



ADDIS ABABA UNIVERSITY

ADDIS ABABA INSTITUTE OF TECHNOLOGY

SCHOOL OF CIVIL AND ENVIRONMENTAL ENGINEERING

**ANALYSIS OF STRIP PLATES ON ELASTIC FOUNDATION USING
GENERALIZED SUBGRADE MODEL**

BY:

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Addis Ababa University

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By

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A thesis submitted to the School of Graduate Studies of Addis Ababa University in Partial fulfillment of the Degree of Master of Science in Geotechnical Engineering.

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UNDERTAKING

I declare that this thesis is my original work performed under the supervision of my research advisor Dr.-ING Asrat Worku and has not been presented as a thesis for a degree in any other university. All sources of materials used for this thesis have also been duly acknowledged.

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LIST OF SYMBOLS AND ABBREVIATIONS

a : width of Strip Plate

D : Flexural rigidity of a plate

DE : Differential equation

Ep : Elasticity modulus of strip plate

Es : Elasticity modulus of soil

FE : Finite Element

G : Shear modulus of an elastic medium

\bar{G}_p : Pasternak shear modulus with units of force per length

\bar{G}_K : Kerr shear modulus with units of force per length

H = Thickness of stratum

Hp =Thickness of strip plate

\bar{k}_l : Kerr lower spring layer constant with units of force per length cube

\bar{k}_u : Kerr Upper spring layer constant with units of force per length cube

Ke : Equivalent spring stiffness with units of force per length cubed

\bar{k}_e : Equivalent spring stiffness with units of force per length squared

K_{sc} : Horvath equivalent modulus of subgrade reaction

K_0 : Lateral earth-Pressure coefficient for at rest condition

\bar{k}_s : Modulus of subgrade reaction with units of force per length cube

\bar{k}_p : Modulus of subgrade reaction with units of force per length cube

M : Moment

Mo : Applied concentrated moment

P : Applied concentrated vertical load

q : Applied vertical surface pressure

T : Applied Tension force on Filonenko-Brodich Model

u : Displacement in the x-direction

V : shear force

v : Displacement in y-direction

w : vertical surface deflection

w_1 : Deflection due to the contraction or extension of Kerr's upper spring layer

w_2 : Deflection due to the contraction or extension of Kerr's Lower spring layer

∇^2 : Laplace operator

σ : Normal stress

τ : Shear Stress

θ : Slope

ν : Poisson's ratio of a soil

ξ : Characteristic width

χ : Calibration factor

I. Abstract

A concrete plate supported directly by the soil continuum is a very common construction form. The response of the plate when it carries external load is influenced by the soil, and the response of the soil is also influenced by the action of the plate under the load. Thus, developing a subgrade model for soil-structure interaction problem is essential in order to predict the response of both components of the system and arrive at an optimum design.

Many subgrade models have been developed in order to improve on the inherent lack of shear interaction among the individual springs found on the long-enduring model of Winkler. Moreover, these models still have shortcomings with the nature of simplifying assumptions they make to ease the mathematical equation. Recently a generalized model is presented for a subgrade idealized as an elastic layer overlying a rigid base. In contrast to previous works no stresses, strains or displacements are neglected a priori.

The main objective of this work is implementing, verifying and calibrating this improved continuum-based generalized subgrade model in the analysis of strip plates on an elastic foundation. The governing differential equation of a strip plate on elastic foundation is formulated. Then, closed form particular solutions, when using Winkler type and Kerr equivalent Pasternak models, are obtained by considering different boundary conditions of long and short length of a strip plate, under different loading conditions. Microsoft excel programs are written for the computation of deflection, moment and shear force. The subgrade models are then calibrated using Finite Element based Plaxis 2D software. Lastly, numerical illustration is provided using these models in comparison with Plaxis 2D model for long and short strip plates subjected to selected symmetrical loading conditions. The result of the comparison shows that the calibrated variants give good outcome in agreement with FE outputs. Consequently, these findings display that the calibrated models can be used in routine analysis of strip plate on elastic foundation and also can be incorporated in commercial software.

Chapter one

1. Introduction

1.1 Background

Structures are constructed and supported on soil or rocks. Foundation is built between the superstructure and the underlying soil to spread the load carried by building members to the supporting soil. The primary concern is to study the behavior of foundation and the underlying soil with soil-structure analysis when they are in contact. There are few analytical methods for the analysis of these structural systems nowadays. Many engineers analyze this problem by assuming the plate being completely rigid or flexible, which are the two extreme cases of the soil-structure interaction. It is important to find a simple and efficient analytical method for plates with sensible elastic properties.

Plates on elastic foundation are regularly used in civil or mechanical engineering works, such as building infrastructures, roads, railroad, storage tanks or silos foundations, aerospace engineering etc. The key issue in the analysis is modeling the contact between the structural elements- the plate or beam-and the soil bed (soil-structure interaction (SSI) problem). Currently, Finite-Element method (FE) based software is often used by geotechnical engineers to solve such problems. But to find FE software, which model both the superstructure and the foundation soil as a unit is mostly unaffordable, especially in routine designs. In addition, most static analysis software uses the simplified Winkler's foundation model to account for the SSI effect. Thus, there is a need for closing the gap between the rather classy FE modeling and the use of highly simplified analytical foundation models such as that of Winkler, by developing simple and strong analytical models that optimize accuracy and effort.

There are two approaches to develop analytical subgrade models, namely continuum and mechanical. Elastic continuum models typically idealize the subgrade as an elastic medium and specifically as a layer overlying a rigid base, and involve three parameters consisting of the elastic modulus, the Poisson's ratio and the layer thickness. Mechanical models, in contrast, simulate the subgrade behavior by using a few mechanical elements like spring, a plate in pure bending, a plate in pure shear or a stretched membrane etc. in various arrangements.

It is apparent that all continuum models available make certain simplifying assumptions to ease the mathematical work involved in the process of devising the models. These models have the advantage that the elastic constants can be established from tests but suffer from a common shortcoming that they are difficult to apply directly. While in mechanical models, the higher order models were devised with the intention of improving on the drawbacks of the simplest and long-enduring Winkler single-spring-bed model by introducing additional elements to ensure shear interaction among the springs that is missing in Winkler's model. However, mechanical models suffer in general from a major common drawback of not suggesting ways of estimating the model parameters. Synthesis of the two approaches has the benefit of using the strengths of both methods. Also, it provides a means of quantifying the mechanical model parameters in terms of the known parameters of the continuum model (Worku, 2013).

Recently, a newly developed generalized continuum subgrade model has been introduced by Worku (2010). It considered all stress, strain and displacement components unlike other models proposed in the past. Based on this model, this paper tries to give analytical solutions for strip plates under basic loading cases. Furthermore, it also tries to calibrate the model parameters and to compare the results with the classical models and selected FE based modeling software.

1.2 Statement of the problem

The problem of beams and plates on deformable foundation soils is a soil-structure interaction (SSI) problem. Currently, to account for SSI, most designers use Winkler's foundation model or FE based software. However, on the one hand Winkler's model has drawbacks that this type of soil idealization has the discontinuous behavior of the surface displacement, and the effect of interaction between springs is not considered. On the other hand, FE method, which idealizes both the superstructure and the foundation soil as a unit, is mostly unaffordable. Furthermore, most static analysis software still uses Winkler's foundation model to account for SSI effect.

The problem of plate/beams resting on an elastic foundation is very often encountered in the analysis of buildings, highway and railroad structures. The complete analysis of the soil-structure interaction problem requires the determination of stresses and strains within the individual bodies in contact, together with information regarding the distribution of displacements and stresses at the contact regions. Thus, the relative complexity of the problem demands simple and yet robust analytical modeling techniques representing

- ❖ the Mechanical behavior of the plate
- ❖ the Mechanical behavior of the soil as elastic subgrade

- ❖ the form of interaction between the plate and the soil.

Calibrating the analytical method based on comparative studies with numerical analyses is also essential in justifying such studies.

1.3 Objectives

- ❖ To study the application of the newly developed generalized continuum model of Worku (2010) to strip plates on an elastic subgrade and eventually calibrate it with the help of FE based software (Plaxis 2D).
- ❖ Check the validity of the model by comparing the results with FE based software outputs.

1.4 Scope of the work

The scope of this study is limited to the analysis of strip plates subjected to selected symmetrical loadings. The principle of superposition is used in the analysis of the plates, which are assumed to be placed on a homogeneous elastic stratum. The study does not cover plates with holes or with some irregularity.

1.5 Methodology

To achieve the objectives of the study, the research is conducted according to the methodology outlined below

1. Appropriate literature review

- ❖ Both existing mechanical and continuum-based models are reviewed to understand the state of the art of the subject.

2. Analytical study

- ❖ General mathematical formulation for strip plate(DE) using the Winkler –Type model and Kerr-Equivalent Pasternak –type model
- ❖ Solving the governing differential equation for various boundary conditions
- ❖ A spreadsheet program (Excel) is used for the closed form solutions.
- ❖ Determination of deflection and internal actions of various cases of strip plates

3. Numerical Analysis

- ❖ Analyze and calibrate the model with FE method (Plaxis 2D)

4. Illustration and comparison of the results obtained by Analytical and Numerical Analysis.

1.6 Organization of the thesis

This document is organized in five chapters. The first chapter deals with the background of the study, statement of the problem, objective, scope, and methodology employed. The second chapter briefly reviews the major literature works conducted related to the topic. Both mechanical and continuum-based subgrade models are presented. The third chapter deals with the analysis of a strip plate on an elastic foundation. In this part of the study, the formulation of the governing differential equation of strip plates using single and two parameter models is presented with different boundary conditions. In the fourth chapter, a newly proposed model is calibrated by the FE-based Plaxis 2D software and the results of this model are compared with other existing models and also with FE model outputs. In the fifth Chapter, conclusions are drawn and recommendations are put forth. In addition, recommended future research works are identified.

Chapter Two

Review of Literature

2.1 Introduction

Plate on elastic supports is a common model for several types of engineering structures and real life applications. For example, plate on elastic foundation models are always used in the analysis of foundations of buildings, reinforced concrete pavements of highways and airport runways.

Ultimately, all structure loads must be transferred to the foundation soil, and both the soil and the structure act together to resist and support the loads. The integral nature of the foundation and soil actions is further complicated by the complexity of the soil medium itself. Soil is truly a non-homogeneous and an anisotropic medium that behaves in a nonlinear manner, while concrete and steel structures can be adequately modeled and analyzed, assuming isotropic and linear behavior. In addition, the properties of structural building materials are well known so that the stiffness of the structure may be readily determined, given member size and structure geometry (Fai, 1988).

On the other hand, soil properties are very difficult to determine because in addition to the previously mentioned characteristics, soils are "soft" materials, which makes it very difficult in obtain samples for testing that will produce laboratory results paralleling actual in situ soil behavior. Two additional complicating factors are that soil material properties are stress-dependent, and the soil formation in practice consists of layers of materials with different constitutive relations and material properties. Because of these factors, the time properties and constitutive relations of soils are essentially unknown and indeterminable. As a result, it is necessary to make simplifying assumptions to solve problems including this improving soil-structure interaction (Worku and Degu, 2010).

2.2 Elastic foundation Models

The elastic modeling of a soil bed is based on an assumption for the behavior of the subgrade reaction under loading. The most popular relation between forces and deformations is linear because of the simplicity of the equations' solution.

The elastic subgrade reaction may be analytically represented by using the following Models.

1. Mechanical Models
2. Continuum Model

2.2.1 Mechanical Models

Mechanical models idealize the subgrade as an assemblage of a few mechanical elements in various arrangements like springs, a plate in pure bending, a plate in pure shear and other arrangements to simulate the subgrade behavior. The development of these models starts from simplest single parameter model and become more complex as the number of parameters increases with additional mechanical elements in order to bring the models closer to reality. Depending on the number of parameters used to describe the foundation behavior, mechanical models may be categorized in to three (one, two and three-parameter models (Worku, 2010).

2.2.1.1 Single Parameter Models

2.2.1.1.1. Winkler model

The simplest model of linear elastic behavior of the supporting soil medium is suggested by Winkler (1867). This model assumed that the surface displacement of the soil medium is directly proportional to the stress applied to it and completely independent of stresses or displacements at other points of the soil-foundation interface (Straughan, 1990).

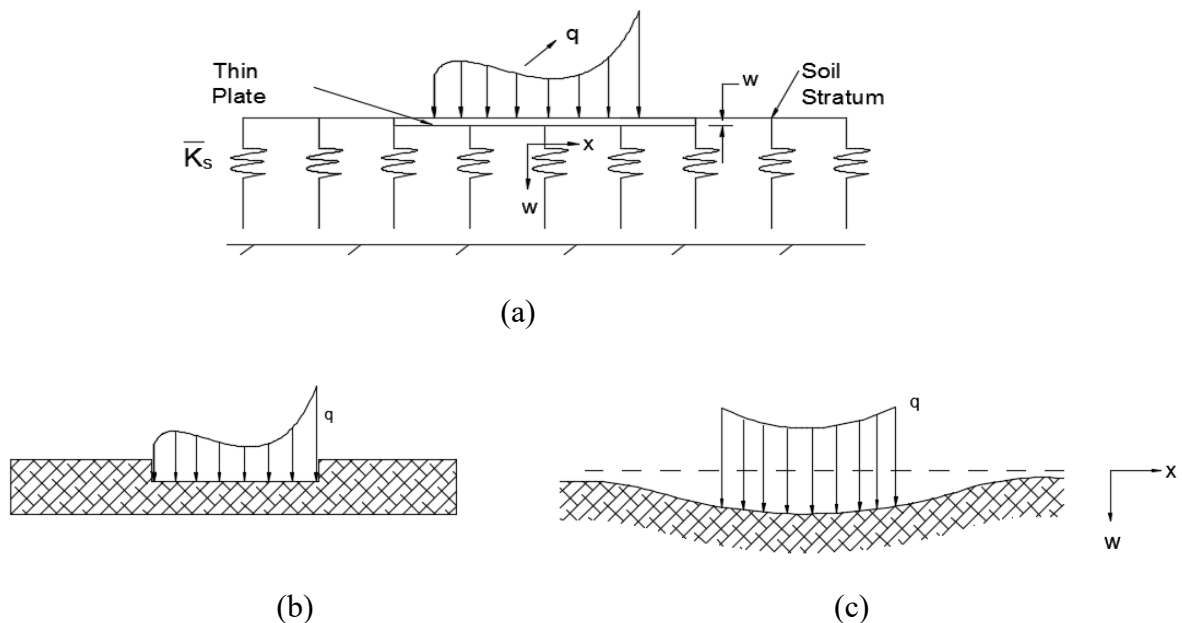


Fig 2.1 Surface displacements due to (a) a distributed load on a rigid foundation-Winkler model, (b) a uniform load on flexible foundation-Winkler model (c) actual dish-shaped deformation,

$$q(x, y) = \bar{k}_s w(x, y) \quad (2.1)$$

Where: - w - the vertical deflection of the soil

q - Stress applied at that point and

\bar{k}_s - modulus of subgrade reaction or the coefficient of subgrade reaction

For plates, application of the Winkler model involves the solution of a fourth-order differential equation.

$$D\nabla^4 w(x, y) + \bar{k}_s w(x, y) = q(x, y) \quad (2.2)$$

Where: - D - plate flexural rigidity

The model consists of linearly elastic springs with a stiffness of \bar{k}_s placed at discrete intervals below the plate. Each of the springs will then be deformed only by the pressure directly applied on it, while adjacent springs remain unaffected. For example, if such a foundation is subjected to a partially distributed surface loading, q , the springs will not be affected beyond the loaded region. Also, it can be seen that the displacements of a loaded region will be constant whether the soil is subjected to an infinitely rigid load or uniform flexible load as shown in Figure 2.1 a& b. However, it was observed that for most materials due to inherent occurrence of load spreading, the displacement of the foundation surface is dish shaped (an overall concave-upward curve) as shown in Figure 2.1c. Hence by comparing the behavior of theoretical mode and actual foundation, it can be seen that this model essentially suffers from a complete lack of continuity in the supporting medium.

In addition to the discontinuity behavior of the model it's also very difficult to determine the value of the modulus of subgrade reaction, \bar{k}_s . A review of the literature Fai (1988), Straughan (1990), Selvaduri (1979) and numerous researchers have worked on developing techniques for the determination of \bar{k}_s . One of the most definitive papers was proposed by Terzaghi who presented tables of recommended values of \bar{k}_s , showing that the modulus of subgrade reaction depends upon the dimensions of the area acted upon by the subgrade reaction, and size effects has been incorporated in the equations.

In attempt to develop a more exact method to determine \bar{k}_s , Biot (1937) solved the problem of an infinite beam with a concentrated load resting on a three-dimensional subgrade by evaluating the maximum bending moment in the beam. Biot found that a good correlation can be obtained with the Winkler model for maximum moment case by setting \bar{k}_s as follows (Straughan, 1990).

$$\bar{k}_s = \frac{0.95 E_s}{(1-\nu_s)^2} \left(\frac{E_s B^4}{(1-\nu_s^2) EI} \right)^{0.108} \quad (2.3)$$

Where: - E_s = modulus of elasticity of the soil, ν_s = Poisson's ratio of the soil, B = width

E = modulus of elasticity of the beam and I = moment of inertia

Later Vesic showed that \bar{k}_s depends up on both the stiffness of the soil, as well as the stiffness of the structure, so that similar size structures of different stiffness's will yield different values of \bar{k}_s for the same applied load. He found the continuum solution correlated with the Winkler model where all terms were previously defined Straughan (1990).

$$\bar{k}_s = \frac{0.65 E_s}{(1-\nu_s^2)} \sqrt[12]{\frac{E_s B^4}{(1-\nu_s^2) EI}} \quad (2.4)$$

2.2.1.2 Two Parameter models

Knowing the essential problem with Winkler model several researchers attempted to eliminate the discontinuity behavior by introducing additional mechanical interaction between the spring's elements of the Winkler medium, which is capable of transferring pure shearing deformation. These additional elements bring other parameter to the subgrade model in addition to the coefficient of subgrade reaction. Some developed two-parameter models are presented below.

2.2.1.2.1 Filonenko-Borodich model

Filonenko-Borodich proposed a model that acquires the continuity between the individual spring elements in Winkler model by connecting the top ends of the springs with an elastic membrane stretched to a constant tension (T). Refer to Figure 2.2 the response function is given by

$$q(x,y) = \bar{k}_s w(x,y) - T \nabla^2 w(x,y)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Where: ∇^2 is the Laplace operator, and all other terms were previously defined

In the case of two-dimensional problems (strip plate and beam) the above equation will be reduced to

$$q(x) = \bar{k}_s w(x) - T \frac{d^2 w(x)}{dx^2} \quad (2.5)$$

The two elastic constants necessary to characterize the soil model are \bar{k}_s and T. However, no method is provided for the computation of \bar{k}_s or T (Selvaduri, 1979).

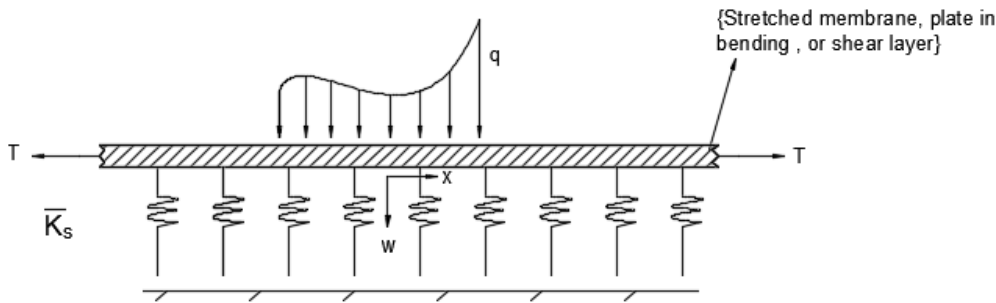


Fig 2.2 Filonenko-Borodich Model

2.2.1.2.2 Hetenyi Model

The model proposed by Hetenyi (1946) created an interaction among the springs in the foundation by imbedding an additional plate with flexural rigidity (D) in the Winkler foundation in a manner shown in Figure 2.2. And it is assumed that the plate or beam deformed in pure bending. According to this model, the response function is given by:

$$q(x,y) = \bar{k}_s w(x,y) - D \nabla^4 w(x,y)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Where: ∇^2 - is the Laplace operator and D -flexural rigidity of the plate

In the case of two-dimensional problems, the above equation will be reduced to

$$q(x) = \bar{k}_s w(x) - D \frac{d^4 w(x)}{dx^4} \quad (2.6)$$

2.2.1.2.3 Pasternak Model

The model proposed by Pasternak improved Winkler model by connecting the ends of the springs to a shear layer consisting of incompressible, vertical elements, which can deform only by lateral shear. This is illustrated in Figure 2.3, furthermore this model assumes the shear layer to be isotropic in x, y plane with shear modules.

$$G_x = G_y = \bar{G}_p \quad (2.7)$$

Where \bar{G}_p = shear modulus of the elastic foundation

$$\tau_{xz} = \bar{G}_p \gamma_{xy}, \quad \bar{G}_p \frac{dw}{dx}, \quad \tau_{yz} = \bar{G}_p \gamma_{yz}, \quad \bar{G}_p \frac{dw}{dy} \quad (2.8)$$

Total shear forces per unit length of the shear layer are

$$Nx = \int_0^1 \tau_{xz} dz = \bar{G}_p \frac{dw}{dx}; Ny = \int_0^1 \tau_{yz} dz = \bar{G}_p \frac{dw}{dy} \quad (2.9)$$

From the force equilibrium in the Z -direction

$$\frac{\partial Nx}{\partial x} + \frac{\partial Ny}{\partial y} + q - \bar{k}_p w = 0 \text{ and } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (2.10)$$

$$q(x, y) = \bar{k}_p w(x, y) - \bar{G}_p \nabla^2 w(x, y) \quad (2.11)$$

Where: \bar{k}_p - modulus of subgrade reaction of Pasternak model

In the case of two-dimensional problems, the above equation will be reduced to

$$q(x, y) = \bar{k}_p w(x, y) - \bar{G}_p \frac{d^2 w(x)}{dx^2} \quad (2.12)$$

It can be seen from equation (2.5) it is identical with (2.12) if T replaced by \bar{G}_p . Thus surface deflection for this model is very similar to those obtained for Filonenko-Borodich model.

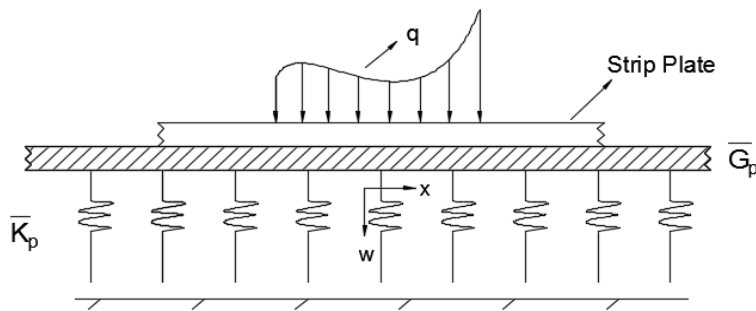


Fig 2.3 A strip plate supported Pasternak foundation

2.2.1.3 Three Parameter model

For making two parameter models more realistic third parameter is introduced. The basic feature of three parameter model is the convenience and flexibility in determination of the level of continuity of the vertical displacements at the boundaries between the loaded and unloaded surface of the soil. From the developed three parameter models, Kerr model is the most widely used.

2.2.1.3.1 Kerr Model

Because of the occurrence of the concentrated reaction along free edges of a structure when the Pasternak model is used, Kerr proposed a generalization of the Pasternak model by adding a spring layer on the top of shearing layer as shown in the Figure 2.4.

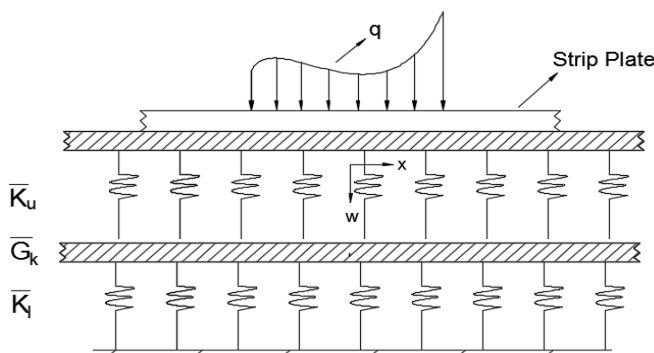


Fig 2.4 a strip plate supported by Kerr Foundation Model

Kerr mechanical model consists of two layers of spring constants separated by a shear layer. The mathematical equation is derived by considering two parts of plate deflection, the

deflection due to the contraction or extension of the upper spring layer w_u and due to the rest of the foundation w_l (Worku and Degu, 2010).

$$W = W_l + w_u, \text{ where } w \text{ is the surface deflection} \quad (2.13)$$

The respective mathematical equation is given by

$$\left(1 + \frac{\bar{k}_u}{\bar{k}_l}\right) P(x,y) - \frac{\bar{G}_k}{\bar{k}_u} \nabla^2 P(x,y) = \bar{k}_l w(x,y) - \bar{G}_k \nabla^2 w(x,y) \quad (2.14)$$

2.2.2 Continuum Models

Continuum models represent the subgrade as an elastic layer overlying a rigid formation characterized by three parameters

- The elastic modules
- Poisson's Ratio and
- Thickness of soft layer

Although the continuum behavior is usually regarded as a more accurate description of soil behavior in general, the analysis of the soil foundation interaction problem is mathematically complex. To simplify the mathematical work almost all continuum models available, make a certain assumption in the process of developing the model. Some developed continuum models are presented below (Worku, 2014).

2.2.2.1 Horvath's Winkler type Model

Horvath used the same physical model as the Reissner's model but he assumed that all stress components except the vertical normal stress (σ_z) were equal to zero. The consequence of this assumption is constant with depth and equal to the magnitude to P and vertical deflection w vary from w at the surface to zero at a depth equal to stratum thickness H. Horvath solved the following cases

$$E_s = A(\text{constant with depth})$$

$$E_s = A + BZ$$

$$E_s = A + BZ^{0.5}$$

Where: E_s -Young's modulus of the subgrade

A - Young's modulus value directly beneath the loaded area

B – the rate of change of Young's modulus with depth, Z

By making use of the equation of equilibrium, the stress-strain and strain –displacement relations he derived the following Winkler-type simplified continuum model for the cases mentioned above.

$$P=k_{sc}w \quad (2.15)$$

Where : k_{sc} -equivalent modulus of subgrade reaction for simplified continuum

$$\left. \begin{aligned} k_{sc} &= \frac{A}{H} && (\text{for } E_s = A) \\ k_{sc} &= \frac{B}{\ln(A+BH) - \ln(A)} && (\text{for } E_s = A + BZ) \\ k_{sc} &= \frac{B^2}{\left((A+BH^{0.5}) - A \ln(A+BH^{0.5}) - A + A \ln(A) \right)} && (\text{for } E_s = A + BZ^{0.5}) \end{aligned} \right\} \quad (2.16)$$

Where H -is the thickness of the stratum

Due to the identity of equation (2.15) of Horvath model and equation (2.1) Winkler subgrade model equation (2.15) is referred as the Winkler-type simplified continuum (WTSC) subgrade mode. Thus \bar{k}_s can be evaluated from k_{sc} of elasticity theory based simplified continuum model from Known soil parameters E_s and H (Horvath , 2011).

2.2.2.2 Vlasov's Model

Vlazon's and Leont'ev's model (commonly known as the "Vlazon model") provided for shear strains within the soil continuum. This model presents two parameter elastic models which is derived by introducing displacement constraints that simplify the basic equation for linear elastic isotropic continuum. The formation of the soil model is based on the application of variation method by imposing certain restrictions upon the possible distribution of displacements in an elastic layer. In this model the state of strain in the foundation is assumed to be such that displacement components are

$$u(x,y)=0, w(x,y)=w(x)h(z) \quad (2.17)$$

Where u and w are the displacements in the X and Z direction and $h(z)$ describes the variation of the displacement $w(x,z)$ in the z-direction .

$$h(z) = \left(1 - \frac{z}{H}\right); h(z) = \frac{\sinh(\gamma(H-Z)/L)}{\sinh(\gamma H/L)} \quad (2.18)$$

Where γ is a constant which characterize the vertical deformation profile within the soil continuum

Applying variational method the resulting relation between the load p and the deflection w is given by:

$$q(x, y) = kw(x) - 2t \frac{d^2w(x)}{dx^2} \quad (2.19)$$

$$k = \frac{E_o}{1 - \nu_o^2} \int_0^H \left(\frac{dh}{dz}\right)^2 dz, \quad t = \frac{E_o}{4(1 + \nu_o)} \int_0^H (h)^2 dz \quad \text{and} \quad E_o = \frac{Es}{(1 + \nu)}, \quad \nu_o = \nu(1 - \nu) \quad (2.20)$$

Where: Es = Elastic modulus

ν = Poisson's Ratio

k = is a measure of the soil medium to deform under applied compressive stress and

t = is a measure of the transmissibility of an applied force to neighboring elements or the load spreading capacity.

For linear variation of $h(z)$ the equation (2.20) reduces to

$$k = \frac{E_o}{H(1 - \nu_o^2)} \quad \text{and} \quad t = \frac{E_o H}{12(1 + \nu_o)}$$

As mentioned above this equation considers shear interactions within the foundation and structure, was developed using variational principles. The real strength of Vlazov's and Leont'ev's approach is in the total elimination of the necessity to determine empirically the values of the modulus of subgrade reaction, k , or even the shear parameter, t , as their values can be computed once the value of γ is determined. This model has the disadvantage of requiring an estimate of the γ parameter since no mechanism was developed for computing the value of γ (Straughan,1990).

2.2.2.3 Reissner's model

The model proposed by Reissner is also based some simplifying assumptions to ease the mathematical work involved in the process of developing the models. The basic assumptions is that the in-plane stresses (in the axis x,y plane) throughout a soil layer of thickness H are negligibly small ($\sigma_{xx}=\sigma_{yy}=\tau_{xy}=0$) and that the displacement components u , v and w in the rectangular Cartesian coordinate directions x,y,z respectively satisfy conditions

$$u = v = w = 0 \quad \text{on } Z=H \quad \text{and} \quad u=v=0 \quad \text{on } Z=0 \quad (2.21)$$

It can be shown that the response function for soil model is given by

$$q - \frac{GH^2}{12E_s} \nabla^2 q = \frac{E_s}{H} w - \frac{GH^2}{12E_s} \nabla^2 w \quad (2.22)$$

Where w = is vertical displacement of the surface of the elastic layer at $Z=0$

q = is external load

G_s = shear modules and

E_s = Elastic modulus

Here again, as a consequence of assuming that the in-plane stress $\sigma_{xx}=\sigma_{yy}=\tau_{xy}$ are zero, the shear stress τ_{xz} and τ_{yz} are independent of Z . These stress are constants throughout the depth of the elastic layer for a given location x,y . such an assumption would prove to be particularly unrealistic for a thick soil layer (Worku & Degu ,2010).

2.2.2.4 Generalized Model

Worku (2010) proposed a new approach of continuum modeling by addressing the major shortcomings other continuum models. The subgrade is idealized as an elastic continuum of finite thickness, H , similar to the simplified continuum of Reissner. This approach of the subgrade by an elastic stratum is convenient in developing the model for both an actual elastic stratum of finite thickness overlaying a rigid base as well as for very thick strata commonly idealized as a uniform half space.

In the latter case, H can be expressed in terms of the foundation width, B , as $H=\chi B$ and the coefficient χ determined from comparison of analytical results with results of finite-element

model. The calibration factor x for beams on elastic foundation has been done by Worku and Degu (2010) and in later section of this work the calibration factor x for strip plate will be done

Worku assumed that the elasticity modulus, E_s and the shear modulus, varies with depth while the Poisson's ratio to be constant with depth as his model is less sensitive to Poisson's ratio. None of stress, strain or displacement components are neglected a prior unlike Reissner's model. In derivation of this model, the lateral normal stresses in terms of the vertical normal stress according to

$$\sigma_x(x, y, z) = g_x(z)\sigma_z(x, y, z); \sigma_y(x, y, z) = g_y(z)\sigma_z(x, y, z) \quad (2.23)$$

Where, g_x and g_y are function of z

Whereas the vertical shear stress components are

$$\tau_{xz}(x, y, z) = I_{zx}(z)\bar{\tau}_{zx}(x, y); \tau_{zy}(x, y, z) = I_{zy}(z)\bar{\tau}_{zy}(x, y) \quad (2.24)$$

In which I_{zx} and I_{zy} are function of Z only, whereas $\bar{\tau}_{zx}$ and $\bar{\tau}_{zy}$ are function of x and y the equilibrium equation for the z -direction without body forces is given by

$$\sigma_{z,z} + \tau_{xz,x} + \tau_{zy,y} = 0 \quad (2.25)$$

The applicable stress-strain and strain-displacement relationships are

$$\frac{\partial w}{\partial z} = \frac{1}{E_s(z)} [\sigma_z - \nu(\sigma_x + \sigma_y)] \quad (2.26)$$

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\tau_{zx}}{G}; \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{\tau_{zy}}{G} \quad (2.27)$$

Which u , v , and w are the deformation components in the x , y , and z directions respectively. The prevailing and assumed boundary conditions are

$$\sigma_z(x, y, 0) = -p \quad (2.28)$$

$$u(x, y, 0) = v(x, y, 0) = 0; w(x, y, 0) = w_0 \quad (2.29)$$

$$u(x, y, H) = v(x, y, H) = w(x, y, H) = 0 \quad (2.30)$$

In the process of derivation of the model the following assumption were taken for reasons of mathematical convenience and this assumption is valid for axis-symmetric problems.

$$I_{zx}(z) = I_{zy}(z) = I_z(z) \quad (2.31)$$

After a lengthy mathematical work, Worku come up with the following differential equation for his model

$$p(x, y) - \frac{G}{E} \frac{1}{k_l} \left(L_{gl} - \frac{k_{gl} L_g}{k_g} \right) \nabla^2 p(x, y) = \frac{E}{k_g} w_o(x, y) - \frac{GL_{gl}}{k_g k_l} \nabla^2 w_o(x, y) \quad (2.32)$$

In which the terms in the coefficients are definite integrals given by

$$K_g = \int_0^H g dz ; \quad K_{gl} = \int_0^H g \tilde{I}_z dz ; \quad K_l = \int_0^H I_z dz$$

$$L_g = \int_0^H [g dz - (\int g dz)_{z=H}]$$

$$L_{gl} = \int_0^H [g \tilde{I}_z dz - (\int g \tilde{I}_z dz)_{z=H}] dz \quad (2.32)$$

$$g(z) = 1 - \nu [g_x(z) + g_y(z)]$$

$$g_x(z) = \frac{\sigma_x}{\sigma_z} ; \quad g_y(z) = \frac{\sigma_y}{\sigma_z}$$

$$\tilde{I}_z(z) = (\int I_z dz)_{z=0} - \int I_z dz$$

$$I_z(z) = \frac{\tau_{xz}(x, y, z)}{\bar{\tau}_{xz}(x, y)} = \frac{\tau_{yz}(x, y, z)}{\bar{\tau}_{yz}(x, y)}$$

Equation (2.24) is the mathematical model for generalized case of an elastic subgrade, the elastic and shear moduli of which may vary with depth, and in which no simplifying assumption has been made with respect to stress, strains or displacements. This formulation shows that the maximum order of the governing differential equation of non-homogeneous, isotropic elastic subgrade is two occurring on the surface deformation, w_o and the surface traction.

Based on generalized model, Worku noted that number of models can be obtained by introducing different assumption on the lateral deformations, the vertical shear stress, and the depth-wise variation of the functions $g_x(z)$, $g_y(z)$ and $I(z)$. Some of his models are presented in the following sections.

2.2.2.4.1 The generalized Winkler-Type Continuum Models

Worku (2013) show that the generalized continuum model reduces to Winkler-type models. A Winkler-type model arises, when the vertical shear stress alone is neglected

$$\tau_{xz} = \tau_{yz} = 0 \quad (2.33)$$

Thus the equilibrium equation (2.25) for the vertical direction simplifies to

$$\sigma_{z,z} = 0 \quad (2.34)$$

Equation (2.34) implies that σ_z is constant with respect to depth. This constant is determined from the boundary condition (2.28)

$$\sigma_z(x, y) = -p(x, y) \quad (2.35)$$

Substituting equation (2.23) in to (2.26) and integrating the resulting equation and applying the boundary condition, the following Winkler-type model is obtained:

This yields

$$p = \frac{E}{kg} w_o \quad (2.36)$$

Where K_g is a definite integral as defined in equation (2.32)

Equation (2.36) is similar to Winkler model given by the equation (2.1) and it is referred as the generalized Winkler-type continuum model. It can be notice that unlike in the previous work Horvath (2011) the lateral normal stresses are not neglected in addition to vertical shear stresses to obtain a Winkler type model.

2.2.2.4.2 The generalized Pasternak-Type Continuum models

The general Pasternak-type continuum model can be obtained when lateral displacements, $u(x,y)$ and $v(x,y)$ are neglected (Worku, 2010).

The resulting mathematical model takes the following form

$$p = \frac{E}{H_{\alpha_0}} w_o - \frac{GL}{H} \nabla^2 w_o \quad (2.37)$$

Where $L = L_x = L_y = \int_0^H \frac{\bar{I}_z G}{E_s I_z} dz$ is a defined in equation (2.32) and $\alpha_o = \frac{1-\nu-2\nu^2}{1-\nu}$

Equation (2.37) is similar in the form and order to mathematical equivalent of Pasternak mechanical model equation (2.12). Thus, the continuum model of equation 2.30 is referred to as the generalized Pasternak-type continuum model.

2.2.2.4.3 The generalized Kerr-Type Continuum models

For homogenous elastic stratum of thickness H over laying a rigid half space a generalized mathematical relationship is obtained by (Worku, 2010).

$$p - \frac{G_k}{E k_{kl}} \left(L_{gl} - \frac{K_{gl} L_g}{K_g} \right) \nabla^2 p = \frac{E}{K_g} w - \frac{G L_{gl}}{K_g K_l} \nabla^2 w \quad (2.38)$$

The approach of this model does not neglect any stress, strain or displacement in advance. The differential equation is similar in form and order to Kerr's mechanical model expressed by the following equation

$$p + \frac{\bar{G}_k}{\bar{k}_u + \bar{k}_l} \nabla^2 p = \frac{\bar{k}_u * \bar{k}_l}{\bar{k}_u + \bar{k}_l} w - \frac{\bar{G}_k * \bar{k}_l}{\bar{k}_u + \bar{k}_l} \nabla^2 w \quad (2.39)$$

Thus this equation is referred to as the generalized Kerr-type continuum model.

2.3 Synthesis of Mechanical and Continuum Models

The approach and origins of mechanical model and simplified continuum models are different. However, there is similarity in their governing differential equations. This similarity can be utilized to synthesize corresponding mode of the two categories so that the mechanical model parameters can be quantified in terms of the continuum` parameters. Unlike mechanical models Continuum models are difficult to visualize and model in commercially available software. Thus, Synthesis of two approaches has benefit of using the strengths of both methods (Worku ,2013).

2.3.1 Synthesis of Winkler model with Winkler-type continuum model

Comparison of the constant coefficient in equation (2.1) of Winkler mechanical model with equation (2.36) of Winkler-type continuum model results in the following equation for the unknown \bar{k}_s .

$$k_s = \frac{E_s}{H} \quad (2.40)$$

This equation is a generalized form of expression in terms of the continuum parameter. The definite integrals involved in this expression and defined in Eq. (2.32) are evaluated once the functions $g_x(z)$, $g_y(z)$ and $I_z(z)$ are specified.

The lateral to vertical normal stress component with depth under rectangular or circular regions subjected to uniformly distributed loads indicates that their variation with depth can be reasonably represented by an exponentially decaying function. Also the distribution of the vertical shear stress can be expressed using bilinear function. Accordingly, Worku (2013) obtain the following relations

$$g(z) = 1 - \nu [g_x(z) + g_y(z)] = 1 - 1.6\nu e^{-(3.96z/H)} \quad (2.41)$$

$$I_z(z) = \begin{cases} 5z/3H, & 0 \leq z/H \leq 0.6 \\ 2.35 - 2.25z/H, & 0.6 \leq z/H \leq 1 \end{cases}$$

The values of the integrals defined in equation (2.32) can be evaluated after the substitution of equation (2.41) in to (2.32) the resulting values will be substituted in (2.40) to obtain the following expression

$$\bar{k}_s = \frac{E}{(1 - 0.4\nu)H} \quad (2.42)$$

The layer thickness, H , can be eliminated by using the substitution $x = H/B$. And can be equivalently expressed as;

$$\bar{k}_s = \frac{E}{(1 - 0.4\nu)\chi_w a} \quad (2.43)$$

Where a = width of the plate under consideration. And χ_w = the calibration factor in Winkler's model.

2.3.2 Synthesis of Pasternak model with Pasternak-type continuum model

Comparison of the constant coefficient in equation (2.12) of Pasternak mechanical model with equation (2.37) of Pasternak-type continuum model results in the following equation for the unknown \bar{k}_p and \bar{G}_p .

$$\bar{k}_p = \frac{E_s}{H}, \bar{G}_p = \frac{GH}{2} \quad (2.44)$$

The above expression gives less accurate result because of the neglected lateral deformation to reduce the generalized Kerr-type model in to generalized Pasternak-type continuum model. Therefore, expressions for the Pasternak model parameters will be dealt after calibration of Kerr model under the section Kerr-equivalent Pasternak-type model (Worku,2010).

2.3.3 Synthesis of Kerr mechanical model with Kerr-type continuum model

Similarly comparing the coefficients of Kerr mechanical model and the generalized Kerr type continuum model in equation (2.14) and (2.32) respectively. The equation for Kerr mechanical parameters are obtained

$$\left. \begin{aligned} \bar{k}_u &= \frac{L_{gl}H}{(k_g L_{gl} - k_{gl} L_g)} \frac{E}{H}; \bar{k}_l = \frac{L_{gl}H}{(k_{gl} L_g)} \frac{E}{H}; \\ \bar{G}_k &= \frac{L_{gl}^2}{(k_l K_{gl} L_g H)} GH \end{aligned} \right\} \quad (2.45)$$

The values of the integrals defined in equation (2.32) can be evaluated after the substitution of equation (2.41) in to (2.32) the resulting values will be substituted in (2.45) to obtain the following expression by

$$\left. \begin{aligned} \bar{k}_u &= \frac{E}{(0.46 - 0.18\nu)H}, \bar{k}_l = \frac{E}{(0.54 - 0.26\nu)H} \\ \bar{G}_k &= \left(\frac{0.33 - 0.15\nu}{0.14 - 0.11\nu} \right) GH \end{aligned} \right\} \quad (2.46)$$

In the above equation the stratum thickness is eliminated by using the substitution of $t = xH/B$

Where B is as defined previously and χ_k = the calibration factor in Kerr's model (Worku,2013).

2.3.4 The new Kerr-Equivalent Pasternak model

For repetitive practical purpose two parameter Pasternak model is more convenient than three parameter Kerr model (Worku, 2014) establish the Kerr- Equivalent Pasternak model. For the Kerr and Pasternak mechanical models to yield equivalent results identical shear interaction has to exhibit (\bar{G}_k must be equal to \bar{G}_p) and their surface deflection under the action of same loading condition must be equal.

$$w = w_u + w_l \quad (2.47)$$

where $w =$ is the surface deflection in both models, and as described earlier w_u and w_l are the deflection in the upper and lower spring beds, respectively, of the Kerr model.

Equation (2.12) can be expressed as:

$$p = p_p + p_{sl} \quad (2.48)$$

Where P_{sl} is the pressure attributed to the shear layer and Pp is the surface pressure shared by the Pasternak spring bed as shown in Fig. (2.3).

Similarly, the pressure sharing in the Kerr model of fig (2.4) can be written as

$$p = p_{sl} + p_l \quad (2.49)$$

Where P_{sl} is the pressure attributed to the shear layer and P_l is the pressure carried by the lower spring bed of Kerr's model.

The upper spring layer of Kerr's model directly transfers the entire contact pressure to the shear layer and the lower spring bed, the pressure shared by this layer is identical to p .

$$p_u = p \quad (2.50)$$

Substituting Eq. (2.48) and (2.49) into Eq. (2.47) together with the definition of a linear spring, the following expression is obtained.

$$\frac{p - p_{sl}}{\bar{k}_p} = \frac{p}{\bar{k}_u} + \frac{p - p_{sl}}{\bar{k}_l} \quad (2.51)$$

The following equation can be attained after dividing by P throughout and rearranging the equation.

$$k_p = \frac{\bar{k}_u \bar{k}_l}{\bar{k}_u + \eta \bar{k}_l} \quad (2.52)$$

Where $\eta = \frac{1}{1 - (p_{sl} / p)}$, but for most actual cases the value of η is closer to unity.

To obtain the normalized equivalent Pasternak spring coefficient, the expressions of k_u and k_l of Equation (2.45) are inserted into Equation (2.52) giving;

$$\bar{k}_p = \frac{k_p}{(E/H)} = \frac{L_{gl}^2 H}{k_{gl} L_g L_{gl} + \eta (K_g L_{gl}^2 - K_{gl} L_g L_{gl})} \quad (2.53)$$

Similarly, the normalized shear parameter for the replacement Pasternak model in Eq. (2.45) can be written as:

$$\bar{G}_p = \frac{G_p}{(GH)} = \frac{L_{gl}^2}{k_l k_{gl} L_g H} \quad (2.54)$$

For $\eta=1$, equation (2.53) simplifies to:

$$\bar{k}_p = \frac{k_p}{(E/H)} = \frac{H}{k_g} \quad (2.55)$$

Thus, the replacement Pasternak model is fully established with the two Pasternak parameters, \bar{k}_p and \bar{G}_p , found as in Eq. (2.53) and Eq. (2.54).

Based on the plot of the normalized spring stiffness of Kerr-equivalent Pasternak model against Poisson's ratio for all practical purposes found that the stiffness is not that much sensitive to Poisson's ratio (Worku, 2014).

Worku also suggest a linear relationship for the normalized stiffness and finally after introducing calibration factor the Kerr equivalent Pasternak model parameters take the following form.

$$\bar{k}_p = \frac{(0.4\nu + 0.67)E}{\chi B} \quad \bar{G}_p = (1.36\nu + 2.28)G\chi B \quad (2.56)$$

The above parameters of Kerr equivalent Pasternak model can be used for analysis of plates and beam for practical cases.

Chapter Three

Analysis of Strip Plate on Elastic foundation

3.1 Introduction

A plate is a flat structural element with large platform dimensions compared to its thickness and is subjected to loads that cause bending deflection in addition to stretching. There are two types of rectangular plates that can be treated as one-dimensional problems beams and cylindrical bending of strip plate (Reddy, 2007).

When the length along the y-axis of a plate is very small compared to the length along the x-axis, it is treated as a beam. In cylindrical bending, the plate is assumed to be a plate strip that is very long along the y-axis and has a finite dimension along the x-axis (see Figure 3.1). The transverse load q is assumed to be uniform at any section parallel to the x-axis, i.e., $q = q(x)$. In such a case, the vertical deflection w_0 and the horizontal displacements (u_0, v_0) of the plate are functions of only x , and all derivatives with respect to y are zero. In this work, we will consider cylindrical bending of plate strips that is subjected to transverse load that does not vary along the width of the plate. The deflected surface of a portion of such a plate at a considerable distance from the ends can be assumed cylindrical, with the axis of the cylinder parallel to the width of the plate. The deflection of this strip plate is given by a differential equation which is similar to the deflection equation of bent beam.

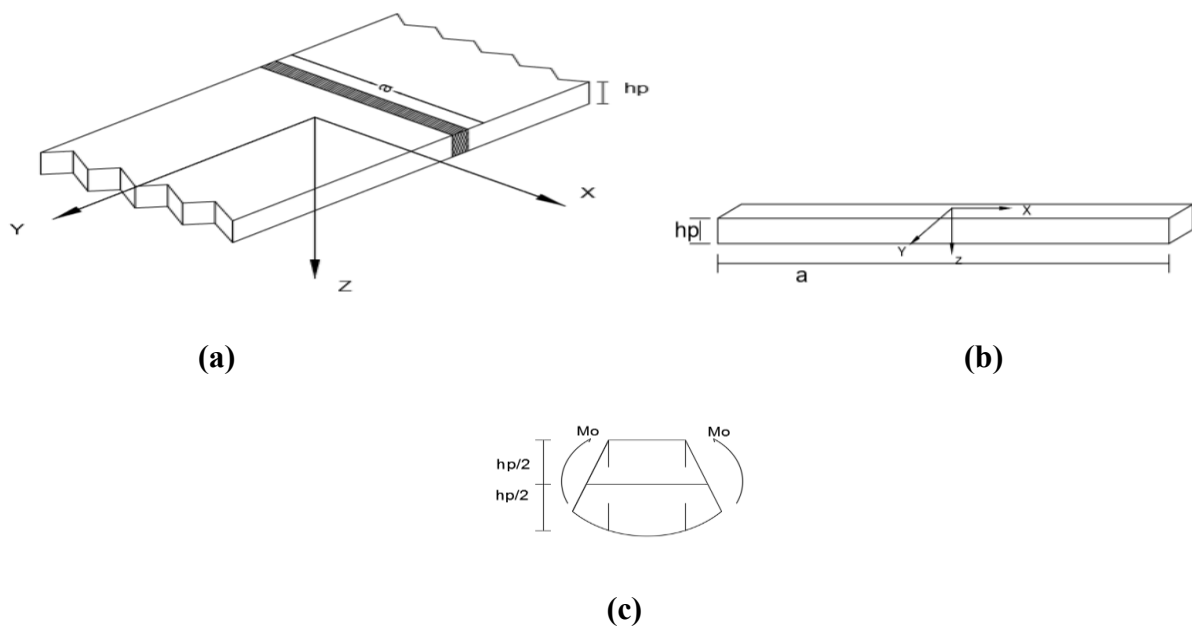


Fig 3.1 a) strip plate b) section of strip plate c) curvature of the deflection

3.2 The Plate Rigidity

In this section, the differential equation (DE) for a strip plate on an elastic subgrade when subjected to a transverse load will be formulated. To obtain the equation for the deflection consider an isotropic rectangular plate of uniform thickness equal to h and let the x and y coordinates be parallel to the edges of the strip. Suppose that the plate is long in the y -direction, has a finite dimension along the x -direction, and is subjected to a transverse load $q(x)$ that is uniform at any section parallel to the x -axis and the plate bends into a cylindrical surface (see Figure 3.1). In Calculating bending stress, we assume the cross section of strip plate remain plane during bending so they will only rotate with respect to their neutral axes. If no normal forces are applied to the end sections of the strip plate the neutral surface coincides with the middle surface of the plate and the unit elongation of a fiber parallel to the x -axis is proportional to its distance z from the middle surface. Temoshenko (1989) clearly stated that the curvature of the deflection curve can be expressed as $-\frac{d^2w}{dx^2}$. Thus, the unit elongation ε_x of the fiber at a distance z from the middle surface becomes

$$\varepsilon_x = \frac{-z d^2w}{dx^2} \quad (3.1)$$

According to Hooke's Law, the unit elongation ε_x and ε_y in terms of the normal stresses σ_x and σ_y is given as:

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu \sigma_y}{E} \quad \text{and} \quad \varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu \sigma_x}{E} \quad (3.2)$$

Where E is the modulus of elasticity of the material and ν is poisson ratio.

The lateral strain in y direction must be zero in order to maintain continuity in the plate during bending, from which by equation of (3.2) one obtains

$$\sigma_y = \nu \sigma_x \quad (3.3)$$

$$\varepsilon_x = \frac{(1-\nu^2)\sigma_x}{E}$$

Then, making use of equation 3.1, the stress σ_x becomes

$$\sigma_x = \frac{-Ez}{1-\nu^2} \left(\frac{d^2w}{dx^2} \right)$$

Moving the expression for bending stress σ_x , Timoshenko (1989) obtain by integration the bending moment in the element strip

$$M = \int_{-h/2}^{h/2} \sigma_x z dz$$

Thus, the equation for the deflection curve of the strip plate of unit length is obtained as follows:-

$$M = \frac{-Eh^3}{12(1-\nu^2)} \frac{d^2w}{dx^2} \tag{3.4}$$

Let $D = \frac{Eh^3}{12(1-\nu^2)}$, the plate flexural stiffness/rigidity, so that

$$M = -D \frac{d^2w}{dx^2} \tag{3.5}$$

In the following section, the Differential equation (DE) of plate on elastic subgrade will be formulated for pertinent subgrade models.

3.3 Formulation of Differential Equations of the Strip Plate

3.3.1 Plate on Single –Parameter Model

In this case consider bending of a uniformly loaded strip plate subjected to transverse load supported over the entire bottom surface by an elastic foundation as it is shown in Figure 3.2a, to formulate the DE of strip plate.

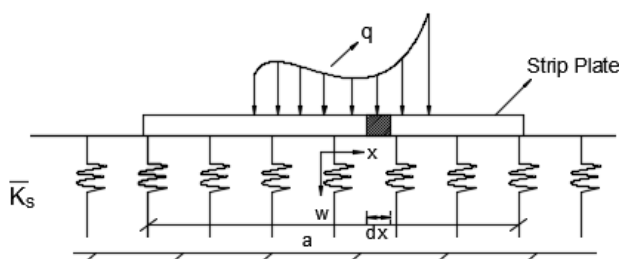
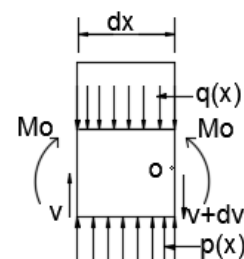


Fig 3.2 (a) A plate on Winkler's mechanical model



(b) strip plate element

Consider the force equilibrium in the z-direction for plate element displayed in Figure 3.2 b

$$p(x) - q(x) = \frac{dV}{dx} \quad (3.6)$$

Where p = contact pressure and q = transvers load

Moment equilibrium requires that

$$V = \frac{dM}{dx} \quad (3.7)$$

Substituting Equation (3.7) in Equation (3.6), one obtains

$$p - q = \frac{d^2w}{dx^2} \quad (3.8)$$

Substituting Equation (3.5) in to Equation (3.8)

$$q(x) - p(x) = D \frac{d^4w}{dx^4}$$

The governing differential equation becomes

$$D\nabla^4 w(x) + p(x) = q(x) \quad (3.9)$$

Note: The governing equation for plate is $D\nabla^4 w(x, y) + p(x, y) = q(x, y)$. However, the governing equation for strip plate will have the form of ODE Equation (3.9). This is because of a strip plate is one dimensional problem (length along y-axis is very long). Therefore, all the derivative with respect to y are zero and the plate will be only functions of x.

Now introducing Winkler model which is

$$p = \bar{k}_s w \quad (3.10)$$

Where p = contact pressure and \bar{k}_s = modulus of subgrade reaction (in N/m³)

Combining equation (3.9) and (3.10) the differential equation for the deflected surface of the plate supported on an elastic Winkler foundation becomes

$$D\nabla^4 w(x) + \bar{k}_s w(x) = q(x) \quad (3.11)$$

The homogenous counterpart is given by

$$D\nabla^4 w(x) + \bar{k}_s w(x) = 0 \quad (3.12)$$

3.3.2 Plate on a Two –Parameter Model

As mentioned earlier, several researchers, recognizing the inherent problems with the Winkler model, attempted to make the model more realistic by assuming some form of interaction among the spring elements that represent the soil continuum. Pasternak improved upon the Winkler model by connecting the ends of the springs to a plate, or "shear layer," consisting of incompressible, vertical elements, which can deform only by lateral shear. A sketch of strip plate on a two-parameter subgrade model is given Figure 3.3.

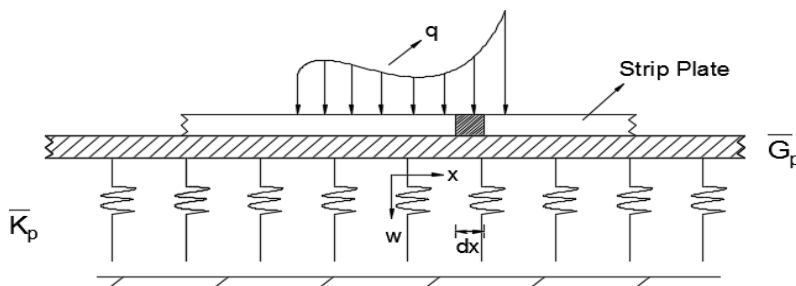


Fig 3.3. A plate on Pasternak's mechanical model

$$p(x) = \bar{k}_p w(x) - \bar{G}_p \nabla^2 w(x) \quad (3.13)$$

Where $p(x)$ = contact pressure

\bar{G}_p = Shear parameter

\bar{k}_p = modulus of subgrade reaction

Substituting Equation (3.9) in to Equation (3.13)

$$D\nabla^4 w(x) + \bar{k}_p w(x) - \bar{G}_p \nabla^2 w(x) = q(x)$$

The homogenous counterpart is

$$D\nabla^4 w(x) + \bar{k}_p w(x) - \bar{G}_p \nabla^2 w(x) = 0 \quad (3.14)$$

3.4 Solution of a differential equation of Strip Plate on Winkler Foundation

3.4.1. General solution for a Single Parameter Model

Here we consider the problem of bending of uniformly loaded plate strip supported over the entire bottom surface by an elastic foundation. The Governing equation of Strip plate on an elastic foundation is given by in Equation (3.11) can be written as equation below.

$$D \frac{d^4 w_0}{dx^4} + \bar{k}_s w_0 = q \quad (3.15)$$

Where D: is the stiffness coefficient

\bar{k}_s : Modules of subgrade reaction (N/m³)

q; is intensity of the traverse load

The homogenous equation becomes

$$D \frac{d^4 w_0}{dx^4} + \bar{k}_s w_0 = 0 \quad (3.16)$$

$$\frac{d^4 w_0}{dx^4} + \frac{\bar{k}_s w_0}{D} = 0 \quad (3.17)$$

Since Equation (3.17) is an ODE with constant coefficients, the solution is obtained as

$$w(x) = \sin \xi x [C_1 \cos h \xi x + C_2 \sin h \xi x] + \cos \xi x [C_3 \cosh \xi x + C_4 \sinh \xi x] \quad (3.18)$$

Where $\xi = 2 \frac{\beta^4}{a} = \sqrt[4]{\frac{\bar{k}_s}{4D}}$

$$\beta^4 = \frac{\bar{k}_s a^4}{64D} \quad a = \text{width of strip plate,}$$

Thus, characteristic width of strip plate becomes $\xi = \sqrt[4]{\frac{\bar{k}_s}{4D}}$

For brevity reasons the detail of the solution are provided in Annex A

3.4.2. Long strip plate

3.4.2.1 A strip plate on a single parameter subgrade subjected to concentrated vertical load

As it is presented in Figure 3.4 a strip plate of long length subjected to a concentrated vertical load resting on Winkler foundation will be solved by introducing the boundary conditions. To simplify the mathematical formulation, the alternative form of the general solution is considered from Annex A (A1.4).

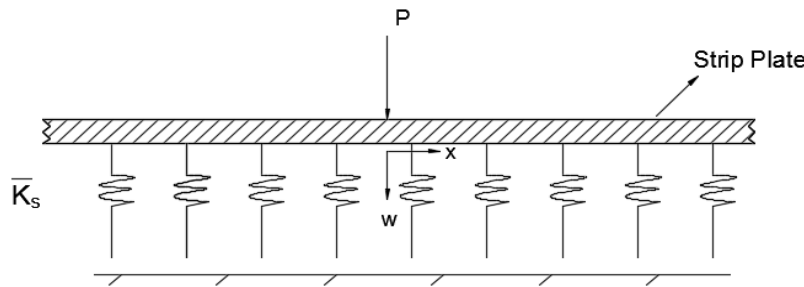


Fig 3.4. Long strip plate on Winkler Foundation subjected to vertical concentrated load

$$w(x) = e^{\xi x} (A_1 \cos \xi x + A_2 \sin \xi x) + e^{-\xi x} (A_3 \cos \xi x + A_4 \sin \xi x) \quad (3.19)$$

Boundary conditions

1. $w'(0) = 0$
2. $w(\infty) = 0$

Applying the 2nd boundary equation the left part of the equation must be zero to satisfy the third boundary condition thus the constant A_1 and A_2 must be zero resulting in

$$w(x) = e^{-\xi x} (A_3 \cos \xi x + A_4 \sin \xi x) \quad (3.20)$$

Applying the 1st boundary Condition

$$A_3 = A_4 = C \quad (3.21)$$

Equation (3.20) can be written as

$$w(x) = C e^{-\xi x} (\cos \xi x + \sin \xi x) \quad (3.22)$$

The constant C can be obtained from the consideration that the sum of the reaction force will keep equilibrium with the load P,

$$2 \int_0^{\infty} \bar{k}_s w dx = P$$

$$2 \bar{k}_s C \int_0^{\infty} e^{-\xi x} (\cos \xi x + \sin \xi x) dx = P, \text{ from which follows}$$

$$C = \frac{\xi P}{2 \bar{k}_s} \tag{3.23}$$

Substituting Equation (3.23) in to Equation (3.22)

$$w(x) = \frac{\xi P}{2 \bar{k}_s} e^{-\xi x} (\cos \xi x + \sin \xi x) \tag{3.24}$$

Successive differential Equation (3.24) gives the slope, moment and shear force

$$\theta(x) = \frac{-P \xi^2}{\bar{k}_s} e^{-\xi x} \sin \xi x \tag{3.25}$$

$$M(x) = \frac{P}{4(\xi)} e^{-\xi x} (\cos \xi x - \sin \xi x) \tag{3.26}$$

$$V(x) = -\frac{P}{2} e^{-\xi x} (\cos \xi x) \tag{3.27}$$

3.4.2.2 A strip plate on single parameter subgrade subjected to Concentrated Moment

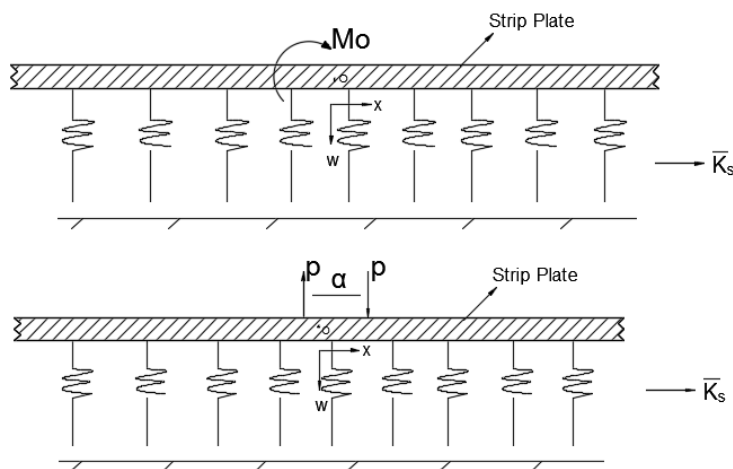


Fig 3.5 Long strip plate on Winkler foundation subjected to concentrated moment

By following Hetenyi's method (Hetenyi, 1946) when long length strip plate resting on a single-parameter subgrade is subjected to a concentrated moment the following relationship is obtained. This is demonstrated on Figure 3.5.

$$w(x) = \frac{\xi p \alpha}{2 \bar{k}_s} \left[\frac{e^{-\xi x + \alpha} (\cos(\xi x + \alpha) + \sin(\xi x + \alpha)) - e^{-\xi x} (\cos \xi x + \sin \xi x)}{\alpha} \right] \quad (3.28)$$

With α approaching zero, the expression in the brackets simplifies to

$$\left[\frac{e^{-\xi x + \alpha} (\cos(\xi x + \alpha) + \sin(\xi x + \alpha)) - e^{-\xi x} (\cos \xi x + \sin \xi x)}{\alpha} \right]_{\alpha \rightarrow 0} = 2 \xi e^{-\xi x} \sin \xi x$$

And as the same time $[p \alpha]_{\alpha \rightarrow 0} = M_0$

Thus the deflection Equation becomes

$$w(x) = \frac{M_0 (\xi)^2}{\bar{k}_s} e^{-\xi x} \sin \xi x \quad (3.29)$$

$$\theta(x) = \frac{M_0 \xi^3}{\bar{k}_s} e^{-\xi x} (\cos \xi x - \sin \xi x) \quad (3.30)$$

$$M(x) = \frac{M_0}{2} e^{-\xi x} (\cos \xi x) \quad (3.31)$$

$$V(x) = \frac{M_0}{2} \xi e^{-\xi x} (\cos \xi x + \sin \xi x) \quad (3.32)$$

3.4.2.3 A strip plate on single parameter subgrade subjected to Uniformly distributed load

By following Hetenyi's method (Hetenyi, 1946) when long strip plate resting on a single parameter subgrade is subjected to a uniformly distributed load, the following expressions are obtained depending on the location of the point of interest with respect to the loaded region as it is shown in Figure 3.6.

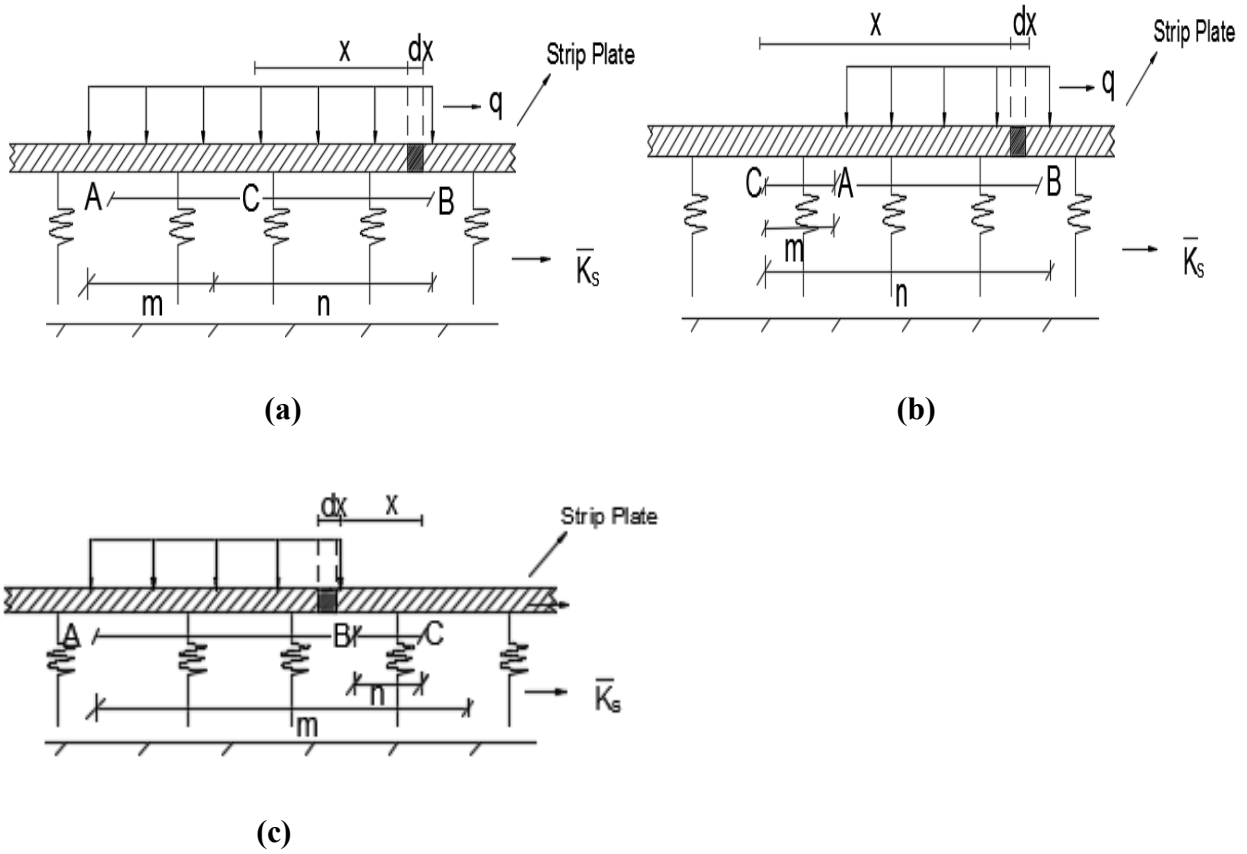


Fig 3.6 Long strip plate on Winkler Foundation subjected to uniformly distributed load a) when point C is under the loaded region b) when point C is to the left side of the loaded region C) when point C is to the right of the loaded region

The Mathematical formulation for the above diagrams is explained below.

1. When Point C is under loaded region

$$w_c = \frac{q}{2k_s} \left(2 - e^{-\xi m} (\cos \xi m) - e^{-\xi n} (\cos \xi n) \right) \quad (3.33a)$$

$$\theta_c = \frac{q(\xi)}{2k_s} \left[e^{-\xi m} (\cos(\xi m) + \sin(\xi m)) - e^{-\xi n} (\cos(\xi n) + \sin(\xi n)) \right] \quad (3.33 b)$$

$$M_c = \frac{q}{4(\xi)^2} \left[e^{-\xi m} \sin(\xi m) + e^{-\xi n} \sin(\xi n) \right] \quad (3.33c)$$

$$V_c = \frac{q}{4(\xi)} \left[e^{-\xi m} (\cos \xi m - \sin \xi m) - e^{-\xi n} (\cos \xi n - \sin \xi n) \right] \quad (3.33d)$$

2. When Point C is to the left of the load

$$w_C = \frac{q}{2k_s} \left(e^{-\xi m} (\cos \xi m) - e^{-\xi n} (\cos \xi n) \right) \quad (3.34 a)$$

$$\theta_C = \frac{q\xi}{2k_s} \left[e^{-\xi m} (\cos(\xi m) + \sin(\xi m)) - e^{-\xi n} (\cos(\xi n) + \sin(\xi n)) \right] \quad (3.34b)$$

$$M_C = -\frac{q}{4\xi^2} \left(e^{-\xi m} \sin \xi m - e^{-\xi n} \sin \xi n \right) \quad (3.34c)$$

$$V_C = \frac{q}{4\xi} \left[e^{-\xi m} (\cos(\xi m) - \sin(\xi m)) - e^{-\xi n} (\cos(\xi n) - \sin(\xi n)) \right] \quad (3.34d)$$

3. When Point C is to the right of the load

$$w_C = -\frac{q}{2k_s} \left(e^{-\xi m} (\cos \xi m) - e^{-\xi n} (\cos \xi n) \right) \quad (3.35a)$$

$$\theta_C = \frac{q\xi}{2k_s} \left[e^{-\xi m} (\cos \xi m + \sin \xi m) - e^{-\xi n} (\cos \xi n + \sin \xi n) \right] \quad (3.35b)$$

$$M_C = \frac{q}{4\xi^2} \left(e^{-\xi m} \sin \xi n - e^{-\xi n} \sin \xi m \right) \quad (3.35c)$$

$$V_C = \frac{q}{4\xi} \left(e^{-\xi m} (\cos \xi m - \sin \xi m) - e^{-\xi n} (\cos \xi n - \sin \xi n) \right) \quad (3.35d)$$

3.4.3 Short strip plate

Hetenyi (1946) solved the case of short beams subjected to different loading condition by using superposition method. In this section by following the superposition principle, the problem of a short strip plate when subjected to a concentrated vertical load and a uniformly distributed vertical load is solved.

3.4.3.1 A strip plate on single parameter subgrade subjected to vertical concentrated load

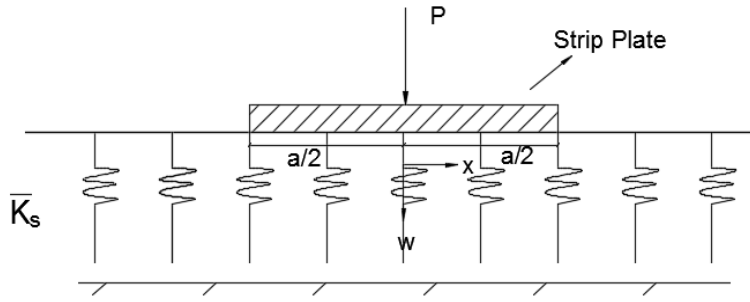


Fig 3.7. A short strip plate on Winkler foundation subjected to Vertical Concentrated load

The solution for the deflection, slope, moment and shear for short strip plate shown in Figure (3.7) are given below, whereas the detailed solution are given in Annex B.

$$w(x) = \frac{p\xi}{2k_s} \frac{1}{\sinh 2\beta + \sin 2\beta} * \left[\frac{\cosh \xi x \cos \xi (a-x) + \cos \xi x \cosh \xi (a-x) - \sinh \xi x \sin \xi (a-x) + \sin \xi x \sinh \xi (a-x)}{2 \cosh \xi x \cos \xi x} \right] \quad (3.36)$$

$$\theta(x) = \frac{p\xi^2}{2k_s} \frac{1}{\sinh 2\beta + \sin 2\beta} \left[\frac{\sinh \xi x (\cos \xi x + \cos \xi (a-x)) - \sin \xi x (\cosh \xi x + \cosh \xi (a-x))}{\sinh \xi x (\cos \xi x + \cos \xi (a-x)) - \sin \xi x (\cosh \xi x + \cosh \xi (a-x))} \right] \quad (3.37)$$

$$M(x) = \frac{p}{4\xi} \frac{1}{\sinh 2\beta + \sin 2\beta} \left[\frac{\sinh \xi x (\sin \xi x - \sin \xi (a-x)) - \cosh \xi x (\cos \xi x + \cos \xi (a-x)) + \sin \xi x (\sinh \xi x - \sinh \xi (a-x)) - \cos \xi x (\cosh \xi x + \cosh \xi (a-x))}{\sinh \xi x (\sin \xi x - \sin \xi (a-x)) - \cosh \xi x (\cos \xi x + \cos \xi (a-x)) + \sin \xi x (\sinh \xi x - \sinh \xi (a-x)) - \cos \xi x (\cosh \xi x + \cosh \xi (a-x))} \right] \quad (3.38)$$

$$v(x) = \frac{p}{2} \frac{1}{\sinh 2\beta + \sin 2\beta} \left[\cosh \xi x (\sin \xi x - \sin \xi (a-x)) + \cos \xi x (\sinh \xi x - \sinh \xi (a-x)) \right] \quad (3.39)$$

3.4.3.2 A strip plate on single parameter subgrade a subjected to symmetrically placed distributed Load

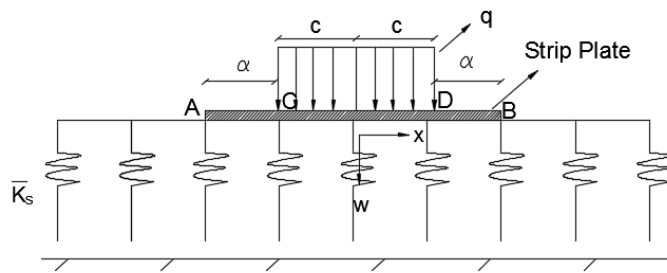


Fig 3.8 A short strip plate on Winkler foundation subjected to uniformly distributed load

By using the superposition method, one can obtain the following expressions for deflection under the portion of (A-C) for the figure shown above.

$$w_{A-C} = \frac{q}{k_s} \frac{1}{\sinh 2\beta + \sin 2\beta} \left(\begin{aligned} & \cosh \xi x \cos \xi x \left(\cosh \xi \alpha \sin \xi (a - \alpha) - \sinh \xi \alpha \cos \xi (a - \alpha) + \right. \\ & \left. \cos \xi \alpha \sinh \xi (a - \alpha) - \sin \xi \alpha \cosh \xi (a - \alpha) \right) + \\ & \left(\cosh \xi x \sin \xi x + \sinh \xi x \cos \xi x \right) \left(\sin \xi \alpha \sinh \xi (a - \alpha) - \sinh \xi \alpha \sin \xi (a - \alpha) \right) \end{aligned} \right) \quad (3.40)$$

The deflection for the portion (C-D) will be

$$w_{C-D} = [w_{A-C}]_{x>a} + \frac{q}{k_s} [1 - \cosh \xi (x - \alpha) \cosh \xi (x - \alpha)] \quad (3.41)$$

The deflection at the middle (point O)

$$w_o = \frac{q}{k_s} \left[1 - \frac{2(\sinh \xi \alpha \cos \xi c \cosh \beta + \sin \xi \alpha \cosh \xi c \cos \beta)}{\sinh 2\beta + \sin 2\beta} \right] \quad (3.42)$$

Due to the complexity of the mathematical formulation, the analytical solutions for the moment and shear force for short strip plate on Winkler model subjected to uniformly distributed load are not included in this study.

3.5 Solution of the Differential Equation of Strip Plate on Kerr-Equivalent Pasternak Foundation

3.5.1 General Solution of Strip Plate resting on Kerr-Equivalent Pasternak Foundation

The differential equation for the deflection $w(x)$ of a strip plate on Pasternak foundation can be determined by combining of Pasternak's model and the strip plate differential equation. The resulting homogenous differential equation is given by

$$\frac{Dd^4w}{dx^4} - \frac{\bar{G}_p d^2w}{dx^2} + \bar{k}_p w = 0 \quad (3.43)$$

To solve this differential equation, the following parameters are assumed

$$\text{Let } \beta = \sqrt[4]{\frac{\bar{k}_p a^4}{64D}}, \gamma = \sqrt{\frac{\bar{k}_p a^2}{\bar{G}_p}} \text{ and } \rho = \left(\frac{\beta}{\gamma}\right)^2 \quad (3.44)$$

$$\rho = \left[\sqrt[4]{\frac{\bar{k}_p a^4}{64D}} * \sqrt{\frac{\bar{G}_p}{\bar{k}_p a^2}} \right]^2, \rho = \frac{\bar{G}_p}{8\sqrt{D\bar{k}_p}} \quad (3.45)$$

Thus the differential equation becomes

$$\frac{d^4w}{dx^4} - \frac{64\beta^2\rho}{a^2} \frac{d^2w}{dx^2} + 64\frac{\beta^4}{a^4} w = 0 \quad (3.46)$$

Since Equation (3.46) is an ODE with constant coefficients the solution is of the form $w = ce^{mx}$ substituting this expression one obtain the following characteristics polynomial

$$m^4 - \frac{64\beta^2\rho}{a^2} m^2 + 64\frac{\beta^4}{a^4} = 0$$

There are three possible cases of the general solution of equation (3.46) depend on whether, $\rho < 1/4$, $\rho = 1/4$ and $\rho > 1/4$. For brevity reasons the final forms of the general solutions are provided here while the detailed are given in Annex C

Case I ($\rho < 1/4$)

$$w(x) = e^{\xi\phi_1 x} (A_1 \cos \xi\phi_2 x + A_2 \sin \xi\phi_2 x) + e^{-\xi\phi_1 x} (A_3 \cos \xi\phi_2 x + A_4 \sin \xi\phi_2 x) \quad (3.47)$$

Where $\phi_1 = \sqrt{1+4\rho}$, $\phi_2 = \sqrt{1-4\rho}$ and A_1, A_2, A_3 and A_4 are open constants

Case II ($\rho = 1/4$)

$$w(x) = e^{\xi\phi_1 x} (B_1 + B_3 x) + e^{-\xi\phi_1 x} (B_2 + B_4 x) \quad (3.48)$$

Where $\phi_1 = \sqrt{2}$ and B_1, B_2, B_3 and B_4 are open constants

Case III ($\rho > 1/4$)

$$w(x) = e^{\xi\phi_1 x} (E_1 \cosh \xi\phi_2 x + E_2 \sinh \xi\phi_2 x) + e^{-\xi\phi_1 x} (E_3 \cosh \xi\phi_2 x + E_4 \sinh \xi\phi_2 x) \quad (3.49)$$

Where $\phi_1 = \sqrt{1+4\rho}$ and $\phi_2 = \sqrt{4\rho-1}$ and E_1, E_2, E_3 and E_4 are open constants

3.5.2 Long strip plate

3.5.2.1. Long strip plate subjected to a vertical Concentrated Load

In this section a strip plate of long length subjected to concentrated load and supported by an elastic foundation represented by Pasternak model is considered. By introducing the suitable boundary conditions, the particular solutions are determined.

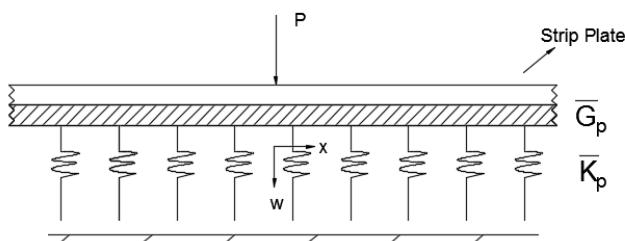


Fig 3.9. Long strip plate on Kerr Equivalent Pasternak model subjected to vertical concentrated load

The elastic soil layer of thickness H is represented by Pasternak mechanical model as illustrated in the Figure 3.9 where \bar{k}_p = modules of subgrade reaction and \bar{G}_p shear parameter

Boundary condition

$$1. w'(0) = 0$$

$$2. Dw'''(0) = P/2$$

$$3. w(\infty) = 0$$

The final expression of deflection, slope, moment and shear force are given below and the detailed are given in Annex D.

Case I $\left(\rho < \frac{1}{4}\right)$

The general solution for deflection is given by

$$w(x) = e^{\xi\phi_1 x} (A_1 \cos(\xi\phi_2 x) + A_2 \sin(\xi\phi_2 x)) + e^{-\xi\phi_1 x} (A_3 \cos(\xi\phi_2 x) + A_4 \sin(\xi\phi_2 x))$$

For the above equation, the constants can be determined by using the boundary condition presented to Figure 3.9 and obtain the following expressions.

$$w(x) = \frac{P}{8D\xi^3} e^{-\xi\phi_1 x} \left\{ \frac{\cos(\xi\phi_2 x)}{\phi_1} + \frac{\sin(\xi\phi_2 x)}{\phi_2} \right\} \quad (3.50)$$

$$\theta(x) = \frac{Pe^{-\xi\phi_1 x}}{4D\xi^2} \left(\frac{\sin \xi\phi_2 x}{\phi_1\phi_2} \right) \quad (3.51)$$

$$M(x) = \frac{-Pe^{-\xi\phi_1 x}}{4\xi^2} \left(\frac{\sin(\xi\phi_2 x)}{\phi_2} - \frac{\cos(\xi\phi_2 x)}{\phi_1} \right) \quad (3.52)$$

$$V(x) = \frac{Pe^{-\xi\phi_1 x}}{2} [\cos(\xi\phi_2 x) - \sin(\xi\phi_2 x)] \quad (3.53)$$

Case II $\left(\rho = \frac{1}{4}\right)$

The general solution is for deflection given is given by

$$w(x) = B_1 e^{\xi\phi_1 x} + B_2 e^{-\xi\phi_1 x} + B_3 x e^{\xi\phi_1 x} + B_4 x e^{-\xi\phi_1 x}$$

Applying the boundary conditions mention in above one can determine the open constant and arrived to the following expressions

$$w(x) = \frac{Pe^{-\xi\phi_1 x}}{4D(\xi\phi_1)^2} \left(\frac{1}{(\xi\phi_1)} + x \right) \quad (3.54)$$

$$\theta(x) = \frac{-Pe^{-\xi\phi_1 x}}{4D} \left(\frac{x}{(\xi\phi_1)} \right) \quad (3.55)$$

$$M(x) = \frac{Pe^{-\xi\phi_1 x}}{4} \left(\frac{1}{(\xi\phi_1)} - x \right) \quad (3.56)$$

$$V(x) = \frac{Pe^{-\xi\phi_1 x}}{2} \left(\frac{x(\xi\phi_1)}{2} - 1 \right) \quad (3.57)$$

Case III ($\rho > 1/4$)

For the general Solution for deflection is given by

$$w(x) = e^{\xi\phi_1 x} (E_1 \cosh \xi\phi_2 x + E_2 \sinh \xi\phi_2 x) + e^{-\xi\phi_1 x} (E_3 \cosh \xi\phi_2 x + E_4 \sinh \xi\phi_2 x)$$

In similar manner the constants can be determined by using the boundary condition presented to Figure (3.9) and obtain the following expressions.

$$w(x) = \frac{Pe^{-\xi\phi_1 x}}{4D\xi^3(\phi_1^2 - \phi_2^2)} \left[\frac{\cosh(\xi\phi_2 x)}{\phi_1} + \frac{\sinh(\xi\phi_2 x)}{\phi_2} \right] \quad (3.58)$$

$$\theta(x) = \frac{-Pe^{-\xi\phi_1 x} \sinh(\xi\phi_2 x)}{4D\xi^2\phi_2\phi_1} \quad (3.59)$$

$$M(x) = \frac{Pe^{-\xi\phi_1 x}}{4\xi} \left[\frac{\cosh \xi\phi_2 x}{\phi_1} - \frac{\sinh \frac{2\beta\phi_2 x}{a}}{\phi_2} \right] \quad (3.60)$$

$$V(x) = \frac{Pe^{-\xi\phi_1 x}}{2} \left[-\cosh(\xi\phi_2 x) + \frac{\sinh(\xi\phi_2 x)}{\phi_2\phi_1} \right] \quad (3.61)$$

3.5.2.2. Long strip plate subjected to a uniformly distributed load

A uniformly distributed loading acting over a portion “AB” of the long length strip plate is considered, as shown in Figure 3.10. The distributed load can be regarded as a series of infinitely small concentrated forces, qdx . Thus, the deflection, rotation, moment and shear force can be obtained by substituting qdx for p in the respective solutions of the vertical concentrated load case. Three cases will be considered depending on the location of point C

1. When point C is within the loaded region
2. When point C is to the left of the loaded region
3. When point C is to the right of the loaded region

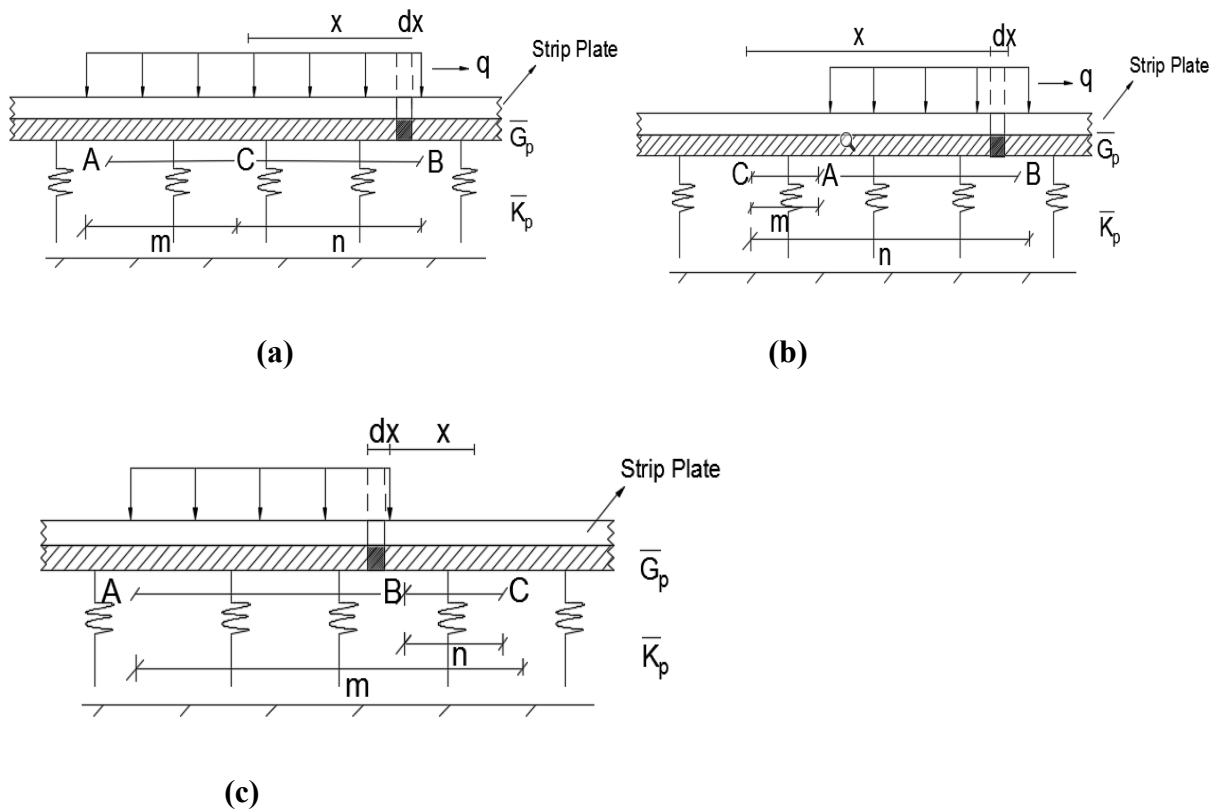


Fig 3.10. Long strip plate on Pasternak model subjected to uniformly distributed load a) when point c is under the loaded region b) when point c is to the left side of the loaded region C) when point C is to the right of the loaded region

Case I ($\rho < \frac{1}{4}$)

1. When point C is within the loaded region

$$wc = \int \delta dw, dw = \int -\frac{qdx}{2D} y(x)$$

Where $y(x) = \frac{P}{8D\xi^3} e^{-\xi\phi_1 x} \frac{\cos(\xi\phi_2 x)}{\phi_1} + \frac{P}{8D\xi^3} e^{-\xi\phi_1 x} \frac{\sin(\xi\phi_2 x)}{\phi_2}$

$$w_C = \frac{q}{8D\xi^3} \left(\frac{\int_0^m e^{-\xi\phi_1 x} \cos \xi\phi_2 x}{\phi_1} + \frac{\int_0^m e^{-\xi\phi_1 x} \sin \xi\phi_2 x}{\phi_2} + \frac{\int_0^n e^{-\xi\phi_1 x} \cos \xi\phi_2 x}{\phi_1} + \frac{\int_0^n e^{-\xi\phi_1 x} \sin \xi\phi_2 x}{\phi_2} \right)$$

$$w_C = \frac{q}{8D\xi^4 (\phi_1^2 + \phi_2^2)} \left(\left(\frac{e^{-\xi\phi_1 m}}{\phi_1} (\phi_2 \sin \xi\phi_2 m - \phi_1 \cos \xi\phi_2 m) + \frac{e^{-\xi\phi_1 n}}{\phi_1} (\phi_2 \sin \xi\phi_2 n - \phi_1 \cos \xi\phi_2 n) + 2\phi_1 \right) + \left(-\frac{e^{-\xi\phi_1 m}}{\phi_2} (\phi_1 \sin \xi\phi_2 m + \phi_2 \cos \xi\phi_2 m) + \frac{e^{-\xi\phi_1 n}}{\phi_2} (\phi_1 \sin \xi\phi_2 n + \phi_2 \cos \xi\phi_2 n) + 2\phi_2 \right) \right) \quad (3.62)$$

In a similar manner substituting qdx for P and integrating within the assigned limits one can obtain the expression of slope, moment and shear force at point C as

$$\theta_C = \frac{q^3}{8D\xi^3} \left[e^{-\xi\phi_1 m} \left(\frac{\cos \xi\phi_2 m}{\phi_1} + \frac{\sin \xi\phi_2 m}{\phi_2} \right) - e^{-\xi\phi_1 n} \left(\frac{\cos \xi\phi_2 n}{\phi_1} + \frac{\sin \xi\phi_2 n}{\phi_2} \right) \right] \quad (3.63)$$

$$M_C = \frac{q}{8\xi^2} \left(\left[\frac{\phi_2}{\phi_1} e^{-\xi\phi_1 m} \sin \xi\phi_2 m + \frac{\phi_1}{\phi_2} e^{-\xi\phi_1 m} \sin \xi\phi_2 m \right] - \left[\frac{\phi_2}{\phi_1} e^{-\xi\phi_1 n} \sin \xi\phi_2 n + \frac{\phi_1}{\phi_2} e^{-\xi\phi_1 n} \sin \xi\phi_2 n \right] \right) \quad (3.64)$$

$$V_C = \frac{-q}{8\xi} \left(\left(-e^{-\xi\phi_1 m} \sin \xi\phi_2 m \left(\phi_2 + \frac{\phi_1^2}{\phi_2} \right) + e^{-\xi\phi_1 m} \cos \xi\phi_2 m \left(\phi_1 + \frac{\phi_2^2}{\phi_1} \right) \right) + \left(e^{-\xi\phi_1 n} \sin \xi\phi_2 n \left(\phi_2 + \frac{\phi_1^2}{\phi_2} \right) - e^{-\xi\phi_1 n} \cos \xi\phi_2 n \left(\phi_1 + \frac{\phi_2^2}{\phi_1} \right) \right) \right) \quad (3.65)$$

2. When point C is to the left of the loaded Region

$$w_C = \frac{q}{8D\xi^3} \left(\frac{\int_0^n e^{-\xi\phi_1 x} \cos \xi\phi_2 x}{\phi_1} + \frac{\int_0^n e^{-\xi\phi_1 x} \sin \xi\phi_2 x}{\phi_2} - \frac{\int_0^m e^{-\xi\phi_1 x} \cos \xi\phi_2 x}{\phi_1} - \frac{\int_0^m e^{-\xi\phi_1 x} \sin \xi\phi_2 x}{\phi_2} \right)$$

$$w_C = \frac{q}{8D\xi^4(\phi_1^2 + \phi_2^2)} \left(\begin{array}{l} \frac{e^{-\xi\phi_1 n}}{\phi_1} (\phi_2 \sin(\xi\phi_2 n) - \phi_1 \cos(\xi\phi_2 n)) + \\ \frac{e^{-\xi\phi_1 n}}{\phi_2} (\phi_1 \sin(\xi\phi_2 n) + \phi_2 \cos(\xi\phi_2 n)) - \\ \frac{e^{-\xi\phi_1 m}}{\phi_1} (\phi_2 \sin(\xi\phi_2 m) - \phi_1 \cos(\xi\phi_2 m)) + \\ \frac{e^{-\xi\phi_1 m}}{\phi_2} (\phi_1 \sin(\xi\phi_2 m) + \phi_2 \cos(\xi\phi_2 m)) \end{array} \right) \quad (3.66)$$

$$\theta_C = \frac{q}{8D\xi^3} \left[e^{-\xi\phi_1 m} \left(\frac{\cos \xi\phi_2 m}{\phi_1} + \frac{\sin \xi\phi_2 m}{\phi_2} \right) - e^{-\xi\phi_1 n} \left(\frac{\cos \xi\phi_2 n}{\phi_1} + \frac{\sin \xi\phi_2 n}{\phi_2} \right) \right] \quad (3.67)$$

$$M_C = \frac{q}{8\xi^2} \left(\begin{array}{l} \left[\frac{\phi_2}{\phi_1} e^{-\xi\phi_1 n} \sin \xi\phi_2 n + \frac{\phi_1}{\phi_2} e^{-\xi\phi_1 n} \sin \xi\phi_2 n \right] - \\ \left[\frac{\phi_2}{\phi_1} e^{-\xi\phi_1 m} \sin \xi\phi_2 m + \frac{\phi_1}{\phi_2} e^{-\xi\phi_1 m} \sin \xi\phi_2 m \right] \end{array} \right) \quad (3.68)$$

$$V_C = \frac{-q}{8\xi} \left(\begin{array}{l} -e^{-\xi\phi_1 m} \sin \xi\phi_2 m \left(\phi_2 + \frac{\phi_1^2}{\phi_2} \right) + e^{-\xi\phi_1 m} \cos \xi\phi_2 m \left(\phi_1 + \frac{\phi_2^2}{\phi_1} \right) + \\ e^{-\xi\phi_1 n} \sin \xi\phi_2 n \left(\phi_2 + \frac{\phi_1^2}{\phi_2} \right) - e^{-\xi\phi_1 n} \cos \xi\phi_2 n \left(\phi_1 + \frac{\phi_2^2}{\phi_1} \right) \end{array} \right) \quad (3.69)$$

3. When point C is to the right of the loaded Region

$$w_C = \frac{q}{8D\xi^3} \left(\frac{\int_0^m e^{-\xi\phi_1 x} \cos \xi\phi_2 x}{\phi_1} + \frac{\int_0^m e^{-\xi\phi_1 x} \sin \xi\phi_2 x}{\phi_2} - \frac{\int_0^n e^{-\xi\phi_1 x} \cos \xi\phi_2 x}{\phi_1} - \frac{\int_0^n e^{-\xi\phi_1 x} \sin \xi\phi_2 x}{\phi_2} \right)$$

$$w_C = \frac{q}{8D\xi^4(\phi_1^2 + \phi_2^2)} \left(\begin{array}{l} \frac{e^{-\xi\phi_1 m}}{\phi_1} (\phi_2 \sin \xi\phi_2 m - \phi_1 \cos \xi\phi_2 m) - \\ \frac{e^{-\xi\phi_1 m}}{\phi_2} (\phi_1 \sin \xi\phi_2 m + \phi_2 \cos \xi\phi_2 m) - \\ \frac{e^{-\xi\phi_1 n}}{\phi_1} (\phi_2 \sin \xi\phi_2 n - \phi_1 \cos \xi\phi_2 n) + \\ \frac{e^{-\xi\phi_1 n}}{\phi_2} (\phi_1 \sin \xi\phi_2 n + \phi_2 \cos \xi\phi_2 n) \end{array} \right) \quad (3.70)$$

$$\theta_C = \frac{q}{8D\xi^3} \left[e^{-\xi\phi_1 m} \left(\frac{\cos \xi\phi_2 m}{\phi_1} + \frac{\sin \xi\phi_2 m}{\phi_2} \right) - e^{-\xi\phi_1 n} \left(\frac{\cos \xi\phi_2 n}{\phi_1} + \frac{\sin \xi\phi_2 n}{\phi_2} \right) \right] \quad (3.71)$$

$$M_C = \frac{q}{8\xi^2} \left(\begin{array}{l} \left[\frac{\phi_2}{\phi_1} e^{-\xi\phi_1 m} \sin \xi\phi_2 m + \frac{\phi_1}{\phi_2} e^{-\xi\phi_1 m} \sin \xi\phi_2 m \right] - \\ \left[\frac{\phi_2}{\phi_1} e^{-\xi\phi_1 n} \sin \xi\phi_2 n + \frac{\phi_1}{\phi_2} e^{-\xi\phi_1 n} \sin \xi\phi_2 n \right] \end{array} \right) \quad (3.72)$$

$$V_C = \frac{-q}{8\xi} \left(\begin{array}{l} -e^{-\xi\phi_1 m} \sin \xi\phi_2 m \left(\phi_2 + \frac{\phi_1^2}{\phi_2} \right) + e^{-\xi\phi_1 m} \cos \xi\phi_2 m \left(\phi_1 + \frac{\phi_2^2}{\phi_1} \right) + \\ e^{-\xi\phi_1 n} \sin \xi\phi_2 n \left(\phi_2 + \frac{\phi_1^2}{\phi_2} \right) - e^{-\xi\phi_1 n} \cos \xi\phi_2 n \left(\phi_1 + \frac{\phi_2^2}{\phi_1} \right) \end{array} \right) \quad (3.73)$$

Case II ($\rho = \frac{1}{4}$)

1. When point C is within the loaded region

$$wc = \int \delta dw, dw = \int -\frac{qdx}{2D} y(x)$$

$$\text{Where } y(x) = \frac{Pe^{-\xi\phi_1 x}}{4D(-\xi\phi_1)^3} + \frac{xPe^{-\xi\phi_1 x}}{4D(\xi\phi_1)^2}$$

$$w_C = \frac{-q}{4D(\xi\phi_1)^4} \left[e^{-\xi\phi_1 m} + e^{-\xi\phi_1 n} + (\xi\phi_1 m e^{-\xi\phi_1 m} + 1) + (\xi\phi_1 n e^{-\xi\phi_1 n} + 1) \right] \quad (3.74)$$

$$\theta_C = \frac{q}{4D(\xi\phi_1)^2} \left[m e^{-\xi\phi_1 n} - n e^{-\xi\phi_1 m} \right] \quad (3.75)$$

$$M_C = \frac{-q}{4(\xi\phi_1)^2} \left[e^{-\xi\phi_1 m} - \xi\phi_1 m e^{-\xi\phi_1 m} + e^{-\xi\phi_1 n} - \xi\phi_1 n e^{-\xi\phi_1 n} \right] \quad (3.76)$$

$$V_C = \frac{q}{\xi\phi_1} \left[\frac{\xi}{2} \phi_1 m e^{-\xi\phi_1 m} - \frac{\xi}{2} \phi_1 n e^{-\xi\phi_1 n} - e^{-\xi\phi_1 m} + e^{-\xi\phi_1 n} \right] \quad (3.77)$$

2. When point C is to the left of the loaded Region

$$w_C = \frac{-q}{4D(\xi\phi_1)^4} \left[e^{-\xi\phi_1 n} - e^{-\xi\phi_1 m} + (\xi\phi_1 n e^{-\xi\phi_1 n}) + (\xi\phi_1 m e^{-\xi\phi_1 m}) \right] \quad (3.78)$$

$$\theta_C = \frac{q}{4D(\xi\phi_1)^2} \left[m e^{-\xi\phi_1 m} - n e^{-\xi\phi_1 n} \right] \quad (3.79)$$

$$M_C = \frac{-q}{4(\xi\phi_1)^2} \left[e^{-\xi\phi_1 n} - \xi\phi_1 n e^{-\xi\phi_1 n} - e^{-\xi\phi_1 m} + \xi\phi_1 m e^{-\xi\phi_1 m} \right] \quad (3.80)$$

$$V_C = \frac{q}{\xi\phi_1} \left[\frac{\xi}{2} \phi_1 m e^{-\xi\phi_1 m} - \frac{\xi}{2} \phi_1 n e^{-\xi\phi_1 n} - e^{-\xi\phi_1 m} + e^{-\xi\phi_1 n} \right] \quad (3.81)$$

3. When point C is to the right of the loaded Region

$$w_C = \frac{-q}{4D(\xi\phi_1)^4} \left[e^{-\xi\phi_1 m} - e^{-\xi\phi_1 n} + (\xi\phi_1 m e^{-\xi\phi_1 m}) - (\xi\phi_1 n e^{-\xi\phi_1 n}) \right] \quad (3.82)$$

$$\theta_C = \frac{q}{4D(\xi\phi_1)^2} \left[m e^{-\xi\phi_1 m} - n e^{-\xi\phi_1 n} \right] \quad (3.83)$$

$$M_C = \frac{-q}{4(\xi\phi_1)^2} \left[e^{-\xi\phi_1 m} - \xi\phi_1 m e^{-\xi\phi_1 m} - e^{-\xi\phi_1 n} + \xi\phi_1 n e^{-\xi\phi_1 n} \right] \quad (3.84)$$

$$V_C = \frac{q}{\xi\phi_1} \left[\frac{\xi}{2} \phi_1 m e^{-\xi\phi_1 m} - \frac{\xi}{2} \phi_1 n e^{-\xi\phi_1 n} - e^{-\xi\phi_1 m} + e^{-\xi\phi_1 n} \right] \quad (3.85)$$

Case III ($\rho > 1/4$)

1. When point C is within the loaded region

$$w_C = \int \delta dw, \quad dw = \int -\frac{q dx}{2D} y(x)$$

$$\text{Where } y(x) = \frac{q}{4D\xi^3(\phi_1^2 - \phi_2^2)} \left(\frac{e^{-\xi\phi_2 x} \cosh \xi\phi_2 x}{\phi_1} + \frac{e^{-\xi\phi_1 x} \sinh \xi\phi_2 x}{\phi_2} \right) \quad (3.86)$$

$$w_C = \frac{q}{4D\xi^4(\phi_1^2 - \phi_2^2)} \left(\begin{array}{l} -e^{-\xi\phi_1 m} \left(\cosh(\xi\phi_2 m) + \frac{\phi_2}{\phi_1} \sinh(\xi\phi_2 m) \right) \\ -e^{-\xi\phi_1 n} \left(\cosh(\xi\phi_2 n) + \frac{\phi_2}{\phi_1} \sinh(\xi\phi_2 n) \right) \\ -e^{-\xi\phi_2 m} \left(\frac{\phi_1}{\phi_2} \sinh(\xi\phi_2 m) + \cosh(\xi\phi_2 m) \right) \\ -e^{-\xi\phi_2 n} \left(\frac{\phi_1}{\phi_2} \sinh(\xi\phi_2 n) + \cosh(\xi\phi_2 n) \right) + 4 \end{array} \right) \quad (3.87)$$

$$\theta_C = \frac{q}{4D\xi^3(\phi_1^2 - \phi_2^2)} \left(\begin{array}{l} \frac{e^{-\xi\phi_2 m} \cosh(\xi\phi_2 m)}{\phi_1} - \frac{e^{-\xi\phi_1 n} \cosh(\xi\phi_2 n)}{\phi_1} \\ + \frac{e^{-\xi\phi_1 m} \sinh(\xi\phi_2 m)}{\phi_2} - \frac{e^{-\xi\phi_2 n} \sinh(\xi\phi_2 n)}{\phi_2} \end{array} \right) \quad (3.88)$$

$$M_C = \frac{q}{4\xi^2} \left(\frac{e^{-\xi\phi_2 m} \sinh(\xi\phi_2 m)}{\phi_2\phi_1} + \frac{e^{-\xi\phi_1 n} \sinh(\xi\phi_2 n)}{\phi_2\phi_1} \right) \quad (3.89)$$

$$V_C = \frac{q}{4\xi} \left(\begin{array}{l} \frac{-e^{-\xi\phi_2 m} \sinh(\xi\phi_2 m)}{\phi_2} + \frac{e^{-\xi\phi_1 m} \cosh(\xi\phi_2 m)}{\phi_1} + \\ \frac{e^{-\xi\phi_1 n} \sinh(\xi\phi_2 n)}{\phi_2} - \frac{e^{-\xi\phi_2 n} \cosh(\xi\phi_2 n)}{\phi_1} \end{array} \right) \quad (3.90)$$

2. When point C is to the left of the loaded Region

$$w_C = \frac{q}{4D\xi^4(\phi_1^2 - \phi_2^2)} \left(\begin{array}{l} -e^{-\xi\phi_1 n} \left(\cosh(\xi\phi_2 n) + \frac{\phi_2}{\phi_1} \sinh(\xi\phi_2 n) \right) \\ + e^{-\xi\phi_2 m} \left(\cosh(\xi\phi_2 m) + \frac{\phi_2}{\phi_1} \sinh(\xi\phi_2 m) \right) \\ -e^{-\xi\phi_2 n} \left(\frac{\phi_1}{\phi_2} \sinh(\xi\phi_2 n) + \cosh(\xi\phi_2 n) \right) \\ + e^{-\xi\phi_1 m} \left(\frac{\phi_1}{\phi_2} \sinh(\xi\phi_2 m) + \cosh(\xi\phi_2 m) \right) \end{array} \right) \quad (3.91)$$

$$\theta_C = \frac{q}{4D\xi^3(\phi_1^2 - \phi_2^2)} \left(\frac{e^{-\xi\phi_2 m} \cosh(\xi\phi_2 m)}{\phi_1} - \frac{e^{-\xi\phi_2 n} \cosh(\xi\phi_2 n)}{\phi_1} + \frac{e^{-\xi\phi_2 m} \sinh(\xi\phi_2 m)}{\phi_2} - \frac{e^{-\xi\phi_2 n} \sinh(\xi\phi_2 n)}{\phi_2} \right) \quad (3.92)$$

$$M_C = \frac{q}{4\xi^2} \left[\frac{e^{-\xi\phi_2 n} \sinh(\xi\phi_2 n)}{\phi_2\phi_1} - \frac{e^{-\xi\phi_2 m} \sinh(\xi\phi_2 m)}{\phi_2\phi_1} \right] \quad (3.93)$$

$$V_C = \frac{q}{4\xi} \left(\frac{-e^{-\xi\phi_2 m} \sinh(\xi\phi_2 m)}{\phi_2} + \frac{e^{-\xi\phi_2 m} \cosh(\xi\phi_2 m)}{\phi_1} + \frac{e^{-\xi\phi_2 n} \sinh(\xi\phi_2 n)}{\phi_2} - \frac{e^{-\xi\phi_2 n} \cosh(\xi\phi_2 n)}{\phi_1} \right) \quad (3.94)$$

3. When point C is to the right of the loaded Region

$$w_C = \frac{q}{4D\xi^4(\phi_1^2 - \phi_2^2)} \left(\begin{aligned} & -e^{-\xi\phi_2 m} \left(\cosh(\xi\phi_2 m) + \frac{\phi_2}{\phi_1} \sinh(\xi\phi_2 m) \right) \\ & + e^{-\xi\phi_2 n} \left(\cosh(\xi\phi_2 n) + \frac{\phi_2}{\phi_1} \sinh(\xi\phi_2 n) \right) \\ & - e^{-\xi\phi_2 m} \left(-\frac{\phi_1}{\phi_2} \sinh(\xi\phi_2 m) + \cosh(\xi\phi_2 m) \right) \\ & + e^{-\xi\phi_2 n} \left(-\frac{\phi_1}{\phi_2} \sinh(\xi\phi_2 n) + \cosh(\xi\phi_2 n) \right) \end{aligned} \right) \quad (3.95)$$

$$\theta_C = \frac{q}{4D\xi^3(\phi_1^2 - \phi_2^2)} \left(\frac{e^{-\xi\phi_2 m} \cosh(\xi\phi_2 m)}{\phi_1} - \frac{e^{-\xi\phi_2 m} \cosh(\xi\phi_2 m)}{\phi_1} + \frac{e^{-\xi\phi_2 n} \sinh(\xi\phi_2 n)}{\phi_2} - \frac{e^{-\xi\phi_2 n} \sinh(\xi\phi_2 n)}{\phi_2} \right) \quad (3.96)$$

$$M_C = \frac{q}{4\xi^2} \left[\frac{e^{-\xi\phi_2 m} \sinh(\xi\phi_2 m)}{\phi_2\phi_1} - \frac{e^{-\xi\phi_2 n} \sinh(\xi\phi_2 n)}{\phi_2\phi_1} \right] \quad (3.97)$$

$$V_C = \frac{q}{4\xi} \left(\frac{-e^{-\xi\phi_2 m} \sinh(\xi\phi_2 m)}{\phi_2} + \frac{e^{-\xi\phi_2 m} \cosh(\xi\phi_2 m)}{\phi_1} + \frac{e^{-\xi\phi_2 n} \sinh(\xi\phi_2 n)}{\phi_2} - \frac{e^{-\xi\phi_2 n} \cosh(\xi\phi_2 n)}{\phi_1} \right) \quad (3.98)$$

3.5.3 Short strip plate

3.5.3.1. A short strip plate subjected to vertical concentrated load

A short strip plate on a Pasternak foundation is shown in Figure below and two regions are considered for analysis purpose, the loaded region and the unloaded region.

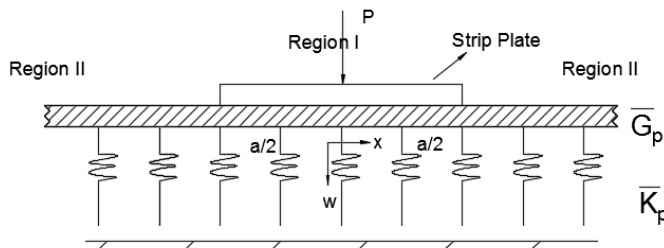


Fig 3.11 A short strip plate resting on a Kerr Equivalent Pasternak foundation and subjected to a vertical concentrated load

The deflection of loaded region I is governed by fourth order differential equation

$$\frac{d^4 w}{dx^4} - \frac{64\beta^2 \rho}{a^2} \frac{d^2 w}{dx^2} + 64 \frac{\beta^4}{a^4} w = 0$$

$$\text{Where } \beta = \sqrt[4]{\frac{k_p a^4}{64D}} \quad \gamma = \sqrt{\frac{k_p a^2}{G_p}} \quad \rho = \left(\frac{\beta}{\gamma}\right)^2$$

The deflection of unloaded region (II) is governed by second order differential equation given by

$$\bar{k}_p w_f(x) - \bar{G}_p \frac{d^2 w}{dx^2} = 0 \quad (3.99)$$

$$w_f(x) = C e^m x$$

The general solution for unloaded area will be

$$w_f(x) = C_5 e^{\sqrt{\frac{ks}{G_p}} x} + C_6 e^{-\sqrt{\frac{ks}{G_p}} x} \quad (3.100)$$

The following boundary conditions are drawn from the loading arrangement shown in Figure (3.10).

At x=0

1. $w'(x)=0$

2. $w'''(x) = \frac{p}{2D}$

At x=a/2

3. $w(a/2) = w_f(a/2)$

4. $w'(a/2) = w_f'(a/2)$

5. $w''(a/2)=0$

6. $w'''(a/2)=0$

At x=∞

7. $w_f(x)=0$

This boundary conditions are employed to general solution given in strip plate resting on Kerr-Equivalent Pasternak subgrade model

Case I $\rho < 1/4$

The general solution is given by

$$w(x) = e^{\xi\phi_2 x} (C_1 \cos \xi\phi_2 x + C_2 \sin \xi\phi_2 x) + e^{-\xi\phi_2 x} (C_3 \cos \xi\phi_2 x + C_4 \sin \xi\phi_2 x)$$

Case II ($\rho = 1/4$)

$$w(x) = e^{\xi\phi_2 x} (C_1 + C_3 x) + e^{-\xi\phi_2 x} (C_2 + C_4 x)$$

Case III ($\rho > 1/4$)

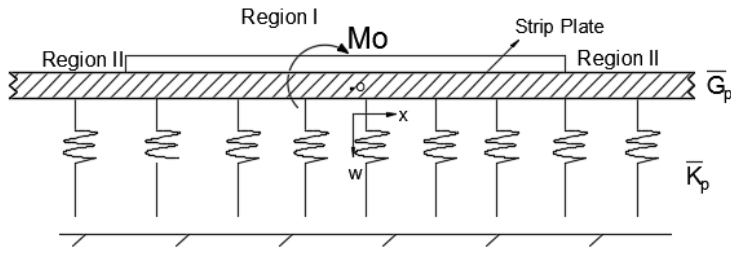
$$w(x) = (C_1 e^{\xi\phi_2 x} \cosh \xi\phi_2 x + C_2 e^{\xi\phi_2 x} \sinh \xi\phi_2 x + C_3 e^{-\xi\phi_2 x} \cosh \xi\phi_2 x + C_4 e^{-\xi\phi_2 x} \sinh \xi\phi_2 x)$$

By using the above boundary conditions one can determined the constants and obtain the expression for deflection, slope, moment and shear force. For brevity reasons, the detailed of the solution are provided in Annex E.

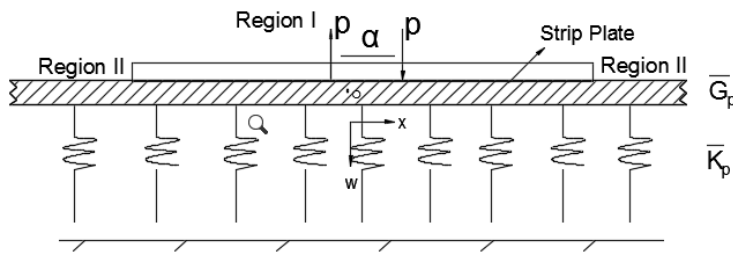
3.5.3.2. A strip plate on Pasternak foundation when subjected to concentrated moment

A short length strip plate resting on Pasternak foundation and subjected to a concentrated moment is dealt in this part of the work. A similar approach to Hetnyi (1946) is used to solve this condition, the concentrated moment in Figure 3.12a can be regarded as the limiting cases of the loading shown in Figure 3.12 b.

$$\lim_{\alpha \rightarrow x} p\alpha = M_0$$



(a)



(b)

Fig 3.12: (a) A short strip plate resting on Kerr Equivalent Pasternak foundation and subjected to a concentrated moment (b) A short strip plate subjected to oppositely directed forces

Only the final expression of deflection is provided in here whereas the expression for internal actions of strip plate is provided in Annex F.

Case I ($\rho < 1/4$)

$$w(x) = \frac{-p}{2D} y(x)$$

Where

$$y(x) = K_1 e^{\xi \phi_2 x} \cos \xi \phi_2 x + K_2 e^{\xi \phi_2 x} \sin \xi \phi_2 x + K_3 e^{-\xi \phi_2 x} \cos \xi \phi_2 x + K_4 e^{-\xi \phi_2 x} \sin \xi \phi_2 x$$

$$k_i = \frac{-2D}{p} C_i \quad \text{and} \quad C_i = L^{-1}$$

For loading showing in the Figure (3.12b):

$$w(x) = \frac{pa}{2D} \frac{y(x+\alpha) - y(\alpha)}{\alpha}$$

$$w(x) = \frac{M_o}{2D} \begin{pmatrix} k_1 \xi \phi_1 e^{\xi \phi_1 x} \cos(\xi \phi_2 x) - k_1 \xi \phi_2 e^{\xi \phi_1 x} \sin(\xi \phi_2 x) + \\ k_2 \xi \phi_1 e^{\xi \phi_1 x} \sin(\xi \phi_2 x) + k_2 \xi \phi_2 e^{\xi \phi_1 x} \cos(\xi \phi_2 x) \\ -k_3 \xi \phi_1 e^{-\xi \phi_1 x} \cos(\xi \phi_2 x) - k_3 \xi \phi_2 e^{-\xi \phi_1 x} \sin(\xi \phi_2 x) \\ -k_4 \xi \phi_1 e^{-\xi \phi_1 x} \sin(\xi \phi_2 x) + k_4 \xi \phi_2 e^{-\xi \phi_1 x} \cos(\xi \phi_2 x) \end{pmatrix} \quad (3.101)$$

Case II ($\rho=1/4$)

$$w(x) = \frac{M_o}{2D} \begin{pmatrix} k_1 \xi \phi_1 e^{\xi \phi_1 x} - k_2 \xi \phi_1 e^{-\xi \phi_1 x} + k_3 e^{\xi \phi_1 x} + \\ x k_3 \xi \phi_1 e^{\xi \phi_1 x} + k_4 e^{-\xi \phi_1 x} - x k_4 \xi \phi_1 e^{-\xi \phi_1 x} \end{pmatrix} \quad (3.102)$$

Case III ($\rho > 1/4$)

$$w(x) = \frac{M_o}{2D} \begin{pmatrix} k_1 \xi \phi_1 e^{\xi \phi_1 x} \cosh(\xi \phi_2 x) + k_1 \xi \phi_2 e^{\xi \phi_1 x} \sinh(\xi \phi_2 x) + \\ k_2 \xi \phi_1 e^{\xi \phi_1 x} \sinh(\xi \phi_2 x) + k_2 \xi \phi_2 e^{\xi \phi_1 x} \cosh(\xi \phi_2 x) \\ -k_3 \xi \phi_1 e^{-\xi \phi_1 x} \cosh(\xi \phi_2 x) + k_3 \xi \phi_2 e^{-\xi \phi_1 x} \sinh(\xi \phi_2 x) \\ -k_4 \xi \phi_1 e^{-\xi \phi_1 x} \sinh(\xi \phi_2 x) + k_4 \xi \phi_2 e^{-\xi \phi_1 x} \cosh(\xi \phi_2 x) \end{pmatrix} \quad (3.103)$$

3.5.3.3. A short length strip plate subjected to Uniformly Distributed load

A short length strip plate resting on subgrade and subjected to a uniformly load, as shown in Figure 3.13 is presented in this section

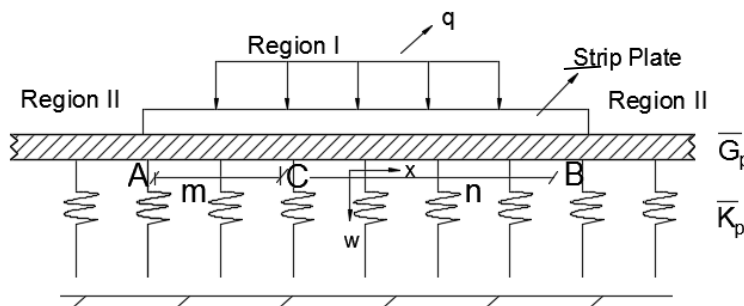


Fig 3.13. A short strip plate resting on Kerr Equivalent Pasternak foundation and subjected to a uniformly distributed load.

Only the final expression of deflection is provided in here whereas the expression for internal actions of strip plate is provided in Annex G.

Case I ($\rho < 1/4$)

1. When point C is within the loaded region

$$w_C = \int \delta dw$$

$$= \int -\frac{q dx}{2D} y(x)$$

$$y(x) = k_1 e^{\xi\phi_1 x} \cos \xi\phi_2 x + k_2 e^{\xi\phi_1 x} \sin \xi\phi_2 x + k_3 e^{-\xi\phi_1 x} \cos \xi\phi_2 x + k_4 e^{-\xi\phi_1 x} \sin \xi\phi_2 x$$

$$k_i = -\frac{2D}{q} C_i$$

The integration constants C_i are then obtained from the inverse matrix relation

$$C = L^{-1}M$$

$$w_C = \frac{-q}{2D} \left[\int_0^m y(x) dx + \int_0^n y(x) dx \right]$$

$$w_C = \frac{-q}{2D\xi(\phi_1^2 + \phi_2^2)} \left(\begin{aligned} & \left[k_1 (e^{\xi\phi_1 m} (\phi_1 \cos \xi\phi_2 m + \phi_2 \sin \xi\phi_2 m) + e^{\xi\phi_1 n} (\phi_1 \cos \xi\phi_2 n + \phi_2 \sin \xi\phi_2 n) - 2\phi_1) \right] + \\ & \left[k_2 (e^{\xi\phi_1 m} (\phi_1 \sin \xi\phi_2 m - \phi_2 \cos \xi\phi_2 m) + e^{\xi\phi_1 n} (\phi_1 \sin \xi\phi_2 n - \phi_2 \cos \xi\phi_2 n) + 2\phi_2) \right] + \\ & \left[k_3 (e^{-\xi\phi_1 m} (-\phi_1 \cos \xi\phi_2 m + \phi_2 \sin \xi\phi_2 m) + e^{-\xi\phi_1 n} (-\phi_1 \cos \xi\phi_2 n + \phi_2 \sin \xi\phi_2 n) + 2\phi_1) \right] + \\ & \left[k_4 (e^{-\xi\phi_1 m} (-\phi_1 \sin \xi\phi_2 m - \phi_2 \cos \xi\phi_2 m) + e^{-\xi\phi_1 n} (-\phi_1 \sin \xi\phi_2 n - \phi_2 \cos \xi\phi_2 n) + 2\phi_2) \right] \end{aligned} \right) \quad (3.104)$$

2. When point C is to the left of loaded region

$$w_C = \frac{-q}{2D} \int_0^n y(x) dx - \int_0^m y(x) dx$$

$$w_C = \frac{-q}{2D\xi(\phi_1^2 + \phi_2^2)} \left(\begin{aligned} & \left[k_1 (e^{\xi\phi_1 n} (\phi_1 \cos \xi\phi_2 n + \phi_2 \sin \xi\phi_2 n) - (e^{\xi\phi_1 m} (\phi_1 \cos \xi\phi_2 m + \phi_2 \sin \xi\phi_2 m))) \right] + \\ & \left[k_2 (e^{\xi\phi_1 n} (\phi_1 \sin \xi\phi_2 n - \phi_2 \cos \xi\phi_2 n) - (e^{\xi\phi_1 m} (\phi_1 \sin \xi\phi_2 m - \phi_2 \cos \xi\phi_2 m))) \right] + \\ & \left[k_3 (e^{-\xi\phi_1 n} (-\phi_1 \cos \xi\phi_2 n + \phi_2 \sin \xi\phi_2 n) - (e^{-\xi\phi_1 m} (-\phi_1 \cos \xi\phi_2 m + \phi_2 \sin \xi\phi_2 m))) \right] + \\ & \left[k_4 (e^{-\xi\phi_1 n} (-\phi_1 \sin \xi\phi_2 n - \phi_2 \cos \xi\phi_2 n) - (e^{-\xi\phi_1 m} (-\phi_1 \sin \xi\phi_2 m - \phi_2 \cos \xi\phi_2 m))) \right] \end{aligned} \right) \quad (3.105)$$

3. When point C is to the right of loaded region

$$w_C = \frac{-q}{2D} \int_0^m y(x) dx - \int_0^n y(x) dx$$

$$w_C = \frac{-q}{2D\xi(\phi_1^2 + \phi_2^2)} \left(\begin{aligned} & \left[k_1 (e^{\xi\phi_1 m} (\phi_1 \cos \xi\phi_2 m + \phi_2 \sin \xi\phi_2 m) - (e^{\xi\phi_1 n} (\phi_1 \cos \xi\phi_2 n + \phi_2 \sin \xi\phi_2 n)) \right] + \\ & \left[k_2 (e^{\xi\phi_1 m} (\phi_1 \sin \xi\phi_2 m - \phi_2 \cos \xi\phi_2 m) - (e^{\xi\phi_1 n} (\phi_1 \sin \xi\phi_2 n - \phi_2 \cos \xi\phi_2 n)) \right] + \\ & \left[k_3 (e^{-\xi\phi_1 m} (-\phi_1 \cos \xi\phi_2 m + \phi_2 \sin \xi\phi_2 m) - (e^{-\xi\phi_1 n} (-\phi_1 \cos \xi\phi_2 n + \phi_2 \sin \xi\phi_2 n)) \right] + \\ & \left[k_4 (e^{-\xi\phi_1 m} (-\phi_1 \sin \xi\phi_2 m - \phi_2 \cos \xi\phi_2 m) - e^{-\xi\phi_1 n} (-\phi_1 \sin \xi\phi_2 n - \phi_2 \cos \xi\phi_2 n)) \right] \end{aligned} \right) \quad (3.106)$$

Case II ($\rho = 1/4$)

1. When point C is within the loaded region

$$w_C = \frac{-q}{2D\xi\phi_1} \left(\begin{aligned} & k_1 (e^{\xi\phi_1 m} + e^{\xi\phi_1 n} - 2) - k_2 (e^{-\xi\phi_1 m} + e^{-\xi\phi_1 n} - 2) + \\ & k_3 \left(me^{\xi\phi_1 m} + ne^{\xi\phi_1 n} - \frac{e^{\xi\phi_1 m}}{\xi\phi_1} - \frac{e^{\xi\phi_1 n}}{\xi\phi_1} + \frac{2}{\xi\phi_1} \right) + \\ & k_4 \left(-me^{-\xi\phi_1 m} - ne^{-\xi\phi_1 n} + \frac{e^{\xi\phi_1 m}}{\xi\phi_1} + \frac{e^{\xi\phi_1 n}}{\xi\phi_1} - \frac{2}{\xi\phi_1} \right) \end{aligned} \right) \quad (3.107)$$

2. When point C is to the left of loaded region

$$w_C = \frac{-q}{2D\xi\phi_1} \left(\begin{aligned} & k_1 (e^{\xi\phi_1 m} + e^{\xi\phi_1 n}) - k_2 (e^{-\xi\phi_1 n} - e^{-\xi\phi_1 m}) + \\ & k_3 \left(Be^{\xi\phi_1 n} - Ae^{\xi\phi_1 m} - \frac{e^{\xi\phi_1 n}}{\xi\phi_1} + \frac{e^{\xi\phi_1 m}}{\xi\phi_1} \right) + \\ & k_4 \left(-ne^{-\xi\phi_1 n} + me^{-\xi\phi_1 m} + \frac{e^{\xi\phi_1 n}}{\xi\phi_1} - \frac{e^{\xi\phi_1 m}}{\xi\phi_1} \right) \end{aligned} \right) \quad (3.108)$$

3. When point C is to the right of loaded region

$$w_C = \frac{-q}{2D\xi\phi_1} \left(\begin{array}{l} k_1 (e^{\xi\phi_1 m} - e^{\xi\phi_1 n}) - k_2 (e^{-\xi\phi_1 m} - e^{-\xi\phi_1 n}) + \\ k_3 \left(me^{\xi\phi_1 m} - ne^{\xi\phi_1 n} - \frac{e^{\xi\phi_1 m}}{\xi\phi_1} + \frac{e^{\xi\phi_1 n}}{\xi\phi_1} \right) + \\ k_4 \left(-me^{-\xi\phi_1 m} + ne^{-\xi\phi_1 n} + \frac{e^{\xi\phi_1 m}}{\xi\phi_1} - \frac{e^{\xi\phi_1 n}}{\xi\phi_1} \right) \end{array} \right) \quad (3.109)$$

Case III $\rho > \frac{1}{4}$

1. When point C is within the loaded region

$$w_C = \frac{-q}{2D\xi(\phi_1^2 + \phi_2^2)} \left(\begin{array}{l} \left[k_1 (e^{\xi\phi_1 m} (\phi_1 \cosh \xi\phi_2 m - \phi_2 \sinh \xi\phi_2 m) + e^{\xi\phi_1 n} (\phi_1 \cosh \xi\phi_2 n - \phi_2 \sinh \xi\phi_2 n) - 2\phi_1 \right] + \\ \left[k_2 (e^{\xi\phi_1 m} (\phi_1 \sinh \xi\phi_2 m - \phi_2 \cosh \xi\phi_2 m) + e^{\xi\phi_1 n} (\phi_1 \sinh \xi\phi_2 n - \phi_2 \cosh \xi\phi_2 n) + 2\phi_2 \right] + \\ \left[-k_3 (e^{-\xi\phi_1 m} (\phi_1 \cosh \xi\phi_2 m + \phi_2 \sinh \xi\phi_2 m) + e^{-\xi\phi_1 n} (\phi_1 \cosh \xi\phi_2 n + \phi_2 \sinh \xi\phi_2 n) - 2\phi_1 \right] + \\ \left[k_4 (e^{-\xi\phi_1 m} (-\phi_1 \sinh \xi\phi_2 m - \phi_2 \cosh \xi\phi_2 m) + e^{-\xi\phi_1 n} (-\phi_1 \sinh \xi\phi_2 n - \phi_2 \cosh \xi\phi_2 n) - 2\phi_2 \right] \end{array} \right) \quad (3.110)$$

2. When point C is to the left of loaded region

$$w_C = \frac{-q}{2D\xi(\phi_1^2 + \phi_2^2)} \left(\begin{array}{l} \left[k_1 (e^{\xi\phi_1 n} (\phi_1 \cosh \xi\phi_2 n - \phi_2 \sinh \xi\phi_2 n) - (e^{\xi\phi_1 m} (\phi_1 \cosh \xi\phi_2 m - \phi_2 \sinh \xi\phi_2 m)) \right] + \\ \left[k_2 (e^{\xi\phi_1 n} (\phi_1 \sinh \xi\phi_2 n - \phi_2 \cosh \xi\phi_2 n) - (e^{\xi\phi_1 m} (\phi_1 \sinh \xi\phi_2 m - \phi_2 \cosh \xi\phi_2 m)) \right] + \\ \left[-k_3 (e^{-\xi\phi_1 n} (\phi_1 \cosh \xi\phi_2 n + \phi_2 \sinh \xi\phi_2 n) - (e^{-\xi\phi_1 m} (\phi_1 \cosh \xi\phi_2 m + \phi_2 \sinh \xi\phi_2 m)) \right] + \\ \left[k_4 (e^{-\xi\phi_1 n} (-\phi_1 \sinh \xi\phi_2 n - \phi_2 \cosh \xi\phi_2 n) - (e^{-\xi\phi_1 m} (-\phi_1 \sinh \xi\phi_2 m - \phi_2 \cosh \xi\phi_2 m)) \right] \end{array} \right) \quad (3.111)$$

3. When point C is to the right of loaded region

$$w_C = \frac{-q}{2D\xi(\phi_1^2 + \phi_2^2)} \left(\begin{array}{l} \left[k_1 (e^{\xi\phi_1 m} (\phi_1 \cosh \xi\phi_2 m - \phi_2 \sinh \xi\phi_2 m) - (e^{\xi\phi_1 n} (\phi_1 \cosh \xi\phi_2 n - \phi_2 \sinh \xi\phi_2 n)) \right] + \\ \left[k_2 (e^{\xi\phi_1 m} (\phi_1 \sinh \xi\phi_2 m - \phi_2 \cosh \xi\phi_2 m) - (e^{\xi\phi_1 n} (\phi_1 \sinh \xi\phi_2 n - \phi_2 \cosh \xi\phi_2 n)) \right] + \\ \left[-k_3 (e^{-\xi\phi_1 m} (\phi_1 \cosh \xi\phi_2 m + \phi_2 \sinh \xi\phi_2 m) - (e^{-\xi\phi_1 n} (\phi_1 \cosh \xi\phi_2 n + \phi_2 \sinh \xi\phi_2 n)) \right] + \\ \left[-k_4 (e^{-\xi\phi_1 m} (-\phi_1 \sinh \xi\phi_2 m + \phi_2 \cosh \xi\phi_2 m) - (e^{-\xi\phi_1 n} (-\phi_1 \sinh \xi\phi_2 n + \phi_2 \cosh \xi\phi_2 n)) \right] \end{array} \right) \quad (3.112)$$

Chapter Four

4. Numerical Analysis and Discussion

In this chapter a numerical study is conducted using the different solutions obtained for long and short strip plates under several loading conditions. The solutions based on Winkler's model and Kerr-equivalent Pasternak models developed by Worku and the FE based plaxis 2D model are used. Then the outputs of these models are compared amongst each other. The results of the FE analyses are used to calibrate the analytical methods.

In the previous chapter, different solution cases were obtained for strip plates on a two parameter subgrade model depending on the value of the parameter ρ . It is important to identify which case represents the most likely scenario for real problems. This can be done by plotting ρ against another carefully selected parameter, incorporating all factors influencing ρ for selected values of thickness of stratum, H . As it can be seen from equation (3.45), ρ is a function of the modulus of elasticity of the strip plate (E_p), shear modulus of the soil (G), depth of strip plate (h_p), Poisson's ratio of the soil (ν) and the thickness of the stratum, H . The effect of these parameters on ρ can be seen by introducing the use of the following dimensionless stiffness factor or relative rigidity of the soil-plate system as suggested by (Rajapakse and Selvadurai ,1991). Plots of ρ against K_r are given in Fig. 4.1. & 4.2. for clayey and granular soils for a range of H/a values.

$$K_r = \frac{E_p * \left(\frac{h_p}{a}\right)^3}{G} \quad (4.1)$$

where K_r = relative rigidity of the soil-plate system

h_p = depth of strip plate

E_p =modulus of elasticity of strip plate

a =width of strip plate

G =shear modules of the soil

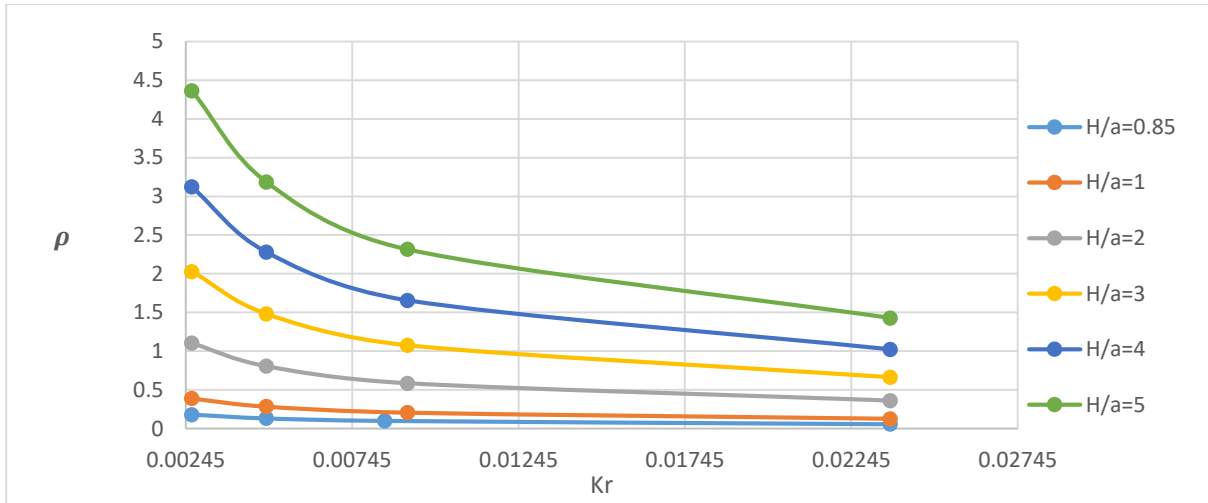


Fig 4.1 Effects of K_r and H on ρ for clay soil

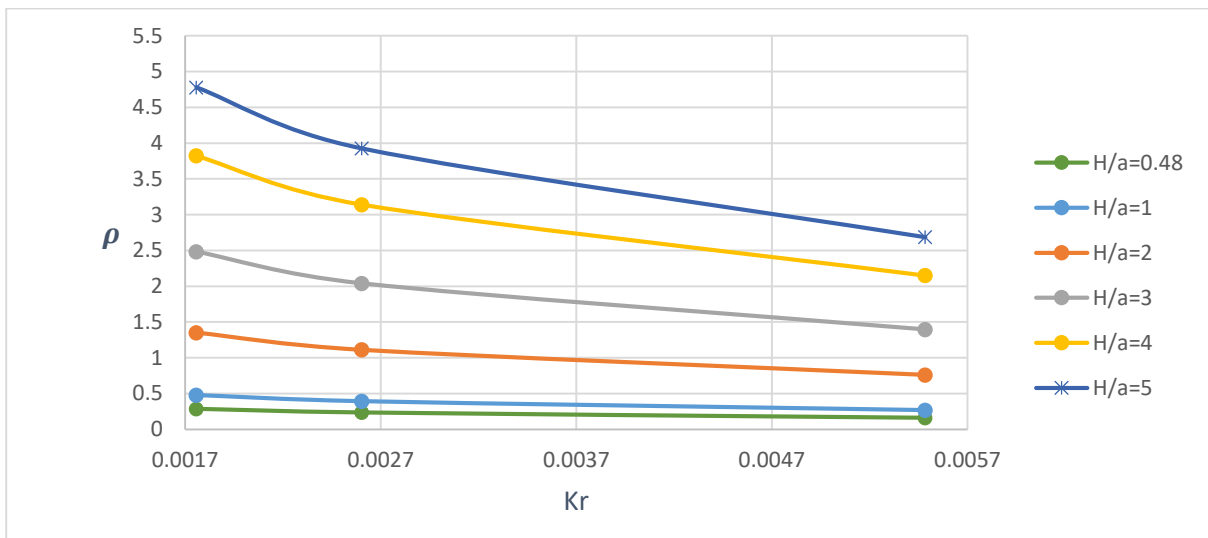


Fig 4.2 Effects of K_r and H on ρ for Sand and gravel soil

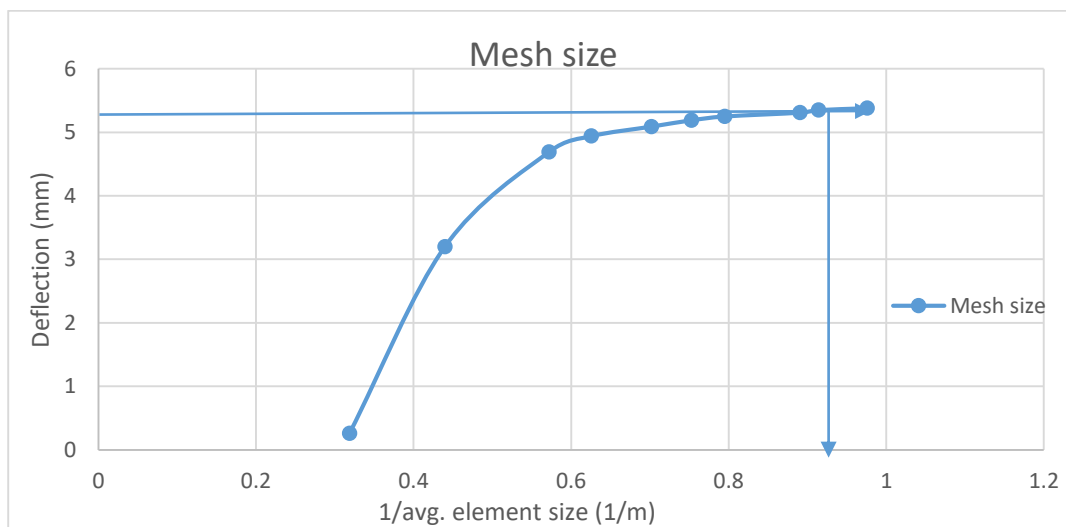
The above plot revealed that the value of ρ lies in case III ($\rho > \frac{1}{4}$) when the ratio of H/a is greater than 0.85 for clay soil and 0.48 for sand soil. On the other hand, when H/a is less than 0.85 and 0.48 for clay and sand soil respectively, case I ($\rho < \frac{1}{4}$) should be considered.

Due to the fact that the ratio H/a is less than 0.48 is rare, the most realistic scenario is seen in case III. Therefore, it can be concluded that for all practical purposes it is sufficient to consider only case three with $\rho > \frac{1}{4}$.

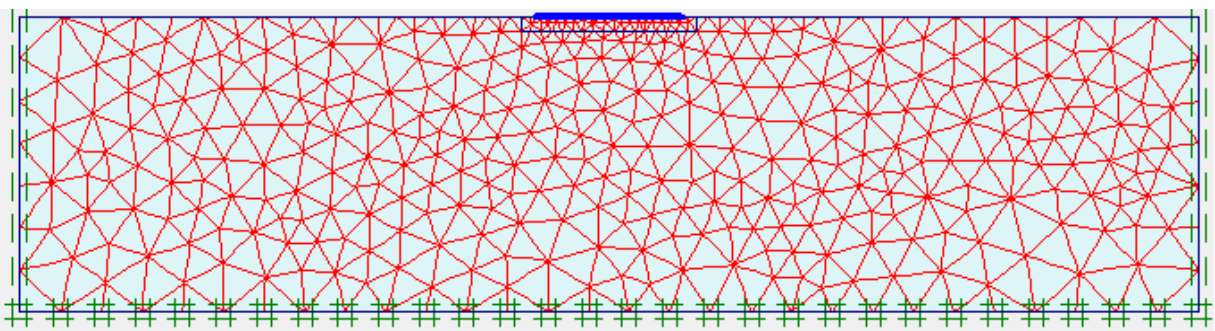
4.1 Calibration of the Winkler –type and Kerr equivalent Pasternak-type models

All continuum models including the generalized continuum models developed by Worku are sensitive to the thickness, H , of the stratum. When the thickness of the soil layer increases, it may give unrealistic or excessive deformation. Therefore, it is important to calibrate this model by using Finite Element based model. Whereby a calibration factor associated with H is introduced, Plaxis 2D software is used as a yardstick.

The first step in this process is to establish the optimum mesh size for the Finite element modeling (Plaxis 2D). This is due to the fact that all the output of the Plaxis are affected by the mesh size. As the mesh size is decreased, more accurate results will be obtained but it will take a longer time to get the results. Hence it is necessary to find the optimum mesh size to balance between the computation time and the accuracy. This is obtained by plotting the maximum deflection against the mesh size.



(a)



(b)

Fig 4.3 (a) Determination of optimum mesh size (b) plot of average mesh size

As it is shown in the Figure 4.3(a), the average mesh size is determined by drawing a horizontal tangent line where the plot becomes almost constant and finally it resembles like the diagram shown in Figure 4.3(b). After fixing the average mesh size, the next step is calibration of the Winkler-Type and Kerr- equivalent Pasternak-type model. This is attained after a number of analysis involving different types of soils, with various relative thickness of the stratum with respect to strip plate Width, $\chi=H/a$. Subsequently, the maximum deflection, W_{max} , will be plotted against χ for long and short length strip plate subjected to different loading cases.

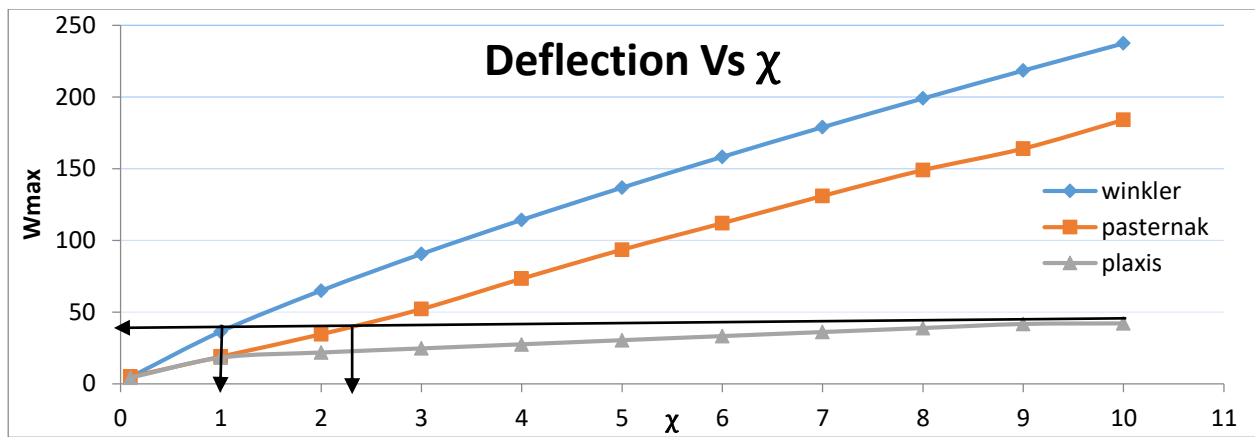


Fig 4.4 Determination of χ value for long strip plate subjected to selected basic loading types

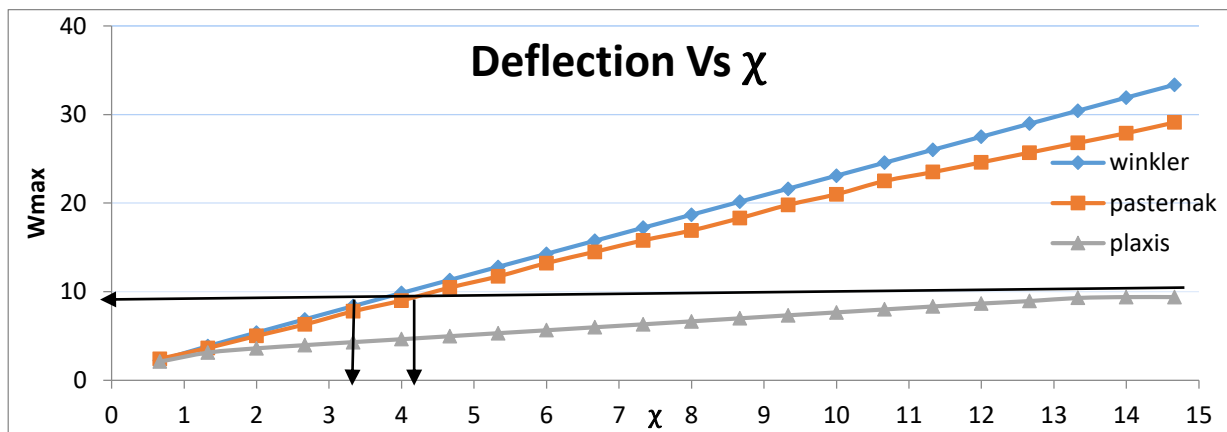


Fig 4.5 Determination of χ value for short strip plate subjected to selected basic loading types

In order to find the value of χ , first tangent line is drawn to the Plaxis plot, where it becomes almost constant. Then the intersection point between the tangent line and the corresponding graph of the models becomes the range value of χ . This step is illustrated in Fig 4.4 and 4.5. Finally, a calibration factor for each combination of representative type of soil and loading condition is obtained by establishing best fitting trend between the identified range value of χ against characteristic length(ξ). This is shown in the graphs found below.

A. Best fitting curves for long strip plate

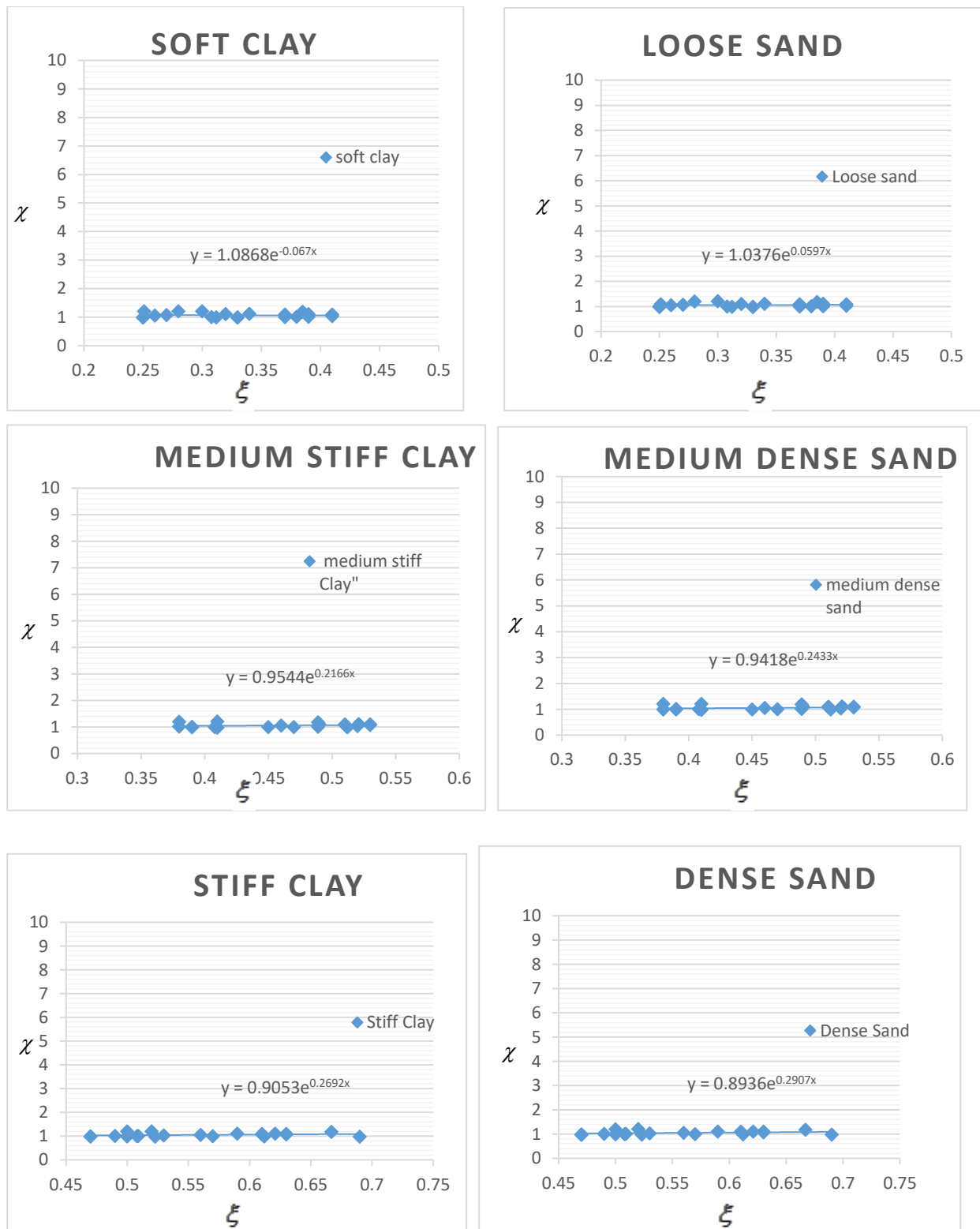


Fig 4.6 Calibration of Winkler type model for long strip plate subjected to vertical concentrated load for different soil types

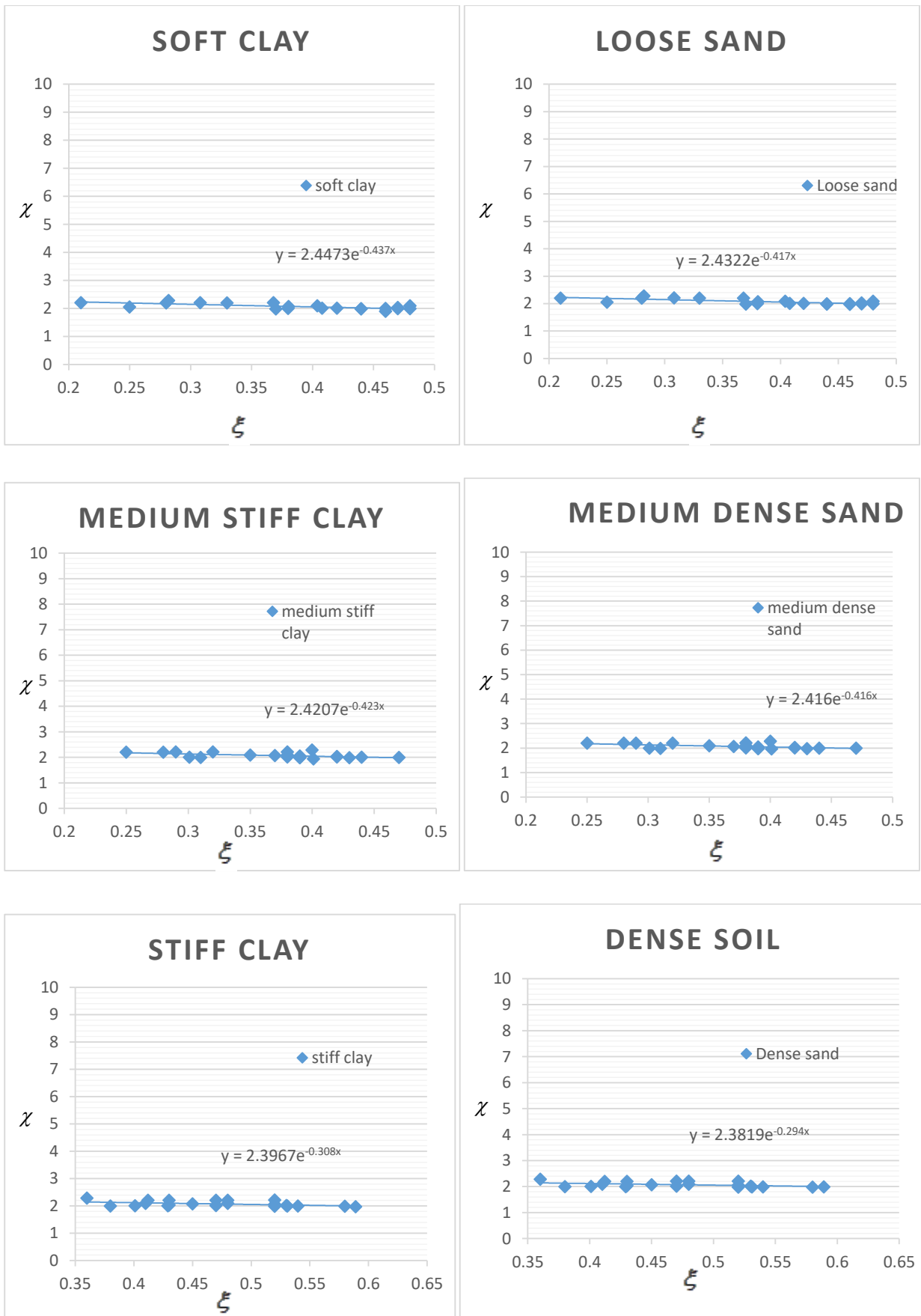


Fig 4.7 Calibration of Pasternak type model for long strip plate subjected to vertical concentrated load for different soil types

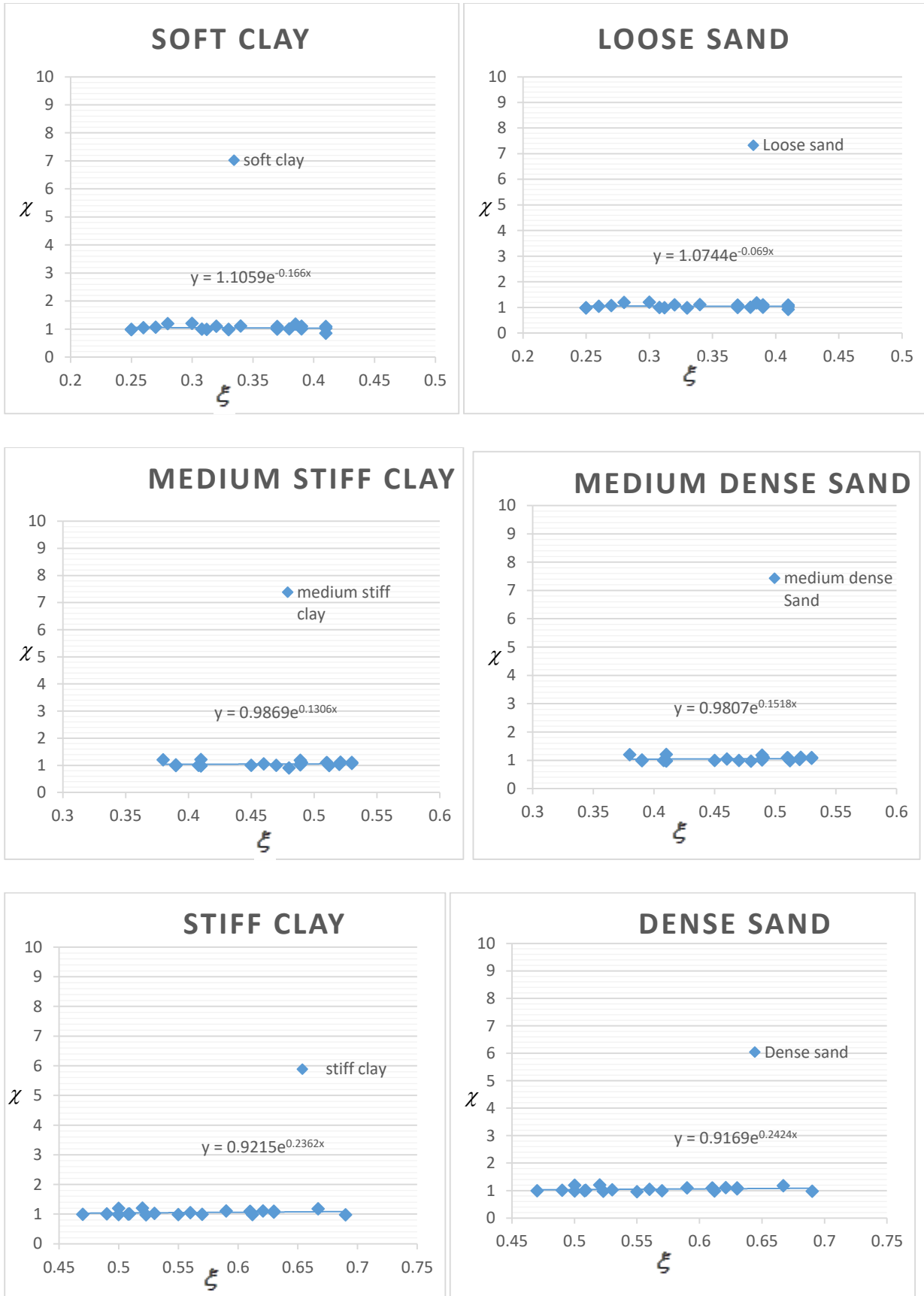


Fig 4.8 Calibration of Winkler type model for long strip plate subjected to uniformly distributed load for different soil types

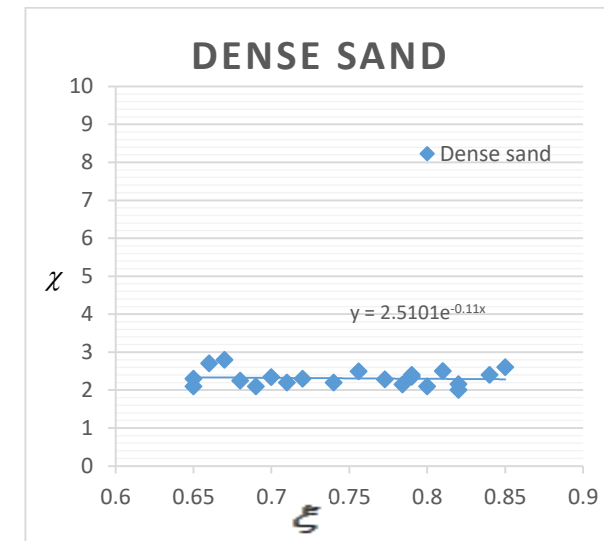
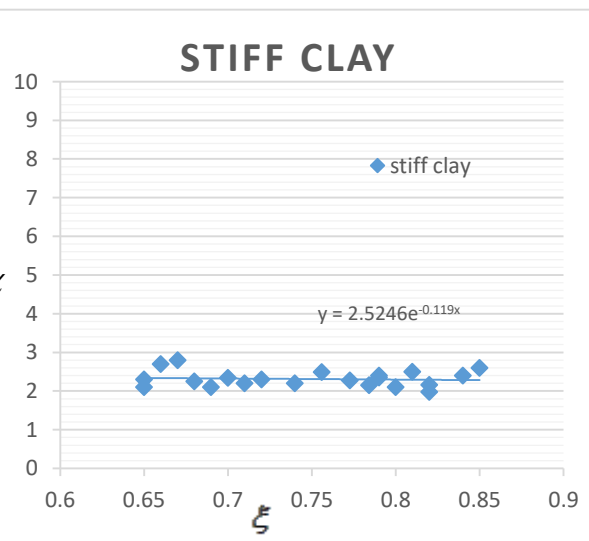
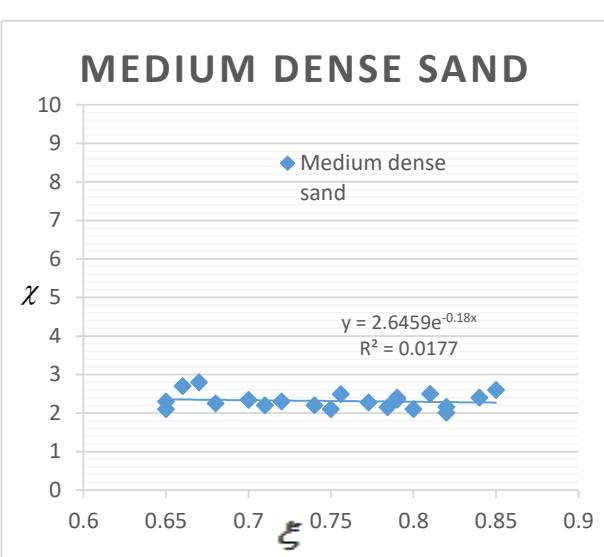
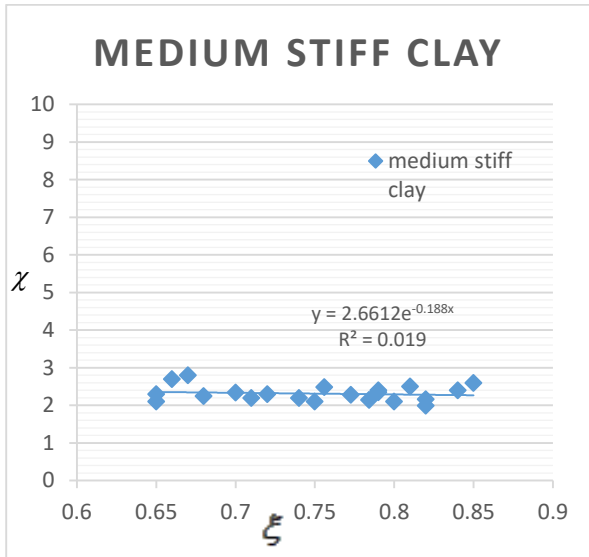
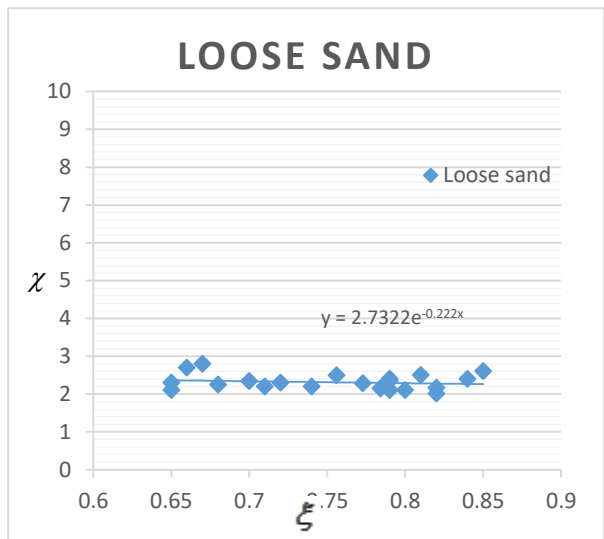
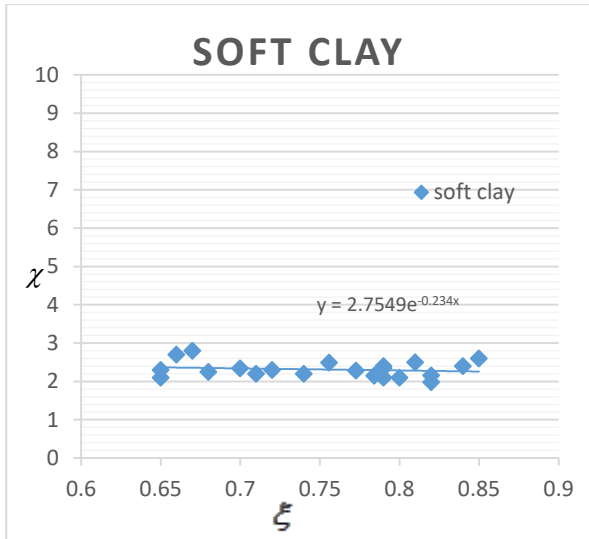


Fig 4.9 Calibration of Pasternak type model for long strip plate subjected to uniformly distributed load for different soil types

B. Best fitting curves for short strip plate

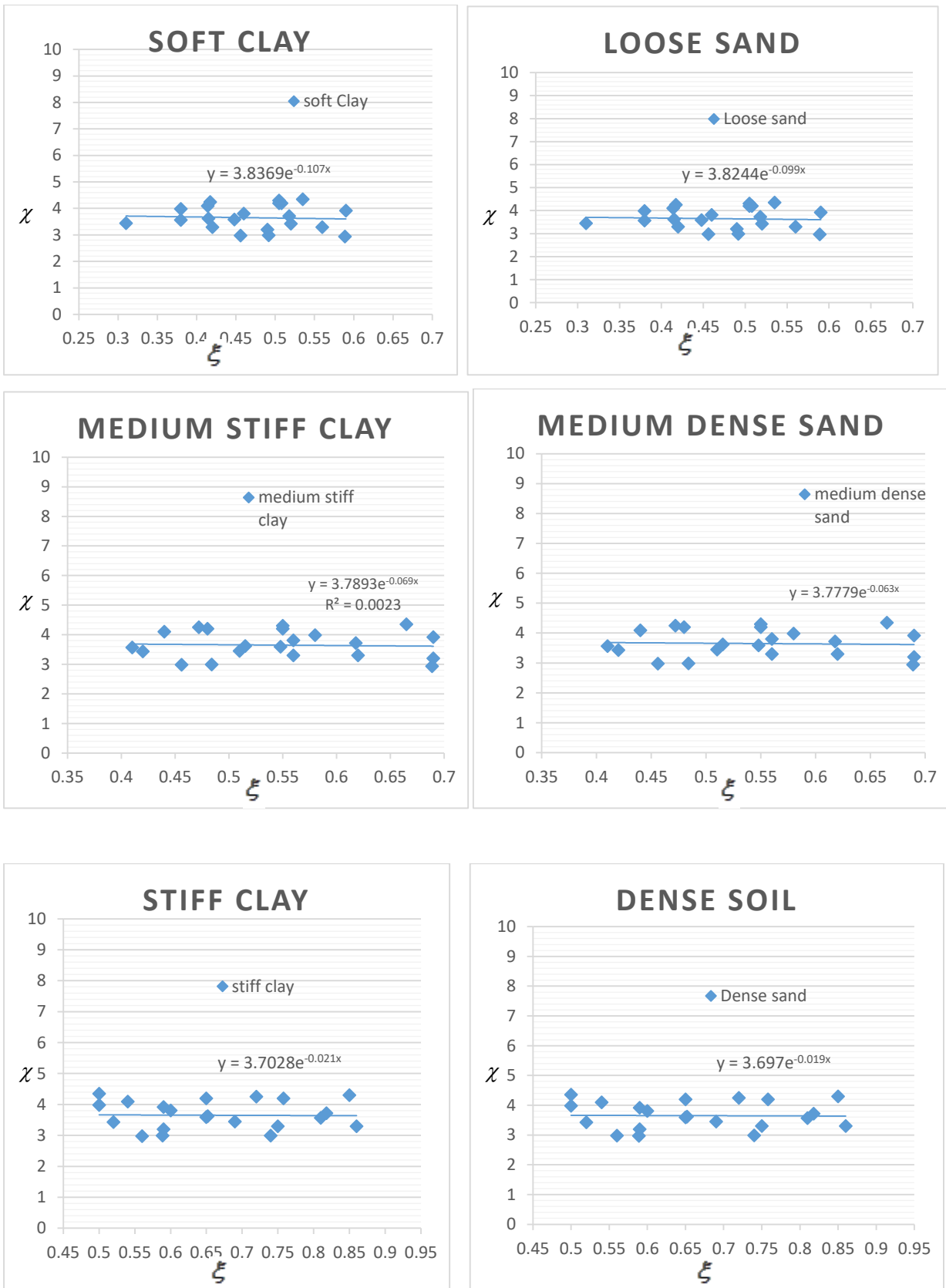


Fig 4.10 Calibration of Winkler type model for short strip plate subjected to vertical concentrated load for different soil types

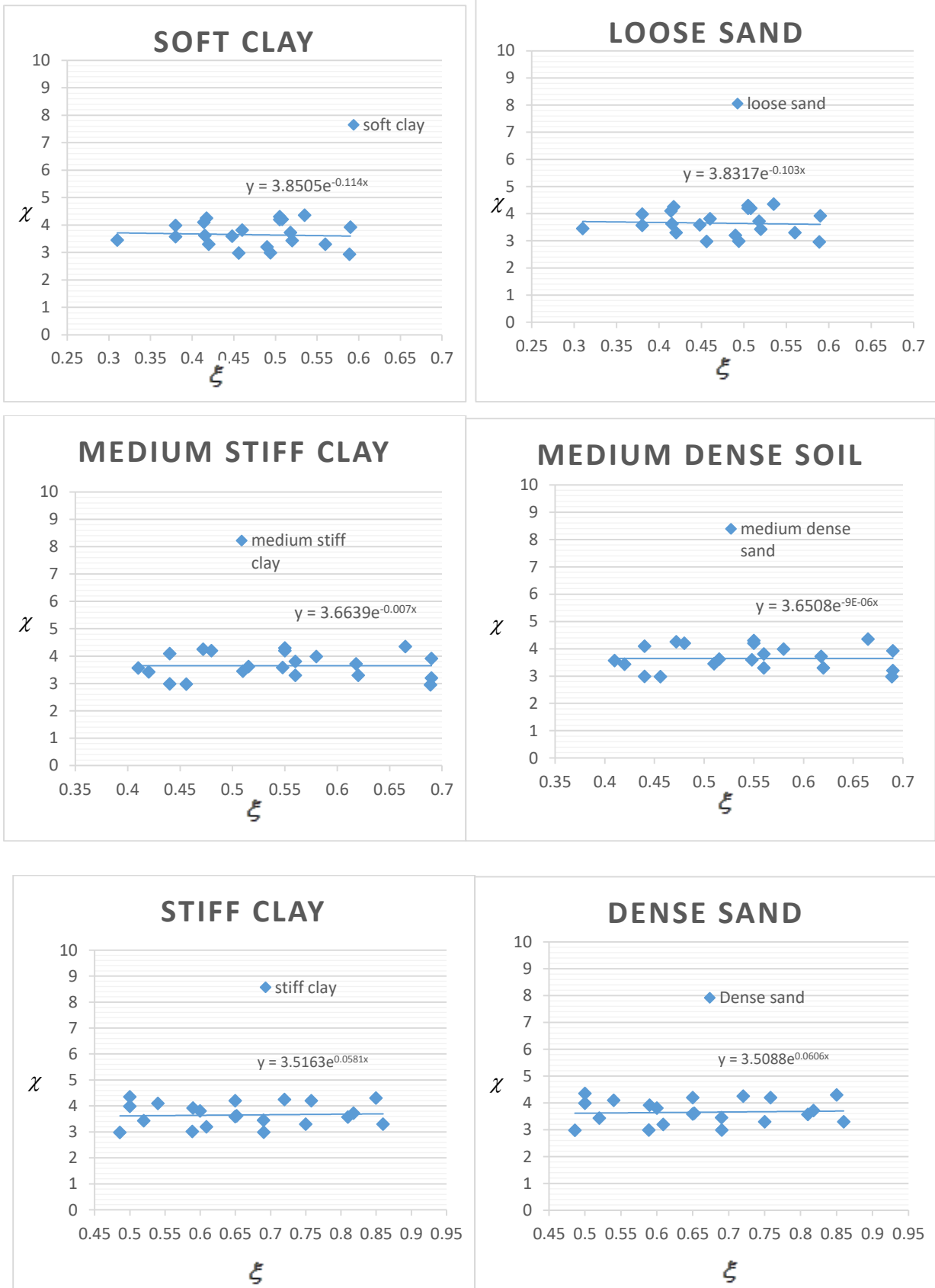


Fig 4.11 . Calibration of Pasternak type model for short strip plate subjected to vertical concentrated load for different soil types

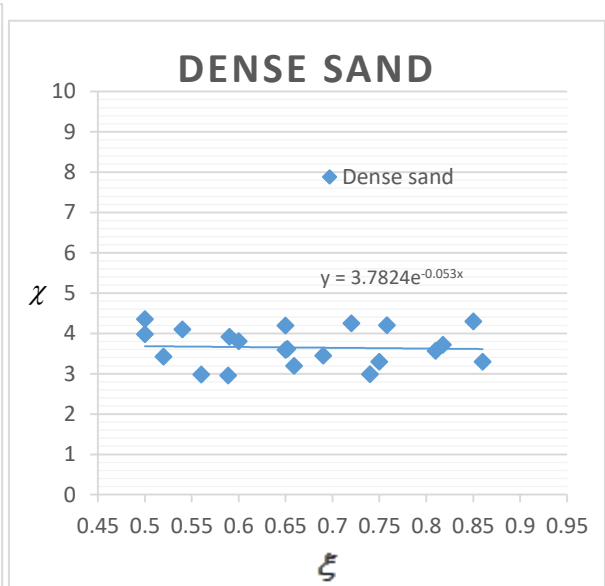
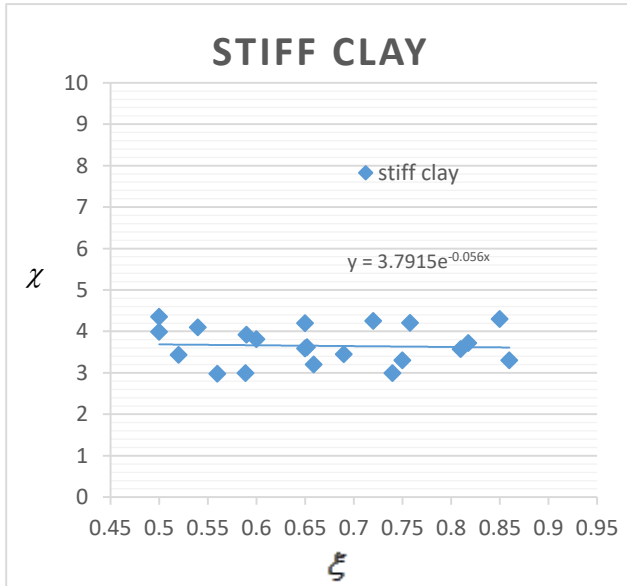
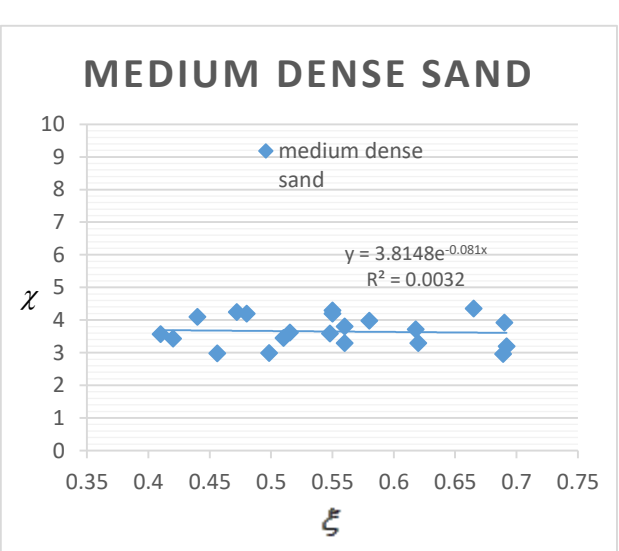
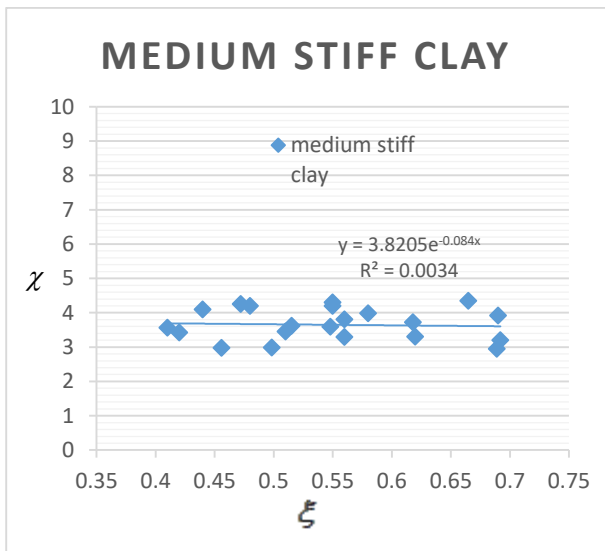
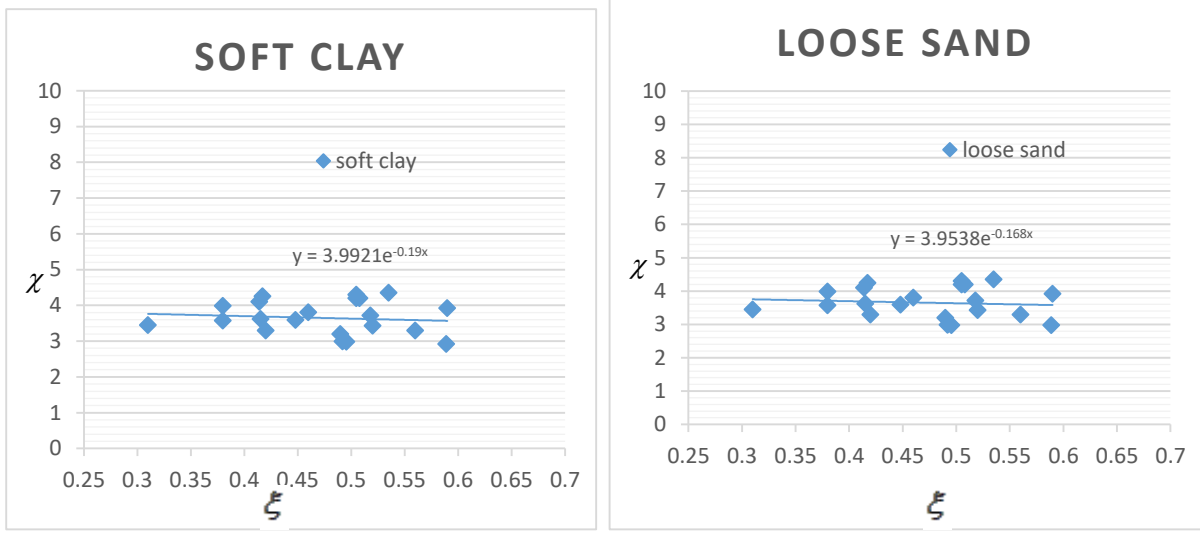


Fig 4.12. Calibration of Winkler type model for short strip plate subjected to uniformly distributed load for different soil types

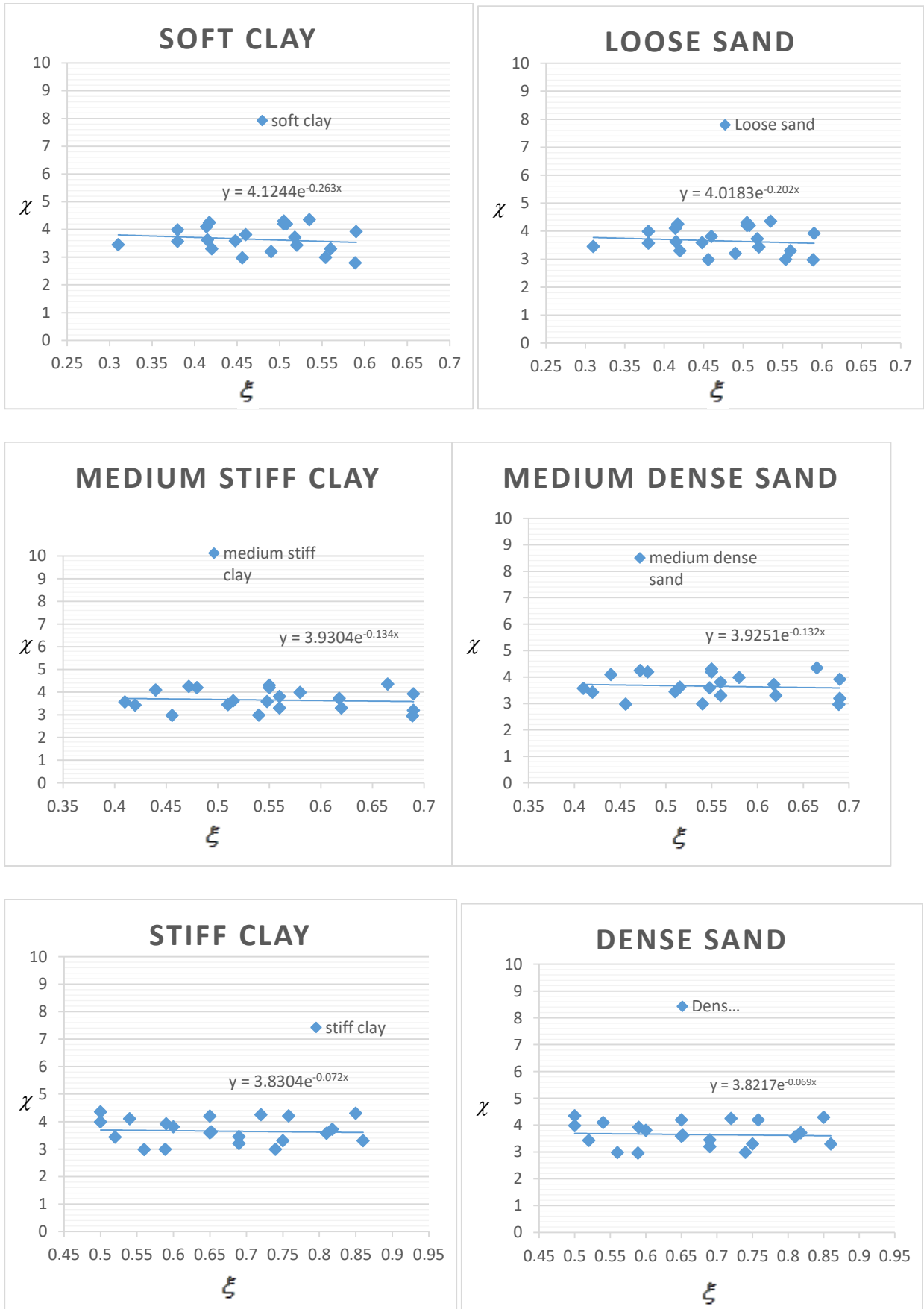


Fig 4.13 Calibration of Pasternak type model for short strip plate subjected to uniformly distributed load for different soil types

As shown in the plots, all discrete points in the plots lie on a narrow horizontal band. This is true irrespective of the different loading conditions. The recommended calibration factors for a strip plate resting on different soil types and subjected to selected loading cases as established from the plots are presented in the following tables.

Table 4.1 Summary of calibration factor for Winkler type and Kerr- equivalent Pasternak type model for the case of point load for long strip plate.

Recommended value of calibration factor for both models in the case of point load		
Soil types	χ_w	χ_p
Soft clay	1.09	2.45
Loose sand	1.04	2.43
Medium stiff clay	0.95	2.42
Medium dense sand	0.94	2.41
Stiff Clay	0.9	2.39
Dense sand	0.89	2.38

Table 4.2 Summary of calibration factor for Winkler type and Kerr- equivalent Pasternak type model for the case of uniformly distributed load for long strip plate.

Recommended value of calibration factor for both models in the case of uniformly distributed vertical load		
Soil types	χ_w	χ_p
Soft clay	1.1	2.75
Loose sand	1.07	2.73
Medium stiff clay	0.99	2.66
Medium dense sand	0.98	2.64
Stiff clay	0.92	2.52
Dense sand	0.91	2.51

Table 4.3 Summary of calibration factor for Winkler type and Kerr- equivalent Pasternak type model for the case of point load for short strip plates.

Recommended value of calibration factor for both models in the case of point load		
Soil types	χ_w	χ_p
Soft clay	3.83	3.85
Loose sand	3.82	3.83
Medium stiff clay	3.79	3.67
Medium dense sand	3.78	3.65
Stiff Clay	3.7	3.52
Dense sand	3.69	3.50

Table 4.4 Summary of calibration factor for Winkler type and Kerr- equivalent Pasternak type model for the case of uniformly distributed load for short strip plates.

Recommended value of calibration factor for both Models in the case of uniformly distributed vertical load		
Soil types	χ_w	χ_p
Soft clay	3.99	4.12
Loose sand	3.95	4.02
Medium stiff clay	3.82	3.93
Medium dense sand	3.81	3.92
Stiff clay	3.79	3.83
Dense sand	3.78	3.82

For an actual stratum $H > \chi a$, it is recommended to take the calibration factors given in (Tables 4.1 - 4.4) whereas when $H < \chi a$, the actual soil stratum depth may be taken as it is.

4.2. Calibrated Model parameters

As stated above, a calibrating parameter is introduced through the relation:

$$H = \chi a \tag{4.2}$$

Where H is the thickness of the stratum, a is the width of strip plate and χ is the calibrated factor for the subgrade model given in Tables 4.1-4.4. Introducing Eq.(4.2) in Eq.(2.43) &(2.56), one obtains the calibrated model parameters given below

Winkler Type model

$$\overline{k_s} = \frac{E}{(1-0.4\nu)\chi_w a} \tag{4.3}$$

Kerr-equivalent Pasternak model

$$\overline{k_p} = \frac{(0.4\nu+0.67)E}{\chi_p a} \quad \text{and} \quad \overline{G_p} = (1.36\nu+2.28) G\chi_p a \tag{4.4}$$

The analysis of short and long strip plate presented in the following sections by using the above parameters, and it gives results in good agreement with those obtained by Finite element based Plaxis 2D software.

4.3 Numerical Illustrations

In this part of the work, Winkler Type model and Kerr equivalent Pasternak Type model will be compared against each other and with Finite element based Plaxis 2D software results. The comparison is done for long and short length of strip plates under selected loading conditions. The following classification suggested by (Hetenyi , 1946) can be adopted for strip plate resting on elastic foundation.

If $(\xi)*a > \pi$Long strip plate

If $\frac{\pi}{4} < (\xi)*a < \pi$intermediate strip plate

If $(\xi)*a < \frac{\pi}{4}$ short strip plate

Where ξ - characteristic width and a- is width of strip plate

In this work Intermediate and short strip plates are treated as short length strip plates

4.3.1 Long Strip plates

Numerical calculations for strip plate of long length subjected to vertical concentrated and uniformly distributed load are carried out for different types of soils. Subsequently, the results obtained from Winkler type model, Kerr equivalent Pasternak type model and Finite Element based Plaxis 2D model are compared. The analysis for Winkler type models and Kerr equivalent Pasternak type model are done by using an excel program prepared for this purpose.

A. Plate Properties (Concrete)

- Elastic modulus of the plate (E_p) = 25 GPa
- Width of the strip plate (a) = 20 m
- Depth of the strip plate (h_p) = 0.15m
- Poisson's ratio (ν_p) = 0.2

B. Soil Property

For all types of soil, the stratum H=60m (thick Stratum) is used. All the soil data described below are taken from Bowels (1997).

1. Soft clay

- Elastic modulus of the soil (E_s) = 15000 kN/m²
- Poisson's ratio (ν_s) = 0.4

2. Loose sand

- Elastic modulus of the soil (E_s) = 20000 kN/m²
- Poisson's ratio (ν_s) = 0.3

3. Medium stiff clay

- Elastic modulus of the soil (E_s) = 30000 kN/m²
- Poisson's ratio (ν_s) = 0.3

4. Medium dense soil

- Elastic modulus of the soil (E_s) = 40000 kN/m²
- Poisson's ratio (ν_s) = 0.25

5. Stiff clay

- Elastic modulus of the soil (E_s) = 80000 kN/m²
- Poisson's ratio (ν_s) = 0.25

6. Dense sand soil

- Elastic modulus of the soil (E_s) = 81000 kN/m²
- Poisson's ratio (ν_s) = 0.2

C. Loading Conditions

1. Vertical Concentrated load

- $P=100$ kN/m

2. Uniformly Concentrated Load

- $q= 50$ kN/m², Loaded region =2m

D. Finite Element (Plaxis) Inputs

1. Types of element

- 15 node

2. Plate model

- Plain strain model

3. Soil model

- Linear elastic model

The results of the deflection, moment and shear force are evaluated and presented graphically for the strip plate and soil parameters is given above. Furthermore, a sample example is selected and its Plaxis outputs (deflection, moment and shear force), for strip plate subjected to point load, is presented in the following section.

1. Vertical concentrated load

A. Plaxis Analysis

1. Plaxis model

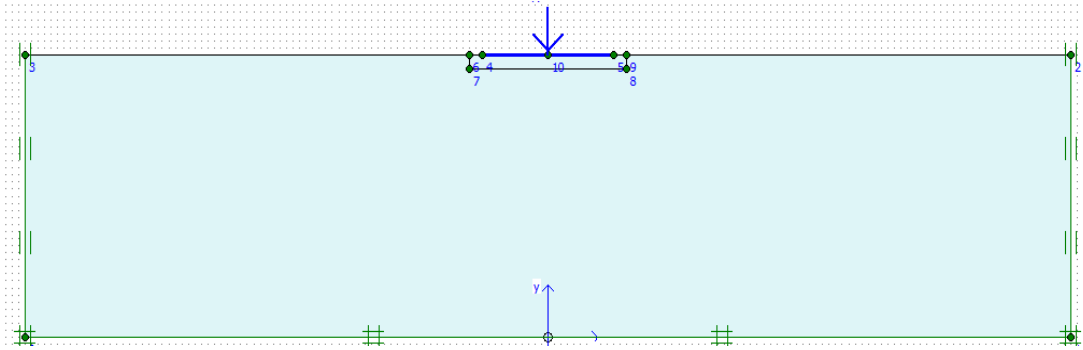


Fig 4.14 Plaxis geometry model

2. Deformed mesh

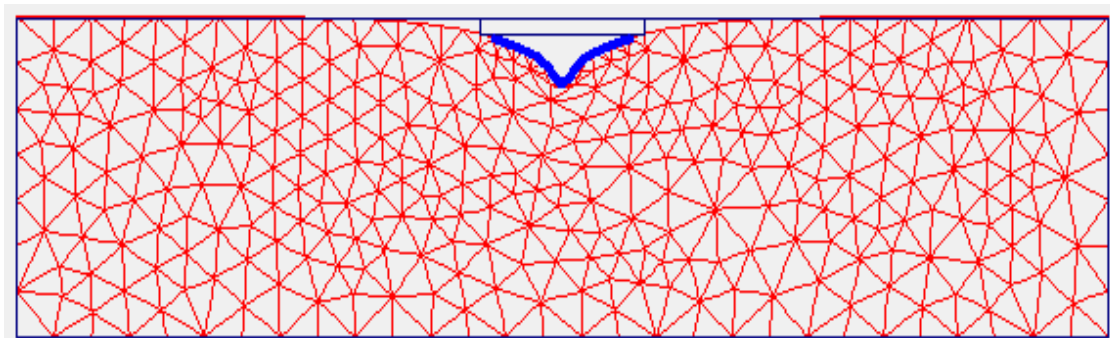


Fig 4.15 Plaxis model output of deformation for strip plate resting on soft clay soil and subjected to point load

3. Total displacement

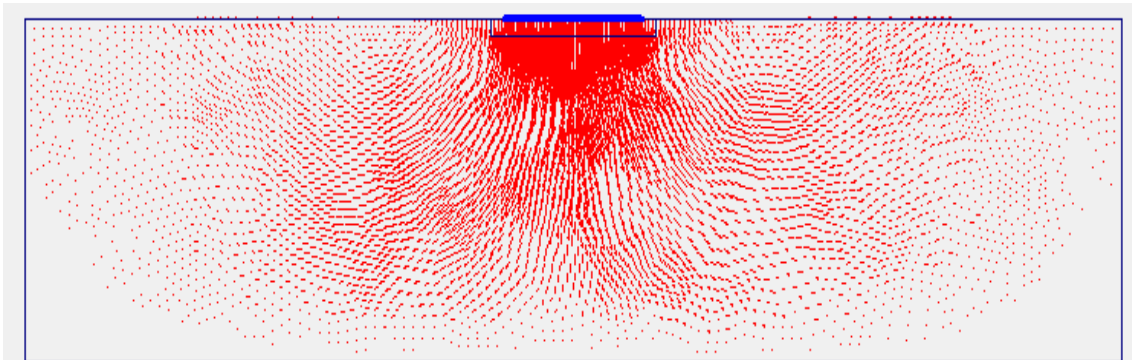


Fig 4.16. Plaxis model output total displacement of strip plate resting on soft clay soil and subjected to point load

4. Cartesian total stresses

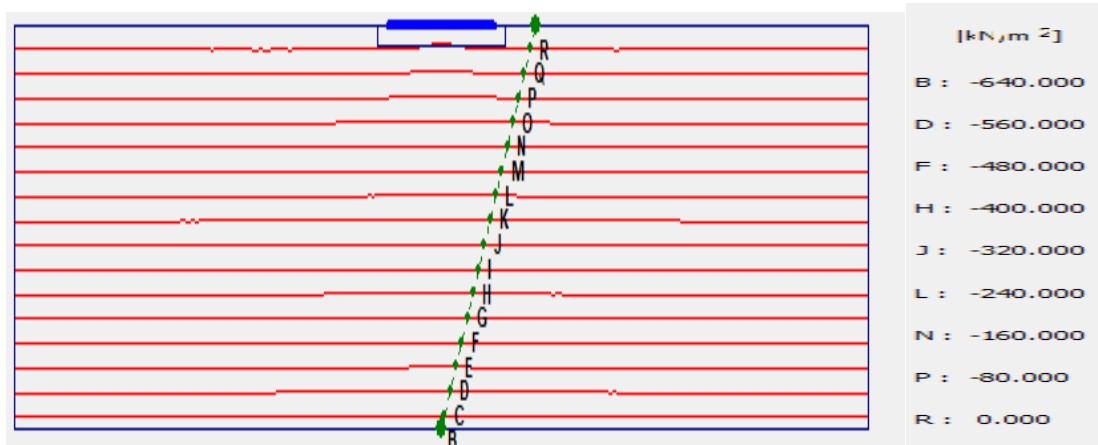


Fig 4.17. Plaxis model output cartesian total stresses of strip plate resting on soft clay soil and subjected to point load

5.Plaxis outputs

Table 4.5 Plaxis outputs of deformation, moment and shear force

a(m)	Deflection (mm)	Moment (KN.m/m)	shear force (KN/M)
-10	-2.07	0	-0.455
-9	-2.198	-0.169	-0.059
-8	-2.2845	-0.067	0.205
-7	-2.807	0.21465	0.317
-6	-4.5163	0.6476	0.525
-5	-6.212	1.427	0.97
-4	-8.1865	2.268	1.357
-3	-10.158	3.935	1.514
-2	-12.043	4.449	-1.557
-1	-14.518	-3.13	-15.241
0	-16.217	-36.001	-50.141
0	-16.217	-36.001	50.141
1	-14.518	-3.13	15.272
2	-12.043	4.448	1.558
3	-10.158	3.933	-1.523
4	-8.1865	2.267	-1.388
5	-6.212	1.427	-0.975
6	-4.5163	0.647	-0.523
7	-2.807	0.214	-0.316
8	-2.2845	-0.066	-0.205
9	-2.198	-0.17	0.059
10	-2.07	0	0.455

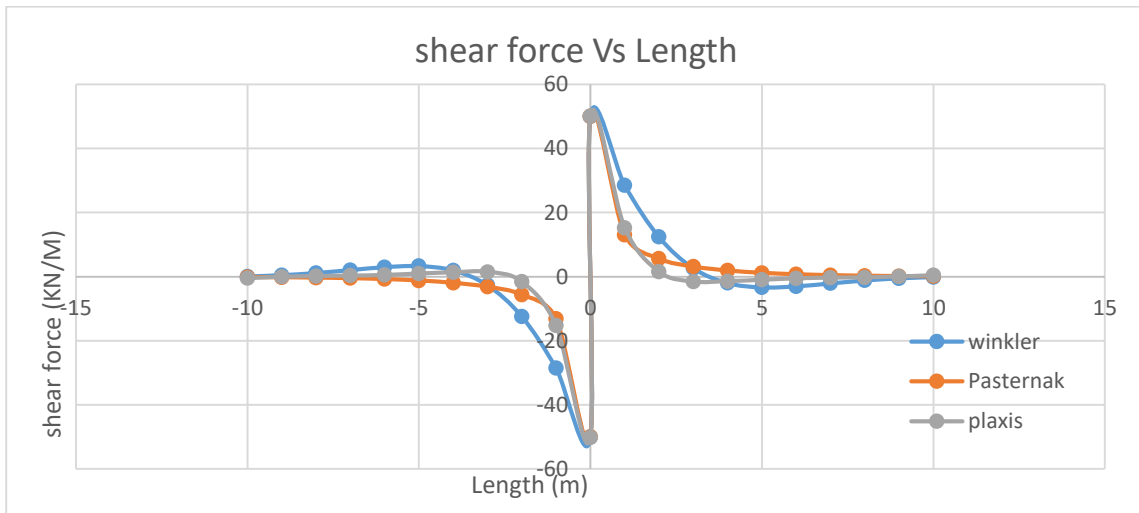
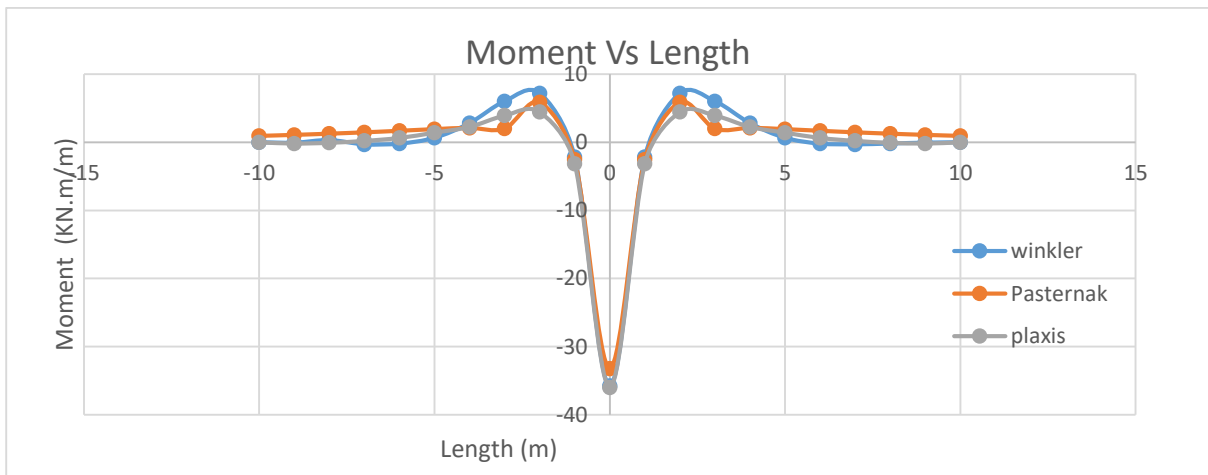
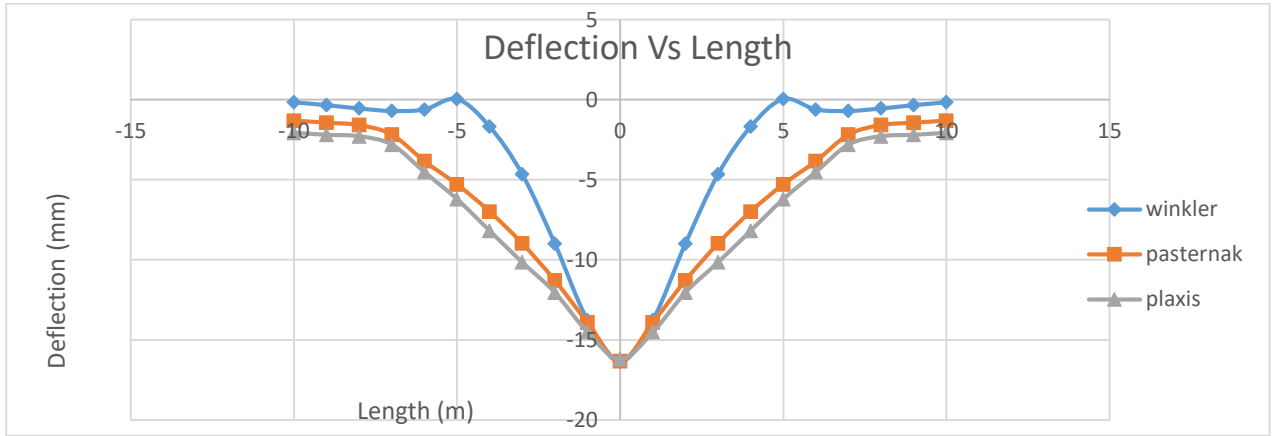
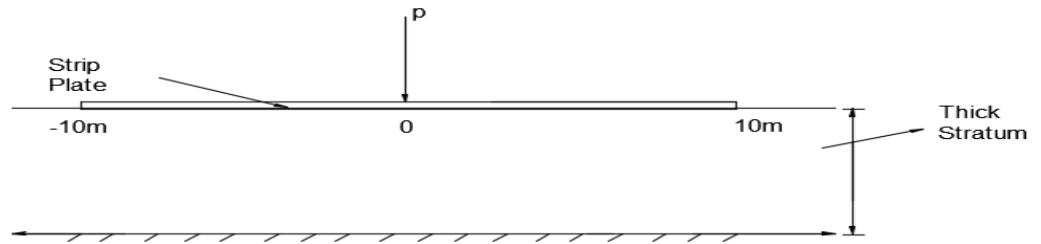


Fig 4.18. Long strip plate on soft clay soil subjected to a point load

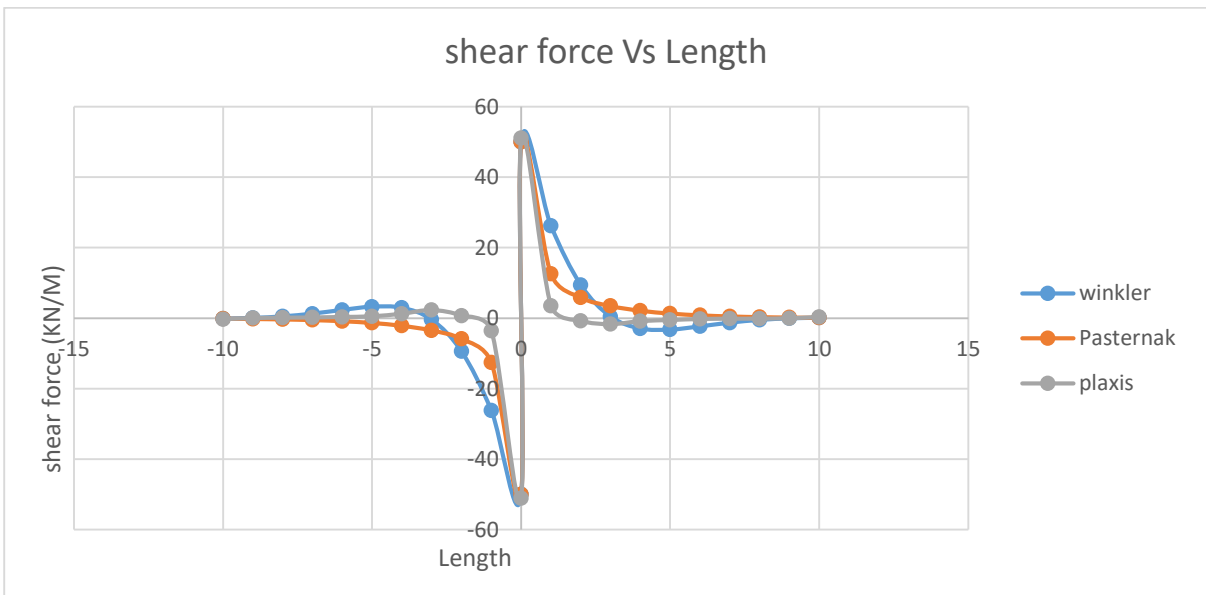
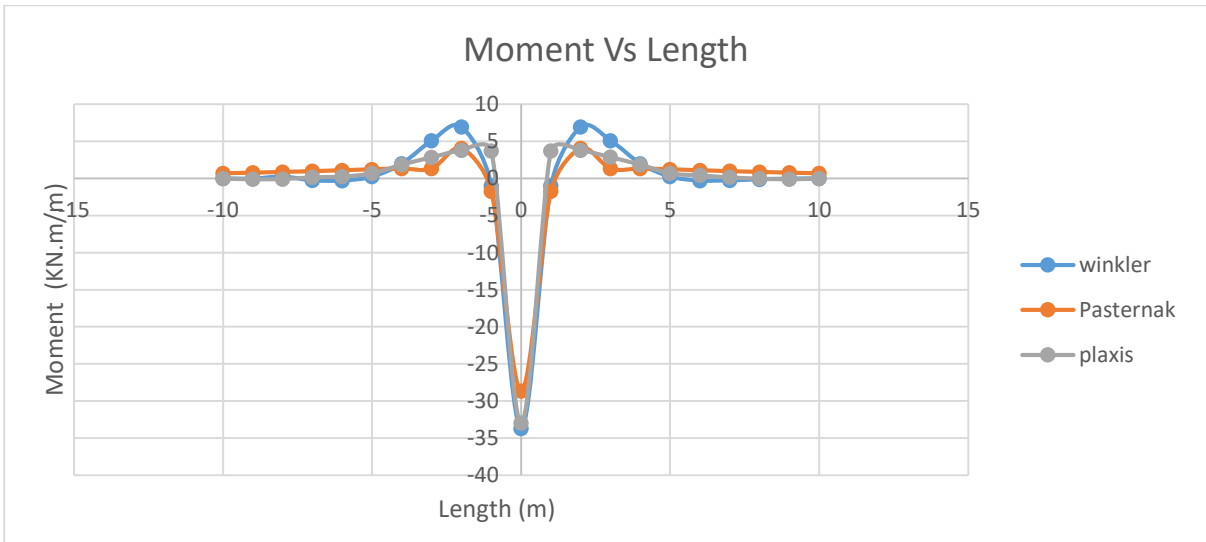
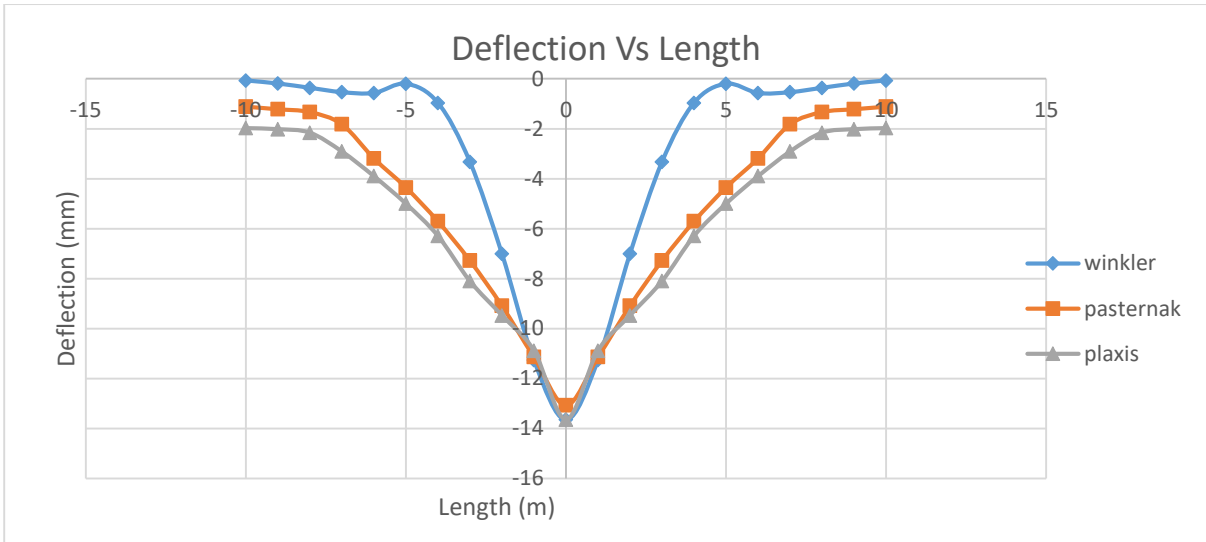


Fig 4.19. Long strip plate on loose sand soil subjected to a point load

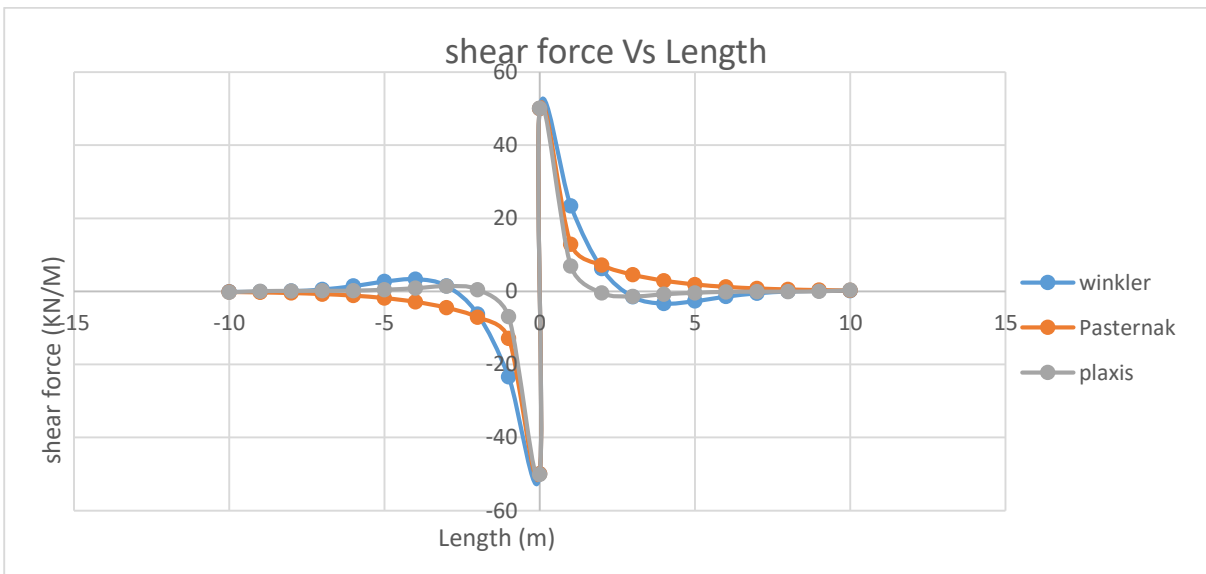
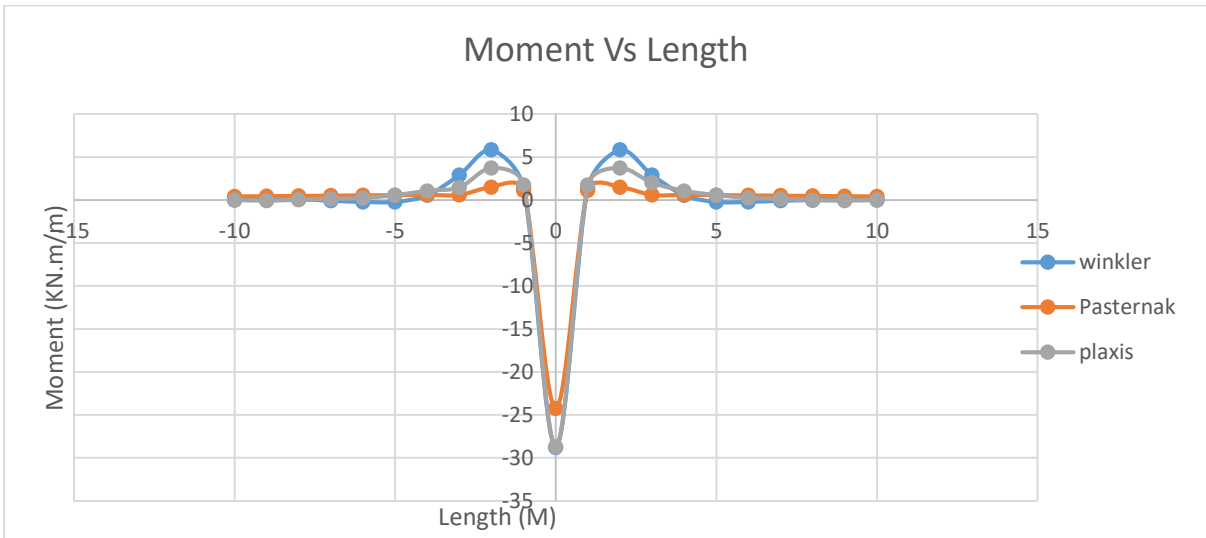
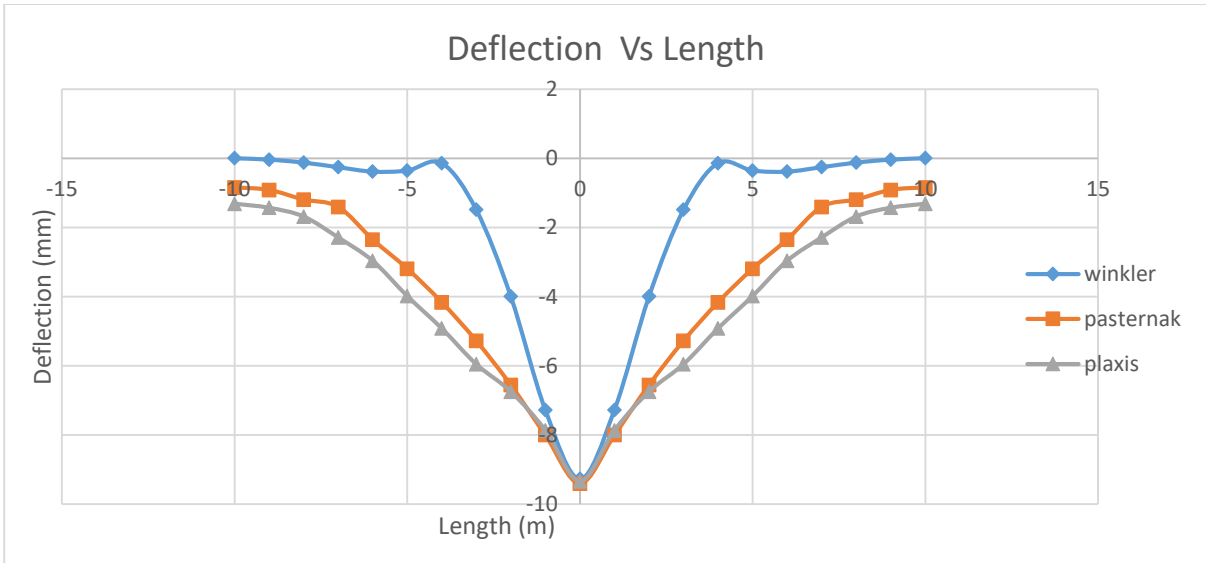


Fig 4.20. Long strip plate on medium stiff clay soil subjected to point load

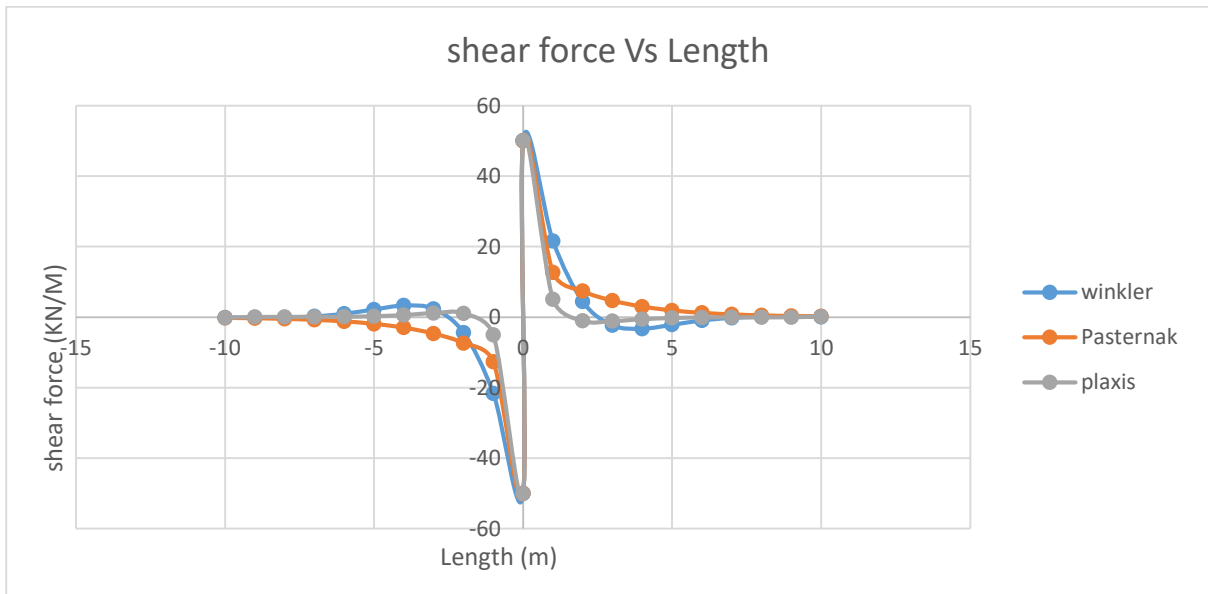
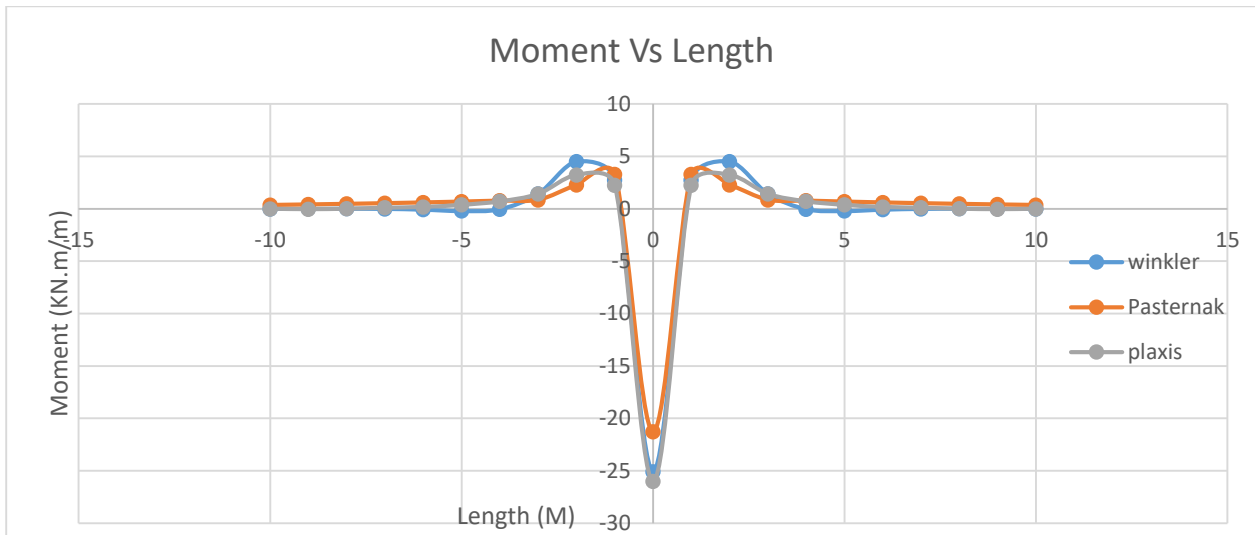
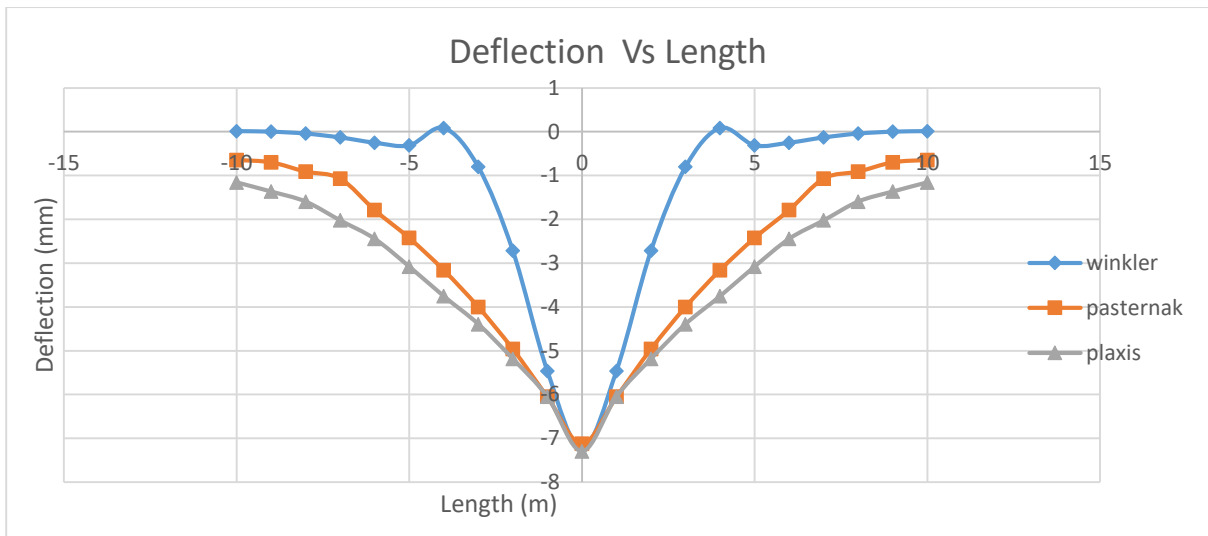


Fig 4.21. Long strip plate on medium dense sand soil subjected to point load

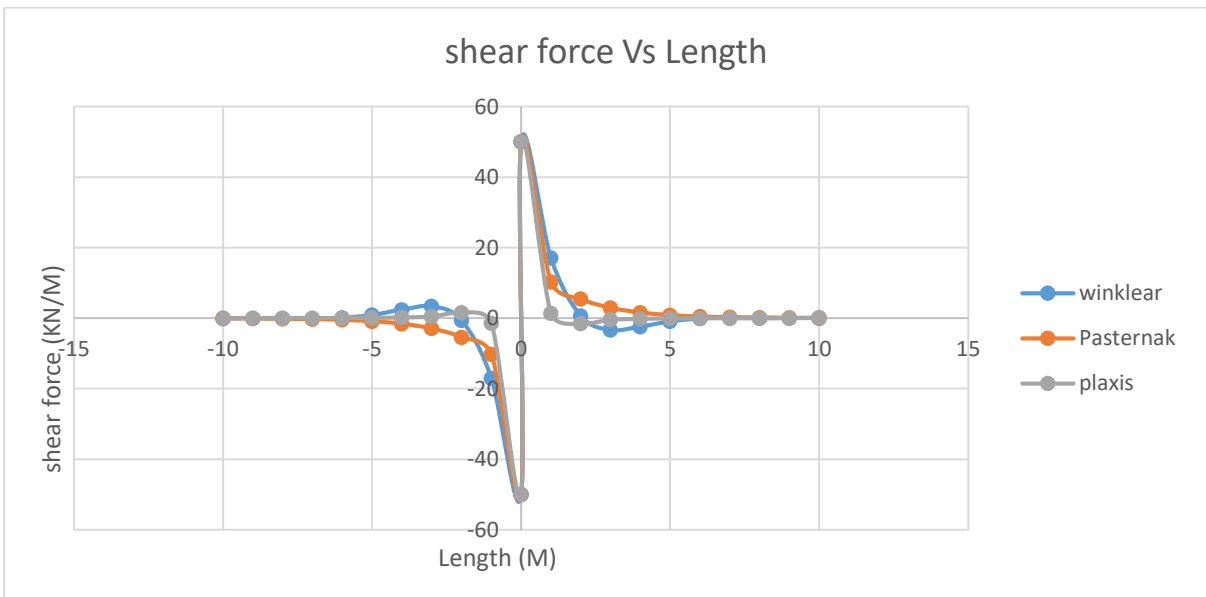
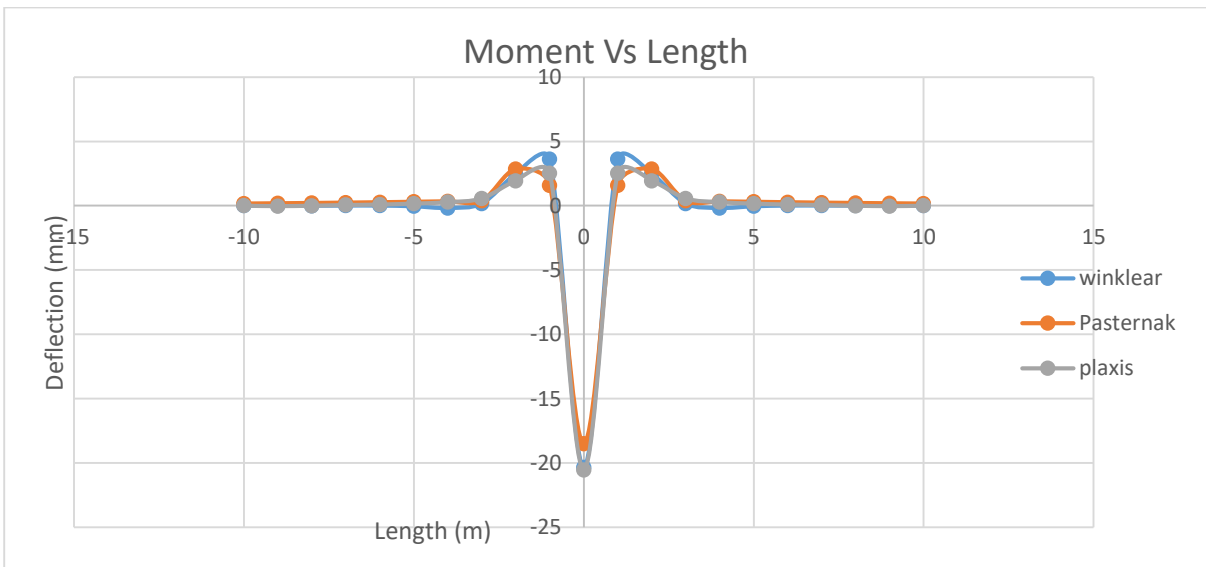
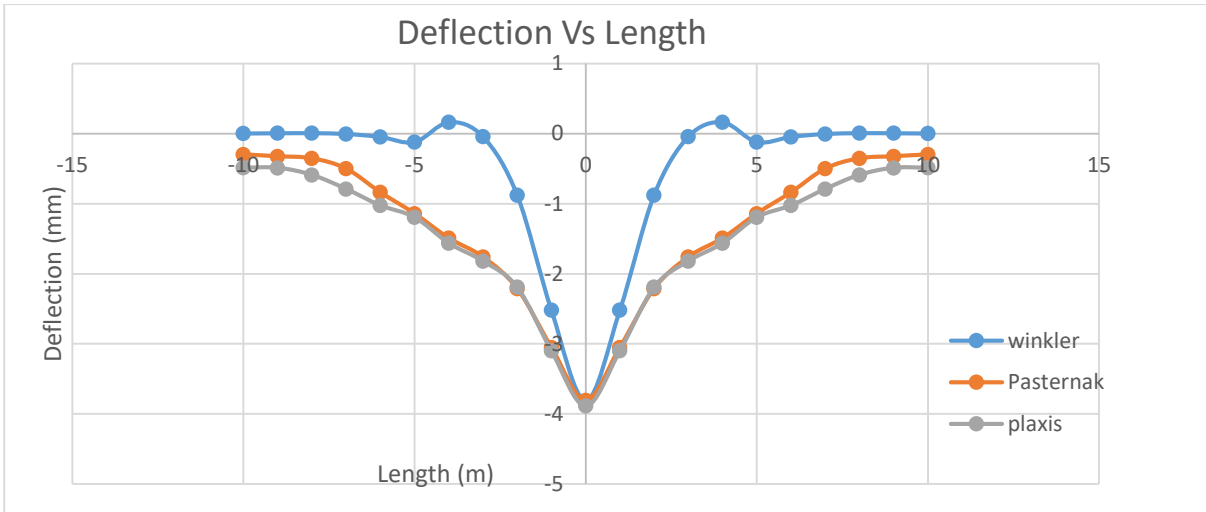


Fig 4.22. Long strip plate on stiff clay soil subjected to point load

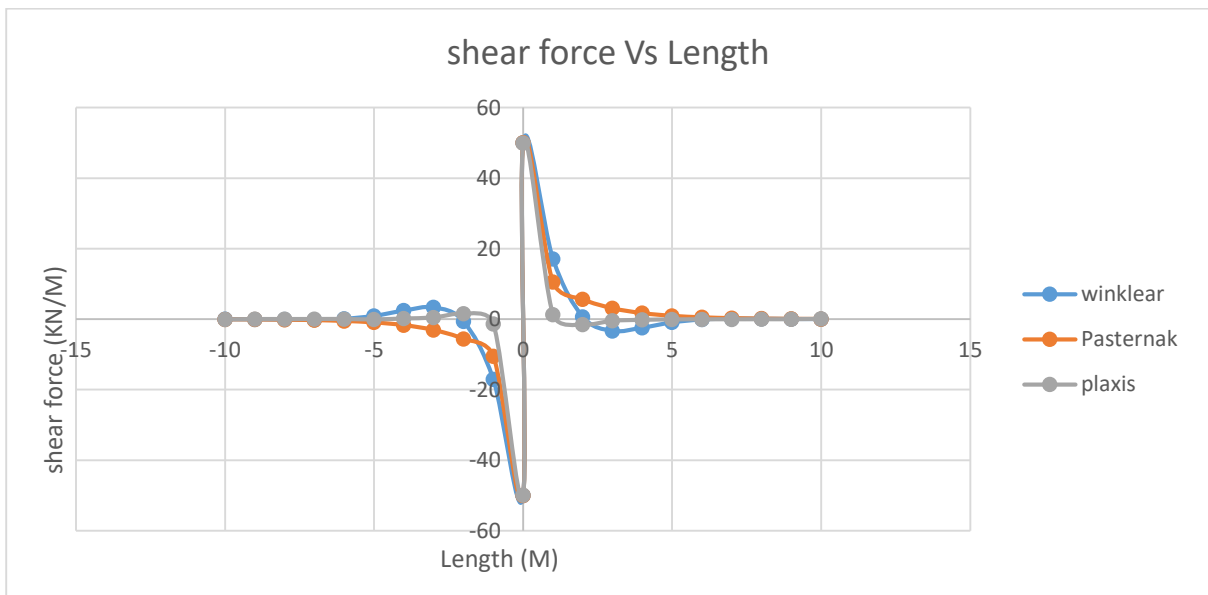
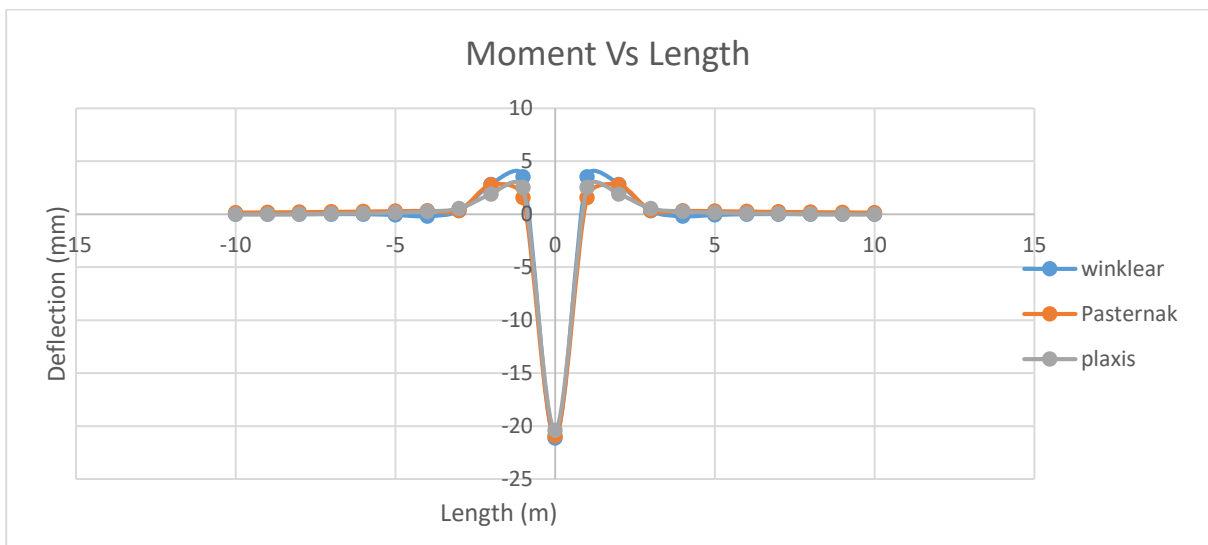
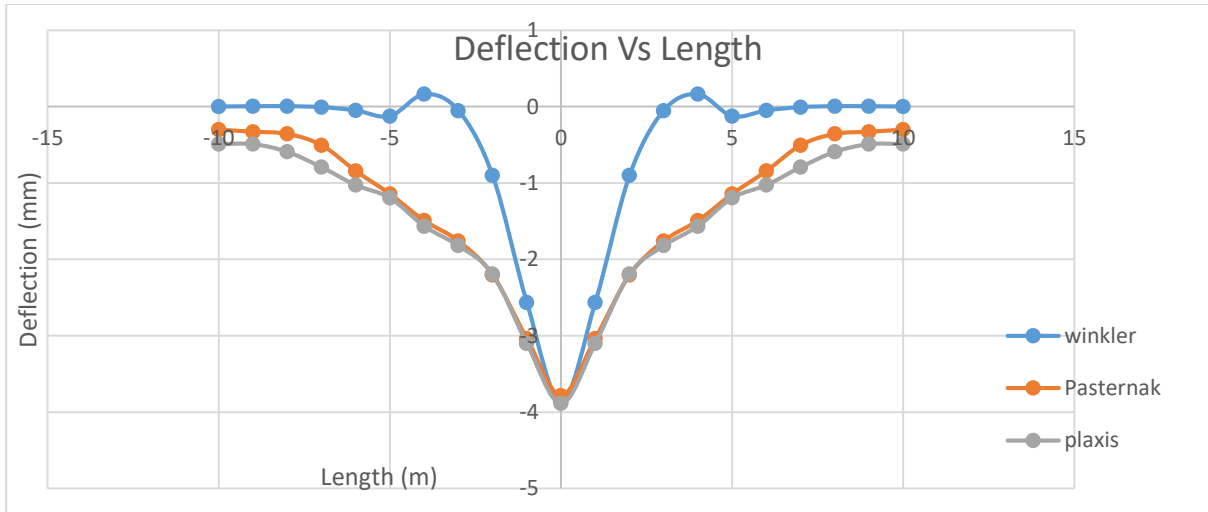


Fig 4.23. Long strip plate on dense sand soil subjected to point load

2. Uniformly distributed load

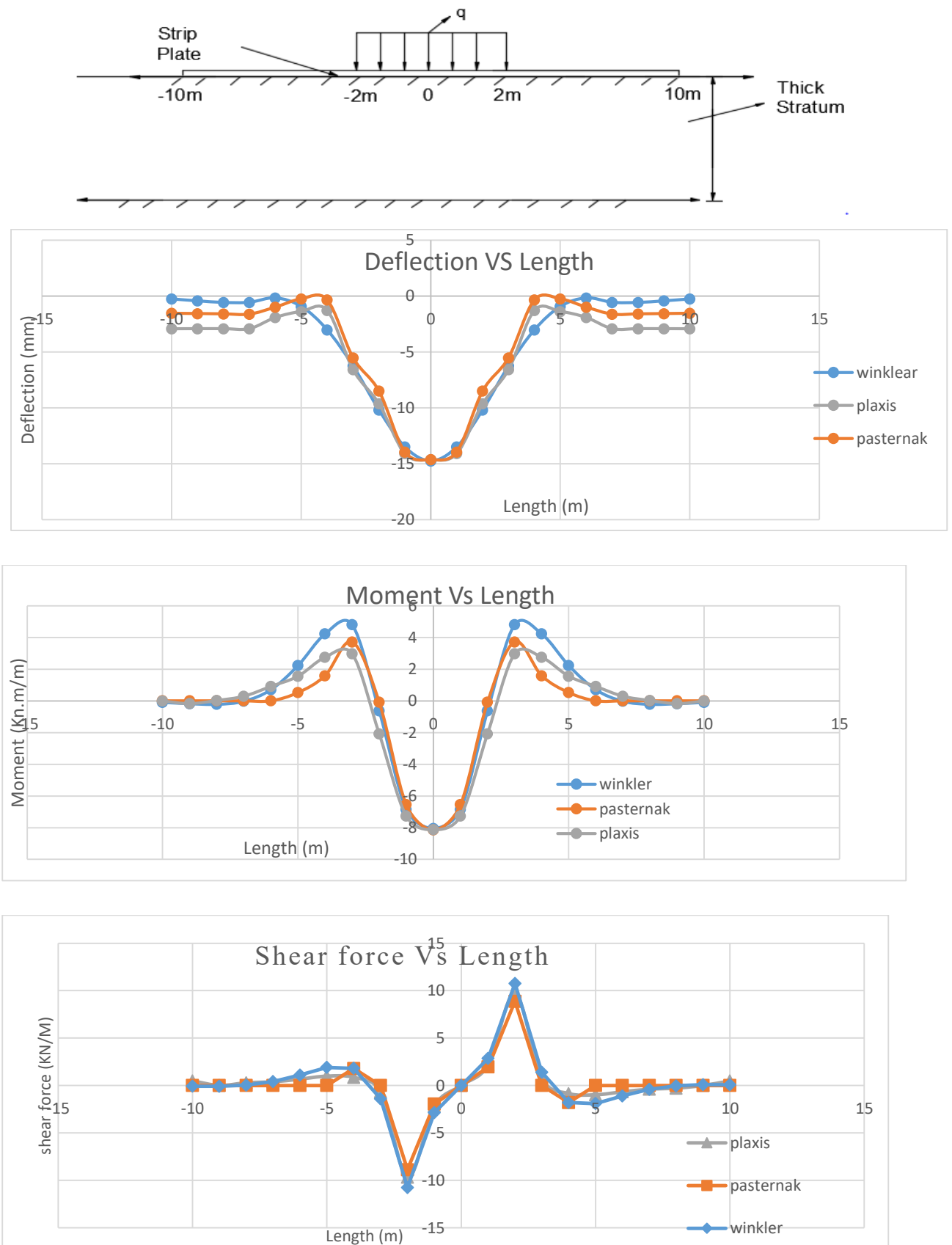


Fig 4.24. Long strip plate on soft clay soil subjected to uniformly distributed Load

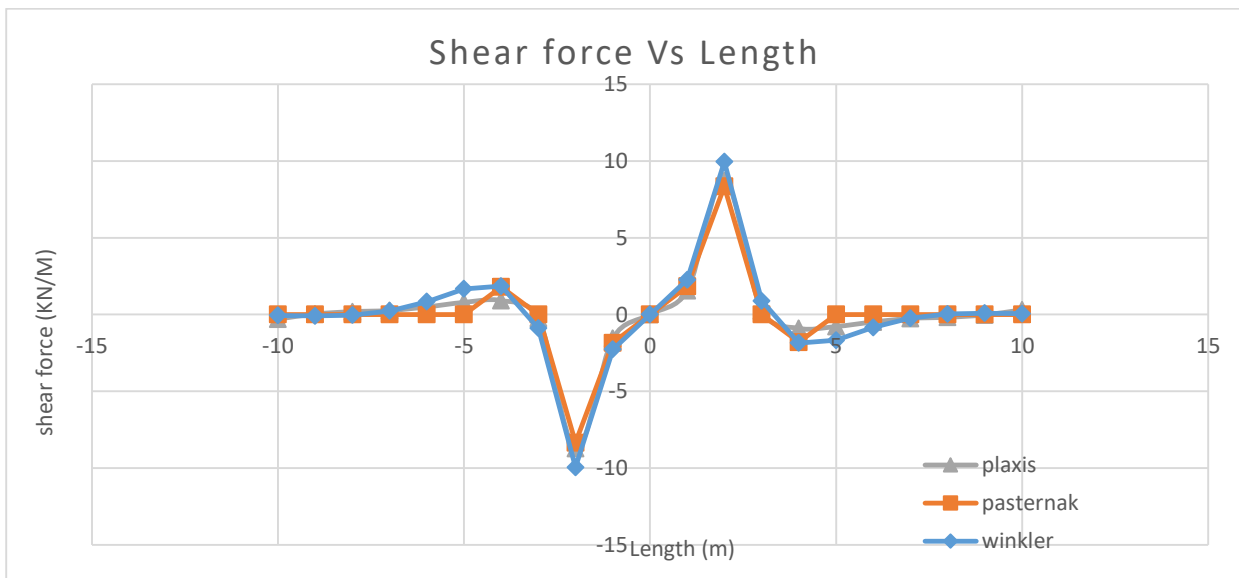
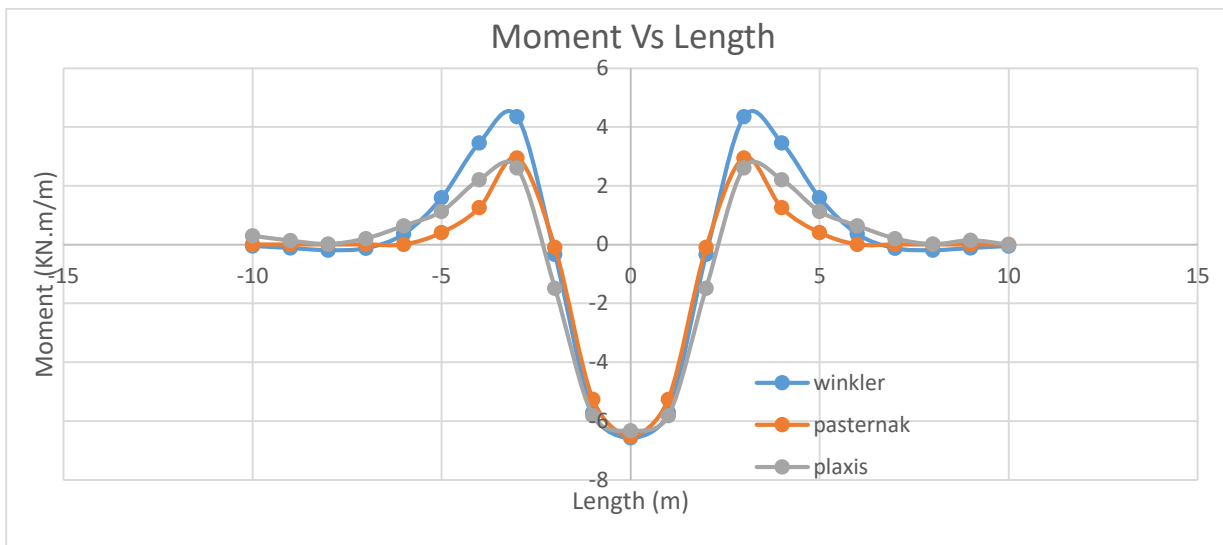
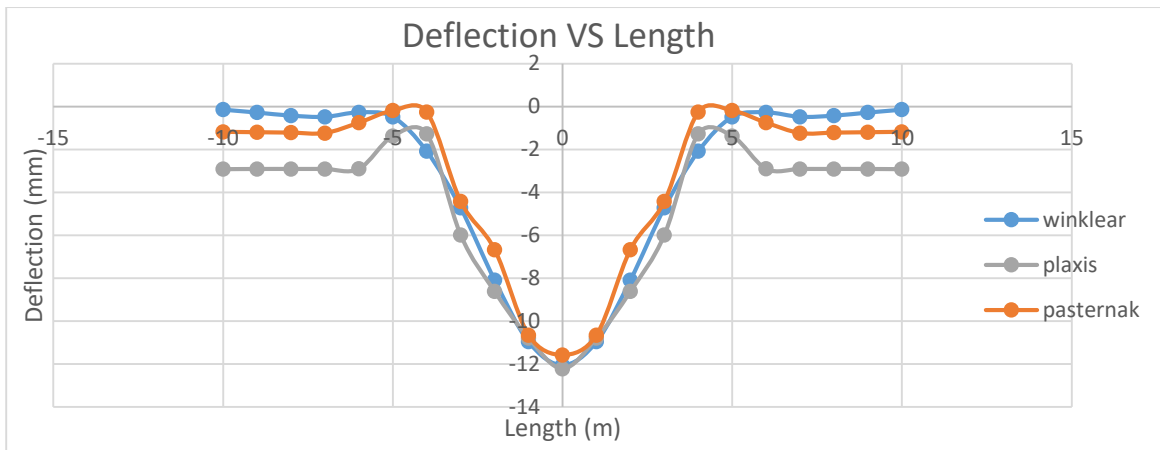


Fig 4.25. Long strip plate on loose sand soil subjected to uniformly distributed Load.

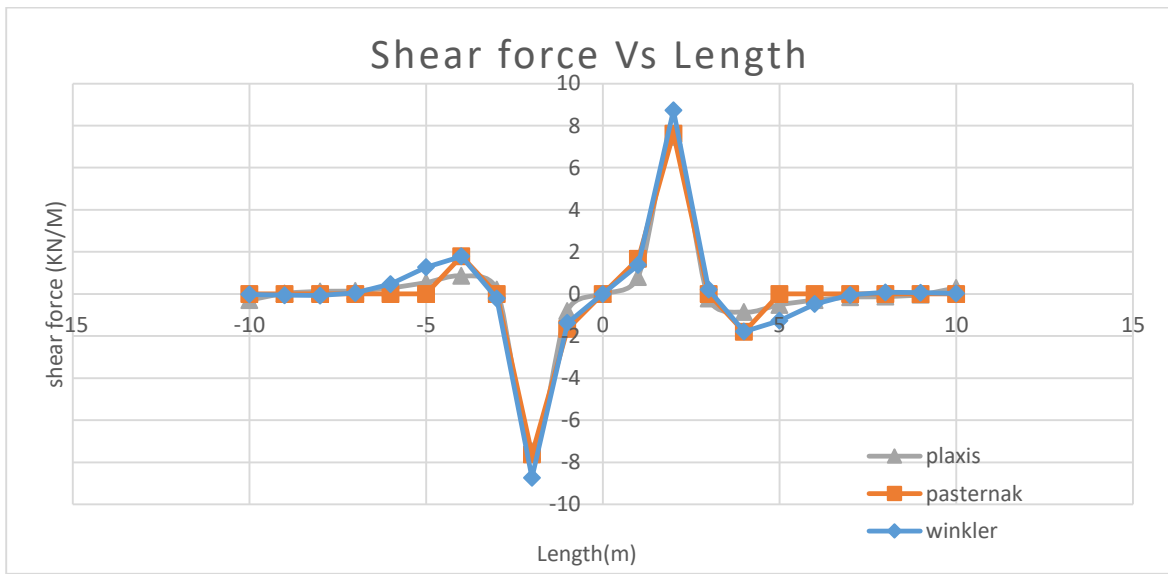
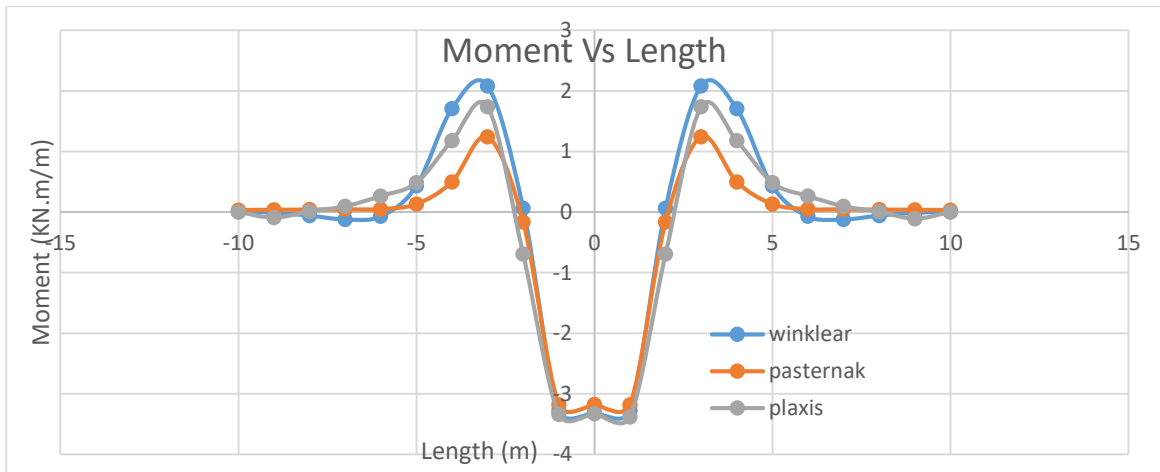
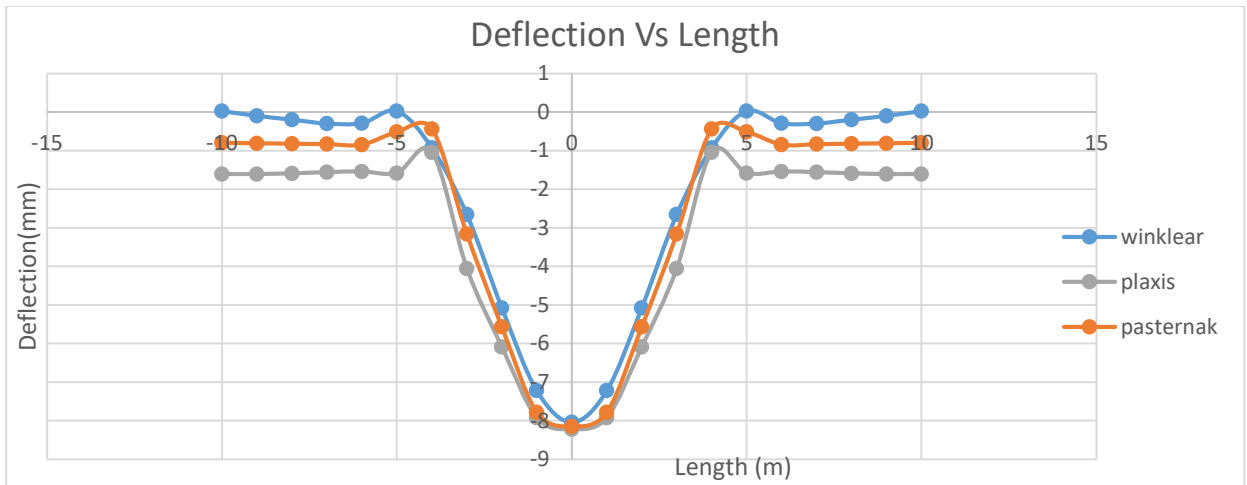


Fig 4.26. Long strip plate on medium stiff clay soil subjected to uniformly distributed load

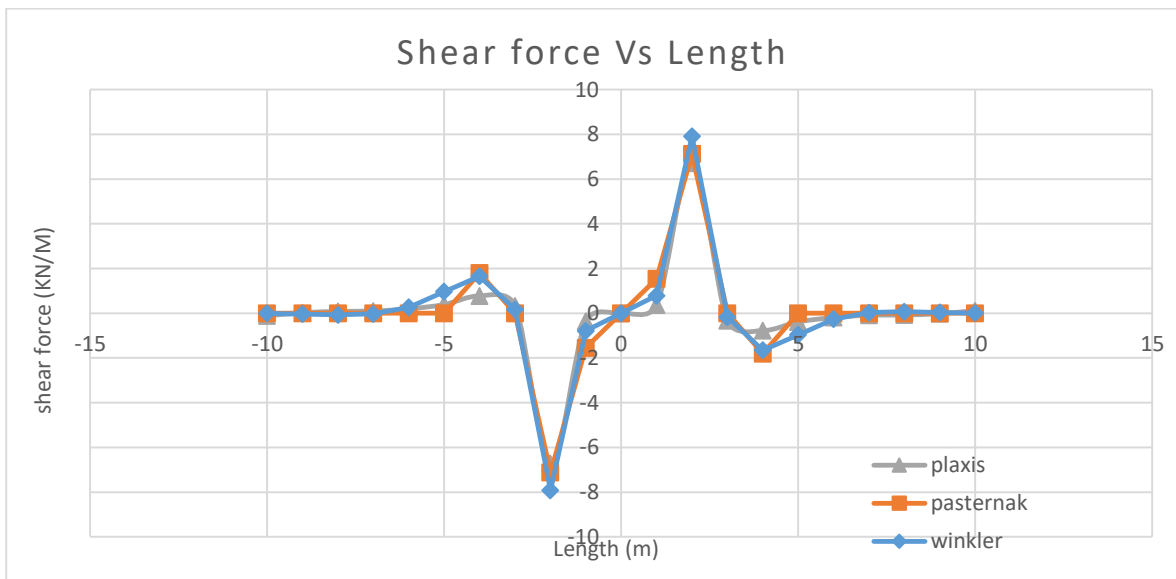
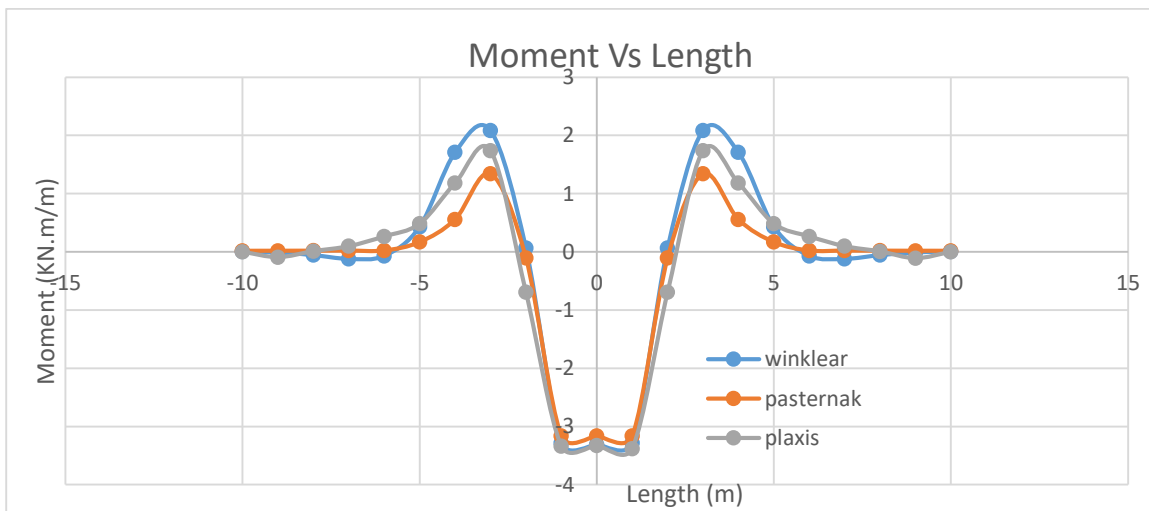
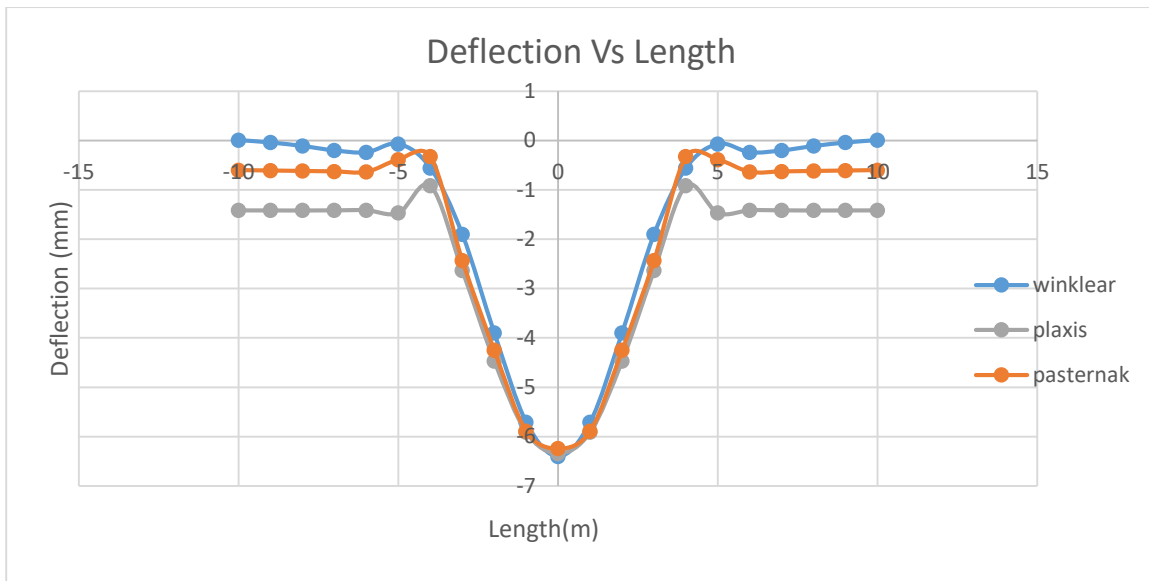


Fig 4.27 Long strip plate on medium dense sand soil subjected to uniformly distributed load

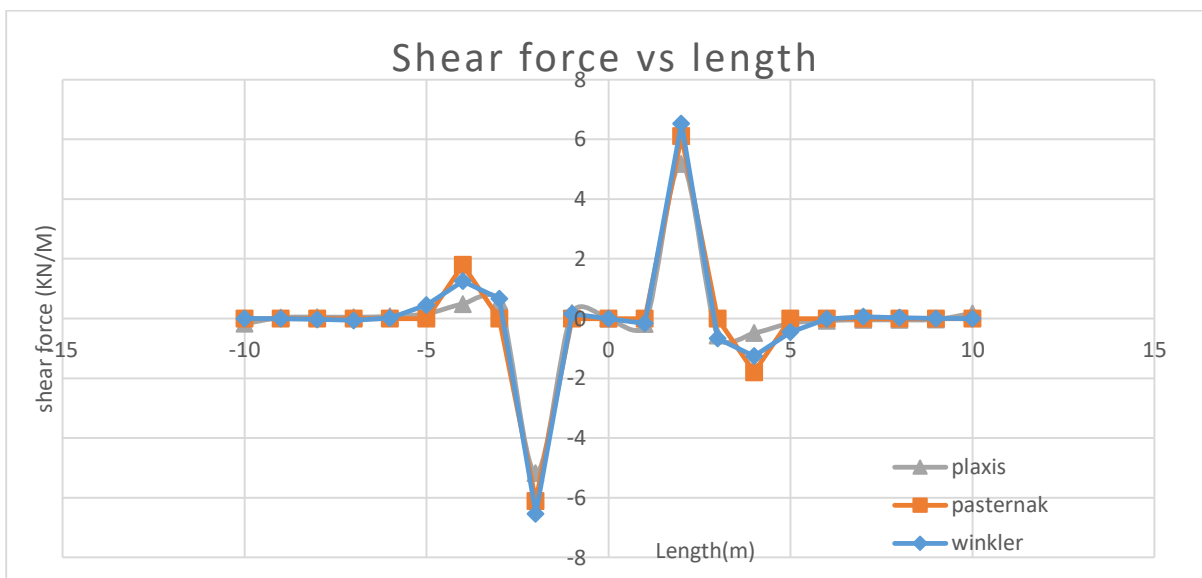
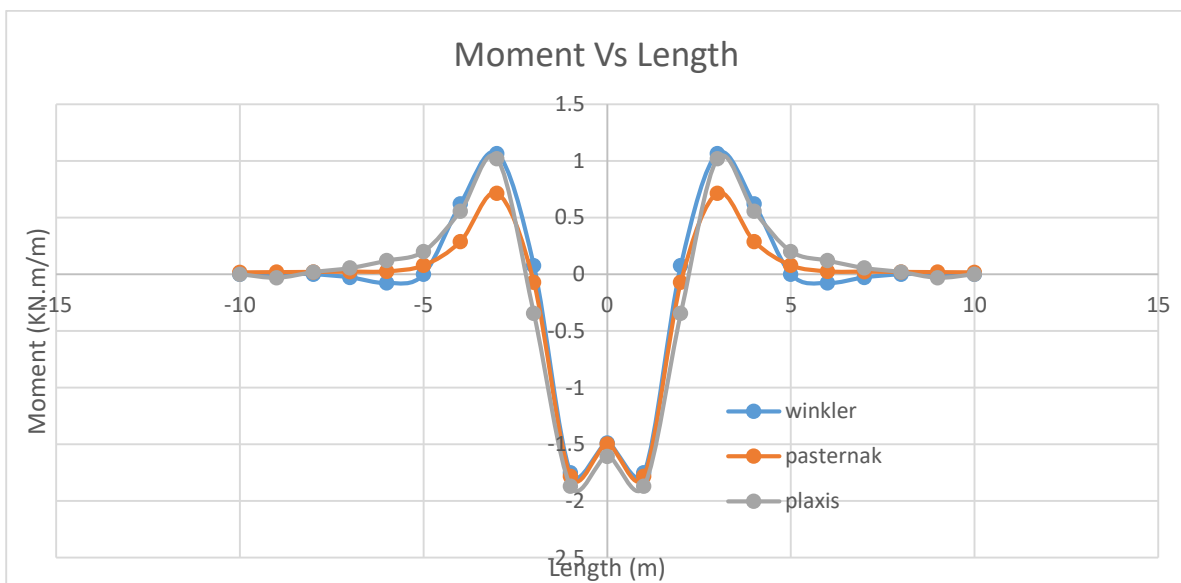
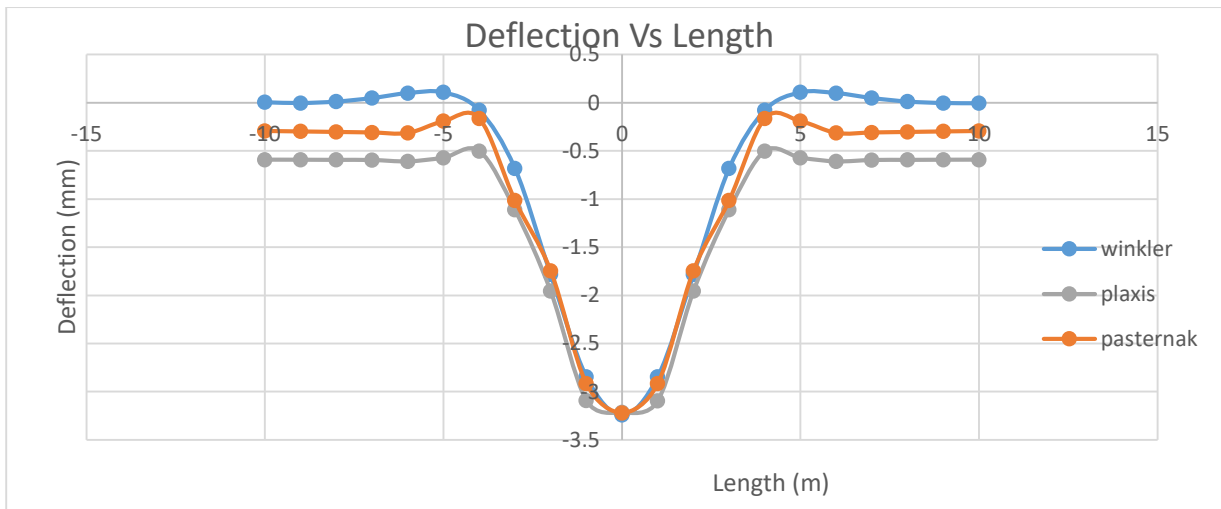


Fig 4.28. Long strip plate on stiff clay soil subjected to uniformly distributed load

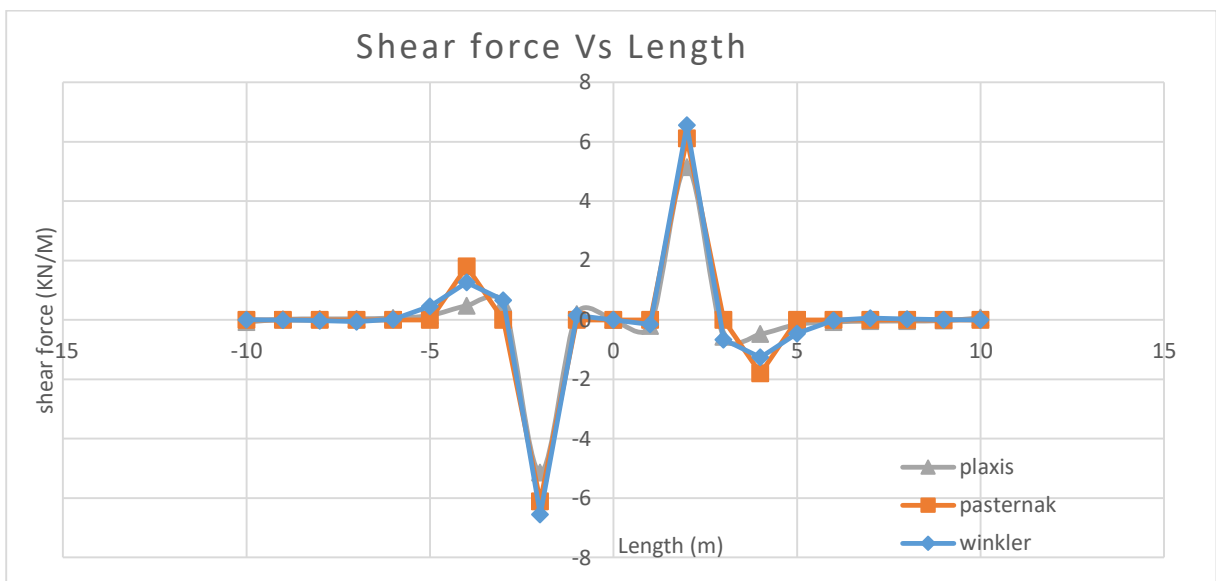
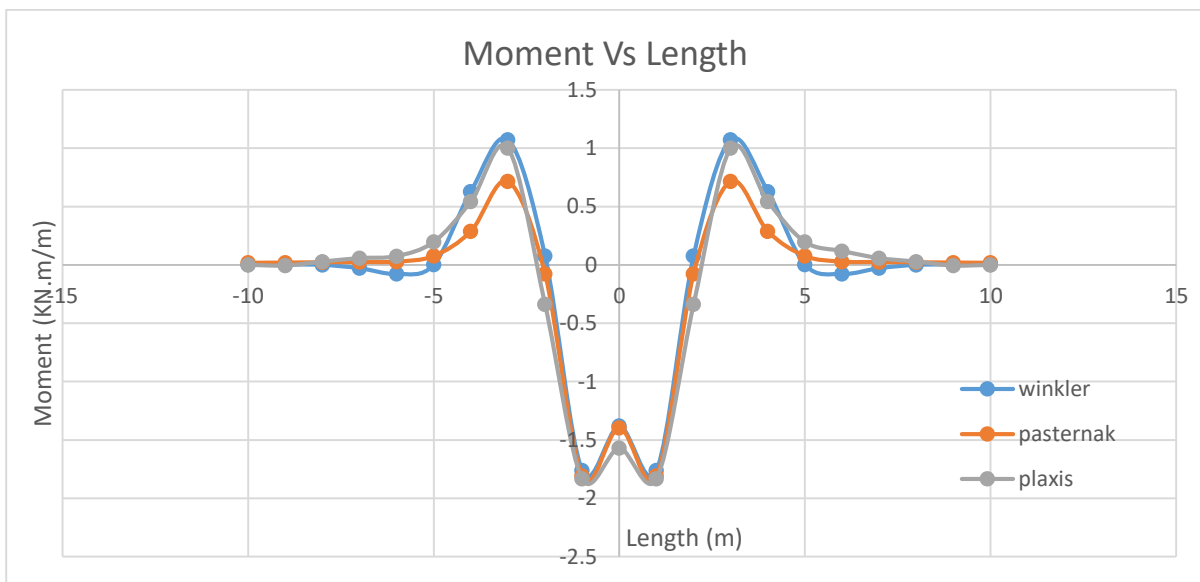
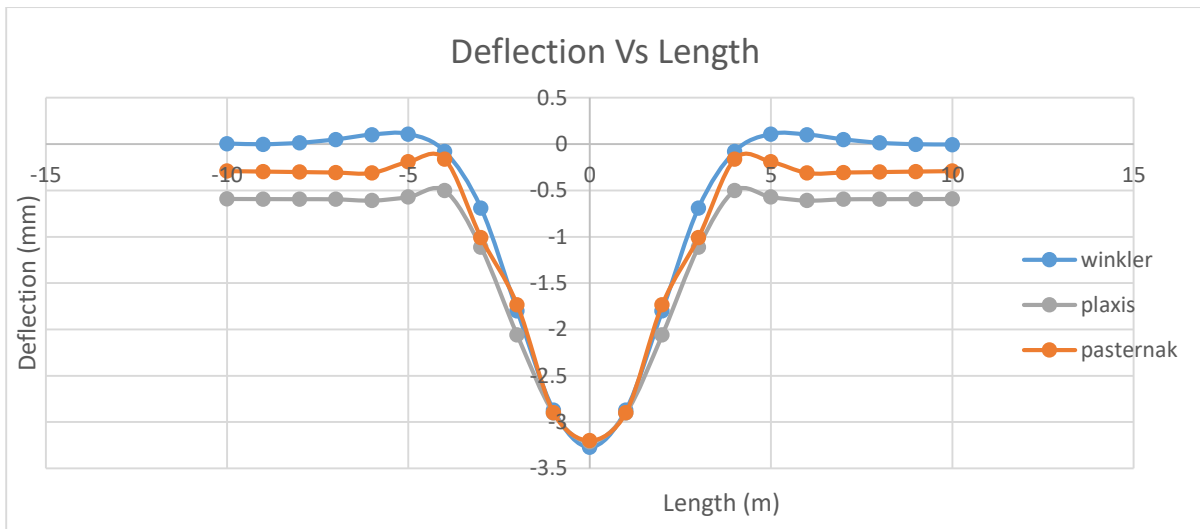


Fig 4.29. Long strip plate on dense sand soil subjected to uniformly distributed load

A review of the curves reveals a number of significant observations. The maximum deflection obtained by the two models is in good agreement with the FE based Plaxis model. However, there is a deviation of deflection when moving away from the mid span. This becomes more pronounced in the Winkler-Type model especially in soft soils where deviations may amount to up to 45% in the case of concentrated load and 20 % in uniformly distributed cases. This is due to the fact that Winkler-Type model does not consider the shear interaction behavior of the soil, from the outset despite the calibration. Additionally, because the calibration factor is solely dependent on the maximum deflection, it is certain to have deviations when moving away from the maximum deflection. As far as the Kerr- Equivalent Pasternak type model is concerned, a very good agreement with the reference FE based analysis result is obtained over the majority of the plate width.

Unlike the deflection, the bending moment and shear force show little deviation between the models and the Plaxis 2D outputs, in both cases of loading conditions. As it can be seen in the above bending moment diagrams, the deviations increase when moving away from the mid span but ultimately decrease before reaching the edge of the plate. Furthermore, in the case of plate on high modulus soils subjected a uniformly distributed load results of a reduced shear force are observed. Whereas in the case of a concentrated vertical load, the shear force is invariably the same for all models at the midpoint. This implies that midpoint span shear force is independent of soil rigidity in the case of the concentrated load. For all cases, a stiffer soil (higher modulus of elasticity) reduces the plate deflection and the maximum moment experienced in the plate.

In general, the Kerr-Equivalent Pasternak model gives results in very good agreement with Finite Element based Plaxis 2D output, for the considered mesh size (Fig 4.3) in both loading cases. Thus this models can be proposed as an efficient means to analyze long length strip plate on elastic foundation.

4.3.2 Short strip plates

In this section, numerical computations are carried out for short length strip plate subjected to concentrated and uniformly distributed vertical loads to compare the result obtained from Worku's models and FE-based Plaxis 2D software. The same type of strip plate properties and soil parameters are used to plot deflection, bending moment and shear force curves.

A comparison of moment and shear force plots for a short strip plate on Winkler model subjected to uniformly distributed load is not included due to the complexity of the mathematical formulation.

A. Plate properties (concrete)

- Elastic modulus of the plate (E_p) = 25 GPa
- Width of the strip plate (a) = 3m
- Depth of the strip plate (h_p) = 0.15m
- Poisson's ratio (ν_p) = 0.2

B. Soil Properties

For all types of soil the stratum H=40m is used. All the soil data described below is taken from Bowles (1997).

1. Soft clay

- Elastic modulus of the soil (E_s) = 15000 kN/m²
- Poisson's ratio (ν_s) = 0.4

2. Loose sand

- Elastic modulus of the soil (E_s) = 20000 kN/m²
- Poisson's ratio (ν_s) = 0.3

3. Medium stiff clay

- Elastic modulus of the soil (E_s) = 30000 kN/m²
- Poisson's ratio (ν_s) = 0.3

4. Medium dense soil

- Elastic modulus of the soil (E_s) = 40000 kN/m²
- Poisson's ratio (ν_s) = 0.25

5. Stiff clay

- Elastic modulus of the soil (E_s) = 80000 kN/m²
- Poisson's ratio (ν_s) = 0.25

6. Dense sand soil

- Elastic modulus of the soil (E_s) = 81000 kN/m²
- Poisson's ratio (ν_s) = 0.2

C. Loading Conditions

1. Vertical Concentrated load

- $P=100$ kN/m

2. Uniformly Concentrated Load

- $q= 50$ kN/m², Loaded region =2m

D. Finite Element (Plaxis) Inputs

1.Types of element

- 15 node

2.Plate model

- Plain strain model

3.Soil model

- Linear elastic model

The results of the deflection, moment and shear force are evaluated and presented graphically for the strip plate and soil parameters given above.

1. Vertical concentrated load

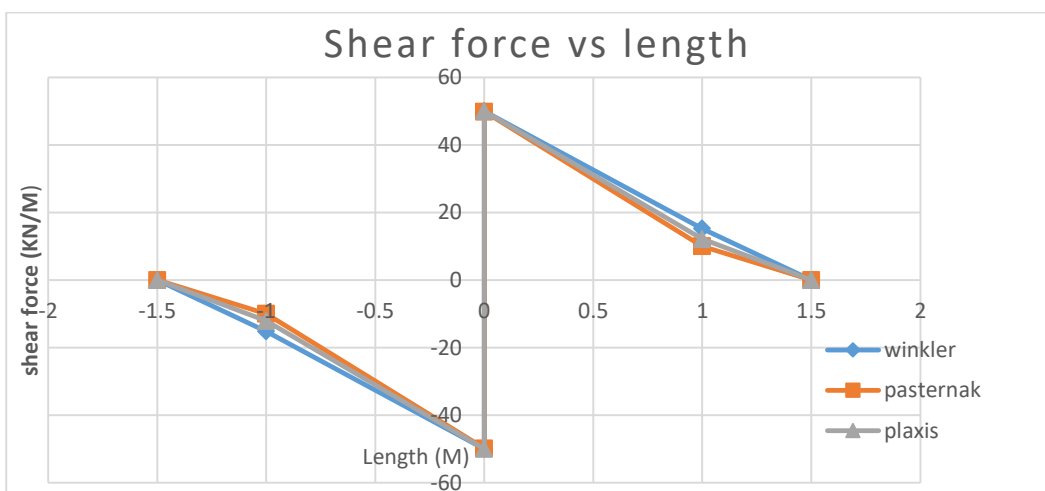
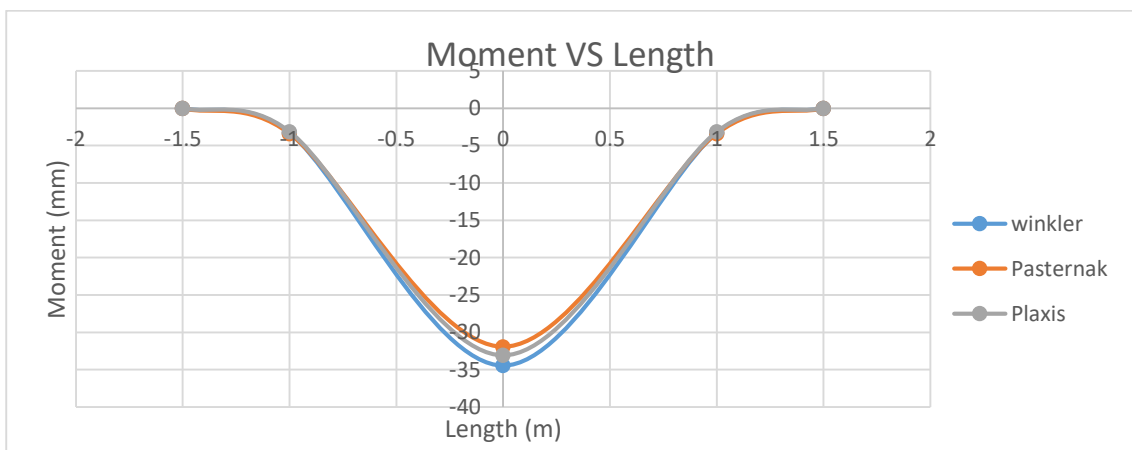
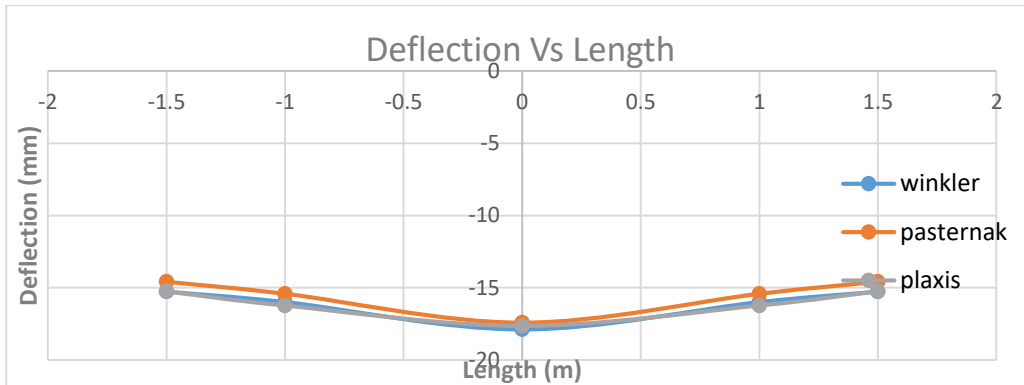
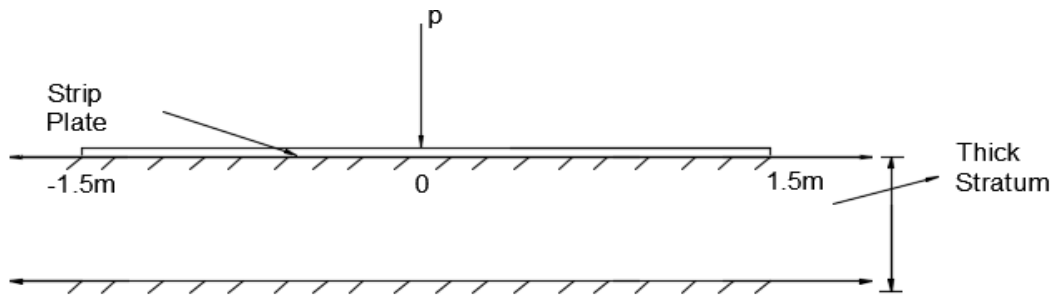


Fig 4.30. Short strip plate on soft clay soil subjected to point load.

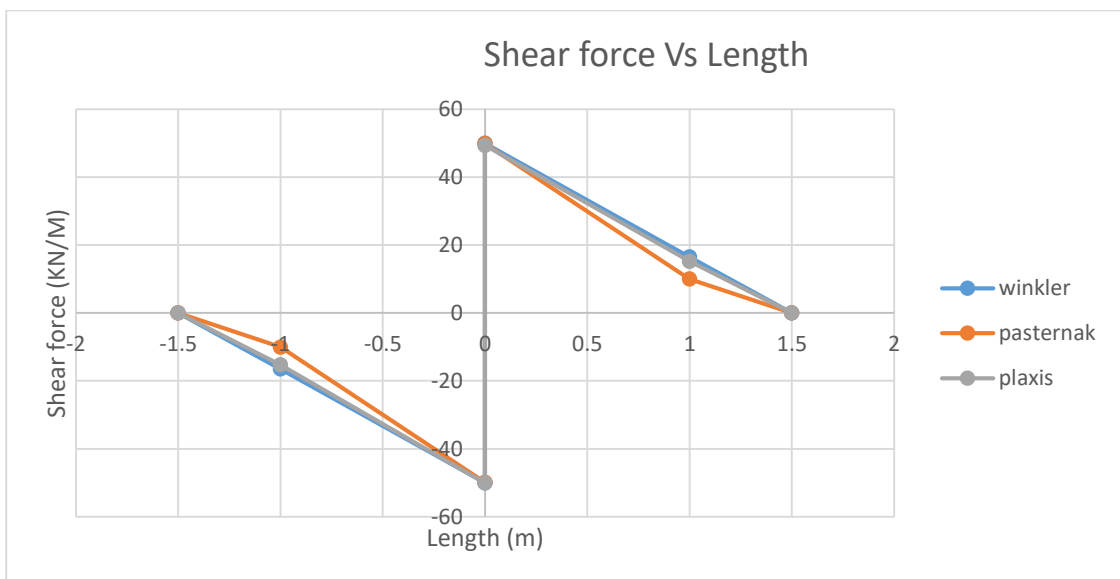
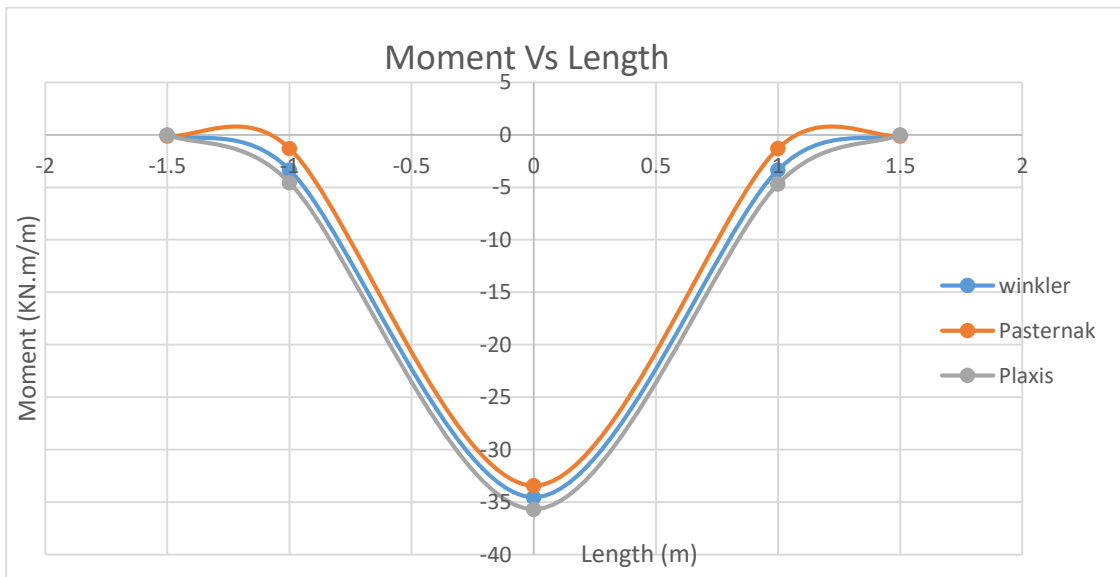
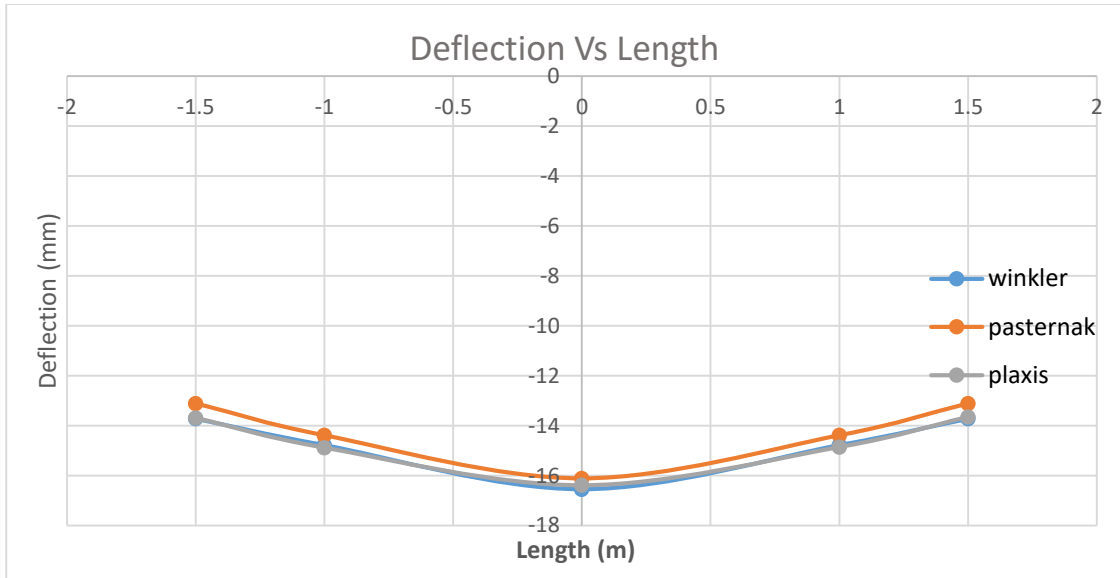


Fig 4.31. Short strip plate on loose sand soil subjected to point load.

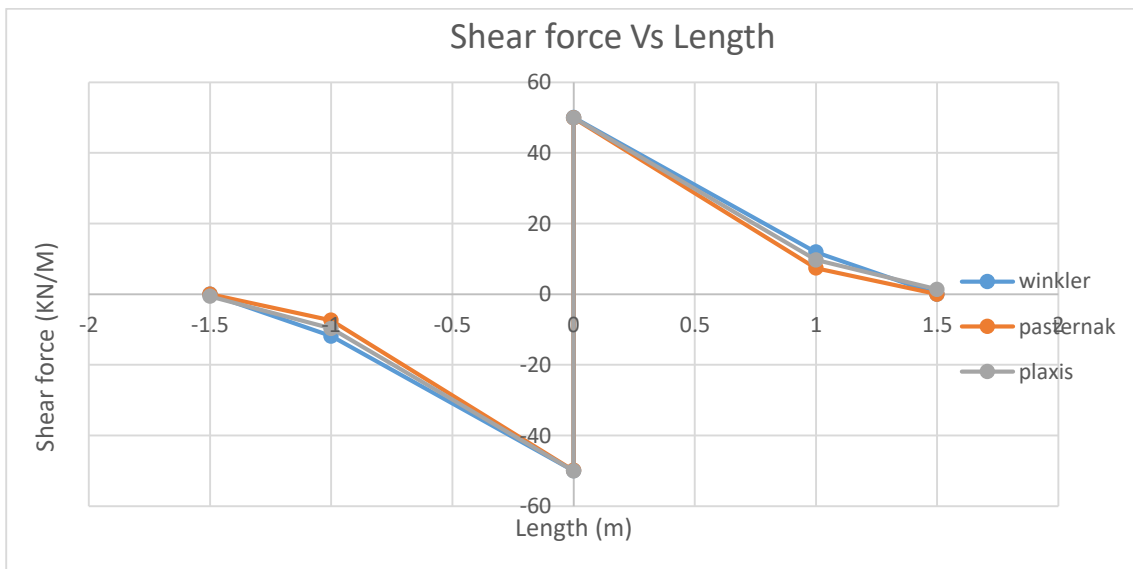
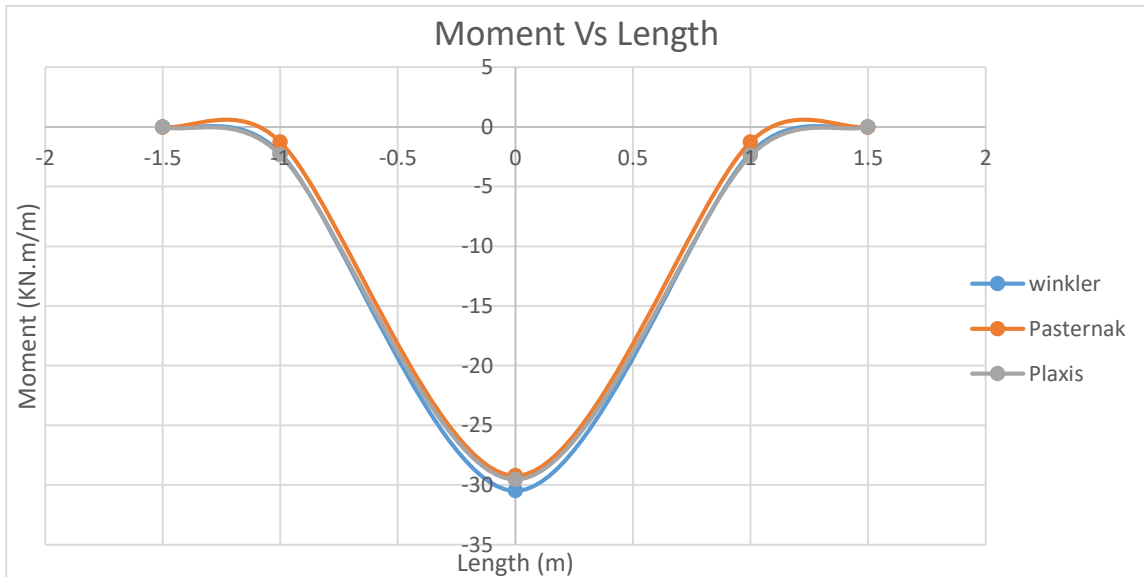
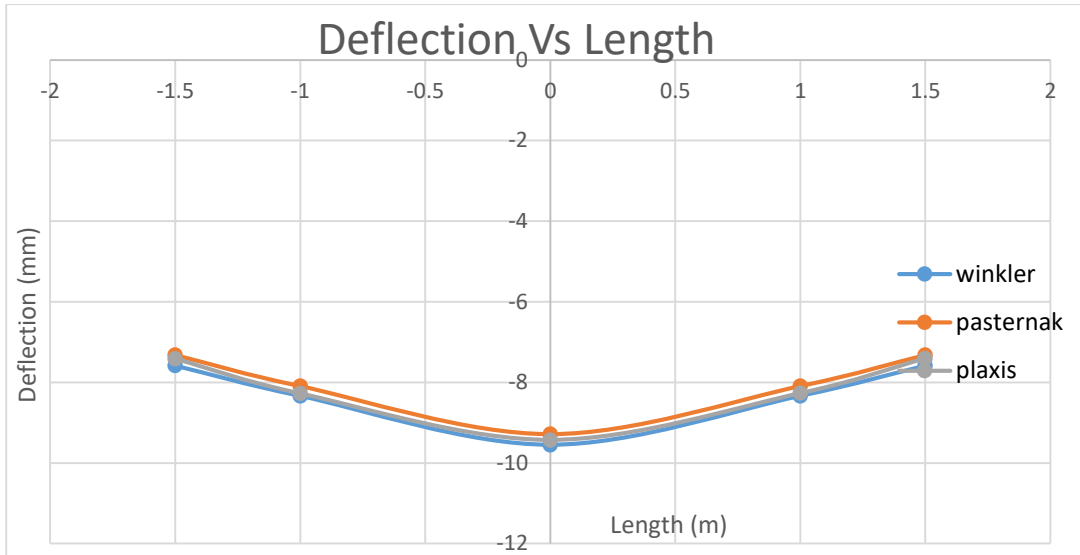


Fig 4.32. Short strip plate on medium stiff clay soil subjected to point load.

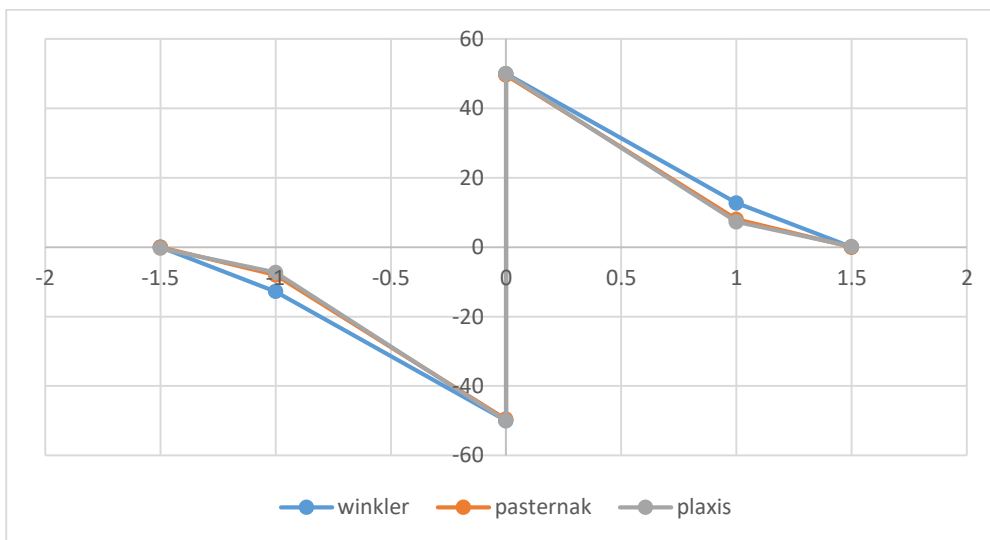
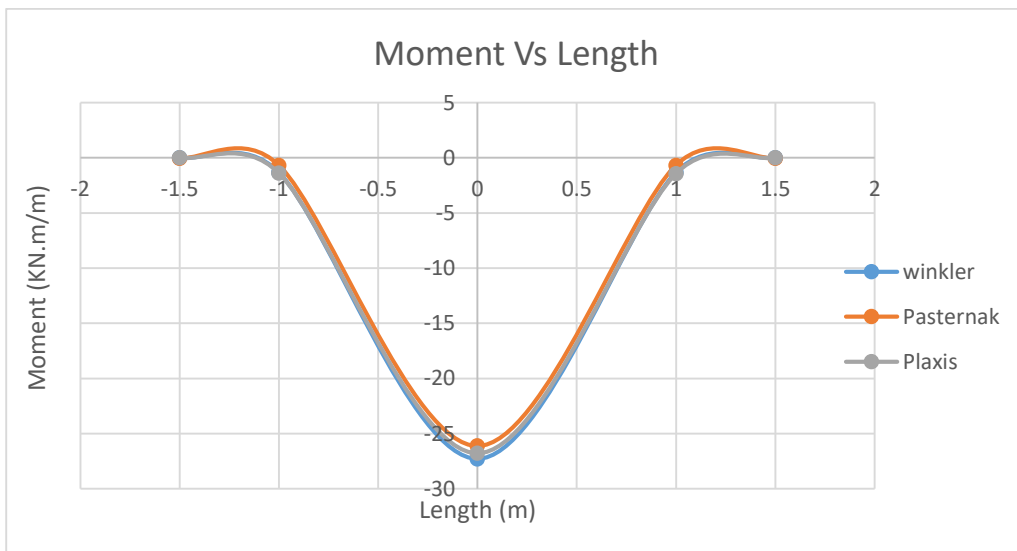
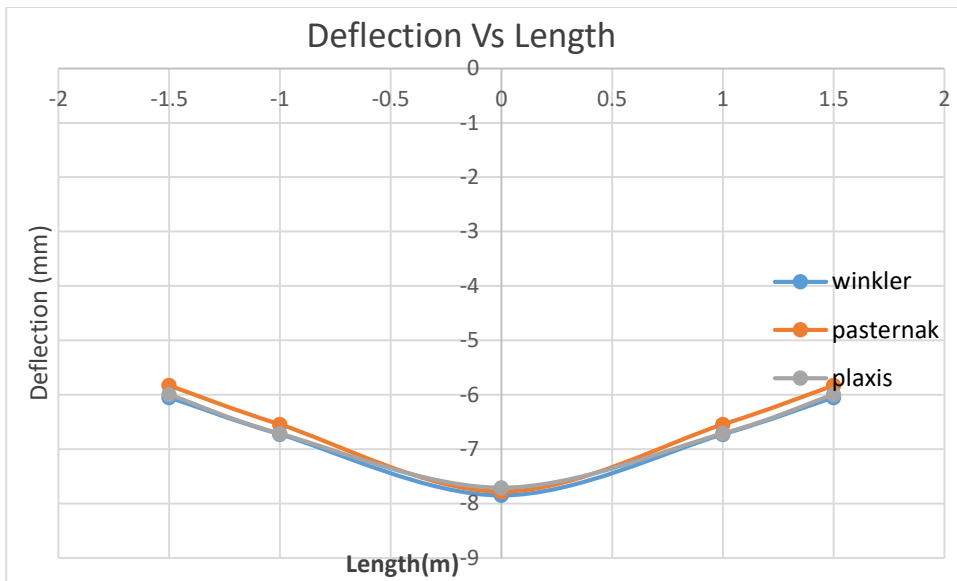


Fig 4.33 Short strip plate on medium dense sand soil subjected to point load

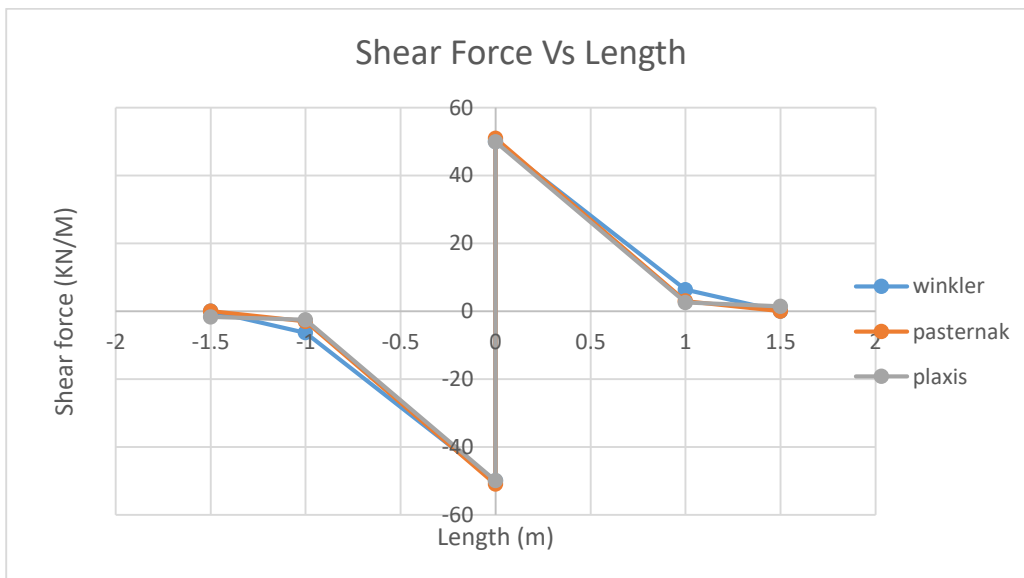
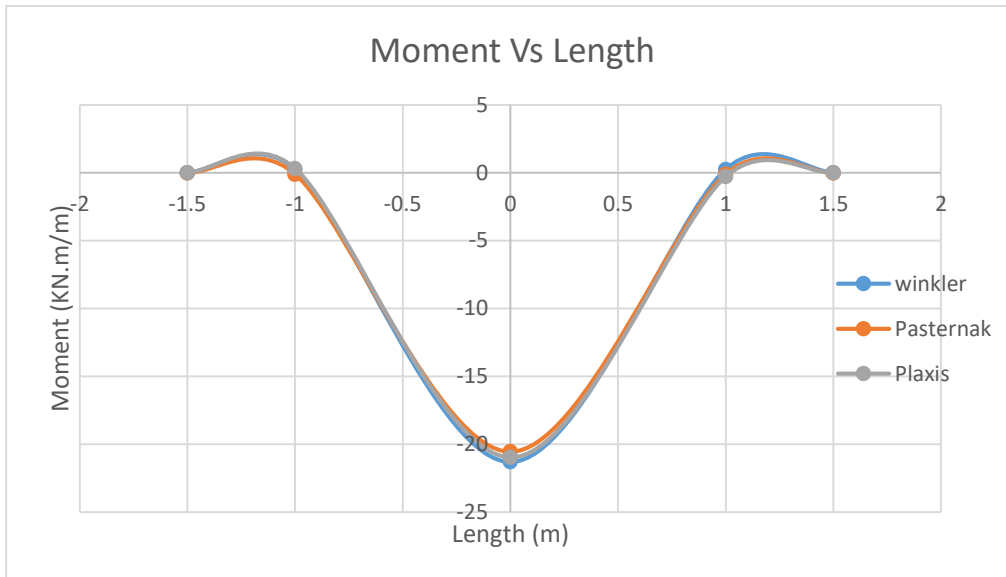
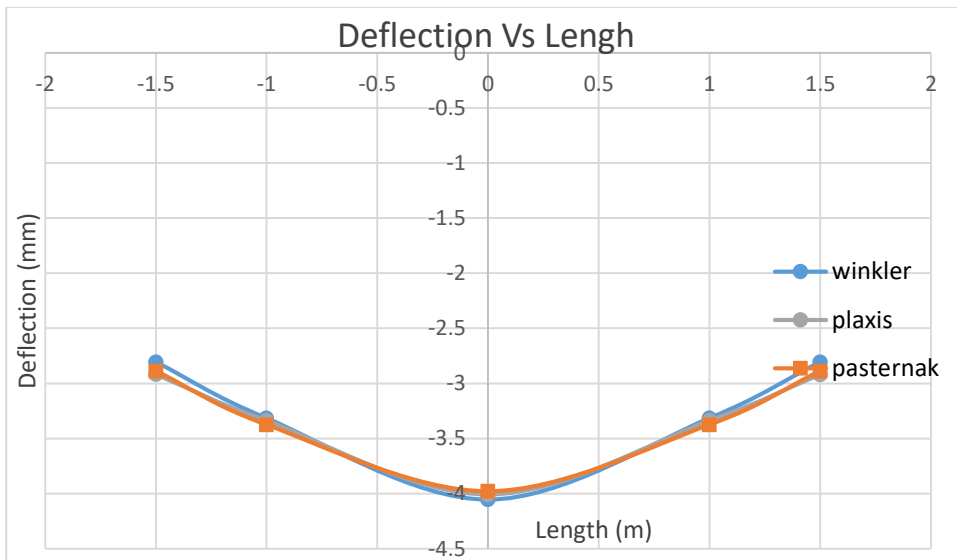


Fig 4.34. Short strip plate on stiff clay soil subjected to point load

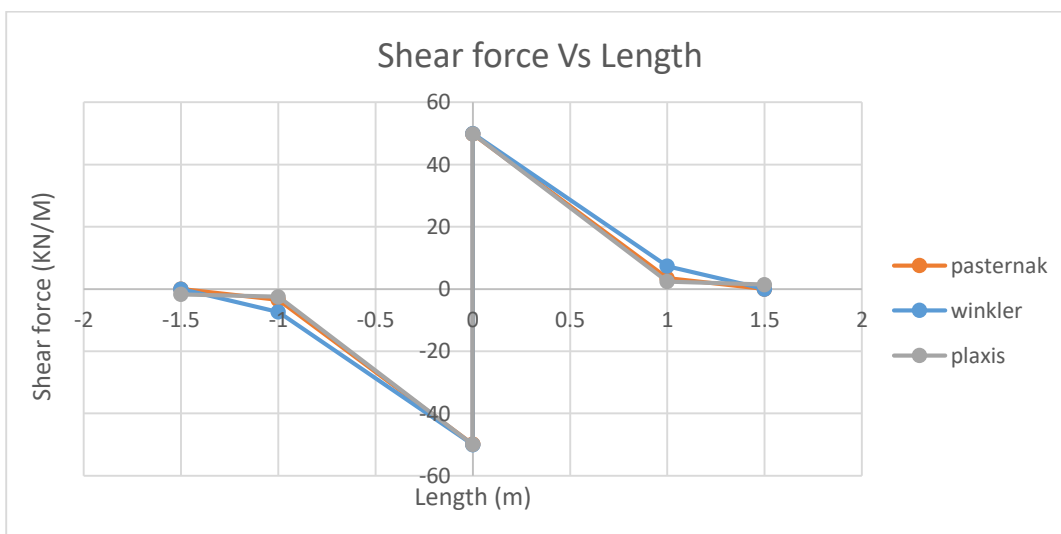
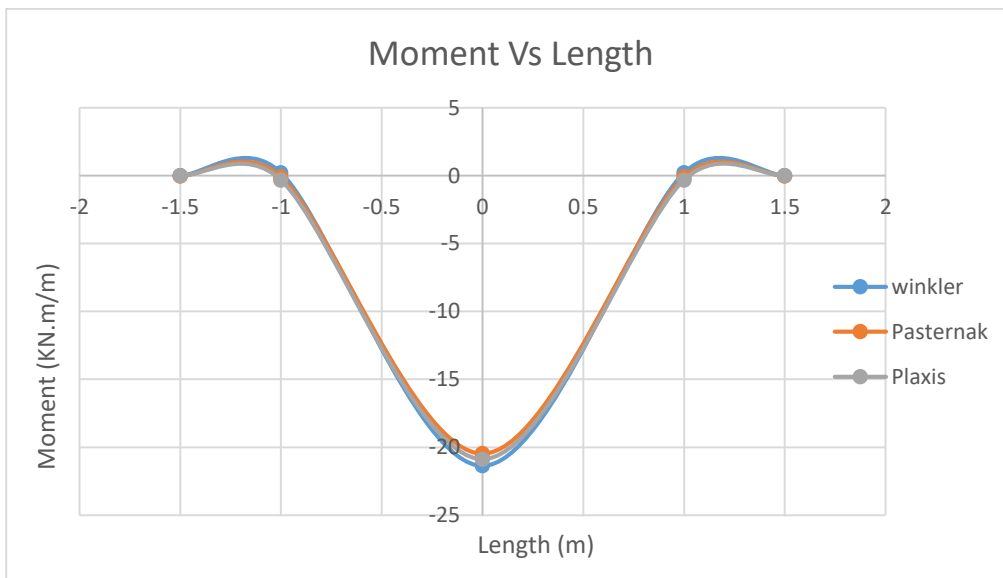
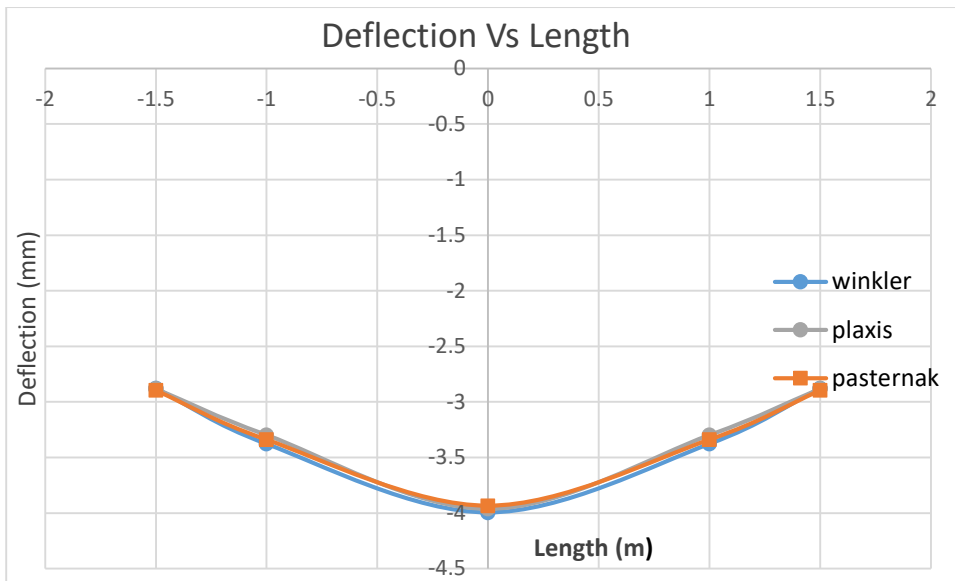


Fig 4.35. Short strip plate on dense sand soil subjected to point load

2. Uniformly Distributed load

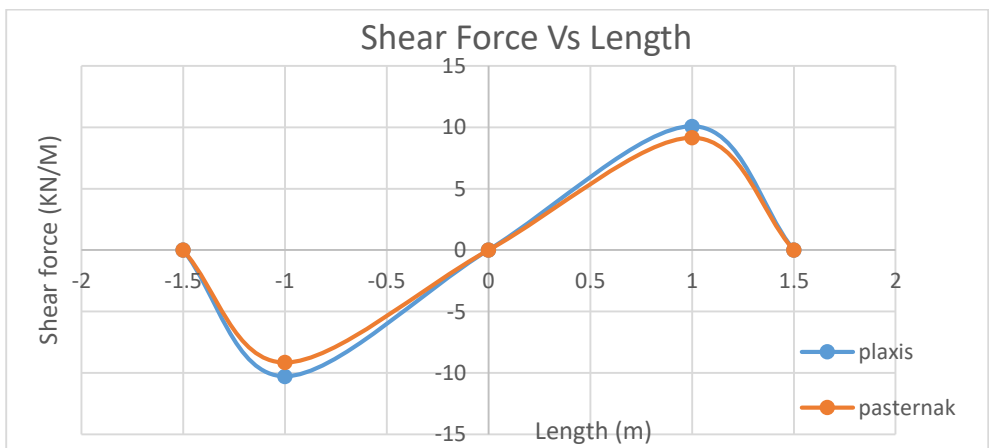
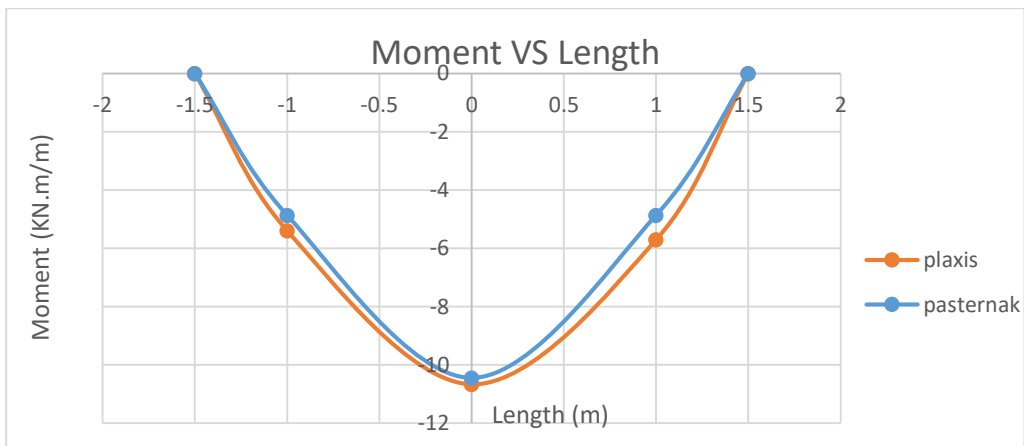
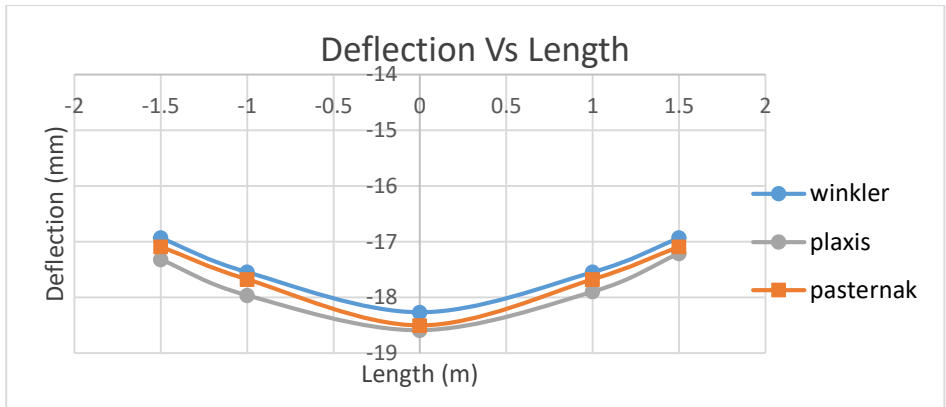
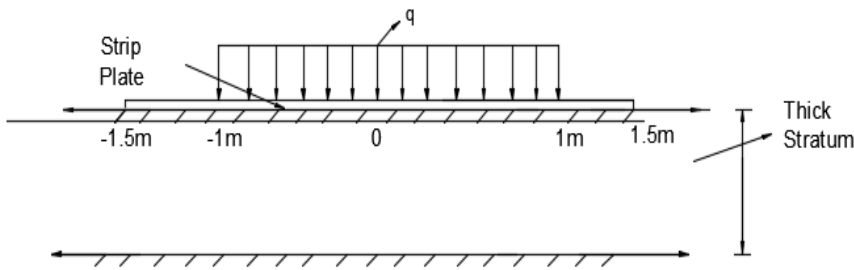


Fig 4.36 Short strip plate on soft clay soil subjected to uniformly distributed load

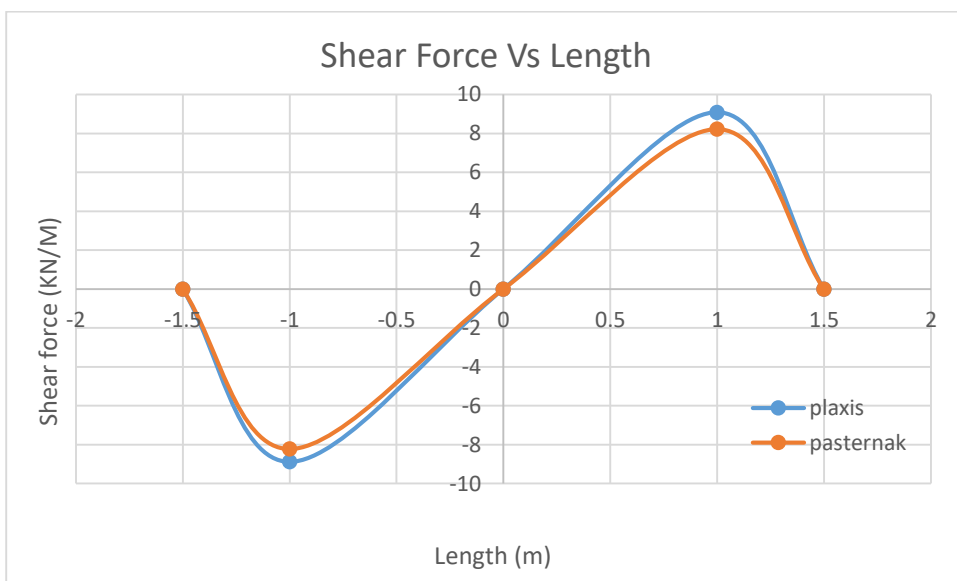
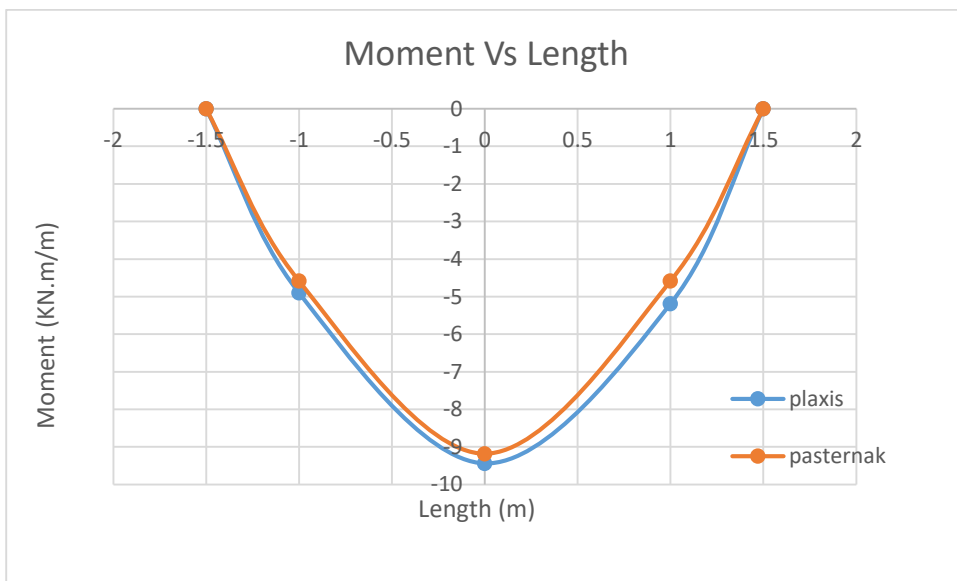
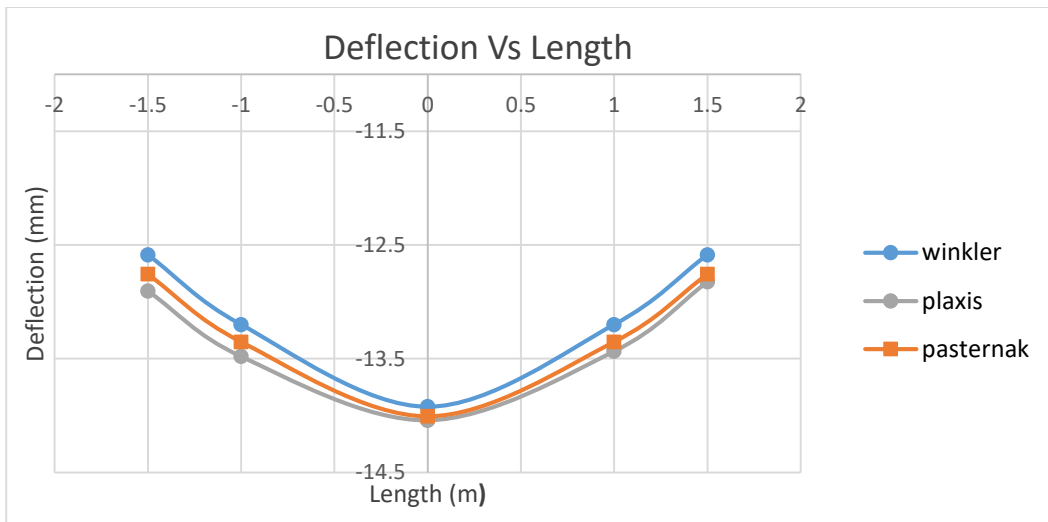


Fig 4.37 Short strip plate on loose sand soil subjected to uniformly distributed load

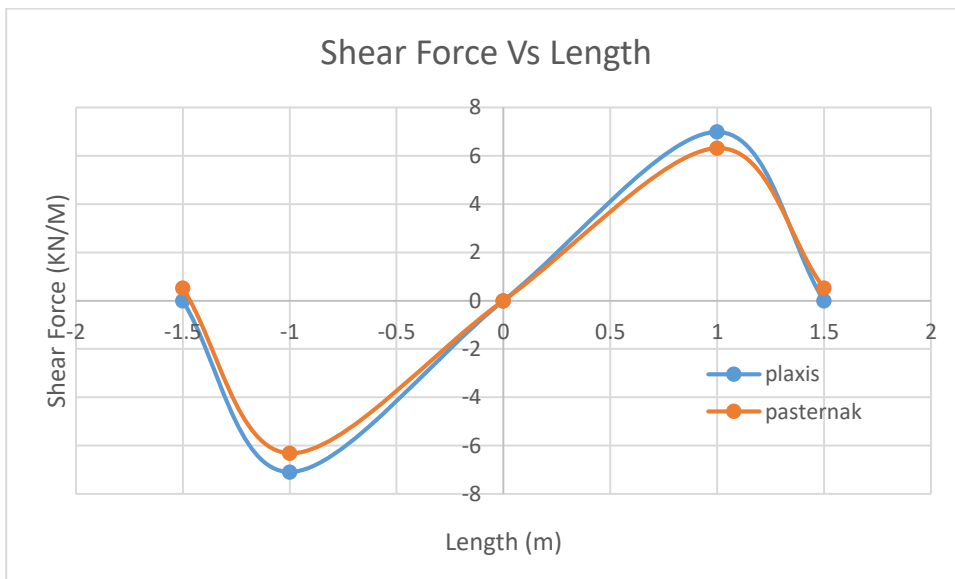
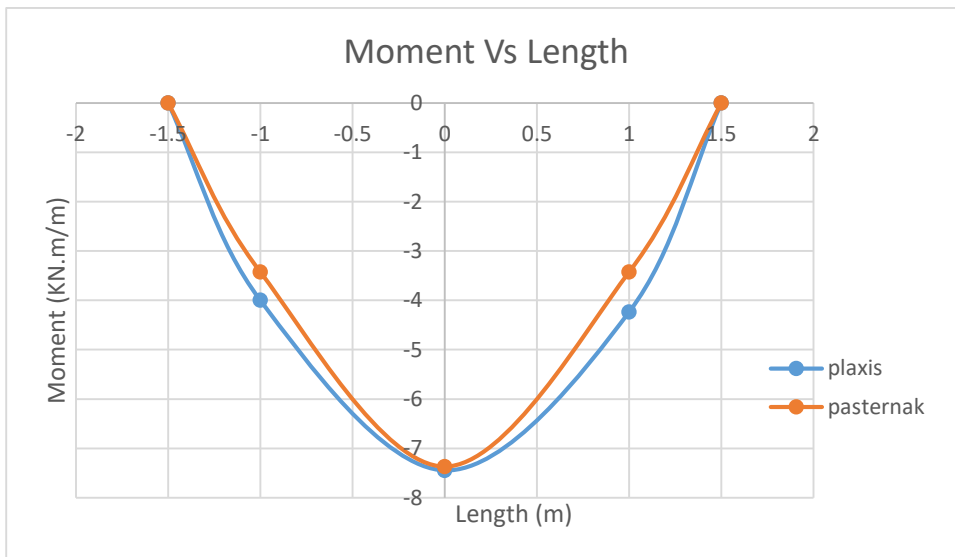
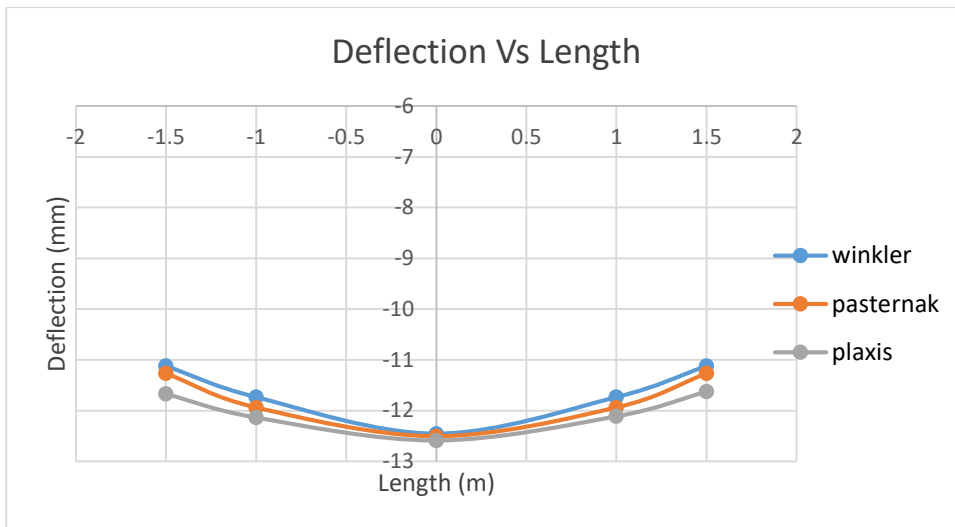


Fig 4.38 Short strip plate on medium stiff clay soil subjected to uniformly distributed load

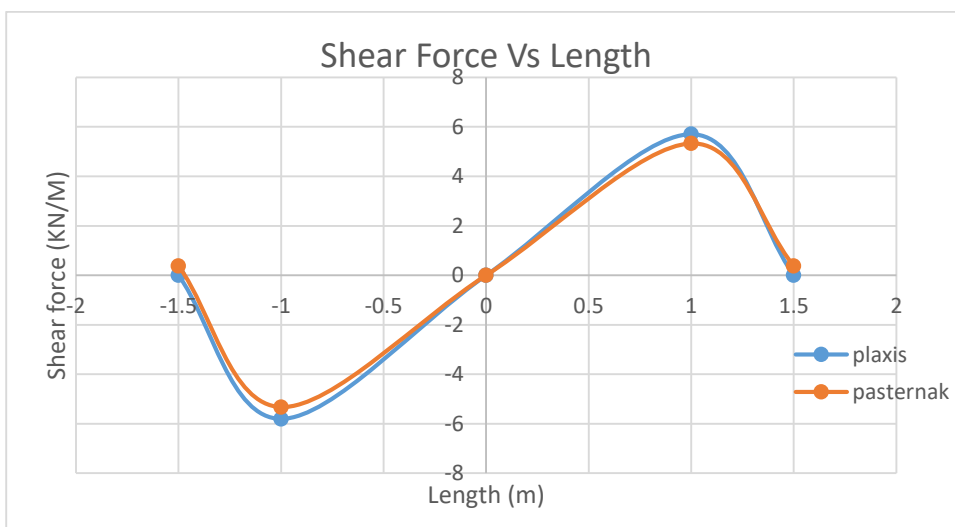
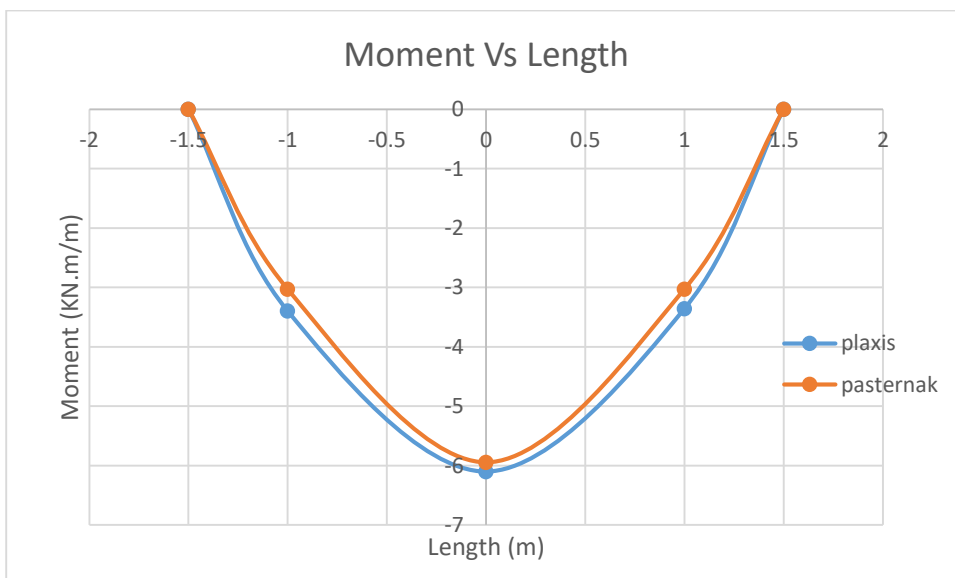
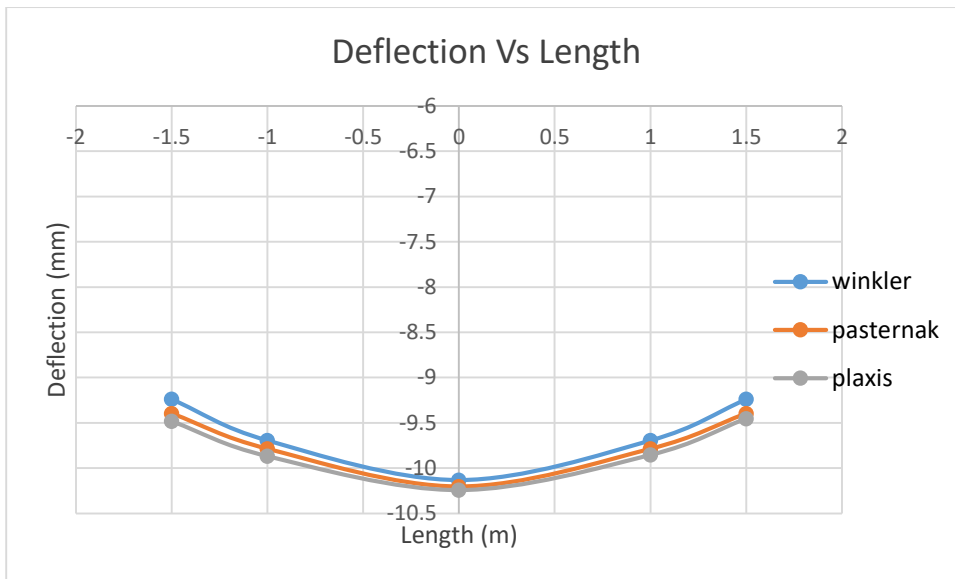


Fig 4.39 Short strip plate on medium dense sand soil subjected to uniformly distributed load

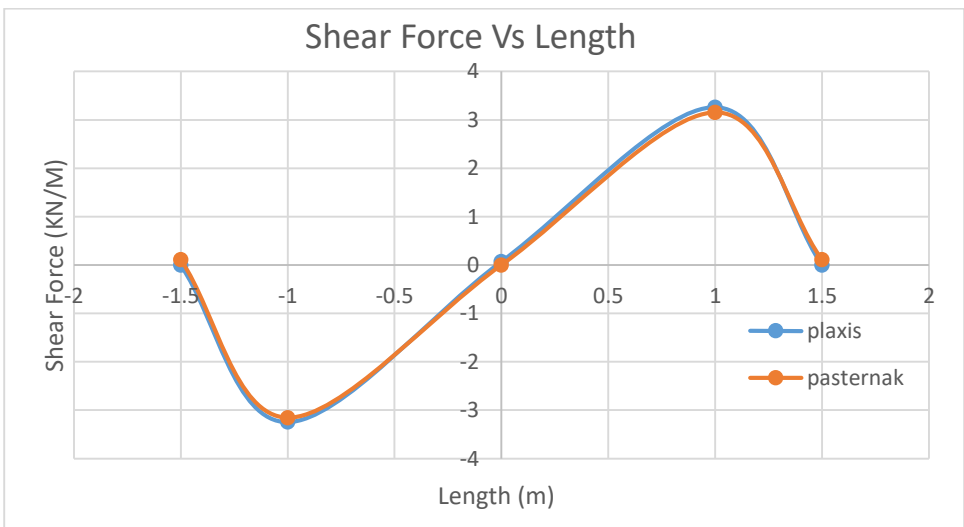
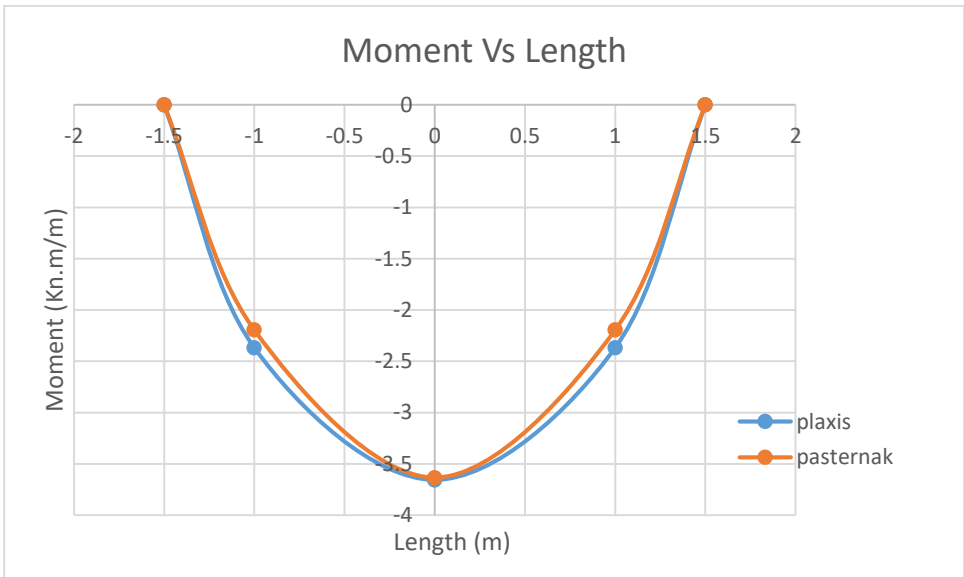
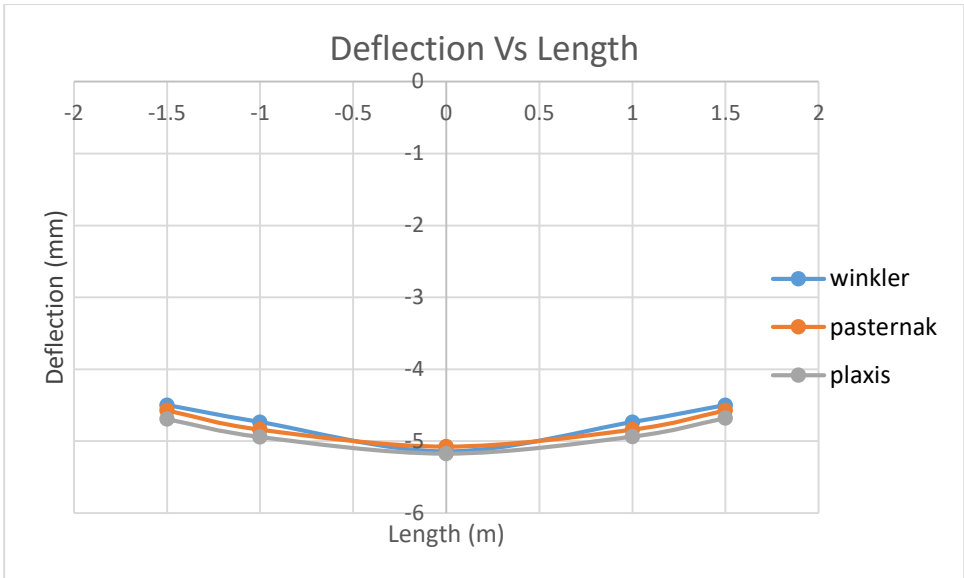


Fig 4.40 Short strip plate on stiff clay soil subjected to uniformly distributed load

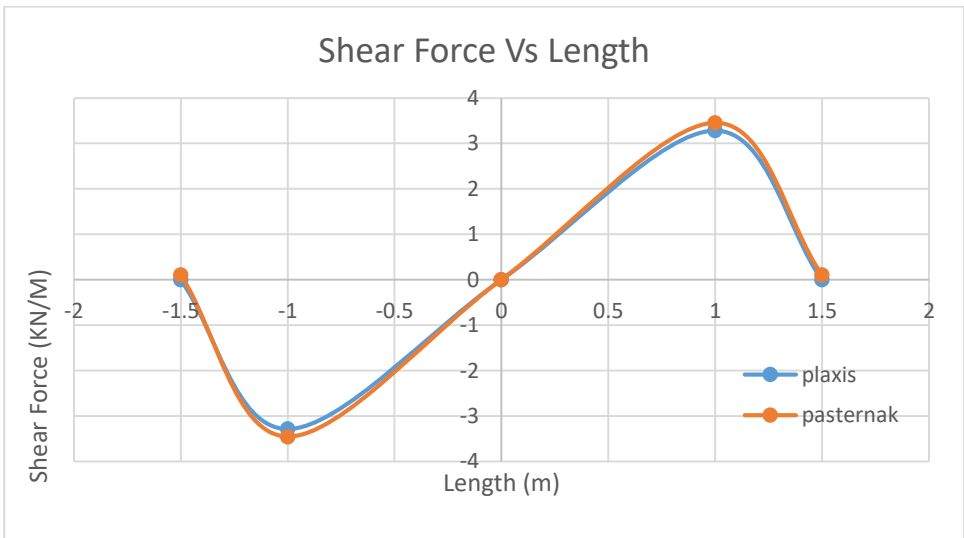
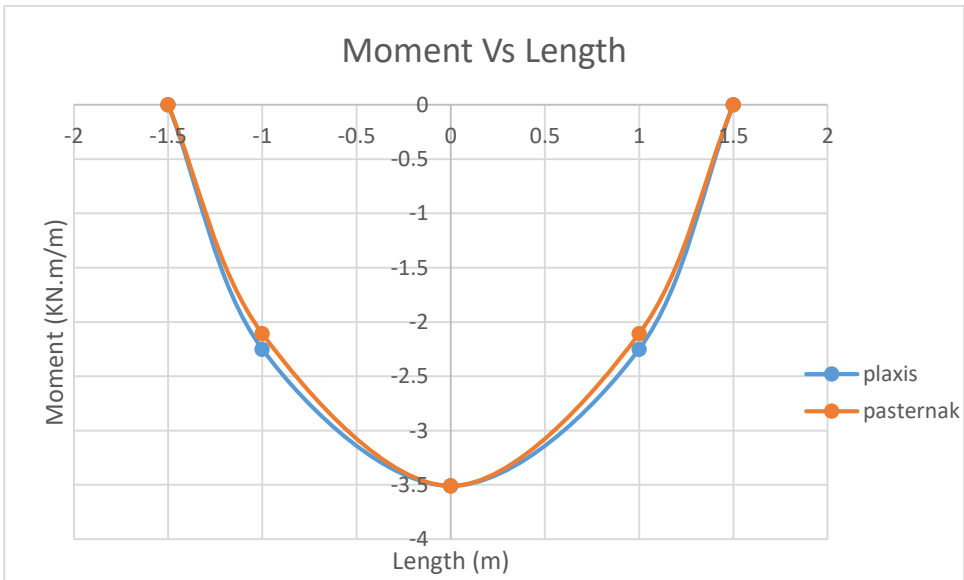
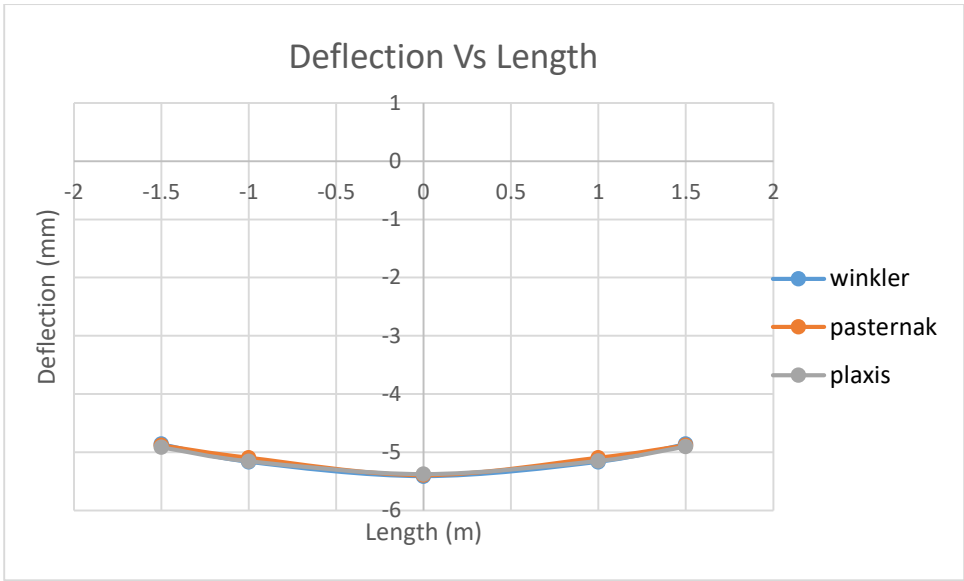


Fig 4.41 Short strip plate on dense sand soil subjected to uniformly distributed load

As could be observed from the figures, the deflection shows little deviation throughout the plate in all models. For some particular soils, the maximum deflection obtained by Winkler type model gives a slightly higher value in the case of concentrated vertical load. Whereas, the maximum deflection is slightly lower in the case of uniformly distributed load. It should be noted that the deflection for the Kerr Equivalent Pasternak model is in an excellent compliance with the Plaxis 2D model outputs.

When considering the maximum bending moments, the deviations are substantially smaller than those of the maximum deflections. As it was seen for long length plates, for soils with higher modulus of elasticity the deviations in deflection and bending moment is smaller for both models. Both models show little, if any, divergence in the case of shear force.

Chapter Five

5. Conclusion and recommendation

Certainly the major part of this study has been centered around the application of recent generalized continuum model developed by Worku (Winkler-type and Kerr-Equivalent Pasternak model), for the analysis of long and short length of strip plates. Closed form relations for the plate response under different loading conditions have been obtained, which at times are lengthy. A spreadsheet program is developed to compute the internal actions for selected loading conditions. The resulting classical models are compared with Finite –Element based model results (Plaxis 2D). In the process, the analytical models are calibrated to obtain the adjusted formulas for the model parameters.

5.1 Conclusions

The following major conclusions are drawn:

1. Closed form solutions are obtained for the deflection, bending moment and shear force for strip plate supported by elastic foundation under selected basic loading types.
2. For the two parameter subgrade model, three possible cases of the general solution were obtained. Only one case ($\rho > \frac{1}{4}$) is important for all practical cases that would be encountered.
3. A stiffer soil (higher modulus of elasticity of soil) reduces the plate deflection and bending moment.
4. For long strip plates, there is very small deviation in the maximum deflection for both models but eventually increases when moving away from mid span. This is more pronounced in the Winkler-Type model, due to the fact that the assumption when formulating the model does not include shear interaction behavior of the soil. Moreover, since the calibration factor is linked solely to the maximum deflection, it is certain to have deviations when moving away from the maximum deflection.
5. In short strip plate's deviation of deflection between the models and the Plaxis 2D is very small. Thus, for practical purpose one can also use a Winkler Type Model for analysis purposes.
6. The deviation in the bending moment modestly increases when moving away from the mid span but ultimately decreases before reaching the edge of the long strip plate. In

contrary, for short strip plates the models are concurrent with slight deviation with FE based software.

7. The shear force is invariably the same for all models. It implies that it is little affected by the type of model used.
8. Generally, in all cases using the recommended calibration factors provided in Table 4.1-4.4, the Kerr Equivalent Pasternak type model gives more compatible results to the Plaxis 2D output than the Winkler Type model. Therefore, by using this model and the accompanying spreadsheet program, practical analysis of strip plates on an elastic foundation can be easily performed.

5.2 Recommendation

As the scope of this work is limited to thin strip plates subjected to selected symmetrical loading conditions, the following actions could possibly strengthen, enhance and expand the application of this research for future work.

1. A homogeneous, isotropic soil is considered for this research but soils are inherently anisotropic. Thus this model would be strengthened through the provision for anisotropic soil behavior.
2. Selected symmetrical loading types are considered to analyze the strip plate. For future works other types loading (e.g. unsymmetrical loading) are recommended. This can be done by using the principles of superposition method.
3. This research can be expanded by developing a technique to facilitate the analysis of thick rectangular plates with holes or with some irregularity.
4. Furthermore, mathematical formulation has been done for strip plate subjected to concentrated moments. However, a numerical illustration was not conducted for this problem. For future research, it is best advised to conduct numerical illustration based on the existing mathematical formulation.
5. A Finite Element program can be written incorporating with the developed mathematical model for strip plate, that can aid future works in the field.

REFERENCES

- Bowles, J.E. (1979). "Foundation Analysis and Design," *Fifth Edition*, New York.
- Degu, Y. (2008). "Analysis of Beams on Elastic Foundations Using Newly Developed Generalized Continuum Based Sub-Grade Model," *Department of Civil and Environmental Engineering, Addis Ababa Institute of Technology*.
- Fai, M.K. (1988). "Elastic Solution for Rectangular and Circular Plates on Non-Homogeneous Soil Foundation," *University of Hong Kong*.
- Hetényi. M. (1946). "Beams on Elastic Foundation," *University of Michigan Press, Ann Arbor, Michigan*.
- Horvath J.S. (1983). "Modulus of Subgrade Reaction: New Perspective," *Journal of Geotechnical Engineering*, Vol. 109, No. 12.
- Horvath J.S. (2011). "A Practical Subgrade Model for Improved Soil-Structure Interaction Analysis: Parameter Assessment," *Manhattan College School of Engineering Civil and Environmental Engineering Department Bronx, New York, U.S.A.*
- Huang, M.H. and Thambiratnam, D.P. (2001). "Analysis of Plate on Elastic Supports and Elastic Foundation by Finite Strip Method," *Journal of Computers and Structures*.
- Kerr A.D. (1985). "On the Determination of Foundation Model Parameters," *Journal of Geotechnical Engineering*, Vol. 1, No. 11.
- Kneifati M.C. (1985). "Analysis of plate on Kerr Foundation Model," *Journal of Engineering Mechanics*, Vol.111, No. 11.
- Kreysig, E. (2006). "Advance Engineering Mathematics," *Ninth Edition, John Wiley & Sons Inc., Columbus, Ohio, U.S.A.*
- Reddy, J.N. (2007). "Theory and Analysis of Elastic Plates and Shells," *Second Edition, London New York*.
- Rajapakse and Selvaduri. (1991). "Response of Circular Footings and Anchor Plates in Non - Homogenous Elastic Soils".

- Selvaduri, A.P.S. (1979). "Elastic Analysis of Soil Foundation Interaction," *Elsevier scientific Publishing Company*, Vol. 17, Amsterdam, The Netherlands.
- Straughan, W.T. (1990). "Analysis of Plates on Elastic Foundation," *Texas Tech University*.
- Timoshenko, S. (1989). "Theory of Plates and Shells," *Second Edition, McGraw-Hill book company*.
- Worku, A. (2010). "Part I: A Generalized Formulation of Continuum Models for Elastic Foundations. *Advances in Analysis, Modeling & Design*, GeoFlorida.
- Worku, A. and Degu Y. (2010). "Part II: Application of Newly Derived and Calibrated Subgrade Models in the Analysis of Beams on Elastic Foundations," *Journal of Eng. Mechanics*, GeoFlorida.
- Worku, A. (2013). "Calibrated Analytical Formulas for Foundation Model Parameters," *International Journal of Geomechanics*, Vol. 13, No,4.
- Worku, A. (2014). "Development of a Calibrated Pasternak Foundation Model for Practical Use," *International Journal of Geotechnical Engineering*, Vol. 8, No. 1.
- Worku, A. (2017). "Effective Use of Winkler's Single-Parameter Subgrade Model to Capture the Missing Shear Interaction," *Not Published*.

ANNEX

ANNEX A

A.1 General Solution of strip Plate on a single parameter subgrade

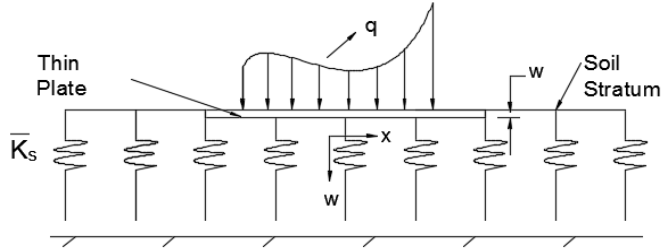


Fig A1 strip plate resting on Winkler's model

The homogenous differential equation for displacement w of strip plate on Winkler foundation is given by

$$D \frac{d^4 w_0}{dx^4} + \bar{k}_s w_0 = 0 \quad (\text{A1.1})$$

Since equation (A1.1) is ODE the Solution form will be

$$w = e^{\lambda x}$$

$$(w)^{iv} = \lambda^4 e^{\lambda x}$$

Substituting back in Equation (A1.1)

$$\lambda^4 e^{\lambda x} + \frac{\bar{k}_s}{D} e^{\lambda x} = 0$$

$$\lambda^4 = -\frac{\bar{k}_s}{D}$$

$$\text{Let } \lambda^2 = \mu, \text{ Then } \mu_{1,2} = \pm i \sqrt{\frac{\bar{k}_s}{D}}$$

Since $\lambda^2 = \mu$

$$\lambda = \pm \sqrt{i} \sqrt[4]{\frac{\bar{k}_s}{D}} \quad \text{and} \quad \lambda = \pm \sqrt{-i} \sqrt[4]{\frac{\bar{k}_s}{D}}$$

Therefore

$$\lambda_{1,2} = \sqrt{i} \sqrt[4]{\frac{\bar{k}_s}{D}} \quad \text{and} \quad -\sqrt{i} \sqrt[4]{\frac{\bar{k}_s}{D}} \quad (\text{A1.2})$$

$$\lambda_{3,4} = \sqrt{-i}^4 \sqrt[4]{\frac{k_s}{D}} \text{ and } -\sqrt{-i}^4 \sqrt[4]{\frac{k_s}{D}} \quad (\text{A1.3})$$

$$\begin{aligned} \text{With } \sqrt{i} &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i & \text{and } \sqrt{-i} &= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i \\ &= \frac{1}{\sqrt{2}}(1+i) & &= \frac{1}{\sqrt{2}}(1-i) \end{aligned}$$

$$\text{Thus } \lambda_1 = \frac{1}{\sqrt{2}}(1+i) \sqrt[4]{\frac{k_s}{D}} \rightarrow \lambda_1 = (1+i) \sqrt[4]{\frac{k_s}{D}}$$

$$\lambda_2 = -(1+i) \sqrt[4]{\frac{k_s}{D}}, \quad \lambda_3 = (1-i) \sqrt[4]{\frac{k_s}{D}} \text{ and } \lambda_4 = (1-i) \sqrt[4]{\frac{k_s}{D}}$$

So the general Solution will be

$$w(x) = c_1 e^{(1+i)\sqrt[4]{\frac{k_s}{D}}x} + c_2 e^{-(1+i)\sqrt[4]{\frac{k_s}{D}}x} + c_3 e^{-(1-i)\sqrt[4]{\frac{k_s}{D}}x} + c_4 e^{(1-i)\sqrt[4]{\frac{k_s}{D}}x}$$

$$\text{Let } \beta^4 = \frac{k_s a^4}{64D} \text{ Thus } \frac{2\beta}{a} = \sqrt[4]{\frac{k_s}{D}} = \xi$$

$$w(x) = a_1 e^{(1+i)\xi x} + a_2 e^{(-1-i)\xi x} + a_3 e^{(1-i)\xi x} + a_4 e^{(i-1)\xi x}$$

$$\text{Where } e^{i\lambda} = \cos \lambda x + i \sin \lambda x \quad e^{-i\lambda} = \cos \lambda x - i \sin \lambda x$$

$$w(x) = a_1 e^{\xi x} e^{\xi x i} + a_2 e^{-\xi x} e^{-\xi x i} + a_3 e^{\xi x} e^{-\xi x i} + a_4 e^{-\xi x} e^{\xi x i}$$

$$w(x) = e^{\xi x} (a_1 e^{\xi x i} + a_3 e^{-\xi x i}) + e^{-\xi x} (a_2 e^{-\xi x i} + a_4 e^{\xi x i})$$

$$w(x) = \left(e^{\xi x} (a_1 \cos \xi x + i a_1 \sin \xi x + a_3 \cos \xi x - i a_3 \sin \xi x) + e^{-\xi x} (a_2 \cos \xi x - i a_2 \sin \xi x + a_4 \cos \xi x - i a_4 \sin \xi x) \right)$$

$$w(x) = \left(e^{\xi x} ((a_1 + a_3) \cos \xi x + (a_1 i - a_3 i) \sin \xi x) + e^{-\xi x} ((a_2 + a_4) \cos \xi x + (a_4 i - a_2 i) \sin \xi x) \right)$$

$$\text{Let } a_1 + a_3 = A_1 \quad a_2 + a_4 = A_3$$

$$a_1 i - a_3 i = A_2 \quad a_4 i - a_2 i = A_4$$

$$w(x) = e^{\xi x} (A_1 \cos \xi x + A_2 \sin \xi x) + e^{-\xi x} (A_3 \cos \xi x + A_4 \sin \xi x)$$

(A1.4)

With the introduction of

$$e^{\xi x} = \cosh \xi x + \sinh \xi x \quad \text{and} \quad e^{-\xi x} = \cosh \xi x - \sinh \xi x$$

$$w(x) = \begin{pmatrix} A_1 \cosh \xi x \cos \xi x + A_1 \sinh \xi x \cos \xi x + A_2 \cosh \xi x \sin \xi x + A_2 \sinh \xi x \sin \xi x \\ A_3 \sinh \xi x \cos \xi x - A_3 \cosh \xi x \cos \xi x + A_4 \cosh \xi x \sin \xi x - A_4 \sinh \xi x \sin \xi x \end{pmatrix}$$

$$w(x) = \begin{pmatrix} \sin \xi x ((A_2 - A_4) \sinh \xi x + (A_2 + A_4) \cosh \xi x) + \\ \cos \xi x ((A_1 - A_3) \sinh \xi x + (A_1 + A_3) \cosh \xi x) \end{pmatrix}$$

$$A_2 + A_4 = C_1 \quad A_1 + A_3 = C_3$$

$$A_2 - A_4 = C_2 \quad A_1 - A_3 = C_4$$

$$w(x) = \sin \xi x (C_1 \cosh \xi x + C_2 \sinh \xi x) + \cos \xi x (C_3 \cosh \xi x + C_4 \sinh \xi x) \quad (\text{A1.5})$$

Hence, Equation (A1.4) and Equation (A1.5) are two alternative forms of solution to homogenous differential Equation (A1.1)

ANNEX B

B.1 Finite strip plate on Winkler foundation subjected to vertical load

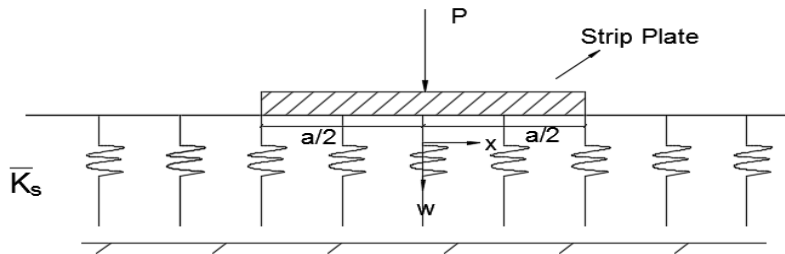


Fig B1. Finite Strip plate subjected to vertical loading

Due to the complexity of the mathematical equation of a finite strip plate on single parameter Hetenyi (1946) superposition method is used to solve this equation. As it is shown in Figure B.1 we assume that an infinite strip plate is subjected to a given loading, as a result of this loading certain values of deflection, slope, Moment and shear will be produced at the end points. To fulfilled the end condition, the elastic curve can be modified by superposing the loaded strip plate two pairs of concentrated forces and moment (M_{oA} , P_{oA} , and M_{oB} , P_{oB}).

Where M_{oA} , P_{oA} , and M_{oB} , P_{oB} are end condition forces.

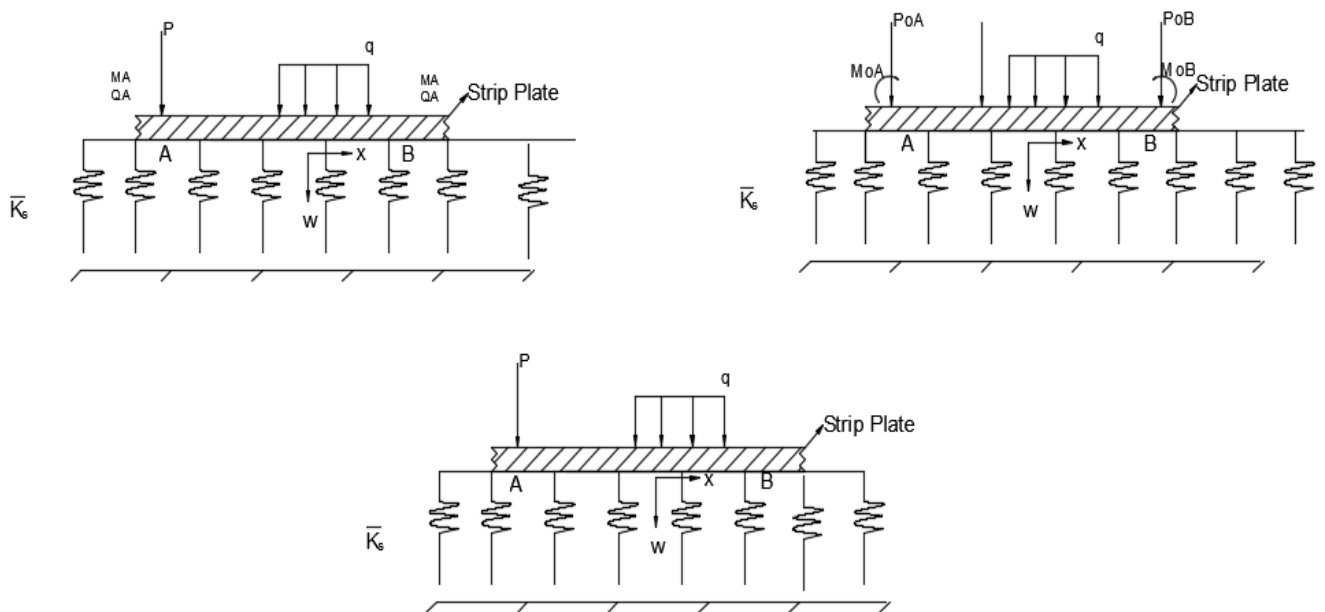


Fig B.2 End condition of the original case

The corrective value of this terms must be determined from simultaneous equations, representing the simultaneous fulfillment of the end conditions at point A and B (moment and share at point A and B must be zero). By using equation (3.26),(3.27),(3.31) and (3.32) the end conditions can be written as

$$\left. \begin{aligned} M_A + \xi + \frac{P_{OB}}{4\xi} e^{-\xi x} (\cos \xi x - \sin \xi x) + \frac{M_{oA}}{2} + \frac{M_{oB}}{2} e^{-\xi x} (\cos \xi x) &= 0 \\ Q_A - \frac{P_{OA}}{4\xi} + \frac{P_{OB}}{4\xi} e^{-\xi x} (\cos \xi x) - \xi \frac{M_{oA}}{2} - \xi \frac{M_{oB}}{2} e^{-\xi x} (\cos \xi x + \sin \xi x) &= 0 \\ M_B + \frac{P_{OA}}{4\xi} e^{-\xi x} (\cos \xi x - \sin \xi x) + \frac{P_{OB}}{4\xi} + \frac{M_{oA}}{2} e^{-\xi x} (\cos \xi x) + \frac{M_{oB}}{2} &= 0 \\ Q_B - \frac{P_{OA}}{2} e^{-\xi x} (\cos \xi x) + \frac{P_{OB}}{2} - \frac{2\beta}{a} \frac{M_{oA}}{2} e^{-\xi x} (\cos \xi x + \sin \xi x) - \xi \frac{M_{oB}}{2} &= 0 \end{aligned} \right\} \quad (B1.1)$$

In order to simplify the solution of these equations the original loading will be resolved into two parts, symmetric part and antisymmetrical one

Hetenyi (1946) solve this case and come up the following expression

$$\begin{aligned} M_A &= M'_A + M''_A, & M_B &= M'_B - M''_B \\ Q_A &= Q'_A + Q''_A, & Q_B &= -Q'_B + Q''_B \end{aligned} \quad (B1.2)$$

Where: M_A, Q_A, M_B and Q_B moment and shear at point A and B due to original loading case

M'_A, Q'_A, M'_B & Q'_B moment and shear at at point A and B due symmetrical loading case

M''_A, Q''_A, M''_B & Q''_B moment and shear at point A due to antisymmetrical loading case

By superposing the symmetrical and antisymmetrical components the following end-condition forces will be obtained

$$\begin{aligned} P_{oA} &= P'_O + P''_O & P_{oB} &= P'_O + P''_O \\ M_{oA} &= M'_O + M''_O & M_{oB} &= M'_O + M''_O \end{aligned} \quad (B1.3)$$

In order to remove the moment and shear at A and B on infinite beams P_{oA}, M_{oA}, P_{oB} and M_{oB} should produce all together $-M_A, -Q_A$ at A and $-M_B, -Q_B$ at pint B, if P'_O and M'_O produce altogether $-M'_A, -Q'_A$ at A and $-M'_B, Q'_B$ at B, and P''_O and M''_O produce $-M''_A, -Q''_A$ at A and $-M''_B, +Q''_B$ at B. From this relation the end conditioning forces can be determined separately for symmetrical and antisymmetrical cases

For symmetrical case

$$\left. \begin{aligned} p'_{o} &= \frac{4e^{2\beta}}{2(\sinh 2\beta + \sin 2\beta)} \left(Q'_{A} (1 + e^{-2\beta} (\cos 2\beta)) + \xi M'_{A} (1 - e^{-2\beta} (\cos 2\beta + \sin 2\beta)) \right) \\ M'_{o} &= \frac{1}{\xi} \frac{-e^{2\beta}}{(\sinh 2\beta + \sin 2\beta)} \left(Q'_{A} (1 + e^{-2\beta} (\cos 2\beta - \sin 2\beta)) + \xi M'_{A} (1 - e^{-2\beta} (\cos 2\beta)) \right) \end{aligned} \right\} \quad (\text{B1.4})$$

For antisymmetrical case

$$\left. \begin{aligned} p''_{o} &= \frac{4e^{2\beta}}{2(\sinh 2\beta - \sin 2\beta)} \left(Q''_{A} (1 - e^{-2\beta} (\cos 2\beta)) + \xi M''_{A} (1 + e^{-2\beta} (\cos 2\beta + \sin 2\beta)) \right) \\ M''_{o} &= \frac{1}{\xi} \frac{-e^{2\beta}}{(\sinh 2\beta - \sin 2\beta)} \left(Q''_{A} (1 - e^{-2\beta} (\cos 2\beta - \sin 2\beta)) + \xi M''_{A} (1 + e^{-2\beta} (\cos 2\beta)) \right) \end{aligned} \right\} \quad (\text{B1.5})$$

By applying equation (B1.3) one can obtain the end condition forces

$$\begin{aligned} P_{oA} &= \frac{4e^{2\beta}}{2(\sinh 4\beta^2 - \sin 4\beta^2)} \left[\begin{aligned} &\sinh 2\beta(Q_A) - \sinh 2\beta^*(e^{-2\beta} \cos 2\beta)(Q_B) + \\ &\xi \sinh 2\beta(M_A) - \xi \sinh 2\beta^* e^{-2\beta} (\cos 2\beta + \sin 2\beta)(M_B) + \\ &\sin 2\beta(Q_B) - \sin 2\beta^*(e^{-2\beta} \cos 2\beta)(Q_A) - \\ &\xi \sin 2\beta(M_B) + \xi \sin 2\beta e^{-2\beta} (\cos 2\beta + \sin 2\beta)(M_A) \end{aligned} \right] \\ P_{oB} &= \frac{4e^{2\beta}}{2(\sinh 4\beta^2 - \sin 4\beta^2)} \left[\begin{aligned} &-\sinh 2\beta(Q_B) + \sinh 2\beta^*(e^{-2\beta} \cos 2\beta)(Q_A) + \\ &\xi \sinh 2\beta(M_B) - \xi \sinh 2\beta^* e^{-2\beta} (\cos 2\beta + \sin 2\beta)(M_A) - \\ &-\sin 2\beta(Q_A) + \sin 2\beta^*(e^{-2\beta} \cos 2\beta)(Q_B) - \\ &\xi \sin 2\beta(M_A) + \xi \sin 2\beta e^{-2\beta} (\cos 2\beta + \sin 2\beta)(M_B) \end{aligned} \right] \\ M_{oA} &= \frac{1}{\xi} \frac{e^{2\beta}}{(\sinh 2\beta - \sin 2\beta)} \left[\begin{aligned} &-\sinh 2\beta(Q_B) + \sinh 2\beta^* e^{-2\beta} (\cos 2\beta - \sin 2\beta)(Q_A) + \\ &2\xi \sinh 2\beta(M_B) - 2\xi \sinh 2\beta^* e^{-2\beta} (\cos 2\beta)(M_A) - \\ &\sin 2\beta(Q_A) + \sin 2\beta^* e^{-2\beta} (\cos 2\beta - \sin 2\beta)(Q_B) - \\ &\xi \sin 2\beta(M_A) + \xi \sin 2\beta e^{-2\beta} (\cos 2\beta)(M_B) \end{aligned} \right] \\ M_{oB} &= \frac{1}{\xi} \frac{e^{2\beta}}{(\sinh 2\beta - \sin 2\beta)} \left[\begin{aligned} &-\sinh 2\beta(Q_A) - \sinh 2\beta^* e^{-2\beta} (\cos 2\beta - \sin 2\beta)(Q_B) + \\ &2\xi \sinh 2\beta(M_A) - 2\xi \sinh 2\beta^* e^{-2\beta} (\cos 2\beta)(M_B) + \\ &\sin 2\beta(Q_B) - \sin 2\beta^* e^{-2\beta} (\cos 2\beta - \sin 2\beta)(Q_A) - \\ &\xi \sin 2\beta(M_A) + \xi \sin 2\beta e^{-2\beta} (\cos 2\beta)(M_B) \end{aligned} \right] \end{aligned}$$

By substituting M_A, M_B, Q_A and Q_B from equation (3.27,3.28,3.32 and 3.33) the end force will be

$$P_{oA} = \frac{-e^{2\beta}}{\xi(\sinh 2\beta + \sin 2\beta)} \left[\begin{aligned} &\frac{p}{2} e^{-\beta} \cos \beta + \frac{p}{2} e^{-\beta} \cos \beta * e^{-2\beta} \cos 2\beta + \\ &\frac{p}{4} e^{-\beta} (\cos \beta - \sin \beta) - \frac{p}{4} e^{-\beta} (\cos \beta - \sin \beta) * e^{-2\beta} (\cos 2\beta + \sin 2\beta) \end{aligned} \right] \quad (B1.6)$$

$$M_{oA} = \frac{4e^{2\beta}}{2(\sinh 4\beta^2 - \sin 4\beta^2)} \left[\begin{aligned} &\frac{p}{2} e^{-\beta} \cos \beta + \frac{p}{2} e^{-\beta} \cos \beta * e^{-2\beta} (\cos 2\beta - \sin 2\beta) + \\ &\frac{p}{4} e^{-\beta} (\cos \beta - \sin \beta) - \frac{p}{4} e^{-\beta} (\cos \beta - \sin \beta) * e^{-2\beta} (\cos 2\beta) \end{aligned} \right] \quad (B1.7)$$

$$P_{oB} = \frac{-e^{2\beta}}{\xi(\sinh 2\beta + \sin 2\beta)} \left[\begin{aligned} &\frac{p}{2} e^{-\beta} \cos \beta + \frac{p}{2} e^{-\beta} \cos \beta * e^{-2\beta} \cos 2\beta - \\ &\frac{p}{4} e^{-\beta} (\cos \beta - \sin \beta) - \frac{p}{4} e^{-\beta} (\cos \beta - \sin \beta) * e^{-2\beta} (\cos 2\beta + \sin 2\beta) \end{aligned} \right] \quad (B1.8)$$

$$M_{oB} = \frac{4e^{2\beta}}{2(\sinh 4\beta^2 - \sin 4\beta^2)} \left[\begin{aligned} &\frac{p}{2} e^{-\beta} \cos \beta + \frac{p}{2} e^{-\beta} \cos \beta * e^{-2\beta} (\cos 2\beta - \sin 2\beta) - \\ &\frac{p}{4} e^{-\beta} (\cos \beta - \sin \beta) + \frac{p}{4} e^{-\beta} (\cos \beta - \sin \beta) * e^{-2\beta} (\cos 2\beta) \end{aligned} \right] \quad (B1.9)$$

Deflection at any point x will be the summation of deflections due to vertical load (Wp), due to end force at A (Wp_{oA}), due to end force at B (Wp_{oB}), due to end force at A (WM_{oA}), due to end force at B (WM_{oB}),

Where

$$\left. \begin{aligned}
w_p &= \frac{\xi}{2k_s} e^{-\frac{2\beta}{a}x} (\cos \xi + \sin \xi) \\
w_{P_{oA}} &= \frac{\xi P_{oB}}{2k_s} e^{-\xi(a/2+x)} (\cos \xi(a/2+x) + \sin \xi(a/2+x)) \\
w_{P_{oB}} &= \frac{\xi P_{oB}}{2k_s} e^{-\xi(a/2-x)} (\cos \xi(a/2-x) + \sin \xi(a/2-x)) \\
w_{M_{oA}} &= \frac{\xi^2 M_{oB}}{2k_s} e^{-\xi(a/2+x)} + \sin \xi(a/2+x) \\
w_{M_{oB}} &= \frac{\xi^2 M_{oB}}{2k_s} e^{-\xi(a/2-x)} + \sin \xi(a/2-x)
\end{aligned} \right\} \quad (B1.10)$$

After summation one can obtain the deflection equation at any point x will be

$$w(x) = \frac{p\xi}{2k_s} \frac{1}{\sinh 2\beta + \sin 2\beta} * \left[\begin{array}{l} \cosh \xi x \cos \xi(a-x) + \cos \xi x \cosh \xi(a-x) - \\ \sinh \xi x \sin \xi(a-x) + \sin \xi x \sinh \xi(a-x) + \\ 2 \cosh \xi x \cos \xi x \end{array} \right] \quad (B1.11)$$

Once the deflection is obtained the slop, moment and shear can be determined easily

$$\theta(x) = \frac{p\xi^2}{2k_s} \frac{1}{\sinh 2\beta + \sin 2\beta} \left[\begin{array}{l} \sinh \xi x (\cos \xi x + \cos \xi(a-x)) - \\ \sin \xi x (\cosh \xi x + \cosh \xi(a-x)) \end{array} \right] \quad (B1.12)$$

$$M(x) = \frac{p}{4\xi} \frac{1}{\sinh 2\beta + \sin 2\beta} \left[\begin{array}{l} \sinh \xi x (\sin \xi x - \sin \xi(a-x)) - \cosh \xi x (\cos \xi x + \cos \xi(a-x)) + \\ \sin \xi x (\sinh \xi x - \sinh \xi(a-x)) - \cos \xi x (\cosh \xi x + \cosh \xi(a-x)) \end{array} \right] \quad (B1.13)$$

$$v(x) = \frac{p}{2} \frac{1}{\sinh 2\beta + \sin 2\beta} \left[\cosh \xi x (\sin \xi x - \sin \xi(a-x)) + \cos \xi x (\sinh \xi x - \sinh \xi(a-x)) \right] \quad (B1.14)$$

ANNEX C

C.1 General Solution of a strip plate on Kerr Equivalent Pasternak Foundations

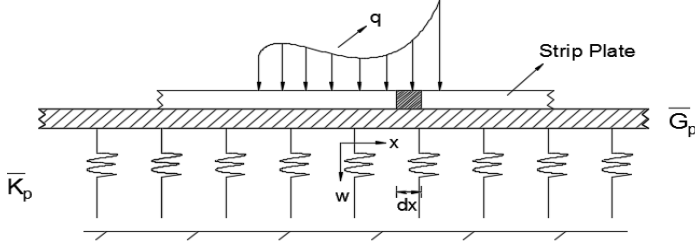


Fig C1 strip plate resting on Kerr Equivalent Pasternak model

The homogenous differential equation for displacement, w of strip plate on Kerr-Equivalent Pasternak model is illustrated in the figure (C.1) is given by

$$\frac{Dd^4w}{dx^4} - \frac{\bar{G}_p d^2w}{dx^2} + \bar{k}_p w = 0 \quad (C1.1)$$

$$\text{Where } D = \frac{Eh^3}{12(1-\nu^2)}$$

To solve this differential equation we assume some parameters

$$\left. \begin{aligned} \beta &= \sqrt[4]{\frac{\bar{k}_p a^4}{64D}}, \xi = \frac{2\beta}{a} = \sqrt[4]{\frac{\bar{k}_p}{D}}, \gamma = \sqrt{\frac{\bar{k}_p a^2}{\bar{G}_p}} \text{ and } \rho = \left(\frac{\beta}{\gamma}\right)^2 \\ \text{Let= } \rho &= \left[\sqrt[4]{\frac{\bar{k}_p a^4}{64D}} * \sqrt{\frac{\bar{G}_p}{\bar{k}_p a^2}} \right]^2, \rho = \frac{\bar{G}_p}{8\sqrt{D\bar{k}_p}} \end{aligned} \right\} \quad (C1.2)$$

Thus the differential equation becomes

$$\frac{d^4w}{dx^4} - \frac{64\beta^2 \rho}{a^2} \frac{d^2w}{dx^2} + 64 \frac{\beta^4}{a^4} w = 0 \quad (C1.3)$$

Since Equation (3.46) is an ODE with constant coefficients the solution is of the form $w = ce^{mx}$ substituting this expression one obtain the following characteristics polynomial

$$m^4 - \frac{64\beta^2 \rho}{a^2} m^2 + 64 \frac{\beta^4}{a^4} = 0$$

Let $m^2 = n$

$$n^2 - \frac{64\beta^2\rho}{a^2}n + 64\frac{\beta^4}{a^4} = 0$$

From quadratic equation

$$n_{1,2} = \frac{32\beta^2\rho}{a^2} \pm \frac{8\beta^2}{a^2} \sqrt{(16\rho^2 - 1)}$$

But $m^2 = n$

$$\begin{aligned} m_{1,2,3,4} &= \pm\sqrt{n} \\ &= \pm 2\sqrt{2} \frac{\beta}{a} \sqrt{\left(4\rho \pm \sqrt{(16\rho^2 - 1)}\right)} \end{aligned}$$

There are three possible cases of general solution which depends on whether

$$\rho > \frac{1}{4}, \rho = \frac{1}{4} \text{ or } \rho < \frac{1}{4}$$

Case I ($\rho < 1/4$)

In this case the term $\sqrt{(16\rho^2 - 1)}$ is complex number we can write it like this form

$$m_{1,2,3,4} = \pm 2\sqrt{2} \frac{\beta}{a} \sqrt{\left(4\rho \pm i\sqrt{(1-16\rho^2)}\right)}$$

$$\text{From algebra } \sqrt{y + iy} = \left[\frac{1}{2}(r + y) + \text{sign } zi \sqrt{\frac{1}{2}(r - y)} \right] \quad (\text{C1.4})$$

Where $r = \sqrt{y^2 + z^2}$

$$y = 4\rho \quad z = \sqrt{1 - 16\rho^2}$$

$$r = \sqrt{(4\rho)^2 + (\sqrt{(1 - 16\rho^2)})^2} = 1$$

$$m_{1,2,3,4} = \pm 2\sqrt{2} \frac{\beta}{a} \left[\sqrt{\frac{1}{2}(r + y)} \pm (\text{sign}(z))i \sqrt{\frac{1}{2}(r - y)} \right]$$

$$= \frac{2\beta}{a} \left[\sqrt{(1 + 4\rho)} \pm i\sqrt{(1 - 4\rho)} \right]$$

$$\text{Let } \phi_1 = \sqrt{(1+4\rho)} \text{ and } \phi_2 = \sqrt{(1-4\rho)}$$

$$m_{1,2,3,4} = \pm \frac{2\beta}{a} [\phi_1 \pm i\phi_2]$$

$$\text{Where } \frac{2\beta}{a} = \xi$$

$$m_{1,2,3,4} = \pm \xi [\phi_1 \pm i\phi_2] \tag{C1.5}$$

Substituting equation (C1.5) in to $w = ce^{mx}$

$$w(x) = \sum_{i=1}^4 c_i e^{m_i x}$$

$$w(x) = e^{\xi\phi_1 x} (F_1 e^{\xi\phi_2 x i} + F_2 e^{-\xi\phi_2 x i}) + e^{-\xi\phi_1 x} (F_3 e^{-\xi\phi_2 x i} + F_4 e^{\xi\phi_2 x i}) \tag{C1.7}$$

Where $e^{i\lambda} = \cos \lambda x + i \sin \lambda x$

$$e^{-i\lambda} = \cos \lambda x - i \sin \lambda x$$

$$w(x) = \begin{pmatrix} e^{\xi\phi_1 x} ((F_1 + F_2) \cos \xi\phi_2 x + (iF_1 - iF_2) \sin \xi\phi_2 x) + \\ e^{-\xi\phi_1 x} ((F_3 + F_4) \cos \xi\phi_2 x + (iF_4 - iF_3) \sin \xi\phi_2 x) \end{pmatrix}$$

$$\text{Let } F_1 + F_2 = A_1 \quad F_3 + F_4 = A_3$$

$$iF_1 - iF_2 = A_2 \quad iF_4 - iF_3 = A_4$$

$$w(x) = e^{\xi\phi_1 x} (A_1 \cos \xi\phi_2 x + A_2 \sin \xi\phi_2 x) + e^{-\xi\phi_1 x} (A_3 \cos \xi\phi_2 x + A_4 \sin \xi\phi_2 x) \tag{C1.8}$$

The above equation can further be simplified

$$e^x = \cos hx + \sin hx$$

$$e^{-x} = \cos hx - \sin hx$$

$$w(x) = \begin{pmatrix} [\cos h(\xi\phi_1 x) + \sin h(\xi\phi_1 x)] * [A_1 \cos(\xi\phi_2 x) + A_2 \sin(\xi\phi_2 x)] \\ [\cos h(\xi\phi_1 x) - \sin h(\xi\phi_1 x)] * [A_3 \cos(\xi\phi_2 x) + A_4 \sin(\xi\phi_2 x)] \end{pmatrix}$$

$$w(x) = \begin{pmatrix} \sin(\xi\phi_2 x) * [(A_2 + A_4) \cos h(\xi\phi_1 x) + (A_2 - A_4) \sin h(\xi\phi_1 x) +] \\ \cos(\xi\phi_2 x) * [(A_1 + A_3) \cos h(\xi\phi_1 x) + (A_1 - A_3) \sin h(\xi\phi_1 x)] \end{pmatrix}$$

$$\text{Let } A_2 + A_4 = C_1 \quad A_1 + A_3 = C_3$$

$$A_2 - A_4 = C_2 \quad A_1 - A_3 = C_4$$

$$w(x) = \begin{pmatrix} \sin(\xi\phi_2 x)(C_1 \cos h(\xi\phi_1 x) + C_2 \sin h(\xi\phi_1 x)) \\ + \cos(\xi\phi_2 x)(C_3 \cos h(\xi\phi_1 x) + C_4 \sin h(\xi\phi_1 x)) \end{pmatrix} \quad (C1.9)$$

Equation (C1.8) & (C1.9) are two alternative forms of the solution to the homogenous differential Equation of (C1.1). Since Equations (C1.9) is very complicated, Equation (C1.8) is used as a general solution in this work.

Case II ($\rho = 1/4$)

In this particular case $\rho = 1/4$ will be substitute in the equation below

$$m_{1,2,3,4} = \pm \sqrt{2}\xi \sqrt{(4\rho - \sqrt{(16\rho^2 - 1)})}$$

$$m_{1,2,3,4} = \pm \sqrt{2}\xi$$

$$\text{Let } \phi_1 = \sqrt{2} \quad (C1.10)$$

Thus the general solution becomes

$$w(x) = \sum_{i=1}^4 c_i e^{m_i x}$$

$$w(x) = B_1 e^{\xi\phi_1 x} + B_2 e^{-\xi\phi_1 x} + B_3 x e^{\xi\phi_1 x} + B_4 x e^{-\xi\phi_1 x}$$

$$w(x) = e^{\xi\phi_1 x} (B_1 + B_3 x) + e^{-\xi\phi_1 x} (B_2 + B_4 x) \quad (C1.12)$$

Case III $\rho > \frac{1}{4}$

Since $\rho > \frac{1}{4}$ all the roots are real

$$m_{1,2,3,4} = \pm \sqrt{2}\xi \sqrt{(4\rho - \sqrt{(16\rho^2 - 1)})}$$

Introducing Equitation (C1.4) yields

$$m_{1,2,3,4} = \pm 2\xi [\sqrt{4\rho + 1} \pm \sqrt{4\rho - 1}] \quad (C1.13)$$

Let $\phi_1 = \sqrt{4\rho+1}$ and $\phi_2 = \sqrt{4\rho+1}$

General Solution is given by

$$w(x) = e^{\xi\phi_2 x} (F_1 e^{\xi\phi_2 x} + F_2 e^{-\xi\phi_2 x}) + e^{-\xi\phi_2 x} (F_3 e^{-\xi\phi_2 x} + F_4 e^{\xi\phi_2 x}) \quad (C1.14)$$

$$e^x = \cos hx + \sin hx \quad \text{and} \quad e^{-x} = \cos hx - \sin hx$$

$$w(x) = \begin{pmatrix} e^{\xi\phi_2 x} (\cosh \xi\phi_2 (F_1 + F_2) + \sinh \xi\phi_2 (F_1 - F_2)) + \\ e^{-\xi\phi_2 x} (\cosh \xi\phi_2 (F_3 + F_4) + \sinh \xi\phi_2 (F_4 - F_3)) \end{pmatrix}$$

$$\text{Let } F_1 + F_2 = E_1 \quad \text{and} \quad F_3 + F_4 = E_3$$

$$F_1 - F_2 = E_2 \quad F_4 - F_3 = E_4$$

$$w(x) = e^{\xi\phi_2 x} (E_1 \cosh \xi\phi_2 x + E_2 \sinh \xi\phi_2 x) + e^{-\xi\phi_2 x} (E_3 \cosh \xi\phi_2 x + E_4 \sinh \xi\phi_2 x) \quad (C1.15)$$

ANNEX D

D.1 Infinite strip plate subjected to vertical concentrated load

An Infinite strip plate on Kerr Equivalent Pasternak foundation subjected to concentrated vertical load is considered in this section. The particular solutions are determined by introducing the suitable boundary conditions from Figure (D1).

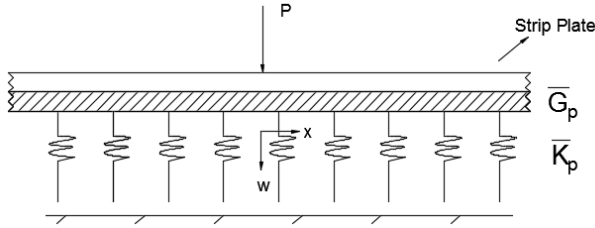


Fig D.1 An infinite strip plate on Kerr Equivalent Pasternak model subjected to vertical concentrated load

Boundary condition

$$1. w'(0) = 0$$

$$2. Dw'''(0) = P/2$$

$$3. w(\infty) = 0$$

Case I ($\rho < \frac{1}{4}$)

The general Solution for deflection is given by

$$w(x) = e^{\xi\phi_1 x} (A_1 \cos \xi\phi_2 x + A_2 \sin \xi\phi_2 x) + e^{-\xi\phi_1 x} (A_3 \cos \xi\phi_2 x + A_4 \sin \xi\phi_2 x)$$

Where $\phi_1 = \sqrt{1+4\rho}$, $\phi_2 = \sqrt{1-4\rho}$ and A_1, A_2, A_3 and A_4 are open constants

Applying the 3rd boundary equation the left part of the equation must be zero to satisfy the third boundary condition so that A_1 and $A_2 = 0$

So the equation becomes

$$w(x) = A_3 e^{-\xi\phi_1 x} \cos \xi\phi_2 x + A_4 e^{-\xi\phi_1 x} \sin \phi_2 x \quad (D1.1)$$

Apply the 1st boundary Condition

$$w'(x) = 0 \text{ at } x=0$$

$$w'(x) = \begin{pmatrix} -\xi\phi_1 A_3 e^{-\xi\phi_1 x} \cos \xi\phi_2 x - \xi\phi_2 A_3 e^{-\xi\phi_1 x} \sin \xi\phi_2 x \\ -\xi\phi_1 A_4 e^{-\xi\phi_1 x} \sin \xi\phi_2 x + \xi\phi_2 A_4 e^{-\xi\phi_1 x} \cos \xi\phi_2 x = 0 \end{pmatrix}$$

$$A_3 = \frac{\phi_2}{\phi_1} A_4 \quad (D1.2)$$

Applying 2nd Boundary Condition

$$Dw'''(0) = \frac{P}{2}$$

$$w(x)'''' = \begin{pmatrix} -(\xi\phi_1)^3 A_3 + 2(\xi)^3 \phi_1 \phi_2^2 A_3 + (\xi)^3 \phi_1 \phi_2^2 A_3 \\ +(\xi)^3 \phi_2 \phi_1^2 A_4 + 2(\xi)^3 \phi_2 \phi_1^2 A_4 - (\xi\phi_2)^3 A_4 \end{pmatrix} = \frac{P}{2} \quad (D1.3)$$

Substituting equation (D1.2) in to in to (D1.3)

$$\left. \begin{aligned} A_4 &= \frac{P}{4D(\xi)^3 (\phi_2^3 + \phi_1^2 \phi_2)} \\ A_3 &= \frac{P}{4D(\xi)^3 (\phi_1^3 + \phi_1 \phi_2^2)} \end{aligned} \right\} \quad (D1.4)$$

By placing equation (D1.4) in to equation (D1.1) the following Deflection equation can be obtained

$$w(x) = \frac{P}{8D(\xi)^3} e^{-\xi\phi_1 x} \left\{ \frac{\cos \xi\phi_2 x}{\phi_1} + \frac{\sin \xi\phi_2 x}{\phi_2} \right\} \quad (D1.5)$$

Successive differential of Equation D1.5 yield the following expression for slop, moment and shear forces.

$$\theta(x) = \frac{Pe^{-\xi\phi_1 x}}{4D(\xi)^2} \left(\frac{\sin \xi\phi_2 x}{\phi_1 \phi_2} \right) \quad (D1.6)$$

$$M(x) = \frac{-Pe^{-\xi\phi_1 x}}{4(\xi)^2} \left(\frac{\sin \xi\phi_2 x}{\phi_2} - \frac{\cos \xi\phi_2 x}{\phi_1} \right) \quad (D1.7)$$

$$V(x) = \frac{Pe^{-\xi\phi_1 x}}{2} (\cos \xi\phi_2 x - \sin \xi\phi_2 x) \quad (D1.8)$$

Case II ($\rho = \frac{1}{4}$)

The general solution is given by

$$w(x) = B_1 e^{\xi\phi_1 x} + B_2 e^{-\xi\phi_1 x} + B_3 x e^{\xi\phi_1 x} + B_4 x e^{-\xi\phi_1 x}$$

Where $\phi_1 = \sqrt{2}$ and B_1, B_2, B_3 and B_4 are open constants

Applying the 3rd Boundary Condition $w(\infty) = 0$, B_1 and B_3 will be zero

$$w(x) = B_2 e^{-\xi\phi_1 x} + B_4 x e^{-\xi\phi_1 x} \quad (D1.9)$$

Applying the 1st Boundary Condition, $w'(x) = 0$ at $X=0$

$$w(x) = -\xi\phi_1 B_2 e^{-\xi\phi_1 x} + B_4 x e^{-\xi\phi_1 x} - \xi\phi_1 B_4 x e^{-\xi\phi_1 x}$$

$$C_4 = \frac{2\beta\phi_1}{a} C_2 \quad (D1.10)$$

Applying the 2nd Boundary Condition

$$w'''(0) = \frac{P}{2D}$$

$$w(x)''' = (-\xi\phi_1)^3 C_2 e^{-\xi\phi_1 x} + 3(\xi\phi_1)^2 C_4 e^{-\xi\phi_1 x} - (\xi\phi_1)^3 x C_4 e^{-\xi\phi_1 x}$$

At $X=0$

$$(-\xi\phi_1)^3 B_2 + 3(\xi\phi_1)^2 B_4 = \frac{P}{2D} \quad (D1.11)$$

Substituting Equation (D1.10) in to Equation (D1.11) one can obtain the following open constants

$$\left. \begin{aligned} B_2 &= \frac{P}{4D(\xi\phi_1)^3} \\ B_4 &= \frac{P}{4D(\xi\phi_1)^2} \end{aligned} \right\} \quad (D1.12)$$

Thus the deflection equation becomes

$$w(x) = \frac{Pe^{-\xi\phi_1 x}}{4D(\xi\phi_1)^2} \left(\frac{1}{(\xi\phi_1)} + x \right) \quad (D1.13)$$

Successive differential equation (3.63) gives slope, moment and shear force

$$\theta(x) = \frac{-Pe^{-\xi\phi_1 x}}{4D} \left(\frac{x}{(\xi\phi_1)} \right) \quad (D1.14)$$

$$M(x) = \frac{Pe^{-\xi\phi_1 x}}{4} \left(\frac{1}{(\xi\phi_1)} - x \right) \quad (D1.15)$$

$$V(x) = \frac{Pe^{-\xi\phi_1 x}}{2} \left(\frac{x(\xi\phi_1)}{2} - 1 \right) \quad (D1.16)$$

Case III ($\rho > 1/4$)

.The general Solution for deflection is given by

$$w(x) = e^{\xi\phi_1 x} (E_1 \cosh \xi\phi_2 x + E_2 \sinh \xi\phi_2 x) + e^{-\xi\phi_1 x} (E_3 \cosh \xi\phi_2 x + E_4 \sinh \xi\phi_2 x)$$

Where $\phi_1 = \sqrt{1+4\rho}$ and $\phi_2 = \sqrt{4\rho-1}$ and E_1, E_2, E_3 and E_4 are open constants

Applying the 3rd boundary equation the left part of the equation must be zero to satisfy the third Boundary condition thus the constant E_1 and E_2 will be zero. Thus, the equation becomes

$$w(x) = E_3 e^{-\xi\phi_1 x} \cosh \xi\phi_2 x + E_4 e^{-\xi\phi_1 x} \sinh \xi\phi_2 x \quad (D1.17)$$

Apply the 1st boundary Condition ($w'(x) = 0$ at $X = 0$)

$$w'(x) = \left(\begin{array}{l} -\xi\phi_1 E_3 e^{-\xi\phi_1 x} \cosh \xi\phi_2 x + \xi\phi_2 E_3 e^{-\xi\phi_1 x} \sinh \xi\phi_2 x \\ -\xi\phi_1 E_4 e^{-\xi\phi_1 x} \sinh \xi\phi_2 x + \xi\phi_2 E_4 e^{-\xi\phi_1 x} \cosh \xi\phi_2 x = 0 \end{array} \right)$$

$$E_3 = \frac{\phi_2}{\phi_1} E_4 \quad (D1.18)$$

Applying the 2nd boundary Condition

$$w'''(0) = \frac{P}{2D}$$

$$\left(\begin{array}{l} -(\xi\phi_1)^3 E_3 - 2(\xi)^3 \phi_1\phi_2^2 E_3 - (\xi)^3 \phi_1\phi_2^2 E_3 + \\ (\xi)^3 \phi_2\phi_1^2 E_4 + 2(\xi)^3 \phi_2\phi_1^2 E_4 + (\xi\phi_2)^3 E_4 \end{array} \right) = 0 \quad (D1.19)$$

Substituting Equation (D1.18) in to (D1.19)

$$\left. \begin{array}{l} E_4 = \frac{P}{4D(\xi)^3 \phi_2 (\phi_1^2 - \phi_2^2)} \\ E_3 = \frac{P}{4D(\xi)^3 \phi_1 (\phi_1^2 - \phi_2^2)} \end{array} \right\} \quad (D1.20)$$

Thus the deflection equation becomes

$$w(x) = \frac{Pe^{-\xi\phi_1 x}}{4D(\xi)^3 (\phi_1^2 - \phi_2^2)} \left[\frac{\cosh(\xi\phi_2 x)}{\phi_1} + \frac{\sinh(\xi\phi_2 x)}{\phi_2} \right] \quad (D1.21)$$

$$\theta(x) = \frac{-Pe^{-\xi\phi_1 x} \sinh \xi\phi_2 x}{4D(\xi)^2 \phi_2 \phi_1} \quad (D1.22)$$

$$M(x) = \frac{Pe^{-\xi\phi_1 x}}{4(\xi)} \left[\frac{\cosh \xi\phi_2 x}{\phi_1} - \frac{\sinh \xi\phi_2 x}{\phi_2} \right] \quad (D1.23)$$

$$V(x) = \frac{Pe^{-\xi\phi_1 x}}{2} \left[-\cosh \xi\phi_2 x + \frac{\sinh \xi\phi_2 x}{\phi_2 \phi_1} \right] \quad (D1.24)$$

ANNEX E

E.1 Finite strip plate subjected to vertical concentrated load

A finite strip plate on a Pasternak foundation subjected to concentrated load and two regions are considered as shown in Figure (E.1), the loaded and unloaded region.

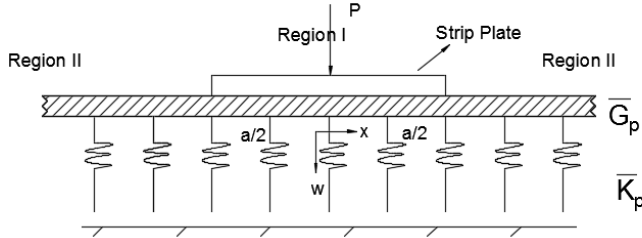


Fig E.1 A finite strip plate resting on a Kerr Equivalent Pasternak foundation and subjected to a vertical concentrated load

The deflection of loaded region I is governed by fourth order differential equation

$$\frac{d^4 w}{dx^4} - \frac{64B^2 \rho}{a^2} \frac{d^2 w}{dx^2} + 64 \frac{\beta^4}{a^4} w = 0$$

$$\text{Where } \beta = \sqrt[4]{\frac{k_p a^4}{64D}} \quad \gamma = \sqrt{\frac{k_p a^4}{G_p}} \quad \rho = \left(\frac{\beta}{\gamma}\right)^2$$

$$\xi = \sqrt[4]{\frac{k_s}{D}}$$

The deflection of unloaded region (II) is governed by second order differential equation given by

$$\bar{k}_p w_f(x) - \bar{G}_p \frac{d^2 w}{dx^2} = 0 \tag{E1.1}$$

$$w_f(x) = C e^m x$$

The general solution for unloaded area will be

$$w_f(x) = C_5 e^{\sqrt{\frac{k_s}{G_p}} x} + C_6 e^{-\sqrt{\frac{k_s}{G_p}} x} \tag{E1.2}$$

The following boundary conditions are drawn from the loading arrangement shown in Figure (E.1)

At X=0	At X=L	At X=∞
1. $W'(x)=0$	3. $W(L)=W_f(L)$	7. $W_f(x)=0$
2. $W'''(x) = \frac{p}{2D}$	4. $W'(L)=W'_f(L)$	
	5. $W''(L)=0$	
	6. $W'''(L)=0$	

These boundary conditions are employed to general solution given in Equation (3.47-3.49)

Case I $\rho < \frac{1}{4}$

The general solution is given by

$$w(x) = e^{\xi\phi_1 x} (C_1 \cos \xi\phi_2 x + C_2 \sin \xi\phi_2 x) + e^{-\xi\phi_1 x} (C_3 \cos \xi\phi_2 x + C_4 \sin \xi\phi_2 x)$$

Where C is open constant , $\phi_1 = \sqrt{1 + 4\rho}$ and $\phi_2 = \sqrt{1 - 4\rho}$

➤ Applying the 7th boundary equation

$$W_f(x) = 0 \quad \text{at } x=0$$

$$w_f(x) = C_5 e^{\sqrt{\frac{k_p}{G_p}} x} + C_6 e^{-\sqrt{\frac{k_p}{G_p}} x}$$

$$C_5 e^{\sqrt{\frac{k_p}{G_p}} x} + C_6 e^{-\sqrt{\frac{k_p}{G_p}} x} = 0 \tag{E1.3}$$

In order to make this equation zero C_6 must be zero

$$w_f(x) = C_5 e^{\sqrt{\frac{k_p}{G_p}} x} = 0$$

➤ Applying the 1st boundary equation

$$w'(x) = 0 \quad \text{at } x = 0$$

$$w'(x) = \xi\phi_1 C_1 + \xi\phi_2 C_2 - \xi\phi_1 C_3 + \xi\phi_2 C_4 = 0 \quad (\text{E1.4})$$

➤ Applying the 2nd boundary equation

$$w'''(x) = \xi^3 \left(\begin{aligned} &(\phi_1^3 - 3\phi_1\phi_2)C_1 - (3\phi_1^2\phi_2 - \phi_2^3)C_2 - \\ &(-\phi_1^3 + 3\phi_1\phi_2^2)C_3 - (3\phi_1^2\phi_2 - \phi_2^3 +)C_4 \end{aligned} \right) = \frac{p}{2D} \quad (\text{E1.5})$$

Applying the 3rd boundary equation

Continuity of Deflection at the ends of the beam at $x = \frac{a}{2}$ yields

$$w\left(\frac{a}{2}\right) = w_f\left(\frac{a}{2}\right)$$

$$C_1 e^{\beta\phi_1} \cos(\beta\phi_2) + C_2 e^{\beta\phi_1} \sin(\beta\phi_2) + C_3 e^{-\beta\phi_1} \cos(\beta\phi_2) + C_4 e^{-\beta\phi_1} \sin(\beta\phi_2) - C_5 e^{-\sqrt{\frac{k_p}{G_p}} \frac{a}{2}} = 0 \quad (\text{E1.6})$$

➤ Applying the 4th boundary equation

Continuity of Slope at the ends of the beam at $x = \frac{a}{2}$ yields

$$w'\left(\frac{a}{2}\right) = w_f'\left(\frac{a}{2}\right)$$

$$\left(\begin{aligned} &C_1 \xi e^{\beta\phi_1} (\phi_1 \cos(\beta\phi_2) - \phi_2 \sin(\beta\phi_2)) + C_2 \xi e^{\beta\phi_1} (\phi_1 \sin(\beta\phi_2) + \phi_2 \cos(\beta\phi_2)) \\ &- C_3 \xi e^{-\beta\phi_1} (\phi_1 \cos(\beta\phi_2) - \phi_2 \sin(\beta\phi_2)) + C_4 \xi e^{-\beta\phi_1} (-\phi_1 \sin(\beta\phi_2) + \phi_2 \cos(\beta\phi_2)) \\ &+ C_5 \sqrt{\frac{k_p}{G_p}} e^{-\sqrt{\frac{k_p}{G_p}} x} \end{aligned} \right) = 0 \quad (\text{E1.7})$$

➤ Applying the 5th boundary equation

Moment at the ends of the beam at yields

$$w''(x) = 0$$

$$\begin{pmatrix} C_1 \xi^2 e^{-\beta \phi_1} \left((\phi_1^2 - \phi_2^2) \cos(\beta \phi_2) - 2\phi_1 \phi_2 \sin(\beta \phi_2) \right) + \\ C_2 \xi^2 e^{\beta \phi_1} \left((\phi_1^2 - \phi_2^2) \sin(\beta \phi_2) + 2\phi_1 \phi_2 \cos(\beta \phi_2) \right) + \\ C_3 \xi^2 e^{-\beta \phi_1} \left((\phi_1^2 - \phi_2^2) \cos(\beta \phi_2) + 2\phi_1 \phi_2 \sin(\beta \phi_2) \right) + \\ C_4 \xi^2 e^{-\beta \phi_1} \left((\phi_1^2 - \phi_2^2) \sin(\beta \phi_2) - 2\phi_1 \phi_2 \cos(\beta \phi_2) \right) \end{pmatrix} = 0 \quad (\text{E1.8})$$

Equation (E1.4) (E1.5) ,(E1.6) (E1.7) &(E1.8)can be represented in matrix form

$L \times C = M$

$$\text{Where } C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{bmatrix} \text{ M} = \begin{bmatrix} 0 \\ \frac{p}{2D} \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } L = \begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ L_{21} & L_{22} & L_{23} & L_{24} & L_{25} \\ L_{31} & L_{32} & L_{33} & L_{34} & L_{35} \\ L_{41} & L_{42} & L_{43} & L_{44} & L_{45} \\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} \end{bmatrix}$$

$$C = L^{-1}M$$

$$L_{11} = \xi \phi_1, \quad L_{12} = 2 \frac{\beta}{a} \phi_2, \quad L_{13} = \xi \phi_1, \quad L_{14} = -\xi \phi_2, \quad L_{15} = 0$$

$$L_{21} = \xi^3 (\phi_1^3 - 3\phi_1 \phi_2^2), \quad L_{22} = \xi^3 (3\phi_1^2 \phi_2 - \phi_2^3),$$

$$L_{23} = \left[\frac{2\beta}{a} \right]^3 (-\phi_1^3 + 3\phi_1 \phi_2^2), \quad L_{24} = \xi^3 (3\phi_1^2 \phi_2 - \phi_2^3), \quad L_{25} = 0$$

$$L_{31} = e^{\beta \phi_1} \cos(\beta \phi_2), \quad L_{32} = e^{\beta \phi_1} \sin(\beta \phi_2),$$

$$L_{33} = e^{-\beta \phi_1} \cos(\beta \phi_2), \quad L_{34} = e^{-\beta \phi_1} \sin(\beta \phi_2), \quad L_{35} = e^{-\sqrt{\frac{ks}{Gp}} \frac{a}{2}}$$

$$L_{41} = \xi e^{\beta \phi_1} \{ \phi_1 \cos(\beta \phi_2) - \phi_2 \sin(\beta \phi_2) \}, \quad L_{42} = \xi e^{\beta \phi_1} \{ \phi_1 \sin(\beta \phi_2) + \phi_2 \cos(\beta \phi_2) \}$$

$$L_{43} = \xi \frac{2\beta}{a} e^{-\beta \phi_1} \{ \phi_1 \cos(\beta \phi_2) - \phi_2 \sin(\beta \phi_2) \} + \dots, \quad L_{44} = \xi$$

$$e^{-\beta \phi_1} \{ -\phi_1 \sin(\beta \phi_2) + \phi_2 \cos(\beta \phi_2) \} L_{45} = \sqrt{\frac{ks}{Gp}} e^{-\sqrt{\frac{ks}{Gp}}}$$

$$L_{51} = \left[\xi \right]^2 e^{\beta \phi_1} \{ (\phi_1^2 - \phi_2^2) \cos(\beta \phi_2) - 2\phi_1 \phi_2 \sin(\beta \phi_2) \}$$

$$L_{52} = [\xi]^2 e^{\beta\phi_1} \{(\phi_1^2 - \phi_2^2) \sin(\beta\phi_2) + 2\phi_1\phi_2 \cos(\beta\phi_2)\}$$

$$L_{53} = [\xi]^2 e^{-\beta\phi_1} \{(\phi_1^2 - \phi_2^2) \cos(\beta\phi_2) + 2\phi_1\phi_2 \sin(\beta\phi_2)\}$$

$$L_{54} = [\xi]^2 e^{-\beta\phi_1} \{(\phi_1^2 - \phi_2^2) \sin(\beta\phi_2) - 2\phi_1\phi_2 \cos(\beta\phi_2)\}$$

$$L_{55} = 0$$

The integral constants C_i can be determined by inverting L

$$C = L^{-1}M$$

Case II ($\rho = 1/4$)

$$w(x) = e^{\xi\phi_1 x} (C_1 + C_3 x) + e^{-\xi\phi_1 x} (C_2 + C_4 x) \quad (E1.9)$$

➤ Applying the 1st boundary equation

$$w'(x) = 0 \quad \text{at } x=0$$

$$w(x) = e^{\xi\phi_1 x} (C_1 + C_3 x) + e^{-\xi\phi_1 x} (C_2 + C_4 x)$$

$$\xi\phi_1 C_1 - \xi\phi_1 C_2 + C_3 - C_4 \quad (E1.10)$$

➤ Applying the 2nd boundary equation

$$w'''(x) = \frac{p}{2D} \quad \text{at } x=0$$

$$(\xi\phi_1)^3 C_1 - (\xi\phi_1)^3 C_2 + 3(\xi\phi_1)^2 C_3 + 3(\xi\phi_1)^2 C_4 = 0 \quad (E1.11)$$

➤ Applying the 3rd boundary equation

Continuity of Deflection at the ends of the beam at $x = \frac{a}{2}$ yields

$$w\left(\frac{a}{2}\right) = w_f\left(\frac{a}{2}\right)$$

$$e^{\beta\phi_1} C_1 + e^{-\beta\phi_1} C_2 + \left(\frac{a}{2}\right) e^{\beta\phi_1} C_3 + \left(\frac{a}{2}\right) e^{-\beta\phi_1} C_4 - e^{-\sqrt{\frac{k_p}{G_p}} \frac{a}{2}} C_5 = 0 \quad (\text{E1.12})$$

➤ Applying the 4th boundary equation

Continuity of slope at the ends of the beam at $x = \frac{a}{2}$ yield

$$w' \left(\frac{a}{2} \right) = w_f' \left(\frac{a}{2} \right)$$

$$\xi\phi_1 e^{\beta\phi_1} C_1 - \xi\phi_1 e^{-\beta\phi_1} C_2 + e^{\beta\phi_1} C_3 + \beta\phi_1 e^{\beta\phi_1} C_3 + e^{-\beta\phi_1} C_4 - \beta\phi_1 e^{-\beta\phi_1} C_4 + \sqrt{\frac{k_p}{G_p}} e^{-\sqrt{\frac{k_p}{G_p}} x} C_5 \quad (\text{E1.13})$$

➤ Applying the 5th boundary equation

Moment at the ends of the beam at $X=a/2$ yields

$$w'' \left(\frac{a}{2} \right) = 0$$

$$\left(\begin{aligned} & \left((\xi\phi_1)^2 e^{\beta\phi_1} C_1 + (\xi\phi_1)^2 e^{-\beta\phi_1} C_2 + 2(\xi\phi_1) e^{\beta\phi_1} C_3 + \right. \\ & \left. \frac{a}{2} (\xi\phi_1)^2 e^{\beta\phi_1} C_3 - 2(\xi\phi_1) e^{-\beta\phi_1} C_4 - \frac{a}{2} (\xi\phi_1)^2 e^{-\beta\phi_1} C_4 = 0 \right) \end{aligned} \right) \quad (\text{E1.14})$$

Equation (E1.10) (E1.11) ,(E1.12) (E1.13) &(E1.14) can be represented in matrix form

$L \times C = M$

$$\text{Where } C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{bmatrix} \text{ M} = \begin{bmatrix} 0 \\ \frac{p}{2D} \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } L = \begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ L_{21} & L_{22} & L_{23} & L_{24} & L_{25} \\ L_{31} & L_{32} & L_{33} & L_{34} & L_{35} \\ L_{41} & L_{42} & L_{43} & L_{44} & L_{45} \\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} \end{bmatrix}$$

$$L_{11} = \xi \phi_1, \quad L_{12} = \xi \phi_1, \quad L_{13} = 1, \quad L_{14} = -1, \quad L_{15} = 0$$

$$L_{21} = [\xi\phi_1]^3, \quad L_{22} = [\xi\phi_1]^3, \quad L_{23} = 3[\xi\phi_1]^2, \quad L_{24} = 3[\xi\phi_1]^2, \quad L_{25} = 0$$

$$L_{31} = e^{\beta\phi_1} \quad , \quad L_{32} = e^{-\beta\phi_1} \quad , \quad L_{33} = \frac{a}{2} e^{\beta\phi_1} \quad , \quad L_{34} = \frac{a}{2} e^{-\beta\phi_1} F4 \quad , \quad L_{35} = e^{-\sqrt{\frac{ks}{Gp}} \frac{a}{2}} F5$$

$$L_{41} = \xi\phi_1 e^{\beta\phi_1} \quad , \quad L_{42} = \xi\phi_1 e^{-\beta\phi_1} \quad , \quad L_{43} = (1+\beta\phi_1) e^{\beta\phi_1} \quad , \quad L_{44} = (1-\beta\phi_1) e^{-\beta\phi_1} \quad , \quad L_{45} = \sqrt{\frac{ks}{Gp}} e^{-\sqrt{\frac{ks}{Gp}} \frac{a}{2}}$$

$$L_{51} = \{\xi\}^2 \quad L_{52} = \{\xi\}^2 e^{-\beta\phi_1} \quad L_{53} = \left[2\{\xi\} + \frac{a}{2}\{\xi\}^2 \right] e^{\beta\phi_1}$$

$$L_{54} = \left[-2\{\xi\} + \frac{a}{2}\{\xi\}^2 \right] e^{-\beta\phi_1} \quad L_{55} = 0$$

The integral constants C_i can be determined by inverting L

$$C = L^{-1}M$$

Case III ($\rho > 1/4$)

$$w(x) = \begin{pmatrix} C_1 e^{\xi\phi_1 x} \cosh \xi\phi_2 x + C_2 e^{\xi\phi_1 x} \sinh \xi\phi_2 x \\ + C_3 e^{-\xi\phi_1 x} \cosh \xi\phi_2 x + C_4 e^{-\xi\phi_1 x} \sinh \xi\phi_2 x \end{pmatrix} \quad (E1.15)$$

➤ Applying the 1st boundary equation

$$W'(x) = 0 \quad \text{at } x=0$$

$$\xi\phi_1 C_1 + \xi\phi_2 C_2 - \xi\phi_1 C_3 + \xi\phi_2 C_4 = 0 \quad (E1.16)$$

➤ Applying the 2nd boundary equation

$$w'''(x) = \frac{p}{2D} \quad \text{at } x=0$$

$$w'''(x) = (\xi)^3 \begin{pmatrix} (\phi_1^3 + 3\phi_1\phi_2^2)C_1 + (3\phi_1^2\phi_2 + \phi_2^3)C_2 - \\ (\phi_1^3 + 3\phi_1\phi_2^2)C_3 + (3\phi_1^2\phi_2 + \phi_2^3)C_4 \end{pmatrix} = \frac{p}{2D} \quad (E1.17)$$

Applying the 3rd boundary equation

Continuity of Deflection at the ends of the beam at $x = \frac{a}{2}$ yields

$$w\left(\frac{a}{2}\right) = w_f\left(\frac{a}{2}\right)$$

$$C_1 e^{\beta\phi_1} \cosh(\beta\phi_2) + C_2 e^{\beta\phi_1} \sinh(\beta\phi_2) + C_3 e^{-\beta\phi_1} \cosh(\beta\phi_2) + C_4 e^{-\beta\phi_1} \sinh(\beta\phi_2) - C_5 e^{-\sqrt{\frac{k_p}{G_p}} \frac{a}{2}} = 0$$

➤ Applying the 4th boundary equation

Continuity of Slope at the ends of the beam at $x = \frac{a}{2}$ yields

$$w'\left(\frac{a}{2}\right) = w_f'\left(\frac{a}{2}\right)$$

$$\left(\begin{array}{l} C_1 \xi e^{\beta\phi_1} (\phi_1 \cosh(\beta\phi_2) + \phi_2 \sinh(\beta\phi_2)) + C_2 \xi e^{\beta\phi_1} (\phi_1 \sinh(\beta\phi_2) + \phi_2 \cosh(\beta\phi_2)) \\ + C_3 \xi e^{-\beta\phi_1} (-\phi_1 \cosh(\beta\phi_2) + \phi_2 \sinh(\beta\phi_2)) + C_4 \xi e^{-\beta\phi_1} (-\phi_1 \sinh(\beta\phi_2) + \phi_2 \cosh(\beta\phi_2)) \\ + C_5 \sqrt{\frac{k_p}{G_p}} e^{-\sqrt{\frac{k_p}{G_p}} \frac{a}{2}} \end{array} \right) \quad (E1.18)$$

➤ Applying the 5th boundary equation

Moment at the ends of the beam at yields

$$w''(x) = 0$$

$$\left(\begin{array}{l} C_1 (\xi)^2 e^{\beta\phi_1} \left((\phi_1^2 + \phi_2^2) \cosh(\beta\phi_2) + 2\phi_1\phi_2 \sinh(\beta\phi_2) \right) + \\ C_2 (\xi)^2 e^{\beta\phi_1} \left((\phi_1^2 + \phi_2^2) \sinh(\beta\phi_2) + 2\phi_1\phi_2 \cosh(\beta\phi_2) \right) + \\ C_3 (\xi)^2 e^{-\beta\phi_1} \left((\phi_1^2 + \phi_2^2) \cosh(\beta\phi_2) - 2\phi_1\phi_2 \sinh(\beta\phi_2) \right) + \\ C_4 (\xi)^2 e^{-\beta\phi_1} \left((\phi_1^2 - \phi_2^2) \sinh(\beta\phi_2) - 2\phi_1\phi_2 \cosh(\beta\phi_2) \right) \end{array} \right) \quad (E1.19)$$

Equation (E1.15) (E1.16) ,(E1.17) (E1.18) &(E1.19) can be represented in matrix form

$$L \times C = M$$

$$\text{Where } C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{bmatrix} \quad M = \begin{bmatrix} 0 \\ \frac{p}{2D} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and } L = \begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ L_{21} & L_{22} & L_{23} & L_{24} & L_{25} \\ L_{31} & L_{32} & L_{33} & L_{34} & L_{35} \\ L_{41} & L_{42} & L_{43} & L_{44} & L_{45} \\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} \end{bmatrix}$$

$$L_{11} = \xi\phi_1 \quad L_{12} = \xi\phi_2 \quad L_{13} = -\xi\phi_1 \quad L_{14} = \xi\phi_2 \quad L_{15} = 0$$

$$L_{21} = [\xi]^3 \phi_1 (\phi_1^2 + 3\phi_2^2) \quad L_{22} = [\xi]^3 \phi_2 (\phi_2^2 + 3\phi_1^2) \quad L_{23} = -[\xi]^3 \phi_1 (\phi_1^2 + 3\phi_2^2)$$

$$L_{24} = [\xi]^3 \phi_2 (\phi_2^2 + 3\phi_1^2) \quad L_{25} = 0$$

$$L_{31} = \cosh \beta \phi_2 e^{\beta\phi_1} \quad L_{32} = \sinh \beta \phi_2 e^{\beta\phi_1}$$

$$L_{33} = \cosh \beta \phi_2 e^{-\beta\phi_1}$$

$$L_{34} = \sinh \beta \phi_2 e^{-\beta\phi_1} \quad L_{35} = -e^{-\sqrt{\frac{ks}{Gp}} \frac{a}{2}}$$

$$L_{41} = e^{\beta\phi_1} [\xi] (\phi_2 \sinh \beta \phi_2 + \phi_1 \cosh \beta \phi_2) \quad L_{42} = e^{\beta\phi_1} [\xi] (\phi_2 \cosh \beta \phi_2 + \phi_1 \sinh \beta \phi_2)$$

$$L_{43} = e^{-\beta\phi_1} [\xi] (\phi_2 \sinh \beta \phi_2 - \phi_1 \cosh \beta \phi_2) \quad L_{44} = e^{-\beta\phi_1} [\xi] (\phi_2 \cosh \beta \phi_2 - \phi_1 \sinh \beta \phi_2)$$

$$L_{45} = \sqrt{\frac{ks}{Gp}} e^{-\sqrt{\frac{ks+a}{Gp}} \frac{a}{2}}$$

$$L_{51} = [\xi]^2 e^{\beta\phi_1} \{ (\phi_1^2 + \phi_2^2) \cosh \beta \phi_2 + 2\phi_1 \phi_2 \sinh \beta \phi_2 \}$$

$$L_{52} = [\xi]^2 e^{\beta\phi_1} \{ (\phi_1^2 + \phi_2^2) \sinh \beta \phi_2 + 2\phi_1 \phi_2 \cosh \beta \phi_2 \}$$

$$L_{53} = [\xi]^2 e^{-\beta\phi_1} \{ (\phi_1^2 + \phi_2^2) \cosh \beta \phi_2 - 2\phi_1 \phi_2 \sinh \beta \phi_2 \}$$

$$L_{54} = [\xi]^2 e^{-\beta\phi_1} \{ (\phi_1^2 + 3\phi_2^2) \sinh \beta \phi_2 - 2\phi_1 \phi_2 \cosh \beta \phi_2 \}$$

$$L_{55} = 0$$

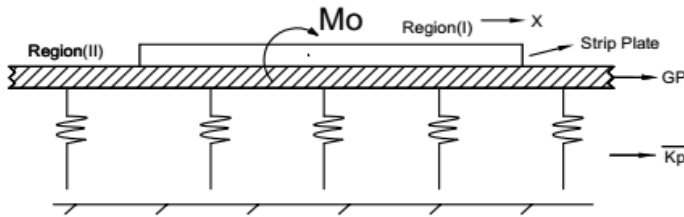
The integral constants C_i can be determined by inverting L

ANNEX F

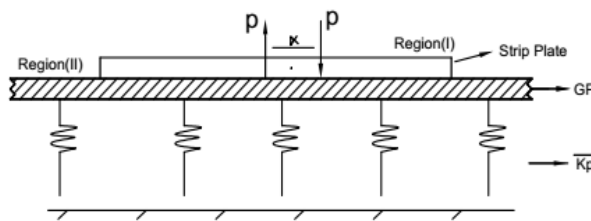
F.1 Finite strip plate subjected to concentrated Moment

A finite length strip plate resting on Pasternak foundation and subjected to a concentrated moment is considered as shown in the Figure.

$$\lim_{\alpha \rightarrow x} p\alpha = Mo$$



(a)



(b)

Fig (F.1); (a) A finite strip plate resting on Kerr Equivalent Pasternak foundation and subjected to a concentrated moment (b) A finite strip plate subjected to oppositely directed forces

Case I ($\rho < 1/4$)

$$w(x) = \frac{-p}{2D} y(x)$$

Where

$$y(x) = K_1 e^{\xi\phi_2 x} \cos \xi\phi_2 x + K_2 e^{\xi\phi_2 x} \sin \xi\phi_2 x + K_3 e^{-\xi\phi_2 x} \cos \xi\phi_2 x + K_4 e^{-\xi\phi_2 x} \sin \xi\phi_2 x$$

$$k_i = \frac{-2D}{p} C_i \quad \text{and} \quad C_i = L^{-1}$$

For loading showing in the Figure (3.12b)

$$w(x) = \frac{pa}{2D} \frac{y(x+\alpha) - y(\alpha)}{\alpha} \quad (\text{F1.1})$$

$$w(x) = \frac{M_o}{2D} \begin{pmatrix} k_1 \xi \phi_1 e^{\xi \phi_1 x} \cos(\xi \phi_2 x) - k_1 \xi \phi_2 e^{\xi \phi_1 x} \sin(\xi \phi_2 x) + \\ k_2 \xi \phi_1 e^{\xi \phi_1 x} \sin(\xi \phi_2 x) + k_2 \xi \phi_2 e^{\xi \phi_1 x} \cos(\xi \phi_2 x) \\ -k_3 \xi \phi_1 e^{-\xi \phi_1 x} \cos(\xi \phi_2 x) - k_3 \xi \phi_2 e^{-\xi \phi_1 x} \sin(\xi \phi_2 x) \\ -k_4 \xi \phi_1 e^{-\xi \phi_1 x} \sin(\xi \phi_2 x) + k_4 \xi \phi_2 e^{-\xi \phi_1 x} \cos(\xi \phi_2 x) \end{pmatrix} \quad (\text{F1.2})$$

$$\theta(x) = \frac{M_o}{2D} \begin{pmatrix} k_1 (\xi \phi_1)^2 e^{\xi \phi_1 x} \cos(\xi \phi_2 x) - 2k_1 (\xi)^2 \phi_1 \phi_2 e^{\xi \phi_1 x} \sin(\xi \phi_2 x) \\ k_1 (\xi \phi_2)^2 e^{\xi \phi_1 x} \cos(\xi \phi_2 x) + k_2 (\xi \phi_1)^2 e^{\xi \phi_1 x} \sin(\xi \phi_2 x) \\ 2k_2 (\xi)^2 \phi_1 \phi_2 e^{\xi \phi_1 x} \cos(\xi \phi_2 x) - k_2 (\xi \phi_1)^2 e^{\xi \phi_1 x} \sin(\xi \phi_2 x) + \\ k_3 (\xi \phi_1)^2 e^{-\xi \phi_1 x} \cos(\xi \phi_2 x) + 2k_3 (\xi)^2 \phi_1 \phi_2 e^{-\xi \phi_1 x} \sin(\xi \phi_2 x) - \\ k_3 (\xi \phi_2)^2 e^{-\xi \phi_1 x} \cos(\xi \phi_2 x) + k_4 (\xi \phi_1)^2 e^{-\xi \phi_1 x} \sin(\xi \phi_2 x) - \\ (2k_4 (\xi)^2 \phi_1 \phi_2 e^{-\xi \phi_1 x} \cos(\xi \phi_2 x) - k_4 (\xi \phi_2)^2 e^{-\xi \phi_1 x} \sin(\xi \phi_2 x)) \end{pmatrix} \quad (\text{F1.3})$$

$$M(x) = \frac{-M_o}{2} \begin{pmatrix} k_1 (\xi \phi_1)^3 e^{\xi \phi_1 x} \cos(\xi \phi_2 x) - 3k_1 (\xi)^3 \phi_1^2 \phi_2 e^{\xi \phi_1 x} \sin(\xi \phi_2 x) - \\ 3k_1 (\xi)^3 \phi_1 \phi_2^2 e^{\xi \phi_1 x} \cos(\xi \phi_2 x) + k_1 (\xi \phi_2)^3 e^{\xi \phi_1 x} \sin(\xi \phi_2 x) + \\ k_2 (\xi \phi_1)^3 e^{\xi \phi_1 x} \sin(\xi \phi_2 x) + 3k_2 (\xi)^3 \phi_1^2 \phi_2 e^{\xi \phi_1 x} \cos(\xi \phi_2 x) - \\ 3k_2 (\xi)^3 \phi_1 \phi_2^2 e^{\xi \phi_1 x} \sin(\xi \phi_2 x) - k_2 (\xi \phi_1)^3 e^{\xi \phi_1 x} \cos(\xi \phi_2 x) - \\ k_3 (\xi \phi_1)^3 e^{-\xi \phi_1 x} \cos(\xi \phi_2 x) - 3k_3 (\xi)^3 \phi_1^2 \phi_2 e^{-\xi \phi_1 x} \sin(\xi \phi_2 x) + \\ 3k_3 (\xi)^3 \phi_1 \phi_2^2 e^{-\xi \phi_1 x} \cos(\xi \phi_2 x) + k_3 (\xi \phi_2)^3 e^{-\xi \phi_1 x} \sin(\xi \phi_2 x) - \\ k_4 (\xi \phi_1)^3 e^{\xi \phi_1 x} \sin(\xi \phi_2 x) + 3k_4 (\xi)^3 \phi_1^2 \phi_2 e^{\xi \phi_1 x} \cos(\xi \phi_2 x) - \\ 3k_4 (\xi)^3 \phi_1 \phi_2^2 e^{\xi \phi_1 x} \sin(\xi \phi_2 x) - k_4 (\xi \phi_2)^3 e^{\xi \phi_1 x} \cos(\xi \phi_2 x) \end{pmatrix} \quad (\text{F1.4})$$

$$v(x) = \frac{-M_o}{2} \left(\begin{aligned} &k_1 (\xi\phi_1)^4 e^{\xi\phi_1 x} \cos(\xi\phi_2 x) - 4k_1 (\xi)^4 \phi_1^3 \phi_2 e^{\xi\phi_1 x} \sin(\xi\phi_2 x) \\ &6k_1 (\xi)^4 \phi_1^2 \phi_2^2 e^{\xi\phi_1 x} \cos(\xi\phi_2 x) + 4k_1 (\xi)^4 \phi_1 \phi_2^3 e^{\xi\phi_1 x} \sin(\xi\phi_2 x) + \\ &k_1 (\xi\phi_2)^4 e^{\xi\phi_1 x} \cos(\xi\phi_2 x) + k_2 (\xi\phi_1)^4 e^{\xi\phi_1 x} \sin(\xi\phi_2 x) + \\ &4k_2 (\xi)^4 \phi_1^3 \phi_2 e^{\xi\phi_1 x} \cos(\xi\phi_2 x) - 6k_2 (\xi)^4 \phi_1^2 \phi_2^2 e^{\xi\phi_1 x} \sin(\xi\phi_2 x) - \\ &4k_2 (\xi)^4 \phi_1 \phi_2^3 e^{\xi\phi_1 x} \cos(\xi\phi_2 x) + k_2 (\xi\phi_2)^4 e^{\xi\phi_1 x} \sin(\xi\phi_2 x) + \\ &k_3 (\xi\phi_1)^4 e^{-\xi\phi_1 x} \cos(\xi\phi_2 x) + 4k_3 (\xi)^4 \phi_1^3 \phi_2 e^{-\xi\phi_1 x} \sin(\xi\phi_2 x) - \\ &6k_3 (\xi)^4 \phi_1^2 \phi_2^2 e^{-\xi\phi_1 x} \cos(\xi\phi_2 x) - 4k_3 (\xi)^4 \phi_1 \phi_2^3 e^{-\xi\phi_1 x} \sin(\xi\phi_2 x) + \\ &k_3 (\xi\phi_2)^4 e^{-\xi\phi_1 x} \cos(\xi\phi_2 x) + k_4 (\xi\phi_1)^4 e^{-\xi\phi_1 x} \sin(\xi\phi_2 x) - \\ &4k_4 (\xi)^4 \phi_1^3 \phi_2 e^{-\xi\phi_1 x} \cos(\xi\phi_2 x) - 6k_4 (\xi)^4 \phi_1^2 \phi_2^2 e^{-\xi\phi_1 x} \sin(\xi\phi_2 x) + \\ &4k_4 (\xi)^4 \phi_1 \phi_2^3 e^{-\xi\phi_1 x} \cos(\xi\phi_2 x) + k_4 (\xi\phi_2)^4 e^{-\xi\phi_1 x} \sin(\xi\phi_2 x) \end{aligned} \right) \quad (F1.5)$$

Case II ($\rho=1/4$)

$$w(x) = \frac{-p}{2D} y(x)$$

$$\text{where } y(x) = k_1 e^{\xi\phi_1 x} + k_2 e^{-\xi\phi_1 x} + xk_3 e^{\xi\phi_1 x} + xk_4 e^{-\xi\phi_1 x}$$

$$k_i = \frac{-2D}{p} C_i \quad \text{and } C_i = L^{-1}$$

$$w(x) = \frac{M_o}{2D} \left(k_1 \xi\phi_1 e^{\xi\phi_1 x} - k_2 \xi\phi_1 e^{-\xi\phi_1 x} + k_3 e^{\xi\phi_1 x} + xk_3 \xi\phi_1 e^{\xi\phi_1 x} + k_4 e^{-\xi\phi_1 x} - xk_4 \xi\phi_1 e^{-\xi\phi_1 x} \right) \quad (F1.6)$$

$$\theta(x) = \frac{M_o}{2D} \left(\begin{aligned} &k_1 (\xi\phi_1)^2 e^{\xi\phi_1 x} + k_2 (\xi\phi_1)^2 e^{-\xi\phi_1 x} + 2k_3 (\xi\phi_1) e^{\xi\phi_1 x} + \\ &xk_3 (\xi\phi_1)^2 e^{\xi\phi_1 x} - 2k_4 (\xi\phi_1) e^{-\xi\phi_1 x} + xk_4 (\xi\phi_1)^2 e^{-\xi\phi_1 x} \end{aligned} \right) \quad (F1.7)$$

$$M(x) = \frac{-M_o}{2} \left(\begin{aligned} &k_1 (\xi\phi_1)^3 e^{\xi\phi_1 x} - k_2 (\xi\phi_1)^3 e^{-\xi\phi_1 x} + 3k_3 (\xi\phi_1)^2 e^{\xi\phi_1 x} + \\ &xk_3 (\xi\phi_1)^3 e^{\xi\phi_1 x} + 3k_4 (\xi\phi_1)^2 e^{-\xi\phi_1 x} - xk_4 (\xi\phi_1)^3 e^{-\xi\phi_1 x} \end{aligned} \right) \quad (F1.8)$$

$$v(x) = \frac{-M_o}{2} \left(\begin{aligned} &k_1 (\xi\phi_1)^4 e^{\xi\phi_1 x} + k_2 (\xi\phi_1)^4 e^{-\xi\phi_1 x} + 4k_3 (\xi\phi_1)^3 e^{\xi\phi_1 x} + \\ &k_3 (\xi\phi_1)^4 e^{\xi\phi_1 x} - 4k_4 (\xi\phi_1)^3 e^{-\xi\phi_1 x} + xk_4 (\xi\phi_1)^4 e^{-\xi\phi_1 x} \end{aligned} \right) \quad (F1.9)$$

Case III ($\rho > 1/4$)

$$w(x) = \frac{-p}{2D} y(x)$$

$$y(x) = k_1 e^{\xi \phi_1 x} \cosh \xi \phi_2 x + k_2 e^{\xi \phi_1 x} \sinh \xi \phi_2 x + k_3 e^{-\xi \phi_1 x} \cosh \xi \phi_2 x + k_4 e^{-\xi \phi_1 x} \sinh \xi \phi_2 x$$

$$k_i = \frac{-2D}{p} C_i \quad \text{and} \quad C_i = L^{-1}$$

$$w(x) = \frac{M_o}{2D} \begin{pmatrix} k_1 \xi \phi_1 e^{\xi \phi_1 x} \cosh(\xi \phi_2 x) + k_1 \xi \phi_2 e^{\xi \phi_1 x} \sinh(\xi \phi_2 x) \\ k_2 \xi \phi_1 e^{\xi \phi_1 x} \sinh(\xi \phi_2 x) + k_2 \xi \phi_2 e^{\xi \phi_1 x} \cosh(\xi \phi_2 x) \\ -k_3 \xi \phi_1 e^{-\xi \phi_1 x} \cosh(\xi \phi_2 x) - k_3 \xi \phi_2 e^{-\xi \phi_1 x} \sinh(\xi \phi_2 x) \\ -k_4 \xi \phi_1 e^{-\xi \phi_1 x} \sinh(\xi \phi_2 x) - k_4 \xi \phi_2 e^{-\xi \phi_1 x} \cosh(\xi \phi_2 x) \end{pmatrix} \quad (F1.10)$$

$$\theta(x) = \frac{M_o}{2D} \begin{pmatrix} k_1 (\xi \phi_1)^2 e^{\xi \phi_1 x} \cosh(\xi \phi_2 x) - 2k_1 (\xi)^2 \phi_1 \phi_2 e^{\xi \phi_1 x} \sinh(\xi \phi_2 x) \\ k_1 (\xi \phi_2)^2 e^{\xi \phi_1 x} \cosh(\xi \phi_2 x) + k_2 (\xi \phi_1)^2 e^{\xi \phi_1 x} \sinh(\xi \phi_2 x) \\ 2k_2 (\xi)^2 \phi_1 \phi_2 e^{\xi \phi_1 x} \cosh(\xi \phi_2 x) - k_2 (\xi \phi_1)^2 e^{\xi \phi_1 x} \sinh(\xi \phi_2 x) + \\ k_3 (\xi \phi_1)^2 e^{-\xi \phi_1 x} \cosh(\xi \phi_2 x) + 2k_3 (\xi)^2 \phi_1 \phi_2 e^{-\xi \phi_1 x} \sinh(\xi \phi_2 x) - \\ k_3 (\xi \phi_2)^2 e^{-\xi \phi_1 x} \cosh(\xi \phi_2 x) + k_4 (\xi \phi_1)^2 e^{-\xi \phi_1 x} \sinh(\xi \phi_2 x) - \\ 2k_4 (\xi)^2 \phi_1 \phi_2 e^{-\xi \phi_1 x} \cosh(\xi \phi_2 x) - k_4 (\xi \phi_2)^2 e^{-\xi \phi_1 x} \sinh(\xi \phi_2 x) \end{pmatrix} \quad (F1.11)$$

$$M(x) = \frac{-M_o}{2} \begin{pmatrix} k_1 (\xi \phi_1)^3 e^{\xi \phi_1 x} \cosh(\xi \phi_2 x) + 3k_1 (\xi)^3 \phi_1^2 \phi_2 e^{\xi \phi_1 x} \sinh(\xi \phi_2 x) \\ 3k_1 (\xi)^3 \phi_1 \phi_2^2 e^{\xi \phi_1 x} \cosh(\xi \phi_2 x) + k_1 (\xi \phi_2)^3 e^{\xi \phi_1 x} \sinh\left(\frac{2\beta}{a} \phi_2 x\right) + \\ k_2 (\xi \phi_1)^3 e^{\xi \phi_1 x} \sinh(\xi \phi_2 x) + 3k_2 (\xi)^3 \phi_1^2 \phi_2 e^{\xi \phi_1 x} \cosh\left(\frac{2\beta}{a} \phi_2 x\right) \\ 3k_2 (\xi)^3 \phi_1 \phi_2^2 e^{\xi \phi_1 x} \sinh(\xi \phi_2 x) + k_2 (\xi \phi_2)^3 e^{\xi \phi_1 x} \cosh\left(\frac{2\beta}{a} \phi_2 x\right) - \\ k_3 (\xi \phi_1)^3 e^{-\xi \phi_1 x} \cosh(\xi \phi_2 x) + 3k_3 (\xi)^3 \phi_1^2 \phi_2 e^{-\xi \phi_1 x} \sinh(\xi \phi_2 x) - \\ 3k_3 (\xi)^3 \phi_1 \phi_2^2 e^{-\xi \phi_1 x} \cosh(\xi \phi_2 x) + k_3 (\xi \phi_2)^3 e^{-\xi \phi_1 x} \sinh(\xi \phi_2 x) - \\ k_4 (\xi \phi_1)^3 e^{\xi \phi_1 x} \sinh(\xi \phi_2 x) + 3k_4 \left(\frac{2\beta}{a}\right)^3 \phi_1^2 \phi_2 e^{\xi \phi_1 x} \cos(\xi \phi_2 x) - \\ 3k_4 (\xi)^3 \phi_1 \phi_2^2 e^{\xi \phi_1 x} \sinh(\xi \phi_2 x) + k_4 (\xi \phi_2)^3 e^{\xi \phi_1 x} \cosh(\xi \phi_2 x) \end{pmatrix} \quad (F1.11)$$

$$v(x) = \frac{-M_o}{2} \left(\begin{aligned} &k_1 (\xi \phi_1)^4 e^{\xi \phi_1 x} \cosh(\xi \phi_2 x) + 4k_1 (\xi)^4 \phi_1^3 \phi_2 e^{\xi \phi_1 x} \sinh(\xi \phi_2 x) \\ &6k_1 (\xi)^4 \phi_1^2 \phi_2^2 e^{\xi \phi_1 x} \cosh(\xi \phi_2 x) + 4k_1 (\xi)^4 \phi_1 \phi_2^3 e^{\xi \phi_1 x} \sinh(\xi \phi_2 x) + \\ &k_1 (\xi \phi_2)^4 e^{\xi \phi_1 x} \cosh(\xi \phi_2 x) + k_2 (\xi \phi_1)^4 e^{\xi \phi_1 x} \sinh(\xi \phi x) + \\ &4k_2 (\xi)^4 \phi_1^3 \phi_2 e^{\xi \phi_1 x} \cosh(\xi \phi_2 x) + 6k_2 (\xi)^4 \phi_1^2 \phi_2^2 e^{\xi \phi_1 x} \sinh(\xi \phi_2 x) + \\ &4k_2 (\xi)^4 \phi_1 \phi_2^3 e^{\xi \phi_1 x} \cosh(\xi \phi_2 x) + k_2 (\xi \phi_2)^4 e^{\xi \phi_1 x} \sinh(\xi \phi_2 x) + \\ &k_3 (\xi \phi_1)^4 e^{-\xi \phi_1 x} \cosh(\xi \phi_2 x) - 4k_3 (\xi)^4 \phi_1^3 \phi_2 e^{-\xi \phi_1 x} \sinh(\xi \phi_2 x) + \\ &6k_3 (\xi)^4 \phi_1^2 \phi_2^2 e^{-\xi \phi_1 x} \cosh(\xi \phi_2 x) - 4k_3 (\xi)^4 \phi_1 \phi_2^3 e^{-\xi \phi_1 x} \sinh(\xi \phi_2 x) + \\ &k_3 (\xi \phi_2)^4 e^{-\xi \phi_1 x} \cosh(\xi \phi_2 x) + k_4 (\xi \phi_1)^4 e^{-\xi \phi_1 x} \sinh(\xi \phi_2 x) - \\ &4k_4 (\xi)^4 \phi_1^3 \phi_2 e^{-\xi \phi_1 x} \cosh(\xi \phi_2 x) + 6k_4 (\xi)^4 \phi_1^2 \phi_2^2 e^{-\xi \phi_1 x} \sinh(\xi \phi_2 x) - \\ &4k_4 (\xi)^4 \phi_1 \phi_2^3 e^{-\xi \phi_1 x} \cosh(\xi \phi_2 x) + k_4 (\xi \phi_2)^4 e^{-\xi \phi_1 x} \sinh(\xi \phi x) \end{aligned} \right) \quad (\text{F1.12})$$

ANNEX G

G.1 A finite length strip plate subjected to Uniformly Distributed load

A finite length strip plate resting on subgrade and subjected to a uniformly load is presented in this section

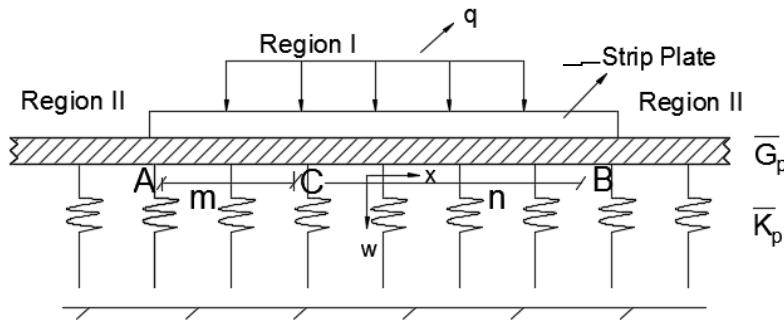


Fig G.1. A finite strip plate resting on Kerr Equivalent Pasternak foundation and subjected to a uniformly distributed load.

Case I ($\rho < 1/4$)

1. When point C is within the loaded region

$$w_c = \int \delta dw = \int -\frac{q dx}{2D} y(x)$$

$$y(x) = k_1 e^{\xi\phi_1 x} \cos \xi\phi_2 x + k_2 e^{\xi\phi_1 x} \sin \xi\phi_2 x + k_3 e^{-\xi\phi_1 x} \cos \xi\phi_2 x + k_4 e^{-\xi\phi_1 x} \sin \xi\phi_2 x$$

(G1.1)

$$k_i = -\frac{2D}{q} C_i$$

The integration constants C_i are then obtained from the inverse matrix relation

$$C = L^{-1}M$$

$$= \frac{-q}{2D} \left[\int_0^m y(x) dx + \int_0^n y(x) dx \right]$$

$$w_c = \frac{-q}{2D\xi(\phi_1^2 + \phi_2^2)} \left(\begin{aligned} & \left[k_1(e^{\xi\phi_2 m}(\phi_1 \cos \xi\phi_2 m + \phi_2 \sin \xi\phi_2 m) + e^{\xi\phi_2 n}(\phi_1 \cos \xi\phi_2 n + \phi_2 \sin \xi\phi_2 n) - 2\phi_1 \right] + \\ & \left[k_2(e^{\xi\phi_2 m}(\phi_1 \sin \xi\phi_2 m - \phi_2 \cos \xi\phi_2 m) + e^{\xi\phi_2 n}(\phi_1 \sin \xi\phi_2 n - \phi_2 \cos \xi\phi_2 n) + 2\phi_2 \right] + \\ & \left[k_3(e^{-\xi\phi_2 m}(-\phi_1 \cos \xi\phi_2 m + \phi_2 \sin \xi\phi_2 m) + e^{-\xi\phi_2 n}(-\phi_1 \cos \xi\phi_2 n + \phi_2 \sin \xi\phi_2 n) + 2\phi_1 \right] + \\ & \left[k_4(e^{-\xi\phi_2 m}(-\phi_1 \sin \xi\phi_2 m - \phi_2 \cos \xi\phi_2 m) + e^{-\xi\phi_2 n}(-\phi_1 \sin \xi\phi_2 n - \phi_2 \cos \xi\phi_2 n) + 2\phi_2 \right] \end{aligned} \right) \quad (G1.2)$$

Once the deflection is obtained the slop, moment and shear can be determined easily

$$\theta_c = \frac{-q}{2D} \left(\begin{aligned} & k_1(e^{\xi\phi_2 m} \cos \xi\phi_2 m - e^{\xi\phi_2 n} \cos \xi\phi_2 n) + k_2(e^{\xi\phi_2 m} \sin \xi\phi_2 m - e^{\xi\phi_2 n} \sin \xi\phi_2 n) + \\ & k_3(e^{-\xi\phi_2 m} \cos \xi\phi_2 m - e^{-\xi\phi_2 n} \cos \xi\phi_2 n) + k_4(e^{-\xi\phi_2 m} \sin \xi\phi_2 m - e^{-\xi\phi_2 n} \sin \xi\phi_2 n) \end{aligned} \right) \quad (G1.3)$$

$$M(c) = -D \frac{-q}{2D} \left[\int_0^m \frac{d^2 y}{dx^2} dx + \int_0^n \frac{d^2 y}{dx^2} dx \right]$$

$$Mc = \frac{q}{2} \left(\begin{aligned} & k_1 \left(\xi\phi_1 e^{\xi\phi_2 m} \cos \xi\phi_2 m - \xi\phi_2 e^{\xi\phi_2 m} \sin \xi\phi_2 m + \xi\phi_1 e^{\xi\phi_2 n} \cos \xi\phi_2 n - \xi\phi_2 e^{\xi\phi_2 n} \sin \xi\phi_2 n - 2\phi_1 \right) \\ & + k_2 \left(\xi\phi_1 e^{\xi\phi_2 m} \sin \xi\phi_2 m + \xi\phi_2 e^{\xi\phi_2 m} \cos \xi\phi_2 m + \xi\phi_1 e^{\xi\phi_2 n} \sin \xi\phi_2 n + \xi\phi_2 e^{\xi\phi_2 n} \cos \xi\phi_2 n - 2\phi_2 \right) \\ & + k_3 \left(-\xi\phi_1 e^{-\xi\phi_2 m} \cos \xi\phi_2 m - \xi\phi_2 e^{-\xi\phi_2 m} \sin \xi\phi_2 m - \xi\phi_1 e^{-\xi\phi_2 n} \cos \xi\phi_2 n - \xi\phi_2 e^{-\xi\phi_2 n} \sin \xi\phi_2 n - 2\phi_1 \right) \\ & + k_4 \left(-\xi\phi_1 e^{-\xi\phi_2 m} \sin \xi\phi_2 m + \xi\phi_2 e^{-\xi\phi_2 m} \cos \xi\phi_2 m - \xi\phi_1 e^{-\xi\phi_2 n} \sin \xi\phi_2 n + \xi\phi_2 e^{-\xi\phi_2 n} \cos \xi\phi_2 n - 2\phi_2 \right) \end{aligned} \right) \quad (G1.4)$$

$$Vc = D \frac{-q}{2D} \left[\int_0^m \frac{d^3 y}{dx^3} dx - \int_0^n \frac{d^3 y}{dx^3} dx \right]$$

$$V_c = q\xi^2 \left(\begin{array}{l} K1 \left(\left(\phi_1^2 e^{\xi\phi_2 m} \cos \xi\phi_2 m - 2\phi_1\phi_2 e^{\xi\phi_2 m} \sin \xi\phi_2 m - \phi_2^2 e^{\xi\phi_2 m} \cos \xi\phi_2 m \right) - \right. \\ \left. \left(\phi_1^2 e^{\xi\phi_2 n} \cos \xi\phi_2 n - 2\phi_1\phi_2 e^{\xi\phi_2 n} \sin \xi\phi_2 n - \phi_2^2 e^{\xi\phi_2 n} \cos \xi\phi_2 n \right) + \right) \\ K2 \left(\left(\phi_1^2 e^{\xi\phi_2 m} \sin \xi\phi_2 m + 2\phi_1\phi_2 e^{\xi\phi_2 m} \cos \xi\phi_2 m - \phi_2^2 e^{\xi\phi_2 m} \sin \xi\phi_2 m \right) - \right. \\ \left. \left(\phi_1^2 e^{\xi\phi_2 n} \sin \xi\phi_2 n + 2\phi_1\phi_2 e^{\xi\phi_2 n} \cos \xi\phi_2 n - \phi_2^2 e^{\xi\phi_2 n} \sin \xi\phi_2 n \right) + \right) \\ K3 \left(\phi_1^2 e^{-\xi\phi_2 m} \cos \xi\phi_2 m + 2\phi_1\phi_2 e^{-\xi\phi_2 m} \sin \xi\phi_2 m - \phi_2^2 e^{-\xi\phi_2 m} \cos \xi\phi_2 m - \right. \\ \left. \left(\phi_1^2 e^{-\xi\phi_2 n} \cos \xi\phi_2 n + 2\phi_1\phi_2 e^{-\xi\phi_2 n} \sin \xi\phi_2 n - \phi_2^2 e^{-\xi\phi_2 n} \cos \xi\phi_2 n \right) + \right) \\ K4 \left(\phi_1^2 e^{-\xi\phi_2 m} \sin \xi\phi_2 m - 2\phi_1\phi_2 e^{-\xi\phi_2 m} \cos \xi\phi_2 m - \phi_2^2 e^{-\xi\phi_2 m} \sin \xi\phi_2 m - \right. \\ \left. \left(\phi_1^2 e^{-\xi\phi_2 n} \sin \xi\phi_2 n - 2\phi_1\phi_2 e^{-\xi\phi_2 n} \cos \xi\phi_2 n - \phi_2^2 e^{-\xi\phi_2 n} \sin \xi\phi_2 n \right) \right) \end{array} \right) \quad (G1.5)$$

2. When point C is to the left of loaded region

$$w_c = \frac{-q}{2D} \left[\int_0^n y(x) dx - \int_0^m y(x) dx \right]$$

$$w_c = \frac{-q}{2D\xi(\phi_1^2 + \phi_2^2)} \left(\begin{array}{l} \left[k_1 (e^{\xi\phi_2 n} (\phi_1 \cos \xi\phi_2 n + \phi_2 \sin \xi\phi_2 n) - (e^{\xi\phi_2 m} (\phi_1 \cos \xi\phi_2 m + \phi_2 \sin \xi\phi_2 m) \right) \right. \\ \left[k_2 (e^{\xi\phi_2 n} (\phi_1 \sin \xi\phi_2 n - \phi_2 \cos \xi\phi_2 n) (e^{\xi\phi_2 m} (\phi_1 \sin \xi\phi_2 m - \phi_2 \cos \xi\phi_2 m) \right] \\ \left[k_3 e^{-\xi\phi_2 n} (-\phi_1 \cos \xi\phi_2 n + \phi_2 \sin \xi\phi_2 n) - (e^{-\xi\phi_2 m} (-\phi_1 \cos \xi\phi_2 m + \phi_2 \sin \xi\phi_2 m) \right] + \\ \left[k_4 e^{-\xi\phi_2 n} (-\phi_1 \sin \xi\phi_2 n - \phi_2 \cos \xi\phi_2 n) - (e^{-\xi\phi_2 m} (-\phi_1 \sin \xi\phi_2 m - \phi_2 \cos \xi\phi_2 m) \right] \end{array} \right) \quad (G1.6)$$

Once the deflection is obtained the slop, moment and shear can be determined easily

$$\theta_c = \frac{-q}{2D} \left[\int_0^n \frac{dy}{dx} dx - \int_0^m \frac{dy}{dx} dx \right]$$

$$\theta_c = \frac{-q}{2D} \left(\begin{array}{l} k_1 (e^{\xi\phi_2 m} \cos \xi\phi_2 m - e^{\xi\phi_2 n} \cos \xi\phi_2 n) + k_2 (e^{\xi\phi_2 m} \sin \xi\phi_2 m - e^{\xi\phi_2 n} \sin \xi\phi_2 n) + \\ k_3 (e^{-\xi\phi_2 m} \cos \xi\phi_2 m - e^{-\xi\phi_2 n} \cos \xi\phi_2 n) + k_4 (e^{-\xi\phi_2 m} \sin \xi\phi_2 m - e^{-\xi\phi_2 n} \sin \xi\phi_2 n) \end{array} \right) \quad (G1.7)$$

$$M_c = -D \frac{-q}{2D} \left[\int_0^m \frac{d^2y}{dx^2} dx - \int_0^n \frac{d^2y}{dx^2} dx \right]$$

$$Mc = \frac{q}{2} \begin{pmatrix} k_1 \xi \left(e^{\xi \phi_1 n} (\phi_1 \cos \xi \phi_2 n - \phi_2 \sin \xi \phi_2 n) - e^{\xi \phi_1 m} (\phi_1 \cos \xi \phi_2 m - \phi_2 \sin \xi \phi_2 m) \right) \\ k_2 \xi \left(e^{\xi \phi_1 n} (\phi_1 \sin \xi \phi_2 n + \phi_2 \cos \xi \phi_2 n) - e^{\xi \phi_1 m} (\phi_1 \sin \xi \phi_2 m + \phi_2 \cos \xi \phi_2 m) \right) \\ -k_3 \xi \left(e^{-\xi \phi_1 n} (\phi_1 \cos \xi \phi_2 n + \phi_2 \sin \xi \phi_2 n) - e^{-\xi \phi_1 m} \frac{2\beta \phi_1 m}{a} (\phi_1 \cos \xi \phi_2 m + \phi_2 \sin \xi \phi_2 m) \right) \\ +k_4 \xi \left(e^{-\xi \phi_1 n} (-\phi_1 \sin \xi \phi_2 n + \phi_2 \cos \xi \phi_2 n) - e^{-\xi \phi_1 m} (-\phi_1 \sin \xi \phi_2 m + \phi_2 \cos \xi \phi_2 m) \right) \end{pmatrix} \quad (G1.8)$$

$$vc = -D \frac{-q}{2D} \left[\int_0^n \frac{d^3 y}{dx^3} dx - \int_0^m \frac{d^3 y}{dx^3} dx \right]$$

$$Vc = q \xi^2 \begin{pmatrix} K1 \left(\left(\phi_1^2 e^{\xi \phi_1 m} \cos \xi \phi_2 m - 2\phi_1 \phi_2 e^{\xi \phi_1 m} \sin \xi \phi_2 m - \phi_2^2 e^{\xi \phi_1 m} \cos \xi \phi_2 m \right) - \right. \\ \left. \left(\phi_1^2 e^{\xi \phi_1 n} \cos \xi \phi_2 n - 2\phi_1 \phi_2 e^{\xi \phi_1 n} \sin \xi \phi_2 n - \phi_2^2 e^{\xi \phi_1 n} \cos \xi \phi_2 n \right) + \right) \\ K2 \left(\left(\phi_1^2 e^{\xi \phi_1 m} \sin \xi \phi_2 m + 2\phi_1 \phi_2 e^{\xi \phi_1 m} \cos \xi \phi_2 m - \phi_2^2 e^{\xi \phi_1 m} \sin \xi \phi_2 m \right) - \right. \\ \left. \left(\phi_1^2 e^{\xi \phi_1 n} \sin \xi \phi_2 n + 2\phi_1 \phi_2 e^{\xi \phi_1 n} \cos \xi \phi_2 n - \phi_2^2 e^{\xi \phi_1 n} \sin \xi \phi_2 n \right) + \right) \\ K3 \left(\phi_1^2 e^{-\xi \phi_1 m} \cos \xi \phi_2 m + 2\phi_1 \phi_2 e^{-\xi \phi_1 m} \sin \xi \phi_2 m - \phi_2^2 e^{-\xi \phi_1 m} \cos \xi \phi_2 m - \right. \\ \left. \left(\phi_1^2 e^{-\xi \phi_1 n} \cos \xi \phi_2 n + 2\phi_1 \phi_2 e^{-\xi \phi_1 n} \sin \xi \phi_2 n - \phi_2^2 e^{-\xi \phi_1 n} \cos \xi \phi_2 n \right) + \right) \\ K4 \left(\phi_1^2 e^{-\xi \phi_1 m} \sin \xi \phi_2 m - 2\phi_1 \phi_2 e^{-\xi \phi_1 m} \cos \xi \phi_2 m - \phi_2^2 e^{-\xi \phi_1 m} \sin \xi \phi_2 m - \right. \\ \left. \left(\phi_1^2 e^{-\xi \phi_1 n} \sin \xi \phi_2 n - 2\phi_1 \phi_2 e^{-\xi \phi_1 n} \cos \xi \phi_2 n - \phi_2^2 e^{-\xi \phi_1 n} \sin \xi \phi_2 n \right) \right) \end{pmatrix} \quad (G1.9)$$

3. When point C is to the right of loaded region

$$wc = \frac{-q}{2D} \left[\int_0^m y(x) dx - \int_0^n y(x) dx \right]$$

$$wc = \frac{-q}{2D \xi (\phi_1^2 + \phi_2^2)} \begin{pmatrix} \left[k_1 (e^{\xi \phi_1 m} (\phi_1 \cos \xi \phi_2 m + \phi_2 \sin \xi \phi_2 m) - (e^{\xi \phi_1 n} (\phi_1 \cos \xi \phi_2 n + \phi_2 \sin \xi \phi_2 n) \right) \\ \left[k_2 (e^{\xi \phi_1 m} (\phi_1 \sin \xi \phi_2 m - \phi_2 \cos \xi \phi_2 m) - (e^{\xi \phi_1 n} (\phi_1 \sin \xi \phi_2 n - \phi_2 \cos \xi \phi_2 n) \right) \\ \left[k_3 (e^{-\xi \phi_1 m} (-\phi_1 \cos \xi \phi_2 m + \phi_2 \sin \xi \phi_2 m) - (e^{-\xi \phi_1 n} (-\phi_1 \cos \xi \phi_2 n + \phi_2 \sin \xi \phi_2 n) \right] + \\ \left[k_4 (e^{-\xi \phi_1 m} (-\phi_1 \sin \xi \phi_2 m - \phi_2 \cos \xi \phi_2 m) - e^{-\xi \phi_1 n} (-\phi_1 \sin \xi \phi_2 n - \phi_2 \cos \xi \phi_2 n) \right] \end{pmatrix} \quad (G1.10)$$

Once the deflection is obtained the slop, moment and shear can be determined

$$\theta_c = \frac{-q}{2D} \left[\int_0^m \frac{dy}{dx} dx - \int_0^n \frac{dy}{dx} dx \right]$$

$$\theta_c = \frac{-q}{2D} \left(k_1 (e^{\xi\phi_2 m} \cos \xi\phi_2 m - e^{\xi\phi_2 n} \cos \xi\phi_2 n) + k_2 (e^{\xi\phi_2 m} \sin \xi\phi_2 m - e^{\xi\phi_2 n} \sin \xi\phi_2 n) + k_3 (e^{-\xi\phi_2 m} \cos \xi\phi_2 m - e^{-\xi\phi_2 n} \cos \xi\phi_2 n) + k_4 (e^{-\xi\phi_2 m} \sin \xi\phi_2 m - e^{-\xi\phi_2 n} \sin \xi\phi_2 n) \right) \quad (G1.11)$$

$$M_c = -D \frac{-q}{2D} \left[\int_0^m \frac{d^2y}{dx^2} dx - \int_0^n \frac{d^2y}{dx^2} dx \right]$$

$$M_c = \frac{q}{2} \left(\begin{aligned} & k_1 \xi \left(e^{\xi\phi_2 m} (\phi_1 \cos \xi\phi_2 m - \phi_2 \sin \xi\phi_2 m) - e^{\xi\phi_2 n} (\phi_1 \cos \xi\phi_2 n - \phi_2 \sin \xi\phi_2 n) \right) \\ & k_2 \xi \left(e^{\xi\phi_2 m} (\phi_1 \sin \xi\phi_2 m + \phi_2 \cos \xi\phi_2 m) - e^{\xi\phi_2 n} (\phi_1 \sin \xi\phi_2 n + \phi_2 \cos \xi\phi_2 n) \right) \\ & -k_3 \xi \left(e^{-\xi\phi_2 m} (\phi_1 \cos \xi\phi_2 m + \phi_2 \sin \xi\phi_2 m) - e^{-\xi\phi_2 n} (\phi_1 \cos \xi\phi_2 n + \phi_2 \sin \xi\phi_2 n) \right) \\ & +k_4 \xi e^{-\xi\phi_2 m} (-\phi_1 \sin \xi\phi_2 m + \phi_2 \cos \xi\phi_2 m) - \left(e^{-\xi\phi_2 n} (-\phi_1 \sin \xi\phi_2 n + \phi_2 \cos \xi\phi_2 n) \right) \end{aligned} \right) \quad (G1.12)$$

$$V_c = -D \frac{-q}{2D} \left[\int_0^m \frac{d^3y}{dx^3} dx - \int_0^n \frac{d^3y}{dx^3} dx \right]$$

$$V_c = q\xi^2 \left(\begin{aligned} & K1 \left(\left(\phi_1^2 e^{\xi\phi_2 m} \cos \xi\phi_2 m - 2\phi_1\phi_2 e^{\xi\phi_2 m} \sin \xi\phi_2 m - \phi_2^2 e^{\xi\phi_2 m} \cos \xi\phi_2 m \right) - \right. \\ & \left. \left(\phi_1^2 e^{\xi\phi_2 n} \cos \xi\phi_2 n - 2\phi_1\phi_2 e^{\xi\phi_2 n} \sin \xi\phi_2 n - \phi_2^2 e^{\xi\phi_2 n} \cos \xi\phi_2 n \right) + \right. \\ & K2 \left(\left(\phi_1^2 e^{\xi\phi_2 m} \sin \xi\phi_2 m + 2\phi_1\phi_2 e^{\xi\phi_2 m} \cos \xi\phi_2 m - \phi_2^2 e^{\xi\phi_2 m} \sin \xi\phi_2 m \right) - \right. \\ & \left. \left(\phi_1^2 e^{\xi\phi_2 n} \sin \xi\phi_2 n + 2\phi_1\phi_2 e^{\xi\phi_2 n} \cos \xi\phi_2 n - \phi_2^2 e^{\xi\phi_2 n} \sin \xi\phi_2 n \right) + \right. \\ & K3 \left(\phi_1^2 e^{-\xi\phi_2 m} \cos \xi\phi_2 m + 2\phi_1\phi_2 e^{-\xi\phi_2 m} \sin \xi\phi_2 m - \phi_2^2 e^{-\xi\phi_2 m} \cos \xi\phi_2 m - \right. \\ & \left. \left(\phi_1^2 e^{-\xi\phi_2 n} \cos \xi\phi_2 n + 2\phi_1\phi_2 e^{-\xi\phi_2 n} \sin \xi\phi_2 n - \phi_2^2 e^{-\xi\phi_2 n} \cos \xi\phi_2 n \right) + \right. \\ & K4 \left(\phi_1^2 e^{-\xi\phi_2 m} \sin \xi\phi_2 m - 2\phi_1\phi_2 e^{-\xi\phi_2 m} \cos \xi\phi_2 m - \phi_2^2 e^{-\xi\phi_2 m} \sin \xi\phi_2 m \right) - \\ & \left. \left(\phi_1^2 e^{-\xi\phi_2 n} \sin \xi\phi_2 n - 2\phi_1\phi_2 e^{-\xi\phi_2 n} \cos \xi\phi_2 n - \phi_2^2 e^{-\xi\phi_2 n} \sin \xi\phi_2 n \right) \right) \quad (G1.13)$$

Case II ($\rho = 1/4$)

1. When point C is within the loaded region

$$w_c = \int \delta dw$$

$$= \int -\frac{q dx}{2D} y(x)$$

$$y(x) = k_1 e^{\xi \phi_1 x} + k_2 e^{-\xi \phi_1 x} + k_3 x e^{\xi \phi_1 x} + k_4 x e^{-\xi \phi_1 x}$$

$$k_i = -C_i \frac{2D}{q} \quad w_c = \frac{-q}{2D} \left[\int_0^m y(x) dx + \int_0^n y(x) dx \right]$$

$$w_c = \frac{-q}{2D \xi \phi_1} \left(\begin{array}{l} k_1 \left(e^{\xi \phi_1 m} + e^{\xi \phi_1 n} - 2 \right) - k_2 \left(e^{-\xi \phi_1 m} + e^{-\xi \phi_1 n} - 2 \right) + \\ k_3 \left(m e^{\xi \phi_1 m} + n e^{\xi \phi_1 n} - \frac{e^{\xi \phi_1 m}}{\xi \phi_1} - \frac{e^{\xi \phi_1 n}}{\xi \phi_1} + \frac{2}{\xi \phi_1} \right) \\ + k_4 \left(-m e^{-\xi \phi_1 m} - n e^{\xi \phi_1 n} + \frac{e^{\xi \phi_1 m}}{\xi \phi_1} + \frac{e^{\xi \phi_1 n}}{\xi \phi_1} - \frac{2}{\xi \phi_1} \right) \end{array} \right) \quad (G1.14)$$

The integration constants C_i are then obtained from the inverse matrix relation

$$C = L^{-1} M$$

Once the deflection is obtained the slop, moment and shear can be determined easily

$$\theta_c = \frac{-q}{2D} \left[\int_0^m \frac{dy}{dx} dx - \int_0^n \frac{dy}{dx} dx \right]$$

$$\theta_c = \frac{-q}{2D} \left(\begin{array}{l} k_1 \left(e^{\xi \phi_1 m} - e^{\xi \phi_1 n} \right) + k_2 \left(e^{-\xi \phi_1 m} - e^{-\xi \phi_1 n} \right) \\ + k_3 \left(e^{\xi \phi_1 m} - e^{\xi \phi_1 n} \right) + k_4 \left(\left(\frac{2e^{\xi \phi_1 m}}{\xi \phi_1} - e^{\xi \phi_1 m} \right) - \left(\frac{2e^{\xi \phi_1 n}}{\xi \phi_1} - e^{-\xi \phi_1 n} \right) \right) \end{array} \right) \quad (G1.15)$$

$$M_c = -D \frac{-q}{2D} \left[\int_0^m \frac{d^2 y}{dx^2} dx + \int_0^n \frac{d^2 y}{dx^2} dx \right]$$

$$M_c = \frac{-q}{2} \left(\begin{array}{l} k_1 \left(\xi \phi_1 e^{\xi \phi_1 m} + \xi \phi_1 e^{\xi \phi_1 n} - 2 \right) - k_2 \left(\xi \phi_1 e^{-\xi \phi_1 m} + \xi \phi_1 e^{-\xi \phi_1 n} - 2 \right) \\ + k_3 \left(\xi \phi_1 e^{\xi \phi_1 m} + \xi \phi_1 e^{\xi \phi_1 n} - 2 \right) + k_4 \left(\left(-2e^{-\xi \phi_1 m} + \xi \phi_1 e^{-\xi \phi_1 m} \right) + \left(-2e^{-\xi \phi_1 n} + \xi \phi_1 e^{-\xi \phi_1 n} \right) - 2 \right) \end{array} \right) \quad (G1.16)$$

$$V_c = -D \frac{-q}{2D} \left[\int_0^m \frac{d^3 y}{dx^3} dx + \int_0^n \frac{d^3 y}{dx^3} dx \right]$$

$$V_c = \frac{q}{2} \left(\begin{array}{l} k_1 \left(2\xi^2 \phi_1^2 e^{\xi \phi_1 m} - 2\xi^2 \phi_1^2 e^{\xi \phi_1 n} \right) + k_2 \left(2\xi^2 \phi_1^2 e^{-\xi \phi_1 m} - 2\xi^2 \phi_1^2 e^{-\xi \phi_1 n} \right) \\ k_3 \left(2\xi^2 \phi_1^2 e^{\xi \phi_1 m} - 2\xi^2 \phi_1^2 e^{\xi \phi_1 n} \right) + k_4 \left(\left(2\xi \phi_1 e^{-\xi \phi_1 m} - 2\xi^2 \phi_1^2 e^{-\xi \phi_1 m} \right) \right. \\ \left. - \left(2\xi \phi_1 e^{-\xi \phi_1 n} - 2\xi^2 \phi_1^2 e^{-\xi \phi_1 n} \right) \right) \end{array} \right) \quad (G1.17)$$

2. When point C is to the left of loaded region

$$w_c = \frac{-q}{2D} \left[\int_0^n y(x) dx - \int_0^m y(x) dx \right]$$

$$w_c = \frac{-q}{2\xi D \phi_1} \left(\begin{array}{l} k_1 \left(e^{\xi \phi_1 m} + e^{\xi \phi_1 n} \right) - k_2 \left(e^{-\xi \phi_1 n} - e^{-\xi \phi_1 m} \right) + \\ k_3 \left(n e^{\xi \phi_1 n} - m e^{\xi \phi_1 m} - \frac{e^{\xi \phi_1 n}}{\xi \phi_1} + \frac{e^{\xi \phi_1 m}}{\xi \phi_1} \right) + \\ k_4 \left(-n e^{\xi \phi_1 n} + m e^{-\xi \phi_1 m} + \frac{e^{\xi \phi_1 n}}{\xi \phi_1} - \frac{e^{\xi \phi_1 m}}{\xi \phi_1} \right) \end{array} \right) \quad (G1.18)$$

Once the deflection is obtained the slop, moment and shear can be determined easily

$$\theta_c = \frac{-q}{2D} \left[\int_0^n \frac{dy}{dx} dx - \int_0^m \frac{dy}{dx} dx \right]$$

$$\theta_c = \frac{-q}{2D} \left(\begin{array}{l} k_1 \left(e^{\xi \phi_1 m} - e^{\xi \phi_1 n} \right) + k_2 \left(e^{-\xi \phi_1 m} - e^{-\xi \phi_1 n} \right) \\ + k_3 \left(e^{\xi \phi_1 m} - e^{\xi \phi_1 n} \right) + k_4 \left(\left(\frac{2e^{\xi \phi_1 m}}{\xi \phi_1} - e^{\xi \phi_1 m} \right) - \left(\frac{2e^{\xi \phi_1 n}}{\xi \phi_1} - e^{-\xi \phi_1 n} \right) \right) \end{array} \right) \quad (G1.19)$$

$$M_c = -D \frac{-q}{2D} \left[\int_0^m \frac{d^2y}{dx^2} dx - \int_0^n \frac{d^2y}{dx^2} dx \right]$$

$$M_c = \frac{-q}{2} \left(\begin{array}{l} k_1 \left(\xi \phi_1 e^{\xi \phi_1 m} - \xi \phi_1 e^{\xi \phi_1 n} \right) - k_2 \left(\xi \phi_1 e^{-\xi \phi_1 m} - \xi \phi_1 e^{-\xi \phi_1 n} \right) \\ + k_3 \left(\xi \phi_1 e^{\xi \phi_1 n} - \xi \phi_1 e^{\xi \phi_1 m} \right) + k_4 \left(\left(\xi \phi_1 e^{-\xi \phi_1 n} - 2e^{-\xi \phi_1 n} \right) + \left(\xi \phi_1 e^{-\xi \phi_1 m} - 2e^{-\xi \phi_1 m} \right) \right) \end{array} \right) \quad (G1.20)$$

$$v_c = -D \frac{-q}{2D} \left[\int_0^n \frac{-d^3y}{dx^3} dx - \int_0^m \frac{-d^3y}{dx^3} dx \right]$$

$$V_c = \frac{q}{2} \left(\begin{array}{l} k_1 \left(2\xi^2 \phi_1^2 e^{\xi \phi_1 m} - 2\xi^2 \phi_1^2 e^{\xi \phi_1 n} \right) + k_2 \left(2\xi^2 \phi_1^2 e^{-\xi \phi_1 m} - 2\xi^2 \phi_1^2 e^{-\xi \phi_1 n} \right) \\ k_3 \left(2\xi^2 \phi_1^2 e^{\xi \phi_1 n} - 2\xi^2 \phi_1^2 e^{\xi \phi_1 m} \right) + k_4 \left(\left(2\xi \phi_1 e^{-\xi \phi_1 m} - 2\xi^2 \phi_1^2 e^{-\xi \phi_1 m} \right) \right. \\ \left. - \left(2\xi \phi_1 e^{-\xi \phi_1 n} - 2\xi^2 \phi_1^2 e^{-\xi \phi_1 n} \right) \right) \end{array} \right) \quad (G1.21)$$

3. When point C is to the right of loaded region

$$w_c = \frac{-q}{2D} \left[\int_0^m y(x) dx - \int_0^n y(x) dx \right]$$

$$w_c = \frac{-q}{2\xi D \phi_1} \left(\begin{array}{l} k_1 \left(e^{\xi \phi_1 m} - e^{\xi \phi_1 n} \right) - k_2 \left(e^{-\xi \phi_1 m} - e^{-\xi \phi_1 n} \right) + \\ k_3 \left(m e^{\xi \phi_1 m} - n e^{\xi \phi_1 n} - \frac{e^{\xi \phi_1 m}}{\xi \phi_1} + \frac{e^{\xi \phi_1 n}}{\xi \phi_1} \right) + \\ k_4 \left(-m e^{-\xi \phi_1 m} + n e^{\xi \phi_1 n} + \frac{e^{\xi \phi_1 m}}{\xi \phi_1} - \frac{e^{\xi \phi_1 n}}{\xi \phi_1} \right) \end{array} \right) \quad (G1.22)$$

Once the deflection is obtained the slop, moment and shear can be determined easily

$$\theta_c = \frac{-q}{2D} \left[\int_0^m \frac{dy}{dx} dx - \int_0^n \frac{dy}{dx} dx \right]$$

$$\theta_c = \frac{-q}{2D} \left(\begin{array}{l} k_1 \left(e^{\xi \phi_1 m} - e^{\xi \phi_1 n} \right) + k_2 \left(e^{-\xi \phi_1 m} - e^{-\xi \phi_1 n} \right) \\ + k_3 \left(e^{\xi \phi_1 m} - e^{\xi \phi_1 n} \right) + k_4 \left(\left(\frac{2e^{\xi \phi_1 m}}{\xi \phi_1} - e^{\xi \phi_1 m} \right) - \left(\frac{2e^{\xi \phi_1 n}}{\xi \phi_1} - e^{-\xi \phi_1 n} \right) \right) \end{array} \right) \quad (G1.23)$$

$$M_c = -D \frac{-q}{2D} \left[\int_0^m \frac{d^2 y}{dx^2} dx - \int_0^n \frac{d^2 y}{dx^2} dx \right]$$

$$M_c = \frac{-q}{2} \left(\begin{array}{l} k_1 \left(\xi \phi_1 e^{\xi \phi_1 m} - \xi \phi_1 e^{\xi \phi_1 n} \right) - k_2 \left(\xi \phi_1 e^{-\xi \phi_1 m} - \xi \phi_1 e^{-\xi \phi_1 n} \right) \\ + k_3 \left(\xi \phi_1 e^{\xi \phi_1 m} - \xi \phi_1 e^{\xi \phi_1 n} \right) + k_4 \left(\left(\xi \phi_1 e^{-\xi \phi_1 m} - 2e^{-\xi \phi_1 m} \right) + \left(\xi \phi_1 e^{-\xi \phi_1 n} - 2e^{-\xi \phi_1 n} \right) \right) \end{array} \right) \quad (G1.24)$$

$$V_c = -D \frac{-q}{2D} \left[\int_0^m \frac{d^3 y}{dx^3} dx - \int_0^n \frac{d^3 y}{dx^3} dx \right]$$

$$V_c = \frac{q}{2} \left(\begin{array}{l} k_1 \left(2\xi^2 \phi_1^2 e^{\xi \phi_1 m} - 2\xi^2 \phi_1^2 e^{\xi \phi_1 n} \right) + k_2 \left(2\xi^2 \phi_1^2 e^{-\xi \phi_1 m} - 2\xi^2 \phi_1^2 e^{-\xi \phi_1 n} \right) \\ k_3 \left(2\xi^2 \phi_1^2 e^{\xi \phi_1 m} - 2\xi^2 \phi_1^2 e^{\xi \phi_1 n} \right) + k_4 \left(\left(2\xi \phi_1 e^{-\xi \phi_1 m} - 2\xi^2 \phi_1^2 e^{-\xi \phi_1 m} \right) \right. \\ \left. - \left(2\xi \phi_1 e^{-\xi \phi_1 n} - 2\xi^2 \phi_1^2 e^{-\xi \phi_1 n} \right) \right) \end{array} \right) \quad (G1.25)$$

Case III $\rho > \frac{1}{4}$

1. When point C is within the loaded region

$$w_c = \int \delta dw$$

$$= \int -\frac{q dx}{2D} y(x)$$

Where $y(x) = k_1 e^{\xi \phi_1 x} \cosh \xi \phi_2 x + k_2 e^{\xi \phi_1 x} \sinh \xi \phi_2 x + k_3 e^{-\xi \phi_1 x} \cosh \xi \phi_2 x + k_4 e^{-\xi \phi_1 x} \sinh \xi \phi_2 x$

And $k_i = -i C \frac{2D}{q}$

$$w_c = \frac{-q}{2D} \left[\int_0^m y(x) dx + \int_0^n y(x) dx \right]$$

$$w_c = \frac{-q}{2D\xi(\phi_1^2 + \phi_2^2)} \left(\begin{aligned} & \left[k_1 (e^{\xi \phi_1 m} (\phi_1 \cosh \xi \phi_2 m - \phi_2 \sinh \xi \phi_2 m) + e^{\xi \phi_1 n} (\phi_1 \cosh \xi \phi_2 n - \phi_2 \sinh \xi \phi_2 n) - 2\phi_1) + \right. \\ & \left[k_2 (e^{\xi \phi_1 m} (\phi_1 \sinh \xi \phi_2 m - \phi_2 \cosh \xi \phi_2 m) + e^{\xi \phi_1 n} (\phi_1 \sinh \xi \phi_2 n - \phi_2 \cosh \xi \phi_2 n) + 2\phi_2) + \right. \\ & \left. \left[-k_3 (e^{-\xi \phi_1 m} (\phi_1 \cosh \xi \phi_2 m + \phi_2 \sinh \xi \phi_2 m) + e^{-\xi \phi_1 n} (\phi_1 \cosh \xi \phi_2 n + \phi_2 \sinh \xi \phi_2 n) - 2\phi_1) + \right. \right. \\ & \left. \left. \left[k_4 (e^{-\xi \phi_1 m} (-\phi_1 \sinh \xi \phi_2 m - \phi_2 \cosh \xi \phi_2 m) + e^{-\xi \phi_1 n} (-\phi_1 \sinh \xi \phi_2 n - \phi_2 \cosh \xi \phi_2 n) - 2\phi_2) \right] \right) \right] \end{aligned} \right) \quad (G1.26)$$

Once the deflection is obtained the slop, moment and shear can be determined easily

$$\theta_c = \frac{-q}{2D} \left[\int_0^m \frac{dy}{dx} dx - \int_0^n \frac{dy}{dx} dx \right]$$

$$\theta_c = \frac{-q}{2D} \left(\begin{aligned} & \left(k_1 (e^{\xi \phi_1 m} \cosh \xi \phi_2 m - e^{\xi \phi_1 n} \cosh \xi \phi_2 n) + k_2 (e^{\xi \phi_1 m} \sinh \xi \phi_2 m - e^{\xi \phi_1 n} \sinh \xi \phi_2 n) + \right. \\ & \left. k_3 (e^{-\xi \phi_1 m} \cosh \xi \phi_2 m - e^{-\xi \phi_1 n} \cosh \xi \phi_2 n) + k_4 (e^{-\xi \phi_1 m} \sinh \xi \phi_2 m - e^{-\xi \phi_1 n} \sinh \xi \phi_2 n) \right) \end{aligned} \right) \quad (G1.27)$$

$$M_c = -D \frac{-q}{2D} \left[\int_0^m \frac{d^2 y}{dx^2} dx + \int_0^n \frac{d^2 y}{dx^2} dx \right]$$

$$M_c = \frac{q}{2} \left(\begin{array}{l} k_1 \left(\xi \phi_1 e^{\xi \phi_1 m} \cosh \xi \phi_2 m - \xi \phi_2 e^{\xi \phi_1 m} \sinh \xi \phi_2 m + \xi \phi_1 e^{\xi \phi_1 n} \cosh \xi \phi_2 n - \xi \phi_2 e^{\xi \phi_1 n} \sinh \xi \phi_2 n - 2\phi_1 \right) \\ + k_2 \left(\xi \phi_1 e^{\xi \phi_1 m} \sinh \xi \phi_2 m + \xi \phi_2 e^{\xi \phi_1 m} \cosh \xi \phi_2 m + \xi \phi_1 e^{\xi \phi_1 n} \sinh \xi \phi_2 n + \xi \phi_2 e^{\xi \phi_1 n} \cosh \xi \phi_2 n - 2\phi_2 \right) \\ + k_3 \left(-\xi \phi_1 e^{-\xi \phi_1 m} \cosh \xi \phi_2 m - \xi \phi_2 e^{-\xi \phi_1 m} \sinh \xi \phi_2 m - \xi \phi_1 e^{-\xi \phi_1 n} \cosh \xi \phi_2 n - \xi \phi_2 e^{-\xi \phi_1 n} \sinh \xi \phi_2 n - 2\phi_1 \right) \\ + k_4 \left(-\xi \phi_1 e^{-\xi \phi_1 m} \sinh \xi \phi_2 m + \xi \phi_2 e^{-\xi \phi_1 m} \cosh \xi \phi_2 m - \xi \phi_1 e^{-\xi \phi_1 n} \sinh \xi \phi_2 n + \xi \phi_2 e^{-\xi \phi_1 n} \cosh \xi \phi_2 n - 2\phi_2 \right) \end{array} \right) \quad (G1.28)$$

After few steps one can obtain this equation

$$V_c = -D \frac{-q}{2D} \left[\int_0^m \frac{d^3 y}{dx^3} dx - \int_0^n \frac{d^3 y}{dx^3} dx \right]$$

$$V_c = q \xi^2 \left(\begin{array}{l} K1 \left(\left(\phi_1^2 e^{\xi \phi_1 m} \cosh \xi \phi_2 m + 2\phi_1 \phi_2 e^{\xi \phi_1 m} \sinh \xi \phi_2 m + \phi_2^2 e^{\xi \phi_1 m} \cosh \xi \phi_2 m \right) - \right. \\ \left. \left(\phi_1^2 e^{\xi \phi_1 n} \cosh \xi \phi_2 m + 2\phi_1 \phi_2 e^{\xi \phi_1 n} \sinh \xi \phi_2 n + \phi_2^2 e^{\xi \phi_1 n} \cosh \xi \phi_2 n \right) + \right) \\ K2 \left(\left(\phi_1^2 e^{\xi \phi_1 m} \sinh \xi \phi_2 m + 2\phi_1 \phi_2 e^{\xi \phi_1 m} \cosh \xi \phi_2 m + \phi_2^2 e^{\xi \phi_1 m} \sinh \xi \phi_2 m \right) - \right. \\ \left. \left(\phi_1^2 e^{\xi \phi_1 n} \sinh \xi \phi_2 n + 2\phi_1 \phi_2 e^{\xi \phi_1 n} \cosh \xi \phi_2 n + \phi_2^2 e^{\xi \phi_1 n} \sinh \xi \phi_2 n \right) + \right) \\ K3 \left(\phi_1^2 e^{-\xi \phi_1 m} \cosh \xi \phi_2 m - 2\phi_1 \phi_2 e^{-\xi \phi_1 m} \sinh \xi \phi_2 m + \phi_2^2 e^{-\xi \phi_1 m} \cosh \xi \phi_2 m - \right. \\ \left. \left(\phi_1^2 e^{-\xi \phi_1 n} \cosh \xi \phi_2 n - 2\phi_1 \phi_2 e^{-\xi \phi_1 n} \sinh \xi \phi_2 n + \phi_2^2 e^{-\xi \phi_1 n} \cosh \xi \phi_2 n \right) + \right) \\ K4 \left(\phi_1^2 e^{-\xi \phi_1 m} \sinh \xi \phi_2 m - 2\phi_1 \phi_2 e^{-\xi \phi_1 m} \cosh \xi \phi_2 m + \phi_2^2 e^{-\xi \phi_1 m} \sinh \xi \phi_2 m - \right. \\ \left. \left(\phi_1^2 e^{-\xi \phi_1 n} \sinh \xi \phi_2 n - 2\phi_1 \phi_2 e^{-\xi \phi_1 n} \cosh \xi \phi_2 n + \phi_2^2 e^{-\xi \phi_1 n} \sinh \xi \phi_2 n \right) \right) \end{array} \right) \quad (G1.29)$$

2. When point C is to the left of loaded region

$$w_c = \frac{-q}{2D} \left[\int_0^n y(x) dx - \int_0^m y(x) dx \right]$$

$$w_c = \frac{-q}{2D \xi (\phi_1^2 + \phi_2^2)} \left(\begin{array}{l} \left[k_1 (e^{\xi \phi_1 n} (\phi_1 \cos \xi \phi_2 n - \phi_2 \sin \xi \phi_2 n) - (e^{\xi \phi_1 m} (\phi_1 \cosh \xi \phi_2 m - \phi_2 \sin \xi \phi_2 m))) \right] + \\ \left[k_2 (e^{\xi \phi_1 n} (\phi_1 \sinh \xi \phi_2 n - \phi_2 \cosh \xi \phi_2 n) - (e^{\xi \phi_1 m} (\phi_1 \sinh \xi \phi_2 m - \phi_2 \cosh \xi \phi_2 m))) \right] + \\ \left[-k_3 (e^{-\xi \phi_1 n} (\phi_1 \cosh \xi \phi_2 n + \phi_2 \sinh \xi \phi_2 n) - (e^{-\xi \phi_1 m} (\phi_1 \cosh \xi \phi_2 m + \phi_2 \sinh \xi \phi_2 m))) \right] + \\ \left[k_4 (e^{-\xi \phi_1 n} (-\phi_1 \sinh \xi \phi_2 n - \phi_2 \cosh \xi \phi_2 n) - (e^{-\xi \phi_1 m} (-\phi_1 \sinh \xi \phi_2 m - \phi_2 \cosh \xi \phi_2 m))) \right] \end{array} \right) \quad (G1.30)$$

Once the deflection is obtained the slop, moment and shear can be determined

$$\theta_c = \frac{-q}{2D} \left[\int_0^n \frac{dy}{dx} dx - \int_0^m \frac{dy}{dx} dx \right]$$

$$\theta_c = \frac{-q}{2D} \left(k_1 (e^{\xi\phi_2 m} \cosh \xi\phi_2 n - e^{\xi\phi_2 n} \cosh \xi\phi_2 m) + k_2 (e^{\xi\phi_2 m} \sinh \xi\phi_2 n - e^{\xi\phi_2 n} \sinh \xi\phi_2 m) + \right. \\ \left. k_3 (e^{-\xi\phi_2 m} \cosh \xi\phi_2 n - e^{-\xi\phi_2 n} \cosh \xi\phi_2 m) + k_4 (e^{-\xi\phi_2 m} \sinh \xi\phi_2 n - e^{-\xi\phi_2 n} \sinh \xi\phi_2 m) \right) \quad (G1.31)$$

$$Mc = -D \frac{-q}{2D} \left[\int_0^n \frac{d^2y}{dx^2} dx - \int_0^m \frac{d^2y}{dx^2} dx \right]$$

$$Mc = \frac{q}{2} \left(k_1 \left((\xi\phi_1 e^{\xi\phi_2 n} \cosh \xi\phi_2 m + \xi\phi_2 e^{\xi\phi_2 n} \sinh \xi\phi_2 m) - (\xi\phi_1 e^{\xi\phi_2 m} \cosh \xi\phi_2 n + \xi\phi_2 e^{\xi\phi_2 m} \sinh \xi\phi_2 n) \right) \right. \\ + k_2 \left((\xi\phi_1 e^{\xi\phi_2 n} \sinh \xi\phi_2 m + \xi\phi_2 e^{\xi\phi_2 n} \cosh \xi\phi_2 m) - (\xi\phi_1 e^{\xi\phi_2 m} \sinh \xi\phi_2 n + \xi\phi_2 e^{\xi\phi_2 m} \cosh \xi\phi_2 n) \right) \\ + k_3 \left((\xi\phi_1 e^{-\xi\phi_2 n} \cosh \xi\phi_2 m - \xi\phi_2 e^{-\xi\phi_2 n} \sinh \xi\phi_2 m) - (\xi\phi_1 e^{-\xi\phi_2 m} \cosh \xi\phi_2 n - \xi\phi_2 e^{-\xi\phi_2 m} \sinh \xi\phi_2 n) \right) \\ \left. + k_4 \left((\xi\phi_1 e^{-\xi\phi_2 n} \sinh \xi\phi_2 m - \xi\phi_2 e^{-\xi\phi_2 n} \cosh \xi\phi_2 m) - (\xi\phi_1 e^{-\xi\phi_2 m} \sinh \xi\phi_2 n - \xi\phi_2 e^{-\xi\phi_2 m} \cosh \xi\phi_2 n) \right) \right) \quad (G1.32)$$

$$Vc = -D \frac{-q}{2D} \left[\int_0^n \frac{-d^3y}{dx^3} dx - \int_0^m \frac{-d^3y}{dx^3} dx \right]$$

$$Vc = q\xi^2 \left(K1 \left((\phi_1^2 e^{\xi\phi_2 m} \cosh \xi\phi_2 n + 2\phi_1\phi_2 e^{\xi\phi_2 m} \sinh \xi\phi_2 n + \phi_2^2 e^{\xi\phi_2 m} \cosh \xi\phi_2 m) - \right. \right. \\ \left. (\phi_1^2 e^{\xi\phi_2 n} \cosh \xi\phi_2 m + 2\phi_1\phi_2 e^{\xi\phi_2 n} \sinh \xi\phi_2 m + \phi_2^2 e^{\xi\phi_2 n} \cosh \xi\phi_2 n) \right) + \\ K2 \left((\phi_1^2 e^{\xi\phi_2 m} \sinh \xi\phi_2 n + 2\phi_1\phi_2 e^{\xi\phi_2 m} \cosh \xi\phi_2 n + \phi_2^2 e^{\xi\phi_2 m} \sinh \xi\phi_2 m) - \right. \\ \left. (\phi_1^2 e^{\xi\phi_2 n} \sinh \xi\phi_2 m + 2\phi_1\phi_2 e^{\xi\phi_2 n} \cosh \xi\phi_2 m + \phi_2^2 e^{\xi\phi_2 n} \sinh \xi\phi_2 n) \right) + \\ K3 \left((\phi_1^2 e^{-\xi\phi_2 m} \cosh \xi\phi_2 n - 2\phi_1\phi_2 e^{-\xi\phi_2 m} \sinh \xi\phi_2 n + \phi_2^2 e^{-\xi\phi_2 m} \cosh \xi\phi_2 m) - \right. \\ \left. (\phi_1^2 e^{-\xi\phi_2 n} \cosh \xi\phi_2 m - 2\phi_1\phi_2 e^{-\xi\phi_2 n} \sinh \xi\phi_2 m + \phi_2^2 e^{-\xi\phi_2 n} \cosh \xi\phi_2 n) \right) + \\ K4 \left((\phi_1^2 e^{-\xi\phi_2 m} \sinh \xi\phi_2 n - 2\phi_1\phi_2 e^{-\xi\phi_2 m} \cosh \xi\phi_2 n + \phi_2^2 e^{-\xi\phi_2 m} \sinh \xi\phi_2 m) - \right. \\ \left. (\phi_1^2 e^{-\xi\phi_2 n} \sinh \xi\phi_2 m - 2\phi_1\phi_2 e^{-\xi\phi_2 n} \cosh \xi\phi_2 m + \phi_2^2 e^{-\xi\phi_2 n} \sinh \xi\phi_2 n) \right) \right) \quad (G1.33)$$

3. When point C is to the right of loaded region

$$w_c = \frac{-q}{2D} \left[\int_0^m y(x) dx - \int_0^n y(x) dx \right]$$

$$= \frac{-q}{2D\xi(\phi_1^2 + \phi_2^2)} \left(\begin{aligned} & \left[k_1 (e^{\xi\phi_1 m} (\phi_1 \cosh \xi\phi_2 m - \phi_2 \sin \xi\phi_2 m) - (e^{\xi\phi_1 n} (\phi_1 \cos \xi\phi_2 n - \phi_2 \sin \xi\phi_2 n)) \right] + \\ & \left[k_2 (e^{\xi\phi_1 m} (\phi_1 \sinh \xi\phi_2 m - \phi_2 \cosh \xi\phi_2 m) - (e^{\xi\phi_1 n} (\phi_1 \sinh \xi\phi_2 n - \phi_2 \cosh \xi\phi_2 n)) \right] + \\ & \left[-k_3 (e^{-\xi\phi_1 m} (\phi_1 \cosh \xi\phi_2 m + \phi_2 \sinh \xi\phi_2 m) - (e^{-\xi\phi_1 n} (\phi_1 \cosh \xi\phi_2 n + \phi_2 \sinh \xi\phi_2 n)) \right] + \\ & \left[-k_4 (e^{-\xi\phi_1 m} (-\phi_1 \sinh \xi\phi_2 m + \phi_2 \cosh \xi\phi_2 m) - (e^{-\xi\phi_1 n} (-\phi_1 \sinh \xi\phi_2 n + \phi_2 \cosh \xi\phi_2 n)) \right] \end{aligned} \right)$$

(G1.34)

Once the deflection is obtained the slop, moment and shear can be determined

$$\theta_c = \frac{-q}{2D} \left[\int_0^m \frac{dy}{dx} dx - \int_0^n \frac{dy}{dx} dx \right]$$

$$\theta_c = \frac{-q}{2D} \left(\begin{aligned} & k_1 (e^{\xi\phi_1 m} \cosh \xi\phi_2 m - e^{\xi\phi_1 n} \cosh \xi\phi_2 n) + k_2 (e^{\xi\phi_1 m} \sinh \xi\phi_2 m - e^{\xi\phi_1 n} \sinh \xi\phi_2 n) + \\ & k_3 (e^{-\xi\phi_1 m} \cosh \xi\phi_2 m - e^{-\xi\phi_1 n} \cosh \xi\phi_2 n) + k_4 (e^{-\xi\phi_1 m} \sinh \xi\phi_2 m - e^{-\xi\phi_1 n} \sinh \xi\phi_2 n) \end{aligned} \right)$$

(G1.35)

$$M_c = -D \frac{-q}{2D} \left[\int_0^m \frac{d^2 y}{dx^2} dx - \int_0^n \frac{d^2 y}{dx^2} dx \right]$$

After few steps one can obtain this equation

$$M_c = \frac{q}{2} \left(\begin{aligned} & k_1 (-\xi\phi_1 e^{\xi\phi_1 m} \cosh \xi\phi_2 m + \xi\phi_2 e^{\xi\phi_1 m} \sinh \xi\phi_2 m) - ((\xi\phi_1 e^{\xi\phi_1 n} \cosh \xi\phi_2 n + \xi\phi_2 e^{\xi\phi_1 n} \sinh \xi\phi_2 m)) \\ & + k_2 ((\xi\phi_1 e^{\xi\phi_1 m} \sinh \xi\phi_2 m + \xi\phi_2 e^{\xi\phi_1 m} \cosh \xi\phi_2 m) - (\xi\phi_1 e^{\xi\phi_1 n} \sinh \xi\phi_2 n + \xi\phi_2 e^{\xi\phi_1 n} \cosh \xi\phi_2 m) -) \\ & + k_3 ((\xi\phi_1 e^{-\xi\phi_1 m} \cosh \xi\phi_2 m - \xi\phi_2 e^{-\xi\phi_1 m} \sinh \xi\phi_2 m) - (\xi\phi_1 e^{-\xi\phi_1 n} \cosh \xi\phi_2 n - \xi\phi_2 e^{\xi\phi_1 n} \sinh \xi\phi_2 m)) \\ & + k_4 ((\xi\phi_1 e^{-\xi\phi_1 m} \sinh \xi\phi_2 m - \xi\phi_2 e^{-\xi\phi_1 m} \cosh \xi\phi_2 m) - (\xi\phi_1 e^{-\xi\phi_1 n} \sinh \xi\phi_2 n - \xi\phi_2 e^{\xi\phi_1 n} \cosh \xi\phi_2 m) -) \end{aligned} \right)$$

(G1.36)

$$v_c = -D \frac{-q}{2D} \left[\int_0^m \frac{d^3 y}{dx^3} dx - \int_0^n \frac{d^3 y}{dx^3} dx \right]$$

After few steps one can obtain this equation

$$Vc = q\xi^2 \left(\begin{array}{l} K1 \left(\left(\phi_1^2 e^{\xi\phi_1 m} \cosh \xi\phi_2 m + 2\phi_1\phi_2 e^{\xi\phi_1 m} \sinh \xi\phi_2 m + \phi_2^2 e^{\xi\phi_1 m} \cos \xi\phi_2 m \right) - \right. \\ \left. \left(\phi_1^2 e^{\xi\phi_1 n} \cosh \xi\phi_2 m + 2\phi_1\phi_2 e^{\xi\phi_1 n} \sinh \xi\phi_2 n + \phi_2^2 e^{\xi\phi_1 n} \cosh \xi\phi_2 n \right) + \right) \\ K2 \left(\left(\phi_1^2 e^{\xi\phi_1 m} \sinh \xi\phi_2 m + 2\phi_1\phi_2 e^{\xi\phi_1 m} \cosh \xi\phi_2 m + \phi_2^2 e^{\xi\phi_1 m} \sinh \xi\phi_2 m \right) - \right. \\ \left. \left(\phi_1^2 e^{\xi\phi_1 n} \sinh \xi\phi_2 n + 2\phi_1\phi_2 e^{\xi\phi_1 n} \cosh \xi\phi_2 n + \phi_2^2 e^{\xi\phi_1 n} \sinh \xi\phi_2 n \right) + \right) \\ K3 \left(\phi_1^2 e^{-\xi\phi_1 m} \cosh \xi\phi_2 m - 2\phi_1\phi_2 e^{-\xi\phi_1 m} \sinh \xi\phi_2 m + \phi_2^2 e^{-\xi\phi_1 m} \cosh \xi\phi_2 m - \right. \\ \left. \left(\phi_1^2 e^{-\xi\phi_1 n} \cosh \xi\phi_2 n - 2\phi_1\phi_2 e^{-\xi\phi_1 n} \sinh \xi\phi_2 n + \phi_2^2 e^{-\xi\phi_1 n} \cosh \xi\phi_2 n \right) + \right) \\ K4 \left(\phi_1^2 e^{-\xi\phi_1 m} \sinh \xi\phi_2 m - 2\phi_1\phi_2 e^{-\xi\phi_1 m} \cosh \xi\phi_2 m + \phi_2^2 e^{-\xi\phi_1 m} \sinh \xi\phi_2 m - \right. \\ \left. \left(\phi_1^2 e^{-\xi\phi_1 n} \sinh \xi\phi_2 n - 2\phi_1\phi_2 e^{-\xi\phi_1 n} \cosh \xi\phi_2 n + \phi_2^2 e^{-\xi\phi_1 n} \sinh \xi\phi_2 n \right) \right) \end{array} \right)$$

(G1.37)