# STUDIES ON THE ROLE OF PION DEGREES OF FREEDOM IN NUCLEAR INTERACTIONS

A Thesis

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## ABSTRACT

In this thesis, we work out the low-energy (∼300MeV) form of the proton-proton (p-p), neutron-neutron (n-n), and proton-neutron (p-n) interaction potentials taking the field-theoretic view that these interactions are mediated through the exchange of $\pi^0$, $\pi^{\pm}$ mesons. In this process we reproduce the exact form of Yukawa potential. We extend these calculations to obtain an approximate form of two-pion exchange potential. We then generalize the formalism to include isospin and arrive at the correct form of the low-energy nucleon-nucleon (N-N) potential. In the process we establish the charge independent hypothesis of nuclear forces. Further we observe that our model for the long-range part of N-N forces correctly predicts that for $I = 0$, the two nucleon system exists as a bound state (deuteron nucleus) while for $I = 1$, it exists as scattering states.

## Contents

# LIST OF FIGURES

CHAPTER 1

# Introduction

During the middle of $20^{th}$ century, the ideas of relativity, quantum mechanics and classical electrodynamics had been combined by J. Schwinger, F. Dyson, S. Tomonaga, R. P. Feynman and others, into a field theory called quantum electrodynamics (QED) [1], which successfully accounts for the electromagnetic interactions of relativistically moving charged particles such as electrons and muons. The predictions of QED [2, 3] are in complete agreement with experiment. It is well known that in quantum electrodynamics the electromagnetic field is considered as if it consists of photons which are quanta of this field. This basic idea of QED can be also extended to nuclear interactions.

Protons and neutrons were collectively termed as nucleons by Heisenberg. Analogous to the field-theoretic description of electromagnetic interaction between two relativistically moving electrons, Yukawa pictured the strong nucleon-nucleon (N-N) interaction as arising from one nucleon emitting a strong field quantum which is promptly absorbed by the other nucleon. Yukawa proposed a field theory of nuclear forces which predicted the existence of particles with mass intermediate between the mass of electron and proton. Such particles were named by Yukawa as mesons [3]. These mesons were regarded as the carriers of the strong force between the two nucleons (i.e. proton–proton (p-p), proton-neutron (p-n) or neutron-neutron (n-n)) just as photons are the carriers of the electromagnetic force between two electrons. Later on these mesons were experimentally discovered in cosmic rays with mass [3] exactly the same as the mass which Yukawa had predicted.

According to our present knowledge, the nuclear force is not simply due to one kind of meson, but it is a complex process involving the exchange of many kinds of mesons of various masses, spins and charges. Since the range of any force field is inversely proportional to the mass of the quanta of the force field, the lightest meson ($\pi-$meson) with mass $m_\pi = 140MeV$ is the major contributor to the long-range ($\approx 1fm - 2fm$) part of nuclear forces [3 − 5].

In the non-relativistic limit, nuclear interactions can be described by a potential [6]. As mentioned earlier, the long-range part of nuclear force arises due to the exchange of $\pi-$meson between the interacting nucleons just as the force between two electrons arises due to the exchange of a photon between the interacting electrons. Whereas the short-range ($< 1fm$) part of N-N force arises due to the exchange of quarks and gluons between the two interacting nucleons which can be regarded as quark "bags". For a recent work on the exact treatment of this problem of the short-range part of N-N force, see ref. [7].

Analysis of N-N scattering experiments gives information on the character of the nuclear forces. In this connection we have evaluated the proton-proton scattering cross-section in chapter 3. The fact that the deuteron has a finite electric quadrupole moment is a confirmation of the fact that the nuclear interaction potential is non-central, that is, it depends on the orientations of the spins of the two interacting particles with the line joining the two particles [6, 8].

Further it is experimentally seen that the nuclear forces are charge independent. That is, the nuclear forces between two protons, two neutrons and between a proton and a neutron are the same. This charge independence can be described by introducing the concept of isospin. As charge independence is equivalent to invariance of the nuclear interaction under rotations in fictitious charge space (isospin space), the interaction should be a scalar in isospin space [9, 10].

The main goal of this thesis is to study the form of low-energy (long-range part) N-N potential arising due to the exchange of $\pi-$meson, using field-theoretic techniques in the language of Feynman propagators. In this process the techniques for calculation of the differential cross-section for p-p scattering process are shown.

The outline of the thesis is as follows. In chapter 2, the expression for first-order correction to the transition amplitude for e-e interaction mediated through the exchange of a photon is calculated and then this result is used to derive the low-energy e-e potential. Chapter 3 deals with calculation of one-pion exchange potential (OPEP), two-pion exchange potential (TPEP) for low-energy p-p interaction, and differential scattering cross-section for p-p scattering. In chapter 4, we analyze p-n scattering. Next we introduce the isospin formalism in which we can have a single unified formulation of p-n, p-p, and n-n interactions in terms of N-N interaction. We then obtain the long-range part of N-N potential. Finally, the main results of the thesis are summarized in chapter five.

CHAPTER 2

# Field-Theoretic Derivation of Low-Energy Electron-Electron (e-e) Potential

In non-relativistic quantum mechanics and in classical mechanics, interactions can be described by analyzing the corresponding potential. On the other hand, in relativistic field theory, interactions are understood to be caused by exchange of field quanta between the interacting particles. The connection between the two concepts can be made, for low-energy interactions involving lowest-order processes, using the Born approximation [2].

From quantum field theory of electromagnetic interactions, an electron constantly emits and absorbs virtual photons. As a result, two electrons interact by exchanging the photons they emit.

The purpose of this chapter is to first demonstrate the techniques to obtain the form of e-e potential based on the quantum field theory descriptions of electromagnetic interactions. To this end, we need to obtain the non-relativistic limit of the differential cross-section for Moller scattering, and then compare it with that obtained applying Fermi's Golden rule.

In the first section of this chapter, we wish to obtain the transition amplitude for a given QED process. Next we apply the result obtained to compute the differential cross-section for Moller scattering. Finally the potential equivalent to the interaction of two low-energy electrons due to the exchange of a photon will be derived using Born approximation in the Schrodinger theory.

### 2.1 First-Order Correction to The Transition Amplitude

The interaction of two electrons through the exchange of a photon is described by first-order correction to the transition amplitude, $S_{fi}^{(1)}$, which is the transition amplitude that an electron in the initial state $|i\rangle$ is scattered to a final state $|f\rangle$. To obtain the expression for the transition amplitude, $S_{fi}^{(1)}$, we start with the Dirac equation of an electron interacting with electromagnetic field, $A_\mu$, as expressed below:

$$(\gamma_\mu \frac{\partial}{\partial x_\mu} + m)\Psi = ie\gamma_\mu A_\mu \Psi \tag{2.1}$$

where $\Psi$ is the four-component Dirac wave function of the electron, e is the electron charge, $\gamma_\mu$ ($\mu$ =1, 2, 3, 4) are 4x4 traceless Hermitian Dirac matrices, and m is rest mass of the electron. The solution of (2.1) can be obtained by considering the unit source problem

$$(\gamma_\mu \frac{\partial}{\partial x_\mu} + m)K(x, x') = -i\delta^{(4)}(x - x'), \tag{2.2}$$

where $\delta^{(4)}(x-x')$ is the four- dimensional delta function and $K(x, x')$ is the Green's function for the free-particle Dirac equation. Consequently, the solution of (2.1) can be written as

$$\Psi(x) = -\int d^4x' K(x, x') e\gamma_\mu A_\mu(x')\Psi(x'). \tag{2.3}$$

Letting $\Psi_0(x)$ be a solution of the free, homogeneous Dirac equation, the general solution of (2.1) can be expressed as a superposition of $\Psi_0(x)$ and $\Psi(x)$ from

(2.3). Thus,

$$\Psi(x) = \Psi_0(x) - \int d^4x' K(x,x') e\gamma_\mu A_\mu(x')\Psi(x'). \tag{2.4}$$

We see that the solution of (2.1) can be given to any order of the interaction parameter, $e$, using the iteration method since we can not have a closed form of (2.4). Hence we have

$$\Psi(x) = \Psi_0(x) + \int d^4x' K(x,x')[-e\gamma_\mu A_\mu(x')]\Psi_0(x')$$

$$+ \int\int d^4x' d^4x'' K(x,x')[-e\gamma_\mu A_\mu(x')]K(x',x'')[-e\gamma_\nu A_\nu(x'')]\Psi_0(x'') + ... \tag{2.5}$$

As it can be observed from (2.5), one can obtain the total wave function of an electron interacting with electromagnetic field if we know the explicit form of the electron propagator, $K(x,x')$. This can be done by Fourier transforming $K(x,x')$ into momentum space. We note that $K(x,x')$ represents the amplitude of a wave at $x$, originating at $x'$ and so $K$ is a function of the difference $x - x'$. Hence, the Fourier transform of $K(x,x')$ can be written as

$$K(x-x') = \frac{1}{(2\pi)^4}\int d^4p e^{ip\cdot(x-x')}k(p). \tag{2.6}$$

Setting $x' = 0$ into (2.6), one finds

$$K(x) = \frac{1}{(2\pi)^4}\int d^4p e^{ip\cdot x}k(p). \tag{2.7}$$

Inserting this into (2.2) and setting $x' = 0$, we have

$$\frac{1}{(2\pi)^4}\int d^4p(\gamma_\mu \frac{\partial}{\partial x_\mu} + m)e^{ip\cdot x}k(p) = \frac{-i}{(2\pi)^4}\int d^4p e^{ip\cdot x},$$

so that

$$k(p) = \frac{-i\gamma_\mu p_\mu + m}{i(p^2 + m^2)}. \tag{2.8}$$

Feeding (2.8) into (2.7), we find

$$K(x) = \frac{-i}{(2\pi)^4} \int d^4p \frac{(-i\gamma_\mu p_\mu + m)}{p^2 + m^2} e^{ip\cdot x},$$

which can be generalized to

$$K(x - x') = \frac{-i}{(2\pi)^4} \int d^4p \frac{(-i\gamma_\mu p_\mu + m)}{p^2 + m^2} e^{ip\cdot(x-x')}. \tag{2.9}$$

Spliting into space and time integrations, (2.9) can be written as

$$K(x - x') = \frac{i}{(2\pi)^3} \int d^3\vec{p} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} \frac{1}{2\pi} \int_{-\infty}^{\infty} dp_0 \frac{(-i\gamma_\mu p_\mu + m)}{(p_0 - E)(p_0 + E)} e^{-ip_0\cdot(t-t')}. \tag{2.10}$$

Treating $p_0$ as a complex variable, (2.10) can be integrated over $p_0$ by applying the method of residues. For $t < t'$, we integrate over the contour in the upper half complex $p_0$-plane, as indicated below in Fig. 2.1a.

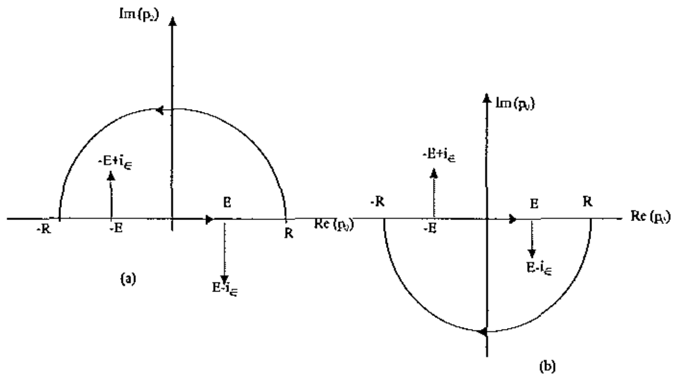


Figure 2.1: *Contour of integration for* (a) $t < t'$ *and* (b) $t > t'$.

Upon integration, one can verify that

$$\int_{-\infty}^{\infty} dp_0 \frac{(-i\gamma_\mu p_\mu + m)}{(p_0 - E)(p_0 + E)} e^{-ip_0(t-t')} = 2\pi i \frac{(i\gamma_i p_i + \gamma_4 E - m)}{2E} e^{iE(t-t')}. \qquad (2.11)$$

Combining (2.11) with (2.10), we get

$$K(x - x') = \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{2E} e^{-i\vec{p}\cdot(\vec{x}-\vec{x}')} (i\gamma_i p_i - \gamma_4 E + m) e^{iE(t-t')}, for\ t < t'. \qquad (2.12)$$

In the case of $t > t'$, the contour of integration will be the lower half plane, as shown in Fig. 2.1b. Following similar procedure, one readily arrives at

$$K(x - x') = \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{2E} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} (-i\gamma_i p_i + \gamma_4 E + m) e^{-iE(t-t')}, for\ t > t'. \qquad (2.13)$$

Replacing

$$\frac{1}{(2\pi)^3} \int d^3\vec{p} \ \ by \ \ \lim_{V\to\infty} \frac{1}{V} \sum_{\vec{p}}$$

in (2.12) and (2.13) results in

$$K(x - x') = \lim_{V\to\infty} \frac{1}{V} \sum_{\vec{p}} e^{-i\vec{p}\cdot(\vec{x}-\vec{x}')} \frac{(i\gamma_i p_i - \gamma_4 E + m)}{2E} e^{iE(t-t')}, for\ t < t', \qquad (2.14)$$

and

$$K(x - x') = \lim_{V\to\infty} \frac{1}{V} \sum_{\vec{p}} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} \frac{(-i\gamma_i p_i + \gamma_4 E + m)}{2E} e^{-iE(t-t')}, \ for\ t > t' \qquad (2.15)$$

respectively. Employing the relations

$$\sum_{s=1}^{2} v^{(s)}(\vec{p}) \bar{v}^{(s)}(\vec{p}) = \frac{-i\gamma_i p_i + \gamma_4 E - m}{2m}$$

and

$$\sum_{s=1}^{2} u^{(s)}(\vec{p}) \bar{u}^{(s)}(\vec{p}) = \frac{-i\gamma_i p_i + \gamma_4 E + m}{2m},$$

in which $v^{(s)}(\vec{p})$ and $u^{(s)}(\vec{p})$ are negative- and positive-energy $4 \times 1$ Dirac spinors, in (2.14) and (2.15) respectively leads to

$$K(x-x') = \lim_{V\to\infty} \sum_{\vec{p}} \sum_{s=1}^{2} \left[-(\frac{m}{EV}) v^{(s)}(\vec{p}) \bar{v}^{(s)}(\vec{p}) e^{-ip\cdot(x-x')}\right], \ for\ t < t', \quad (2.16)$$

and

$$K(x-x') = \lim_{V\to\infty} \sum_{\vec{p}} \sum_{s=1}^{2} \left[(\frac{m}{EV}) u^{(s)}(\vec{p}) \bar{u}^{(s)}(\vec{p}) e^{ip\cdot(x-x')}\right], \ for\ t > t' \quad (2.17)$$

For first-order scattering, the perturbative expansion (2.5) can be written as

$$\Psi(x) \simeq \int d^4x' K(x,x')[-e\gamma_\mu A_\mu(x')]\Psi_0(x'). \quad (2.18)$$

This means the incident electron with wave function $\Psi_0(x')$ is scattered at $x'$ by the electromagnetic field $A_\mu$ and then propagates to $x$ with amplitude $K(x,x')$. Hence, $K(x,x')$ is a function of momentum $p'$, spin $s'$, and energy $E'$ of the scattered electron. With this in mind, inserting (2.16) and (2.17) into (2.18), we can write

$$\Psi(x) = \lim_{V\to\infty} \sum_{\vec{p}',s'} C^{+}_{\vec{p}',s'}(t) \sqrt{\frac{m}{E'V}} u^{(s')}(\vec{p}') e^{ip'\cdot x} + \lim_{V\to\infty} \sum_{\vec{p}',s'} C^{-}_{\vec{p}',s'}(t) \sqrt{\frac{m}{E'V}} v^{(s')}(\vec{p}') e^{-ip'\cdot x}; \quad (2.19)$$

where

$$C^{+}_{\vec{p}',s'}(t) = -e \int d^3\vec{x}' \int_{-\infty}^{t} dt' \sqrt{\frac{m}{E'V}} \bar{u}^{(s')}(\vec{p}') e^{-ip'\cdot x'} \gamma_\mu A_\mu(x') \Psi_0(x'), \quad (2.20)$$

and

$$C^{-}_{\vec{p}',s'}(t) = e \int d^3\vec{x}' \int_{t}^{\infty} dt' \sqrt{\frac{m}{E'V}} \bar{v}^{(s')}(\vec{p}') e^{ip'\cdot x'} \gamma_\mu A_\mu(x') \Psi_0(x'). \quad (2.21)$$

Suppose $\Psi_0(x')$ be normalized plane-wave function of a free positive-energy incident electron with momentum $p$, energy $E$, and spin $s$. In other words,

$$\Psi_0(x') = \sqrt{\frac{m}{EV}} u^{(s)}(\overrightarrow{p}) e^{ip\cdot x'}. \tag{2.22}$$

Making use of (2.22) in (2.20), we find that

$$C^{+}_{\vec{p}',s'}(t) = -e \int d^3\vec{x}' \int_{-\infty}^{t} dt' \sqrt{\frac{m}{E'V}} \overline{u}^{(s')}(\vec{p}') e^{-ip'\cdot x'} [\gamma_\mu A_\mu(x')] \sqrt{\frac{m}{EV}} u^{(s)}(\overrightarrow{p}) e^{ip\cdot x'}, \tag{2.23a}$$

or

$$C^{+}_{\vec{p}',s'}(\infty) = -e \int d^3\vec{x}' \int_{-\infty}^{\infty} dt' \sqrt{\frac{m}{E'V}} \overline{u}^{(s')}(\vec{p}') e^{-ip'\cdot x'} [\gamma_\mu A_\mu(x')] \sqrt{\frac{m}{EV}} u^{(s)}(\overrightarrow{p}) e^{ip\cdot x'}, \tag{2.23b}$$

$$= -e \int d^3\vec{x}' \int_{-\infty}^{\infty} dt' \overline{\Psi}_f(x') \gamma_\mu A_\mu(x') \Psi_i(x') \tag{2.23c}$$

where

$$\overline{\Psi}_f(x') = \sqrt{\frac{m}{E'V}} \overline{u}^{(s')}(\vec{p}') e^{-ip'\cdot x'}$$

and

$$\Psi_i(x') = \sqrt{\frac{m}{EV}} u^{(s)}(\overrightarrow{p}) e^{ip\cdot x'}$$

are normalized plane-wave functions of scattered and incident free-electrons respectively.

Now we know that the first-order correction to the scattering matrix is given by [2, 11]

$$S^{(1)}_{fi} = -i \int_{-\infty}^{\infty} dt' \langle f | H_I(t) | i \rangle, \tag{2.23d}$$

in which the time-dependent perturbing Hamiltonian $H_I(t)$ is expressed in terms of the Hamiltonian density, $\mathcal{H}(t')$, describing the interaction of free Dirac fields with electromagnetic field $A_\mu$, as

$$H_I(t) = \int d^3\overrightarrow{x}'\mathcal{H}(t') \tag{2.23e}$$

where

$$\mathcal{H}(t') = -ie\overline{\Psi}\gamma_\mu\Psi A_\mu(x'). \tag{2.23f}$$

Substituting (2.23e) along with (2.23f) into (2.23d) results in

$$S_{fi}^{(1)} = -e\int d^3\vec{x}' \int_{-\infty}^{\infty} dt'\overline{\Psi}_f(x')\gamma_\mu A_\mu(x')\Psi_i(x'). \tag{2.23g}$$

Comparing (2.23g) with (2.23c), we observe that $C^{+}_{\vec{p}',s'}(\infty) = S^{(1)}_{fi}$, indicating that $C'^{+}_{\vec{p}',s'}(\infty)$ is the first-order correction to the transition amplitude that a positive-energy incident electron with momentum $p$ and spin $s$ is scattered to a positive-energy electron state with momentum $p'$and spin $s'$. From (2.20), one can see that $C^{+}_{\vec{p}',s'}(-\infty) = 0$, implying that an incident free electron can not be scattered to a positive-energy free electron back ward in time. Based on the same reasoning,

$$C^{-}_{\vec{p}',s'}(-\infty) = e\int d^3\vec{x}' \int_{-\infty}^{\infty} dt' \sqrt{\frac{m}{E'V}}\bar{v}^{(s')}(\vec{p}')e^{ip'\cdot x'}\gamma_\mu A_\mu(x')\sqrt{\frac{m}{EV}}u^{(s)}(\overrightarrow{p})e^{ip\cdot x'}$$

is the first-order correction to the transition amplitude that an incident free positive-energy electron is scattered to a free negative-energy electron state. We can also see that $C^{-}_{\vec{p}',s'}(+\infty) = 0$, indicating that an incident electron can not be scattered to a negative-energy electron forward in time.

## 2.2 Moller Scattering

The electromagnetic interaction between two electrons mediated through the exchange of a photon is depicted by the Feynman diagrams shown in Fig. 2.2.

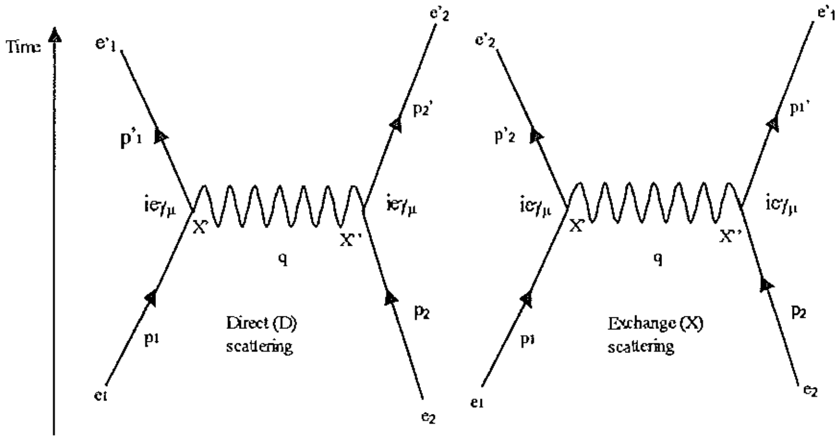


Figure 2.2: *Moller scattering of two electrons:* $p_1$ *and* $p_2$ *are 4-momenta of incident electrons* $e_1$ *and* $e_2$, $p_1'$ *and* $p_2'$ *are 4-momenta of scattered electrons* $e_1'$ *and* $e_2'$ *and* $q$ *is 4-momentum of the exchanged photon, respectively. The wavy line represents the exchanged photons.*

The overall amplitude, $S_{fi}^{(1)}$, for this process is given by

$$S_{fi}^{(1)} = S_{fi}^{(1)}|_D - S_{fi}^{(1)}|_X, \tag{2.24}$$

where $S_{fi}^{(1)}|_D$ and $S_{fi}^{(1)}|_X$ are respectively transition amplitudes for the direct and exchange scattering. We recall that the first-order correction to the transition amplitude for the direct scattering is given by

$$S_{fi}^{(1)}|_D = -e\int d^3\vec{x}' \int_{-\infty}^{\infty} dt' \sqrt{\frac{m}{E_1'V}}\overline{u}^{(s_1')}(\vec{p}_1')e^{-ip_1'\cdot x'}[\gamma_\mu A_\mu(x')]\sqrt{\frac{m}{E_1V}}u^{(s_1)}(\overrightarrow{p}_1)e^{ip_1\cdot x'}$$

or

$$S_{fi}^{(1)}|_D = ie\int d^4x' \sqrt{\frac{m}{E_1'V}}\overline{u}^{(s_1')}(\vec{p}_1')e^{-ip_1'\cdot x'}[\gamma_\mu A_\mu(x')]\sqrt{\frac{m}{E_1V}}u^{(s_1)}(\overrightarrow{p}_1)e^{ip_1\cdot x'}, \tag{2.25}$$

where $A_\mu(x')$ is the electromagnetic field produced by the second moving electron

$e_2$ and it satisfies the Maxwell's equations

$$\Box^2 A_\mu(x) = j_\mu(x), \tag{2.26}$$

in which

$$\Box^2 = \vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}.$$

Equation (2.26) can be solved by considering again a unit source problem,

$$\Box^2 D_F(x - x') = \delta^{(4)}(x - x'), \tag{2.27}$$

where $D_F(x - x')$ is Green's function. Employing the Fourier transform method with $x' = 0$ in (2.27), we can write

$$D_F(x) = \frac{1}{(2\pi)^4}\int d^4q e^{iq\cdot x} D_F(q). \tag{2.28}$$

Combining (2.28) and (2.27) and upon differentiation, one can verify that

$$D_F(q) = -\frac{1}{q^2} \tag{2.29}$$

where the Fourier transform of $\delta^{(4)}(x)$ has been used. Substituting (2.29) into (2.28) yields

$$D_F(x) = \frac{1}{(2\pi)^4}\int d^4q e^{iq\cdot x}[\frac{-1}{q^2}],$$

which can be generalized to

$$D_F(x - x') = \frac{1}{(2\pi)^4}\int d^4q e^{iq\cdot(x-x')}[\frac{-1}{q^2}]. \tag{2.30}$$

This relation is known as photon propagator. Using this relation, the electromagnetic field $A_\mu(x')$ due to the electromagnetic transition current, $j_\mu(x'')$, produced by the second moving electron $e_2$ can be written as

$$A_\mu(x') = \frac{1}{(2\pi)^4}\int\int d^4q d^4x''[\frac{-e^{iq\cdot(x'-x'')}}{q^2}]j_\mu(x''). \tag{2.31}$$

Feeding (2.31) into (2.25), one can verify that

$$
\begin{aligned}
S_{fi}^{(1)}|_D &= \frac{1}{(2\pi)^4}\int\int\int d^4q d^4x' d^4x'' \sqrt{\frac{m}{E_1'V}}\overline{u}^{(s_1')}(\overrightarrow{p}_1')e^{-ip_1'\cdot x'}(ie\gamma_\mu) \\
&\quad \times\sqrt{\frac{m}{E_1V}}u^{(s_1)}(\overrightarrow{p}_1)e^{ip_1\cdot x'}[\frac{-e^{iq\cdot(x'-x'')}}{q^2}] \\
\times &\quad \sqrt{\frac{m}{E_2'V}}\overline{u}^{(s_2')}(\overrightarrow{p}_2')e^{-ip_2'\cdot x''}(ie\gamma_\mu)\sqrt{\frac{m}{E_2V}}u^{(s_2)}(\overrightarrow{p}_2)e^{ip_2\cdot x''}, \qquad (2.32)
\end{aligned}
$$

where the four-current density, $j_\mu(x'') = ie\overline{\Psi}_{2'}(x'')\gamma_\mu\Psi_2(x'')$ has been used. Carrying out the integration, we obtain

$$
\begin{aligned}
S_{fi}^{(1)}|_D &= (ie)^2\sqrt{\frac{m^4}{E_1'E_1E_2'E_2V^4}}(2\pi)^4\delta^{(4)}(p_2'+p_1'-p_2-p_1)[\overline{u}^{(s_1')}(\overrightarrow{p}_1')\gamma_\mu u^{(s_1)}(\overrightarrow{p}_1)] \\
&\quad \times(\frac{-1}{q_D^2})[\overline{u}^{(s_2')}(\overrightarrow{p}_2')\gamma_\mu u^{(s_2)}(\overrightarrow{p}_2)], \qquad (2.33)
\end{aligned}
$$

in which

$$q_D = p_1' - p_1 = p_2 - p_2'.$$

The relation corresponding to (2.33) for the exchange scattering can easily be obtained by making the following interchanges in (2.33):

$$p_1' \rightleftarrows p_2',\ u^{(s_1')} \rightleftarrows u^{(s_2')},\ and\ E_1' \rightleftarrows E_2'.$$

Hence we have

$$
\begin{aligned}
S_{fi}^{(1)}|_X &= (ie)^2\sqrt{\frac{m^4}{E_1'E_1E_2'E_2V^4}}(2\pi)^4\delta^{(4)}(p_2'+p_1'-p_2-p_1)[\overline{u}^{(s_2')}(\overrightarrow{p}_2')\gamma_\mu u^{(s_1)}(\overrightarrow{p}_1)] \\
&\quad \times(\frac{-1}{q_x^2})[\overline{u}^{(s_1')}(\overrightarrow{p}_1')\gamma_\mu u^{(s_2)}(\overrightarrow{p}_2)], \qquad (2.34)
\end{aligned}
$$

where

$$q_x = p_2' - p_1 = p_2 - p_1'.$$

Finally, the overall transition amplitude

$$S_{fi}^{(1)} = S_{fi}^{(1)}|_D - S_{fi}^{(1)}|_X \tag{2.35}$$

becomes

$$S_{fi}^{(1)} = \sqrt{\frac{m^4}{E_1' E_1 E_2' E_2 V^4}}(2\pi)^4 \delta^{(4)}(p_2' + p_1' - p_2 - p_1) M_{fi}, \tag{2.36}$$

where

$$M_{fi} = \left\{ \begin{array}{l} e^2 [ \frac{\left[\overline{u}^{(s_1')}(\overrightarrow{p}'_1)\gamma_\mu u^{(s_1)}(\overrightarrow{p}_1)\right]\left[\overline{u}^{(s_2')}(\overrightarrow{p}'_2)\gamma_\mu u^{(s_2)}(\overrightarrow{p}_2)\right]}{q_D^2} \\ - \frac{\left[\overline{u}^{(s_2')}(\overrightarrow{p}'_2)\gamma_\mu u^{(s_1)}(\overrightarrow{p}_1)\right]\left[\overline{u}^{(s_1')}(\overrightarrow{p}'_1)\gamma_\mu u^{(s_2)}(\overrightarrow{p}_2)\right]}{q_x^2} ]. \end{array} \right\} \tag{2.37}$$

The transition probability thus takes the form

$$|S_{fi}^{(1)}|^2 = (\frac{m^4}{E_1' E_1 E_2' E_2 V^4})[(2\pi)^4 \delta^{(4)}(p_2' + p_1' - p_2 - p_1)]^2 |M_{fi}|^2,$$

$$= (\frac{m^4}{E_1' E_1 E_2' E_2 V^4}) VT (2\pi)^4 \delta^{(4)}(p_2' + p_1' - p_2 - p_1) |M_{fi}|^2, \tag{2.38}$$

where $(2\pi)^4 \delta^{(4)}(p_2'+p_1'-p_2-p_1) = VT$ has been used with $T$, representing the interaction time and $V$ the interaction volume.

The differential cross-section, $d\sigma$, is defined as

$$d\sigma = \frac{|S_{fi}^{(1)}|^2}{T|\overrightarrow{j}_{inc}|} \frac{V d^3 \overrightarrow{p}'_1}{(2\pi)^3} \frac{V d^3 \overrightarrow{p}'_2}{(2\pi)^3}, \tag{2.39}$$

in which $|\overrightarrow{j}_{inc}|$ is the incident flux given by $|\overrightarrow{j}_{inc}| = \frac{|\overrightarrow{v}_{rel}|}{V}$, with $\overrightarrow{v}_{rel}$ the relative velocity of the incident electrons and $\frac{V d^3 \overrightarrow{p}'_1}{(2\pi)^3} \frac{V d^3 \overrightarrow{p}'_2}{(2\pi)^3}$ the number of available final states in the momentum intervals $d^3 \overrightarrow{p}'_1$ and $d^3 \overrightarrow{p}'_2$. In the center-of-mass frame, $|\overrightarrow{v}_{rel}| = 2v$, where $v$ is the speed of the incident electrons, and so $|\overrightarrow{j}_{inc}| = \frac{2v}{V} = \frac{2\beta}{V}$. Upon combining (2.38) and (2.39), one arrives at

$$d\sigma = (\frac{m^4}{E_1' E_1 E_2' E_2}) \frac{(2\pi)^4}{2\beta} \delta^{(4)}(p_2' + p_1' - p_2 - p_1)\, |M_{fi}|^2 \frac{d^3 \overrightarrow{p}'_1}{(2\pi)^3} \frac{d^3 \overrightarrow{p}'_2}{(2\pi)^3}. \tag{2.40}$$

The total cross-section can be obtained by integrating (2.40) as,

$$\sigma = (\frac{m^4}{E_1 E_2' E_2})|M_{fi}|^2 \frac{|\vec{p}'_1|}{2\beta(2\pi)^2} \int d\Omega \int dE_1' \int d^3\vec{p}'_2 \delta^{(4)}(p_2' + p_1' - p_2 - p_1)$$

where

$$d^3\vec{p}'_1 = |\vec{p}'_1|^2 |d\vec{p}'_1| d\Omega = |\vec{p}'_1| E_1' dE_1' d\Omega$$

has been used in (2.40). Integrating over $\vec{p}'_2$ using the $\delta^{(3)}$–function leads to

$$\sigma = (\frac{m^4}{E_1 E_2' E_2})|M_{fi}|^2 \frac{|\vec{p}'_1|}{2\beta(2\pi)^2} \int d\Omega \int dE_1' \delta(E_2' + E_1' - E_2 - E_1). \tag{2.41}$$

The differential cross-section per unit solid angle thus takes the form

$$\frac{d\sigma}{d\Omega} = (\frac{m^4}{E_1 E_2' E_2})|M_{fi}|^2 \frac{|\vec{p}'_1|}{2\,|\vec{v}|\,(2\pi)^2} \int dE_1' \delta(E_2' + E_1' - E_2 - E_1) \tag{2.42}$$

or

$$\frac{d\sigma}{d\Omega} = (\frac{m^4}{E_1 E_2' E_2})|M_{fi}|^2 \frac{|\vec{p}'_1|}{4\,|\vec{v}|\,(2\pi)^2}. \tag{2.43}$$

We know that in the center-of-mass frame, $E_1 = E_2$ and $E_1' = E_2'$; and from conservation of energy, $E_1 = E_2 = E_1' = E_2' = E$. Further $|\vec{p}'_1| = m\,|\vec{v}'_1|$ and $E_2' \simeq m$ in the non-relativistic limit. Hence the differential cross-section per unit solid angle becomes

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{m^4}{4E^2} \frac{|M_{fi}^{(NR)}|^2}{(2\pi)^2}. \tag{2.44}$$

## 2.3 Derivation of e-e Potential

As stated at the beginning of this chapter, our aim is to obtain the form of the potential corresponding to the interaction of two electrons mediated through the exchange of a photon. We recall that in non-relativistic scattering theory, the differential cross-section for elastic scattering of two particles having mass $m_1$ and $m_2$, in the center-of-mass frame, is given by [2]

$$(\frac{d\sigma}{d\Omega})_{cm} = |\frac{\mu_{red}}{2\pi} \int d^3\overrightarrow{r} V e^{-i\overrightarrow{q}\cdot\overrightarrow{r}}|^2. \tag{2.45}$$

Here, $\mu_{red}$ is the reduced mass, and $V$ is the interaction potential. Relation (2.45) is what we call differential cross-section in the Born approximation. Now using $E^2 = m^2$ in the non-relativistic limit and the reduced mass, $\mu_{red} = \frac{m}{2}$ for two identical particles, (2.44) can be written as

$$(\frac{d\sigma}{d\Omega})_{cm} = (\frac{m}{2})^2 \frac{|M_{fi}^{(NR)}|^2}{(2\pi)^2} \tag{2.46}$$

or

$$(\frac{d\sigma}{d\Omega})_{cm} = (\frac{\mu_{red}}{2\pi})^2 |M_{fi}^{(NR)}|^2. \tag{2.47}$$

Here, $|M_{fi}|^2$ contains the entire physics of the process under consideration. On comparing (2.47) with (2.45), we observe that

$$M_{fi}^{(NR)} = \int d^3\overrightarrow{r} V e^{-i\overrightarrow{q}\cdot\overrightarrow{r}}, \tag{2.48}$$

where $\overrightarrow{q} = \overrightarrow{p}'_1 - \overrightarrow{p}_1$ is the momentum transferred between the interacting electrons. Fourier transforming (2.48) results in

$$V = \frac{1}{(2\pi)^3} \int d^3\overrightarrow{q} M_{fi}^{(NR)} e^{i\overrightarrow{q}\cdot\overrightarrow{r}}. \tag{2.49}$$

Thus, we see that the three-dimensional potential can be constructed once we know the non-relativistic limit of the matrix element, $M_{fi}^{(NR)}$, in (2.37). For scattering of two identical fermions, we take the $M_{fi}^{(NR)}$-matrix element that corresponds to the direct scattering, since the contribution of exchange scattering is included by using the antisymmetric wave functions in the non-relativistic quantum mechanics [2]. Therefore from (2.37) we obtain

$$M_{fi}^{(NR)} = \frac{e^2[\overline{u}^{(s'_1)}(\vec{p}'_1)\gamma_\mu u^{(s_1)}(\vec{p}_1)][\overline{u}^{(s'_2)}(\vec{p}'_2)\gamma_\mu u^{(s_2)}(\vec{p}_2)]}{|\vec{q}|^2}. \tag{2.50}$$

Splitting (2.50) into space and time components, the matrix-element, $M_{fi}$, takes the form

$$\begin{aligned} M_{fi}^{(NR)} &= \frac{e^2[\overline{u}^{(s'_1)}(\vec{p}'_1)\gamma_i u^{(s_1)}(\vec{p}_1)][\overline{u}^{(s'_2)}(\vec{p}'_2)\gamma_i u^{(s_2)}(\vec{p}_2)]}{|\vec{q}|^2} \\ &+ \frac{e^2[\overline{u}^{(s'_1)}(\vec{p}'_1)\gamma_4 u^{(s_1)}(\vec{p}_1)][\overline{u}^{(s'_2)}(\vec{p}'_2)\gamma_4 u^{(s_2)}(\vec{p}_2)]}{|\vec{q}|^2} \end{aligned} \tag{2.51}$$

Writing the matrix structure of the positive-energy spinor, $u^{(s)}(\vec{p})$, as

$$u^{(s)}(\vec{p}) = \sqrt{\frac{E+mc^2}{2mc^2}} \begin{pmatrix} \chi_s \\ \frac{c\vec{\sigma}\cdot\vec{p}}{E+mc^2}\chi_s \end{pmatrix}$$

[where $\chi_s$ is two-component Pauli spinor and $\vec{\sigma}$ is Pauli matrix] and

$$\overline{u}^{(s)}(\vec{p}) = u^{\dagger(s)}(\vec{p})\gamma_4,$$

we note that

$$\begin{aligned} \overline{u}^{(s'_1)}(\vec{p_1}')\gamma_i u^{(s_1)}(\vec{p}_1) &= \frac{E_1+mc^2}{2mc^2} \begin{pmatrix} \chi^\dagger_{s'_1} & \chi^\dagger_{s'_1}\frac{c\vec{\sigma}^{(1)}\cdot\vec{p}'_1}{E_1+mc^2} \end{pmatrix} \\ &\times \begin{pmatrix} 0 & -i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix} \begin{pmatrix} \chi_{s_1} \\ \frac{c\vec{\sigma}^{(1)}\cdot\vec{p}_1}{E_1+mc^2}\chi_{s_1} \end{pmatrix} \end{aligned}$$

or

$$\overline{u}^{(s'_1)}(\vec{p}'_1)\gamma_i u^{(s_1)}(\vec{p}_1) = \frac{\chi^\dagger_{s'_1}}{2m}[-i(\vec{p}_1+\vec{p}'_1) + \vec{\sigma}^{(1)} \times \vec{q}_1]\chi_{s_1} \tag{2.52}$$

where $\overrightarrow{q}_1 = \overrightarrow{p}'_1 - \overrightarrow{p}_1$ is momentum transferred between the electrons, $m$ is electron rest mass, and the speed of light, $c$, is taken to be one in natural units.

Similarly we can write

$$\overline{u}^{(s'_2)}(\overrightarrow{p}'_2)\gamma_i u^{(s_2)}(\overrightarrow{p}_2) = \frac{\chi^{\dagger}_{s'_2}}{2m}[-i(\overrightarrow{p}_2 + \overrightarrow{p}'_2) + \overrightarrow{\sigma}^{(2)} \times \overrightarrow{q}_2]\chi_{s_2} \tag{2.53}$$

where $\overrightarrow{q_2} = \overrightarrow{p_2}' - \overrightarrow{p_2}$. Following the same procedure, one can readily verify that

$$\overline{u}^{(s'_1)}(\overrightarrow{p}'_1)\gamma_4 u^{(s_1)}(\overrightarrow{p}_1) = \frac{E_1 + mc^2}{2mc^2}\left(\chi^{\dagger}_{s'_1}\chi_{s_1} + \chi^{\dagger}_{s'_1}c^2\frac{\overrightarrow{p}_1 \cdot \overrightarrow{p}'_1 + i\overrightarrow{\sigma}^{(1)} \cdot (\overrightarrow{p}'_1 \times \overrightarrow{p}_1)}{(E_1 + mc^2)^2}\chi_{s_1}\right) \tag{2.54}$$

and

$$\overline{u}^{(s'_2)}(\overrightarrow{p}'_2)\gamma_4 u^{(s_2)}(\overrightarrow{p}_2) = \frac{E_2 + mc^2}{2mc^2}\left(\chi^{\dagger}_{s'_2}\chi_{s_2} + \chi^{\dagger}_{s'_2}c^2\frac{\overrightarrow{p}_2 \cdot \overrightarrow{p}'_2 + i\overrightarrow{\sigma}^{(2)} \cdot (\overrightarrow{p}'_2 \times \overrightarrow{p}_2)}{(E_2 + mc^2)^2}\chi_{s_2}\right) \tag{2.55}$$

In the limit $\overrightarrow{p}'_1$, $\overrightarrow{p}_1$, $\overrightarrow{p}'_2$ and $\overrightarrow{p}_2$ are very small, expressions (2.52), (2.53), (2.54), and (2.55) can approximately be expressed respectively as

$$\overline{u}^{(s'_1)}(\overrightarrow{p}'_1)\gamma_i u^{(s_1)}(\overrightarrow{p}_1) = \frac{\chi^{\dagger}_{s'_1}}{2m}[-i(\overrightarrow{p}_1 + \overrightarrow{p}'_1) + \overrightarrow{\sigma}^{(1)} \times \overrightarrow{q}_1]\chi_{s_1} \simeq 0,$$

$$\overline{u}^{(s'_2)}(\overrightarrow{p}'_2)\gamma_i u^{(s_2)}(\overrightarrow{p}_2) = \frac{\chi^{\dagger}_{s'_2}}{2m}[-i(\overrightarrow{p}_2 + \overrightarrow{p}'_2) + \overrightarrow{\sigma}^{(2)} \times \overrightarrow{q}_2]\chi_{s_2} \simeq 0,$$

$$\overline{u}^{(s'_1)}(\overrightarrow{p}'_1)\gamma_4 u^{(s_1)}(\overrightarrow{p}_1) \simeq \chi^{\dagger}_{s'_1}\chi_{s_1} = \delta_{s'_1 s_1} = 1,$$

and

$$\overline{u}^{(s'_2)}(\overrightarrow{p}'_1)\gamma_4 u^{(s_2)}(\overrightarrow{p}_2) \simeq \chi^{\dagger}_{s'_2}\chi_{s_2} = \delta_{s'_2 s_2} = 1.$$

In view of these, the matrix element (2.51) can be expressed as

$$M_{fi} = \frac{e^2\delta_{s'_1 s_1}\delta_{s'_2 s_2}}{|\overrightarrow{q}|^2},$$

or

$$M_{fi} = \frac{e^2}{|\vec{q}|^2}. \tag{2.56}$$

The electron-electron potential corresponding to this limit is then

$$V = \frac{1}{(2\pi)^3} \int d^3\vec{q}\, M_{fi} e^{i\vec{q}\cdot\vec{r}}$$

which, upon integration, gives

$$V = \frac{e^2}{4\pi r}, \tag{2.57}$$

which is the expected coulomb potential between two stationary electrons. The first term,

$$\frac{e^2[\overline{u}^{(s_1')}(\vec{p}\,'_1)\gamma_i u^{(s_1)}(\vec{p}_1)][\overline{u}^{(s_2')}(\vec{p}\,'_2)\gamma_i u^{(s_2)}(\vec{p}_2)]}{|\vec{q}|^2}$$

in (2.51) has contribution to the potential when the electrons are not static and their speed is not relativistic. Let us proceed to compute the contribution of this term to the electron-electron potential. From (2.52) and (2.53), we observe that

$$\frac{e^2[\overline{u}^{(s_1')}(\vec{p}\,'_1)\gamma_i u^{(s_1)}(\vec{p}_1)][\overline{u}^{(s_2')}(\vec{p}\,'_2)\gamma_i u^{(s_2)}(\vec{p}_2)]}{|\vec{q}|^2}$$

involves dipole-dipole, and charge-dipole interactions. The dipole-dipole interaction involves

$$(\vec{\sigma}^{(1)} \times \vec{q}_1) \cdot (\vec{\sigma}^{(2)} \times \vec{q}_2).$$

The corresponding potential is thus given by

$$V_{dd} = \frac{-e^2}{(2m)^2(2\pi)^3} \int d^3\vec{q}\, \frac{(\vec{\sigma}^{(1)} \times \vec{q}) \cdot (\vec{\sigma}^{(2)} \times \vec{q})}{|\vec{q}|^2} e^{i\vec{q}\cdot\vec{r}}. \tag{2.58}$$

Carrying out the integration yields

$$V_{dd} = \frac{e^2}{(2m)^2} \vec{\sigma}^{(1)} \cdot [\vec{\nabla} \times (\vec{\sigma}^{(2)} \times \vec{\nabla})](\frac{1}{4\pi r}). \tag{2.59}$$

Recalling that the magnetic moment, $\vec{\mu}$, of an electron due to its spin is given by

$$\vec{\mu} = \frac{e}{m}\vec{s} = \frac{e}{2m}\vec{\sigma}$$

and the vector potential due to this magnetic moment is

$$\vec{A}_{dip} = \frac{\vec{\mu} \times \hat{r}}{4\pi r^2},$$

relation (2.59) can be written as

$$V_{dd} = -\vec{\mu}_1 \cdot \vec{B}_2, \tag{2.60}$$

in which $\vec{\mu}_1$ is the intrinsic magnetic moment of the first electron $e_1$ and $\vec{B}_2$ is the magnetic field produced by the intrinsic magnetic moment of the second electron $e_2$. As it can be seen, relation (2.60) is the interaction energy between the magnetic moments of the two electrons.

The charge-dipole interaction arises from

$$-i(\frac{e}{2m})^2 \left[\frac{(\vec{\sigma}^{(1)} \times \vec{q}_1) \cdot (\vec{p}_2 + \vec{p}'_2) + (\vec{\sigma}^{(2)} \times \vec{q}_2) \cdot (\vec{p}_1 + \vec{p}'_1)}{|\vec{q}|^2}\right]$$

in which

$$(\vec{\sigma}^{(1)} \times \vec{q}_1) \cdot (\vec{p}_2 + \vec{p}'_2) = 2\vec{\sigma}^{(1)} \cdot (\vec{p}_1 \times \vec{p}'_1),$$

and

$$(\vec{\sigma}^{(2)} \times \vec{q}_2) \cdot (\vec{p}_1 + \vec{p}'_1) = 2\vec{\sigma}^{(2)} \cdot (\vec{p}_1 \times \vec{p}'_1).$$

As a result, the corresponding potential to the charge-dipole interaction is

$$V_{dc} = \frac{2e^2}{(2m)^2}\frac{-i}{(2\pi)^3}\int \vec{\sigma} \cdot (\vec{p}_1 \times \vec{p}'_1) d^3\vec{q}\,\frac{e^{i\vec{q}\cdot\vec{r}}}{|\vec{q}|^2}, \tag{2.61}$$

where

$$\vec{\sigma} = \vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}.$$

Upon integration, we obtain

$$V_{dc} = \frac{-2e^2}{(2m)^2} \vec{\sigma} \cdot (\vec{p}_1 \times \vec{\nabla})(\frac{1}{4\pi r}). \tag{2.62}$$

Setting $\vec{\nabla} = \hat{r}\frac{d}{dr}$, one finds

$$V_{dc} = \frac{-2e^2}{(2m)^2} \vec{\sigma} \cdot \vec{p}_1 \times \vec{r} \frac{1}{r} \frac{d}{dr}(\frac{1}{4\pi r}),$$

which can be written in the form

$$V_{dc} = \frac{-4}{(2m)^2} \frac{1}{r} \frac{d}{dr}(\frac{e^2}{4\pi r}) \vec{L} \cdot \frac{(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)})}{2}, \tag{2.63a}$$

as the potential due to spin-orbit interaction. The complete e-e potential can be expressed, using the relations (2.60) and (2.63a) as:

$$V = V_{dd} + V_{dc}$$

$$= -\vec{\mu}_1 \cdot \vec{B}_2 + \frac{-4}{(2m)^2} \frac{1}{r} \frac{d}{dr}(\frac{e^2}{4\pi r}) \vec{L} \cdot \frac{(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)})}{2}. \tag{2.63b}$$

Hence, we have reproduced the coulomb potential between two static electrons arising due to the exchange of virtual photon between them. We have also correctly reproduced the spin-orbit potential using the techniques of quantum field theory, which is the underlying dynamical framework governing the electromagnetic interactions of charged particles, instead of writing the mathematical forms of this potentials through heuristic arguments, as is done in non-relativistic theories.

We have thus demonstrated the power and elegance of field-theoretic techniques to arrive at the correct form of electromagnetic potentials. These methods will later be extended to obtain the correct form of nucleon-nucleon potentials in the following chapters.

CHAPTER 3

# Proton-Proton (p-p) Scattering

Most of the quantitative informations about the nature of nuclear forces have been obtained from the scattering of protons by protons. From historical point of view, proton-proton interaction gave the first good estimate of the range of the forces between two nucleons [12].

In this chapter, we deal with the long-range part of strong interaction, which is mediated by pions. This $\pi-$meson is responsible for the major portion of the long-range ($\approx 1fm - 2fm$) part of the nucleon-nucleon potential [8, 13]. This part of the nuclear force is described by the theory called Quantum-Hadrodynamics, QHD. Quantum-Chromodynamics (QCD) is the correct gauge theory [7] of strong interactions and hence, all nuclear phenomenon at the most fundamental level.

The short-range ($< 1fm$) part of nucleon-nucleon (N-N) interaction arises due to the exchange of quarks and gluons between the two nucleons. While the long-range part arises due to the exchange of $\pi-$mesons between the nucleons. For a recent work on the exact treatment of the problem of short-range part of N-N force using the quark structure of nucleons, see ref. [7]. In this thesis, we are primarily investigating the role of $\pi-$mesons in generating the long-range part of N-N interactions by considering $\pi-$mesons as elementary particles in the framework of Quantum-Hadrodynamics (QHD).

This chapter is organized as follows. In section 3.1, the potential equivalent to the long-range part of proton-proton (p-p) interaction through the exchange of one $\pi$-meson will be discussed. For this purpose, we need first to obtain the non-relativistic form of the matrix element, $M_{fi}$, of the process under consideration, which is related to the potential via the Born approximation. The second section of the chapter is devoted to calculation of the approximate form of the two-pion exchange potential (TPEP). Finally the differential cross-section for p-p scattering is computed.

### 3.1 One-Pion Exchange Potential (OPEP)

The interaction of two protons through the exchange of one $\pi-$meson can be described by the Feynman diagrams shown in Fig. 3.1.

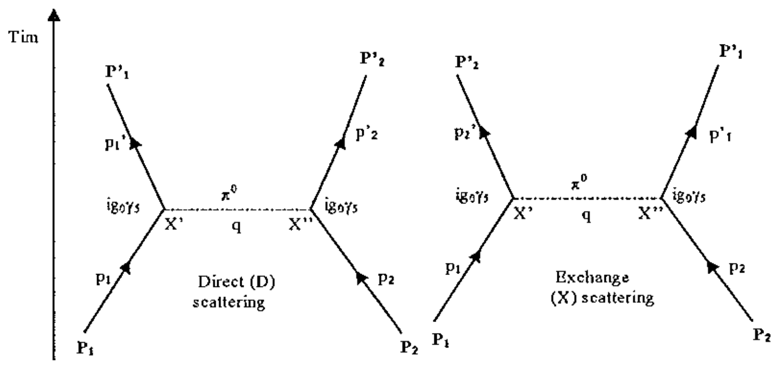


Figure 3.1: *Feynman diagrams denoting interaction of two protons through the exchange of a* $\pi^0-$*meson:* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *stand for incident protons, and* $\mathbf{P}_1'$ *and* $\mathbf{P}_2'$ *for scattered protons, whereas* $\mathrm{p}_1$ *and* $\mathrm{p}_2$; $\mathrm{p}_1'$ *and* $\mathrm{p}_2'$ *denote 4-momenta of incident, and scattered protons respectively.* $\mathrm{g}_0$ *is the coupling constant between proton and neutral pion, and* $q$ *is 4-momenta of the exchanged pion. The dotted line stands for the exchanged* $\pi-$*meson.*

The Dirac equation representing the interaction of an electron with electromagnetic field $A_\mu$ can easily be generalized to the proton interacting with a $\pi^0-$meson field $\Phi_0(x)$ as

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m_p\right) \Psi_p(x) = ig_0 \Phi_0(x) \gamma_5 \Psi_p(x) \tag{3.1}$$

in which $m_p$ is mass of the proton, $\Psi_p(x)$ is the Dirac wave function of the proton, $\Phi_0(x)$ is a neutral scalar ($\pi^0-$meson) field, and $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. Since $\pi^0-$meson is a spinless particle, the free field $\Phi_0(x)$ satisfies the Klein-Gordon equation

$$\left(\Box^2 - \mu^2\right) \Phi_0(x) = 0. \tag{3.2}$$

The interaction of $\Phi_0(x)$ field with the Dirac field of the proton can thus be described by the equation

$$\left(\Box^2 - \mu^2\right) \Phi_0(x) = ig_0 \overline{\Psi}_f(x) \gamma_5 \Psi_i(x), \tag{3.3}$$

where $\mu$ is rest mass of the $\pi^0-$meson, $\overline{\Psi}_f(x)$ and $\Psi_i(x)$ are respectively the Dirac wave functions of the scattered and incident protons, and the term on the right-hand side is the transition current due to the moving proton.

In solving (3.3), we consider as usual the unit source case

$$\left(\Box^2 - \mu^2\right) \Delta_F(x - x') = \delta^{(4)}(x - x') \tag{3.4}$$

where $\Delta_F(x - x')$ is meson propagator. Using the translational symmetry of $\Delta_F(x - x')$ in both space and time, we can put $x' = 0$ so that the Fourier transform of $\Delta_F(x)$ can be written as

$$\Delta_F(x) = \frac{1}{(2\pi)^4} \int d^4q e^{iq\cdot x} \Delta_F(q). \tag{3.5}$$

Substituting (3.5) into (3.4) with $x' = 0$, one easily obtains

$$\Delta_F(q) = \frac{-1}{q^2 + \mu^2} \tag{3.6}$$

which when replaced into (3.5) gives

$$\Delta_F(x) = \frac{-1}{(2\pi)^4} \int d^4q \frac{e^{iq\cdot x}}{q^2 + \mu^2}, \tag{3.7}$$

so that the expression of the meson propagator has the form

$$\Delta_F(x - x') = \frac{-1}{(2\pi)^4} \int d^4q \frac{e^{iq\cdot(x-x')}}{q^2 + \mu^2}. \tag{3.8}$$

Finally, the $\pi^0$−meson field, $\Phi_0(x')$ can be expressed as

$$\Phi_0(x') = \int d^4x'' \Delta_F(x' - x'') \left[ig_0 \overline{\Psi}_{2'}(x'')\gamma_5 \Psi_2(x'')\right] \tag{3.9a}$$

$$= \frac{-1}{(2\pi)^4} \int\int d^4q d^4x'' \frac{e^{iq\cdot(x'-x'')}}{q^2 + \mu^2} \left[ig_0 \overline{\Psi}_{2'}(x'')\gamma_5 \Psi_2(x'')\right], \tag{3.9b}$$

where

$$\overline{\Psi}_{2'}(x'') = \sqrt{\frac{m_p}{E_2' V}} \overline{u}^{(s_2')}(\overrightarrow{p}'_2) e^{-ip_2'\cdot x''},$$

and

$$\Psi_2(x'') = \sqrt{\frac{m_p}{E_2 V}} u^{(s_2)}(\overrightarrow{p}_2) e^{ip_2\cdot x''}$$

are respectively the Dirac wave functions of the scattered proton $\mathbf{P}_2'$ and incident proton $\mathbf{P}_2$.

Having obtained the expression for the scalar propagator $\Delta_F(x - x')$, and for the $\pi^0$−meson field $\Phi_0(x)$, we now proceed to calculate the transition amplitude for p-p scattering. The overall transition amplitude, $S_{fi}^{(1)}$, for p-p scattering is given by

$$S_{fi}^{(1)} = S_{fi}^{(1)}|_D - S_{fi}^{(1)}|_X \tag{3.10}$$

in which $S_{fi}^{(1)}|_D$ and $S_{fi}^{(1)}|_X$ stand for first-order corrections to the transition amplitude corresponding to the direct and exchange scattering respectively. We recall that the first-order transition amplitude, $S_{fi}^{(1)}|_D$, can be written as

$$S_{fi}^{(1)}|_D = \int d^4x' \sqrt{\frac{m_p}{E_1'V}}\overline{u}^{(s_1')}(\overrightarrow{p}'_1)e^{-ip_1'\cdot x'}\,[ig_0\gamma_5\Phi_0(x')]\,\sqrt{\frac{m_p}{E_1V}}u^{(s_1)}(\overrightarrow{p}_1)e^{ip_1\cdot x'}, \tag{3.11}$$

so that with the help of (3.9b), $S_{fi}^{(1)}|_D$ takes the form,

$$\begin{aligned} S_{fi}^{(1)}|_D \;=\; & \frac{-1}{(2\pi)^4}\int\int\int d^4x'd^4x''d^4q\sqrt{\frac{m_p}{E_1'V}}\overline{u}^{(s_1')}(\overrightarrow{p}'_1)e^{-ip_1'\cdot x'}\;(ig_0\gamma_5) \\ & \times\sqrt{\frac{m_p}{E_1V}}u^{(s_1)}(\overrightarrow{p}_1)e^{ip_1\cdot x'}\;\left[\frac{e^{iq(x'-x'')}}{q^2+\mu^2}\right]\sqrt{\frac{m_p}{E_2'V}}\overline{u}^{(s_2')}(\overrightarrow{p}'_2)e^{-ip_2'\cdot x''} \\ & \times(ig_0\gamma_5)\;\sqrt{\frac{m_p}{E_2V}}u^{(s_2)}(\overrightarrow{p}_2)e^{ip_2\cdot x''} \end{aligned} \tag{3.12}$$

Upon integrating over the space-time co-ordinates of the interaction vertices $x'$, $x''$, and the exchanged momentum $q$, one finds

$$\begin{aligned} S_{fi}^{(1)}|_D \;=\; & g_0^2\sqrt{\frac{m_p^4}{E_1'E_1E_2'E_2V^4}}(2\pi)^4\delta^{(4)}\,(p_2'+p_1'-p_2-p_1) \\ & \times\frac{[\,\overline{u}^{(s_1')}(\overrightarrow{p}'_1)\gamma_5u^{(s_1)}(\overrightarrow{p}_1)][\overline{u}^{(s_2')}(\overrightarrow{p}'_2)\gamma_5u^{(s_2)}(\overrightarrow{p}_2)]}{q_D^2+\mu^2}. \end{aligned} \tag{3.13}$$

Making the interchanges

$$p_2' \rightleftharpoons p_1' \;\; and \;\; s_2' \rightleftharpoons s_1'$$

in (3.13), the first-order correction to the transition amplitude, $S_{fi}^{(1)}|_X$, corresponding to the exchange scattering can be written as

$$S_{fi}^{(1)}|_X \;=\; g_0^2\sqrt{\frac{m_p^4}{E_1'E_1E_2'E_2V^4}}(2\pi)^4\delta^{(4)}\,(p_2'+p_1'-p_2-p_1)$$

$$\times \frac{[\,\overline{u}^{(s_2')}(\overrightarrow{p}'_2)\gamma_5 u^{(s_1)}(\overrightarrow{p}_1)][\overline{u}^{(s_1')}(\overrightarrow{p}'_1)\gamma_5 u^{(s_2)}(\overrightarrow{p}_2)]}{q_x^2+\mu^2}. \tag{3.14}$$

Putting (3.14) and (3.13) into (3.10), the expression for the transition amplitude becomes

$$S_{fi}^{(1)} = (2\pi)^4\delta^{(4)}\left(p_2'+p_1'-p_2-p_1\right)\sqrt{\frac{m_p^4}{E_1'E_1E_2'E_2V^4}}M_{fi} \tag{3.15a}$$

where

$$M_{fi} = g_0^2\left\{\begin{array}{c}\frac{[\,\overline{u}^{(s_1')}(\overrightarrow{p}'_1)\gamma_5 u^{(s_1)}(\overrightarrow{p}_1)][\overline{u}^{(s_2')}(\overrightarrow{p}'_2)\gamma_5 u^{(s_2)}(\overrightarrow{p}_2)]}{q_D^2+\mu^2} \\ -\frac{[\,\overline{u}^{(s_2')}(\overrightarrow{p}'_2)\gamma_5 u^{(s_1)}(\overrightarrow{p}_1)][\overline{u}^{(s_1')}(\overrightarrow{p}'_1)\gamma_5 u^{(s_2)}(\overrightarrow{p}_2)]}{q_x^2+\mu^2}\end{array}\right\}. \tag{3.15b}$$

Since the contribution of the exchange scattering is inclded by using antisymmetric wave functions in the non-relativistic quantum mechanics, we take the matrix element, $M_{fi}$, corresponding to the direct scattering [2]. That is,

$$M_{fi} = \frac{g_0^2\,[\,\overline{u}^{(s_1')}(\overrightarrow{p}'_1)\gamma_5 u^{(s_1)}(\overrightarrow{p}_1)][\overline{u}^{(s_2')}(\overrightarrow{p}'_2)\gamma_5 u^{(s_2)}(\overrightarrow{p}_2)]}{q_D^2+\mu^2-i\epsilon}. \tag{3.16}$$

Employing the relation

$$u^{(s)}(\overrightarrow{p}) = \left(\frac{E+m_pc^2}{2m_pc^2}\right)\left(\begin{array}{c}\chi_s \\ \frac{c\overrightarrow{\sigma}\cdot\overrightarrow{p}}{E+m_pc^2}\chi_s\end{array}\right)$$

one finds in the non-relativistic limit that

$$\overline{u}^{(s_1')}(\overrightarrow{p}'_1)\gamma_5 u^{(s_1)}(\overrightarrow{p}_1) = \frac{\overrightarrow{\sigma}^{(1)}\cdot\overrightarrow{q}_1}{2m_p} \tag{3.17}$$

and

$$\overline{u}^{(s_2')}(\overrightarrow{p}'_2)\gamma_5 u^{(s_2)}(\overrightarrow{p}_2) = \frac{-\overrightarrow{\sigma}^{(2)}\cdot\overrightarrow{q}_1}{2m_p}, \tag{3.18}$$

where use has been made of the relation $\overrightarrow{q}_2 = -\overrightarrow{q}_1$. Inserting (3.18) and (3.17) into (3.16) and considering the non-relativistic limit, the matrix element, $M_{fi}$, can be expressed as

$$M_{fi} = \frac{-g_0^2}{(2m_p)^2} \frac{(\vec{\sigma}^{(1)} \cdot \vec{q})(\vec{\sigma}^{(2)} \cdot \vec{q})}{|\vec{q}|^2 + \mu^2}. \tag{3.19}$$

Noting that the low-energy nucleon-nucleon potential is the three-dimensional Fourier transform of the matrix element, $M_{fi}$, the one-pion exchange potential (OPEP)

$$V = \frac{1}{(2\pi)^3} \int d^3\vec{q}\, M_{fi} e^{i\vec{q}\cdot\vec{r}} \tag{3.20a}$$

can be written with the use of (3.19) in the form,

$$V = \frac{-g_0^2}{(2m_p)^2(2\pi)^3} \int d^3\vec{q} \frac{(\vec{\sigma}^{(1)} \cdot \vec{q})(\vec{\sigma}^{(2)} \cdot \vec{q})}{|\vec{q}|^2 + \mu^2} e^{i\vec{q}\cdot\vec{r}} \tag{3.20b}$$

$$= \frac{g_0^2}{(2m_p)^2(2\pi)^3} \int d^3\vec{q} \frac{(\vec{\sigma}^{(1)} \cdot \vec{\nabla})(\vec{\sigma}^{(2)} \cdot \vec{\nabla})}{|\vec{q}|^2 + \mu^2} e^{i\vec{q}\cdot\vec{r}} \tag{3.20c}$$

$$= \frac{g_0^2}{(2m_p)^2(2\pi)^3} (\vec{\sigma}^{(1)} \cdot \vec{\nabla})(\vec{\sigma}^{(2)} \cdot \vec{\nabla}) \int d^3\vec{q} \frac{e^{i\vec{q}\cdot\vec{r}}}{|\vec{q}|^2 + \mu^2}. \tag{3.20d}$$

Now

$$\int d^3\vec{q} \frac{e^{i\vec{q}\cdot\vec{r}}}{|\vec{q}|^2 + \mu^2} = \int\int d\Omega d|\vec{q}||\vec{q}|^2 \frac{e^{i|\vec{q}|r\cos\theta}}{|\vec{q}|^2 + \mu^2} \tag{3.20e}$$

$$= \int d|\vec{q}||\vec{q}|^2 \int\int d(\cos\theta) d\phi \frac{e^{i|\vec{q}|r\cos\theta}}{|\vec{q}|^2 + \mu^2} \tag{3.20f}$$

$$= 2\pi \int_0^\infty d|\vec{q}||\vec{q}|^2 \int_{-1}^{1} d(\cos\theta) \frac{e^{i|\vec{q}|r\cos\theta}}{|\vec{q}|^2 + \mu^2} \tag{3.20g}$$

$$= \frac{4\pi}{r} \int_0^\infty d|\vec{q}| \frac{|\vec{q}|}{|\vec{q}|^2 + \mu^2} \sin|\vec{q}|r. \tag{3.21}$$

Treating $|\vec{q}|$ as a complex variable and integrating (3.21) using Cauchy's theorem results in

$$\int d^3\vec{q} \frac{e^{i\vec{q}\cdot\vec{r}}}{|\vec{q}|^2 + \mu^2} = 2\pi^2 \frac{e^{-\mu r}}{r}. \tag{3.22}$$

Hence relation (3.20d) with the use of (3.22) becomes,

$$V = \frac{g_0^2}{(2m_p)^2 4\pi}(\overrightarrow{\sigma}^{(1)} \cdot \overrightarrow{\nabla})(\overrightarrow{\sigma}^{(2)} \cdot \overrightarrow{\nabla})\frac{e^{-\mu r}}{r} \tag{3.23}$$

As the functions operated on are functions of $r$ only, $\overrightarrow{\nabla}$ can be replaced by $\hat{r}\frac{d}{dr}$.

Thus we have

$$(\overrightarrow{\sigma}^{(1)} \cdot \overrightarrow{\nabla})(\overrightarrow{\sigma}^{(2)} \cdot \overrightarrow{\nabla}) = (\overrightarrow{\sigma}^{(1)} \cdot \hat{r})\frac{d}{dr}(\overrightarrow{\sigma}^{(2)} \cdot \hat{r})\frac{d}{dr} \tag{3.24a}$$

$$= (\overrightarrow{\sigma}^{(1)} \cdot \hat{r})\left[\overrightarrow{\sigma}^{(2)} \cdot \left(\frac{d\hat{r}}{dr}\frac{d}{dr} + \hat{r}\frac{d^2}{dr^2}\right)\right] \tag{3.24b}$$

$$= (\overrightarrow{\sigma}^{(1)} \cdot \hat{r})(\overrightarrow{\sigma}^{(2)} \cdot \frac{d\hat{r}}{dr})\frac{d}{dr} + (\overrightarrow{\sigma}^{(1)} \cdot \hat{r})(\overrightarrow{\sigma}^{(2)} \cdot \hat{r}\frac{d^2}{dr^2}) \tag{3.24c}$$

$$= (\overrightarrow{\sigma}^{(1)} \cdot \hat{r})(\overrightarrow{\sigma}^{(2)} \cdot \hat{r})\frac{d^2}{dr^2} + [(\overrightarrow{\sigma}^{(1)} \cdot \hat{r}\frac{d}{dr})(\overrightarrow{\sigma}^{(2)} \cdot \hat{r})]\frac{d}{dr} \tag{3.24d}$$

$$= (\overrightarrow{\sigma}^{(1)} \cdot \hat{r})(\overrightarrow{\sigma}^{(2)} \cdot \hat{r})\frac{d^2}{dr^2} + [(\overrightarrow{\sigma}^{(1)} \cdot \overrightarrow{\nabla})(\overrightarrow{\sigma}^{(2)} \cdot \hat{r})]\frac{d}{dr}, \tag{3.24e}$$

where $\overrightarrow{\nabla}$ acts only on $\overrightarrow{\sigma}^{(2)} \cdot \hat{r}$. Furthermore, the last term in the bracket on the right-hand side of (3.24e) can be simplified as follows:

$$(\overrightarrow{\sigma}^{(1)} \cdot \overrightarrow{\nabla})(\overrightarrow{\sigma}^{(2)} \cdot \hat{r}) = (\overrightarrow{\sigma}^{(1)} \cdot \frac{\overrightarrow{r}}{r})\frac{d}{dr}(\overrightarrow{\sigma}^{(2)} \cdot \frac{\overrightarrow{r}}{r}) \tag{3.25a}$$

$$= \frac{1}{r}(\overrightarrow{\sigma}^{(1)} \cdot \overrightarrow{r})\left[\overrightarrow{\sigma}^{(2)} \cdot (\frac{1}{r}\frac{d\overrightarrow{r}}{dr} - \frac{\overrightarrow{r}}{r^2})\right] \tag{3.25b}$$

$$= -\frac{1}{r}(\overrightarrow{\sigma}^{(1)} \cdot \overrightarrow{r})(\overrightarrow{\sigma}^{(2)} \cdot \frac{\overrightarrow{r}}{r^2}) + \frac{1}{r}(\overrightarrow{\sigma}^{(1)} \cdot \overrightarrow{r})(\overrightarrow{\sigma}^{(2)} \cdot \frac{1}{r}\frac{d\overrightarrow{r}}{dr}) \tag{3.25c}$$

$$= -\frac{1}{r^3}(\vec{\sigma}^{(1)} \cdot \vec{r})(\vec{\sigma}^{(2)} \cdot \vec{r}) + \frac{1}{r}(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}) \qquad (3.25d)$$

Replacing (3.25d) into (3.24e) gives

$$(\vec{\sigma}^{(1)}.\vec{\nabla})(\vec{\sigma}^{(2)}.\vec{\nabla}) = (\vec{\sigma}^{(1)}.\hat{r})(\vec{\sigma}^{(2)}.\hat{r})\frac{d^2}{dr^2}+\left[\frac{1}{r}(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}) - \frac{1}{r}(\vec{\sigma}^{(1)} \cdot \hat{r})(\vec{\sigma}^{(2)} \cdot \hat{r})\right]\frac{d}{dr} \qquad (3.26a)$$

$$= (\vec{\sigma}^{(1)} \cdot \hat{r})(\vec{\sigma}^{(2)} \cdot \hat{r})\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right] + (\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)})\frac{1}{r}\frac{d}{dr} \qquad (3.26b)$$

$$= \frac{1}{3}\left(S_{12} + \vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}\right)\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right] + (\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)})\frac{1}{r}\frac{d}{dr}, \qquad (3.26c)$$

where

$$S_{12} = 3(\vec{\sigma}^{(1)} \cdot \hat{r})(\vec{\sigma}^{(2)} \cdot \hat{r}) - \vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}$$

is the tensor operator. Using (3.26c) into

$$V = \frac{g_0^2}{(2m_p)^2 4\pi}(\vec{\sigma}^{(1)} \cdot \vec{\nabla})(\vec{\sigma}^{(2)} \cdot \vec{\nabla})\frac{e^{-\mu r}}{r}$$

and carrying out the differentiation gives

$$V = \frac{g_0^2}{4\pi}\left(\frac{\mu}{2m_p}\right)^2\left\{\frac{1}{3}(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}) + \left[\frac{1}{(\mu r)^2} + \frac{1}{\mu r} + \frac{1}{3}\right]S_{12}\right\}\frac{e^{-\mu r}}{r}. \qquad (3.27)$$

Hence, it can be observed that the proton-proton potential due to the exchange of a $\pi^0$−meson contains spin-spin and tensor forces. In other words, we can express the total p-p potential, $V$, as

$$V = V_1(r)\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)} + V_2(r)S_{12}, \qquad (3.28)$$

where

$$V_1(r)\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)} = [\frac{g_0^2}{4\pi}\left(\frac{\mu}{2m_p}\right)^2\frac{1}{3}(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)})]\frac{e^{-\mu r}}{r} \qquad (3.29)$$

and

$$V_2(r)S_{12} = \left[\frac{g_0^2}{4\pi}\left(\frac{\mu}{2m_p}\right)^2\left(\frac{1}{(\mu r)^2}+\frac{1}{\mu r}+\frac{1}{3}\right)S_{12}\right]\frac{e^{-\mu r}}{r}. \tag{3.30}$$

Relation (3.29) represents the spin dependent part of the p-p force, while (3.30) represents the tensor part of the p-p force. We further notice that we have obtained the exact form of Yukawa potential.

A simple argument in terms of the uncertainty principle shows that the range of nuclear forces due to the exchange of one $\pi-$meson appears as $\frac{1}{m_\pi}$. Viewed in this way, the nature of the Yukawa potential gives a short-range ($< 2fm$) to the nuclear force due to the finite mass of the exchanged $\pi-$meson between the two protons. The tensor part of the potential, $V_2(r)S_{12}$ (relation (3.30)), is the non-central part of p-p potential which depends on the orientation of the spins of the two interacting protons with the line joining them.

Thus in this section we have successfully reproduced the precise mathematical form of the long-range (low-energy) part of p-p potential arising due to the exchange of $\pi^0-$meson between the two protons using the field-theoretic approach.

### 3.2 Two-Pion Exchange Potential (TPEP) For p-p Interaction

The approach discussed for OPEP can be extended to compute the contribution of two-pion exchange to p-p potential. To begin with, we consider Feynman diagrams denoting the interaction of two protons through the exchange of two pions as shown in Fig. 3.2 a and Fig. 3.2b.

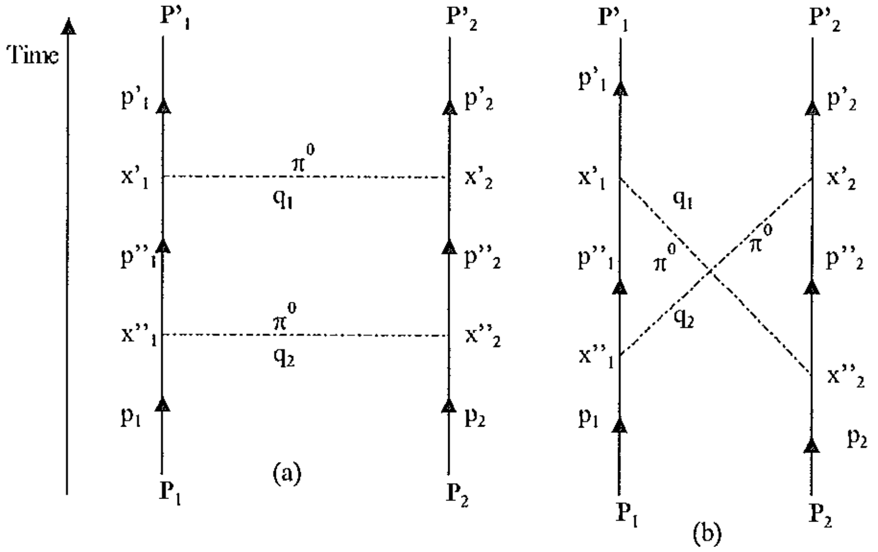


Figure 3.2: *Feynman diagrams for p-p interaction through the exchange of two-pions, (a) Ladder and (b) Crossed diagrams. Here* $x_1'$, $x_1''$, $x_2'$ *and* $x_2''$ *are space-time co-ordinates of the interaction vertices;* $q_1$ *and* $q_2$ *are 4-momenta transferred between the two protons;* $p_1$ *and* $p_2$, $p_1''$ *and* $p_2''$, *and* $p_1'$ and $p_2'$ *are momenta of the incident, intermediate, and scattered protons respectively.*

This interaction involves the exchange of two $\pi-$mesons and thus is described by second-order correction to the transition amplitude, $S_{fi}^{(2)}$. Corresponding to Fig. 3.2a, this amplitude can be written as

$$\begin{aligned} S_{fi}^{(2)}|_a &= \int\int\int\int d^4x_1'' d^4x_1' d^4x_2'' d^4x_2' \left[\overline{\Psi}_{1'}(x_1')(ig_0\gamma_5)\Psi_{1''}(x_1')\right] \left[\overline{\Psi}_{1''}(x_1'')(ig_0\gamma_5)\Psi_1(x_1'')\right] \\ &\quad \times \Delta_F(x_1''-x_2'')\Delta_F(x_1'-x_2') \left[\overline{\Psi}_{2'}(x_2')(ig_0\gamma_5)\Psi_{2''}(x_2')\right] \left[\overline{\Psi}_{2''}(x_2'')(ig_0\gamma_5)\Psi_2(x_2'')\right] \end{aligned}$$

where

$$\Delta_F(x_1''-x_2'') = \frac{-1}{(2\pi)^4}\int d^4q_2 \frac{e^{iq_2\cdot(x_1''-x_2'')}}{q_2^2+\mu^2}$$

and

$$\Delta_F(x_1' - x_2') = \frac{-1}{(2\pi)^4} \int d^4q_1 \frac{e^{iq_1\cdot(x_1'-x_2')}}{q_1^2 + \mu^2}$$

are the usual meson propagators and $\Psi_{1'}$ and $\Psi_{2'}$, $\Psi_{1''}$ and $\Psi_{2''}$, $\Psi_1$ and $\Psi_2$ are Dirac wave functions of scattered, intermediate, and incident protons, respectively.

Using the explicit form of the wave functions and carrying out the integrations over the interactions vertices, $x_1''$, $x_1'$, $x_2''$, $x_2'$ and the exchanged four-momenta $q_1$ and $q_2$, we can write (3.31) as

$$\begin{aligned} S_{fi}^{(2)}|_a &= g_0^4 \sqrt{\frac{m^8}{E_1' E_1''^2 E_1 E_2' E_2''^2 E_2 V^8}} (2\pi)^4 \delta^{(4)}(p_2'' - p_2' - p_1' + p_1'') \\ &\times (2\pi)^4 \delta^{(4)}(p_2 - p_2'' - p_1'' + p_1) \\ &\times \frac{\left[\overline{u}^{(s_1')}(\vec{p}'_1)\gamma_5 u^{(s_1'')}(\vec{p}''_1)\right] \left[\overline{u}^{(s_1'')}(\vec{p}''_1)\gamma_5 u^{(s_1)}(\vec{p}_1)\right]}{q_1^2 + \mu^2} \\ &\times \frac{\left[\overline{u}^{(s_2')}(\vec{p}'_2)\gamma_5 u^{(s_2'')}(\vec{p}''_2)\right] \left[\overline{u}^{(s_2'')}(\vec{p}''_2)\gamma_5 u^{(s_2)}(\vec{p}_2)\right]}{q_2^2 + \mu^2}. \end{aligned} \quad (3.32)$$

Similarly, corresponding to Fig. 3.2b, we have

$$\begin{aligned} S_{fi}^{(2)}|_b &= g_0^4 \sqrt{\frac{m^8}{E_1' E_1''^2 E_1 E_2' E_2''^2 E_2 V^8}} (2\pi)^4 \delta^{(4)}(p_2'' - p_2' + p_1 - p_1'') \\ &\times (2\pi)^4 \delta^{(4)}(p_2 - p_2'' - p_1' + p_1'') \\ &\times \frac{\left[\overline{u}^{(s_1')}(\vec{p}'_1)\gamma_5 u^{(s_1'')}(\vec{p}''_1)\right] \left[\overline{u}^{(s_1'')}(\vec{p}''_1)\gamma_5 u^{(s_1)}(\vec{p}_1)\right]}{q_1^2 + \mu^2} \\ &\times \frac{\left[\overline{u}^{(s_2')}(\vec{p}'_2)\gamma_5 u^{(s_2'')}(\vec{p}''_2)\right] \left[\overline{u}^{(s_2'')}(\vec{p}''_2)\gamma_5 u^{(s_2)}(\vec{p}_2)\right]}{q_2^2 + \mu^2}. \end{aligned} \quad (3.33)$$

The overall transition amplitude for the two diagrams is given by

$$S_{fi}^{(2)} = S_{fi}^{(2)}|_a + S_{fi}^{(2)}|_b \quad (3.34a)$$

$$= g_0^4 \sqrt{\frac{m^8}{E_1' E_1''^2 E_1 E_2' E_2''^2 E_2 V^8}} \frac{\left[\overline{u}^{(s_1')}(\vec{p}'_1)\gamma_5 u^{(s_1'')}(\vec{p}''_1)\right] \left[\overline{u}^{(s_1'')}(\vec{p}''_1)\gamma_5 u^{(s_1)}(\vec{p}_1)\right]}{q_1^2 + \mu^2}$$

$$\times \frac{\left[\overline{u}^{(s_2')}(\vec{p}_2')\gamma_5 u^{(s_2'')}(\vec{p}_2'')\right]\left[\overline{u}^{(s_2'')}(\vec{p}_2'')\gamma_5 u^{(s_2)}(\vec{p}_2)\right]}{q_2^2+\mu^2}$$

$$\times[(2\pi)^4\delta^{(4)}(p_2''-p_2'-p_1'+p_1'')(2\pi)^4\delta^{(4)}(p_2-p_2''-p_1'+p_1)$$

$$+(2\pi)^4\delta^{(4)}(p_2''-p_2'+p_1-p_1'')(2\pi)^4\delta^{(4)}(p_2-p_2''-p_1'+p_1'')]. \tag{3.34b}$$

Now we proceed to obtain the approximate form of the potential corresponding to Fig. 3.2 using the Born approximation on similar lines as we worked out the one-pion exchange potential. We know that,

$$V=\frac{1}{(2\pi)^3}\int d^3\vec{q}\, M_{fi}e^{i\vec{q}\cdot\vec{r}}, \tag{3.34c}$$

where the matrix element $M_{fi}$ is,

$$\begin{aligned} M_{fi} &= g_0^4\frac{\left[\overline{u}^{(s_1')}(\vec{p}_1')\gamma_5 u^{(s_1'')}(\vec{p}_1'')\right]\left[\overline{u}^{(s_1'')}(\vec{p}_1'')\gamma_5 u^{(s_1)}(\vec{p}_1)\right]}{q_1^2+\mu^2} \\ &\times\frac{\left[\overline{u}^{(s_2')}(\vec{p}_2')\gamma_5 u^{(s_2'')}(\vec{p}_2'')\right]\left[\overline{u}^{(s_2'')}(\vec{p}_2'')\gamma_5 u^{(s_2)}(\vec{p}_2)\right]}{q_2^2+\mu^2} \end{aligned} \tag{3.35}$$

which can be written in the non-relativistic limit as

$$M_{fi}=\frac{g_0^4}{(2m)^4}\frac{(\vec{\sigma}^{(1)}\cdot\vec{q}_1)(\vec{\sigma}^{(2)}\cdot\vec{q}_1)}{|\vec{q}_1|^2+\mu^2}\frac{(\vec{\sigma}^{(1)}\cdot\vec{q}_2)(\vec{\sigma}^{(2)}\cdot\vec{q}_2)}{|\vec{q}_2|^2+\mu^2}. \tag{3.36}$$

Putting (3.36) into (3.34c) leads to

$$\begin{aligned} V &= \frac{g_0^4}{(2m)^4}\frac{1}{(2\pi)^3}\int d^3\vec{q}_1\frac{(\vec{\sigma}^{(1)}\cdot\vec{q}_1)(\vec{\sigma}^{(2)}\cdot\vec{q}_1)}{|\vec{q}_1|^2+\mu^2}e^{i\vec{q}_1\cdot\vec{r}_1} \\ &\times\frac{1}{(2\pi)^3}\int d^3\vec{q}_2\frac{(\vec{\sigma}^{(1)}\cdot\vec{q}_2)(\vec{\sigma}^{(2)}\cdot\vec{q}_2)}{|\vec{q}_2|^2+\mu^2}e^{i\vec{q}_2\cdot\vec{r}_2} \end{aligned} \tag{3.37a}$$

or

$$\begin{aligned} V &= \frac{g_0^4}{(2m)^4}\frac{1}{(2\pi)^3}(\vec{\sigma}^{(1)}\cdot\vec{\nabla}_1)(\vec{\sigma}^{(2)}\cdot\vec{\nabla}_1)\int d^3\vec{q}_1\frac{e^{i\vec{q}_1\cdot\vec{r}_1}}{|\vec{q}_1|^2+\mu^2} \\ &\times\frac{1}{(2\pi)^3}(\vec{\sigma}^{(1)}\cdot\vec{\nabla}_2)(\vec{\sigma}^{(2)}\cdot\vec{\nabla}_2)\int d^3\vec{q}_2\frac{e^{i\vec{q}_2\cdot\vec{r}_2}}{|\vec{q}_2|^2+\mu^2}. \end{aligned} \tag{3.37b}$$

Upon integration, the general form of the TPEP can be expressed as

$$V = \frac{g_0^4}{(2m)^4}\left(\frac{\mu^2}{4\pi}\right)^2 f(r, \vec{\sigma}^{(1)}, \vec{\sigma}^{(2)})\frac{e^{-2\mu r}}{r}; \tag{3.38a}$$

where

$$f(r, \vec{\sigma}^{(1)}, \vec{\sigma}^{(2)}) = \left\{\frac{1}{3}(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}) + \left[\frac{1}{(\mu r)^2} + \frac{1}{\mu r} + \frac{1}{3}\right] S_{12}\right\}^2 \tag{3.38b}$$

is a function depending on the separation $r$ between the protons and the spins of the protons. Hence we observe that the approximate form of the two pion-exchange potential worked out using the Born approximation varies as $f(r, \vec{\sigma}^{(1)}, \vec{\sigma}^{(2)})\frac{e^{-2\mu r}}{r}$. However we see that we have correctly predicted the range of the two-pion exchange potential $\sim \frac{1}{2\mu}$ which is much shorter than the range of the one-pion exchange force, where the range $\sim \frac{1}{\mu}$.

### 3.3 Proton-Proton (p-p) Differential Cross-section

In section 3.1, the two possible lowest order Feynman diagrams corresponding to the interaction of two protons mediated by the exchange of one $\pi-$meson were described in Fig. 3.1. Here we wish to compute the differential cross-section for proton-proton scattering in the center-of-mass frame. We know that the differential cross-section, $d\sigma$, is related to the transition probability, $|S_{fi}^{(1)}|^2$, by (2.39) as,

$$d\sigma = \frac{|S_{fi}^{(1)}|^2}{T|\vec{j}_{inc}|}\frac{Vd^3\vec{p}'_1}{(2\pi)^3}\frac{Vd^3\vec{p}'_2}{(2\pi)^3} \tag{3.39a}$$

$$= \frac{|S_{fi}^{(1)}|^2}{T\frac{(2\beta)}{V}}\frac{Vd^3\vec{p}'_1}{(2\pi)^3}\frac{Vd^3\vec{p}'_2}{(2\pi)^3} \tag{3.39b}$$

$$= \frac{|S_{fi}^{(1)}|^2 V}{2T} \frac{V d^3 \vec{p}'_1}{(2\pi)^3} \frac{V d^3 \vec{p}'_2}{(2\pi)^3}, \tag{3.39c}$$

where the value of the incident flux, $|\vec{j}_{inc}|$, in the center-of-mass frame has been employed.

The transition probability, $|S_{fi}^{(1)}|^2$ for the proton-proton scattering can be obtained from (3.15a) as,

$$|S_{fi}^{(1)}|^2 = g_0^4 [(2\pi)^4 \delta^{(4)} (p'_2 + p'_1 - p_2 - p_1)]^2 \frac{m_p^4}{E'_1 E_1 E'_2 E_2 V^4} |M'_{fi}|^2, \tag{3.40}$$

in which

$$M'_{fi} = \left\{ \begin{array}{c} \frac{[\,\overline{u}^{(s'_1)}(\vec{p}'_1)\gamma_5 u^{(s_1)}(\vec{p}_1)][\overline{u}^{(s'_2)}(\vec{p}'_2)\gamma_5 u^{(s_2)}(\vec{p}_2)]}{q_D^2+\mu^2} \\ -\frac{[\,\overline{u}^{(s'_2)}(\vec{p}'_2)\gamma_5 u^{(s_1)}(\vec{p}_1)][\overline{u}^{(s'_1)}(\vec{p}'_1)\gamma_5 u^{(s_2)}(\vec{p}_2)]}{q_x^2+\mu^2} \end{array} \right\}. \tag{3.41}$$

Employing

$$[(2\pi)^4 \delta^{(4)} (p'_2 + p'_1 - p_2 - p_1)]^2 = VT (2\pi)^4 \delta^{(4)} (p'_2 + p'_1 - p_2 - p_1),$$

the expression for the transition probability can be written as

$$|S_{fi}^{(1)}|^2 = g_0^4 (2\pi)^4 \delta^{(4)} (p'_2 + p'_1 - p_2 - p_1)\, VT \frac{m^4}{E'_1 E_1 E'_2 E_2 V^4} |M'_{fi}|^2. \tag{3.42}$$

Substituting (3.42) into (3.39c) leads to

$$d\sigma = g_0^4 \frac{m^4}{2E'_1 E_1 E'_2 E_2} (2\pi)^4 \delta^{(4)}(p'_2 + p'_1 - p_1 - p_2) |M'_{fi}|^2 \frac{d^3 \vec{p}'_1}{(2\pi)^3} \frac{d^3 \vec{p}'_2}{(2\pi)^3}. \tag{3.43}$$

Using $d^3\vec{p}'_1 = |\vec{p}'_1|^2 d|\vec{p}'_1| d\Omega = |\vec{p}'_1| E'_1 dE'_1 d\Omega$, the total cross-section $\sigma$ is thus given by

$$\sigma = g_0^4 \frac{m_p^4 |\vec{p}'_1|}{2E_1 E'_2 E_2} \frac{|M'_{fi}|^2}{(2\pi)^2} \int d\Omega \int dE'_1 \int d^3\vec{p}'_2 \delta^{(4)}(p'_2 + p'_1 - p_1 - p_2) \tag{3.44a}$$

Integrating over $\overrightarrow{p}'_2$ gives us,

$$\frac{d\sigma}{d\Omega} = g_0^4 \frac{m^4 |\overrightarrow{p}'_1|}{2E_1 E'_2 E_2} \frac{|M'_{fi}|^2}{(2\pi)^2} \int dE'_1 \delta(E'_2 + E'_1 - E_1 - E_2). \tag{3.44b}$$

Noting that $E_1 = E_2$ and $E'_1 = E'_2$ in the center-of-mass frame and $|\overrightarrow{p}'_1| \simeq E'_1$ in the ultra-relativistic limit, and carrying out the integration, relation (3.44b) can be written as,

$$\frac{d\sigma}{d\Omega} = g_0^4 \frac{m^4 |M'_{fi}|^2}{4E^2 (2\pi)^2}. \tag{3.44c}$$

If the incident protons are not polarized and furthermore if the polarizations of the final state protons are not observed [11], then we sum over the final spin states and average over the initial spin states in computing $|M'_{fi}|^2$. Thus

$$|M'_{fi}|^2 = \frac{1}{4} \sum_{spins} \left| \frac{[\overline{u}^{(s'_1)}(\overrightarrow{p}'_1)\gamma_5 u^{(s_1)}(\overrightarrow{p}_1)][\overline{u}^{(s'_2)}(\overrightarrow{p}'_2)\gamma_5 u^{(s_2)}(\overrightarrow{p}_2)]}{q_D^2 + \mu^2} - \frac{[\overline{u}^{(s'_2)}(\overrightarrow{p}'_2)\gamma_5 u^{(s_1)}(\overrightarrow{p}_1)][\overline{u}^{(s'_1)}(\overrightarrow{p}'_1)\gamma_5 u^{(s_2)}(\overrightarrow{p}_2)]}{q_x^2 + \mu^2} \right|^2 \tag{3.45a}$$

$$= \frac{1}{4} \sum_{spins} \left[ \frac{[\overline{u}^{(s'_1)}(\overrightarrow{p}'_1)\gamma_5 u^{(s_1)}(\overrightarrow{p}_1)][\overline{u}^{(s'_2)}(\overrightarrow{p}'_2)\gamma_5 u^{(s_2)}(\overrightarrow{p}_2)]}{q_D^2 + \mu^2} - \frac{[\overline{u}^{(s'_2)}(\overrightarrow{p}'_2)\gamma_5 u^{(s_1)}(\overrightarrow{p}_1)][\overline{u}^{(s'_1)}(\overrightarrow{p}'_1)\gamma_5 u^{(s_2)}(\overrightarrow{p}_2)]}{q_x^2 + \mu^2} \right]^\dagger$$
$$\times \left[ \frac{[\overline{u}^{(s'_1)}(\overrightarrow{p}'_1)\gamma_5 u^{(s_1)}(\overrightarrow{p}_1)][\overline{u}^{(s'_2)}(\overrightarrow{p}'_2)\gamma_5 u^{(s_2)}(\overrightarrow{p}_2)]}{q_D^2 + \mu^2} - \frac{[\overline{u}^{(s'_2)}(\overrightarrow{p}'_2)\gamma_5 u^{(s_1)}(\overrightarrow{p}_1)][\overline{u}^{(s'_1)}(\overrightarrow{p}'_1)\gamma_5 u^{(s_2)}(\overrightarrow{p}_2)]}{q_x^2 + \mu^2} \right] \tag{3.45b}$$

$$= \frac{1}{4} \sum_{spins} \left[ \frac{[\overline{u}^{(s_2)}(\overrightarrow{p}_2)\gamma_5 u^{(s'_2)}(\overrightarrow{p}'_2)][\overline{u}^{(s_1)}(\overrightarrow{p}_1)\gamma_5 u^{(s'_1)}(\overrightarrow{p}'_1)]}{q_D^2 + \mu^2} - \frac{[\overline{u}^{(s_2)}(\overrightarrow{p}_2)\gamma_5 u^{(s'_1)}(\overrightarrow{p}'_1)][\overline{u}^{(s_1)}(\overrightarrow{p}_1)\gamma_5 u^{(s'_2)}(\overrightarrow{p}'_2)]}{q_x^2 + \mu^2} \right]$$
$$\times \left[ \frac{[\overline{u}^{(s'_1)}(\overrightarrow{p}'_1)\gamma_5 u^{(s_1)}(\overrightarrow{p}_1)][\overline{u}^{(s'_2)}(\overrightarrow{p}'_2)\gamma_5 u^{(s_2)}(\overrightarrow{p}_2)]}{q_D^2 + \mu^2} - \frac{[\overline{u}^{(s'_2)}(\overrightarrow{p}'_2)\gamma_5 u^{(s_1)}(\overrightarrow{p}_1)][\overline{u}^{(s'_1)}(\overrightarrow{p}'_1)\gamma_5 u^{(s_2)}(\overrightarrow{p}_2)]}{q_x^2 + \mu^2} \right] \tag{3.45c}$$

$$|M'_{fi}|^2 = \frac{1}{4}\left[\frac{I_1}{(q_D^2+\mu^2)^2} + \frac{I_2}{(q_x^2+\mu^2)^2} - \frac{I_3}{(q_D^2+\mu^2)(q_x^2+\mu^2)} - \frac{I_4}{(q_D^2+\mu^2)(q_x^2+\mu^2)}\right], \tag{3.45d}$$

in which

$$\begin{aligned} I_1 &= \sum_{spins} [\,\overline{u}^{(s_2)}(\overrightarrow{p}_2)\gamma_5 u^{(s_2')}(\overrightarrow{p}'_2)][\overline{u}^{(s_2')}(\overrightarrow{p}'_2)\gamma_5 u^{(s_2)}(\overrightarrow{p}_2)] \\ &\quad \times[\overline{u}^{(s_1)}(\overrightarrow{p}_1)\gamma_5 u^{(s_1')}(\overrightarrow{p}'_1)][\overline{u}^{(s_1')}(\overrightarrow{p}'_1)\gamma_5 u^{(s_1)}(\overrightarrow{p}_1)], \end{aligned} \tag{3.45e}$$

$$\begin{aligned} I_2 &= \sum_{spins}[\overline{u}^{(s_2)}(\overrightarrow{p}_2)\gamma_5 u^{(s_1')}(\overrightarrow{p}'_1)][\overline{u}^{(s_1')}(\overrightarrow{p}'_1)\gamma_5 u^{(s_2)}(\overrightarrow{p}_2)] \\ &\quad \times[\overline{u}^{(s_2')}(\overrightarrow{p}'_2)\gamma_5 u^{(s_1)}(\overrightarrow{p}_1)],[\overline{u}^{(s_1)}(\overrightarrow{p}_1)\gamma_5 u^{(s_2')}(\overrightarrow{p}'_2)] \end{aligned} \tag{3.45f}$$

$$\begin{aligned} I_3 &= \sum_{spins} [\,\overline{u}^{(s_2)}(\overrightarrow{p}_2)\gamma_5 u^{(s_2')}(\overrightarrow{p}'_2)][\overline{u}^{(s_2')}(\overrightarrow{p}'_2)\gamma_5 u^{(s_1)}(\overrightarrow{p}_1)] \\ &\quad \times[\overline{u}^{(s_1)}(\overrightarrow{p}_1)\gamma_5 u^{(s_1')}(\overrightarrow{p}'_1)][\overline{u}^{(s_1')}(\overrightarrow{p}'_1)\gamma_5 u^{(s_2)}(\overrightarrow{p}_2)], \end{aligned} \tag{3.45g}$$

and

$$\begin{aligned} I_4 &= \sum_{spins}[\overline{u}^{(s_2)}(\overrightarrow{p}_2)\gamma_5 u^{(s_1')}(\overrightarrow{p}'_1)][\overline{u}^{(s_1')}(\overrightarrow{p}'_1)\gamma_5 u^{(s_1)}(\overrightarrow{p}_1)] \\ &\quad \times[\overline{u}^{(s_1)}(\overrightarrow{p}_1)\gamma_5\; u^{(s_2')}(\overrightarrow{p}'_2)][\overline{u}^{(s_2')}(\overrightarrow{p}'_2)\gamma_5 u^{(s_2)}(\overrightarrow{p}_2)]. \end{aligned} \tag{3.45h}$$

The values of $I_1, I_2, I_3$ and $I_4$ can be computed using trace technique along with the relation

$$\sum_{s=1}^{2} u^{(s)}(\vec{p})\overline{u}^{(s)}(\vec{p}) = \frac{-i\gamma\cdot p + m}{2m} = \frac{-i\widetilde{p}+m}{2m},$$

where $\widetilde{p} \equiv \gamma\cdot p$. It then follows that

$$I_1 = Tr\left[\left(\frac{-i\widetilde{p}_2+m}{2m}\right)\gamma_5\left(\frac{-i\widetilde{p}'_2+m}{2m}\right)\gamma_5\right] Tr\left[\left(\frac{-i\widetilde{p}_1+m}{2m}\right)\gamma_5\left(\frac{-i\widetilde{p}'_1+m}{2m}\right)\gamma_5\right] \tag{3.46a}$$

$$= \frac{1}{16m^4} Tr\left[-\widetilde{p}_2\gamma_5\widetilde{p}'_2\gamma_5 + m^2\right] Tr\left[-\widetilde{p}_1\gamma_5\widetilde{p}'_1\gamma_5 + m^2\right] \tag{3.46b}$$

$$= \frac{1}{m^4}\left\{(p_2\cdot p_2')(p_1\cdot p_1') + m^2[p_2\cdot p_2' + p_1\cdot p_1'] + m^4\right\}. \qquad (3.46c)$$

Following similar procedure, one can readily verify that

$$I_2 = \frac{1}{m^4}\left\{(p_2\cdot p_1')(p_1\cdot p_2') + m^2[p_2\cdot p_1' + p_1\cdot p_2'] + m^4\right\}, \qquad (3.47)$$

$$I_3 = \frac{1}{4m^4}\left\{\begin{array}{c}(p_2\cdot p_2')(p_1\cdot p_1') + (p_2\cdot p_1')(p_1\cdot p_2') - (p_2\cdot p_1)(p_1'\cdot p_2') \\ +m^2\left[p_2\cdot p_2' - p_2\cdot p_1 + p_2\cdot p_1' + p_1\cdot p_2' - p_1'\cdot p_2' + p_1\cdot p_1'\right] + m^4\end{array}\right\}, \qquad (3.48)$$

and

$$I_4 = \frac{1}{4m^4}\left\{\begin{array}{c}(p_2\cdot p_2')(p_1\cdot p_1') + (p_2\cdot p_1')(p_1\cdot p_2') - (p_2\cdot p_1)(p_1'\cdot p_2') \\ +m^2\left[p_2\cdot p_2' - p_2\cdot p_1 + p_2\cdot p_1' + p_1\cdot p_2' - p_1'\cdot p_2' + p_1\cdot p_1'\right] + m^4\end{array}\right\}. \qquad (3.49)$$

In the center-of-mass frame, the momenta of the incident and scattered protons are given by

$$p_1 = (\overrightarrow{p}_1, iE_1), \qquad (3.50a)$$

$$p_2 = (-\overrightarrow{p}_1, iE_2), \qquad (3.50b)$$

$$p_1' = (\overrightarrow{p}'_1, iE_1'), \qquad (3.50c)$$

$$p_2' = (-\overrightarrow{p}'_1, iE_2'), \qquad (3.50d)$$

and

$$E_1 = E_1' = E_2 = E_2' = E. \qquad (3.50e)$$

Using relations (3.50a-e) together with $|\overrightarrow{p}'_1| \simeq E_1'$ and $|\overrightarrow{p}_1| \simeq E_1$, the expressions for $I_1, I_2, I_3$, and $I_4$ can be written respectively as

$$I_1 = \frac{1}{m^4}\left[4E^4\sin^4\frac{\theta}{2} - 4m^2E^2\sin^2\frac{\theta}{2} + m^4\right], \qquad (3.51)$$

$$I_2 = \frac{1}{m^4}\left[4E^4\cos^4\frac{\theta}{2} - 4m^2E^2\cos^2\frac{\theta}{2} + m^4\right], \tag{3.52}$$

and

$$I_3 = I_4 = 4. \tag{3.53}$$

Recalling that

$$q_D = p_1 - p_1' \tag{3.54}$$

and

$$q_x = p_1 - p_2', \tag{3.55}$$

we find that

$$q_D^2 = 4E_1E_1'\sin^2\frac{\theta}{2} \tag{3.56}$$

and

$$q_x^2 = 4E_1E_1'\cos^2\frac{\theta}{2}. \tag{3.57}$$

Making use of the relations (3.51), (3.52), (3.53), (3.56), and (3.57) into the expression of the square of the matrix element $M'_{fi}$,(Equation 3.45d), one obtains

$$|M'_{fi}|^2 = \frac{1}{4}\left\{\begin{array}{c} \frac{1}{m^4}\left[\frac{4E^4\sin^4\frac{\theta}{2} - 4m^2E^2\sin^2\frac{\theta}{2} + m^4}{16E^4\sin^4\frac{\theta}{2} + 8E^2\mu^2\sin^2\frac{\theta}{2} + \mu^4}\right] + \frac{1}{m^4}\left[\frac{4E^4\cos^4\frac{\theta}{2} - 4m^2E^2\cos^2\frac{\theta}{2} + m^4}{16E^4\cos^4\frac{\theta}{2} + 8E^2\mu^2\cos^2\frac{\theta}{2} + \mu^4}\right] \\ -\frac{8}{(4E^2\sin^2\frac{\theta}{2} + \mu^2)(4E^2\cos^2\frac{\theta}{2} + \mu^2)} \end{array}\right\}. \tag{3.58}$$

Using (3.58) into (3.44d), the expression for the differential cross-section can be written as

$$\frac{d\sigma}{d\Omega} = \frac{g_0^4}{64E^2\pi^2}\left\{\begin{array}{c} \frac{4m^2E\sin^2\frac{\theta}{2} - 4E^4\sin^4\frac{\theta}{2} - m^4}{16E^4\sin^4\frac{\theta}{2} + 8E^2\mu^2\sin^2\frac{\theta}{2} + \mu^4} + \frac{4m^2E^2\cos^2\frac{\theta}{2} - 4E^4\cos^4\frac{\theta}{2} - m^4}{16E^4\cos^4\frac{\theta}{2} + 8E^2\mu^2\cos^2\frac{\theta}{2} + \mu^4} \\ +\frac{8m^4}{\left[4E^4\sin^2\frac{\theta}{2} + \mu^2\right]\left[4E^2\cos^2\frac{\theta}{2} + \mu^2\right]} \end{array}\right\}. \tag{3.59}$$

Here we observe that the differential scattering cross-section can be measured as a function of the scattering angle for a given value of the incident energy, E.

CHAPTER 4

# Nucleon-Nucleon (N-N) Interaction

The nucleon-nucleon interaction is a basic problem of nuclear and particle physics. In nuclear physics, a clear understanding of the nucleon-nucleon force leads to a better understanding of the properties of nuclei. And in particle physics, the nucleon-nucleon interaction is intimately related to an understanding of the structure of hadrons [3, 12]. The nature of nuclear forces is best revealed in the interactions between nucleons.

This chapter is organized as follows: Section 4.1 is aimed at determining the connection between the coupling constants of neutral and charged $\pi-$mesons with nucleon by studying proton-neutron scattering. In section 4.2, we introduce the concept of isospin to describe the charge-independence of nuclear forces. Finally, the long-range part of nucleon-nucleon potential will then be derived.

## 4.1 Proton-Neutron (p-n) Scattering

In this section, we consider the interaction of a proton and a neutron. This process is more complicated than the p-p scattering process considered in the previous chapter since protons and neutrons interact through the exchange of both neutral ($\pi^0-$) and charged ($\pi^+, \pi^-$) mesons. These two processes are shown in Fig. 4.1a and Fig. 4.1b by the Feynman diagrams shown below.

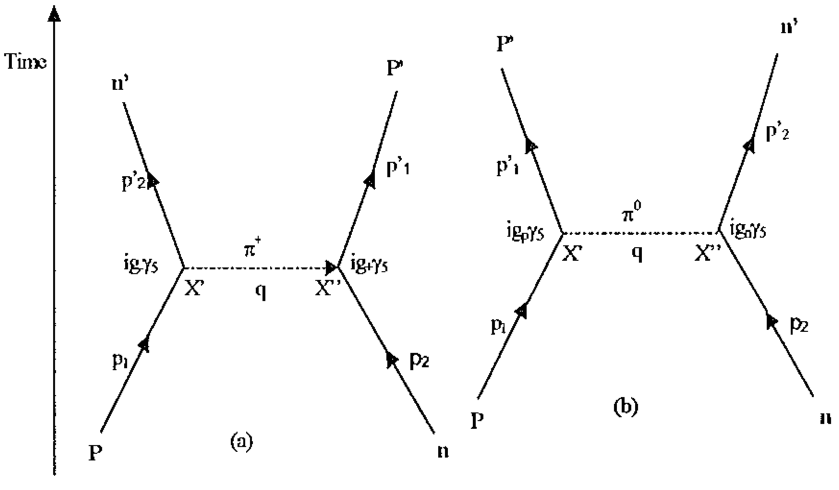


Figure 4.1: *Feynman diagrams for* p-n *scattering through the exchange of* (a) *charged* $\pi-$*meson and* (b) *neutral* $\pi-$*meson: Here* $g_- = g_+$ *is coupling constant between charged* $\pi-$*meson and nucleon, whereas* $g_p$ *is the coupling constant between neutral* $\pi-$*meson and proton,* $g_n$ *is coupling constant between neutral* $\pi^0-$*meson and neutron.*

From the requirement of charge independence of nucleon-nucleon force, we note that the interaction between proton and $\pi^-$−meson field is the same as that between neutron and $\pi^+$−meson field. Hence the coupling constant between proton and $\pi^-$−meson field, and between neutron and $\pi^+$−meson field are the same, that is, $g_- = g_+$. If both charged and neutral mesons are to be involved in the production of nuclear forces there must be a consequent relationship between the coupling strengths of neutral and charged mesons with nucleons [14]. Thus we first study the relationship between $g_+$ and $g_0$. To this end, we compare the first-order transition amplitudes for p-n scattering and p-p scattering on the account of charge independence of nuclear forces, which tells us that for a given value of isospin $\overrightarrow{I}$, $V_{p-p} = V_{n-n} = V_{n-p}$ (i.e. the nuclear potentials between two protons, between two neutrons, and between a proton and a neutron are the same.).

The Dirac equations of a proton and a neutron and Klein-Gordon equation for pion fields in the case of p-n scattering can be obtained employing the total Lagrangian density $\mathcal{L}$ for this process, which is given by

$$\mathcal{L} = \mathcal{L}_{Dirac} + \mathcal{L}_{scalar} + \mathcal{L}_{int} \tag{4.1}$$

where

$$\mathcal{L}_{Dirac} = \overline{\Psi}_p(\gamma_\mu \frac{\partial}{\partial x_\mu} + m)\Psi_p + \overline{\Psi}_n(\gamma_\mu \frac{\partial}{\partial x_\mu} + m)\Psi_n \tag{4.2}$$

is the Lagrangian density for free Dirac fields,

$$\mathcal{L}_{scalar} = -\frac{1}{2}\left(\frac{\partial \Phi_0}{\partial x_\mu}\frac{\partial \Phi_0}{\partial x_\mu} + \mu^2\Phi_0^2\right) - \left(\frac{\partial \Phi_-}{\partial x_\mu}\frac{\partial \Phi_+}{\partial x_\mu} + \mu^2\Phi_-\Phi_+\right) \tag{4.3}$$

is the Lagrangian density for free scalar fields (neutral and charged scalar), and

$$\mathcal{L}_{int} = -ig_0\overline{\Psi}_p\gamma_5\Psi_p\Phi_0 + ig_0\overline{\Psi}_n\gamma_5\Psi_n\Phi_0 - ig_+\overline{\Psi}_n\gamma_5\Psi_p\Phi_- - ig_+\overline{\Psi}_p\gamma_5\Psi_n\Phi_+, \tag{4.4a}$$

is the interaction Lagrangian density. We put $\mathcal{L}$ into Euler-Lagrange equations in the generalized coordinates $\overline{\Psi}_p$, $\overline{\Psi}_n$, $\Phi_-$, $\Phi_+$ and $\Phi_0$,

$$\frac{\partial}{\partial x_\mu}\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \overline{\Psi}_p}{\partial x_\mu}\right)}\right] - \frac{\partial \mathcal{L}}{\partial \overline{\Psi}_p} = 0, \tag{4.4b}$$

$$\frac{\partial}{\partial x_\mu}\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \overline{\Psi}_n}{\partial x_\mu}\right)}\right] - \frac{\partial \mathcal{L}}{\partial \overline{\Psi}_n} = 0, \tag{4.4c}$$

$$\frac{\partial}{\partial x_\mu}\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \Phi_-}{\partial x_\mu}\right)}\right] - \frac{\partial \mathcal{L}}{\partial \Phi_-} = 0, \tag{4.4d}$$

$$\frac{\partial}{\partial x_\mu}\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \Phi_+}{\partial x_\mu}\right)}\right] - \frac{\partial \mathcal{L}}{\partial \Phi_+} = 0, \tag{4.4e}$$

and

$$\frac{\partial}{\partial x_\mu}\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \Phi_0}{\partial x_\mu}\right)}\right] - \frac{\partial \mathcal{L}}{\partial \Phi_0} = 0, \tag{4.4f}$$

respectively. Here, $\Phi_- = \Phi_+^\dagger$. This leads to Dirac equation of a proton and a neutron respectively as

$$(\gamma_\mu \frac{\partial}{\partial x_\mu} + m_p)\Psi_p = ig_0\gamma_5\Psi_p\Phi_0 + ig_+\gamma_5\Psi_n\Phi_+ \tag{4.5}$$

$$(\gamma_\mu \frac{\partial}{\partial x_\mu} + m_n)\Psi_n = -ig_0\gamma_5\Psi_n\Phi_0 + ig_+\gamma_5\Psi_p\Phi_-; \tag{4.6}$$

and Klein-Gordon equations for $\Phi_+$, $\Phi_-$ and $\Phi_0$ fields as,

$$(\Box^2 - \mu^2)\Phi_+(x) = ig_+\overline{\Psi}_n(x)\gamma_5\Psi_p(x), \tag{4.7}$$

$$(\Box^2 - \mu^2)\Phi_-(x) = ig_+\overline{\Psi}_p(x)\gamma_5\Psi_n(x) \tag{4.8}$$

and

$$(\Box^2 - \mu^2)\Phi_0(x) = ig_0\overline{\Psi}_p(x)\gamma_5\Psi_p(x) - ig_0\overline{\Psi}_n(x)\gamma_5\Psi_n(x). \tag{4.9}$$

respectively. Here, $\Phi_0$, $\Phi_+$ and $\Phi_+^\dagger$ are $\pi^0-$meson, $\pi^+-$meson and $\pi^--$meson fields; $g_0$ and $g_+$ are coupling constants between a nucleon and neutral and charged $\pi-$mesons, respectively.

From the Feynman diagrams shown in Fig. 4.1, expressing the lowest-order possible p-n scattering, the first-order transition amplitude is given by

$$S_{fi}^{(1)}|_{p-n} = S_{fi}^{(1)}|_a + S_{fi}^{(1)}|_b \tag{4.10}$$

where $S_{fi}^{(1)}|_a$ and $S_{fi}^{(1)}|_b$ are, respectively, the transition amplitudes corresponding to Fig. 4.1a and Fig. 4.1b.

Let $p_1$ and $p_1'$ be the 4-momenta of the incident and scattered protons respectively, and $p_2$ and $p_2'$ be the 4-momenta of the incident and scattered neutrons. Then

$$S_{fi}^{(1)}|_a = \int\int d^4x' d^4x'' \overline{\Psi}_{p_2'}(x')(ig_+\gamma_5)\Psi_{p_1}(x') \triangle_F (x' - x'')\overline{\Psi}_{p_1'}(x'')(ig_+\gamma_5)\Psi_{p_2}(x'') \tag{4.11}$$

and

$$S_{fi}^{(1)}|_b = \int\int d^4x' d^4x'' \overline{\Psi}_{p_1'}(x')(ig_0\gamma_5)\Psi_{p_1}(x') \triangle_F (x' - x'')\overline{\Psi}_{p_2'}(x'')(-ig_0\gamma_5)\Psi_{p_2}(x''), \tag{4.12}$$

where

$$\triangle_F(x' - x'') = \frac{-1}{(2\pi)^4}\int d^4q \frac{e^{iq\cdot(x'-x'')}}{q^2 + \mu^2 - i\epsilon}. \tag{4.13}$$

Substituting (4.12) and (4.11) into (4.10) leads to

$$S_{fi}^{(1)}|_{D(p-n)} = (ig_+)^2 \int\int d^4x' d^4x'' \overline{\Psi}_{p_2'}(x')(\gamma_5)\Psi_{p_1}(x')\triangle_F(x'-x'')\overline{\Psi}_{p_1'}(x'')(\gamma_5)\Psi_{p_2}(x'')$$

$$-(ig_0)^2 \int\int d^4x' d^4x'' \overline{\Psi}_{p_1'}(x')(\gamma_5)\Psi_{p_1}(x') \triangle_F (x' - x'')\overline{\Psi}_{p_2'}(x'')(\gamma_5)\Psi_{p_2}(x''). \tag{4.14}$$

As far as strong interactions are concerned, protons and neutrons are identical fermions. Viewed in this way, we can have the exchange diagrams corresponding to each of the diagrams in Fig. 4.1. As a result, the transition amplitude for the exchange diagrams can be written as

$$S_{fi}^{(1)}|_{X(p-n)} = (ig_+)^2 \int\int d^4x' d^4x'' \overline{\Psi}_{p_1'}(x')(\gamma_5)\Psi_{p_1}(x')\triangle_F(x'-x'')\overline{\Psi}_{p_2'}(x'')(\gamma_5)\Psi_{p_2}(x'')$$

$$-(ig_0)^2 \int\int d^4x' d^4x'' \overline{\Psi}_{p_2'}(x')(\gamma_5)\Psi_{p_1}(x') \triangle_F (x' - x'')\overline{\Psi}_{p_1'}(x'')(\gamma_5)\Psi_{p_2}(x''). \tag{4.15}$$

Consequently, the overall transition amplitude for p-n scattering can be written using relations (4.14) and (4.15) as follows:

$$S_{fi}^{(1)}|_{p-n} = S_{fi}^{(1)}|_{D(p-n)} - S_{fi}^{(1)}|_{X(p-n)} \tag{4.16}$$

$$= (ig_+)^2 \int\int d^4x' d^4x'' \overline{\Psi}_{p_2'}(x')(\gamma_5)\Psi_{p_1}(x') \bigtriangleup_F (x' - x'')\overline{\Psi}_{p_1'}(x'')(\gamma_5)\Psi_{p_2}(x'')$$

$$-(ig_0)^2 \int\int d^4x' d^4x'' \overline{\Psi}_{p_1'}(x')(\gamma_5)\Psi_{p_1}(x') \bigtriangleup_F (x' - x'')\overline{\Psi}_{p_2'}(x'')(\gamma_5)\Psi_{p_2}(x'')$$

$$-(ig_+)^2 \int\int d^4x' d^4x'' \overline{\Psi}_{p_1'}(x')(\gamma_5)\Psi_{p_1}(x') \bigtriangleup_F (x' - x'')\overline{\Psi}_{p_2'}(x'')(\gamma_5)\Psi_{p_2}(x'')$$

$$+(ig_0)^2 \int\int d^4x' d^4x'' \overline{\Psi}_{p_2'}(x')(\gamma_5)\Psi_{p_1}(x') \bigtriangleup_F (x' - x'')\overline{\Psi}_{p_1'}(x'')(\gamma_5)\Psi_{p_2}(x''). \tag{4.17}$$

We recall from section 3.1 that the lowest-order Feynman diagrams for p-p scattering looks as shown in Fig. 3.1. Then the transition amplitude for this process can be written as

$$S_{fi}^{(1)}|_{p-p} = (ig_0)^2 \int\int d^4x' d^4x'' \overline{\Psi}_{p_1'}(x')(\gamma_5)\Psi_{p_1}(x') \bigtriangleup_F (x' - x'')\overline{\Psi}_{p_2'}(x'')(\gamma_5)\Psi_{p_2}(x'')$$

$$- (ig_0)^2 \int\int d^4x' d^4x'' \overline{\Psi}_{p_2'}(x')(\gamma_5)\Psi_{p_1}(x') \bigtriangleup_F (x' - x'')\overline{\Psi}_{p_1'}(x'')(\gamma_5)\Psi_{p_2}(x'') \tag{4.18}$$

It is known from experiments that nuclear force between two protons is the same as that between a proton and a neutron. Thus equating (4.18) and (4.17) yields

$$(ig_0)^2 = -(ig_0)^2 - (ig_+)^2, \tag{4.19}$$

or

$$g_+^2 = -2g_0^2. \tag{4.20}$$

However the coupling constant $g_+$ has to be real. Thus we take

$$g_+ = \sqrt{2} g_0. \tag{4.21}$$

This is also supported by $QCD$ calculations and experiments. The arbitrariness in the sign of $g_+^2$ in (4.20) arises due to the effective nature of meson theory which is not gauge theory in the sense of $QCD$.

Once the relation between the coupling constants of nucleon with charged and neutral $\pi-$meson is known, we can deal with nucleon-nucleon scattering regardless of the charge of the interacting nucleons and the charge of the exchanged $\pi-$mesons.

### 4.2 Isospin Formulation

The charge independence of nuclear force can be further investigated using the concept of isospin. This section is, therefore, devoted to introducing this concept. The neutron and proton have identical spin and almost the same mass. Moreover, they were found to behave similarly in nuclear reactions. And the binding energies of mirror nuclei were observed to be the same. Motivated by this evidence, Heisenberg suggested that protons and neutrons can be regarded as two states of a single particle called a nucleon. To quantify his suggestion, Heisenberg introduced an internal degree of freedom, the isospin $\overrightarrow{I}$ [9, 10, 12] analogous to the ordinary intrinsic spin $\overrightarrow{S}$.

Analogous to the case that an electron has spin $\overrightarrow{S} = \overrightarrow{\frac{1}{2}}$ in ordinary spin space, the nucleon is said to have isospin $\overrightarrow{I} = \overrightarrow{\frac{1}{2}}$ in the fictitious isospin space. The third component, $I_3$, of the isospin specifies whether a nucleon is a proton or a neutron. Just as spin up state and spin down state of an electron are described by $|\ \frac{1}{2},+\frac{1}{2}\ \rangle$ and $|\ \frac{1}{2},-\frac{1}{2}\ \rangle$ respectively in the spin space, the proton state $|p\rangle$ and neutron state $|n\rangle$ of a nucleon can be described respectively in the isospin space as,

$$|p\rangle \equiv |\frac{1}{2},+\frac{1}{2}\rangle \ and\ |n\rangle \equiv |\frac{1}{2},-\frac{1}{2}\rangle,$$

with $I_3 = +\frac{1}{2}$ for a proton state and $I_3 = -\frac{1}{2}$ for a neutron state.

Let us have a look at the isospin states of a system consisting of two nucleons each with isospin $\overrightarrow{\frac{1}{2}}$. The quantum number $I$ of the two nucleons can be 1 or 0. The isospin states $|I, I_3\rangle$ corresponding to the isospin triplet $I = 1$ are

$$|1,+1\rangle = |\frac{1}{2},+\frac{1}{2}\rangle|\frac{1}{2},+\frac{1}{2}\rangle = |p\rangle|p\rangle, \tag{4.22}$$

$$\begin{aligned} |1,0\rangle &= \frac{1}{\sqrt{2}}\left(|\frac{1}{2},+\frac{1}{2}\rangle|\frac{1}{2},-\frac{1}{2}\rangle + |\frac{1}{2},-\frac{1}{2}\rangle|\frac{1}{2},+\frac{1}{2}\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left(|p\rangle|n\rangle + |n\rangle|p\rangle\right), \end{aligned} \tag{4.23}$$

$$|1,-1\rangle = |\frac{1}{2},-\frac{1}{2}\rangle|\frac{1}{2},-\frac{1}{2}\rangle = |n\rangle|n\rangle, \tag{4.24}$$

and for the isospin singlet $I = 0$ the corresponding state is,

$$\begin{aligned} |0,0\rangle &= \frac{1}{\sqrt{2}}\left(|\frac{1}{2},+\frac{1}{2}\rangle|\frac{1}{2},-\frac{1}{2}\rangle - |\frac{1}{2},-\frac{1}{2}\rangle|\frac{1}{2},+\frac{1}{2}\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left(|p\rangle|n\rangle - |n\rangle|p\rangle\right). \end{aligned} \tag{4.25}$$

We can see that the isospin triplet states are symmetric under the exchange of the two particles, whereas the isospin singlet state is antisymmetric under this exchange. This implies that the nuclear forces between two protons, between a neutron and a proton, and between two neutrons depend only on the value of the total isospin, $I$, and not on the third component $I_3$. That is the interaction of a neutron and a proton in the isospin triplet state $I = 1$ is different from that in the isospin singlet state $I = 0$. In this sense we can regard the neutron-proton system in the state of $I = 1$ as a system of two identical nucleons having the same nuclear potential as the two protons or two neutrons in this state [9, 10, 12]. In the isospin formalism, the charge-independence of nuclear force states that the forces of interaction between two nucleons with the same total isospin $\overrightarrow{I}$ are independent of the third component $I_3$. Therefore, the isospin concept enables us to treat any two nucleons with the same total isospin $\overrightarrow{I}$ as identical fermions and thereby we can apply the generalized Pauli principle to the two nucleon system. This principle states that the total wave function, $\Psi$, of the two nucleon system, which can be expressed as the product of wave functions in orbital, spin and isospin space;

$$\Psi = \phi(orbital)\chi(spin)I(isospin) \tag{4.26}$$

must be antisymmetric under the combined exchange of space, spin, and isospin coordinates. As a consequence the isospin singlet state ($I = 0$) will denote symmetry of the space-spin wave function $\phi(space)\chi(spin)$, whereas the isospin triplet ($I = 1$) denotes antisymmetry of $\phi(orbital)\chi(spin)$.

In a similar manner to the ordinary spin, we can define the isospin lowering and isospin raising operators $I_-$ and $I_+$ , respectively as follows:

$$I_- = I_1 - iI_2 = \frac{1}{2}(\tau_1 - i\tau_2) \tag{4.27}$$

and

$$I_+ = I_1 + iI_2 = \frac{1}{2}(\tau_1 + i\tau_2), \tag{4.28}$$

where $\tau_1$ and $\tau_2$ are the isospin matrices which are analogous to Pauli spin matrices and are given by

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.29}$$

and

$$\tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{4.30}$$

Substituting (4.29) and (4.30) into (4.27) and (4.28), we get

$$I_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{4.31}$$

and

$$I_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4.32}$$

The proton and neutron states can also be represented by column vectors, $|p\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|n\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.

Note that

$$I_-|p\rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |n\rangle \tag{4.33}$$

and

$$I_+|n\rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |p\rangle. \tag{4.34}$$

In other words the lowering operator, $I_-$, converts a proton to a neutron and the raising operator, $I_+$, converts a neutron to a proton.

Now we consider the Feynman diagram (Fig. 4.2) depicting proton-neutron interaction through the exchange of charged pion.

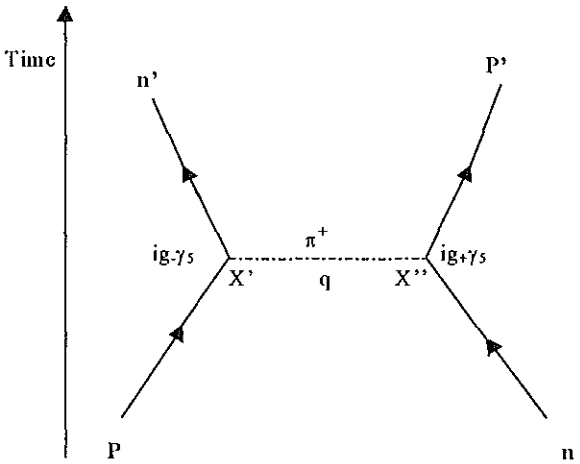


Figure 4.2: *p-n interaction mediated by charged* $\pi-$*meson exchange:*

At the interaction vertex $x'$, an incident proton is converted to a neutron and the $\pi^-$−meson field, $\Phi_-$, is absorbed. Similarly at the vertex $x''$, an incident neutron is transformed to a proton and the $\pi^+$−meson field, $\Phi_+$, is absorbed. Since the lowering operator $I_-$ transforms a proton to a neutron, and the raising operator $I_+$ converts a neutron to a proton, we expect the nucleon-meson field coupling need to contain $g_+ \left(I_-\Phi_- + I_+\Phi_+\right)$. From two independent neutral scalar fields $\Phi_1$ and $\Phi_2$, we can construct two independent charged scalar fields, $\Phi_+$ and $\Phi_-$ as

$$\Phi_- = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2) \tag{4.35}$$

and

$$\Phi_+ = \frac{1}{\sqrt{2}}(\Phi_1 - i\Phi_2), \tag{4.36}$$

such that

$$\Phi_-(x) = \Phi_+^\dagger(x).$$

Now,

$$I_-\Phi_- + I_+\Phi_+ = \frac{1}{2}(\tau_1 - i\tau_2)\frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2) + \frac{1}{2}(\tau_1 + i\tau_2)\frac{1}{\sqrt{2}}(\Phi_1 - i\Phi_2) \tag{4.37a}$$

$$= \frac{1}{\sqrt{2}}(\tau_1\Phi_1 + \tau_2\Phi_2). \tag{4.37b}$$

Relation (4.37b) is obtained for interaction of nucleons through the exchange of charged $\pi-$mesons. However, proton and neutron can also interact through the exchange of neutral $\pi-$mesons. Thus, in view of (4.37b), the nucleon-meson field coupling should also include $\tau_3\Phi_3$, with $\Phi_3$ representing the scalar $\pi^0-$meson field and $\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ the third component of the isospin operator $\overrightarrow{\tau}$. This implies that regardless of the charge of the nucleons and mesons, the meson-nucleon coupling can be written as $g_0\overrightarrow{\tau}\cdot\overrightarrow{\Phi}$. In other words, if the meson-nucleon coupling is of the form $g_0\overrightarrow{\tau}\cdot\overrightarrow{\Phi}$ i.e., a scalar in isospin space, then the interaction will be charge independent, i.e., invariant under the 3-D rotations in isospin space [10, 14]. With this in mind, the Dirac equation of a nucleon in the presence of a $\pi-$ meson field can be written as

$$\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + M\right)\Psi_N = ig_0\gamma_5(\overrightarrow{\tau}\cdot\overrightarrow{\Phi})\ \Psi_N, \tag{4.38a}$$

and the Klein-Gordon equation for the meson field $\overrightarrow{\Phi}$ is

$$(\Box^2 - \mu^2)\overrightarrow{\Phi} = ig_0\overline{\Psi}_N \overrightarrow{\tau}\gamma_5\Psi_N \tag{4.38b}$$

where $M$, is rest mass of the nucleon, $\Psi_N = \begin{pmatrix} \Psi_p \\ \Psi_n \end{pmatrix}$ is nucleon Dirac spinor and the meson field $\overrightarrow{\Phi} = (\Phi_1, \Phi_2, \Phi_3)$ is a vector in the isospin space, corresponding to the fact that the $\pi-$mesons form a charge triplet of isospin $I = 1$.

## 4.3 Nucleon-Nucleon (N-N) Potential

In this section, we wish to obtain nucleon-nucleon potential irrespective of the charge of the nucleons and mesons. The lowest-order Feynman diagrams for nucleon-nucleon scattering is given in Fig. 4.3.

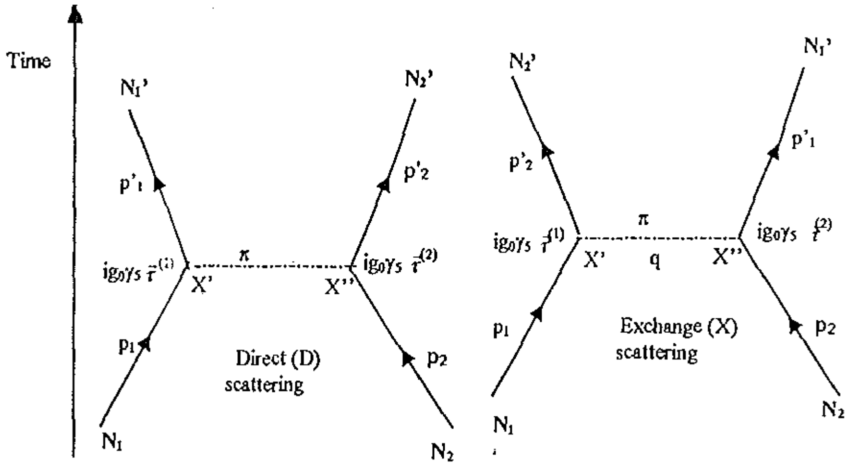


Figure 4.3: *Feynman diagrams for* N-N *scattering. Here* $\vec{\tau}^{(1)}$ *and* $\vec{\tau}^{(2)}$ *are isospin operators corresponding to nucleon* $N_1$ *and* $N_2$ *respectively.*

The first-order correction to the transition amplitude for this process is

$$S_{fi}^{(1)} = S_{fi}^{(1)}|_D - S_{fi}^{(1)}|_X, \tag{4.39}$$

where $S_{fi}^{(1)}|_D$ and $S_{fi}^{(1)}|_X$ represent first-order correction to the transition amplitude for the direct and exchange diagram respectively. We recall that

$$S_{fi}^{(1)}|_D = \int\int d^4x'd^4x''[\overline{\Psi}_{1'}(x')(ig_0\gamma_5\overrightarrow{\tau}^{(1)})\Psi_1(x')]\cdot[\triangle_F(x'-x'')\overline{\Psi}_{2'}(x'')(ig_0\gamma_5\overrightarrow{\tau}^{(2)})\Psi_2(x'')] \tag{4.40}$$

in which

$$\triangle_F(x'-x'') = \frac{-1}{(2\pi)^4}\int d^4q\frac{e^{iq\cdot(x'-x'')}}{q^2+\mu^2-i\epsilon} \tag{4.41}$$

is meson propagator, $\overline{\Psi}_{1'}$ and $\Psi_1$ are wave functions of scattered nucleon $N_1'$ and incident nucleon $N_1$; $\overline{\Psi}_{2'}$ and $\Psi_2$ are wave functions of scattered nucleon $N_2'$ and incident nucleon $N_2$ respectively. Using the explicit expressions of these wave functions along with (4.41), (4.40) can be written as

$$\begin{aligned} S_{fi}^{(1)}|_D &= \frac{(ig_0)^2}{(2\pi)^4}\int\int\int d^4x'd^4x''d^4q[\sqrt{\frac{M}{E_1'V}}\overline{u}^{(s_1')}(\overrightarrow{p}'_1)e^{-ip_1'\cdot x'}(\gamma_5\overrightarrow{\tau}^{(1)}) \\ &\times\sqrt{\frac{M}{E_1V}}u^{(s_1)}(\overrightarrow{p}_1)e^{ip_1\cdot x'}][\frac{-e^{iq\cdot(x'-x'')}}{q^2+\mu^2-i\epsilon}] \\ &\cdot[\sqrt{\frac{M}{E_2'V}}\overline{u}^{(s_2')}(\overrightarrow{p}'_2)e^{-ip_2'\cdot x''}(\gamma_5\overrightarrow{\tau}^{(2)})\sqrt{\frac{M}{E_2V}}u^{(s_2)}(\overrightarrow{p}_2)e^{ip_2\cdot x''}]. \end{aligned} \tag{4.42}$$

After carrying out the integration over the interaction vertices $x'$ and $x''$, and the four-momentum of the exchanged meson, one can show that

$$\begin{aligned} S_{fi}^{(1)}|_D &= (2\pi)^4\delta^{(4)}(p_1'+p_2'-p_2-p_1)\sqrt{\frac{M^4}{E_1'E_1E_2'E_2V^4}} \\ &\times\frac{g_0^2(\overrightarrow{\tau}^{(1)}\cdot\overrightarrow{\tau}^{(2)})\left[\overline{u}^{(s_1')}(\overrightarrow{p}'_1)\gamma_5u^{(s_1)}(\overrightarrow{p}_1)\right]\left[\overline{u}^{(s_2')}(\overrightarrow{p}'_2)\gamma_5u^{(s_2)}(\overrightarrow{p}_2)\right]}{q_D^2+\mu^2-i\epsilon}. \end{aligned} \tag{4.43}$$

Similarly the transition amplitude, $S^{(1)}_{fi}|_X$, corresponding to the exchange scattering becomes

$$S^{(1)}_{fi}|_X = (2\pi)^4\delta^{(4)}(p'_1 + p'_2 - p_2 - p_1)\sqrt{\frac{M^4}{E'_1E_1E'_2E_2V^4}}$$

$$\times\frac{g_0^2(\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)})\left[\overline{u}^{(s'_2)}(\vec{p}\,'_2)\gamma_5u^{(s_1)}(\vec{p}_1)\right]\left[\overline{u}^{(s'_1)}(\vec{p}\,'_1)\gamma_5u^{(s_2)}(\vec{p}_2)\right]}{q_x^2+\mu^2-i\epsilon} \tag{4.44}$$

Putting (4.43) and (4.44) in (4.39) leads to

$$S^{(1)}_{fi} = (2\pi)^4\delta^{(4)}(p'_1 + p'_2 - p_2 - p_1)\sqrt{\frac{M^4}{E'_1E_1E'_2E_2V^4}}M_{fi} \tag{4.45}$$

where

$$M_{fi} = g_0^2(\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)})\left[\begin{array}{c}\frac{\left[\overline{u}^{(s'_1)}(\vec{p}\,'_1)\gamma_5u^{(s_1)}(\vec{p}_1)\right]\left[\overline{u}^{(s'_2)}(\vec{p}\,'_2)\gamma_5u^{(s_2)}(\vec{p}_2)\right]}{q_D^2+\mu^2-i\epsilon}\\ -\frac{\left[\overline{u}^{(s'_2)}(\vec{p}\,'_2)\gamma_5u^{(s_1)}(\vec{p}_1)\right]\left[\overline{u}^{(s'_1)}(\vec{p}\,'_1)\gamma_5u^{(s_2)}(\vec{p}_2)\right]}{q_x^2+\mu^2-i\epsilon}\end{array}\right]. \tag{4.46}$$

Following similar methods used in the case of p-p potential, the low-energy nucleon-nucleon potential becomes

$$V = \frac{1}{(2\pi)^3}\int d^3\vec{q}\,M_{fi}e^{i\vec{q}\cdot\vec{r}} \tag{4.47a}$$

$$= \frac{-g_0^2}{(2M)^2}\frac{(\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)})}{(2\pi)^3}\int d^3\vec{q}(\vec{\sigma}^{(1)}\cdot\vec{q})(\vec{\sigma}^{(2)}\cdot\vec{q})\frac{e^{i\vec{q}\cdot\vec{r}}}{|\vec{q}|^2+\mu^2} \tag{4.47b}$$

$$= \frac{g_0^2}{(2M)^2}\frac{(\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)})}{(2\pi)^3}\int d^3\vec{q}(\vec{\sigma}^{(1)}\cdot\vec{\nabla})(\vec{\sigma}^{(2)}\cdot\vec{\nabla})\frac{e^{i\vec{q}\cdot\vec{r}}}{|\vec{q}|^2+\mu^2} \tag{4.47c}$$

$$= \frac{g_0^2}{(2M)^2}\frac{(\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)})}{(2\pi)^2}(\vec{\sigma}^{(1)}\cdot\vec{\nabla})(\vec{\sigma}^{(2)}\cdot\vec{\nabla})\frac{e^{-\mu r}}{r}. \tag{4.47d}$$

Employing the relation

$$(\vec{\sigma}^{(1)}.\vec{\nabla})(\vec{\sigma}^{(2)}.\vec{\nabla}) = \frac{1}{3}S_{12}\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)+\frac{1}{3}(\vec{\sigma}^{(1)}.\vec{\sigma}^{(2)})\left(\frac{d^2}{dr^2}+2\frac{d}{dr}\right) \tag{4.48a}$$

with the tensor operator $S_{12}$ given by

$$S_{12} = 3(\vec{\sigma}^{(1)}\cdot\hat{r})(\vec{\sigma}^{(2)}\cdot\hat{r}) - \vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}, \tag{4.48b}$$

and carrying out the differentiation results in

$$V = \frac{g_0^2}{4\pi}(\overrightarrow{\tau}^{(1)} \cdot \overrightarrow{\tau}^{(2)})(\frac{\mu}{2M})^2 \left\{ S_{12} \left[ \frac{1}{(\mu r)^2} + \frac{1}{\mu r} + \frac{1}{3} \right] + \frac{1}{3}(\overrightarrow{\sigma}^{(1)} \cdot \overrightarrow{\sigma}^{(2)}) \right\} \frac{e^{-\mu r}}{r}. \tag{4.49}$$

It can be observed that the one pion exchange nucleon-nucleon potential has central and tensor parts. The presence of $\overrightarrow{\tau}^{(1)} \cdot \overrightarrow{\tau}^{(2)}$ guarantees the charge-independence of nucleon-nucleon interaction. Since $\overrightarrow{\tau}^{(1)} \cdot \overrightarrow{\tau}^{(2)} = +1$ for symmetric isospin triplet states, one can infer that the nuclear interaction potentials between two protons, between two neutrons and between a proton and a neutron ( which have the same total isospin) are the same. Thus it can easily be shown that for $I = 1$,

$$V_{p-p} = V_{n-n} = V_{p-n} = \frac{g_0^2}{4\pi}(\frac{\mu}{2M})^2 \left\{ S_{12} \left[ \frac{1}{(\mu r)^2} + \frac{1}{\mu r} + \frac{1}{3} \right] + \frac{1}{3}(\overrightarrow{\sigma}^{(1)} \cdot \overrightarrow{\sigma}^{(2)}) \right\} \frac{e^{-\mu r}}{r}, \tag{4.50}$$

whereas for $I = 0$,

$$V_{p-n} = \frac{g_0^2(-3)}{4\pi}(\frac{\mu}{2M})^2 \left\{ S_{12} \left[ \frac{1}{(\mu r)^2} + \frac{1}{\mu r} + \frac{1}{3} \right] + \frac{1}{3}(\overrightarrow{\sigma}^{(1)} \cdot \overrightarrow{\sigma}^{(2)}) \right\} \frac{e^{-\mu r}}{r}. \tag{4.51}$$

Here relation (4.50) explicitly demonstrates the charge-independence of nuclear interactions. Further from (4.50) and (4.51) we can see that the p-p, n-n and p-n system (for $I = 1$) can exist only as scattering states. Whereas only the p-n system (for $I = 0$) can exist as a bound state (Deuteron nucleus). This validates the charge independence of nuclear forces. Thus in nature we only observe neutron-proton bound states.

# CHAPTER 5

## Conclusion

By solving the Dirac equation for an electron interacting with electromagnetic field $A_\mu$, we know that the first-order correction to the transition amplitude, $S_{fi}^{(1)}$, that an electron in the initial state $|i\rangle$ is transferred to a final state $|f\rangle$ as a result of interacting with the field, is expressed as,

$$S_{fi}^{(1)} = -e \int d^3\vec{x}' \int_{-\infty}^{\infty} dt' \overline{\Psi}_f(x')\gamma_\mu A_\mu(x')\Psi_i(x'). \tag{5.1}$$

Comparing the differential cross-section in the non-relativistic Schrodinger theory and the non-relativistic limit of the differential cross-section obtained using quantum field theory, the low-energy electron-electron potential has been expressed as a 3-D Fourier transform [2] of the non-relativistic limit of the matrix element $M_{fi}$. Here we have calculated the potential equivalent to the interaction of two (low-energy) moving electrons through the exchange of a photon employing the field-theoretic techniques in the language of Feynman propagators.

In the static limit (stationary electrons), the e-e interaction potential due to the exchange of one photon is obtained to be the coulomb poteutial. However, when the speed of the interacting electrons is not zero and not relativistic, the interaction of the electrons through the exchange of a photon involves interaction between their magnetic dipole moments and spin-orbit interaction and the corresponding potential is expressed as,

$$V = -\overrightarrow{\mu}_1 \cdot \overrightarrow{B}_2 + \frac{-4}{(2m)^2}\frac{1}{r}\frac{d}{dr}\left(\frac{e^2}{4\pi r}\right)\overrightarrow{L} \cdot \frac{(\overrightarrow{\sigma}^{(1)} + \overrightarrow{\sigma}^{(2)})}{2}. \qquad (5.2)$$

We have tried to model the long-range part of nuclear forces on the same lines worked out for low-energy electromagnetic interaction of two electrons. We take the long-range part of N-N force arises from the exchange of $\pi-$mesons between the two nucleons. We have thus determined the long-range part of nuclear potential corresponding to nuclear interaction of two low-energy protons through the exchange of one $\pi-$meson, applying field-theoretic techniques, which is given as,

$$V = \frac{g_0^2}{4\pi}\left(\frac{\mu}{2m_p}\right)^2 \left\{\frac{1}{3}(\overrightarrow{\sigma}^{(1)} \cdot \overrightarrow{\sigma}^{(2)}) + \left[\frac{1}{(\mu r)^2} + \frac{1}{\mu r} + \frac{1}{3}\right] S_{12}\right\} \frac{e^{-\mu r}}{r}. \qquad (5.3)$$

It can be seen that the OPEP has spin-spin and non-central (tensor) parts. In the process we have correctly reproduced the exact form of the Yukawa potential, $V \sim \frac{e^{-\mu r}}{r}$, for the low-energy interaction between two protons mediated by the exchange of a $\pi^0-$meson. However if we calculate the two-pion exchange potential (TPEP) for p-p scattering using the Born approximation, we get an approximate form of this potential, $V \sim f(r, \overrightarrow{\sigma}^{(1)}, \overrightarrow{\sigma}^{(2)})\frac{e^{-2\mu r}}{r}$. However, we still predict the range of the TPEP to be $r \sim \frac{1}{2\mu}$, which is much shorter than the range ($r \sim \frac{1}{\mu}$) of the OPEP. Furthermore, the TPEP is much stronger (it is $\sim g_0^4$ ) than the OPEP (which is $\sim g_0^2$).

One of the most important calculations in particle and nuclear physics is calculation of differential cross-section. The differential cross-section (in the center-of-mass frame) for p-p scattering through the exchange of one $\pi-$meson has also been worked out in chapter 3.

Based on the charge independence of nuclear forces, the coupling constant $g_0$ between a nucleon and a neutral $\pi-$meson field is related to the coupling constant $g_+$ ($g_-$) between a nucleon and charged $\pi-$meson field and is expressed by $g_- = g_+ = \sqrt{2}g_0$. However there is a need to have a unified framework in which one can describe n-n, p-n, and p-p interactions in an integrated manner. This necessitates us to introduce the concept of isospin where a neutron and a proton are considered as two charge states of a single particle called nucleon, which is assigned isospin $I = \overrightarrow{\frac{1}{2}}$. Using this isospin language, we can deal with N-N interaction without regard to the charge of the nucleons and of the exchanged $\pi-$meson. As a consequence we have obtained the nuclear potential corresponding to low-energy N-N interaction through the exchange of one $\pi-$meson. This is given as

$$V = \frac{g_0^2}{4\pi}(\overrightarrow{\tau}^{(1)} \cdot \overrightarrow{\tau}^{(2)})(\frac{\mu}{2M})^2 \left\{S_{12}\left[\frac{1}{(\mu r)^2} + \frac{1}{\mu r} + \frac{1}{3}\right] + \frac{1}{3}(\overrightarrow{\sigma}^{(1)} \cdot \overrightarrow{\sigma}^{(2)})\right\} \frac{e^{-\mu r}}{r}. \tag{5.4}$$

. From the results of chapter 4, we observe that for the isospin triplet ($I = 1$) states, $\overrightarrow{\tau}^{(1)}.\overrightarrow{\tau}^{(2)} = +1$. Thus the nuclear potentials between two protons, between two neutrons and between a proton and a neutron are the same ( i.e. $V_{p-p} = V_{n-n} = V_{p-n}$). This validates the charge independent hypothesis of nuclear forces. Further for $I = 0$ we see that $\overrightarrow{\tau}^{(1)} \cdot \overrightarrow{\tau}^{(2)} = -3$. This shows that our model for N-N forces correctly predicts the existence of a bound N-N state (Deuteron nucleus) for $I = 0$. Whereas for $I = 1$, $\overrightarrow{\tau}^{(1)} \cdot \overrightarrow{\tau}^{(2)} = +1$. This fact explains

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