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Combinatorial Design

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INTRODUCTION

In broad terms, the study of combinatorial design is the study of the structure of collections of subsets of a finite setwhen these collections of subsets satisfy certain prescribed properties. In particular, a block design has the property that every one of these subsets has the same size k and every pair of points in the set is in exactly the same number of these subsets.

In this paper, we will introduce the idea of balanced, incomplete block design (BIBD) and look at some special families of such designs, namely: finite projective planes, affine plane, Steiner triple systems and Kirkman triple systems. We will also describe the importance of "difference system" in the construction of designs. This design theory is the most interesting and applicable part of combinatorics. Block design can be applicable in agricultural experiments in order to minimize the effect of chance on decision, to design league schedules, security programs etc.

The discussion of this paper comprises two sections in which the first is dedicated to introducing the theoretical aspects of block design. The second section gives life for the first section by identifying and applying the theoretical aspects to the real physical systems.

Contents

Acronyms

ERTA	Ethiopian Radio and Television Agency
BIBD	Balanced incomplete block design
FPP	Finite projective plane
STS(v)	Steiner triple system of order v
KTS(v)	Kirkeman triple system of order v
RBIBD	Resolvable balanced incomplete block design
SBIBID	Symmetric balanced incomplete block design
vOrder	
b	*Block number*
r	*Replication number*
k	*Block size*
λ	*Index*

PART I: DESIGN THEORY

1.1 BALANCED INCOMPLETE BLOCK DESIGN

Definition 1.1.1 A Balanced incomplete block design (BIBD) with parameters (v, b, r, k, λ) is a space (X, A) where X is a set, A is a collection of subsets of X, the five parametersare non-negative integers with $v>k>O$ and the parameters represent the following.

- v (**order**):The size of X (Elements of X are points)
- b (***block number***):The number of elements of A(elements of A are blocks)
- **r** (***replication number***): the number of blocks to which every point belongs;
- **k** (***block size***): the common size of each block.
- λ (***Index***): the number of blocks to which every pair of distinct points belongs.

Balanced incomplete block designs are used in the design of experiments when the total number *(v)* of objects to be tested is greater than the number (k) that can be tested at any one time. They are used to design experiments where the subjects must be divided in to subsets (blocks) of some size to receive different treatments. Such that each subject tested the same number of times and every pair of subjects appears in the same number of subjects.

The word "*Balanced*" refers to the factthat λ remains constant. If λchanges, depending on the pairofpointschosen, the designis not balanced.

The word "*incomplete*" refers to the fact the $k<v$ that is the size of each block is less than the number of varieties.Different notationsare used for balanced incomplete block designs. Such as (v, b, r, k, λ) BIBD, (v, k, λ)BIBD and $S_{\lambda}(2, k, \lambda)$ design.

The origins of the theory of blocks designs can be traced back to the problem of designing certain types of statistical experiment. It is therefore not insignificant that the name of the distinguished statistician fisher is attached to one of the first results in the subject (Theorem 1.5.2).

The idea behind a block design can be seen in the following type of problem. Suppose that a number of brands of instant coffee are to be tested among a number of housewives, the object of the experiment being to let the ladies compare the different brands and decide on their relative merits. To make the tests as fair as possible, it is decided that the following conditions should satisfied:

(1) Each housewife should taste the same number of brands;
(2) Each pair of brands should be compared by the same number of housewives.

Clearly, one way of achieving this would be to give every housewife every brand of coffee, but this is wasteful and time consuming. The problem is to achieve the aim more economically. Mathematically, all that is involved is a set of varieties (the brands of coffee), and a collection of subsets of S (each subset consisting of those varieties which a particular housewife tastes) called blocks, with properties:

(a) Each block has the same number of elements;
(b) Each pair of varieties is contained in the same number of blocks.
Moreover a block design is a family of b subsets of asset X of elements
Such that, for some fixed k and λ,
(1) Each subset has k elements ,
(2) Each pair of elements of X occur together in exactlyλ subsets,

.

Lemma 1.1.2 In a BIBD each variety is contained in

$$r = \frac{\lambda(v-1)}{k-1} \text{ blocks.}$$

Proof .we use the important technique of counting in two ways and then equating the two counts. Let x_i be any one of the varieties, and suppose that x_i is contained in r blocks

$B_{i1}, B_{i2}, \ldots, B_{ir}$ (1)

Since each block contains k elements, each of these blocks contains k-1 varieties other than x_i. We now consider each of the v-1 pairs $\{x_i, y\}$, where y is variety different from x_i, and for each such pair, we count the number of blocks in which both varieties are contained.

Each pair $\{x_i, y\}$ is contained in λ blocks(these blocks must be λ of the blocks in (1) since they are all the blocks containing x_i).

Adding,we get λ(v-1).

On the other hand, each of the blocks in (1) contains k-1 pairs one element of which is x_i .Adding .we now get (k-1) r.equating these two counts, we obtain

λ(v-1) =(k-1) r.

Hence, x_i is contained in λ(v-1) / (k-1) blocks .This is true for each variety x_i, and thus each variety is contained in r= λ(v-1)/(k-1) blocks.

Corollary 1.1.3 In a BIBD,we have

bk=vr

Proof we have already observed that counting by rows, the number of 1's in the incidence matrix A of a BIBD is bk. By lemma 1.1.2, we know that each column of A contains r1's. Thus, counting by columns, the number of 1's in A equals vr. Equating the two counts, we obtain

bk= vr.

If a (v, b, r, k, λ) design exists then we have a relation.

$$b = \frac{\lambda v(v-1)}{k(k-1)} \text{ and } r = \frac{\lambda(v-1)}{k-1}$$

In view of this b and r are dependent parameters and by considering the above relation *a* (v, b, r, k, λ) design are commonly referred by three parameters *v*, *k*, and λ as *a* (v, k, λ) design

1.1.4 Necessary conditions for existence

1. For (v, b, r, k, λ) Design

I) vr = bk

II) $\lambda(v-1) = r(k-1)$

III) $b \geq v$ (Fishers inequality)

2. For (v, k, λ) Design

I) $\lambda(v-1) \equiv 0 \bmod (k-1)$

II) $\lambda v(v-1) \equiv 0 \bmod k(k-1)$

Example 1.1.5 Consider the set X= {1, 2, 3, 4} with the following subsets (blocks)

$B_1 = \{1, 2, 3\}$ $B_2 = \{2, 3, 4\}$

$B_3 = \{3, 4, 1\}$ $B_4 = \{4, 1, 2\}$

Those five parameters discussed are as follows

v(the order of X) =4

b(Block number b)=4

k(Block size) =3

r (Replicationnumber) = 3

λ(Index)=2

*Thisdesign is **(4, 4, 3, 3, 2)** or **(4, 3, 2)** design.*

Definition 1.1.6 *The incidence matrix of a (v,k,λ)design is a b by v matrix. $A=(a_{ij})$ defined*

by $a_{ij}=\begin{cases} 1 \text{ if the } i^{th} \text{block contain the } j^{th} \text{element} \\ 0 \text{ otherwise} \end{cases}$

for the blocks $B_1, B_2, B_3 \ldots B_n$ of (v, k, λ)design .

Example 1.1.7 Consider the above (4, 3, 2) design.Its incidence matrix representation is the following matrix.

$$\begin{array}{c} \\ \mathbf{B1} \rightarrow \\ \mathbf{B2} \rightarrow \\ \mathbf{B3} \rightarrow \\ \mathbf{B4} \rightarrow \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

The rows arelabeled with varieties of the design and the columns arelabeled with blocks.

Example 1.1.8 Gebeta is the knownboard gameof Ethiopia. Seven Gebeta Players areselected from different parts of Ethiopia for cultural sport competition. The committee decided that each player should play Gebeta each day and also decided that each day they should split in to two groups to play two known Gebeta games, the so called □ □ tooth" and "lame weledech".The group of size three to play "tooth□and size four to play "lame weledech□satisfying the following two conditions

1.Each pair plays together once in a group of size three.

2.Each pair plays together twice in a group of size four.

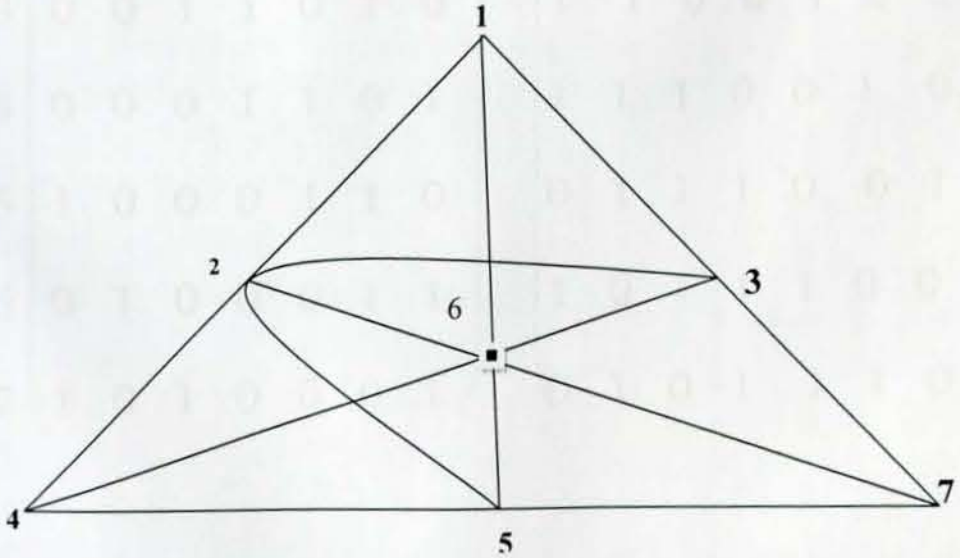
Question: Is it possible to arrange each pair of once in a group of size four just like in a group of size three?

SolutionHere is one solution of the groups for each day as shown. It can be easily checked that each pair plays once in a group of size threeand twice in a group of size four.

Table1.1.9

DAY	**GROUP(Size 3)**	**GROUP(Size 4)**
Day 1	{1, 2, 4}	{3,5,6,7}
Day 2	{3, 6, 4}	{3,5,6,7}
Day 3	{3, 2, 5}	{5,7,1,2}
Day 4	{4, 5, 7}	{6,1,2,3}
Day 5	{1, 6, 5}	{7,2,3,4}
Day 6	{2, 6, 7}	{1,3,4,5}
Day 7	{7, 3, 1}	{2,4,5,6}
	,3,1) design	**(7,4,2) design**

What we have done is to make use of the configuration known as the ***seven point plane*** or ***(Fano Plane)***as shown the figure below. In it are seven points and seven lines, with each line containing three points. And eachpairof points being inexactly one line. The blocks inthe group of size four is the complementsof the line of this configuration



To answer the above question we must have to check that the design (7, 4, 1) exists or not by using the necessary conditions for existence of (v, k,λ) design

i. $\lambda(v-1) \equiv 0 k-1$

$1x(7-1) \equiv 0 \mod 4-1$

$6 \equiv 0 \mod 3$ satisfied

ii. But $\lambda v\,(v-1) \equiv 0 \mod k(k-1)$

$\Rightarrow 1.7(7-1) \equiv 0 \mod 4(4-1)$

$7x6 \equiv 0 \mod 12$

$42 \equiv 0 \mod 12$ is*not satisfied*

is shows that (7, 4, 1) design does not exist

incidence matrix representation of the above designs is as follows

$$\begin{array}{l} \text{Day 1} \\ \text{Day 2} \\ \text{Day 3} \\ \text{Day 4} \\ \text{Day 5} \\ \text{Day 6} \\ \text{Day 7} \end{array} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Example 1.1.10

A slightly larger example (8,14,7,4,3)design on the set X={1,2,3,....8} with blocks.

{1,3,7,8}	{4,5,7,8}	{1,2,5,7}
{1,2,4,8}	{1,5,6,8}	{1,3,4,5}
{2,3,5,8}	{2,6,7,8}	{1,4,6,7}
{3,4,6,8}	{1,2,3,6}	{2,3,4,7}

{3, 5, 6, 7}{2, 4, 5, 6}

Example1.1.11 show that the following designs donot exist

a) (11,6,2)
b) (25, 10, 3)

solution

a. Since it would require $\lambda(v-1) = r(k-1)$

$\Rightarrow 2(11-1) = r(6-1)$ then r=4.

Since $bk = vr$

$\Rightarrow 6b = 44$

This is clearly impossible

b. $\lambda(v-1)=r(k-1) \Rightarrow 3(25-1)=r(10-1)$. Then r=8

And vr=bk

25r=bx10

10b=200

b=20

Which is a contradiction to the fact that v<*b*.

1.2 COMPLEMENTARY DESIGN

Given (v k λ) design D we can obtain another design $\overline{D}$ from it in which the blocks are the complements of the blocks of D then $\overline{D}$ is called the complementary design of D.

Theorem 1.2.1: Let D be a (v, k, λ) design on the set S with blocks $B_1, B_2, \ldots B_b$ then the subsets (blocks) $\bar{B}_i = S \setminus B_i$ from a(v, v-k, λ')design where $\lambda' = b-2r+\lambda$ provided that $\lambda>0$ such design D□ is called complementary design of D

Proof Since $|B_i|=k$ for each i $|\bar{B}_i|=|S|-|B_i|=v-k$ for each i We want to show that every pair of elements of S lie in exactly λ' blocks of B_i Now if x,y belongs to S(i.e. x,y εS) then x,y $\varepsilon\bar{B}_i$ Precisely neither x nor y belongs to B_i but by inclusion - exclusion principle The no of blocks containing neither x nor y is given by ***b- (no of blocks containing x) - (no of blocks containing y)+ (no of blocks containing x and y)***

$= b-2r+\lambda$

$\Rightarrow \underline{\lambda' = b-2r+\lambda}$

Example 1.2.2 Find the complementary design for (7, 3, 1) design

Solution

v=7 k=3 λ=1

i) $k^1 = v-k$

$= 7-3=4$

ii) λ^{1} = b-2r+λ since $b = \frac{\lambda v(v-1)}{k(k-1)}$ and $r = \frac{\lambda(v-1)}{k-1}$

$\Rightarrow \lambda^{1}$=b-2r +λ

$\Leftrightarrow \frac{\lambda V(V-1)}{k(k-1)} - 2\,\frac{\lambda(V-1)}{k-1} + 1$

$\Leftrightarrow \frac{1\times 7(7-1)}{3(3-1)} - \frac{1\times(7-1)}{3-1} + 1$

$\Leftrightarrow$7-6 +1 = 2

λ^{1}=2

The complementary design $\overline{D}$ is (7, 4, 2) design (see example 1.1.6). The design for a group of size four is the complementary design for design of group of size three.

Definition 1.2.3 A balanced incomplete block design is symmetric (SBIBD) if the number of points equals the number of blocks. That is v=b and its incidence matrix representation is a v x v matrix.

Definition 1.2.4 a (4m- 1, 2m-1, m-1) design is called handmade design.

1.3. DEFERENCE METHODS

The seven point plane can be constructed by starting with{1,2,4}and obtaining further blocks by adding 1 (mod7) successively, to obtain the Other blocks {2,3,5},{3,4,5},{3,4,6},{4,5,7},{5,6,1},{6,7,2} and {7,1,3}:as shown in example 1.1.6.
What is special about the choice of {1, 2, and 4} which makes the method work? similarly the block{1,3,4,5,9} was used in example(1.2.5) to obtain Handmard Design: what is special about this choice?
Consider the difference between elements of {1 2 4} modulo 7 they are ±(2-1), ±(4-2), ±(4-1) i.e. ±1, ± 2, ± 3 i.e. 1, 2, 3,4,5,6 all non-zero elements of Z_7 occurring once. Similarly consider the difference between element of {1,2,4,5,9} (mod 11) they are ± 1,± 2, ±3, ±4, ±8, , ±2 ,± 6,± 1, ±5, ±4 i.e. all non-zero number occurring twice.

Note the first gives rise to a design λ=1 and the second gives rise to a design with λ=2

Consider the following subtraction table mod V.

	0	2	4
0	0	6	4
2	1	0	5
4	3 2	0	

Table 1.3.1

-Mod11	1	3	4	5	9
1	0	9	8	7	3
3	2	0	10	9	5
4	3 1	0	10	6	
5	4	2	1	0	7
9	8	6	5	4	0

Table 1.3.2

	0	1	4
	0	6	**3**
	04		
	4 **3**	0	

Table 1.3.3

From the above three tables we observe the following:

I. Every non zero element occurs once in the table.1.3.1

II. Every non zero element occurs twice in the table 1.3.2

III. The number 1 occurs once in the table but the number 4 occurs twice in that table 1.3.3

The blocks {1,2,4}and {1,3,4,5,9}leads to a cyclic design but{0,1,4}doesnot

Definition 1.3.4A subset B of {0, 1,...,v-1}is a perfect difference set mod v if each of 1,2,...,v-1occurs the same number of times in its subtraction table.

Definition 1.3.5Let$Z_v$denotes the integers modulo v.A k-element subset D= $\{d_1,\ldots,d_k\}$ of Z_v is called cyclic perfect difference set.If D is a difference set, the set D+a=$\{d_1+a_1,\ldots\ldots,d_k+a\}$is called atranslate of D and $D_0,D_1,\ldots,D_v$-1 from the design.Such type of design is called cyclic design.

Example 1.3.6

Consider Z_7= {0,1,.....,6} by considering perfect difference set D={1 2 4}

D_0 =D+0= {1, 2, 4}

D_1 =D+1= {2, 3, 5}

D_2 =D+2= {3, 4, 6}

D_3 =D+3= {4, 5, 0}

D_4 =D+4= {5, 6, 1}

D_5 =D+5= {6, 0, 2}

D_6 =D+6= {0, 1, 3}

***Theorem*1.3.7** If D = $\{d_1,\ldots,d_k\}$ is a cyclic (v,k,λ) difference set then the translatesD+i, $0 \leq i<v-1$are blocks of symmetric (v, k,λ) design.

Proof

Here $D+i = \{d_1,\ldots,d_k+i\}$.

Clearly there are v translates each size k. So we have only to check the balance property. Two elements x, y are in the same translate D+a if and only if $x=a+d_i$ and $y=a+d_j$ for some $j\neq i$ we have

$\Rightarrow x-a= d_i$ and $y-a= d_j$

i.e. (x-a,y-a) is one of λ pairs (d_i, d_j) such that $d_i- d_j =x-y$

Example 1.3.8

1) {1,2,4, 10} is starter block for (13, 4, 1) design.

2) {1,2,5,15,17} is a (21,5, 1) difference set in Z_{21}.

3) {1,2,7,19,23,30}is a (31,6,1) difference set.

4) {1,3, 4, 5, 9} is (11, 5, 2) deference set.***The complementof a difference set is alsoa difference set.***

1.4 STEINER TRIPLE SYSTEM

Definition 1.4.1 Where λ=1 a (v,3,1) design is called Steiner triple system of order V STS (v).

The blocks are triples and each pair of varieties appears in exactly one triple.

Theorem1.4.2 (KirkMan1840)

Their existsSteiner triple system of order vvarietiesif and only if$v\geq 3$and $v \equiv 1 \mod 6$ or $v\equiv 3 \mod 6$

Proof

A single triple contains 3 pairshence there are a total ofv (v-1)/2 pairs of elements

Since each pair appears in exactly one triple

$b = v(v-1)/2$

$b = v(v-1)/6$

v consider one element x .Every other element is paired with x in the same block But those blocks contain exactly two other elements with no repetition so $r = (v-1)/2$and r is an integer must be odd since b is an integer v is an integer odd v is add either 3 divides v*or*6divides v-1

$\Rightarrow v \equiv 3 (\bmod 6)$ ***or*** $v \equiv 1 (\bmod 6)$

Example1.4.3

Steiner Triple Systems and there triples (blocks)

v=1 ∅

v=3 {1, 2, 3}

v=7 {1, 2, 4} {2, 3, 5} {3, 4, 6} {4, 5, 7}

{5,6,1} {6,7,2} {7,1,3}

v=9 {1,2,3} {2,5, 8} {3,4,8}

{4,5,6} {3,6,9} {1,6,8}

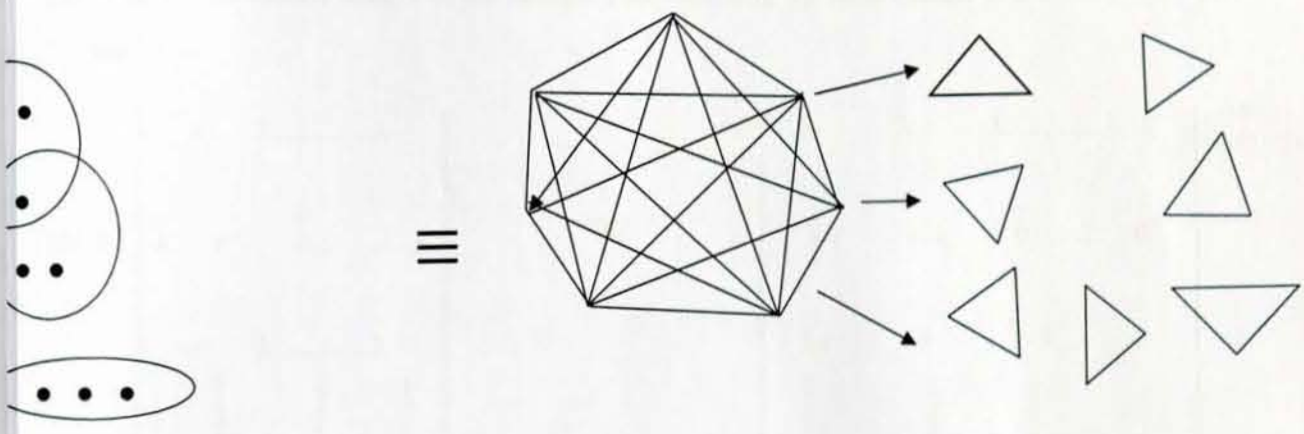
{7,8,9} {1,5,9} {2,4,9}

{1,4,7} {2,6,7} {3,5,7}

The complete graph of order v denoted by K_v is a graph with v vertices in which a pair of vertices is joined by an edge. Steiner triple system is equivalent to the decomposition of K_v.

***Example1.4.4 Decomposition*of K_7.**

A Steiner triple system can be represented graphically as follows. Each variety in the set X represented by a vertex, and each triple$\{a, b, c\}$ is represented by a triangle joiningthe vertices a,b and c. Since each pair of variety in exactly one triple, each edge belongs to exactly one triangle.



1.5 FISHER'S INEQUALITY

Theorem1.5.1 If A be the incidence matrix *of* (v, k, λ) then $A^TA=(r-\lambda)I+\lambda J$ where I is identity matrix and J is v x v matrix with every entry equal to 1.

***Theorem 1.5.2*(*Fisher inequality*)** in a (v, b, r, k, λ) design $b \geq v$

<u>Proof</u> we give a matrix proof rather than giving combinational proof

Let A be the incidence matrix of the design. Then denoting the determinant of a matrix M by |M| we have

$$|A| = \begin{vmatrix} r & \lambda & \lambda \dots\dots\dots \lambda \\ \lambda & r & \lambda \dots\dots\dots \lambda \\ \lambda & r & \lambda \dots\dots\dots \lambda \\ \vdots & \vdots & \vdots \quad\quad \vdots \\ \lambda & \lambda \dots\dots & \lambda \dots\dots r \end{vmatrix} = \begin{vmatrix} \lambda & \lambda & \lambda \dots\dots\dots \lambda \\ \lambda - r & r-\lambda & 0 \dots\dots\dots 0 \\ \lambda - r & 0 & r-\lambda \dots\dots 0 \\ \vdots & \vdots & \vdots \quad\quad \vdots \\ \lambda - r & 0 & 0 \quad\quad \lambda - r \end{vmatrix}$$

on subtracting the first row from each of the other rows. We know add to the first columns the sum of all other columns to obtain

$$|A^TA| = \begin{vmatrix} r+(v-1)\lambda & \lambda & \lambda \dots\dots\dots\dots \lambda \\ 0 & r-\lambda & 0 \dots\dots\dots\dots 0 \\ 0 & 0 & r-\lambda \dots\dots\dots\dots 0 \\ \vdots & \vdots & \vdots \quad\quad \vdots \\ 0 \dots\dots\dots\dots & 0 \dots\dots\dots\dots & 0 \dots\dots\dots\dots r-\lambda \end{vmatrix}$$

$=\{r+(v-1)\lambda\}(v-\lambda)^{v-1} = rk(r-\lambda)^{v-1}$

Since by theorem1.5.1 $r+(v-1)\lambda = r+r(k-1) = rk$

Now $k<v$ so by theorem1.5.1 $r>\lambda$ and so

$|A^TA| \neq 0$ NowA^TA is vxv matrix and so it's of rank $\rho(A^TA)$ must be v.But $\rho(A^TA) \leq \rho(A)$, where $\rho(A) \leq$ numberof rows of A; so $\rho(A^TA) \leq b$ thus $v < b$as required.

1.6 FINITE PROJECTIVE AND AFFINE PLANES

Definition *1.6.1*A finite projective plane (of order n) is a finite set of points and the collection of subsets of that set(each subset being called lines)satisfying.

i) Any two points lie on precisely one line.

ii) Any two lines meet in precisely one point.

iii) Each line contains n + 1 points.

iv) Each point lies on n+ 1 lines.

So the previous example (seven point plane) is a finite projective plane of order 2.In searching for symmetric design it is natural to look for such designs with $\lambda=1$.The seven point plane is such a design. In general, if we have a design with $k=n+1$and$\lambda=1$ then $\lambda(v-1)=k(k-1)$ gives $v-1 = n^2+n$ so that the design is a $(n^2+n+1, n+1, 1)$ design. ***Conversely***,any $(n^2+n+1, n+1, 1)$ design must satisfy$\lambda (v-1)=r(k-1)$ and$bk=vr$.We have $n^2+n=rn$ and $b(n+1)=(n^2+n+1)r$ so that $r= n+1=k$ and $b=n^2+n+1=v$, so the design must be symmetric .Thus in FPP there are equal number blocks and elements (symmetric).

Definition 1.6.2 For $n \geq 2$, a finite projective plane (FPP)of order n is a $(n^2+n+1, n+1, 1)$ design

Example 1.6.3 seven point plane FPP of order 2 is (7,3, 1) design and the blocks are just the blocks in example 1.1.6 of group of size 3.

Example 1.6.4FPP of order 3 is (13, 4,1) design with its blocks

{1,2,4,10},{2,3,5,11},{6,4,6,12},{4,5,7,13},{5,6,8,1},...,{10,11,13,6},{11,12,1,7},{12,13,2,8}.It has the cyclic nature of the design(modulo 13)

It is now known that such cyclic FPP of order P exists for all primes p (modulo p^2+p+1).

Definition 1.6.5

An affine plane (order n) is a finite set of points and the set of subsets of points(lines)satisfy the following

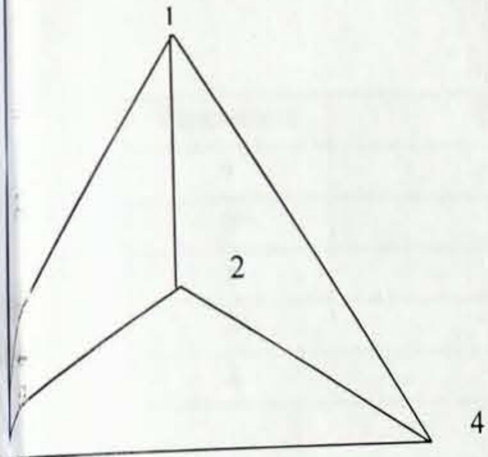
i) One line contains n points
ii) Every point ison exactly n+1 lines
iii) There are exactly n^2 points
iv) There are exactly $n^2 + \mathrm{n}$ lines and n+1 parallel classes.*(AParallel classes is the collection of lines that partition the plane)*

Definition 1.6.6:- For some a positive integer n $(n^2, n, 1)$ design is affine plane of order n.

Example 1.6.7:- Affine plane of order 2 has a set of points X={ 1,2,3,4}.Each line represents a block of the design.

Three parallel classes and 6 lines (block) as shown below

The lines are{1, 2} ,{3, 4}, {1, 3} , {2, 4}, {1, 4}, {3,2}



Those three parallel classes are

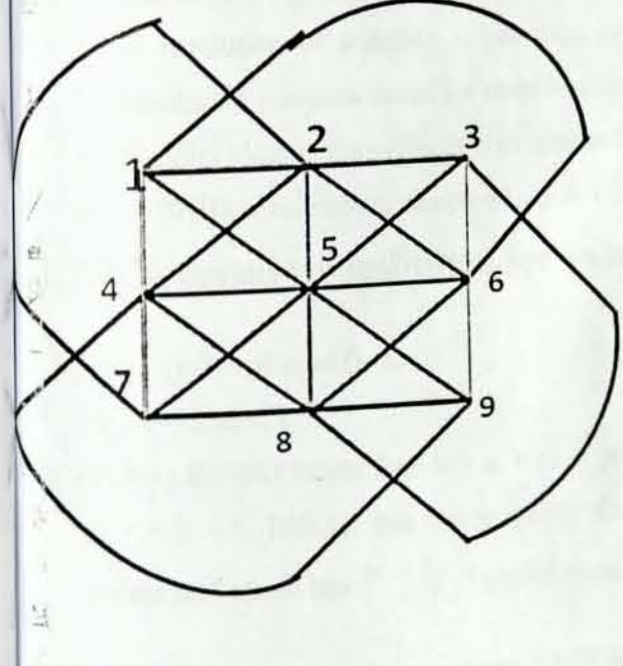
{{1,2}{3,4}}

{{1,3} {2,4}}

{{2,3}{1,4}}

Figure 1.6.8

Example 1.6.9:- Affine plane of order 3 has a set of points X = {1,2,..,9}and 12 lines with four parallel classes are listed below



{{1,2,3}, {4,5,6, }, {7,8,9}}

{{1,5,9}, {6,2,7}, {4,8,3}}

{{1,4,7}, {2,5,8}, {3,6,9}}

{{3,5,7}, {2,4,9}, {8,6,1}}

Figure 1.6.10

Parameters	FPP order (n)	Affine plane order n
v	$n^2 + n + 1$	n^2
b	$n^2 + n + 1$	$n^{2+}n$
r	$n + 1$	$n + 1$
k	$n + 1$	n
λ	1	1

Table 1.6.11

1.7 Resolvable design

Definitions 1.7.1

- A (v, k, **λ**) design on the set S is resolvable if the blocks can be arranged in to **r** groups so that each group forms partition of X.
- A resolution of a BIBD is partition of the family of blocks in to parallel classes. A resolution contains exactly **r** parallel classes
- Parallel class is the collection of blocks that partition a finite set X.
- A BIBD is resolvable denoted by RBIBD if it has at least one resolution.

1.7.2 Necessary conditions for existence of RBIBD

- $\lambda (v-1) = 0 \bmod (k-1)$
- $v = 0 \bmod k$

If a (v.k.λ) RBIBD exists then $b \geq v + r - 1$ where b is the number of blocks and $b = v + r - 1$ or $r = k + \lambda$ .RBIBD has the propriety that two non-parallel lines intersect in exactly k^2/v points and which has $v - 1/_2$Parallel class with each parallel class contains $^v/_3$ triples.

1.7.3 Kirkmantriple system (KTS)

xample 1.7.4:- A KTS (9) with parallel classes' $\pi_1, \pi_2, \pi_3, \pi_4$

n be defined as follows

π_1	π_2	π_3	π_4
2 3	1 4 7	1 5 9	1 6 8
5 6	2 5 8	2 6 7	2 4 9
8 9	3 6 9	3 4 8	3 5 7

xample 1.7.5

7.2 Kirkman's school girl problem

In 1980 Kirkman posses the following problem. Fifteen young ladies walk out to to school three abreast for seven days in succession. It is required to arrange them daily so that no two walk twice abreast. This is equivalent to finding the resolution of some (15, 3, 1) design or KTS (15).The following is the solution of Kirkmans school girl problem

A (V, 3, 1) RBIBD together with resolution of it is called Kirkeman triple system of order v.

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
9,10,12	10,11,13	11,12,14	12,13,15	13,14,9	14,15,10	15,9,11
15,8,1	9,8,2	10,8,3	11,8,4	12,8,5	13,8,6	14,8,7
13,2,7	14,3,1	15,4,2	9,5,3	10,6,4	11,7,5	12,1,6
11,3,6	12,4,7	13,5,1	14,6,2	15,7,3	9,1,4	10,2,5
14,4,5	13,5,6	9,6,7	10,7,1	11,1,2	12,2,3	13,3,4

.8 THE WILSON CONSTRUCTION

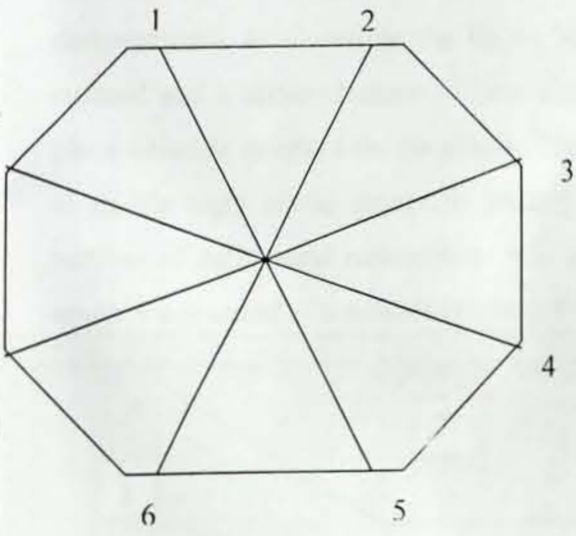
efore giving the Wilson construction we will need to see a few preliminaries.

A 1- factor of a graph G is a pair wise disjoints edges which partition the vertex set. We will eed 1-factorization of a very particular graph. The so called deficiency graph of (Z_n, +). The eficiency graph of (Z_n, +) where $n \equiv 1 \bmod 6$ or $n = 5 \bmod 6$ is defined to be the graph G = V,E) with vertex set $X = Z_n \setminus \{0\}$ and edge set $E = \{ \{x, -x\}, \{x, -2x\} \mid x \in X \}$ .What is important or us is the fact that the deficiency graph always has a1- factorization.

A wheel is a graph consisting a cycle of even length and a 1-factor in which the edges join the pposite vertices of the cycle (*called the spokes*).A wheel can be 1-factored in to three 1-factores y taking alternate edges of cycle as two of a 1-factor and the spokes for the third.

biwheel is a graph consisting of the union of two vertex disjoint cycles C_1 and C_2 of the same ength n and a 1-factor F consisting n edges such that: (i) each edges in F incident with one ertex in C_1 and one vertex in C_2 and (ii) C_1 and C_2 are isomorphic.

<u>**Example**</u> **:-** (wheel on 8 vertices and1-factores)



F_1		F_2		F_3	
1	2	2	3	1	5
3	4	4	5	2	6
5	6	6	7	3	7
7	8	1	8	4	8

1- Factorization of well

<u>**xample1.8.2**</u> Biwheel of 20 edges

2.4(*figure 2.4.1*)

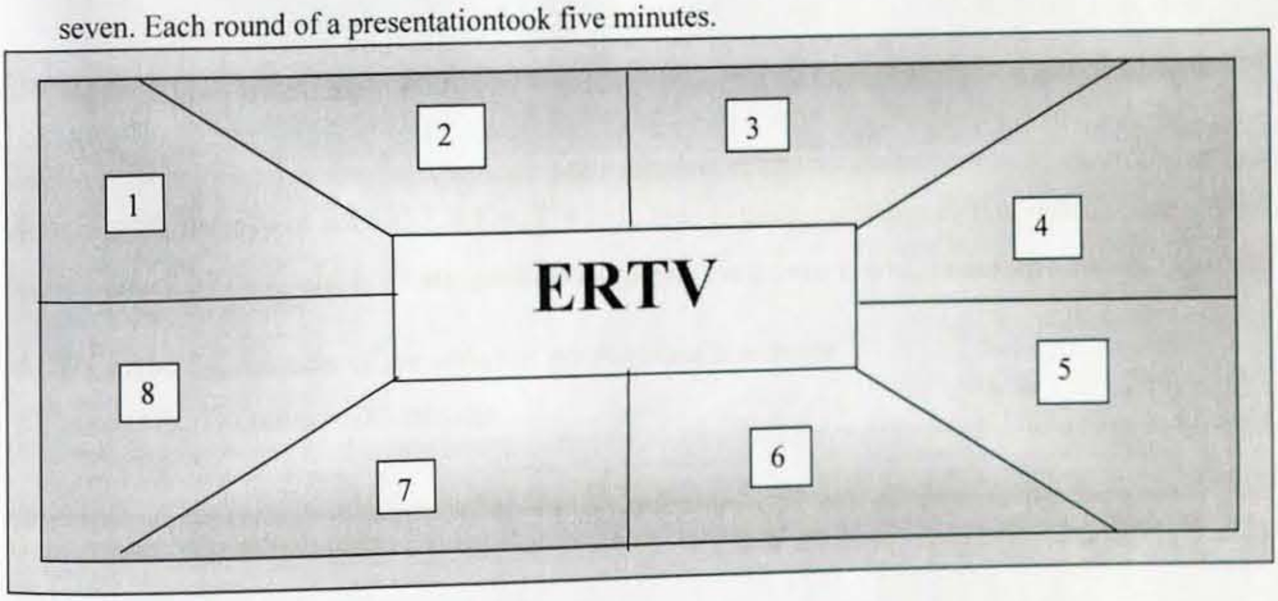
2.4(*figure 2.4.1*)

PART II: APPLICATION OF BLOCK DESIGN

2.1 November 8 2010: Dire Dawa

Celebration day of Ethiopian nations and nationalities

The daygives an opportunity to both Ethiopians and foreigners to observe and appreciate the culture of different nations, nationalities and peoples of Ethiopia in one place.Therefore the program designing committee worked hard to prepare a program that could fulfill the following conditions: The primary objective of the programmers was insuring equal time for each nations, nationalities and peoples on the stage. The stadium had been made to have 8 compartments as shown in the figure below. The eight compartments were reserved for cultural and traditional shows. These eight compartments were made to circle around the place which is reserved for the media. The eight compartments in the stadium were designed to enable eight ethnic groups to present their shows to the spectators at once. The total number of nationsand nationalities who presented their presentation on the date was fifty-seven. Each round of a presentationtook five minutes.



To make what is described above the programmers had to answer the following key questions

1. How many rounds of presentation were supposed to be handled on that day?
2. What about the number of presentation that each ethnic group may participate in
3. What type of BIBD is it?
4. How long did the program take?
5. How does one construct the blocks?

These should be answered satisfactorily

Solution We have v=57 k=8 and $\lambda = 1$

***1.** To find the number of rounds of presentation (block number)*

$$b = \frac{\lambda v(v-1)}{k(k-1)}$$

$$= \frac{1 \cdot 57(57-1)}{8(8-1)} = 57$$

***2.** To find the number of blocks in which each nation participate (replication number)*

$$r = \frac{\lambda(v-1)}{(k-1)}$$

$$= \frac{1(57-1)}{(8-1)} = 7$$

3.To identify the type of BIBD(57, 8,1)= $(7^2 +7+1, 7+1, 1)$ design which is FPP of order7. This is because FPP of order n for any positive integer n>1 is $(n^2+n+1, n+1, 1)$ design

***4.** We have 57 rounds of presentation each takes 5 minute*

We need (57x5) minutes = 285 minutes

5. *We can use starter block of* Z_{57}

   If we check any subset of $X = \{0,2,3,\ldots,56\}$ *of size eight is perfect difference set (mod) 57 then we can use it to generate all other blocks of a* $(57,8,1)$*design.*

$B_o = \{1, 6, 7, 9, 19, 38, 42, 49\}$ which is perfect difference set other blocks (translates) generated by Bo are

$B_0 = Bo+0 = \{ 1,6,7,9,19,38 , 42,49 \}$

$B_1 = Bo+1 = \{2,7,8,10,20,39,43,50\}$

$B_2 = Bo=2 = \{3,8,9,11,21,40,44,51 \}$

$B_3 = Bo+3 = \{4,9,10,12,22,41,45,52\}$

. . . .

. . . .

. . . .

. . . .

$B_{56} = Bo+36 = \{1 ,5, 6, 8,18,36,41, 48\}$

2.2 GROUPING

At the beginning of the academic year all teachers divide their students in to different groups to give assignmentsand use these grouping throughout the year. The problem is one or two students are doing their assignment throughout the year and submit under the name of the group. Due to this condition more than half of the students in the group may not participate in the assignment.

The resolvable design can be used to solve such kind of challenges:

***Example2.3.1*:-**For the case of 21 student we can use KTS (21) design with its resolution as follows

π_1	π_2	π_3	π_4	π_5
1 2 3	4 5 6	7 8 9	10 11 12	13 14 15
4 7 13	7 10 16	10 13 19	1 13 16	4 16 19
5 8 14	8 11 17	11 14 20	12 14 20	5 17 20
6 9 15	9 12 18	12 15 21	3 15 18	6 18 21
10 17 21	3 13 2	2 6 16	5 9 19	1 8 12
11 18 19	1 14 21	3 4 17	6 7 20	2 9 10
12 16 20	2 15 19	1 5 18	4 8 21	3 7 11

π_6	π_7	π_8	π_9	π_{10}
16 17 18	19 20 21	10 18 20	11 16 21	12 17 19
1 7 19	1 4 10	2 13 21	3 14 19	1 15 20
2 8 20	2 5 11	3 5 16	1 6 17	2 4 18
3 9 21	3 6 12	6 8 19	4 9 20	5 7 21
4 11 15	7 14 18	1 9 11	2 7 12	3 8 10
5 12 13	8 15 16	4 12 14	5 10 15	6 11 13
6 10 14	9 13 17	7 15 17	8 13 18	9 14 16

In order to get the above grouping, list all triples of $X = \{1,2,3,\ldots,21\}$those satisfy the condition that $\lambda = 1$ and portioned it into 10 parallel classes.

We can use the above resolutions (USE thepartition $\pi_i$for i^{th}month)

If we have fifteen students we can use Kirkman school girl problem of7 parallelclasses

Moreover for any n >1 KTS (6n+3) exists

2.3 LEAGUE SCHEDULE

1. Ethiopian Premier League

There are 18 teams which is (18,2,2) design. Two teams have got a chance to meet two times within a season

Generalization: -such League design for even number of teams is(2n, 2, 2)design

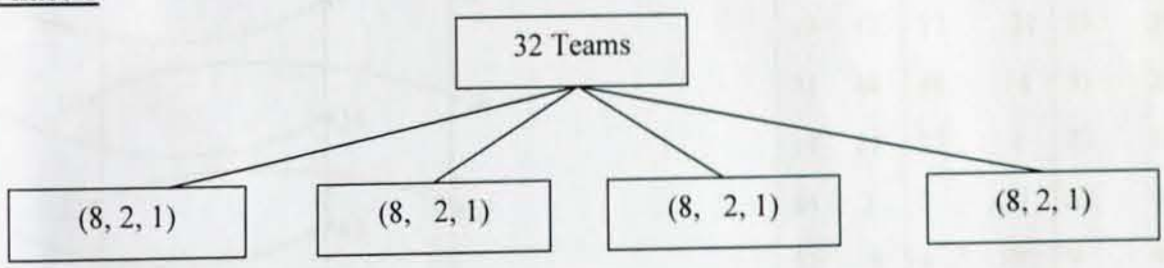
2. English Premier League

Thereare 20 teamswhich is (20, 2, 1) design. Two teams meet once in a season

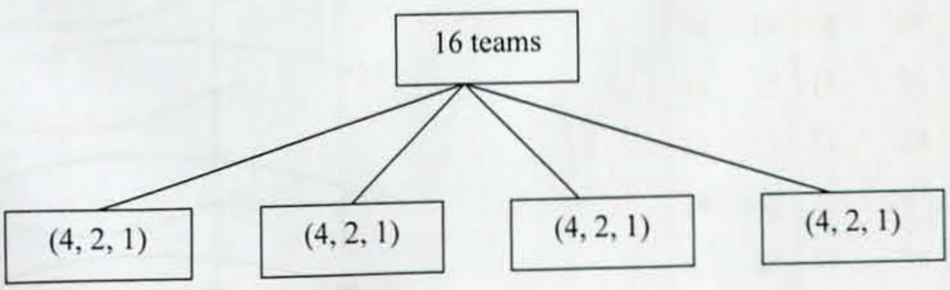
3. Champions League

There are 32 teams. And here we need five different types of design

Phase 1



Phase 2



Phase 3 8 Teams (8, 2, 1)

Phase 4 4 Teams (4, 2, 1)

2.4 Traffic police Placement

The deficiency graph (Z_{55} ,+) have the following three components for each component weproduce a1-factorization usingthe deficiency graph algorithm.F_1 and F_2 are alternate edges of cycles $C_1$and C_2 respectively

and$\boldsymbol{F_3} = \{\{x, y\}/x + y \equiv 0 \; mod \; 55,$

$x \in E_1, y \in \; E_2 \; , wherE_1 andE_2 \; are \; the \; edge \; set \; of \; C_1 andC_2 \; respectively\}$

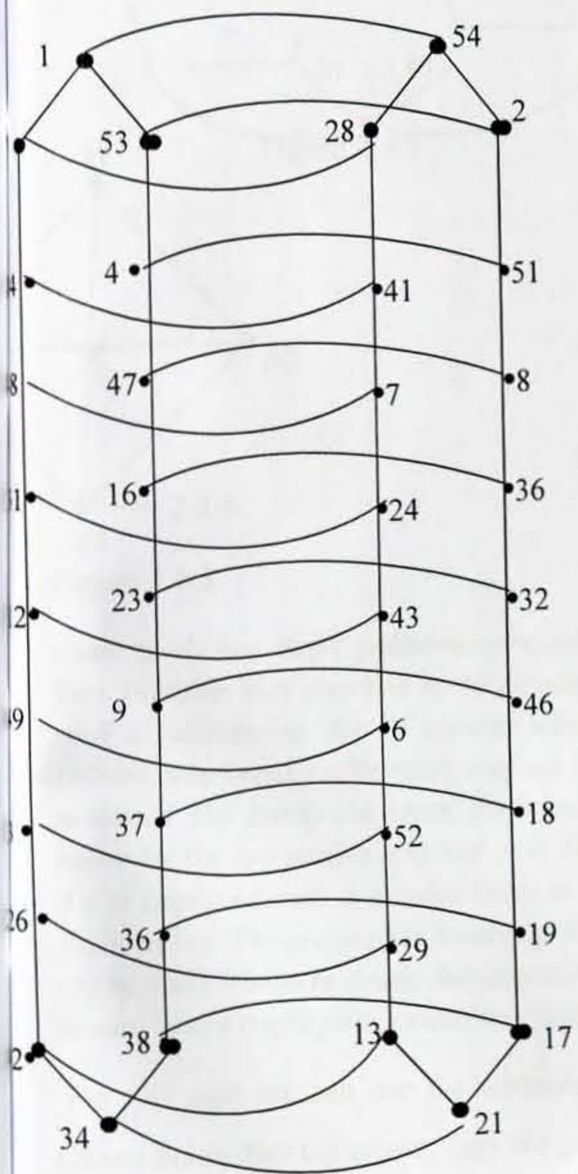


Figure2.4.1

F_1		F_2		F_3	
1	53	53	4	1	54
4	47	47	16	53	2
16	23	23	19	27	28
9	37	37	36	4	51
36	38	38	34	14	41
34	42	42	26	47	8
26	3	3	49	48	7
49	12	12	31	16	39
31	48	48	14	31	24
14	27	27	1	23	32
54	2	2	51	12	43
51	8	8	39	9	46
39	32	32	46	49	6
46	18	18	19	37	18
19	17	17	21	3	52
21	13	13	29	36	19
29	52	52	6	26	29
6	43	43	24	38	17
24	7	7	41	42	13
41	28	28	54	34	21

Table2.4.2

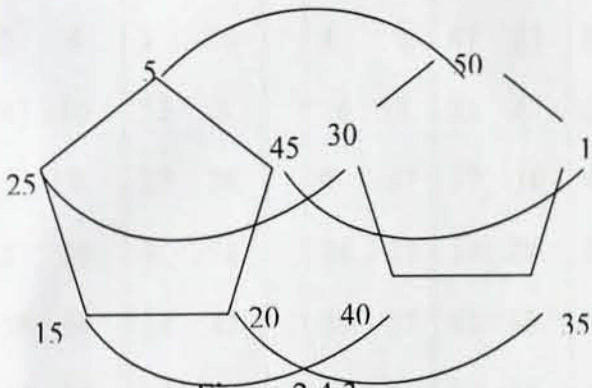
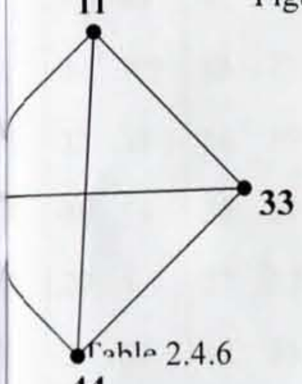


Figure 2.4.3

5	45	45	20	25	5
20	15	15	25	30	50
50	10	10	35	45	10
35	40	40	30	20	35
25	30	5	50	15	40

Table 2.4.4



11	33	33	44	11	44
44	22	22	11	22	33

Table 2.4.6

Figure 2.4.5

Customarily two traffic policemen are assigned to work together in one venue for a long time. Such tradition may give rise to the development of emotional attachment and solidarity. In turn such a relationship due to poverty which we all share in common corruption may result. Drivers, who break traffic rules, may opt to bribe the traffic police for 100 Birr instead of being penalized 300 Birrby the court. Such emotional attachment is more likely to make corruption easier for the two parties involved in it. The following program is used to reduce such problems if it is employed well. It is more likely to reduce corruption by reshuffling the placement of the traffic police. The program is developed for the case of placement of traffic police in Arada Sub-city of Addis Ababa.In Arada Sub-city there are about forty traffic police and twenty high traffic venues, where traffic police stand and facilitate the movement of vehicles and humans.

For this case we can use the tableon page 32 and turning to the right step by step as shown below bay the arrow, then we get the program for 60 days

D_2 D_3D_4 D_5 D_6 $D_{58}D_{59}D_{60}$

	↷	↷
3	53 4	1 54
7	47 16	53 2
3	23 19	27 28
7	37 36	4 51
8	38 34	14 41
2	42 26	47 8
	3 49	48 7
2	12 31	16 39
8	48 14	31 24
7	27 1	23 32
	2 51	12 43
	8 39	9 46
2	32 46	49 6
8	18 19	37 18
7	17 21	3 52
3	13 29	36 19
2	52 6	26 29
3	43 24	38 17
7	7 41	42 13
8	28 54	34 21

↷	↷	↷	↷
4 1	47 53	53 1	.....
16 53	23 4	27 54	.....
9 47	37 16	4 2	.....
36 23	38 19	14 28	.....
34 37	42 36	47 51	.....
26 38	3 34	48 41	.....
49 42	12 26	16 8	.....
31 3	48 49	31 7	.....
14 12	27 31	23 39	.....
54 48	2 14	12 24	.....
51 27	8 1	9 32	.....
39 2	32 51	49 43	.....
46 8	18 39	37 46	.....
19 32	17 46	3 6	.....
21 18	13 19	36 18	.....
29 17	52 21	26 52	.....
6 13	43 29	28 19	.....
24 52	7 6	42 29	.....
41 43	28 24	34 17	.....
28 7	54 41	21 13	.....

53 47	4 16	54 2
1 23	53 19	1 28
4 37	47 36	53 51
16 38	23 34	27 41
9 42	37 26	4 8
36 3	38 49	14 7
34 12	42 31	47 39
26 48	3 14	48 24
49 27	12 1	16 32
31 2	48 51	31 43
14 8	27 39	23 46
54 32	2 46	12 6
51 18	8 19	9 18
39 17	32 21	49 52
46 13	18 29	37 19
19 52	17 6	3 29
21 43	13 24	36 17
29 7	52 41	26 13
6 28	43 54	38 21
24 41	7 28	42 34

APPENDICES

v	k	λ	DIFFERENCE SET
7	3	1	1 2 4
13	4	1	0 1 3 9
11	5	2	1 3 4 5 9
21	5	1	3 6 7 12 14
15	7	3	0 1 2 4 5 8 10
31	6	1	1 5 11 24 25 27
19	9	4	1 4 5 6 7 9 11 16 17
23	11	5	1 2 3 4 6 8 9 12 13 16 18
57	8	1	1 6 7 9 19 38 42 49
73	9	1	1 2 4 8 16 32 37 55 64
31	15	7	1 2 3 4 6 8 12 15 16 17 23 24 27 29 30
91	10	1	0 1 3 9 27 49 56 61 77 81
40	13	4	1 2 3 5 6 9 14 15 18 20 25 27 35
35	17	8	0 1 3 4 5 9 11 12 13 14 16 17 21 27 28 29 33
133	12	1	1 11 16 4 41 43 52 60 74 78 121 128
43	21	10	1 4 6 9 10 11 13 14 15 16 17 21 23 24 25 31 35 3638 40 41
47	14	1	1 2 3 4 6 7 8 9 12 14 16 17 18 21 24 25 27 28 32 34 36 37 42
183	29	14	0 2 3 10 26 39 43 61 109 121 130 136 141 155

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