PULSAR ALIGNMENT DUE TO RADIATION REACTION TORQUE

By

Endris Mohammed Endris

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE IN PHYSICS
AT
ADDIS ABABA UNIVERSITY
ADDIS ABABA, ETHIOPIA
JULY 2010

ADDIS ABABA UNIVERSITY
DEPARTMENT OF
PHYSICS

Supervisor: ____________________
Dr.Legesse Wetro

Examiners: ____________________
PROF. P. SINGH

____
Dr. Tilahun

ADDIS ABABA UNIVERSITY

Date: **July 2010**

Author: **Endris Mohammed Endris**

Title: **Pulsar Alignment due to Radiation Reaction Torque**

Department: **Physics**

Degree: **M.Sc.** Convocation: **June** Year: **2010**

Signature of Author

Table of Contents

Table of Contents **iv**

List of Figures **v**

Abstract **vi**

Acknowledgements **vii**

INTRODUCTION **1**

1 NEUTRON STAR AND PULSAR **5**

1.1 Neutron Star 5
1.2 Pulsar 7

2 THE EFFECTIVE TORQUES **12**

2.1 Introduction 12
2.2 Electromagnetic Torques on the Magnetic Dipole 12
2.3 Frictional Torque 19
2.4 Gravitational Torque 21

3 THE EQUATION OF MOTION **24**

CONCLUSION **35**

Bibliography **36**

List of Figures

1.1 the mass of stars and their collapse 6

1.2 The inner structure of a neutron star 7

1.3 A schematic representation of a possible cross-section of a 1.4 solar mass neutron star 8

1.4 the beam of the pulsar passes over the the earth,the pulsar can be seen when the direction of the magnetic dipole moment is pointing to the eath 9

1.5 conservation of angular mommentum 10

1.6 The pole of the pulsar are not aligned with its axis of rotation 11

2.1 r_2, r' are the retarded position path of the charged particle $r_2 \gg r_1$ 13

3.1 crust angular velocity $\vec{\Omega}_c$, which defines the z-axis.the magnetic moment M is in the y-z plane,with an angle θ with respect to the z-direction.the core angular velocity $\vec{\Omega}_o$ forms an angle α with $\vec{\Omega}_c$ and β with M 24

Abstract

We calculate the time of alignment of the magnetic dipole moment with the axis of rotation. For alignment we take into acount the electromagnetic torque acting on the magnetic dipole moment, the frictional torque between the crust and the core and gravitational effects. The anomalous electromagnetic torque, usually neglected in the a rigid star model, here plays a crucial role for the alignment of the magnetic dipole.

Acknowledgements

Above all, I would like to thank the almighty; Allah Subhanehu Wete'Alla,. I am deeply indebted to my advisor, Dr.legesse Wotro, whose stimulating suggestions and encouragement helped me go though the thesis research. I would like to express my gratitude to my family who gave me the encouragement and support to complete this thesis.

I would like to give my special thanks to my wife Habiba, her fiend Ayisha and my friend Jemal for their support in giving Laptop to complete this work.

INTRODUCTION

Neutron stars are one of the possible ends for a star and pulsars are identified with rapidly rotating, highly magnetized neutron stars. The basic information we receive from dead stars of this kind is a sequence of electromagnetic pulses with a very stable frequency, which is interpreted as directly related to the star rotation. The angular velocity generally decreases gradually as a result of the torque exerted on the star by the radiation reaction. In the vacuum dipole model, the star magnetic field is assumed to be a magnetic dipole M that forms an angle with the rotation axis,so that the star loses energy by electromagnetic radiation because of its rotation (Pacini 1967, 1968; Gunn and Ostriker 1969). This turns out to be the main source of the star energy loss. The evolution of the angular velocity for a neutron star with a moment of inertia I_t is

$$\dot{\Omega} = -\frac{2}{3}\frac{M^2\Omega^3}{I_t c^3}\sin^3\theta \tag{0.0.1}$$

Or, more generally, $\dot{\Omega} = -K\Omega^3$. Several characteristics of the dynamics of the K parameter have been observed. After glitches, a sudden increase of K has been noticed, which does not completely relax back (16). There is also evidence that for old pulsars, with an age of $\Omega/2\dot{\Omega} \sim 10^7 yr$, K is smaller than for younger pulsars (26). The first behavior can be interpreted as an increase of the external torque after some glitches, whereas the second one suggests a slow decrease of the torque with the age of the star.

A number of factors might affect the braking index. One of them is the presence of mechanisms of energy loss different from the dipolar electromagnetic radiation that could change the exponent in equation (0.0.1). For example, higher order multipole radiation gives n≥5, gravitational quadrupole radiation n=5 (Manchester and Taylor 1977), and early neutrino emission $n < 0$ (Alpar and gelman 1990). These effects are expected to be relatively weak for the pulsars with a measured braking index, and in particular the first two would increase n and thus are unable to explain the observed values smaller than 3. The most natural explanation seems to require variations of the parameters I, M, and θ, which, change the braking index in the following way:

$$n = \frac{\Omega\ddot{\Omega}}{\dot{\Omega}^2} \tag{0.0.2}$$

$$\dot{\Omega} = -K\Omega^3 \tag{0.0.3}$$

where K is $K = \frac{2}{3}\frac{M^2}{I_t C^3}\sin^3\theta$ the derivation of equ.(0.0.2) is

$$\ddot{\Omega} = -\dot{K}\Omega^3 - 3K\Omega^2\dot{\Omega} \tag{0.0.4}$$

after substituting equ.(0.0.4) in to equ.(0.0.2) we get

$$n = 3 + \frac{\Omega}{\dot{\Omega}}\frac{\dot{K}}{K} \tag{0.0.5}$$

the derivation of K is

$$\dot{K} = \frac{4}{3}\frac{M\dot{M}}{I_t C^3}\sin\theta^2 - \frac{2}{3}\frac{M^2\dot{I}_t}{I_t^2 C^3}\sin\theta^2 + \frac{4}{3}\frac{M^2}{I_t C^3}\dot{\theta}\sin\theta\cos\theta \tag{0.0.6}$$

$$\frac{\dot{K}}{K} = 2\frac{\dot{\theta}}{\tan\theta} + 2\frac{\dot{M}}{M} - \frac{\dot{I}_t}{I_t} \tag{0.0.7}$$

substitute in to equ.(0.0.5) and we get

$$n = 3 + \frac{\Omega}{\dot{\Omega}}(2\frac{\dot{\theta}}{\tan\theta} + 2\frac{\dot{M}}{M} - \frac{\dot{I}_t}{I_t}) \tag{0.0.8}$$

The parameters responsible for the variation should have characteristic evolution times of the order of the pulsar age$=\frac{\Omega}{2\dot{\Omega}}$, which is around 1000 yr in the case of the stars of known braking index.

With respect to the moment of inertia, sensible variations during such times due to changes in the structure, sphericity, or the coupling of the internal superfluids to the crust seem to be unable to explain the observed deviations (15).

Significant changes in the magnetic moment also seem to be inhibited by the high crust conductivity, which can keep the magnetic fields unchanged and thus fixed to the crust along times of order 10^7 yr or more (Lamb 1991; Chanmugam 1992; Phynney and Kulkarni 1994; Goldreich and Reisenegger 1992), and hence the braking index n is unaffected in young pulsars. Furthermore, the decay of the magnetic field reduces the radiation rate and, hence, would lead to $n > 3$. In this work we adopt this hypothesis and simply assume a stable external magnetic field rigidly fixed to the crust, except perhaps in very short periods at the glitches.

Under the above assumptions, the only mechanism that could be responsible for the anomalous braking index is the variation of the angle between the magnetic dipole moment and the rotation axis. In this case, equation (0.0.8) becomes

$$n - 3 = 2\frac{\Omega}{\dot{\Omega}}\frac{\dot{\theta}}{\tan\theta}. \tag{0.0.9}$$

The angular momentum of the star decreases during its lifetime by radiation, and $\dot{\Omega}$ thus is negative. Therefore, a value $n < 3$ requires an increase in time of θ, e.g., $\dot{\theta} > 0$. In brief, for a rigid star in vacuum the observed dynamics is related to the behavior of the angle θ. The evolution θ is controlled by three characteristic times. There is a fast dynamics, associated with the reorientation of the magnetic dipole during the glitches and having a characteristic time less than or of the order of 50 days; an intermediate dynamics, which dominates during the early stages of the stars, where the magnetic moment slides toward the equator and gives the braking index values smaller than 3; and finally a slow dynamics, where the magnetic moment slides toward the rotation axis of the star, which involves characteristic times of the order of $\tau_\Omega \sim 10^7 yr$ (Lyne and Manchester 1988). The intermediate dynamics competes with the slow one, and the observational evidence shows that for young pulsars it dominates, leading to the anomalous values of the braking index. On the other hand, for old pulsars ($\tau_\Omega \sim 10^7 yr$), the slow one dominates, giving place to the alignment of the magnetic dipole with the rotation axis. However, if we only take into account the radiation reaction of the magnetic dipole on a rigid star, the evolution of θ would perform a slow alignment during the lifetime of the star (Pandey and Prasad 1996) with a braking index $n > 3$, in contradiction to the features mentioned before.

The star is considered as constituted by two rigid interacting components in the absence of a magnetosphere. One of them is the core, which contains the bulk of the mass with a moment of inertia I_o and an angular velocity Ω_o. Its angular momentum is mainly given by the vortices of superfluid neutrons. The other component is the crust, with a moment of inertia $I_c \ll I_o$ and an angular velocity Ω_c and with a dipole magnetic moment M. The evolution of the system is governed by three kinds of torques:

1. Torques acting on the crust due to electromagnetic interactions.

2. Friction torques between the core and the crust due to the interaction of the neutron vortices, the proton flux tubes, and the electrons in the star.

3. Torques due to gravitational effects.

In the following section we present a review of neutron star's formation, structure, internal composition and the the pulsar and its magnetic dipole moment behavior. In the second chapter we write a review of different effective torques we take in to account in our construction of equation of motion. The equations of motion that describe the dynamics of the neutron star. The calculation of the time taken for the alignment of the magnetic dipole moment is fully dealt in chapter three. Finally the last sections are devoted to the conclusion.

Chapter 1

NEUTRON STAR AND PULSAR

1.1 Neutron Star

Neutron stars are one of the possible ends for massive star ($x5-50M_\odot$). The mass of core is between 1.4 and 3 solar masses this mass is between the mass of the white dwarf and the black hole,[fig(1)]. After these stars have finished burning their nuclear fuel, they undergo a supernova explosion. This explosion blows off the outer layers of a star into supernova remnant. The central region of the star collapses under gravity. The compression from the star's gravity will be so great the electron is extrimely energetic and fuse with the protons in the ^{56}Fe nuclei to form neutron. Neutrons can be packed much closer together than electrons.[33]

A neutron star is a type of remnant that can result from the gravitational collapse of a massive star during supernova event. Such stars are composed almost entirely of neutrons, which are subatomic particles without electrical charge and roughly the same mass as protons. Neutron stars are very hot and are supported against further collapse because of the Pauli exclusion principle. This principle states that no two neutrons (or any other fermionic particle) can occupy the same state [29].

A typical neutron star has a density between 3.7×10^{17} to $5.9\times10^{17}kg/m^3$, with a corresponding radius between 10 and 15 kilometers. neutron star's density varies from below $\approx 10^9 kg/m^3$ in the crust increasing with depth to above 6×10^{17} or $8\times10^{17}kg/m^3$ deeper

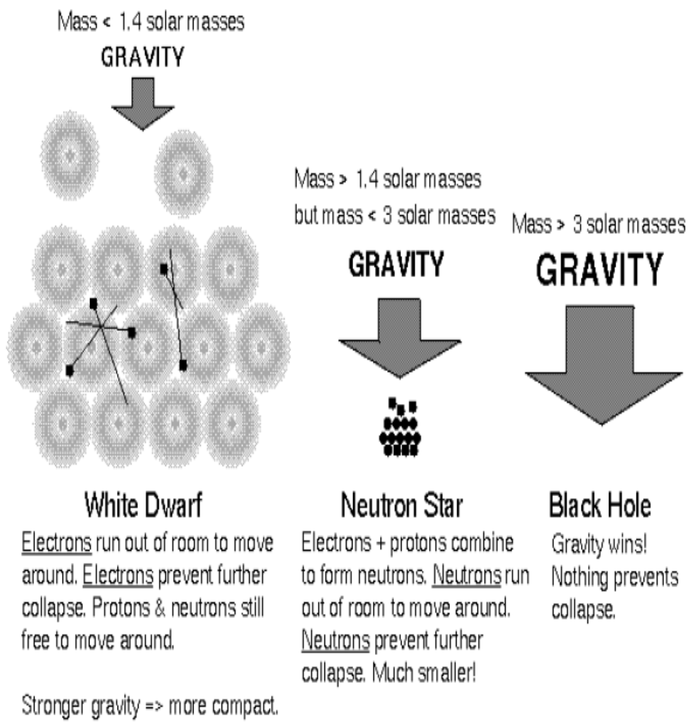

Figure 1.1: the mass of stars and their collapse

inside. In general, compact stars of less than 1.4 solar masses are white dwarfs;above 3 solar masses,a quark star might be created, however this is uncertain. Gravitational collapse will occur on any compact star over 3 solar masses, inevitably producing a black hole.[29]

A neutron star is formed with very high rotation speed, and then gradually slows down. The neutron star's compactness also gives it very high surface gravity, up to 7x$10^{12} m/s^2$ with typical values of a few x$10^{12} m/s^2$ (that is more than 10^{11} times of that of Earth). Matter falling onto the surface of a neutron star would be accelerated to tremendous speed by the star's gravity. Neutron stars can also have magnetic field strength a million times stronger than the strongest magnetic field produced on Earth.

The two component dynamical system composed of the solid crust and the superfluid core constitute the outer and inner parts of Ns. The crust is extremely hard and very smooth (with maximum surface irregularities of ∼5 mm), because of the extreme gravitational field. Outer part is contains ions and electrons and it has a few percent of the neutron star's moment of inertia and the thickness is 0.5km. Also the inner crust has a relatively grater density. Inner crust which consists of electrons, superfluid neutrons and nuclei in the thickness of 1km. The core of the neutron star occupies most of the neutron star's moment of inertia and volume. As shown in the fig.(1.3) the outer region is a mixture of superfluid neutron,superconducting protons and normal electrons which have relatively large thickness. Superdense matter in the inner core is still not well understood.

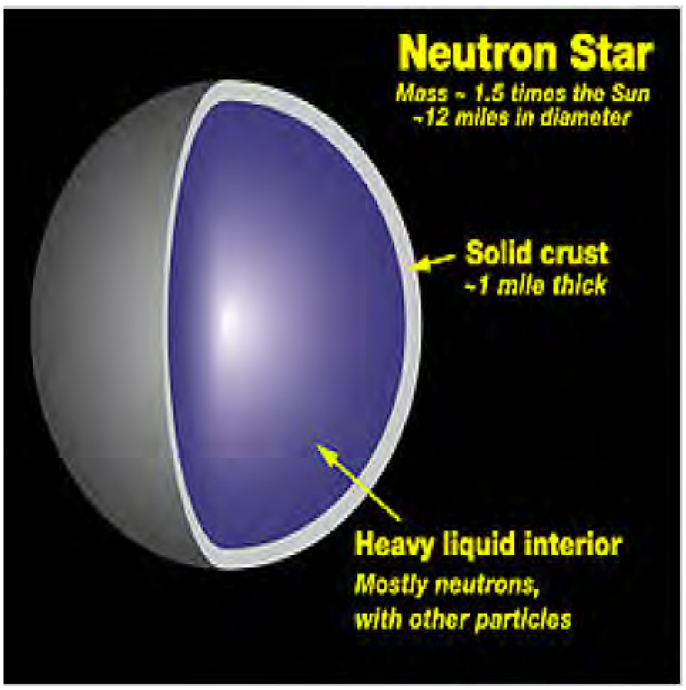

Figure 1.2: The inner structure of a neutron star

1.2 Pulsar

Pulsars are highly magnetized,rotating neutron stars that emit beams of electromagnetic radiation.The observed periods of their pulses range from 1.4 milliseconds to 8.5 seconds. The density of the star determines the pulsation period. Denser stars pulsate more quickly

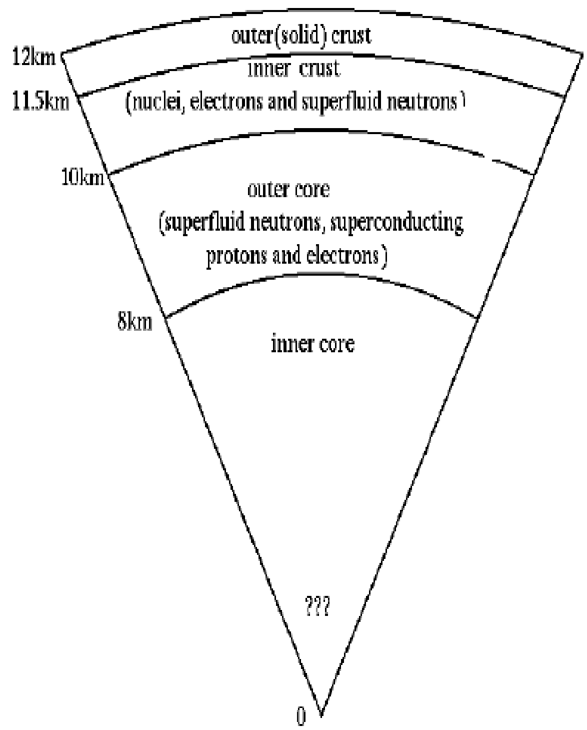

Figure 1.3: A schematic representation of a possible cross-section of a 1.4 solar mass neutron star

than low density old stars. However, normal stars and white dwarfs are not dense enough to pulsate at rates of under one second. Neutron stars would pulsate too quickly because of their huge density, so pulsars must pulsate by a different way than normal variable stars. A rapidly rotating object with a bright spot on it could produce the quick flashes if the bright spot was lined up with the Earth. Normal stars and white dwarfs cannot rotate fast enough because they do not have enough gravity to keep themselves together; they would spin themselves apart. Neutron stars are compact enough and strong enough to rotate that fast. Because neutron stars are very dense objects, the rotation period and thus the interval between observed pulses is very regular.

The events leading to the formation of a pulsar begin when the core of a massive star is compressed during a supernova, which collapses into a neutron star. As shown in the fig.(1.5)The neutron star retains most of its angular momentum, and since it has only

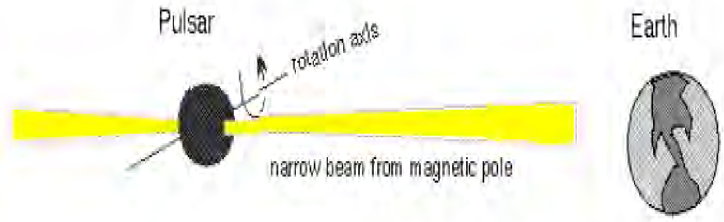

Figure 1.4: the beam of the pulsar passes over the the earth,the pulsar can be seen when the direction of the magnetic dipole moment is pointing to the eath

a tiny fraction of its progenitor's radius (and therefore its moment of inertia is sharply reduced), it is formed with very high rotation speed. The strong magnetic fields of many neutron stars can focus the light, radio waves, and other forms of the radiation that they emit into two narrow beams pointing in the direction of the magnetic field axis (one in the direction of the north magnetic pole, the other in the direction of the south magnetic pole. If the polar field axis of such a neutron star is not aligned with its rotation axis, then its bi-polar beams sweep out two swaths of the sky (as it rotates) – somewhat like the rotating light beam of a lighthouse. If an observer is located in the direction of one of the swaths (e.g., on Earth), the observer would see pulses of radiation each time the beam crosses the observer's line of sight to the remnant.[40] The magnetic poles of a pulsar are not aligned with its axis of rotation.

The beam originates from the rotational energy of the neutron star, which generates

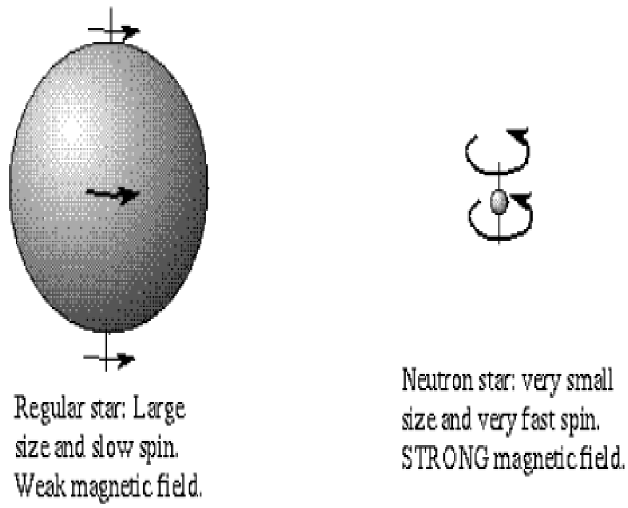

Figure 1.5: conservation of angular mommentum

an electrical field from the movement of the very strong magnetic field, resulting in the acceleration of protons and electrons on the star surface and the creation of an electromagnetic beam emanating from the poles of the magnetic field. This rotation slows down over time as electromagnetic power is emitted. When a pulsar's spin period slows down sufficiently, the radio pulsar mechanism is believed to turn off (the so-called "death line") this happens prier to alignment.

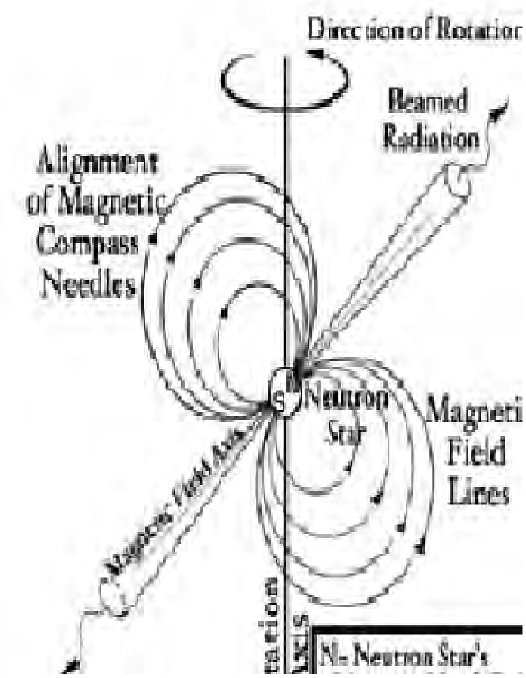

Figure 1.6: The pole of the pulsar are not aligned with its axis of rotation

Chapter 2

THE EFFECTIVE TORQUES

2.1 Introduction

In principle, the coordinates of our work are represented by three vectors, $\vec{M}$, $\vec{\Omega}_o$, and $\vec{\Omega}_c$ which are the magnetic moment, the core angular velocity, and the crust angular velocity, respectively. It should be remarked that the two components considered in the present work are identified with the crust and core. There are 5 degrees of freedom, because we have assumed the magnetic dipole moment constant in modulus and fixed to the crust.

In the following subsections we describe the torques that arise from the different interactions.

2.2 Electromagnetic Torques on the Magnetic Dipole

We have a magnetic dipole M which forms an angle θ with the angular velocity $\vec{\Omega}_c$ of the crust. There are several dipole and quadrupole torque terms that act on the magnetic dipole, and as we have assumed that it is bound to the crust, these torques are directly applied to the latter. The effect of the average torques that involve quadrupole terms are strongly suppressed, and hence it is enough to consider only the dipolar components. There are two types of dipolar torques that act on the crust, the non-anomalous torques of order $\frac{M^2\Omega_c^3}{c^3}$ which are of the same order as the classical spin down torque [35], and the anomalous torques that are $\frac{c}{\Omega_c R}$ times larger than the former ones, where R is the radius of the star. In the following we will use an orthogonal system of reference with $\hat{z}$ in the $\hat{\Omega}_c$ direction, $\hat{M}$ in the plane $(\hat{\Omega}_c, \hat{y})$, and $\hat{X}$ orthogonal to such a plane, according to Fig.3.1. In this plane the retardation torque(alignment torque) can be calculated as,

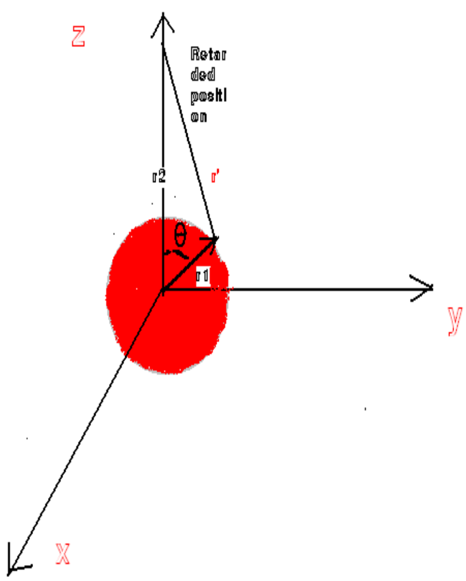

Figure 2.1: r_2, r' are the retarded position path of the charged particle $r_2 \gg r_1$

$r' = r_1 - vt' = c(t - t') = r_{12}$

$$T_{yz} = \int d^3\vec{r}_1 \vec{r}_1 \times \vec{F} \tag{2.2.1}$$

$$T_{yz} = \int d^3\vec{r}_1 \vec{r}_1 \times [\vec{j}Jp(r_1) \times \delta\vec{B}(r_1)] \tag{2.2.2}$$

$$A = \int \frac{\vec{J}(r_1, t_r)}{\vec{r}_{12}} d^3r \tag{2.2.3}$$

$$B = \vec{\nabla} \times \vec{A} = \int \left[\frac{1}{r_{12}}(\vec{\nabla} \times \vec{J}) - \vec{J} \times \nabla(\frac{1}{r_{12}})\right] d^3r \tag{2.2.4}$$

now

$$(\nabla \times \vec{J})_x = \frac{\partial \vec{J}_z}{\partial y} - \frac{\partial \vec{J}_y}{\partial z} \tag{2.2.5}$$

and

$$\frac{\partial \vec{J}_z}{\partial y} = \frac{\partial \vec{J}_z}{\partial t_r}\frac{\partial t_r}{\partial y} = -\frac{1}{c}\frac{\partial \vec{J}_z}{\partial t}\frac{\partial \vec{r}_{12}}{\partial y} \tag{2.2.6}$$

so,

$$(\nabla \times \vec{J})_x = -\frac{1}{c}\left(\frac{\partial \vec{J}_z}{\partial t}\frac{\partial \vec{r}_{12}}{\partial y} - \frac{\partial \vec{J}_y}{\partial t}\frac{\partial \vec{r}_{12}}{\partial z}\right) \tag{2.2.7}$$

$$= \frac{1}{c}\left[\frac{\partial \vec{J}}{\partial t} \times \times \nabla \vec{r}_{12}\right] \tag{2.2.8}$$

But $\vec{\nabla}\vec{r}_{12} = \hat{r_{12}}$

$$\nabla \times \vec{J} = \frac{1}{c}\frac{\partial \vec{J}}{\partial t} \times \hat{r_{12}} \tag{2.2.9}$$

And meanwhile

$$\nabla(\frac{1}{r_{12}}) = -\frac{\hat{r_{12}}}{r_{12}} \tag{2.2.10}$$

The general expression fore the magnetic induction generated by current $\vec{J}(r_2, t')$ is

$$\vec{B}(r_1, t) = \frac{n_o}{4\pi}\int d^3\vec{r}_2[\vec{J}(r_2, t') + \frac{\partial \vec{J}(r_2, t')}{\partial t'}\frac{\vec{r}_{12}}{c}] \times \frac{\hat{r_{12}}}{r_{12}^2} \tag{2.2.11}$$

where, $t' = t - \frac{r_{12}}{c}$ and $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$ are retarded time and position respectively

We now take $Taylor's$ series in t', expanding about $t' = t$

$$\vec{J}(r_2, t') = \vec{J}(r_2, t) - \frac{\partial \vec{J}(r_2, t)}{\partial t}\frac{\vec{r}_{12}}{c} + \frac{1}{2!}\frac{\partial^2 \vec{J}(r_2, t)}{\partial t^2}(\frac{r_{12}}{c})^2 - \frac{1}{3!}\frac{\partial^3 \vec{J}(r_2, t)}{\partial t^3}(\frac{\vec{r}_{12}}{c})^3 + ... \tag{2.2.12}$$

The above expansion converges rapidly if the change of the current density $\vec{J}(r_2, t)$ is small in a characteristic time scale $\tau = \frac{\vec{r}_{12}}{c}$ since $\vec{J}$ varies with t only,Because of the rotation on a time scale of $\frac{2\pi}{\Omega}$ the condition is well satisfied, Differentiating eq.(2.2.12) with respect to t' we get

$$\frac{\partial \vec{J}(r_2, t')}{\partial t} = \frac{\partial \vec{J}(r_2, t)}{\partial t} - \frac{\partial^2 \vec{J}(r_2, t)}{\partial t^2}\frac{\vec{r}_{12}}{c} + \frac{1}{2!}\frac{\partial^3 \vec{J}(r_2, t)}{\partial t^3}(\frac{r_{12}}{c})^2 - \frac{1}{3!}\frac{\partial^4 \vec{J}(r_2, t)}{\partial t^4}(\frac{\vec{r}_{12}}{c})^3 + ... \tag{2.2.13}$$

Multiply this by $\frac{\vec{r}_{12}}{c}$

$$\frac{\partial \vec{J}(r_2, t')}{\partial t}\frac{r_{12}}{c} = \frac{\partial \vec{J}(r_2, t)}{\partial t}\frac{r_{12}}{c} - \frac{\partial^2 \vec{J}(r_2, t)}{\partial t^2}(\frac{\vec{r}_{12}}{c})^2 + \frac{1}{2!}\frac{\partial^3 \vec{J}(r_2, t)}{\partial t^3}(\frac{r_{12}}{c})^3 - \frac{1}{3!}\frac{\partial^4 \vec{J}(r_2, t)}{\partial t^4}(\frac{\vec{r}_{12}}{c})^4 + ... \tag{2.2.14}$$

And add the result to eq.(2.2.12) is proportional to the integrand of eq.(2.2.11)

$$\vec{J}(r_2,t') + \frac{\partial \vec{J}(r_2,t)}{\partial t}\frac{\vec{r}_{12}}{c} = \vec{J}(r_2,t) - \frac{1}{2!}\frac{\partial^2 \vec{J}(r_2,t)}{\partial t^2}(\frac{r_{12}}{c})^2 + \frac{1}{3!}\frac{\partial^3 \vec{J}(r_2,t)}{\partial t^3}(\frac{\vec{r}_{12}}{c})^3 + ... \quad (2.2.15)$$

The $zero's$ order term will just give the self-torque on a static current, which is zero we will keep the second and third order terms and drop the higher one.

$$\delta\vec{B}^{(2)}(r_1) = -\frac{1}{2c^2}\int d^3\vec{r}_2 \frac{\partial^2 \vec{J}(r_2,t)}{\partial^2 t} \times \hat{r_{12}} \quad (2.2.16)$$

$$\delta\vec{B}^{(3)}(r_1) = \frac{1}{3c^3}\int d^3\vec{r}_2 \frac{\partial^3 \vec{J}(r_2,t)}{\partial^3 t} \times \hat{r_{12}} \quad (2.2.17)$$

Takeing the current on the surface as;

$$\vec{J}_p = \vec{K}(r,t)\delta(r-R) \quad (2.2.18)$$

$$\mathbf{K} = K_\theta\hat{\theta} + K_\phi\hat{\phi} \quad (2.2.19)$$

We note that since $\delta\vec{B}$ along $\vec{J}$ contributes no force, and $\delta\vec{B}$ in the surface but perpendicular to $\vec{J}$ contributes a radial force and hence no torque(since $\vec{r}\text{x}\vec{F} = 0$). we need calculate only the component $\delta\vec{B}_{r1}$, which then contributes torque in the the amount $aK\delta\vec{B}_{r1}$ per unit area. In that case $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$ in eq.(2.1.17) replaced by $-\vec{r}_2$

$$d^3\vec{r} = \vec{r}^2 dr d\Omega \quad (2.2.20)$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}.\vec{C}) - (\vec{A}.\vec{B})\vec{C} \quad (2.2.21)$$

by substituting these equation we get

$$\vec{T}_{yz} = \int d\Omega d\vec{r}_1 \vec{r}_1^3[\vec{k}_1\delta(r-R)\delta\vec{B}(r_1)] \quad (2.2.22)$$

$$\int f(x)\delta(x-x')dx = f(x') \quad (2.2.23)$$

so eq.(2.1.22) becomes

$$\vec{T}_{yz} = R^3\int d\vec{\Omega}_1\vec{K}_1\delta\vec{B}_{r1}(r1) \quad (2.2.24)$$

$$\delta\vec{B}^{(3)}(r_1) = -\frac{1}{3c^3}\int \vec{r}_1^2 d\vec{r}_1 d\vec{\Omega}_2 \delta(r_1 - R)\left(\frac{\partial^3 \vec{K}_2}{\partial t^3} \times \vec{r}_2\right).\hat{r}_1 \tag{2.2.25}$$

$$\delta\vec{B}_{r1}^{(3)}(r1) = -\frac{R^2}{3c^3}\int d\vec{\Omega}_2(\frac{\partial^3 \vec{K}_2}{\partial t^3} \times \vec{r}_2).\hat{r}_1 \tag{2.2.26}$$

$$\frac{\partial \vec{K}}{\partial t} = \frac{\partial \vec{k}}{\partial \phi}\frac{\partial \phi}{\partial t} = -\vec{\Omega}_c \frac{\partial \vec{K}_{\theta_2}}{\partial \phi} \tag{2.2.27}$$

$\frac{\partial \phi}{\partial t} = \vec{\Omega}$

$$\frac{\partial^3 \vec{K}}{\partial t^3} = -\vec{\Omega}^3 \frac{\partial^3 \vec{K}_{\theta_2}}{\partial \phi_2^3} \tag{2.2.28}$$

substitute this in to the $\delta\vec{B}_{r_1}^3$ equation.

$$\delta\vec{B}_{r1}^3 = \frac{R^3}{3c^3}\int d\vec{\Omega}_2[(\frac{\partial^3 \vec{K}_{\theta_2}}{\partial t^3})\hat{\phi}_2 - (\frac{\partial^3 \vec{K}_{\phi_2}}{\partial t^3})\hat{\theta}_2].\hat{r}_1 \tag{2.2.29}$$

By integrating three times by part we obtain,

$$\delta\vec{B}_{r1}^{(3)} = \frac{1}{3}(\frac{RvecOmega}{c})^3 \hat{r}_1.\int d\vec{\Omega}_2(\vec{K}_{\theta_2}\frac{\partial^3 \hat{\phi}_2}{\partial \phi_2^3} - \vec{K}_{\phi_2}\frac{\partial^3 \hat{\theta}_2}{\partial \phi_2^3}) \tag{2.2.30}$$

$$\hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j} \tag{2.2.31}$$

$$\hat{\theta} = \cos\theta(\cos\phi\hat{i} + \sin\phi\hat{j}) \tag{2.2.32}$$

$$\frac{\partial^3 \hat{\phi}}{\partial \phi^3} = \cos\phi\hat{i} + \sin\phi\hat{j} \tag{2.2.33}$$

$$\frac{\partial^3 \hat{\theta}}{\partial \phi^3} = -\cos\theta(-\cos\phi\hat{i} + \sin\phi\hat{j}) = -\cos\theta\hat{\phi} \tag{2.2.34}$$

so the integral becomes

$$\delta\vec{B}_{r1}^{(2)} = \frac{1}{3}(\frac{a\vec{\Omega}}{c})^3 \hat{r}_1.\int d\vec{\Omega}_2[\vec{K}_{\theta_2}(\hat{\theta}_2\cos\theta_2 + \hat{r}_2\sin\theta_2) + \vec{K}_{\phi 2}(\hat{\phi}_2\cos\theta_2)] \tag{2.2.35}$$

$$\begin{aligned} \hat{r_1}.[\vec{K}_{\hat{\theta}_2}(\hat{\theta_2}\cos\theta_2 + \hat{r_2}\cos\theta_2) + \vec{K}_{\phi 2}(\hat{\phi_2}\cos\theta_2)] = \quad & \hat{r1}.[(\cos\theta_2)\vec{K}_2 \\ & +(\vec{K}_{\theta_2}\sin\theta_2\hat{r_2})] \\ & = (\hat{r_2})_z(\hat{r_1}.\hat{r_2}) \\ & -(\vec{K}_2)_z(\hat{r_1}.\hat{r_2}) \end{aligned} \tag{2.2.36}$$

$$= [\hat{r_2}(\hat{r_1}.K_2) - \vec{K_2}(\hat{r_1}.\hat{r_2})]_z = [\hat{r_1} \times (\hat{r_2} \times \vec{K_2})]_z \tag{2.2.37}$$

$$\delta \vec{B}^{(3)}_{r_1} = \frac{1}{3}(\frac{R\vec{\Omega}_c}{c})^3[\hat{r_1} \times \int d\vec{\Omega}_2(\hat{r_2} \times \vec{K})]_z \tag{2.2.38}$$

M=$\int d^3r \frac{(\vec{r} \times \vec{j})}{2}$ is the magnetic dipole moment and in the last step we have used $\vec{M_x}$=0 for our choice of coordinate axes. It is clear at this point that only the dipole contribute to the retardation torque. The last equation can be multiplied by 2/2 to substitute the magnetic dipole.and we have,

$$\delta \vec{B}^{(3)}_{r_1} = \frac{2}{3}(\frac{R\vec{\Omega}_c}{c})^3[\hat{r_1} \times \int \frac{d\vec{\Omega}_2(\hat{r_2} \times \vec{K})}{2}]_z \tag{2.2.39}$$

$$= \frac{2}{3}\frac{\Omega_c^3}{c^3}(\hat{r_1} \times \vec{M})_z \tag{2.2.40}$$

$(\hat{r_1} \times \vec{M})_z$, so the term contribute this is the y component of the magnetic dipole and the $\vec{\times}_1$ component of the $\hat{r_1}$ are,$\vec{M} = \vec{M_y} + \vec{M_z}$, $\hat{r_1} = \hat{x_1} + \hat{y_1} + \hat{z_1}$ therefore $\vec{M} = \vec{M_y}$, $\hat{r_1} = \hat{x_1}$

$$\delta \vec{B}^{(3)}_{r_1} = \frac{2}{3}\frac{\vec{\Omega}_c^3}{c^3}(\hat{x}_1 \times \vec{M})_z \tag{2.2.41}$$

$$\delta \vec{B}^{(3)}_{r_1} = \frac{2}{3}\frac{\vec{\Omega}_c^3}{c^3}\hat{x}_1\vec{M_y} \tag{2.2.42}$$

where, $\vec{r_1} = R\hat{r_1}$

$$\delta \vec{B}^{(3)}_{r_1} = \frac{2}{3}\frac{\vec{\Omega}_c^3}{c^3}\frac{R\hat{x}_1}{R}\vec{M_y} = \frac{2}{3}\frac{\vec{\Omega}_c^3}{c^3}\frac{\vec{r_1}}{R}\vec{M_y} \tag{2.2.43}$$

after substitution eq.(2.2.43) in to eq.(2.2.24)

$$\vec{T}_{yz} = \frac{2}{3}(\frac{a\vec{\Omega}_c}{c})^3\vec{M_y}\int d\vec{\Omega}_1(\frac{\vec{x_1}}{R})\vec{K_1} \tag{2.2.44}$$

Now $\vec{K_1}$ is in the tilted coordinate frame: $\vec{K}(r_1) = \hat{x}'K_{x'} + \hat{y}'K_{y'}$ the current in the $\hat{x}'$ has no contribution for the dipole moment, so $K(r_1) = \hat{y}'K_{y'}$

$$\vec{T}_{yz} = \hat{y}'\frac{2}{3}(\frac{a\Omega_c}{c})^3\vec{M_y}\int d\vec{\Omega}_1(\frac{\vec{x_1'}}{R})\vec{K}_{y'} \tag{2.2.45}$$

for any current $\vec{K}_1$ with axial symmetry about z',

$$a^{-1}\int d\vec{\Omega}x'\vec{K}_{y'} = -R^{-1}\int d\vec{\Omega}y'\vec{K}_{x'} = (2R)^{-1}\int d\vec{\Omega}(x'\vec{K}_{y'} - y'\vec{K}_{x'}) = \frac{\vec{M}_{y'}}{R^3} = \frac{\vec{M}}{R^3} \quad (2.2.46)$$

eq.(2.2.45) is simplified after substituting eq.(2.2.46) in to and $\vec{M}_y = \vec{M}\sin\theta$

$$\vec{T}_{yz} = \hat{y'}\frac{2}{3}\frac{\vec{\Omega}_c^3}{c^3}\vec{M}_y\vec{M} = \hat{y'}\frac{2}{3}\frac{\vec{\Omega}_c^3}{c^3}\vec{M}^2\sin\theta \quad (2.2.47)$$

$$\vec{T}_{yz} = \frac{2I_c}{3I_c}\frac{\vec{\Omega}_c^3}{c^3}\vec{M}\times(\vec{M}\times\vec{\Omega}_c) \quad (2.2.48)$$

let $\omega_{yz} = \frac{2}{3}(\frac{\vec{M}^2\vec{\Omega}_c^2}{I_c c^3})$

$$\vec{T}_{yz} = I_c\vec{\omega}_{yz}\hat{M}\times(\hat{M}\times\vec{\Omega}_c) \quad (2.2.49)$$

This torque changes the magnitude and direction of $\vec{\Omega}_c$. The alignment is governed by $\dot{\theta} = -\dot{\vec{\Omega}}_x$. Thus $\vec{T}_z$ causes spin-down T_y, alignment, and T_x causes Ω_y. But $\vec{\Omega}_y$ dose not result in alignment. The reason is that our x-axis is always such that M is in the (yz)-plane, at which instant $\dot{\theta}$ caused by $\vec{\Omega}_y$ is zero. quadrupole-quadrupole torque also have anomalous component, again with $\vec{T}_z = 0$, but this tine, y-components are present, essentially because the y-direction is defined by the dipole moment, and the quadrapole moment can be oriented arbitrarily with respect to dipole. Thus anomalously large quadrupole terms will affect alignment, but not spin-down,for a rigid spherical neutron star.

The dipole torque has x-component, varies as $\sin\theta\cos\theta$. The near-field current have had a large effect, but the z-component is zero, as expected. However, the anomalous dipole influences alignment.

The $\hat{x}$ component has anomalous and normal terms, but the first ones dominate because the normal terms are much smaller than the anomalous ones. The anomalous contribution is given by

$$\vec{T}_x = (I_c\omega_x\cos\theta)\vec{\Omega}_c\times\hat{M}............(9) \quad (2.2.50)$$

where $\vec{\omega}_x = \frac{4}{3}(\frac{\vec{M}^2 Omega\_c}{I_c c^2 R})$ for most pulsars, $\vec{\omega}_x$ is much greater than $\vec{\omega}_{yz}$. Despite this, in the rigid dipole model the anomalous term does not contribute to the alignment of the magnetic dipole with respect to $\vec{\Omega}_c$ or to the spin down of the star. However, in the two-component model considered here the situation changes and the anomalous term acquires a significant role.

2.3 Frictional Torque

The outer core region of a neutron star is a mixture of superfluid neutrons, superconducting protons, and normal electrons. The effects of viscosity and friction due to the scattering of the electrons by the neutron and proton vortices give place to effective torques between the core and the crust. The resulting crust-core friction can be characterized by a time parameter τ_f. The internal dynamics of radio pulsars spinning down may be quite different (Alpar 1993). In the core of the neutron star, the rotating neutron superfluid is coupled to the stellar crust by interactions between the charged particles(electrons and protons) and the quantized vortices.

The period of the crust's precession are affected by the coupling of the crust to the liquid interior. The precession creates time-dependent velocity differences between the crust and liquid that vary over the star's spin period. If the coupling time τ_f between the crust and the liquid interior is much longer than the crust's spin period P, the precession will damp over $\simeq \frac{2\pi\tau_f}{P}$ precession periods. All studies of damping of differential rotation(different parts of a rotating object move with different angular velocity) between the crust and various parts of the liquid interior give $\tau_f \gg P$ (e.g., Alpar and Sauls 1988; Epstein and Baym 1992; Jones 1992; Abney, Epstein, and Olinto 1996; Mendell 1998). To a good approximation, therefore, the liquid interior can be treated as decoupled from the solid (Bondi and Gold 1955; Sedrakian et al. 1999) ; in this regime the crust precesses almost as if the liquid interior were not there.(37)

In current models of pulsar glitches (Pines and Alpar 1985, Link and Epstein 1996)the observed spin up is due to the transfer of angular momentum from the neutron superfluid in the inner-crust to the solid crust. In these crust-initiated models,subsequent exchange of angular momentum between the crust and the core brings these two components into rotational equilibrium. In alternative core-initiated models, the initial spin-up occurs in the core rather than the crust (Sauls 1989) and the crust then catches up to the cores angular velocity.(36)

For either the crust-initiated or core-initiated models, the Vela Christmas glitch strongly constrains the core-crust coupling time-scale. A linear coupling model for crust-interior interaction can illustrate the types of constraints different glitch models yield. If I_c and Ω_c are the moment of inertia and angular velocity of the solid crust, I_o and Ω_o are the moment of inertia and angular velocity of the liquid interior, and. There are several phenomenological estimations on the basis of the glitch dynamics. For example, upper bounds are given for the Vela pulsar of $\tau_f < 10s$ for a crust-initiated glitch and of $\tau_f < 440s$ for a core-initiated glitch (Abney,Epstein, and Olinto 1996). These values are deduced from the 1980 December 24 Christmas glitch data (McCulloch et al.1990). This point has also awakened a great deal of theoretical interest. The interior plasma couples the neutron superfluid due to mixing superfluid effects (Alpar, Langer, and Sauls 1984; Alpar and Sauls 1988), and it is locked to the crust because of Alfvn waves or cyclotron vortex waves. An extensive analysis of these phenomena has been performed by Mendell (1991a, 1991b, 1998c)on the basis of the Newtonian superfluid hydrodynamics,generalized to include dissipation.The characteristic times related to all these couplings are of order 1 s. The average friction torque obtained by considering the scattering of the electron fluid by the neutron and proton vortices has the form.[9]

$$\vec{T}_f = \frac{I_c I_o}{I_c + I_o}\left(\frac{\vec{\Omega}_c - \vec{\Omega}_o}{\tau_f}\right)..........[9] \quad (2.3.1)$$

let f=$\frac{I_c I_o}{(I_c+I_o)\tau_f}$

$$\vec{T}_f = f(\vec{\Omega}_c - \vec{\Omega}_o) \quad (2.3.2)$$

When the system is taken out of equilibrium, such as in a glitch, the time-response frequency is given by $1/\tau_f = \omega_f = f(I_t/I_c I_o)$, where I_c and I_o are the moments of inertia of the components, and $I_t = I_c + I_o$ is the total star moment of inertia. Therefore this torque is relevant for the rapid nucleus spin-up during glitches. The effective friction could depend on the angular velocity. Non dissipative effects give torques of the same kind as the gravitational dragging, which is to be discussed in the next subsection. If they are also characterized by a time of order 1s, they should be smaller than the gravitational torque for a rapidly rotating star.[9]

2.4 Gravitational Torque

The neutron star is essentially constituted by superfluid matter, hence the observed rotation can be achieved only by the presence of vortices. The gravitational fields are strong enough to be relevant. They induce a change in the shape of the vortex lines and also affect the density of vortices (Casini and Montemayor 1997). The main contribution of these effects is a correction on the coefficients of the dissipative torques of the order of with respect to the flat spacetime values, but they do not introduce new terms. An additional term arises from the gravitational dragging, which gives place to a new torque if the components of the star have different angular velocities. The gravitational vector potential

$$\vec{\zeta}(x) = \frac{2G}{R^3 c^2}(\vec{X} \times \vec{J}).....[38] \tag{2.4.1}$$

where $\vec{J}$ is the angular momentum and the gravitational field is

$$\vec{P} = \vec{\nabla} \times \vec{\zeta}(x) \tag{2.4.2}$$

$$\vec{P} = \frac{2G}{R^3 c^2}\vec{\nabla} \times (\vec{X} \times \vec{J}) \tag{2.4.3}$$

by the triple cross product $\vec{\nabla} \times (\vec{A} \times \vec{B}) = \vec{A}(\vec{\nabla}.\vec{B}) - \vec{B}(\vec{\nabla}.\vec{A})$

$$\vec{\nabla} \times (\vec{X} \times \vec{J}) = \vec{X}(\vec{\nabla}.\vec{J}) - \vec{J}(\vec{\nabla}.\vec{X}) \tag{2.4.4}$$

where $\vec{J}$ is independent of position so $\vec{X}(\vec{\nabla}.\vec{J}) = 0$ and also $\vec{\nabla}.\vec{X}$=3

$$\vec{\nabla} \times (\vec{X} \times \vec{J}) = -3\vec{J} \tag{2.4.5}$$

substitute the above equation into eq.(2.3.3)

$$\vec{P} = -\frac{6G}{R^3c^2}\vec{J} \tag{2.4.6}$$

$\vec{J} = I_o\Omega_o$

$$\vec{P} = -\frac{6GI_o}{R^3c^2}\vec{\Omega}_o \tag{2.4.7}$$

$$\vec{F_g} = m_c v \times \vec{P} = m(\vec{\Omega}_c \times \vec{r}) \times \vec{P} \tag{2.4.8}$$

$(\vec{A} \times \vec{B}) \times \vec{C} = (\vec{A}.\vec{C})\vec{B} - (\vec{C}.\vec{B})\vec{A}$

where P, v, r are gravitational field, velocity of the crust and the radius of the region, respectively. The torque acting on the crust is given by

$$\vec{T_g} = r \times \vec{F_g} = m_c\vec{r} \times (\vec{\Omega}_c \times \vec{r}) \times \vec{P} \tag{2.4.9}$$

$$\vec{T_g} = m_c\vec{r} \times (\vec{\Omega}_c \times \vec{r}) \times \vec{P} = m\vec{r} \times [(\vec{\Omega}_c.\vec{P})\vec{r} - (\vec{r}.\vec{P})\vec{\Omega}_c] \tag{2.4.10}$$

$$\vec{T_g} = m_c(\vec{r}.\vec{P})(\vec{r} \times \vec{\Omega}_c) \tag{2.4.11}$$

$$\vec{T_g} = m_c\frac{6GI_o}{R^3c^2}(\vec{r}.\vec{\Omega}_o)(\vec{r} \times \vec{\Omega}_c) \tag{2.4.12}$$

$$\vec{T_g} = m_c\frac{6GI_o}{R^3c^2}\vec{r}.\vec{r}(\vec{\Omega}_c \times \vec{\Omega}_o) \tag{2.4.13}$$

$$\vec{T_g} = m_c r^2 I_o\frac{18G}{3R^3c^2}(\vec{\Omega}_c \times \vec{\Omega}_o) \tag{2.4.14}$$

$I_c = \frac{2}{3}mr^2$

$$\vec{T_g} = z(\vec{\Omega}_c \times \vec{\Omega}_o) \tag{2.4.15}$$

where, $z = I_o I_c \frac{9G}{r^3c^2} \simeq I_c$ For small velocities, this torque could have sizeable effects on the dynamics of the magnetic dipole. The torques $\vec{T_f}$ and $\vec{T_g}$ can be seen as the first terms of the crust-core torque expansion around the point $\vec{\Omega}_o = \vec{\Omega}_o$, and as such the form of the interaction is largely independent of the model. There is only one relevant dipolar magnetic moment present, but it can also be applied when there are several coupled moments with a dynamics much faster than that of the superfluid core-crust interaction, in such a way that the faster dynamics decouples, giving place to a single effective dipole M. In the following we will only consider terms up to first order in the angle between $\vec{\Omega}_c$ and $\vec{\Omega}_o$. As we will discuss later, this is enough for our analysis. In the next section we will use all these torques to construct a set of equations of motion that defines the dynamics of the star components.

Chapter 3

THE EQUATION OF MOTION

The equation of motion for the system described in this section is

$$\dot{\vec{M}} = \vec{\Omega}_c \times \vec{M} \tag{3.0.1}$$

$$\dot{\vec{\Omega}}_c = \vec{\omega}_x \cos\theta \vec{\Omega}_c \times \hat{M} + \vec{\omega}_{yz} \hat{M} \times (\hat{M} \times \vec{\Omega}_c) - \frac{f}{I_c}(\vec{\Omega}_c - \vec{\Omega}_o) + \frac{z}{I_c}\vec{\Omega}_c \times \vec{\Omega}_o \tag{3.0.2}$$

$$\dot{\vec{\Omega}}_o = \frac{f}{I_c}(\vec{\Omega}_c - \vec{\Omega}_o) + \frac{Z}{I_o}\vec{\Omega}_c \times \vec{\Omega}_o \tag{3.0.3}$$

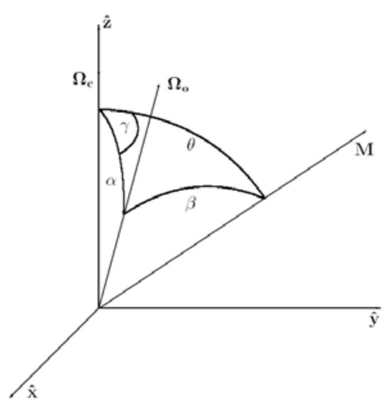

Figure 3.1: crust angular velocity $\vec{\Omega}_c$, which defines the z-axis.the magnetic moment M is in the y-z plane,with an angle θ with respect to the z-direction.the core angular velocity $\vec{\Omega}_o$ forms an angle α with $\vec{\Omega}_c$ and β with M

If $f \longrightarrow 0$ the crust decouples from the core, and if f$\rightarrow \infty$ the two components act as a rigid body. Equation(3.0.1) implies that the magnitude is constant. Therefore, we have only five variables in the system, three angles α, β, and θ, defined in Figure-1 and the two angular velocities, Ω_c and Ω_o. For these variables the equations of motion are Ω_c in the direction of Ω_0 is

$$\vec{\Omega}_o = \vec{\Omega}_o \cos\alpha \hat{\Omega}_c + \vec{\Omega}_o \sin\alpha \hat{y} \tag{3.0.4}$$

$\vec{\Omega}_c$ in the direction of $\vec{\Omega}_0$ is

$$\vec{\Omega}_c = \vec{\Omega}_c \vec{\ } \cos\alpha \hat{\Omega}_o + \vec{\Omega}_c \sin\alpha \hat{\Omega}_o \perp \tag{3.0.5}$$

The first term of eq.(3.0.2) and the second term of eq.(3.0.3) have not the contribution for the magnetude of the alignment of dipole moment. They are perpendicular to Ω_c but that have significant role on the precession of the dipole. so, we drop them. Ω_o in the direction of Ω_c change $\vec{\Omega}_o$ in to $\vec{\Omega}_o \cos\alpha$

$$\dot{\vec{\Omega}} = \vec{\omega}_{yz} \hat{M} \times (\hat{M} \times \vec{\Omega}_c) - \frac{f}{I_c}(\vec{\Omega}_c - \vec{\Omega}_o \cos\alpha) \tag{3.0.6}$$

$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B})$

$$\hat{M} \times (\hat{M} \times \vec{\Omega}_c) = \hat{M}(\hat{M}.\vec{\Omega}_c) - \vec{\Omega}_c(\hat{M}.\hat{M}) \tag{3.0.7}$$

$$\hat{M} \times (\hat{M} \times \vec{\Omega}_c) = \hat{M}(\vec{\Omega}_c \cos\theta) - \vec{\Omega}_c \tag{3.0.8}$$

$\hat{M} = \cos\theta \hat{\Omega}_c + \sin\theta \hat{y}$,$\hat{M}$ in the direction of $\vec{\Omega}_c$ is $\cos\theta \hat{\Omega}_c$ so eq.(3.0.8) becomes

$$\hat{M} \times (\hat{M} \times \vec{\Omega}_c) = \vec{\Omega}_c \cos^2\theta - \vec{\Omega}_c \tag{3.0.9}$$

and replace $\cos^2\theta$ in to the identity equation $\cos^2\theta = 1 - \sin^2\theta$

$$\hat{M} \times (\hat{M} \times \vec{\Omega}_c) = \vec{\Omega}_c(1 - \sin^2\theta) - \vec{\Omega}_c \tag{3.0.10}$$

$$\hat{M} \times (\hat{M} \times \Omega_c) = -\vec{\Omega}_c \sin^2\theta \tag{3.0.11}$$

eq.(3.0.6) becomes

$$\dot{\vec{\Omega}}_c = -\vec{\omega}_{yz} \vec{\Omega}_c \sin^2\theta - \frac{f}{I_c}(\vec{\Omega}_c - \vec{\Omega}_o \cos\alpha) \tag{3.0.12}$$

$$\dot{\vec{\Omega}}_o = \frac{f}{I_c}(\vec{\Omega}_c \cos\alpha - \vec{\Omega}_o) \tag{3.0.13}$$

These two equations are the magnitude of crust and the core angular accelerations. The torques on the crust which affect the magnitude and the orientation of the magnetic dipole can be written as

$$\frac{d\vec{\Omega}_c}{dt} = \hat{M} \times (\hat{M} \times \vec{\Omega}_c) - \frac{f}{I_c}(\vec{\Omega}_c - \vec{\Omega}_o) + \frac{z}{I_o}\vec{\Omega}_c \times \vec{\Omega}_o \tag{3.0.14}$$

we can take the dot product of both side of the above equation and the magnetic dipole $\hat{M}$, and also we ignore the evolution of $\vec{\Omega}_c$

$$\frac{d\vec{\Omega}_c}{dt}.\hat{M} = [\hat{M} \times (\hat{M} \times \vec{\Omega}_c].\hat{M} - \frac{f}{I_c}(\vec{\Omega}_c - \vec{\Omega}_o).\hat{M} + [\frac{z}{I_o}\vec{\Omega}_c \times \vec{\Omega}_o].\hat{M} \tag{3.0.15}$$

$$\cos\theta\frac{d\vec{\Omega}_c}{dt} + \vec{\Omega}_c\frac{d\cos\theta}{dt} = -\vec{\omega}_{yz}\vec{\Omega}_c.\hat{M}\sin^2\theta - \frac{f}{I_c}(\vec{\Omega}_c.\hat{M} - \vec{\Omega}_o.\hat{M}) + \frac{z}{I_o}(\vec{\Omega}_c \times \vec{\Omega}_o).\hat{M} \tag{3.0.16}$$

$$\frac{d\cos\theta}{dt} = -\vec{\omega}_{yz}\cos\theta\sin^2\theta - \frac{f}{I_c}\frac{1}{\vec{\Omega}_c}(\vec{\Omega}_c\cos\theta - \vec{\Omega}_o\cos\beta) - \frac{z}{I_c}\vec{\Omega}_o.(\hat{\Omega}_c \times \hat{M}) \tag{3.0.17}$$

The magnitude of the angular velocity $\vec{\Omega}_c$ in the direction of $\vec{\Omega}_o$ is $\vec{\Omega}_c = \vec{\Omega}_o\cos\alpha$. And after substituting this into the above equation, we get

$$.\frac{d\cos\theta}{dt} = -\vec{\omega}_{yz}\cos\theta\sin^2\theta + \frac{f}{I_c}\frac{\vec{\Omega}_o}{\vec{\Omega}_c}(\cos\beta - \cos\alpha\cos\theta) - \frac{z}{I_c}\vec{\Omega}_o.(\hat{\Omega}_c \times \hat{M}) \tag{3.0.18}$$

$$\frac{d}{dt}\cos\theta = \frac{f}{I_c}(\cos\beta - \cos\alpha\cos\theta) + \vec{\omega}_{xy}\sin^2\theta\cos\theta + \frac{z}{I_c}\mathit{\Omega}_o[\hat{\mathit{\Omega}}_O.(\hat{\mathit{\Omega}}_O \times \hat{M})] \tag{3.0.19}$$

Torques which affect the rotational dynamics of system can be written as

$$\frac{d\vec{\Omega}_o}{dt} = \frac{f}{I_o}(\vec{\Omega}_c - \vec{\Omega}_o) + \frac{z}{I_o}\vec{\Omega}_c \times \vec{\Omega}_o \tag{3.0.20}$$

Take the dot product of both sides of the eq.(3.0.20) with $\hat{M}$ is

$$\frac{d\vec{\Omega}_o}{dt}.\hat{M} = \frac{f}{I_o}(\vec{\Omega}_c - \vec{\Omega}_o).\hat{M} + \frac{z}{I_o}[\vec{\Omega}_c \times \vec{\Omega}_o].\hat{M} \tag{3.0.21}$$

$$\cos\beta\frac{d\vec{\Omega}_o}{dt} + \vec{\Omega}_o\frac{d\cos\beta}{dt} = \frac{f}{I_o}(\vec{\Omega}_c.\hat{M} - \vec{\Omega}_o.\hat{M}) + \frac{z}{I_o}[\vec{\Omega}_c \times \vec{\Omega}_o].\hat{M} \tag{3.0.22}$$

$\cos\beta\frac{d\vec{\Omega}_o}{dt} \simeq \frac{d\vec{\Omega}_c}{dt}$ $\frac{d\vec{\Omega}_c}{dt} = \frac{2\vec{\Omega}_c^2}{3Ic^3}(\vec{\Omega}_c \times \vec{M}) \times \vec{M} + \frac{1}{Rc^2}(\vec{M}.\vec{\Omega}_c)(\vec{\Omega}_c \times \vec{M}) - \vec{\Omega}_c \times \vec{\Omega}_o$...[37] $(\frac{d\vec{\Omega}_c}{dt}).\hat{M} = \frac{2\vec{\Omega}_c^2}{3Ic^3}[(\vec{\Omega}_c \times \vec{M}) \times \vec{M}].\hat{M} + \frac{1}{Rc^2}[(\vec{M}.\vec{\Omega}_c)(\vec{\Omega}_c \times \vec{M})].\hat{M} - (\vec{\Omega}_c \times \vec{\Omega}_o).\hat{M}$ $(\frac{d\vec{\Omega}_c}{dt}).\hat{M} = -(\vec{\Omega}_c \times \vec{\Omega}_o).\hat{M}$

$$\vec{\Omega}_o\frac{d\cos\beta}{dt} = \frac{f}{I_o}(\vec{\Omega}_c.\hat{M} - \vec{\Omega}_o.\hat{M}) - (\frac{d\vec{\Omega}_c}{dt}).\hat{M} + \frac{z}{I_o}(\vec{\Omega}_c \times \vec{\Omega}_o).\hat{M} \tag{3.0.23}$$

$$\vec{\Omega}_o \frac{d\cos\beta}{dt} = \frac{f}{I_o}(\vec{\Omega}_c\cos\theta - \vec{\Omega}_o\cos\theta) - (\vec{\Omega}_c \times \vec{\Omega}_o).\hat{M} + \frac{z}{I_o}(\vec{\Omega}_c \times \vec{\Omega}_o).\hat{M} \tag{3.0.24}$$

The magnitude of $\vec{\Omega}_o$ is $\vec{\Omega}_o = \vec{\Omega}_c \cos\theta$ and substituting in equ.(3.0.24)

$$\frac{d\cos\beta}{dt} = \frac{f}{I_o}\frac{\vec{\Omega}_c}{\vec{\Omega}_o}(\cos\theta - \cos\theta\cos\alpha) + \vec{\Omega}_o.(\vec{\Omega}_c \times \hat{M}) - \frac{z}{I_o}\vec{\Omega}_o.(\vec{\Omega}_c \times \hat{M}) \tag{3.0.25}$$

$$\frac{d}{dt}\cos\beta = \frac{f}{I_c}\frac{\mathit{\Omega}_c}{\mathit{\Omega}_o}(\cos\theta - \cos\beta\cos\alpha) + \mathit{\Omega}_c(1 - \frac{Z}{I_o})[\hat{\mathit{\Omega}}_o.(\hat{\mathit{\Omega}}_c \times \hat{M})] \tag{3.0.26}$$

Torques which affect the rotational dynamics of both the crust and the core is,

$$\frac{d\Omega_c}{dt} = \vec{\omega}_x \cos\theta\Omega_c \times \hat{M} + \vec{\omega}_{yz}\hat{M} \times (\hat{M} \times \vec{\Omega}_c) - \frac{f}{I_c}(\vec{\Omega}_c - \vec{\Omega}_o) \tag{3.0.27}$$

The dot product of eq.(3.0.27) and Ω_c are given as

$$\frac{d\vec{\Omega}_c}{dt}.\hat{\Omega}_o = \vec{\omega}_x \cos\theta[\vec{\Omega}_c \times \hat{M}].\hat{\Omega}_o + \vec{\omega}_{yz}[\hat{M} \times (\hat{M} \times \vec{\Omega}_c)].\hat{\Omega}_o - \frac{f}{I_c}(\vec{\Omega}_c - \Omega_o).\hat{\Omega}_o \tag{3.0.28}$$

$$\begin{aligned}\cos\alpha\frac{d\vec{\Omega}_c}{dt} + \vec{\Omega}_c\frac{d\cos\alpha}{dt} = \quad & \vec{\omega}_x\cos\theta\hat{\Omega}_o.(\vec{\Omega}_c \times \hat{M}) + \\ & \vec{\omega}_{yz}\hat{\Omega}_o.(\hat{M} \times (\hat{M} \times \vec{\Omega}_c)) - \frac{f}{I_c}(\vec{\Omega}_c.\hat{\Omega}_o \\ & -\vec{\Omega}_o.\hat{\Omega}_c) - \frac{f}{I_o}(\vec{\Omega}_c.\hat{\Omega}_c - \vec{\Omega}_o.\hat{\Omega}_o)\end{aligned} \tag{3.0.29}$$

$$\hat{\Omega}_o.[\hat{M} \times (\hat{M} \times \vec{\Omega}_c)] = \hat{\Omega}_o.[(\hat{M}.\vec{\Omega}_c)\hat{M} - (\hat{M}.\hat{M})\vec{\Omega}_c] \tag{3.0.30}$$

$\hat{M} = \vec{M}\cos\theta\hat{\Omega}_c + \vec{M}\sin\theta\hat{\Omega}_c \perp$ we take the Ω_c component

$$\hat{\Omega}_o.[\hat{M} \times (\hat{M} \times \vec{\Omega}_c)] = [\vec{\Omega}_c\cos\theta\hat{\Omega}_o.\hat{M} - \cos^2\theta\hat{\Omega}_o.\vec{\Omega}_c] \tag{3.0.31}$$

$$\hat{\Omega}_o.[\hat{M} \times (\hat{M} \times \vec{\Omega}_c)] = [\vec{\Omega}_c\cos\theta\cos\beta - \cos^2\theta\vec{\Omega}_c\cos\alpha] \tag{3.0.32}$$

$$\cos\alpha\frac{d\vec{\Omega}_c}{dt}\hat{\Omega}_o \simeq \cos\alpha\frac{d\vec{\Omega}_o}{dt}\hat{\Omega}_o \tag{3.0.33}$$

$$\frac{d\vec{\Omega}_o}{dt}\hat{\Omega}_o = \frac{f}{I_o}(\vec{\Omega}_c - \vec{\Omega}_o) \tag{3.0.34}$$

The dot product of eq.(3.0.34)and Ω_c is

$$\cos\alpha\frac{d\vec{\Omega}_o}{dt} = \frac{f}{I_o}(\vec{\Omega}_c.\hat{\Omega}_o - \vec{\Omega}_o.\hat{\Omega}_o) \tag{3.0.35}$$

and also the component of $\vec{\Omega}_c$ along $\vec{\Omega}_o$ is $\cos\alpha\hat{\Omega}_o$

$$\frac{1}{\vec{\Omega}_c}\cos\alpha\frac{d\vec{\Omega}_o}{dt} = \frac{f}{I_o}(\cos\alpha - \frac{\vec{\Omega}_o}{\vec{\Omega}_c}\cos^2\alpha) \tag{3.0.36}$$

The dot product $\frac{f}{I_c}(\vec{\Omega}_c - \vec{\Omega}_o)$ and $\hat{\Omega}_o$ is

$$\frac{f}{I_c\vec{\Omega}_c}(\vec{\Omega}_c.\hat{\Omega}_o - \vec{\Omega}_o.\hat{\Omega}_o) = \frac{f}{I_o\vec{\Omega}_c}(\vec{\Omega}_c\cos\alpha - \vec{\Omega}_o\cos_2\alpha) \tag{3.0.37}$$

and also the component of $\vec{\Omega}_o$ along $\vec{\Omega}_c$ is $\cos\alpha\hat{\Omega}_c$

$$= \frac{f}{I_c}(\cos\alpha - \frac{\vec{\Omega}_o}{\vec{\Omega}_c}\cos^2\alpha) \tag{3.0.38}$$

now we can substitute eq.(3.0.32), eq.(3.0.36) and eq.(3.0.38) into eq.(3.0.29) and we get

$$\begin{aligned}\frac{d\cos\alpha}{dt} = \quad & \vec{\omega}_x\cos\theta[\hat{\Omega}_0.(\hat{\Omega}_c\times\hat{M})] - \vec{\omega}_{yz}\hat{\Omega}_c\cos\theta(\cos\beta - \cos\alpha\cos\theta) \\ & -\frac{f}{I_c}(\cos\alpha - \frac{\Omega_o}{\vec{\Omega}_c}\cos^2\alpha) - \frac{f}{I_o}(\cos\alpha - \frac{\vec{\Omega}_o}{\vec{\Omega}_c}\cos^2\alpha)\end{aligned} \tag{3.0.39}$$

$\cos^2\alpha = 1 - \sin^2\alpha$ and $\cos\alpha \simeq 1$

$$\begin{aligned}\frac{d\cos\alpha}{dt} = \quad & \vec{\omega}_x\cos\theta[\hat{\Omega}_0.(\hat{\Omega}_c\times\hat{M})] - \vec{\omega}_{yz}\hat{\Omega}_c\cos\theta(\cos\beta - \cos\alpha\cos\theta) \\ & -\sin^2(\frac{f\vec{\Omega}_o}{I_c\vec{\Omega}_c} + \frac{f\vec{\Omega}_o}{I_o\vec{\Omega}_c}) + (\frac{f\vec{\Omega}_o}{I_c\vec{\Omega}_c} + \frac{f\vec{\Omega}_o}{I_o\vec{\Omega}_c}) - (\frac{f}{I} + \frac{f}{I})\end{aligned} \tag{3.0.40}$$

$(\frac{f\vec{\Omega}_o}{I_c\vec{\Omega}_c} + \frac{f\vec{\Omega}_o}{I_o\vec{\Omega}_c}) \simeq (\frac{f}{I} + \frac{f}{I})$,thus eq.(3.0.40) becomes,

$$\frac{d\cos\alpha}{dt} = \vec{\omega}_x\cos\theta[\hat{\Omega}_0.(\hat{\Omega}_c\times\hat{M})] - \vec{\omega}_{yz}\hat{\Omega}_c\cos\theta(\cos\beta - \cos\alpha\cos\theta) - \sin^2(\frac{f\vec{\Omega}_o}{I_c\vec{\Omega}_c} + \frac{f\vec{\Omega}_o}{I_o\vec{\Omega}_c}) \tag{3.0.41}$$

Core angular velocity is in xyz plane

$$\hat{\vec{\Omega}}_c = \sin\alpha\sin\gamma\hat{x} + \sin\alpha\cos\gamma\hat{y} + \cos\alpha\hat{z} \tag{3.0.42}$$

$$\hat{M} = \sin\theta\hat{y} + \cos\theta\hat{z} \tag{3.0.43}$$

The dot product of these two equations are

$$\cos\beta = \cos\alpha\cos\theta + \sin\alpha\sin\theta\cos\gamma \tag{3.0.44}$$

In principle, this system of equations would be very difficult to solve, but in fact it contains several dynamics with very different timescales, given by $\vec{\omega}_{yz} \ll \vec{\omega}_x \ll min[\omega_f \simeq (\frac{f}{I_c}), \Omega_c]$ which greatly simplify its treatment. From equations (3.0.12) and (3.0.13), if Ω_c and Ω_o are very different at a given instant, they will attain equilibrium in a relatively short time, of the order of τ_f. The last equation tells us that the transient of α is also characterized by τ_f, and thus, after a time of this order, this variable will acquire a value of the order $\frac{\omega_x}{\omega_f} \ll 1$. Hence, after this transient we will have $\Omega_c \sim \Omega_o, \alpha \ll 1$, and therefore $\theta \sim \beta$. a first-order approximation in α and consider the quasi-stationary regime. To simplify the expressions, instead of dealing with the very similar variables θ and β, we will replace the last angle by a new angle $\gamma\varrho$. It satisfies spherical identity $\cos\beta = \cos\alpha\cos\theta + \sin\alpha\sin\theta\cos\gamma$ and we have $\hat{\Omega}_o.\hat{\Omega}_c \times \hat{M} = \sin\gamma\sin\alpha\sin\theta$ thus, with this substitution and at first order in α, assuming $\frac{I_o}{I_c} \simeq 1, \Omega_o \simeq \Omega_c = \Omega$ we have $\frac{d\cos\theta}{dt} = \frac{f}{I_c}(\cos\beta - \cos\alpha\cos\theta) + \omega_{xy}\sin^2\theta\cos\theta + \frac{z}{I_c}\Omega_o[\hat{\Omega_O}.(\hat{\Omega_O} \times \hat{M})]$

$$\sin\theta\frac{d\theta}{dt} = \frac{f}{I_c}(\cos\beta - \cos\alpha\cos\theta) + \omega_{xy}\sin^2\theta\cos\theta + \frac{z}{I_c}\Omega_o[\hat{\Omega_O}.(\hat{\Omega_O} \times \hat{M})] \tag{3.0.45}$$

$$\hat{\Omega}_o.\hat{\Omega}_c \times \hat{M} = \sin\gamma\sin\alpha\sin\theta \tag{3.0.46}$$

$$\cos\beta = \cos\alpha\cos\theta + \sin\alpha\sin\theta\cos\gamma \tag{3.0.47}$$

substitute eq.(3.0.46) and eq.(3.0.47) in to eq.(3.0.45) and we have

$$\frac{d\theta}{dt} = \frac{-f}{I_c\sin\theta}(\sin\alpha\sin\theta\cos\gamma) + \frac{\omega_{yz}}{-\sin\theta}\sin^2\theta\cos\theta + \frac{-Z\Omega_o}{I_c\sin\theta}\sin\gamma\sin\alpha\sin\theta \tag{3.0.48}$$

$$\frac{d\theta}{dt} = -\frac{f}{I_c}\sin\alpha\sin\gamma - \omega_{yz}\sin\theta\cos\theta - \frac{Z}{I_c}\Omega_o\sin\gamma\sin\alpha \tag{3.0.49}$$

for $\alpha \ll 1$, $\sin\alpha \sim \alpha, \Omega_c \sim \Omega_o \sim \Omega$ and also $\omega_f \equiv \frac{f}{I_c}$

$$\frac{d\theta}{dt} = -\vec{\omega}_f\alpha\cos\gamma - \vec{\omega}_{yz}\sin\theta\cos\theta - \frac{Z}{I_c}\vec{\Omega}_c\alpha\sin\gamma \tag{3.0.50}$$

$\frac{\vec{\Omega}_o}{\vec{\Omega}_c} \simeq \frac{\vec{\Omega}_c}{\vec{\Omega}_o} \simeq 1$, we substitute eq.(3.0.46)and eq.(3.0.47) into eq.(3.0.41)

$$\frac{d\alpha}{dt} = -\frac{\vec{\omega}_x\cos\theta}{\sin\alpha}(\sin\alpha\sin\gamma\sin\theta) - \sin\alpha(\vec{\omega}_f + \frac{f}{I_o}) - \frac{\vec{\omega}_{yz}\cos\theta}{\sin\alpha}(\sin\alpha\sin\theta\cos\gamma) \tag{3.0.51}$$

$$\frac{d\alpha}{dt} = -\vec{\omega}_x \cos\theta \sin\gamma \sin\theta - \sin\alpha(\vec{\omega}_f + \frac{f}{I_o}) - \vec{\omega}_{yz} \cos\theta \sin\theta \cos\gamma \tag{3.0.52}$$

from the above equation $\frac{f}{I_o} \ll 1$ so it s can be approximated to zero

$$.\frac{d\alpha}{dt} = -\vec{\omega}_x \cos\theta \sin\gamma \sin\theta - \alpha\omega_f - \vec{\omega}_{yz} \cos\theta \sin\theta \cos\gamma \tag{3.0.53}$$

$$\frac{d\cos\beta}{dt} = \frac{f}{I_c}\frac{\vec{\Omega}_c}{\vec{\Omega}_o}(\cos\theta - \cos\beta\cos\alpha) + \vec{\Omega}_c(1 - \frac{Z}{I_o})[\hat{\vec{\Omega}}_o(\hat{\Omega}_c \times \hat{M})] \tag{3.0.54}$$

substitute eq.(3.0.46)and eq.(3.0.47)in the above equation and we find

$$\frac{d\cos\beta}{dt} = \frac{f}{I_c}\frac{\vec{\Omega}_c}{\vec{\Omega}_o}(\cos^2\alpha\cos\theta - \sin\alpha\cos\alpha\cos\gamma\sin\theta) + \vec{\Omega}_c(1 - \frac{Z}{I_o})\sin\gamma\sin\alpha\sin\theta \tag{3.0.55}$$

for $\alpha \ll 1, \frac{\vec{\Omega}_o}{\vec{\Omega}_c} \simeq 1, \sin\alpha \simeq \alpha$ and $\cos\alpha \simeq 1$

$$\frac{d\beta}{dt} = -\frac{f}{I_o}\alpha\sin\theta\cos\gamma + \vec{\Omega}_c\alpha\sin\gamma\sin\theta - \frac{z}{I_o}\alpha\sin\gamma\sin\theta \tag{3.0.56}$$

for $\alpha \ll 1, \vec{\Omega}_o \simeq \vec{\Omega}_c = \vec{\Omega}$ first we find the derivation eq.(3.0.47)

$$\frac{d\cos\beta}{dt} = \frac{d(\cos\alpha\cos\theta + \sin\alpha\sin\theta\cos\gamma)}{dt} \tag{3.0.57}$$

$$\frac{d\cos\beta}{dt} = -\frac{d\alpha}{dt}(\sin\theta\cos\alpha\cos\gamma - \sin\alpha\cos\theta) - \frac{d\theta}{dt}(\sin\alpha\cos\theta\cos\gamma - \cos\alpha\sin\theta) - \frac{d\gamma}{dt}\sin\alpha\sin\theta\sin\gamma \tag{3.0.58}$$

when we substitute eq.(3.0.50),eq.(3.0.53)in to eq.(3.0.58) and for $\alpha \ll 1, \sin\alpha \simeq \alpha$ we get

$$\begin{aligned}\frac{d\cos\beta}{dt} = \quad & -\vec{\omega}_x \cos\theta\sin^2\theta\sin\gamma\cos\gamma + \alpha\vec{\omega}_x\cos^2\theta\sin\theta\sin\gamma - \vec{\omega}_f\alpha\sin\theta\cos\gamma + \alpha^2\vec{\omega}_f\cos\theta \\ & -\vec{\omega}_{yz}\cos^2\gamma\sin^2\theta\cos\theta + \alpha\vec{\omega}_{yz}\cos\gamma\sin\theta\cos^2\theta - \omega_f\alpha^2\cos^2\gamma\sin\theta + \omega_f\alpha\cos\gamma\sin\theta \\ & -\omega_{yz}\alpha\sin^2\theta\cos\gamma\cos\theta + \vec{\omega}_{yz}\sin^2\theta\cos\theta - \frac{z}{I_c}\vec{\Omega}\alpha^2\sin\gamma\sin\theta\cos\theta \\ & + \frac{z}{I_c}\vec{\Omega}\alpha\sin\gamma\sin\theta - \dot{\gamma}\sin\theta\sin\alpha\sin\gamma\end{aligned}$$

$$\frac{d\beta}{dt} = -\vec{\omega}_x\cos\theta\sin^2\theta\cos\gamma\sin\gamma - \vec{\omega}_{yz}\sin^2\theta\cos^2\gamma\cos\theta + \vec{\omega}_{yz}\sin^2\theta\cos\theta\theta + \frac{Z}{I_c}\vec{\Omega}\alpha\sin\theta\sin\gamma \tag{3.0.59}$$

we can simplify the above equation by substituting the identity equation

$$\cos^2\gamma = 1 - \sin^2\gamma \tag{3.0.60}$$

$$\frac{d\beta}{dt} = -\vec{\omega}_x \cos\theta \sin^2\theta \cos\gamma \sin\gamma - \vec{\omega}_{yz} \sin^2\theta \sin^2\gamma \cos\theta + \frac{Z}{I_c}\vec{\Omega}\alpha \sin\theta \sin\gamma \tag{3.0.61}$$

by equating eq.(3.0.61)and eq.(3.0.56) we get

$$\dot{\gamma} = -\vec{\Omega}(1 - \frac{Z}{I_c}) - \frac{\vec{\omega}_x}{\alpha} \cos\gamma \sin\theta \cos\theta + \frac{\vec{\omega}_{yz}}{\alpha} \sin\theta \cos\theta \sin\gamma \tag{3.0.62}$$

The solutions for α and γ can be decomposed in a transient dynamics,with a characteristic time τ_f plus a slow-varying time function. This implies that α and γ will reach a quasi-stationary regime in a few seconds. From here on they will be driven by the slow time dependence of θ. The equation for θ contains only small frequencies. The explicit solutions at the fixed point for α and γ are,

$$0 = -\vec{\omega}_x \cos\theta \sin\gamma \sin\theta - \alpha\omega_f - \vec{\omega}_{yz} \cos\theta \sin\theta \cos\gamma \tag{3.0.63}$$

$$0 = -\vec{\Omega}(1 - \frac{Z}{I_c}) - \frac{\vec{\omega}_x}{\alpha} \cos\gamma \sin\theta \cos\theta + \frac{\vec{\omega}_{yz}}{\alpha} \sin\theta \cos\theta \sin\gamma \tag{3.0.64}$$

and square each of them

$$\vec{\omega}_f^2 = \frac{\vec{\omega}_x^2}{\alpha^2} \cos^2\theta \sin^2\theta \sin^2\gamma + \frac{\vec{\omega}_{yz}^2}{\alpha^2} \sin^2\theta \cos^2\theta \cos^2\gamma + 2\frac{\omega_x \vec{\omega}_{yz}}{\alpha^2} \sin\theta \cos\theta \sin\gamma \cos\gamma \tag{3.0.65}$$

$$\vec{\Omega}^2(1 - \frac{z}{I_c})^2 = \frac{\vec{\omega}_x^2}{\alpha^2} \sin^2\theta \cos^2\theta \cos^2\gamma + \frac{\vec{\omega}_{yz}^2}{\alpha^2} \sin^2\theta \cos^2\theta \sin^2\gamma - \frac{\vec{\omega}_{yz}\vec{\omega}_x}{\alpha^2} \sin^2\theta \cos^2\theta \sin\gamma \cos\gamma \tag{3.0.66}$$

adding eq.(3.0.65) and eq.(3.0.66) and we get

$$\vec{\omega}_f^2 + \vec{\Omega}^2(1 - \frac{z}{I_c})^2 = \frac{\vec{\omega}_x^2 + \vec{\omega}_{yz}^2}{\alpha^2} \sin^2\gamma \cos^2\theta \sin^2\theta + \frac{\vec{\omega}_x^2 + \vec{\omega}_{yz}^2}{\alpha^2} \cos^2\theta \sin^2\theta \cos^2\gamma \tag{3.0.67}$$

$$\vec{\omega}_f^2 + \vec{\Omega}^2(1 - \frac{z}{I_c})^2 = \frac{\vec{\omega}_x^2 + \vec{\omega}_{yz}^2}{\alpha^2} \cos^2\theta \sin^2\theta \tag{3.0.68}$$

$$\frac{\vec{\omega}_f^2 + \vec{\Omega}^2(1 - \frac{z}{I_c})^2}{\vec{\omega}_x^2 + \vec{\omega}_{yz}^2} = \frac{\cos^2\theta \sin^2\theta}{\alpha^2} \tag{3.0.69}$$

multiply eq.(3.0.63) by ω_x eq.(3.0.64) by ω_{yz} and we get

$$0 = \frac{\vec{\omega}_x^2}{\alpha} \cos\theta \sin\theta \sin\gamma + \omega_f \vec{\omega}_x + \frac{\vec{\omega}_{yz}\vec{\omega}_x}{\alpha} \sin\theta \cos\theta \cos\gamma \tag{3.0.70}$$

$$0 = -\vec{\Omega}(1 - \frac{z}{I_c})\vec{\omega}_{yz} - \frac{\vec{\omega}_x \vec{\omega}_{yz}}{\alpha} \sin\theta \cos\theta \cos\gamma + \frac{\vec{\omega}_{yz}^2}{\alpha} \sin\theta \cos\theta \sin\gamma \tag{3.0.71}$$

and add these two equations

$$\frac{\vec{\omega}_f\vec{\omega}_x - \vec{\Omega}(1-\frac{z}{I_c})}{\vec{\omega}_{yz}^2+\vec{\omega}_x^2} = -\frac{\sin\theta\cos\theta\sin\gamma}{\alpha} \tag{3.0.72}$$

and square it

$$\left(\frac{\omega_f\vec{\omega}_x - \vec{\Omega}(1-\frac{z}{I_c})}{\vec{\omega}_{yz}^2+\vec{\omega}_x^2}\right)^2 = \frac{\sin^2\theta\cos^2\theta\sin^2\gamma}{\alpha^2} \tag{3.0.73}$$

we get eq.(3.0.74) by substituting eq.(3.0.69) into eq.(3.0.73) and we get

$$\sin\gamma = \pm\frac{\omega_f\vec{\omega}_x - (1-\frac{z}{I_c})\vec{\Omega}\vec{\omega}_{yz}}{\sqrt{(\vec{\omega}_x^2+\vec{\omega}_{yz}^2)[\omega_f^2+(1-\frac{Z}{I_c})^2\Omega^2]}} \tag{3.0.74}$$

from eq.(3.0.63) and eq.(3.0.64) we get

$$\cos\gamma\sin\theta\cos\theta = \frac{\frac{\vec{\omega}_{yz}}{\alpha}\sin\theta\cos\theta\sin\gamma - \vec{\Omega}(1-\frac{z}{I_c})}{\frac{\vec{\vec{\omega}}_x}{\alpha}} \tag{3.0.75}$$

$$\cos\gamma\sin\theta\cos\theta = \frac{-\vec{\omega}_f\alpha - \vec{\omega}_x\cos\theta\sin\theta\sin\gamma}{\vec{\omega}_{yz}} \tag{3.0.76}$$

by equating these two equations,we get

$$\alpha = -\frac{(\vec{\omega}_x^2+\vec{\omega}_{yz}^2)\sin\theta\cos\theta\sin\gamma}{\omega_f\vec{\omega}_x - \vec{\Omega}(1-\frac{Z}{I_c})\vec{\omega}_{yz}} \tag{3.0.77}$$

when we substitute eq.(3.0.74) and eq.(3.0.77) into eq.(3.0.50), $\dot{\theta}$ becomes

$$\dot{\theta} = -\frac{z}{I_c}\vec{\Omega}\frac{\vec{\omega}_x\vec{\omega}_f - \vec{\Omega}(1-\frac{z}{I_c})\vec{\omega}_{yz}}{\vec{\omega}_f^2+\vec{\Omega}^2(1-\frac{Z}{I_c})^2}\sin\theta\cos\theta \tag{3.0.78}$$

substitute into $\vec{\omega}_x$, $\vec{\omega}_{yz}$ and $\frac{Z}{I_c} = \frac{I_c}{I_c} = 1$ into eq.(3.0.78)

$$\dot{\theta} = \frac{2}{3}\frac{M^2\Omega^2}{I_c c^3}\frac{\vec{\Omega}^2}{\vec{\omega}_f^2+(1-\frac{z}{I_c})^2\vec{\Omega}^2}\left(\frac{6\vec{\omega}_f c}{5\vec{\Omega}^2 R} - (1-\frac{z}{I_c})\right)\cos\theta\sin\theta \tag{3.0.79}$$

$(1-\frac{Z}{I_c}) \sim 1$, $\vec{\omega}_f^2 \ll \vec{\Omega}^2$ so $\frac{\vec{\Omega}^2}{\vec{\omega}_f^2+(1-\frac{Z}{I_c})\vec{\Omega}^2} \sim 1$

$$\dot{\theta} = \frac{2}{3}\frac{M^2\vec{\Omega}^2}{I_c c^3}\left(\frac{6\omega_f c}{5\vec{\Omega}^2 R} - 1\right)\sin\theta\cos\theta \tag{3.0.80}$$

$$\dot{\vec{\Omega}} = -\frac{2}{3}\frac{M^2\vec{\Omega}^3}{I_t c^3}\sin^2\theta \tag{3.0.81}$$

The equation of $\dot{\Omega}$ is always negative,which is consistent with the fact that the star is losing energy by electromagnetic radiation and thus the angular velocity is constantly decreasing. The equation of $\dot{\theta}$ implies that the magnetic dipole slides toward the direction of the axis of rotation if $\frac{6\vec{\omega}_f C}{5\vec{\Omega}^2 R} < 1$ or to the the equator if the $\frac{6\omega_f C}{5\vec{\Omega}^2 R} > 1$.For alignment case we take the first case,and we have an equation of

$$\dot{\theta} = -\frac{2}{3}\frac{M^2\vec{\Omega}^2}{I_c c^3}\left(\frac{6\omega_f c}{5\vec{\Omega}^2 R} + 1\right)\sin\theta\cos\theta \tag{3.0.82}$$

from the privies section we have the equation

$$n = 3 + 2\frac{\Omega}{\dot{\vec{\Omega}}}\frac{\dot{\theta}}{\tan\theta} \tag{3.0.83}$$

$$\frac{\dot{\vec{\Omega}}}{\vec{\Omega}} = \frac{2\dot{\theta}}{(n-3)\tan\theta} \tag{3.0.84}$$

from eq.(3.0.80) we have

$$\frac{\dot{\vec{\Omega}}}{\vec{\Omega}} = -\frac{2}{3}\frac{M^2\vec{\Omega}^2}{Ic^3}\sin^2\theta \tag{3.0.85}$$

from the eq.(3.0.84) and eq.(3.0.85) we have

$$\vec{\Omega}^2 = \frac{-3Ic^3}{(n-3)M^2}\frac{\dot{\theta}}{\sin^2\theta\tan\theta} \tag{3.0.86}$$

After substituting this equation into eq.(3.0.80) we have simplified form of

$$\dot{\theta} = \frac{-4\omega_f M^2}{5I_c Rc^2}\cos\theta\sin\theta + \frac{2I_t\dot{\theta}}{I_c(n-3)}\frac{\cos^2\theta}{\sin^2\theta} \tag{3.0.87}$$

$$\dot{\theta}(1 - \frac{2I_t}{I_c(n-3)}\cot^2\theta) = \frac{-4\omega_f M^2}{5I_c Rc^2}\sin\theta\cos\theta \tag{3.0.88}$$

let a=$\frac{2I_t}{I_c(n-3)}$,b=$\frac{4\omega_f M^2}{5I_c Rc^2}$

$$\dot{\theta}(1 - a\cot^2\theta) = -b\sin\theta\cos\theta \tag{3.0.89}$$

$$\frac{d\theta}{\sin\theta\cos\theta} - a\cot^2\theta d\theta = -bdt \tag{3.0.90}$$

$\cot^2\theta = \csc^2\theta - 1$ and $2\cos\theta\sin\theta = \sin 2\theta$ are making the integration simple, by substituting these formulas

$$\frac{2d\theta}{\sin 2\theta} - a(\csc^2\theta - 1)d\theta = -bdt \tag{3.0.91}$$

$$d\theta \csc 2\theta - a(\csc^2\theta - 1)d\theta = -bdt \tag{3.0.92}$$

Integrate the above equation to find the time of alignment

$$\int_{89}^{3} 2d\theta \csc 2\theta - \int_{89}^{3} a(\csc^2\theta - 1)d\theta = -bdt \tag{3.0.93}$$

$$\frac{-1}{2}\ln(\csc\theta + \cot\theta)\Big|_{89}^{3} - a\cot\theta\Big|_{89}^{3} + a\theta\Big|_{89}^{3} = -bt \tag{3.0.94}$$

$$-\frac{1}{2}\ln\left(\frac{1+\cos\theta}{\sin\theta}\right)\Big|_{89}^{3} - a\frac{\cos\theta}{\sin\theta}\Big|_{89}^{3} + a\theta\Big|_{89}^{3} = -bt \tag{3.0.95}$$

The integration limit is at $\theta = 89^o$ to $\theta = 3^o$

$$t = \frac{1}{b}\left[\ln\frac{1+0.9986}{0.0523} + a\frac{0.9986}{0.0523} - 3a - \ln\frac{1+0.0174}{0.9998} - a\frac{0.0174}{0.9998} + 89a\right] \tag{3.0.96}$$

$$t = \frac{1}{b}\left[7.67 + 105.07a\right] \tag{3.0.97}$$

$$t = \frac{5I_cRc^2}{4\omega_f M^2}\left[7.67 + 105.07\frac{2I_o + 2I_c}{I_c(n-3)}\right] \tag{3.0.98}$$

$2I_c + 2I_o \simeq 2I_0$

$$t = \frac{5I_cRc^2}{4\omega_f M^2}\left[7.67 + 105.07\frac{2I_o}{I_c(n-3)}\right] \tag{3.0.99}$$

$I_c = 10^{45}gcm^2$ $I_o = 10^{46}gcm^2$ $\vec{M} = \vec{B}R^3$ M^3=1.296x$10^{57}cm^6 Guass$ n=83 c=10^{10}cm/s $\omega_f = 1.2s^{-1}$ After substituting this constants we get t=4.94x10^{14}se t=1.6x10^7yr

CONCLUSION

In this paper we have considered the interaction between core and the crust with fixed magnetic dipole moment, by taking into account electromagnetic alignment torque(retardation torque) and the anomalous torque acting on the dipole and the friction between the core and the crust. We also have derived the gravitational torque and analyze its effect. We have solved the the equations of motions by considering different characteristic time scale. Finaly,we have calculated the alignment time of the magnetic dipole moment from the equator to the the axis of rotation.

The friction(dissipative)torque causes a very small energy loss. since, the angular velocity of the crust and the core have negligible difference. Because of this,the frictional torque has no significant effect on the alignment time.But, the gravitational torque has an important effect. The anomalous torque is very relevant. We conclude that torques which do not directly affect the dynamics of the magnetic dipole moment have significant effect on the alignment time of magnetic dipole moment.

Bibliography

1.Abney, M., Epstein, R. I., and Olinto, A. V. 1996, ApJ,466,L91

2.Alpar, M. A., Cheng, K. S., and Pines, P. 1989, ApJ, 346, 823

3.Alpar, M. A., Langer, S. A., and Sauls, J. A. 1984,ApJ,282,533

4.Alpar, M. A., and gelman, H. 1990, ApJ, 349, L55

5.lpar,M.A., and Sauls, J. A. 1988, ApJ, 327, 725

6Anderson,P.W.,and Itoh, N. 1975, Nature, 256, 25

7.Blandford,R.D.,Applegate,J.H.,and Hernquist,L.1983,Mnras,204,1025

8.Chanmugam, G. 1992, ARA and A, 30, 143

9.Casini,H.,Montemayor,R.1997

10.Davis,L.,and Goldstein,M.1970,ApJ, 159, L81

11.Goldreich,P.1970,ApJ,160,L11

12.Goldreich,P.,and Julian,W.H.1961,ApJ,157,869

12.H. Casini and R.Montemayor,503:374-380,1998 Aug-10

13.Good, M. L.,and Ng,K.K.1985,ApJ,299,706

14.Gunn,J.E.,and Ostriker, J. P. 1969, Nature, 221, 454

15.Link, B.,and Epstein, R.I.1996 and 1997,ApJ, 457, 844

16.Link,B.,Epstein, R. I.,and Baym, G. 1992, ApJ, 390, L21

17.Lyne,A.G.,and Manchester,R.N.1988,MNRAS,234,477

18.Macy, W. W., Jr. 1973, ApJ, 190, 153

19.Manchester,R.N.,and Taylor,J.H.1977

20.McCulloch,P.M.,Hamilton,P.A.,McConnell,D.,and King,E.A.1990

21.Mendell,G.1991a,ApJ,380,515

22.Mendell,G.1991b, ApJ, 380, 530

23.Muslimov,A.,and Page,D.1995,ApJ,440,L77

24.Pacini,F.1967,Nature,216,567

25.pacini,F.1968, Nature, 219, 145

26.Ruderman,M.,Zhu,T.,and Chen,K.1998,ApJ,492,267

27.Srinivasan,G.,Bhattacharya,D.,Muslimov,A.,andTsygan,A.1990,Curr.Sci.,59,31

28.http://en.wikipedia.org/wiki pulsar 05/03/2010

29.http://en.wikipedia.org/wiki neutron-star 05/03/2010

30.http://www.answers.com/topic/neutron-star 06/05/2010

31.http://imagine.gsfc.nasa.gov/docs/science/known-11/ pulsars.html 03-Feb-2012

32.J.O.Urama.African skies /cieux Africans No.7,may 2002

33.http://www.astronomynotes.com/evolutn/s12.htm june 2,2007

34.M.L.Good and K.K.NGtheAstronomicalJornal,299-722,1985,December 15

35.Davis and Goldstein 1970; Michel and Goldwire 1970 Apj,160,L11

36.Mark Abney,Richard I.Epstein,Algebra V.olinto university of chicago,564 S.chicago,IL60637

37.Richard I.Epstein Jornal,556:392-398,2001 July 20

38.Gravitational and cosmology, principle and application of general theory of Rel- Steven Weinberg(Book)

39.http:www.so/station.com/x.oblects/crab-neb.htm

Signature:

Place and time of submission: Addis Ababa University, June2010

This thesis has been submitted for examination with my approval as University advisor.

Name: Signature: