) \leq x * e, a_i \in F \cup T\}$
$= \{x \in L : \{(a_1*e)o(a_2*e)o...o(a_j*e)\}o\{(b_1*e)o(b_2*e)o...o(b_t*e)\} \leq x*e, a_i \in F, b_i \in T$
(by Theorem 4.1.17, commutative property and renaming)
$= \{x \in L : fot \leq x * e, f = \{(a_1 * e)o(a_2 * e)o...o(a_j * e)\} \in F, t = \{(b_1 * e)o(b_2 * e)0...o(b_t * e)\} \in T, f, t \leq e\ \}$ (by Theorem 2.2.19, Theorem 4.1.3)
Let G be any filter and $T, F \subseteq G$. Assume that $x \in F(F \cup T)$. Then there exists $f \in F, t \in T, f, t \leq e$ and $fot \leq x * e$. Since $f \in F \subseteq G, t \in T \subseteq G$, by Theorem 4.1.3 $fot = (fot) * e \in G$. Hence by definition $x \in G$. Thus $F(F \cup T) \subseteq G$ and so $F \vee T = F(F \cup T)$. □

Theorem 4.1.42. *If L is any BH-lattice, then the set of all filters $\mathbb{F}$ of L forms a complete bounded distributive lattice (under $\subseteq$).*

Proof. Suppose L is any BH-lattice and F and T are filters of L. Take $F \wedge T = F \cap T; F \vee T = F(F \cup T)$. Then clearly it is a bounded lattice with $0 = \bigcap_{F \in \mathbb{F}} F$ and $1 = L$. To show it is distributive it suffices to show $(F \vee G) \wedge (F \vee T) \subseteq F \vee (G \wedge T)$ or $F \wedge (G \vee T) \subseteq (F \wedge G) \vee (F \wedge T)$ for $F, G, T \in \mathbb{F}$. Let $x \in F \wedge (G \vee T) = F \cap (G \vee T)$. This implies that $x \in F$ and $x \in G \vee T$. Hence by Theorem 4.1.41 there exists $t \in T, g \in G$

such that $tog \leq x*e, t, g \leq e$ and $x \in F$. Then by Theorem 4.1.40 and Theorem 2.2.19, $((x*e) \vee t)o((x*e) \vee g) \leq (x*e) \vee (tog) = x*e$.
$x*e, t \leq (x*e) \vee t \Rightarrow (x*e) \vee t \in F \cap T$ and $(x*e), g \leq (x*e) \vee g \Rightarrow (x*e) \vee g \in F \cap G$
Thus by Theorem 4.1.41 $((x*e) \vee t)o((x*e) \vee g) \in (F \cap G) \vee (F \cap T)$. Hence $x \in (F \cap G) \vee (F \cap T)$. Therefore the lattice in theorem is distributive.
Let $\emptyset \neq X \subseteq L$. Any $x \in L$ is in $F(X)$ iff there exists elements $a_1, a_2, ..., a_n \in X$ such that $(a_1 * e)o(a_2 * e)o...o(a_n * e) \leq x * e$. Therefore $x \in F(a_1, a_2, ..., a_n)$ and thus $F(X) = \bigcup\{F(Y), Y \subseteq X, |Y| \leq n, n \in \mathbb{N}\}$. Further, clearly by the definition of a filter generated by a given set and Corollary 4.1.36, for $X, Y \subseteq L, X \subseteq F(X)$ and $X \subseteq Y$ implies that $F(X) \subseteq F(Y)$ and by Theorem 2.2.9, Theorem 4.1.27 and induction $F(F(X)) = F(X)$. Hence the mapping $X \longmapsto F(X)$ is an algebraic closure operator whose closed subsets are filters. Since by Theorem 1.1.36, the set of all closed subsets (with set inclusion as the partial ordering) is a complete lattice, the lattice in the theorem is complete. Hence $(\mathbb{F}, \subseteq)$ is a complete bounded distributive lattice. □

Theorem 4.1.43. *If any BH-lattice L is totally order, then so does* $(\mathbb{F}, \subseteq)$.

Proof. Let $F, T \in \mathbb{F}$ such that $F \neq T$. Then there exists an element $x \in F - T$ or $x \in T - F$. Suppose that $x \in F - T$. Then by theorem 4.1.17, $x * e \in F - T$. Let $y \in T$. Then $y * e \in T$. If $y * e \leq x * e$, then $x \in T$ which is a contradiction. Thus $x * e \leq y * e$ and hence $y \in F$. Hence $T \subseteq F$. Therefore $(\mathbb{F}, \subseteq)$ is totally ordered. □

Theorem 4.1.44. *If L is a BH-lattice, T filter of L and* $a, b \in L$, *then:*

1. $F(T \cup \{a\}) = \{x \in L : (t * e)o(a * e)^n \leq x * e, t \in T, n \geq 0\}$, *where for* $x \in L, x^0 = e$.

2. $F(a) \wedge F(b) \subseteq F(a * e \wedge b * e) = F((a * e)o(b * e)) \subseteq F(a) \vee F(b)$

3. $F(a) \vee F(b) \supseteq F((a * e) \vee (b * e))$

Proof. The first one is trivial from Theorem 4.1.3 and Theorem 4.1.27. For the rest two consider the following. Since by Theorem 2.2.19 $a * e = (a * e) * e$, from Corollary 4.1.30 it is obvious that $F(a) = F(a * e)$. Further, by Theorem 2.2.19 and Theorem

2.2.23, $(a*e)o(b*e) \leq (a*e) \wedge (b*e) \leq a*e, b*e$. Hence by Theorem 2.2.9 and induction we have $((a*e)o(b*e))^n \leq (a*e \wedge b*e)^n \leq (a*e)^n, (b*e)^n, n \in \mathbb{N}$. Hence by Corollary 4.1.30, $F(a), F(b) \subseteq F((a*e) \wedge (b*e)) \subseteq F((a*e)o(b*e))$. Hence $F(a) \wedge F(b) \subseteq F(a), F(b) \subseteq F((a*e) \wedge (b*e)) \subseteq F((a*e)o(b*e))$. Let G be any filter of L such that $F(a), F(b) \subseteq G$ and let $x \in F((a*e)o(b*e))$. Then $((a*e)o(b*e))^n \leq x*e$, for some $n \in \mathbb{N}$. Nevertheless, $a*e, b*e \in G$, thus $(a*e)o(b*e) \in G$. Therefore $x \in G$. Hence $F((a*e)o(b*e)) \subseteq F((a*e) \wedge (b*e)), F((a*e)o(b*e)) \subseteq F(a) \vee F(b)$. Hence (2) holds.

Furthermore, as $a*e, b*e \leq (a*e) \vee (b*e)$, it follows that $(a*e)^n, (b*e)^n \leq ((a*e) \vee (b*e))^n, n \in \mathbb{N}$. Hence it follows that $F((a*e) \vee (b*e)) \subseteq F(a*e), F(b*e) \subseteq F(a) \vee F(b)$. □

In 2 and 3 of Theorem 4.1.44 equality does not hold. To show this consider F_4 and F_5 in example 4.1.11. Observe that $F(2) = \{x \in B : (2*e)^n \leq x*e\} = \{x \in B : 2 \leq x\} = F_4 = \{2, 6, 10, 30\}$. Similarly $F(3) = F_5 = \{3, 6, 15, 30\}$. Then clearly $F(2) \vee F(3) = B = \{1, 2, 3, 5, 6, 10, 15, 30\}$ while $F((2*e) \vee (3*e)) = F(2 \vee 3) = F(6) = \{6, 30\}$. Additionally, $F(2) \wedge F(3) = \{6, 30\}$, while $F((2*e) \wedge (3*e)) = F(1) = \{1, 2, 3, 5, 6, 10, 15, 30\}$.

In any lattice $(P, \leq)$, for $a, b \in P$, the relative Pseudocomplement of a with respect to b is a maximal element c such that $a \wedge c \leq b$ and it is denoted by $b \to a$.

Theorem 4.1.45. *Let L be a BH-lattice and T and K be filters of L. The relative pseudocomplement of T with respect to K is given by* $K \to T = \{x \in L : (x*e) \vee (a*e) \in K, \forall a \in T\}$

Proof. Let $H = \{x \in L : (x*e) \vee (a*e) \in K, \forall a \in T\}$. Since by Theorem 2.2.19 for any $a \in L, a*e \leq e, (e*e) \vee (a*e) = e \in K$. Hence $e \in H$ and so H is non empty set.
Let $x, y \in H$. Then for each $a \in T, (x*e) \vee (a*e), (y*e) \vee (a*e) \in K$.
$(x*e)o(y*e) = \{x \wedge (e \to x)\}o\{y \wedge (e \to y)\} = xoy \wedge yo(e \to x) \wedge xo(e \to y) \wedge (e \to x)o(e \to y)$ (distributive).
$\leq xoy \wedge (e \to x)o(e \to y) \leq (xoy)*e$ (by Theorem 2.2.10)
By Theorem 2.2.19 and Theorem 4.1.40, $\{(x*e) \vee (a*e)\}o\{(y*e) \vee (a*e)\} \leq \{(x*e)o(y*e)\} \vee (a*e) \leq ((xoy)*e) \vee (a*e)$. Thus $((xoy)*e) \vee (a*e) \in K$ and hence

$xoy \in H$. Now let $x \in H$ and $x * e \leq y * e$. Then $(x * e) \vee (a * e) \leq (y * e) \vee (a * e) \in K$. Hence $y \in H$ and consequently H is a filter.

If $x \in T \cap H$, then $x * e = (x * e) \vee (x * e) \in K$ and hence by Theorem 4.1.17 $x \in K$. Hence $T \cap H \subseteq K$. Furthermore, let C be any filter such that $T \cap C \subseteq K$. For $x \in C$ and for each $a \in T$, $x * e, a * e \leq \{(x * e) \vee (a * e)\} * e = (x * e) \vee (a * e) \in T \cap C \subseteq K$. Hence $x \in C$ implies that $x \in H$. Therefore H is the maximal element C such that $T \cap C \subseteq K$. Hence $H = K \rightarrow T$. □

Corollary 4.1.46. *For any BH-lattice L and the set of filters $\mathbb{F}$, the system $(\mathbb{F}, \subseteq)$ forms a Heyting algebra with $K \rightarrow T = \{x \in L : (x*e) \vee (a*e) \in K, \forall a \in T\}, K, T \in \mathbb{F}$.*

Proof. Directly follows from Theorem 4.1.42 and Theorem 4.1.45. □

Observe that for $K, T \in \mathbb{F}, K \rightarrow T = \bigvee\{C \in \mathbb{F} : T \cap C \subseteq K\}$.

4.2 Congruence Relations

In this section we introduce the notion of congruence relations in BH-lattices and prove certain properties.

Definition 4.2.1. *An equivalence relation θ on a BH-lattice $(L, \circ, e, \leq, \rightarrow)$ is called a congruence relation on L if for $(x, y), (c, d) \in \theta, (x \wedge c, y \wedge d), (x \vee c, y \vee d), (xoc, yod), (x \rightarrow c, y \rightarrow d) \in \theta$. That is, a lattice congruence θ on L is a congruence on L if for $(x, y), (c, d) \in \theta, (xoc, yod), (x \rightarrow c, y \rightarrow d) \in \theta$.*

For a congruence relation θ on BH-lattice L, the congruence class containing x is denoted by $[x]$ or $[x]_\theta$ and the set of all congruence classes is denoted by L/θ.

Theorem 4.2.2. *Let F be a filter and define $\theta^F = \{(x, y) \in LXL : xoh \leq y, yoh \leq x$ for some $h \in F\}$. Then θ^F is a congruence relation on L.*

Proof. For any $x \in L, xoe \leq x$ and by Theorem 4.1.2, $e \in F$. so $(x, x) \in \theta^F$. Clearly θ^F is symmetric. Let $(x, y), (y, z) \in \theta^F$. Then there are elements $h, h' \in F$ and $xoh \leq y, yoh \leq x, yoh' \leq z, zoh' \leq y$. By Lemma 1.3.7, $(xoh)oh' \leq yoh' \leq z$ and

$(zoh')oh \leq yoh \leq x$ and $hoh' = h'oh \in F$. So $(x, z) \in \theta^F$. Thus θ^F is an equivalence relation on L.

To establish the substitution properties, let $(a, b), (x, y) \in \theta^F$. Hence there are elements $h, h' \in F$ such that $aoh \leq b$ and $boh \leq a$, $xoh' \leq y$ and $yoh' \leq x$. Let $k = hoh' \in F$, then by Theorem 2.2.9, $(aoh)o(xoh') \leq boy$ and $(boh)o(yoh') \leq aox$. This implies that $(aox)o(hoh') \leq boy$ and $(boy)o(hoh') \leq aox$. Thus, $(aox, boy) \in \theta^F$.

Further, let $j = h \wedge h' \in F$. By Lemma 1.3.7, $aoj \leq aoh \leq b, boj \leq boh \leq a$ and $xoj \leq xoh' \leq y, yoj \leq yoh' \leq x$.

This implies that $a \leq b \to j, b \leq a \to j, x \leq y \to j, y \leq x \to j$. Hence by Lemma 1.3.20 $a \wedge x \leq (b \to j) \wedge (y \to j) = (b \wedge y) \to j$ and $b \wedge y \leq (a \to j) \wedge (x \to j) = (a \wedge x) \to j$. Therefore, $(a \wedge x)oj \leq b \wedge y$ and $(b \wedge y)oj \leq a \wedge x$ and hence $(a \wedge x, b \wedge y) \in \theta^F$. Moreover, by Lemma 1.3.18 $(a \vee x)oj = (aoj) \vee (xoj)$ and $j \leq h, h'$ and this implies that $boj \leq boh \leq a, yoj \leq yoh' \leq x$ and $aoj \leq aoh \leq b, xoj \leq xoh' \leq y$. Hence, $(aoj) \vee (xoj) = (a \vee x)oj \leq b \vee y$ and $(boj) \vee (yoj) = (b \vee y)oj \leq a \vee x$. Thus, $(a \vee x, b \vee y) \in \theta^F$. Therefore, the relation θ^F is a congruence relation. □

Theorem 4.2.3. *Let F be a filter of a BH-lattice L. The relation defined by* $x \equiv y(\theta_F)$ *iff* $x * y = (x \to y) \wedge (y \to x) \in F$ *is a congruence relation.*

Proof. Let F be a filter of BH-lattice L. Define $x \equiv y(\theta_F)$ iff $x * y = (x \to y) \wedge (y \to x) \in F$. Since e belongs to F, by Theorem 1.3.24 it follows that for any element x in L, $x \equiv x(\theta_F)$. So it is reflexive. Let for x, y in L, $x \equiv y(\theta_F)$. Since by Theorem 2.2.19 $x * y = y * x$, it follows that $y \equiv x(\theta_F)$. Hence it is symmetric.

Now let $x \equiv y(\theta_F), y \equiv z(\theta_F)$. Thus $x*y$ and $y*z$ both belong to F. Hence $(x*y)o(y*z)$ belong to F. Using Theorem 2.2.19, $(x * y)o(y * z) \leq x * z \leq e$. Hence using Theorem 2.2.19, $[(x * y)o(y * z)] * e = (x * y)o(y * z) \leq x * z = (x * z) * e$. So, $x * z \in F$ and hence $x \equiv z(\theta_F)$. Thus it is transitive. Hence $x \equiv y(\theta_F)$ is an equivalence relation.

Let $x \equiv y(\theta_F)$ and $t \equiv s(\theta_F)$.

$[(x * y)o(t * s)] * e = (x * y)o(t * s)$(by Theorem 2.2.19)

$= [(x \to y) \wedge (y \to x)]o(t * s) = \{(x \to y)o(t * s)\} \wedge \{(y \to x)o(t * s)\}$ (distributive)

$= \{(x \to y)o(t \to s)\} \wedge \{(y \to x)o(t \to s)\} \wedge \{(y \to x)o(s \to t)\} \wedge \{(x \to y)o(s \to t)\}$

$\leq \{(x \to y)o(t \to s)\} \wedge \{(y \to x)o(s \to t)\}$

$\leq \{(xot) \to (yos)\} \wedge \{(yos) \to (xot)\}$ (by Theorem 2.2.10)
$= (xot) * (yos) = [(xot) * (yos)] * e$ (by Theorem 2.2.19). Hence by the definition of a filter $(xot) * (yos) \in F$. So $(xot) \equiv (yos)(\theta_F)$.
Further, by Lemma 1.3.15, Lemma 1.3.20 and Theorem 2.2.19, $[(x \wedge t) * (y \wedge s)] * e = (x\wedge t)*(y\wedge s) = (x\wedge t \to y\wedge s)\wedge(y\wedge s \to x\wedge t) = (x \to y\wedge s)\wedge(t \to y\wedge s)\wedge(y \to x\wedge t)\wedge (s \to x\wedge t) \geq (x \to y)\wedge(t \to s)\wedge(y \to x)\wedge(s \to t) = (x*y)\wedge(t*s) = [(x*y)\wedge(t*s)]*e$. Since by Theorem 4.1.3, $(x * y) \wedge (t * s)$ belongs to F, $(x \wedge t) * (y \wedge s)$ belongs to F. Hence $x \wedge t \equiv y \wedge s(\theta_F)$.
By Lemma 1.3.14, Lemma 1.3.19 and Theorem 2.2.19, $[(x\vee t)*(y\vee s)]*e = (x\vee t)*(y\vee s) = (x\vee t \to y\vee s)\wedge(y\vee s \to x\vee t) = (x\vee t \to y)\wedge(x\vee t \to s)\wedge(y\vee s \to x)\wedge(y\vee s \to t) \geq (x \to y) \wedge (t \to s) \wedge (y \to x) \wedge (s \to t) = (x * y) \wedge (t * s) = [(x * y) \wedge (t * s)] * e$. Since $(x * y) \wedge (t * s)$ belongs to F, $x \vee t \equiv y \vee s(\theta_F)$.
Finally, Since $s*t, x*y \in F$ and using Theorem 2.2.19, $[x*y]*e = x*y = (x \to y)\wedge(y \to x) \leq (x \to y)*e, (y \to x)*e$ and $[s*t]*e = s*t = (s \to t)\wedge(t \to s) \leq (s \to t)*e, (t \to s)*e$, it follows that $x \to y, y \to x, t \to s, s \to t$ all belong to F. And since F is a filter it follows that $(s \to t)o(x \to y)$ and $(y \to x)o(t \to s)$ belong to F. Now, by Lemma 1.3.9 and Lemma 1.3.17, $to(y \to s)o(s \to t)o(x \to y) \leq to(y \to t)o(x \to y) \leq yo(x \to y) \leq x$. Hence by defining condition of $\to$ it follows that $(x \to t) \to (y \to s) \geq (s \to t)o(x \to y)$. Similarly, as $so(x \to t)o(t \to s)o(y \to x) \leq so(x \to s)o(y \to x) \leq xo(y \to x) \leq y$, it follows that $(y \to s) \to (x \to t) \geq (t \to s)o(y \to x)$.
Hence by Theorem 2.2.19, it follows that $[(x \to t) * (y \to s)] * e = (x \to t) * (y \to s) \geq \{(s \to t)o(x \to y)\}\wedge\{(t \to s)o(y \to x)\} = [\{(s \to t)o(x \to y)\}\wedge\{(t \to s)o(y \to x)\}]*e$. So that $(x \to t)*(y \to s)$ belongs to F, as $\{(s \to t)o(x \to y)\}\wedge\{(t \to s)o(y \to x)\} \in F$. Hence $(x \to t) \equiv (y \to s)(\theta_F)$. Thus if F is a filter of BH-lattice L, then $x \equiv y(\theta_F)$ iff $x * y \in F$ is a congruence relation on L. □

Definition 4.2.4. *If L is a BH-lattice and F is a filter of L, then the congruence relation in Theorem* 4.2.3 *is called congruence relation related to F and is denoted by* θ_F*. For simplicity the set of all congruence classes* L/θ_F *is simply denoted by* L/F*.*

Theorem 4.2.5. *Let* θ *be a congruence relation on BH-lattice L. If e is comparable with each* $x \in L$*, then* $N = \{x \in L : x \equiv e(\theta)\}$ *is a filter of L.*

Proof. Let θ be a congruence relation on L and $N = \{x \in L : x \equiv e(\theta)\}$. Obviously e belongs to N, so N is nonempty subset of L. If x, y belong to N, then by the definition of a congruence relation, it follows that xoy belongs to N. That is, $x, y \in N \Rightarrow xoy \in N$
Let x belongs to N and $x * e \leq y * e$.
$\Rightarrow x \equiv e(\theta)$ and as θ is a congruence relation $e \equiv e(\theta)$
$\Rightarrow x * e = x \wedge (e \rightarrow x) \equiv e(\theta)$.
$\Rightarrow x * e = (x * e) \wedge (y * e) \equiv e(\theta)$
$\Rightarrow [(x * e) \wedge (y * e)] \vee (y * e) = y * e \equiv e(\theta)$ (by Theorem 2.2.19, absorption law and $y * e \equiv y * e(\theta)$).
If $y \leq e$, then by Theorem 2.2.19, $y * e = y$. Hence $y \equiv e(\theta)$. If $e \leq y$, then by Lemma 1.3.15 $e \rightarrow y \leq e \leq y$. Thus $y * e = e \rightarrow y \equiv e(\theta)$. This implies that $yo(e \rightarrow y) \equiv y(\theta)$. Hence $\{yo(e \rightarrow y)\} \vee e \equiv y \vee e(\theta)$. Thus by Lemma 1.3.9 $e \equiv y(\theta)$, so that $y \equiv e(\theta)$. Hence in both cases we have that $y \equiv e(\theta)$. Thus y belongs to N and hence N is a filter. □

Corollary 4.2.6. *If L is a BH-lattice bounded above and θ is a congruence relation on L, then $N = \{x \in L : x \equiv e(\theta)\}$ is filter of L.*

Proof. Follows from Corollary 2.2.22 and Theorem 4.2.5. □

Theorem 4.2.7. *If L is a BH-lattice and e is comparable with each $x \in L$, then there is one to one correspondence between the set of all filters $\mathbb{F}$ of L and the set of all congruence relations of L, Con L.*

Proof. From Theorem 4.2.3 and Theorem 4.2.5, it suffices to show the following. Define a function $f : \mathbb{F} \longrightarrow ConL$ by $f(F) = \theta_F$. For $F, G \in \mathbb{F}$, let $f(F) = f(G)$. Then this implies that $\theta_F = \{(x, y) : x * y \in F\} = \{(x, y); x * y \in G\} = \theta_G$. Hence, $x \in F$ implies that $x * e \in F$, so $x * e \in G$. Thus by Theorem 4.1.17 $x \in G$. Therefore $F \subseteq G$. By a similar argument, $G \subseteq F$. Thus $F = G$ and hence f is one to one function. Let $\theta \in conL$. Then by Theorem 4.2.5 $N = \{x \in L : x \equiv e(\theta)\}$ is a filter. Thus,
$f(N) = \theta_N = \{(x, y) : x * y \in N\}$
$= \{(x, y) : x \rightarrow y, y \rightarrow x \in N\}$ (by Theorem 5.1.9)

$= \{(x, y) : x \equiv y(\theta)\}$

$= \theta$. This is because $x \to y \equiv e(\theta) \Rightarrow (x \wedge y) = \{(x \to y) \wedge e\}oy \equiv y(\theta)$ and $y \to x \equiv e(\theta) \Rightarrow (x \wedge y) = \{(y \to x) \wedge e\}ox \equiv x(\theta)$. Thus as θ is a congruence relation $x \equiv y(\theta)$. Hence f is on to.

Hence, there is a one to one correspondence between congruence relations on L and its filters. □

Corollary 4.2.8. *The filters of a BH-lattice bounded above correspond one to one to its congruence relations.*

Proof. Directly follows from the above Theorem. □

Corollary 4.2.9. *The filters of a Heyting algebra correspond one to one of its congruence relations.*

Proof. Directly follows from the above Theorem. □

Theorem 4.2.10. *In a BH-lattice L, for a filter F in L and* $[x], [y] \in L/F, [x] \leq [y] \Leftrightarrow (y \to x) \wedge e \in F$.

Proof. Let $[x] \leq [y]$. This implies $[x] \wedge [y] = [x] \Rightarrow [x \wedge y] = [x] \Rightarrow (x \wedge y) * x \in F$. As $x \wedge y \leq x$, by Lemma 1.3.15 and Theorem 1.3.24 $x \to x = e \leq x \to x \wedge y$. Hence by Lemma 1.3.20 and Theorem 1.3.24 $(x \wedge y) * x = (x \to (x \wedge y)) \wedge ((x \wedge y) \to x) = (x \to (x \wedge y)) \wedge (x \to x) \wedge (y \to x) = (y \to x) \wedge e$. This implies that $(y \to x) \wedge e \in F$.

Conversely, let $(y \to x) \wedge e \in F$. Since $x \wedge y \leq x$,by Lemma 1.3.15 and Theorem 1.3.24 $x \to x = e \leq x \to x \wedge y$. Hence $x * (x \wedge y) = (y \to x) \wedge e \in F$. Thus it follows that $[x \wedge y] = [x]$ and hence $[x] \leq [y]$. □

Theorem 4.2.11. *If L is a BH-lattice and F is a filter of L, then the system* $(L/F, o, [e], \vee, \wedge, \to)$ *is a BH-lattice (called quotient BH-lattice corresponding to F), where the operations in the quotient are defined by* $[x]o[y] = [xoy], [x] \vee [y] = [x \vee y], [x] \wedge [y] = [x \wedge y], [x] \to [y] = [x \to y]$. *Further, if L is bounded above, so is* $(L/F, o, [e], \vee, \wedge, \to)$.

Proof. Obviously ([10], [8]) from the definition of congruence relation, all the operations in the quotient system are well defined. Also it is clear that $(L/F, o, [e])$ is a commutative monoid with identity element $[e]$ and $(L/F, \vee, \wedge)$ is a lattice.

Let $[x], [a], [b] \in L/F$ such that $[x]o[b] \leq [a] \Leftrightarrow [xob] \leq [a] \Leftrightarrow (a \to xob) \wedge e \in F \Leftrightarrow \{(a \to b) \to x\} \wedge e \in F \Leftrightarrow [x] \leq [a \to b] \Leftrightarrow [x] \leq [a] \to [b]$. Hence the system $(L/F, o, [e], \vee, \wedge, \to)$ is a BH-monoid. Furthermore, $[x]o\{[y] \wedge [z]\} = [xo(y \wedge z)] = [(xoy) \wedge (xoz)] = [xoy] \wedge [xoz] = \{[x]o[y]\} \wedge \{[x]o[z]\}$ and by a similar argument $(([x] \to [y]) \wedge [e])o[y] = [x] \wedge [y]$. Hence all the other axioms of BH-lattices hold. That is, the system $(L/F, o, [e], \vee, \wedge, \to)$ is a BH-lattice.

Finally if L is bounded above, then by Corollary 2.2.22 for $x \in L, x \leq e$

$\Rightarrow e \leq e \to x \leq e$ (by Lemma 1.3.15 and L is bounded)

$\Rightarrow e = (e \to x) \wedge e \in F$. Hence for any $[x] \in L/F, [x] \leq [e]$. Therefore it is bounded above. □

Chapter 5

Some Types of Filters in BH-Lattices

In the previous chapter we introduced the concept of filters in BH-lattices L and furnished examples. In this chapter we will classify filters and discus some properties regarding them.

Notation: Here after $\mathbb{F}$ stands for the set of all filters of a BH-lattice L.

Recall that, a non-empty subset F of L is called a filter of L iff

1. $x, y \in F \Rightarrow xoy \in F$
2. $x * e \leq y * e$ and $x \in F \Rightarrow y \in F$.

5.1 Deductive Filter

Definition 5.1.1. *A nonempty subset F of L is called a deductive filter if the following hold,* $\forall x, y \in L$.

1. $x \in F$ *and* $x * e \leq y * e \Rightarrow y \in F$
2. *If* $x \in F$ *and* $y \to x \in F$*, then* $y \in F$.

Example 5.1.2. *Consider the filter* $F(1) = \{1\}$ *in example 4.1.35. Let for any* $y \in$

$\mathbb{R}^+, y \to 1 \in F. \Rightarrow y \to 1 = 1 \Rightarrow yo1^{-1} = y = 1 \in F.$ *Hence* $F(1) = \{1\}$ *is a deductive filter.*

Theorem 5.1.3. *If the elements of a filter F of a BH-lattice L are invertible, then F is deductive filter.*

Proof. Suppose L is a BH-lattice and F is a filter of L. Let $x \in F, y \in L$ such that $y \to x \in F$. Hence there exists an element $x_0 \in F$ such that $y \to x = x_0 \in F$.
$\Rightarrow yo(e \to x) = x_0$ (by Theorem 1.3.32)
$\Rightarrow y = yo(e \to x)ox = x_0ox \in F$ (by Theorem 1.3.30). Hence F is a deductive filter. □

Corollary 5.1.4. *Every filter F of l-group G is a deductive filter.*

Proof. Since every element in l-group is invertible, every filter F of l-group G is a deductive filter. □

Theorem 5.1.5. *Deductive filter F of a BH-attice L is a filter of L. Moreover, if L is bounded above, then every filter F is a deductive filter of L.*

Proof. Assume a nonempty subset F of the BH-lattice L satisfies the conditions (1) and (2). Since F is non-empty, there exists an element, say $x \in F$. Then by Theorem 2.2.19 $x * e \leq e = e * e$. So $e \in F$. Let $x, y \in F$. By Lemma 1.3.10, $z \to (xoy) = (z \to x) \to y$. For $z = xoy$, by condition (2) in the hypothesis and Theorem 1.3.24, we have $e = (xoy) \to (xoy) = ((xoy) \to x) \to y \in F. \Rightarrow (xoy) \to x \in F \Rightarrow (xoy) \in F$. Hence $x, y \in F \Rightarrow xoy \in F$. Thus F is a filter of L.
Now assume that L is bounded above and F is a filter of L. Clearly, using Corollary 2.2.22 and Theorem 4.1.2, e is the upper bound of L and $e \in F$. Let $x, y \to x \in F \Rightarrow$
$x \wedge y = [(y \to x) \wedge e]ox = xo(y \to x) \in F$
$[xo(y \to x)] * e = \{xo(y \to x)\} \wedge \{e \to [xo(y \to x)]\}$
$= xo(y \to x) \leq y$ (by Lemma 1.3.9, Lemma 1.3.16 and boundedness of L)
$= y * e$ (by Theorem 2.2.19)
$\Rightarrow y \in F$. Hence F is a deductive filter. □

Corollary 5.1.6. *Every filter F of a Heyting algebra is a deductive filter.*

Proof. Since a Heyting algebra is bounded, by Theorem 5.1.5, every filter F of a Heyting algebra is a deductive filter. □

Open Problem 5.1.7. *Is every filter F of a BH-lattice L a deductive filter?*

Theorem 5.1.8. *If a BH-lattice L is bounded above, then a non empty subset F of L is a deductive filter iff*

1. $e \in F$
2. *If $x \in F$ and $y \to x \in F$, then $y \in F$:*

Proof. The forward is trivial by Theorem 5.1.5. Let F be a non empty subset of L and the condition in (1) and (2) are satisfied. Let $x \in F, y \in L$ and $x = x * e \leq y * e = y$. Then by Corollary 2.2.22, Lemma 1.3.15 and Lemma 1.3.16 $e = y * e \to x * e \in F$. Hence by (2) this implies $y = y * e \in F$. Therefore F is a deductive filter. □

Theorem 5.1.9. *Let F be a deductive filter of a BH-lattice L. If $x, x * y \in F$, then $y \in F$, $\forall x, y \in L$.*

Proof. Let F is a deductive filter and $x, x * y \in F$. Using Theorem 2.2.19, $[x * y] * e = x * y \leq (y \to x) * e$. So, by the definition of filter $y \to x \in F$. Hence, by the definition of deductive filter, $y \in F$. □

Definition 5.1.10. *For filter F of L and $a \in L$, define*
$L_a = \{x \in L : x \to a \in F\}$.

Remark: For any $a \in L, a \to a = e \in F, a \in L_a$ and hence $L_a \neq \emptyset$.

Theorem 5.1.11. *If F is a filter of BH-lattice L and $a, b \in L$, then*
$L_a \cap L_b \subseteq L_{a \vee b}$. *Further, if L is bounded above, then $L_a \cap L_b = L_{a \vee b}$.*

Proof. Let F be a filter of L and $a, b \in L$.

$x \in L_a \cap L_b \Rightarrow x \in L_a$ and $x \in L_b \Rightarrow x \to a \in F$ and $x \to b \in F \Rightarrow (x \to a) \wedge (x \to b) \in F$ (using Theorem 4.1.3) $\Rightarrow x \to (a \vee b) = (x \to a) \wedge (x \to b) \in F$ (by Theorem 1.3.19) $\Rightarrow x \in L_{a\vee b}$. Hence $L_a \cap L_b \subseteq L_{a\vee b}$. Furthermore, if L is bounded above, $x \in L_{a\vee b} \Rightarrow x \to (a \vee b) = (x \to a) \wedge (x \to b) \in F$.
Since $(x \to a) \wedge (x \to b) \leq (x \to a), (x \to b)$ and F is a filter, $(x \to a) \in F$ and $(x \to b) \in F$. Hence $x \in L_a \cap L_b$. □

Theorem 5.1.12. *If $F \in \mathbb{F}$ is a deductive filter, then $L_a = F, \forall a \in F$.*

Proof. Suppose a is any element in the deductive filter F. Let $x \in L_a$. Then $x \to a \in F$. Hence by condition 2 in the definition of deductive filter $x \in F$. Hence $L_a \subseteq F$. Suppose $x \in F$. Since, $a \in F$ by Theorem 4.1.3 $x \to a \in F$. Hence $x \in L_a$. Thus $F \subseteq L_a$ and consequently, $L_a = F$. □

5.2 Prime and Maximal Filters

Definition 5.2.1. *A filter M of a BH-lattice L is called maximal (ultrafilter) if M is a proper filter of L and is not a proper subset of any proper filter of L.*

Example 5.2.2. *The filters $F_1 = \{x, z, 1\}$ and $F_2 = \{y, z, 1\}$ of the Heyting algebra given in figure 5.1 are maximal filters.*

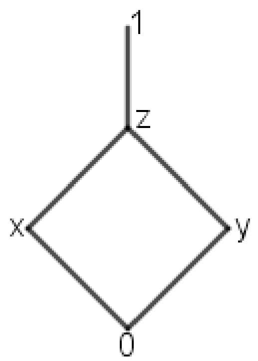


Figure 5.1: Example of Heyting algebra

Example 5.2.3. *The filters $F_3 = \{2, 6, 10, 30\}$ and $F_4 = \{3, 6, 15, 30\}$ of the Boolean algebra $B = \{1, 2, 3, 5, 6, 10, 15, 30\}$ with divisibility are maximal filters of B.*

Remark 5.2.4. *From the examples* 5.2.2 *and* 5.2.3*, we conclude that Maximal filter of L is not necessarily unique.*

Theorem 5.2.5. *A proper filter M of a BH-lattice L is maximal if and only if $F(M\bigcup\{x\}) = L$ for any $x \in L - M$.*

Proof. Let M be a maximal filter of a BH-lattice L and $x \in L - M$. Suppose that $F(M \cup \{x\}) \neq L$. Hence $M \subseteq F(M \cup \{x\}) \subset L$. Since M is maximal we have $M = F(M \cup \{x\})$ and hence $x \in M$, which is a contradiction.
Conversely, assume that $F(M\bigcup\{x\}) = L$ for any $x \in L - M$. Suppose there exists a filter F of L such that $M \subseteq F \subseteq L$ and $M \neq F$. Then $M \subset F$ and there exists an element $x \in F$ and $x \notin M$. Hence by the assumption condition, $F(M\bigcup\{x\}) = L$. Hence $F(M\bigcup\{x\}) \subseteq F(F) \subseteq L$ and $F(M\bigcup\{x\}) = L$. Therefore, $F(F) = F = L$. Hence M is maximal. □

Definition 5.2.6. *A filter P of L is called a prime filter if $(x * e) \vee (y * e) \in P$ implies $x \in P$ or $y \in P, \forall x, y \in L$.*

Example 5.2.7. *The filters F_1, F_2, F_3 and F_4 in example* 5.2.2 *and example* 5.2.3 *are prime.*

Theorem 5.2.8. *For $F \in \mathbb{F}$ and $\forall x, y \in L$, $(x * e) \vee (y * e) \in F \Rightarrow (x \in F$ or $y \in F)$ if and only if $(x \vee y \in F$ and $x \vee y \leq e) \Rightarrow (x \in F$ or $y \in F)$.*

Proof. $(\Rightarrow)$ Suppose $(x * e) \vee (y * e) \in F \Rightarrow (x \in F$ or $y \in F)$. Then by the definition F is a prime filter.
Let $(x \vee y \in F$ and $x \vee y \leq e)$. Then $x, y \leq x \vee y \leq e$ and hence by Theorem 2.2.19, $x = x * e, y = y * e$ and $x \vee y = (x * e) \vee (y * e) \in F$. Hence $x = x * e \in F$ or $y = y * e \in F$.
Conversely, suppose $(x \vee y \in F$ and $x \vee y \leq e) \Rightarrow (x \in F$ or $y \in F)$. Let $(\forall x, y \in L), (x*e)\vee(y*e) \in F$. Since by Theorem 2.2.19, $x*e, y*e \leq e$ and thus $(x*e)\vee(y*e) \leq e$. Hence $x * e \in F$ or $y * e \in F$. Hence by Theorem 2.2.19, $x \in F$ or $y \in F$. □

Theorem 5.2.9. *A prime filter is finitely meet irreducible element in the filter lattice*

$\mathbb{F}$. *That is, if* $P \in \mathbb{F}$ *is prime filter and* $S, T \in \mathbb{F}$ *such that* $S \wedge T = P$ *, then* $S = P$ *or* $T = P$.

Proof. Suppose $P \in \mathbb{F}$ is prime filter and $S, T \in \mathbb{F}$ such that $S \wedge T = P$. Suppose that $S \nsubseteq P$ and $T \nsubseteq P$. Then there exists elements $x \in S - P, y \in T - P$. As $x * e, y * e \leq (x * e) \vee (y * e) = [(x * e) \vee (y * e)] * e$, $(x * e) \vee (y * e) \in S \wedge T = P$. Therefore $x * e \in P$ or $y * e \in P$ and hence by Theorem 4.1.17, $x \in P$ or $y \in P$ which is a contradiction. Hence $S \subseteq P$ or $T \subseteq P$. Thus $S = P$ or $T = P$. □

Definition 5.2.10. *A prime filter* F *is minimal if there exists no prime filter* T *which is properly contained in* F.

Theorem 5.2.11. *Let* $\{P_i\}_{i \in I}$ *be a chain of prime filters of* L. *Then* $P = \bigcap_{i \in I} P_i$ *is a prime filter of* L. *Consequently, every prime filter contains a minimal prime filter.*

Proof. Let for $x, y \in L, (x * e) \vee (y * e) \in P$. Suppose that $x \notin P$. Then there exists some $k \in I$ such that $x \notin P_k$. Hence $x \notin P_j$ for all $j \in I$ such that $P_j \subseteq P_k$. Thus $y \in P_j$ for all $j \in I$ such that $P_j \subseteq P_k$. So $x \in P_k$. Since for any arbitrary P_i in the family either $P_i \subseteq P_k$ or $P_k \subseteq P_i$. Thus $y \in P_i, \forall i \in I$ and hence $y \in P$. Therefore P is prime. Hence every prime filter contains the intersection of chain of prime filters $\{P_i\}_{i \in I}$ such that $P_i \subseteq P$. Therefore every prime filter contains a minimal prime filter. □

Theorem 5.2.12. *If in* L, $xoy = x \wedge y, \forall x, y \in L$, *then Maximal filter is a prime filter.*

Proof. Let F be a maximal filter in L and suppose $(a*e) \vee (b*e) \in F, a, b \in L$. Let $a \notin F$. Since F is maximal filter and $F \subseteq F(F \cup \{a\})$, we have $F(F \cup \{a\}) = L$. Hence by Theorem 4.1.44 there exists an element $x \in F, x \leq e$ such that $xo(a*e)^n \leq b*e, n \in \mathbb{N}$. As $x * e \leq (x * e) \vee (b * e) = [(x * e) \vee (b * e)] * e$, it follows that $(x * e) \vee (b * e) \in F$. By the hypothesis $o = \wedge$, so $(a * e)^n = a * e$. Hence by the hypothesis and distributive property $b * e = (b * e) \vee ((a * e)^n o(x * e)) = (b * e) \vee ((a * e)^n \wedge (x * e)) = \{(a * e)^n \vee (b *$

$e)\} \wedge \{(x*e) \vee (b*e)\} = \{(a*e) \vee (b*e)\} \wedge \{(x*e) \vee (b*e)\} \in F$. Hence by Theorem 4.1.17 $b \in F$ and consequently F is prime filter. □

Corollary 5.2.13. *In a Heyting algebra, every maximal filter is a prime filter.*

Proof. By Theorem 2.2.40, $xoy = x \wedge y$. So by Theorem 5.2.12, every maximal filter in Heyting algebra is prime filter. □

Theorem 5.2.14. *Prime filter of relatively complemented BH-lattice L, is a maximal filter of L.*

Proof. Suppose L is relatively complemented and P is a prime filter of L. Let $F \in \mathbb{F}$ and $P \subset F$. Let $x \in F-P, y \in P$ and $z \in L$. Then by Theorem 4.1.17 $x*e \in F-P, y*e \in P$ and $z \in L$. As $(x*e)\wedge(z*e) \leq x*e \leq (x*e)\vee(y*e)$ and L is relatively complemented, $x*e$ has a complement x' in $[(x*e) \wedge (z*e), (x*e) \vee (y*e)]$. Hence $(x*e) \vee x' = (x*e) \vee (y*e) \in P$ and by Theorem 2.2.19 $x' \leq e$. Since $x*e \notin P, x' \in P \subset F$. Therefore $(x*e) \wedge x' = (x*e) \wedge (z*e) \in F$. AS $(x*e) \wedge (z*e) \leq z*e, z*e \in F$. Hence by Theorem 4.1.17 $z \in F$. Thus $F = L$. So P is maximal filter. □

Theorem 5.2.15. *If L is totally ordered, then $(\mathbb{F}, \subseteq)$ is totally ordered and every filter $F \in \mathbb{F}$ is prime.*

Proof. Let L is totally ordered and suppose that $(\mathbb{F}, \subseteq)$ is not totally ordered. This means, there exists filters $F, T \in \mathbb{F}$ such that $F \nsubseteq T$ and $T \nsubseteq F$. Then there exists elements $x \in F-T$ and $y \in T-F$. Hence by Theorem 4.1.17, $x*e \in F-T$ and $y*e \in T-F$. Further, $x*e \leq y*e$ or $y*e \leq x*e$, without loss of generality let $x*e \leq y*e$. Therefore, $x*e \leq (x*e) \vee (y*e) = y*e \in F$. Hence $y \in F$, which is a contradiction. Hence $\mathbb{F}$ is totally ordered. Moreover, let $(x*e) \vee (y*e) \in F$. Since $x*e \leq y*e$ or $y*e \leq x*e$, it follows that $(x*e) \vee (y*e) = x*e \in F$ or $(x*e) \vee (y*e) = y*e \in F$. Thus F is prime. □

5.3 Implicative Filter

Definition 5.3.1. *A nonempty subset F of L is called an implicative filter if* $\forall x, y, z \in L$

1. $(z \to y) \to x \in F$ *and* $y \to x \in F \Rightarrow z \to x \in F$.

2. $x \in F$ *and* $x * e \leq y * e \Rightarrow y \in F$.

Theorem 5.3.2. *Implicative filter of L is a deductive filter (filter) of L.*

Proof. Let F be an implicative filter of L. Let $x, y \in L$ such that $x, y \to x \in F$. Then $(y \to x) \to e \in F$ and $x = x \to e \in F$. Hence, as F is implicative filter, $y \to e = y \in F$. Hence F is a deductive filter and hence by Theorem 5.1.5 F is a filter. □

The converse of Theorem 5.3.2 is not true. For this consider the following example.

Example 5.3.3. *For* $L = \{0, x, y, 1\}$*, define the binary operations* o *and* $\to$ *by the tables 5.1 and table 5.2 respectively.*

o	0	x	y	1
0	0	0	0	0
x	0	0	x	x
y	0	x	y	y
1	0	x	y	1

Table 5.1: definition of o

$\to$	0	x	y	1
0	1	x	0	0
x	1	1	x	x
y	1	1	1	y
1	1	1	1	1

Table 5.2: definition of $\to$

Then the system $(L, \vee, \wedge, o, 1, \to)$ *is BH-lattice and* $F = \{y, 1\}$ *is filter of L. Since* $(0 \to x) \to x = 1 \in F$ *and* $x \to x = 1 \in F$ *but* $0 \to x = x \notin F$*, F is not an implicative filter of L.*

Theorem 5.3.4. *Filter F of L is an implicative filter if* L_a *is a deductive filter of L,* $\forall a \in L$*. Further, if L is bounded above and F is implicative filter, then* L_a *is a deductive filter,* $\forall a \in L$*.*

Proof. Suppose that L is a BH-lattice, F is a filter and L_a is a deductive filter of L,

$\forall a \in L$. Let $(z \to y) \to x \in F$ and $y \to x \in F$. Then $z \to y \in L_x$ and $y \in L_x$. Since by hypothesis L_x is a deductive filter of L, $z \in L_x$. Hence $z \to x \in F$ and consequently F is implicative filter.

Conversely, let L be bounded above, F be an implicative filter of L and $a \in L$. Clearly $a \in L_a$ and hence $L_a \neq \emptyset$. Suppose $x, y \to x \in L_a$. Hence $x \to a, (y \to x) \to a \in F$. Since, F is implicative filter $y \to a \in F$ and hence $y \in L_a$. Now, let $x \in L_a$ and $x = x * e \leq y * e = y$. Then $x \to a \in F$ and by Theorem 1.3.14, $x \to a \leq y \to a$. Since, F is a filter $y \to a \in F$. Hence $y \in L_a$. Therefore L_a is a deductive filter. □

Theorem 5.3.5. *If F is an implicative filter of L, then* $(x \to y) \to y \in F \Rightarrow x \to y \in F$.

Proof. Suppose F is an implicative filter of L and for $x, y \in L, (x \to y) \to y \in F$. Then by Theorem 5.3.2 F is a filter and by Theorem 4.1.2 $e \in F$. Since by Theorem 1.3.24 and Theorem 4.1.2 $e = y \to y \in F$ and F is implicative filter, we obtain that $x \to y \in F$.

□

Theorem 5.3.6. *If F is an implicative filter of L, then* $((x \to y) \to y) \to z \in F$ *and* $z \in F \Rightarrow x \to y \in F$.

Proof. Suppose F is an implicative filter of L and for $x, y, z \in L, ((x \to y) \to y) \to z \in F, z \in F$. Then by Theorem 4.1.3 and Theorem 5.3.2 F is a filter and hence $\{((x \to y) \to y) \to z\} \to z \in F$. Thus by Theorem 5.3.5, $(x \to y) \to y \in F$ and hence $x \to y \in F$. □

Theorem 5.3.7. *If L is a BH-lattice which is bounded above and F is a filter of L, then the following conditions are equivalent:*

1. *F is an implicative filter.*
2. $(x \to y) \to y \in F \Rightarrow x \to y \in F$
3. $(x \to y) \to z \in F \Rightarrow (x \to z) \to (y \to z) \in F$

4. *$((x \to y) \to y) \to z \in F$ and $z \in F \Rightarrow x \to y \in F$.*

Proof. ($1 \Rightarrow 2$) Presume that F is an implicative filter of L and for $x, y \in L, (x \to y) \to y \in F$. Then by Theorem 5.3.2 F is a filter and by Theorem 4.1.2 $e \in F$. Since by Theorem 1.3.24 $e = y \to y \in F$ and F is implicative filter, we obtain that $x \to y \in F$.
($2 \Rightarrow 3$) Let F be a filter satisfying the condition given in (2) and for $x, y, z \in L, (x \to y) \to z \in F$. Then
$[(x \to (y \to z)) \to z] \to z = [(x \to z) \to (y \to z)] \to z$ (by Theorem 1.3.10)
$\geq (x \to y) \to z \in F$ (by Lemma 1.3.14, Theorem 2.2.4 and L is bounded above)
$\Rightarrow [(x \to (y \to z)) \to z] \to z \in F$ (by definition of a filter.)
$\Rightarrow (x \to (y \to z)) \to z \in F$ (by the hypothesis in (2))
$\Rightarrow (x \to (y \to z)) \to z = (x \to z) \to (y \to z) \in F$ (by Theorem 1.3.10)
($3 \Rightarrow 4$) Suppose that filter F satisfies the condition give in (3). Let $[(x \to y) \to y] \to z \in F$ and $z \in F$. Then using Theorem 1.3.10 and Theorem 5.3.2 $(x \to y) \to y \in F$. Hence by the hypothesis given in (3) it follows that $(x \to y) \to (y \to y) \in F$. This implies that $(x \to y) \to (y \to y) = (x \to y) \to e = x \to y \in F$(by Theorem 1.3.24).
($4 \Rightarrow 1$) Let filter F in L satisfies condition(4). Let $(z \to y) \to x \in F$ and $y \to x \in F$. Then
$(z \to y) \to x = (z \to x) \to y$ (by Theorem 1.3.10)
$\leq [(z \to x) \to x] \to (y \to x)$ (using Theorem 2.2.4)
Hence by the definition of a filter $[(z \to x) \to x] \to (y \to x) \in F$ and consequently using the given hypothesis $z \to x \in F$. So F is an implicative filter. □

Theorem 5.3.8. *If a filter F of a BH-lattice L is an implicative filter, then $x^2 \to x \in F$, for any $x \in L$. Further, if L is bounded above, then $x^2 \to x \in F$ iff F is an implicative filter, $\forall x \in L$.*

Proof. Let F be an implicative filter of BH-lattice L. Then by Theorem 1.3.10 and Theorem 1.3.24 $(x^2 \to x) \to x = x^2 \to x^2 = e \in F$ and $x \to x = e \in F$. So, as F is an implicative filter, $x^2 \to x \in F$
Further, Presume L is bounded above and $x^2 \to x \in F, \forall x \in L$. Let $(z \to y) \to x \in F$ and $y \to x \in F$. Then

$[(z \to y) \to x]o[y \to x]ox^2 = \{[(z \to y) \to x]ox\}o\{[y \to x]ox\}$ (by commutativity of o) $\leq (z \to y)oy \leq z$ (Lemma 1.3.7 and Lemma 1.3.9).

$\Rightarrow [(z \to y) \to x]o[y \to x] \leq z \to x^2$ (by defining condition of $\to$).

$\Rightarrow z \to x^2 \in F$ (by definition of a filter, Theorem 4.1.3 and L is bounded above).

$\Rightarrow (z \to x^2)o(x^2 \to x) \in F$ (by assumption and Theorem 1.3.7) and $(z \to x^2)o(x^2 \to x) \leq z \to x$ (by Lemma 1.3.17).

$\Rightarrow z \to x \in F$. Hence F is an implicative filter of BH-lattice L. □

Corollary 5.3.9. *A filter F of an l-group G is implicative if and only if $G = F$.*

Proof. Let $x \in G$ and F is an implicative filter. Then $x^2 \to x = x^2ox^{-1} = x \in F$. Hence $G \subseteq F$. Hence $F = G$. The converse is trivial. □

Corollary 5.3.10. *Every filter F in a Heyting algebra H is implicative.*

Proof. Let F is a filter and $x \in H$. Then $x^2 \to x = x \wedge x \to x = e \in F$. Hence F is implicative. □

Corollary 5.3.11. *Let L be bounded above and F_1, F_2 filters of L such that $F_1 \subseteq F_2$. If F_1 is implicative filter, then so is F_2.*

Proof. Directly follows from the theorem. □

Theorem 5.3.12. *In any BH-lattice L, the following conditions are equivalent.*

1. *L is idempotent*

2. *Every filter F of L is implicative filter and L is bounded above.*

3. *$\{e\}$ is an implicative filter.*

Proof. $(1 \Rightarrow 2)$ Let L is idempotent and suppose that F be any filter of L. Since $x^2 = x$ by Lemma 1.3.8, we obtain that $x \leq xox \to x = x \to x = e$. Hence L is bounded above by e. Let $(x \to y) \to y \in F$. So, by Lemma 1.3.10 $(x \to y) \to y = x \to y^2 = x \to y \in F$. Hence by Theorem 5.3.7, F is an implicative filter and L is bounded above.

($2 \Rightarrow 3$) Obvious

($3 \Rightarrow 1$) Let $\{e\}$ be an implicative filter of L. By Theorem 5.3.8, $\forall x \in L, x^2 \to x = e$. Hence by Lemma 1.3.10 $e = x^2 \to x^2 = (x^2 \to x) \to x = e \to x$ and hence we attain that $x \leq e$. Hence L is bounded above by e. Thus by Lemma 1.3.7 $x^2 \leq x$. Further, since $x^2 \to x = e, x \leq x^2$. So that $x^2 = x$. Therefore L is idempotent. □

5.4 Positive implicative filter

Definition 5.4.1. *A nonempty subset F of a BH-lattices L is called a positive implicative filter if:*

1. $x \in F$ *and* $x * e \leq y * e \Rightarrow y \in F$ *and*

2. $(y \to (z \to y)) \to x \in F$ *and* $x \in F \Rightarrow y \in F, \forall x, y, z \in L$.

Theorem 5.4.2. *Positive implicative filter of a BH-lattice is a deductive filter (filter).*

Proof. Let $x \in F$ and $y \to x \in F$. Then $y \to x = (y \to e) \to x = (y \to (y \to y)) \to x \in F$. Hence $y \in F$ and consequently using Theorem 5.1.5 F is a filter. □

Corollary 5.4.3. *If F is a positive implicative filter and* $y \to (z \to y) \in F$, *then* $y \in F$.

Proof. Since F is positive implicative filter, by Theorem 5.4.2, F is a filter. Thus $e \in F$ and hence $\{y \to (z \to y)\} \to e \in F$. Therefore $y \in F$. □

Corollary 5.4.4. *Suppose that F is a positive implicative filter of a BH-lattice L which is bounded above. Then* $y \to (z \to y) \in F \Leftrightarrow y \in F$.

Proof. The forward follows from Corollary 5.4.3. Now let $y \in F$ and $z \in L$. Since L is bounded above, by Corollary 2.2.22 $z \leq e$

$\Rightarrow z \to y \leq e \to y$ (by Lemma 1.3.14)

$\Rightarrow y = y \to (e \to y) \leq y \to (z \to y)$ (by Lemma 1.3.15). Hence $y \to (z \to y) \in F$. □

Theorem 5.4.5. *If F is a positive implicative filter of a BH-lattice L which is bounded* *above, then $x \to (x \to x^-) \in F, \forall x \in L$.*

Proof. Suppose L is bounded above and F is a positive implicative filter of L. Then clearly $e \in F$ and $e \to x = e, \forall x \in L$.
$e = e \to (x \to (e \to x)) = e \to (x \to x^-)$ (by Lemma 1.3.12 and $e \to x = e$)
$= (x \to x) \to (x \to x^-) = (x \to (x \to x^-)) \to x$ (by Lemma 1.3.10 and Theorem 1.3.24)
$\leq (e \to x) \to (e \to (x \to (x \to x^-)))$ (by 2 of Theorem 2.2.4)
$= x^- \to (e \to (x \to (x \to x^-)))$
$\leq (x \to (e \to (x \to (x \to x^-)))) \to (x \to x^-)$ (by 2 of Theorem 2.2.4)
$= (x \to (x \to x^-)) \to (e \to (x \to (x \to x^-))) \in F$ (by Lemma 1.3.10, Theorem 2.2.19 and definition of a filter). Hence by Corollary 5.4.3, $x \to (x \to x^-) \in F$. □

Theorem 5.4.6. *A deductive filter F is a positive implicative filter if and only if $x \to (y \to x) \in F \Rightarrow x \in F, \forall x, y \in L$.*

Proof. Let F be a deductive filter of L and satisfies the condition $x \to (y \to x) \in F \Rightarrow x \in F, \forall x, y \in L$. Let $(y \to (z \to y)) \to x \in F$ and $x \in F$. Hence by the definition of a deductive filter $y \to (z \to y) \in F$ and consequently $y \in F$. Thus F is a positive implicative filter.
Conversely, let a deductive filter F is a positive implicative filter and $x \to (y \to x) \in F$. Hence by Theorem 4.1.3 and Theorem 5.1.5 $\{x \to (y \to x)\} \to e, e \in F$. Hence as F is a positive implicative filter $x \in F$. □

Theorem 5.4.7. *Positive implicative filter of a BH-lattice which is bounded above is an implicative filter. But the converse is not true.*

Proof. Suppose L is bounded above and F is a positive implicative filter L. Let for $x, y, z \in L, (z \to y) \to x \in F$ and $y \to x \in F$.
$(z \to y) \to x = [(z \to x) \to y]] * e$ (by Lemma 1.3.10 and Theorem 2.2.19)
$= (z \to x) \to y \leq [(z \to x) \to x] \to (y \to x)$ (by Theorem 2.2.4)
$= [[(z \to x) \to x] \to (y \to x)] * e$ (Theorem 2.2.19).
Hence $[(z \to x) \to x] \to (y \to x) \in F$ and

$[(z \to x) \to x] \to (y \to x) = \{[(z \to x) \to x] \to \{[(z \to x) \to x] \to [(z \to x) \to x]\}\} \to (y \to x)$ (by Lemma 1.3.12 and Theorem 1.3.24).
So by the definition of positive implicative filter $(z \to x) \to x \in F$.
Further, by 2 of Theorem 2.2.4, $(z \to x) \to x \leq (z \to x) \to (z \to (z \to x))$. So, as L is bounded above using Theorem 2.2.19, $(z \to x) \to (z \to (z \to x)) \in F$ and consequently by Theorem 4.1.2 and Theorem 4.1.3 $[(z \to x) \to (z \to (z \to x))] \to e \in F$ and $e \in F$. Hence by the positive implicativness of F, we obtain that $z \to x \in F$. Therefore F is an implicative filter.
For the converse consider the following example. □

Example 5.4.8. *Consider the Heyting algebra* $H = [0, 1]$ *with the usual ordering (which is a BH-lattice by Example* 1.2.18*). Clearly* $F = [\frac{1}{2}, 1]$ *is a filter of H and hence by Corollary* 5.3.10*, it is an implicative filter. Since* $(\frac{1}{4} \to (\frac{1}{5} \to \frac{1}{4})) \to \frac{3}{4} = 1, \frac{3}{4} \in F$ *and* $\frac{1}{4} \notin F$*, F is not a positive implicative filter.*

Remark 5.4.9. *From example 5.4.8 and Theorem 5.3.2, the converse of Theorem 5.4.2 is not true.*

Open Problem 5.4.10. *If a BH-lattice L is not bounded above, under what condition a Positive implicative filter of L is an implicative filter?.*

Theorem 5.4.11. *In a BH-lattice L which is bounded above, implicative filter F is a positive implicative filter iff* $y \to (y \to x) \in F \Rightarrow x \to (x \to y) \in F, \forall x, y \in L$.

Proof. ($\Rightarrow$) Presume that L is a BH-lattice which is bounded above and F is a positive implictive filter of L. Suppose that $y \to (y \to x) \in F$. Since L is bounded above, by Corollary 2.2.22 $x \to y \leq e \Rightarrow x \to e = x \leq x \to (x \to y)$ (by Lemma 1.3.12 and Lemma 1.3.15)

$\Rightarrow y \to [x \to (x \to y)] \leq y \to x$ (by Lemma 1.3.15). $\oplus$

Now $y \to (y \to x) \leq (x \to (y \to x)) \to (x \to y)$ (by 2 of Theorem 2.2.4)

$= (x \to (x \to y)) \to (y \to x)$ (using Lemma 1.3.10)

$\leq (x \to (x \to y)) \to [y \to (x \to (x \to y))]$ (by Theorem 1.3.15 and $\oplus$).

$\Rightarrow (x \to (x \to y)) \to [y \to (x \to (x \to y))] \in F$

$\Rightarrow [(x \to (x \to y)) \to [y \to (x \to (x \to y))]] \to e \in F$ (by Theorem 4.1.2 and 4.1.3).
Hence as F is positive implicative filter and $e \in F$, $x \to (x \to y) \in F$
Conversely, assume that an implicative filter F satisfies the condition $y \to (y \to x) \in F \Rightarrow x \to (x \to y) \in F, \forall x, y \in L$. Let $x \in F$ and $(y \to (z \to y)) \to x \in F$
$\Rightarrow y \to (z \to y) \in F$ (by Theorem 5.3.2).
Moreover $y \to (z \to y) \leq (z \to (z \to y)) \to (z \to y)$(by 2 of Theorem 2.2.4)
$\Rightarrow (z \to (z \to y)) \to (z \to y) \in F$.
$\Rightarrow z \to (z \to y) \in F$ (by Theorem 5.3.5).
$\Rightarrow y \to (y \to z) \in F$ (by hypothesis).
As L is bounded above, $y \leq e$ and using Lemma 1.3.12 and Lemma 1.3.15 $z = z \to e \leq z \to y$. Hence by Theorem 1.3.15, $y \to (z \to y) \leq y \to z \Rightarrow y \to z \in F$ (as $y \to (z \to y) \in F$)$\Rightarrow y \in F$ (as $y \to (y \to z) \in F$ and using Theorem 5.3.2). Hence F is a positive implicative filter. □

Open Problem 5.4.12. *If L is not bounded above, under what condition does an implicative filter be a positive implicative filter?*

Theorem 5.4.13. *In a bounded above BH-lattice L, every $F \in \mathbb{F}$ containing a positive implicative filter is a positive implicative filter.*

Proof. Let T be a positive implicative filter of L contained in a filter F. Then by Theorem 5.4.7, T is an implicative filter. Hence by Corollary 5.3.11, F is an implicative filter. Let $u = x \to (x \to y) \in F$.
$u \to u = \{x \to (x \to y)\} \to u = e \in T$ (by Lemma 1.3.12, Theorem 4.1.2 and Theorem 5.4.2).
$\Rightarrow (x \to u) \to \{(x \to y) \to u\} \in T$ (by Theorem 5.3.7).
$\Rightarrow (x \to u) \to \{(x \to y) \to u\} = (x \to u) \to \{(x \to u) \to y\} \in T$ (by Theorem 1.3.10).
$\Rightarrow y \to \{y \to (x \to u)\} \in T$ (by Theorem 5.4.11)
$\Rightarrow y \to \{y \to (x \to u)\} \in F$ (T is contained in F).
Furthermore, by Theorem 2.2.2 $y \leq x \to (x \to y)$.
$\Rightarrow x \to (x \to y) \leq x \to \{x \to (x \to (x \to y))\}$ (by Lemma 1.3.15 using twice)
$= x \to (x \to u) \leq (y \to (x \to u)) \to (y \to x)$(by Theorem 2.2.4)

$\leq (y \to (y \to x)) \to \{y \to (y \to (x \to u))\}$ (by Theorem 2.2.4).

Hence by the definition of a filter and Theorem 2.2.19 $(y \to (y \to x)) \to \{y \to (y \to (x \to u))\} \in F$. Furthermore, by Theorem 5.3.2, F is a deductive filter and $y \to \{y \to (x \to u)\} \in F$. Consequently $y \to (y \to x) \in F$. Hence by Theorem 5.4.11, F is a positive implicative filter. □

Theorem 5.4.14. *In a bounded above BH-lattice L, the following conditions are equivalent.*

1. *$\{e\}$ is a positive implicative filter of L.*
2. *Every filter of L is a positive implicative filter.*
3. *For any $a \in L$, $L(a) = \{x \in L : a \leq x\}$ is a positive implicative filter of L.*
4. *$x \to (y \to x) = x, \forall x, y \in L$.*

Proof. By Corollary 2.2.22, e is the upper bound of L.
$(1 \Rightarrow 2)$ Follows from Theorem 4.1.7 and Theorem 5.4.13.
$(2 \Rightarrow 3)$ Since, by hypothesis every filter of L is a positive implicative filter and $\{e\}$ is a filter, by Theorem 5.4.7, $\{e\}$ is an implicative filter. Clearly $e \in L(a)$. Let $x, y \to x \in L(a)$. Then $a \leq x, a \leq (y \to x)$. Using Lemma 1.3.14 and Theorem 1.3.24, this implies that $x \to a = e, (y \to x) \to a = e \in \{e\}$. Since, $\{e\}$ is an implicative filter, we have $y \to a = e$ and so $a \leq y$. Thus, $y \in L(a)$ and consequently $L(a)$ is a deductive filter (filter). Therefore by the assumption, $L(a)$ is a positive implicative filter.
$(3 \Rightarrow 4)$ Clearly $a \in L(a)$. So $x \to (y \to x) \in L(x \to (y \to x))$ and by the assumption $L(x \to (y \to x))$ is a positive implicative filter. Thus by Theorem 5.4.2 and Theorem 5.4.6, $x \in L(x \to (y \to x))$. Hence $x \to (y \to x) \leq x$. Since, L is bounded above $y \to x \leq e$. By Lemma 1.3.15 and Lemma 1.3.12, we obtain that $x = x \to e \leq x \to (y \to x)$. Therefore, $x = x \to (y \to x)$.
$(4 \Rightarrow 1)$ Since L is bounded above, clearly $\{e\}$ is a filter (deductive filter). Hence directly follows from Theorem 5.4.6. □

Theorem 5.4.15. *Suppose that L is a BH-lattice bounded above and F is a deductive* *filter of L. Then the following conditions are equivalent:*

1. *F is maximal and positive implicative filter.*

2. *F is maximal and implicative filter.*

3. *F is a proper filter and $x, y \notin F \Rightarrow x \to y, y \to x \in F, \forall x, y \in L$.*

Proof. $(1 \Rightarrow 2)$ Follows from Theorem 5.4.7.
$(2 \Rightarrow 3)$ Suppose that $x, y \notin F$. By Theorem 5.3.4 and Theorem 5.1.5, L_y is a filter. Since L is bounded above $y \leq e$ and hence by Lemma 1.3.15, $a \leq a \to y$ for any $a \in L$. So, if $a \in F$, then $a \in L_y$. Hence, we have $F \subseteq L_y \subseteq L$. Since by the hypothesis $F$is a maximal filter and $x, y \notin F$, we get $L_y = L$. Thus $x \in L_y$ and so that $x \to y \in F$. By a similar argument we can prove that $y \to x \in F$.
$(3 \Rightarrow 1)$ Suppose that (3) hold and F is not a positive implicative filter of L. Hence by Theorem 5.4.6, there exists elements $x, y \in L$ such that $x \to (y \to x) \in F$ but $x \notin F$. If $y \in F$, then clearly $y \leq y \to x$ and so $y \to x \in F$. Hence, as F is deductive filter and $x \to (y \to x) \in F$, we obtain that $x \in F$. If $y \notin F$, then by the hypothesis in (3), $y \to x \in F$ and hence as F is deductive filter and $x \to (y \to x) \in F$. Consequently $x \in F$. Hence, in both the above cases we have a contradiction. Therefore the deductive filter F is a positive implicative filter.
Now we can show that F is maximal filter. Suppose that $a \notin F$. As $a \to a = e \in F, a \in L_a$. By the same argument as in $(2 \Rightarrow 3), F \cup \{a\} \subseteq L_a$. Let T be any filter of L such that $F \cup \{a\} \subseteq T$ and $x \in L_a$. Then $x \to a \in F$. Since $F \subseteq T$, it follows that $x \to a \in T$. As L is bounded above and T is a filter, by Theorem 5.1.5, $x \in T$. Therefore L_a is the smallest filter containing $F \cup \{a\}$.
Let $t \in L$. If $t \in F$, then $t \in L_a$. If $t \notin F$, as $a \notin F$ by the hypothesis we obtain that $t \to a \in F$ and hence we have $t \in L_a$. Therefore we get that $L_a = L$. Now suppose that H be any filter of L such that $F \subseteq H \subseteq L$ and $a \in H - F$. Since L_a is the smallest filter containing $F \cup \{a\}$ we have $F \subseteq L_a \subseteq H \subseteq L$. As $a \notin F$ we have $L_a = L$ and so $H = L$. Therefore F is a maximal filter of L. □

5.5 Boolean Filter

Definition 5.5.1. *For a BH-lattice L with unity l and $x \in L$ denote $x' = l \to x$. A filter F of L is called*

a) a Boolean filter if $x \vee x' \in F$, $\forall x \in L$.

b) semi-Boolean if $(x \wedge x')' \in F$, $\forall x \in L$.

Remark: Every filter is not necessarily a Boolean filter. There are filters which are not Boolean. For this consider the following example.

Example 5.5.2. *Consider example* 5.4.8. *For $x \notin H = [\frac{1}{2}, 1], x \vee (0 \to x) = x$. Hence the filter in example* 5.4.8 *is not a Boolean filter.*

Example 5.5.3. *The filters in example* 5.2.2 *are semi-Boolean filters while the filters in example* 5.2.3 *are Boolean filters.*

Theorem 5.5.4. *In Boolean algebra every filter is Boolean filter.*

Proof. Suppose L is a Boolean algebra and F is a filter. Since Boolean algebra is bounded below, by Corollary 2.2.36, $0 = l$ and for any $x \in L, 0 \to x = 0 \vee x' = x'$ which is the complement of x in Boolean algebra. Hence $x' \vee x = x \vee (0 \to x) = e \in F$. Hence F is Boolean filter. □

Theorem 5.5.5. *In Heyting algebra every filter is semi-Boolean filter.*

Proof. Let L be a Heyting algebra and F be a filter. Since Heyting algebra is bounded below, $0 = l$ and for any $x \in L, (x \wedge x')' = 0 \to (x \wedge (0 \to x)) = 0 \to 0 = e \in F$. Hence F is semi-Boolean filter. □

Theorem 5.5.6. *If L is bounded above, then Boolean filter of a BH-lattice L is semi-Boolean.*

Proof. Suppose that L is bounded above and F is a Boolean filter of L. For any $x \in L$, as $x' \wedge x \leq x', x$, by Theorem 1.3.15, $x', x'' \leq (x \wedge x')'$. Since for any $x, y \in L, x \leq$

$y \to (y \to x)$, $x \leq x'' = l \to (l \to x) \leq (x \wedge x')'$. Hence $x \vee x' \leq (x' \wedge x)'$. Therefore $(x' \wedge x)' \in F$. Hence F is semi-Boolean. $\square$

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Publications

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