# SOME REMARKS ON DUALITY THEORY

# IN

# SEMI-INFINITE LINEAR OPTIMIZATION

## A GRADUATE SEMINAR REPORT

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# Preface

A linear optimization problem is the task of minimizing a linear real valued function of finitely many variables subject to linear constraints; in general there may be infinitely many constraints. This paper is devoted to such problems.

This type of problem often called semi infinite linear optimization problem, which is a generalization of the classical linear optimization problem (with only finitely many constraints), appears in the solution of many concrete examples. In this paper we mention the so-called uniform approximation of functions (as an application of weak duality), which plays a major role in the construction of computer representation of mathematical expression.

The central concept of this paper is that of duality in semi infinite linear optimization. Duality theory here is used as an effective tool for numerical treatment of linear optimization problem.

This paper is written based on the book "Linear optimization and Approximation" by Glash off and Gustafson (cf. [3]), for the methodological formulation, this paper follows the excellent book "Optimization theory I " by Deumlich (cf[1]).

# 0. Preliminaries

## 0.1 Optimization problems:

Optimization problems are encountered in many branches of technology, in science, and in economics as well as in our daily life. They appear in so many different shapes that is useless to attempt a uniform description of them. In the present section we will introduce a few general concepts which occur in all optimization problems.

The general optimization problem is:

Let M be a fixed set and let f be a real valued function defined on M.

We need to find $\bar{x} \in M$ such that

$$f(\bar{x}) \leq f(x), \text{ for all } x \in M .$$

M is called the feasible set and we call f an objective function.

The value of the optimization problem denoted by v is defined as:

$$V = \{\text{in f } f(x) | \, x \in M\}.$$

## 0.2 Some Mathematical Prerequisites.

This paper requires knowledge of some elementary concepts of mathematical analysis as well as linear algebra.

(1) Matrices. An $m \times n$ matrix A ( $m \geq 1$) is a rectangular array of m.n real numbers $a_{ik}$ ($i = 1, 2, \ldots, m$, $k = 1, \ldots, n$) where $a_{ik}$ is situated in row number i and column number k.

To each matrix $A = a_{ik}$, we define its transpose $A^T$, by $A^T = a_{ki}$, with row number k and column number i.We note that $(A^T)^T = A$.

**(2) Linear mappings**. Every m × n matrix A defines a linear mapping of $\Re^n$ into $\Re^m$, where $x \in \Re^n$ is mapped in to $y \in \Re^m$ by

$$y = Ax.$$

For this system of liner equation to be solvable uniquely, The inverse of A must exist.

(3)**Hyperplane.** Let $y \in \Re^n$ and $\eta \in R$ be given. Then the hyperplane denoted by $H(y; \eta)$ is defined as

$$H(y; \eta) := y^T x = y_1 x_1 + \ldots + y_n x_n = \eta, \text{ for all } x \in \Re^n.$$

y is called the normal vector of the hyperplane.

The hyperplane $H(y; \eta)$ separates $\Re^n$, in to three disjoint sets,

i.e. the hyperplane and the two "open half-spaces".

## (3) Scalar product and Euclidean norm

Let $x \in \Re^n$ then norm of x is denoted by $\|x\|$, satisfying

i) $\| x\| \geq 0,\ x \in \Re^n$ and $\|x\| = 0$ for $x = 0$ only

ii) $\|\lambda x\| = |\lambda|\ \|x\|,\ x \in \Re^n,\ \lambda \in \Re$;

iii) $\|x + y\| \leq \|x\| + \|y\|,\ x \in \Re^n,\ y \in \Re^n$.

The scalar product of $x \in \Re^n$ and $y \in \Re^n$ is given by

$$x^T y = y^T x = y_1 y_1 + \ldots + x_n y_n.$$

The real number

$$|x| = \sqrt{x^T x} = \left(x_1^2 + \cdots + x_n^2\right)^{1/2}$$

is called Euclidean norm. (i), (ii) and (iii) can be easily verified.

## (5) Some topological fundamentals

Let $A \subseteq \Re^n$.A point $a \in A$ is said to be an inner point if there is a sphere

$k_r(a)=\{x\in\Re^n \mid |x-a|<a\}$, such that $k(a)\subseteq A$. We denoted the interior of A by Å. A is said to boundary point of A denoted by bdA, if $a\in A\backslash$Å.

(6) Compact sets. Let $A\subseteq\Re^n$. A sis said to be bounded when there is $r>0$ such that $A\subseteq k_r(0)$. Closed bounded subsets of $\Re^n$ are compact.

**Definition**: $A\subseteq\Re^n$ is said to be compact if every infinite sequence $\{x_i\}_{i>1}$ in A has a convergent subsequence $\{x_{i_k}\}_{i>1}$ in A.

If f: $\Re^n\to\Re^m$ is a continuous mapping, then the image f(A) of every compact set A is also compact.

(7) **Theorem of Weierstrass.** Let A be a non empty compact subset of $\Re^n$ and f a real valued function defined on A.Then f assumes its maximum and minimum on A.

i.e., there exists $x_1\in A$ and $x_2\in A$ such that

$$f(x_1)=\max\,\{f(x)\mid x\in A\}$$

and

$$f(x_2)=\min\,\{f(x)\mid x\in A\}.$$

## 1. Weak Duality

### 1.1. Introduction:

In chapter 0, we have defined the general optimization problem by:

To find $\bar{x}\in M$ such that $(\bar{x})\le f(x)$ for all $x\in M$, where M is a fixed set and the objective function f is a real valued function defined on M.

In this paper we will try to study a particular optimization problem called linear optimization.

**Definition 1.1.1.** A linear optimization problem is an optimization problem in which the objective function is linear and the feasible domain is defined by linear constraint functions.

### Linear Optimization Problem:

Given a vector $c = (c_1, \ldots, c_n)^T \in \Re^n$, a non empty index set S, and for every $s \in S$, $a_s \in \Re^n$ and $b_s \in \Re$.

We need to find: $\tilde{y} \in \Re^n$ which solves the problem:

(p) $f(y) = c^T y \rightarrow \min$,

s.t $a_s^T y \geq b_s$, $\forall s \in S'$.

Or we can write (p) by

(p) $\sum_{r=1}^{n} a_r^T (s) c_r y_r \geq b(s)$, $s \in S$.

If we put $S = \{s_1, s_2, \ldots, S_m\}$, where $m \geq 1$, the corresponding linear constraints takes the form.

$$A^T y \geq b_i, \ i = 1, \ldots, m$$

where $A = (a_r(s_i))$, ($r \in \{1, \ldots, n$) and $s \in \{1, \ldots, m\}$.

**Definition 1.1.4** The Linear optimization problem obtained by selecting a finite subset $\{s_1, \ldots, s_m\}$ of s is called discriptization of the original problem.

**Example 1.** Let $S: = [0,1]$. Let $a_s = (1, s)^T$ and $b_s = \sqrt{s}$, $s \in [0, 1]$.

Then the constraints have the form

$$y_1 + sy_2 \geq \sqrt{s}, \ s \in [0,1], \ y \in \Re^2.$$

S may generate infinitely many hyperplane we discritized the problem as follows.

Select a natural number $m > 2$ and put

$$h := \frac{1}{m-1} \text{ and } s_i := (i-1)\,h.$$

We obtain

$$a_{s_i} = (1, (i-1)h)^T, \ i \in \{1, \ldots, m\}.$$

$$b_{s_i} = \sqrt{(i-1)h} \ , \ i \in (1, \ldots m\}.$$

That is we have finitely many constraints and the problem is identified as linear optimization problem with finitly many constraints.If we denote the value of the discetized problem by $v_m(p)$ then clearly we have

$$v_m(p) \leq v(p).$$

The discritization helps us in computational practices to calculate an approximate solution of a linear optimization problem with infinitly many constraints.

## 1.1 Weak Duality

In this section we try to put the foundation for the theoretical as well as computational treatment of linear optimization Problems. We will consider certain examples, which prepare us to understand the central concept of duality theory.

### 1.2.1 Duality Lemma and Dual problem.

Consider the optimization Problem:

$$\text{(p)} \quad f(y) = c^T y \rightarrow \min, \ y \in \Re^n. \tag{1}$$

$$\text{s.t} \ \sum_{r=1}^{n} (a_r(s_i))^T yr \geq b(s_i), \ i \in \{1, \ldots, q\} \tag{2}$$

$$y_r \geq 0, \ r \in \{1, \ldots, n\} \tag{3}$$

As soon as feasible vector y is available, we obtain an upper bound for the value of the problem. Since for any feasible vector y, $v(p) \leq c^T y$.

Now we are interested to determine a good lower bound for the value of the problem. To construct such lower bounds we use the following fundamental lemma and its corollary.

**Lemma 1.2.1** (Duality Lemma): Let $\{s_1, \ldots, s_q\} \subseteq s$, $q \geq 1$, and the non-negative numbers $x_1, x_2, \ldots, x_q$ be such that

$$C = as_1)\, x_1 + a(s_2)\, x_2 + \ldots + a(s_q)x_q \qquad (1)$$

Then for any feasible vector $y \in \Re^n$,

$$b(s_1)\, x_1 + b(s_2)x_2 + \ldots + b(s_q)x_q \leq c^T y \qquad (2)$$

**Proof**:- Since y is feasible for (p) by assumption we have

$$a(s)^T y \geq b(s),\ s \in S.$$

in particular

$$a(s_i)^T y \geq b(s_i),\ i \in \{1, \ldots, q\}$$

Since $x_i \geq 0$, $i \in \{1, \ldots, q\}$

$$b(s_i)x_i \leq (a(s_i)^T y)x_i,\ i \in \{1, 2, \ldots, q\}$$

hence we have

$$\sum_{i=1}^{q} b(s_i)x_i \leq \sum_{i=1}^{q} (a(s_i)^T y)x_i$$

$$= \sum_{i=1}^{q} (a(s_i)\, x_i)^T y$$

$$= c^T y, \text{ by (1)}$$

Therefore $\sum_{i=1}^{q} (b(s_i)x_i \leq c^T y$.

Corollary 1.2.2. Let $\{s_1, s_2, \ldots, s_q\} \subseteq s$, $q \geq 1$ and let the numbers $x_1, \ldots, x_q$ be such that

$$\sum_{i=1}^{q} (a_r(s_i)\, x_i) = c_r,\ r = 1, \ldots, n \tag{3}$$

Then $\sum_{i=1}^{q} (b(s_i)\, x_i \le v(p)$.

The Proof follows immediately from the definition of v(p) and the above lemma.

Now let us consider some examples as an application of the duality lemma and its corollary.

**Example 1.** Find the lower bound for the value of the problem

(P) $\quad y_1 + \frac{1}{2} y_2 \to \min$

subject to

$$y_1 + s y_2 \ge e^s,\ s \in [0, 1].$$

**Solution**: We need to find $\{s_1, s_2, \ldots, s_q\} \subseteq s$ and a non negative numbers $x_1, \ldots, x_q$ for which the assumption of the duality lemma holds.

First let us take q = 1, and find $s_1 \in [0, 1]$ and $x_1 \ge 0$.

$$C = \left(1, \frac{1}{2}\right)^T = a(s_1)\, x_1, \text{ when } a(s) = (1, s)^T.$$

From this

$$1 = 1x_1$$
$$\frac{1}{2} = s_1 x_1$$

This equation is uniquely solvable by $x_1 = 1$, $s_1 = \frac{1}{2}$.

$$b(s_1)x_1 = x_1.e^{s1} = 1.e^{\frac{1}{2}} = \sqrt{e} \le v(p), \qquad \text{by 1.2.2.}$$

To obtain a rough upper bound we need only to find numbers $\tilde{y}_1$, $\tilde{y}_2$ such that the straight line $\tilde{y}_1 + s\tilde{y}_2$ lies above the graph of the function $e^s$ in [0, 1].

For example take $\tilde{y}_1 = 1$, $\tilde{y}_2 = 2$, we get

$$V(p) \leq \tilde{y}_1 + \frac{1}{2}\tilde{y}_2 = 2$$

For a better lower bound we put q = 2

$$\sum_{i=1}^{q} a_r(s_1)x_i = c_r,\ r = 1, 2, \text{ and } c_1 = 1,\ c_2 = \frac{1}{2}.$$

$$a_1(s_1)x_1 + a_1(s_2)x_2 = c_1$$

$$a_2(s_1)x_1 + a_2(s_2)x_2 = c_2$$

$$x_1 + x_2 = c_1 = \frac{1}{2}$$

$$s_1x_1 + s_2x_2 = c_2 = \frac{1}{2}$$

If we put $s_1 = 0$ and $s_2 = 1$, we get

$$x_1 = x_2 = \frac{1}{2}.$$

Hence we have

$$x_1 e^{s_1} + x_2 e^{s_2} = \frac{1}{2} + \frac{1}{2} e \leq v(p) \qquad (*)$$

More over $(\sum_{i=1}^{q} a_r(s_i)x_i)y_r = c_r y_r$, r = 1, 2.

which implies

$$x_1 + x_2 + \ldots x_q = 1$$

$$s_1x_1 + s_2x_2 + \ldots + s_qx_q = \frac{1}{2}$$

One possible solution for the above system of equations is $s_1 = s_2 = \ldots = s_{q-1} = 0$ and $s_q = 1$ which implies $x_q = \frac{1}{2}$.

Thus $x_1 e^{s_1} + x_2 e^{s_2} + \ldots + x_q e^{s_q} = \underbrace{x_1 + \cdots + x_{q-1}}_{=\frac{1}{2}} + \frac{1}{2} e$

$$= \frac{1}{2} + \frac{1}{2} e.$$

There fore we have

$$V(p) \leq \frac{1}{2} (1 + e).$$

From (*) it follows

$$V(p) = \frac{1}{2} (1 + e).$$

Lemma 1.2.3. Let $Y = (y_1,\ldots,y_n)$ be feasible for the problem (p). Assume also that $\{s_1,\ldots,s_q\} \subseteq s$ and the non-negative numbers $x_1, \ldots, x_q$ be such that the assumption of the duality lemma

$$c = a(s_1)\, x_1 + ,\ldots + a(s_q)\, x_q \text{ holds.}$$

If $\sum_{i=1}^{q} b(s_i)x_i = \sum_{r-1}^{n} c_r y_r$ , then y is an optimal solution to (P).

$$V(p) \leq \sum_{r-1}^{n} c_r y_r \qquad (1)$$

On the other hand from corollary 1.2.2, we get

$$\sum_{r-1}^{n} c_r y_r = \sum_{i=1}^{q} b(s_i)x_i \leq V(p) \qquad (2)$$

Hence from (1) and (2)

$V(p) = \sum_{r-1}^{n} c_r y_r$ . It follows y is an optimal solution to (P).

**Linear Optimization**

Consider the particular problem

(P) $C^T Y \to \min$

s.t.

$$A^T Y \geq b$$

where A is an (m.n) matrix with column vectors $a_1, \ldots, a_m$ and $q \leq m$. Then every non negative solution of the system

$$Ax = c, \; x = (x_1, \ldots, x_m)^T$$

will give lower bounds for the value of the objective function of the form

$$b^T \leq V(P\rangle.$$

we can write the equation $Ax = c$ as

$$C = \sum_{i=1}^{m} a_i x_i \, .$$

A natural objective is to select $\{s_1, \ldots, s_q\} \subseteq S$ and a nonnegative numbers $x_1, \ldots, x_q$ in order to maximize the lower bound for the value of the objective function obtained from the duality lemma.

**Dual Problem (D)**

Find $\{s_1, \ldots, s_q\} \subseteq S$ and real numbers $x_1, \ldots, x_q$ such that

$$\sum_{i=1}^{q} x_i b(s_i) \rightarrow \max \qquad (1)$$

s.t

$$\sum_{i=1}^{q} x_i a_r(s_i) \; c_r, \; r = 1, \ldots, n \qquad (2)$$

$$x_i \geq 0, \; i = 1, \ldots, q, \qquad (3)$$

$\{s_1, \ldots, s_q, x_1, x_2, \ldots, x_q\}$ is said to be feasible for (D) When $s_i \in S'$, $i \in \{1, \ldots, q\}$ and (2) and (3) hold.

**Theorem 1.2.4** (Weak duality Theorem).

Let f and $\phi$ denote the objective function of the primal and the dual respectively. Then for any feasible solution of (p), the value of f is never less than the value of $\phi$ for any feasible solution of the dual.

i.e $V(D) \leqq V(P)$.

**Proof**: Let $y\in\Re^n$ be feasible solution for (P)

and $x\in\Re^n$ be feasible for (D).

Then for (P), we have

$a(s)^T y \geq b(s)$, $s\in S$

in particular

$a(s_i)^T y)x_i \geq b(s_i)\, x_i$, i.e $\{1, \ldots, m\}$

by taking summation over all i in both side we get

$$\sum_{i=1}^{m}(a(s_i)^T y)x_i = \left(\sum_{i=1}^{m}(a(s_i))x_i\right)^T y \geq \sum_{i=1}^{m} b(s_i)x_i$$

Since $x\in\Re^m$ is feasible for (D), by definition of (D) we have

$C^T Y \geq \sum_{i=1}^{m} b(s_i)x_i$, for all feasible $y\in\Re^n$ and for all feasible $x\in\Re^m$ of (P) and (D)

respectively which implies

$$\min_{y\in\Re^n}\{C^T Y\} \geq \max_{x\in\Re^m}\left\{\sum_{i=1}^{m} b(s_i)x_i\right.$$

i.e. $V(D) \leq V(P)$

**Lemma 1.2.5** (Complementary Slackness Lemma).

Let $y = (y_1, \ldots, y_n)^T$ be feasible for (p) and $\{s_1, \ldots, s_q, x_1, \ldots x_q\}$ be feasible for (D). Moreover let

$$x_i\left(\sum_{r=1}^{n} a_r(s_i)y_r - b(s_i)\right) = 0,\ i \in \{1, \ldots, q\} \qquad (1)$$

Then Y is a solution of (P) and $\{s_1, \ldots, s_q, x_1, \ldots, x_q\}$ is a solution of (D). Further the values of (P) and (D) coincides.

**Proof**: From (1), $x_i \geq 0$ implies $\sum_{r=1}^{n} a_r(s_i)y_r = b(s_i)$, $i \in \{1, \ldots, q\}$.

Thus we have

$$\sum_{i=1}^{q} b(s_i)x_i = \sum_{r=1}^{n}\left(\sum_{r=1}^{q} a_r(s_i)y_r\right)x_i$$

$$= \sum_{r=1}^{n}\left(\sum_{i=1}^{q} a_r(s_i)x_r\right)y_i$$

$$= \sum_{r=1}^{n} c_r y_r \text{ (by duality lemma and by feasibility of}$$

$\{s_1, \ldots, s_q, x_1, \ldots, x_q\}$).

hence by lemma 1.2.3, y is a solution of (P) and the corresponding values of (D) and (P) coincides. //

Now let S′ = {1, …, M}. The linear optimization Problem given by:

$(P_\ell)$ $\quad C^T Y \rightarrow \min,$

s.t

$A^T Y \geq b$

where c, $y \in \Re^n$, $b \in \Re^n$, A an (m, n) matrix is said to be the primal problem.

To each $(P_\ell)$ we can assign another linear optimization problem called the dual problem given by:

$(P_\ell)$ $\quad b^T x \rightarrow \max$

s.t

$Ax = c$

$x \geq 0,$

where $x \in \Re^m$.

### Transformation of the primal problem to the dual problem

In the primal problem, in order to have equation constraints we introduce slack variables as follows:

$$(P_\ell) \quad C^TY - I_mZ = b, \qquad (1)$$

$$Z \geq 0, \ z \in \Re^m.$$

In the formulation of the dual problem i.e.

$$(\hat{D}_\ell) \qquad - C^TY \rightarrow \max$$

$$\text{s.t} \qquad AY - I_mZ = C$$

we are not guaranteed to have $y \geq 0$.

In order to have it, we may write $y \in \Re^n$ as follows:

$$y = y^+ - y^-, \text{ where } y^+ = \max(y, 0) \geq 0,$$

$$y^- = -\min(y, 0) \geq 0.$$

Hence we can write the dual problem as follows:

$$(\hat{D}_\ell) \quad -(C^TY^+ - C^TY^-) \rightarrow \max$$

$$\text{s.t} \qquad (A^T, -A^T, -I_m)\begin{pmatrix} y+ \\ y- \\ z \end{pmatrix} = b,$$

$$(y^+, y^-, z)^T \geq 0.$$

For the transformation of $(D_\ell)$ to $(P_\ell)$ where $(D_\ell)$ $b^Tx \rightarrow \max$

$$\text{s.t} \qquad Ax = c$$

$$x \geq 0$$

Then we have the transformation

$$(\hat{P}_\ell) \quad - b^Tx \rightarrow \min$$

$$\text{s.t} \qquad \begin{pmatrix} A \\ -A \\ I_m \end{pmatrix} x \geq \begin{pmatrix} C \\ -C \\ D \end{pmatrix}$$

## 1.2 State Diagrams and Duality gaps

Using the weak duality theorem we may derive a first classification table for the dual pair (P), (D).

**Definition: 1.3.1.** The primal problem (P) of a linear optimization is said to be:

1. Inconstant (IC), if there is no feasible vector, which solves (P). In this case we put V (P) = $\infty$.
2. Bounded (B), if there are feasible vectors which solves (P) and V(P) < $\infty$.
3. Unbounded (VB), if there are feasible vectors y such that the value of h the objective function is arbitrary small and we put V(P) = $-\infty$.

For the dual problem (D) we have similarly

1. IC, V(D) = $-\infty$
2. B, V(D) < $\infty$.
3. UB, V(D) = $\infty$.

The statement of the duality theorem may be represented by the state diagram below, where the "numbers" denote possible relation and "x" denote the impossible relation of the dual pair (P), (D).

**State diagram for the dual pair (P), (D)** 1.

| $D/P$ | IC | B | UB |
|---|---|---|---|
| IC | 1 | 2 | 4 |
| B | 3 | 5 | x |
| UB | 6 | x | x |

The case "s" is of main interest for the applications, since both problems are feasible.

**Example:** Consider the linear optimization problem given by

(P) $y_1 \to \min$,

s.t

$sy_1 + s^2y_2 \geq s^2,\ s \in S' = [0, 1]$,

Clearly $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is feasible for (P)

more over, the feasible vector $(y_1, y_2)^T$ must satisfy $y_1 \geq 0$. Which implies

$$V(P) = 0$$

follows (P) is in state B.

The corresponding dual problem is

(D) $\sum_{i=1}^{q} s_i^2 x_i \to \max$, (1)

s.t. $\sum_{i=1}^{q} s_i x_i = 1$ (2)

$\sum_{i=1}^{q} s_i^2 x_i = 0$ (3)

$x_i \geq 0,\ i \in \{1, \ldots, q\}$ (4)

$s_i \in [0, 1],\ i \in \{1, \ldots, q\}$. (5)

From (3), Since $x_i \geq 0$ and $s_i^2 \geq 0$, we have

$s_i = 0$ or $x_i = 0,\ i \in \{1, \ldots, q\}$

But (2) Cannot be satisfied.

which implies

(D) is inconsistent (IC).

Hence we have an example for case "2" in the state diagram (1).

Similar examples can be made for each instant cases.

**Definition 1.3.2:** Let a dual pair (P), (D) be given. The defect of (P), (D) denoted by δ(P, D) is defined as:

$$\delta(P, D) = V(P) - V(D).,$$

Here we use the usual conventions:

$$-\infty - (-(\infty) = +\infty, \qquad (-(-\infty)) = +\infty$$

$$+\infty - (+\infty) = 0, \qquad +\infty - c = +\infty \quad \forall c \in \Re.$$

$$-\infty - (-\infty) = 0,$$

If $\delta(P,D) > 0$, we say that a duality gap has occurred.

From state diagram (1), we can formulate the defect diagram corresponding to the dual pairs.

Defect diagram for the dual pair (P), (D). (2).

| D \| P | IC | B | UB |
|---|---|---|---|
| IC | $+\infty$ | $+\infty$ | 0 |
| B | $+\infty$ | d | |
| UB | 0 | | |

where $0 \le d < \infty$.

**Example 2.** Consider the linear optimization problem

$$\text{(P)} \qquad y_1 \rightarrow \min$$

$$\text{s.t} \quad a(s)^T y \ge b(s), \; s \in S$$

where $S = [0, 1] \cup \{2\}$

$$b(s) = \begin{cases} 0, & if \; s \in [0,1] \\ -10, & if \; = 2 \end{cases}$$

$$a^T(s) = (s, s^2)$$

The corresponding dual problem is:

$$\text{(D)} \qquad \sum_{\varepsilon=1}^{q} b(s_i)x_i \rightarrow \max$$

$$\text{s.t} \qquad \sum_{i=1}^{q-1} \begin{pmatrix} s_i \\ s_i^2 \end{pmatrix} x_i + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \times q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{1}$$

$$x_i \geq 0,\ i \in \{1, \ldots, q\} \tag{2}$$

$$s_i \in [0, 1],\ i \in \{1, \ldots, q-1\} \tag{3}$$

**Solution**: From (1), we have

$$\sum_{i=1}^{q-1} s_i^2 x_i = 0$$

since (2) and (3) must be satisfied, we have

$$x_i = 0 \text{ or } s_i = 0,\ i \in \{1, \ldots, q\}.$$

implies

$$x_q = 1.$$

Hence $\sum_{\varepsilon=1}^{q} b(s_i)x_i = -10.$

i.e $V(D) = -10$

For the solution of the primal

$$sy_1 + s^2y_2 = s(y_1 + sy_2) \geq 0 \text{ implies}$$

$$y_1 + sy_2 \geq 0 \quad \forall s \in [0, 1].$$

it follows

$$y_1 \geq 0.$$

hence $(0, y_2)^T \in \Re^2$, is optimal for (P) $\forall y_2 \geq 0$.

Thus we have

$$V(P) = 0.$$

Therefore

$$\delta(P,D) = V(P) - V(D) = 10 > 0$$

i.e. there is duality gap.

## 2. Application of Weak duality in Uniform Approximation

### 2.1 Uniform Approximation

Let T be an arbitrary set and f: T → R be a real valued function, which is defined on T and bounded there.

The Problem of linear uniform approximation is to determine a linear combination $\sum_{r=1}^{n} y_r v_r$ which is best approximates f in the sense that

$$\sup_{t\in T}\left|\sum_{r=1}^{n} y_r v_r(t) - f(t)\right| \to \min$$

Therefore the problem of uniform approximation can be put as:

$$\text{(PA)} \qquad \sup_{t\in T}\left|\sum_{r=1}^{n} y_r v_r(t) - f(t)\right| \to \min,\ y\in\Re^n.$$

If we put $\sup_{t\in T}\left|\sum_{r=1}^{n} y_r v_r(t) - f(t)\right| = y_{n+1}$

Then $\sup_{t\in T}\left|\sum_{r=1}^{n} y_r v_r(t) - f(t)\right| \le y_{n+1}$, for all t∈T and ∀n∈N.

Hence we can express the problem as:

(PA) $y_{n+1} \to \min,\ (y, y_{n+1})^T \in \Re^{n+1}$

s.t

$$\left|\sum_{r=1}^{n} y_r v_r(t) - f(t)\right| \le y_{n+1},\ \forall t\in T \qquad (1)$$

Observe that: $|\alpha - a| \le \beta \Leftrightarrow \begin{cases} \alpha + \beta \ge a \ \ and \\ -\alpha + p \ge -a,\ \alpha, \beta, a \in \Re \end{cases}$

Similarly we can express (1) as

$$\sum_{r=1}^{n} y_r v_r(t) + y_{n+1} \geq f(t), \ \forall t \in T$$

and $- \sum_{r=1}^{n} y_r v_r(t) + y_{n+1} \geq - f(t), \ \forall t \in T$

Hence the approximation Problem is

$$\text{(PA)} \quad y_{n+1} \to \min \tag{1}$$

$$\text{s.t} \sum_{r=1}^{n} y_r v_r(t) + y_{n+1} \geq f(t), \ \forall t \in T \tag{2}$$

$$- \sum_{r=1}^{n} y_r v_r(t) + y_{n+1} \geq - f(t), \ \forall t \in T \tag{3}$$

Now (PA) has the form of a linear optimization problem (P) in $\Re^{n+1}$ provided that the index set S and the functions a(s) are properly defined. In (PA), we have two different kinds of vectors a(s) since the vectors

$$\begin{pmatrix} v_1(t) \\ \vdots \\ v_n(t) \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -v_1(t) \\ \vdots \\ -v_n(t) \\ 1 \end{pmatrix}, \ t \in T \tag{4}$$

are corresponding to conditions (2) and (3) respectively. The constraints of the dual of the problem (1) – (3) implies the vector $C = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \Re^{n+1}$ which appear in the objective function (1) must be expressed as a nonnegative linear combination of finitely many of the vectors (4). Hence the dual Problem corresponding to (1) – (3) takes the form:

Determine two subsets $\{t_1^+, \ldots, t_{q^+}^+\}$, $\{t_1^-, \ldots, t_q^-\}$ of

$T(q^+ + q^-) \geq 1)$ and real number $x_1^+, \ldots, x_{q^+}^+, x_1^-, \ldots, x_{q^-}^-$

such that the expression

$$\sum_{i=1}^{q} f(t_i^+)x_i^+ - \sum_{i=1}^{q^-} f(t_i^-)x_i^- \qquad \rightarrow \max$$

$$\text{s.t} \qquad \sum_{i=1}^{q^+} v_r(t_i^+)x_i^+ - \sum_{i=1}^{q^-} v_r(t_i^-)x_i^- = 0,\ \mathrm{r} = 1, \ldots, \mathrm{n}$$

$$\sum_{i=1}^{q^+} x_i^+ + \sum_{i=1}^{q^-} x_i^- = 1$$

$$x_i^+ \geq 0,\ \mathrm{i} = 1, \ldots, \mathrm{q}^+,$$

$$x_i^- \geq 0,\ \mathrm{i} = 1, \ldots, \mathrm{q}^-.$$

Using similar argument in the discussion of transformation of (LP) → (LD) we have the following:

The dual problem of (PA) is
determine a subset $\{t_1, \ldots, t_q\}$ of T, and real numbers $x_1, \ldots, x_q$, $(q \geq 1)$, such that

$$\sum_{i=1}^{q} f(t_i)x_i \rightarrow \max$$

$$\text{s.t} \qquad \sum_{i=1}^{q} v_r(t_i)x_i = 0,\ \mathrm{r} = 1, \ldots, \mathrm{n},$$

$$\sum_{i=1}^{q} |x_i| \leq 1.$$

**Lemma 2.1.1** Let $x_1, \dots, x_q \in \Re$, $\{t_1, \dots, t_q\}$ c T, such that

$$\sum_{i=1}^{q} v_r(t_i)x_i = 0 \quad r = 1, \dots, n \tag{1}$$

$$\sum_{i=1}^{q} |x_i| \leq 1. \tag{2}$$

$$\text{Then } \sum_{i=1}^{q} f(t_i)x_i \leq \sup_{t\in T}\left|\sum_{r=1}^{n} y_r v_r(t) - f(t)\right| \text{ for all } y\in\Re^n \tag{3}$$

Proof: - From (1) we have

$$\sum_{r=1}^{n}\left(\sum_{i=1}^{q} v_r(t_i)x_i\right)y_r = 0 \quad \text{implies}$$

$$\sum_{i=1}^{q}\left(\sum_{i=1}^{q} y_r v_r(t_i)\right)x_i = 0$$

hence

$$\sum_{i=1}^{q} f(t_i)x_i = \sum_{i=1}^{q} f(t_i)x_i - \sum_{i=1}^{q}\left(\sum_{i=1}^{n} y_r v_r(t_i)\right)x_i$$

$$= \sum_{i=1}^{q}\left(f(t_i) - \sum_{r=1}^{n} y_r v_r(t_i)\right)x_i$$

$$\leq \sum_{i=1}^{q}\left|f(t_i) - \sum_{r=1}^{n} y_r v_r(t_i)\right||x_i|$$

$$= \sup_{t\in T}\left|f(t) - \sum_{r=1}^{n} y_r v_r\right|\sum_{i=1}^{q}|x_i|$$

$$\leq \sup_{t\in T}\left|f(t) - \sum_{r=1}^{n} y_r v_r(t)\right| \text{ by (2)}$$

Therefore we have

$$\sum_{i=1}^{q} f(t_i)x_i \leq \sup_{t\in T}\left|f(t) - \sum_{r=1}^{n} y_r v_r(t)\right|.$$

If we replace $\sum_{i=1}^{q} f(t_i)x_i$ by $\left|\sum_{i=1}^{q} f(t_i)x_i\right|$, then the result holds true. because

$$\left|\sum_{i=1}^{q} f(t_i)x_i\right| = \left|\sum_{i=1}^{q}\left(f(t_i)-\sum_{r=1}^{n} y_r v_r(t_i)\right)x_i\right|$$

$$\leq \sum_{i=1}^{q}\left|f(t_i)-\sum_{r=1}^{n} y_r v_r(t_i)\right|\left|x_i\right|$$

$$= \sup_{t\in T}\left|f(t)-\sum_{r=1}^{n} y_r v_r(t)\right| \quad //$$

**Remark**: If $q \geq n + 1$, then (1) has a non trivial solution for the choice of elements $t_1, \ldots, t_q$ in T. (1) gives the under determined linear system of equations

$$\begin{pmatrix} v_1(t_1) & \cdots & v_1(t_q) \\ \vdots & & \vdots \\ v_n(t_1) & & v_n(t_q) \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \vdots \\ \hat{x}_q \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

and setting $x = \left(\sum_{i=1}^{q}\left|\hat{x}_i\right|\right)^{-1}\hat{x}$, (3)

$x \in \Re^q$ satisfy (1) and (2) of (DA).

**Example**. Let $f(t) = e^t$, $t \in [-1, 1]$. Find the error of the best approximation of f by a polynomial $p(t) = y_1 + y_2 t$, $t \in [-1, 1]$.

**Solution**. We have to solve the optimization problem:

$$\sup_{t\in T} |e^t - y_1 - y_2 t| \rightarrow \min,\ y = (y_1, y_2) \in \Re^2.$$

By Lemma 2.1.1, we select $q = 3$ and set $t_1 = -1$, $t_2 = 0$, $t_3 = 1$.

The system of equation (1), i.e.

$$\sum_{i=1}^{3} v_r(t_i)x_i = 0,\ r = 1, 2 \qquad \text{becomes}$$

$$\hat{x}_1 + \hat{x}_2 + \hat{x}_3 = 0 \qquad \text{(I)}$$

$$-\hat{x}_1 + \hat{x}_3 = 0 \qquad \text{(II)}$$

from (II) we have $\hat{x}_1 = \hat{x}_3$

and from (I) we get $\hat{x}_2 = -2\hat{x}_1 = -2\hat{x}_3$

hence for any arbitrary $\alpha$, the general solution is

$$\hat{x}_1 \quad = 2 = \hat{x}_3,\ \hat{x}_2 = -2\alpha.$$

The "normalization" (3) gives

$$\mathrm{x} = \left(\sum_{i=1}^{3} |x_i|\right)^{-1} \hat{x} \text{ implies}$$

$$\mathrm{x} = (4\alpha)^{-1} (\alpha, -2\alpha, \alpha) \text{ follows}$$

$$\mathrm{x} = \left(\frac{1}{4}, \frac{-1}{2}, \frac{1}{4}\right)^T.$$

Therefore $t_1 = -1$, $t_2 = 0$, $t_3 = 1$ and $\mathrm{x} = \left(\frac{1}{4}, \frac{-1}{2}, \frac{1}{4}\right)^T$ meets the constraints of (DA).

Therefore we conclude from Lemma 2.1.1 (4) that if $e^t$ is approximated by a straight line over the interval [-1, 1], then the error will be

$$\frac{1}{4}e^{-1} - \frac{1}{2} + \frac{1}{4}e \approx 0.27$$

An upper bound for the smallest possible approximation error is obtained by taking $y_1 = 1.36$ and $y_2 = 1$

$$y_1 + y_2 t = 1.36 + t$$

then $\sup\limits_{t\in[-1,1]} |e^t - 1.36 - t| \approx 0.36.$

**Lemma 2.1.2** Let $\{t_1, \ldots, t_q, x_1, x_q\}$, $t_i \in T$, $i \in \{1, \ldots, q\}$ and $q \geq 1$ be such that

$$\sum_{i=1}^{q} v_r(t_i)x_i = 0,\ r \in \{1, \ldots, n\},\ \sum_{i=1}^{q} |x_i| = 1$$

and let $y \in \Re^n$, $y_{n+1} = \sup\limits_{t\in[-1,1]} \left| f(t) - \sum_{r=1}^{n} y_r v_r(t) \right|,$

Moreover for $i \in \{1, \ldots, q\}$, either $x_i = 0$ or

$$f(t_i) - \sum_{r=1}^{n} y_r v_r(t) = y_{n+1} \operatorname{sgn} s_i, \qquad (1)$$

where $\operatorname{sgn} x_i = \dfrac{x_i}{|x_i|}$

Then $\{t_1, \ldots, t_q, x_1, \ldots, x_q\}$ is an optimal solution of (DA) and Y of (PA), and the value of (PA) and (DA) coincides.

**Proof**: $\sum_{i=1}^{q} f(t_i)x_i = \sum_{i=1}^{q} f(t_i)x_i - \sum_{i=1}^{q} y_r \left( \sum_{r=1}^{q} v_r(t_i)x_i \right)$

$$= \sum_{i=1}^{q} \left[ f(t_i) - \sum_{r=1}^{n} y_r v_i(t_i) \right] x_i$$

from (1) we get

$$\sum_{i=1}^{q} f(t_i)x_i = y_{n+1} \sum_{i=1}^{q} x_i sgb(x_i) = y_{n+1} \sum_{i=1}^{q} |x_i| = y_{n+1}, \quad \text{since } \sum_{i=1}^{q} |x_i| = 1.$$

There fore $\sum_{i=1}^{q} f(t_i)x_i = \sup\limits_{t\in T} \left| f(t) - \sum_{r=1}^{n} y_r v_r(t) \right|$ by Lemma 2.1.1.

$V(D) = \{t_1, \ldots, t_q, x_1, \ldots, x_q\} = V(P)$. //

**Note**: It T is compact and all the functions are assumed to be continuous, then we can write instead of "sup", "Max".

We conclude this section by showing that the approximation problem is soluble under fairly general conditions.

**Theorem 2.1.3** Let $T \subseteq R^k$ be nonempty and compact. Assuming that the function f, $_v$, …, $v_n$ are continuous and linearly independent on T. Then the linear approximation problem (PA) is soluble. i.e. There is a vector $\hat{y} \in \Re^n$ such that

$$\max_{t\in T}\left|f(t)-\sum_{r=1}^{n}\hat{y}_r v_r(t)\right| = \min_{y\in\Re^n}\max_{t\in T}\left|f(t)-\sum_{r=1}^{n}y_r v_r(t)\right|.$$

**Proof**: We define a norm on $\Re^n$ by

$$\|y\|_v = \max_{t\in T}\left|\sum_{r=1}^{n}y_r v_r(t)\right|$$

Putting y = 0, we get

$$\max_{t\in T}\left|f(t)-\sum_{r=1}^{n}y_r v_r(t)\right| = \max_{t\in T}|f(t)|$$

By the above notice and theorem of weistrass, f - $\sum_{r=1}^{n}y_r v_r$ assumes its maximum and minimum – hence

$$\max_{t\in T}\left|f(t)-\sum_{r=1}^{n}y_r v_r(t)\right| = \max_{t\in T}|f(t)| = \Delta.$$

This implies

The optimal value of (PA) lies in $[0, \Delta]$.

Because of the minimization we need only to consider those vectors y which satisfy

$$\max_{t\in T}\left|f(t)-\sum_{r=1}^{n}y_r v_r(t)\right| \le \Delta$$

from triangle inequality

$$\left|\sum_{r=1}^{n} y_r v_r(t)\right| - |\,f(t)\,| \le \left|f(t)-\sum_{r=1}^{n} y_r v_r(t)\right| \text{ implies}$$

$$\left|\sum_{r=1}^{n} y_r v_r(t)\right| \le \left|f(t)-\sum_{r=1}^{n} y_r v_r(t)\right| + |f(t)\,| \le 2\Delta.$$

Thus we need only to minimize for those vectors $y \in \Re^n$ such that

$\|y\|_v \le 2\Delta$; i.e a compact subset of $\Re^n$.

Since $y \to \max_{t\in T} \left|f(t)-\sum_{r=1}^{n} y_r v_r(t)\right|$ is continuous, by Weierstrass theorem the optimal solution exist. Hence (PA) is solvable. //

## 2.2 Polynomial Approximation

This section is devoted to the study of approximation problem in the case when T is real interval and the function is to be approximated by a polynomial.

**Lemma 2.2.1** Let $t_1 < t_2 < \ldots < t_{n+1}$ be fixed real numbers and $(x_1, \ldots, x_{n+1})$ be a non-trivial solution of the homogeneous linear system of equations

$$\sum_{i=1}^{n+1} t_i^{r-1} x_i = 0,\ r = 1, \ldots, n. \qquad (1)$$

Then $x_i\, x_{i+1} < 0,\ i \in \{1, \ldots, n\}$ (2)

**Proof**: Let $1 \le i \le n$, where i be a fixed integer, and $P_n$ be the uniquely determined polynomial given by

$$P_n(t) = \sum_{i=1}^{n} y_r t^{r-1} \text{ satisfying} \qquad (3)$$

$$P_n(t_j) = \begin{cases} 1, j = 1 \\ 0, j = 1, \ldots, n+1, j+i, j+i+1 \end{cases} \qquad (4)$$

(see fig. 1)

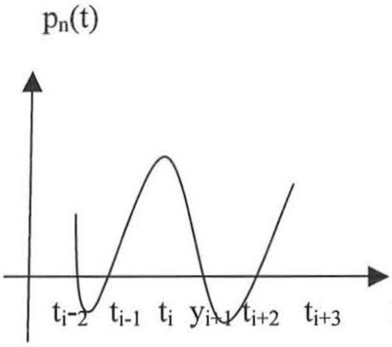


Fig . 1

Consider the vendermonde matrix:

$$V(t_1, \ldots, t_n) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_n \\ t_1^2 & t_2^2 & \cdots & t_n^2 \\ \vdots & & & \\ t_1^{n-1} & t_2^{n-1} & \cdots & t_n^{n-1} \end{pmatrix}$$

Then det $V(t_1, \ldots, t_n) = \pm \prod_{i>j} (t_i - t_j) \neq 0$

i.e. The vendermonde matrix is non singular.

Hence such a $P_n$ exists.

Now, from (1) we have

$$\sum_{i=1}^{n+1} P_n(t_i)x_i = \sum_{r=1}^{n} y_r \sum_{i=1}^{n+1} t_i^{r-1} x_i = 0$$

by construction of $P_n$ (3), we get

$$x_i + P_n(t_{i+1})x_{i+1} = 0$$

$P_n$ cannot vanish in $[t_i, t_{i+1}]$. If it vanishes, then $P_n$ would have n-zeros, which contradicts the construction of $P_n$.

Therefore $P_n(t_{i+1}) > 0$ implies $x_i < 0$ or $x_{i+1} < 0$.

hence we have $x_i \, x_{i+1} < 0$. //

The following theorem, which is due to De La vallee – Poussin, is important for calculating lower bounds for the error of the best approximation without solving the linear system (1) explicitly.

**Theorem 2.2.2.** Let f be continuous on $[\alpha, \beta]$, p be a polynomial of degree less than n and $\alpha \leq t_1 < t_2, \ldots < t_{n+1} \leq \beta$, such that

$$[f(t_i) - p(t_i)]\,[f(t_{i+1}) - p(t_{i+1})] < 0, \; i \in \{1, \ldots, n\}. \qquad (1)$$

The $\min\limits_{i} |f(t_i) - P(t_i)| \le \Delta_n \le \max\limits_{\alpha \le t \le \beta} |f(t) - p(t)|$, where (2)

$$\Delta_n := \inf_{y \in \Re^n} \max_{\alpha \le t \le \beta} |f(t) - \sum_{r=1}^{n} y_r t^{r-1}|.$$

**Proof**: (For illustration of (1) see fig .2)

The right hand side inequality is obvious

Since $\inf\limits_{y \in \Re^n} \max\limits_{\alpha \le t \le \beta} |f(t) - p(t)| \le \max\limits_{\alpha \le t \le \beta} |f(t) - p(t)|$

Let $\rho_1, \ldots, \rho_{n+1}$ be non trivial solution of the system

$$\sum_{i=1}^{n+1} t_i^{r-1} \rho_i = 0, r \in \{1, \ldots, n\}.$$

By Lemma 2.1, we may assume $\rho_i \rho_{i+1} < 0, \in \{1, \ldots, n\}$

Now Put

$$x_i = \rho_i \left\{ \sum_{i=1}^{n+1} |\rho_j| \right\}^{-1}.$$

In this way we get a feasible solution to the dual problem since

$$\sum_{i=1}^{n+1} t_i^{r-1} x_i = 0, r \in \{1, \ldots, n\} \qquad (3)$$

$$\sum_{i=1}^{n+1} |x_i| = 1.$$

From the weak duality lemma, we also have

$$\sum_{i=1}^{n+1} f(t_i) x_i \le \Delta_n. \qquad (4)$$

define $\delta_i = f(t_i) - p(t_i)$, $\delta_i \delta_{i+1} < 0$ by assumption. If for each i, the signs of $x_i$ are change simultaneously, the constraints of (DA) still met.

Hence we get

$$x_i \delta_i > 0 \qquad (5)$$

Since we also have $x_i x_{i+1} < 0$, applying (3) and (5) we get that

$$\sum_{i=1}^{n+1} f(t_i)x_i = \sum_{i=1}^{n+1} x_i|f(t_i) - p(t_i)| = \sum_{i=1}^{n+1} x_i\delta_i$$

$$\geq \min_i |\delta_i| \sum_{i=1}^{n+1} |x_i|$$

$$= \min_i |f(t_i) - p(t_i)|$$

Therefore by (4) we have the desired result. //

**Corollary 2.2.3** Let P be a polynomial of degree lest than n and such that there are n +1 points $\alpha \leq t_1 < t_2 < \ldots < t_{n+1} \leq \beta$ with the properties $|\delta_i| = |f(t_i) - p(t_i)| = \max_{\alpha \leq t \leq \beta} |f(t) - p(t)|$, $i \in \{1, \ldots, n+1\}$ and $\delta_i \, \delta_{i+1} < 0$, $i \in \{1, \ldots, n\}$.

Then P is a polynomial of degree less than n which best approximates f in the uniform norm.

The Proof follows directly from the above theorem.

**Remark:** In the special case when

$|\delta_1| = |\delta_2| = \ldots = |\delta_{n+1}|$, we get

$$\sum_{i=1}^{n+1} f(t_i)\, x_i = \min_i |f(t_i) - p(t_i)|$$

Hence (2) and (4) in Theorem 3.2.2 gives the same lower bound for the attainable approximation error in this case.

Determination of a polynomial satisfying (1) in Theorem 2.2.2

Let $\alpha \leq t_1 < t_2 < \ldots < t_{n+1} \leq \beta$ be given

Define $\delta$ by $\delta(t_i) = (-1)^i$, $i \in \{1, \ldots n, n+1\}$.

Now we have to find a polynomial P of degree less than n and a constant $\varepsilon$ such that

$$P(t_i) = f(t_i) + \varepsilon\delta(t_i),\ i \in \{1, \ldots, n+1\} \qquad (6)$$

which is a linear system of equation with ε and the coefficients of P as unknowns. Using the vandermonde matrix, P and ε are uniquely determined.

Since $P(t_1, \ldots, t_{n+1}) = 0$, from (6) we have

$$\varepsilon = \frac{-f[t_1,\ldots,t_{n+1}]}{\delta[t_1,\ldots,t_{n+1}]},$$ where we use the usual notation for divided differences.

P may be represented in the "Newton" form

$$P(t) = p[t_1] + P[t_1, t_2]\,(t - t_1) + \ldots + P[t_1, t_2, \ldots, t_n] \prod_{i=1}^{n-1} (t - t_i).$$

By theorem 3.2.2, $|\varepsilon|$ is a lower bound for $\Delta_n$.

**Example 1.** Calculate the lower bound for the smallest error, when $f(t) = (1 + t)^{-1}$ be approximated in the uniform over [0, 1].

**Solution**: Consider $n = 2$, $t_1 = 0$, $t_2 = \frac{1}{2}$, $t_3 = 1$ and $[\alpha, \beta] = [0, 1]$.

The difference schemes for f and δ are:

| $t_i$ | $f(t_i)$ | $f[t_i, t_{i+1}]$ | $f[t_1, t_2, t_3]$ | $\delta(t_i)$ | $\delta(t_i, t_{i+1})$ | $\delta[t_1, t_2, t_3]$ |
|---|---|---|---|---|---|---|
| 0 | 1 | | | -1 | | |
| | | $-\frac{2}{3}$ | | | 4 | |
| $\frac{1}{2}$ | $\frac{2}{3}$ | | $\frac{1}{3}$ | 1 | | -8 |
| | | $-\frac{1}{3}$ | | | -4 | |
| 1 | $\frac{1}{2}$ | | | -1 | | |

hence we have

$$\varepsilon = \frac{-f[t_1,t_2,t_3]}{\delta[t_1,t_2,t_3]} = \frac{-\frac{1}{3}}{-8} = \frac{1}{24}$$

i.e f(t) cannot be approximated in the uniform norm over [0, 1] by a straight line with an error less than $^1\!/_{24}$.

**Example 2**. Let f be twice continuously differentiable on [α, β]. Such that $f''(t) > 0$, $t \in [\alpha, \beta]$. Let $\ell$ be the straight line which interpolates f at the end points α and β.

Show that $\ell$ which best approximates f has the representation

$$\ell(t) - {}^{\delta}\!/_{2}, \text{ where } \frac{8}{2} \text{ is the approximation error.}$$

**Solution:** Since $\ell$ interpolates f at the end points, we have

$$f(t) - \ell(t) = \frac{(t-t_1)(t-t_2)}{2} f''(t)$$

Putting $t = \beta$, $t_1 = \alpha$, $t_2 = \dfrac{\alpha+\beta}{2}$, we get

$$2(f(t) - \ell(t)) = (\beta - \alpha) \left(\beta - \frac{(\alpha+\beta)}{2}\right) f''(t).$$

taking absolute values on both sides

$$2|\, f(t) - \ell(t)\,| = \left|\frac{(\beta+\alpha)^2}{2}\right|. \; f''(t), \text{ since } f''(t) > 0.$$

we have

$$2\,(\ell(t) - f(t)) = \frac{(\beta+\alpha)^2}{2}. \; f''(t)$$

which follows

$$f(t) = \ell(t) - -\frac{1}{2}, \frac{(\beta+\alpha)^2}{2}. \, f''(t).$$

Since $\delta \max\limits_{\alpha \le t \le \beta} |f(t) - \ell(t)| = \dfrac{(\beta+\alpha)^2}{2}. \, f''(t),$

we have $f(t) = \ell(t) - \frac{\delta}{2}$.

Hence $\ell$ has the representation $\ell(t) - \frac{\delta}{2}$ which approximates f with an error $\frac{\delta}{2}$.

**Example 2.** Give the approximation error, if $f(t) = t^2$, were $\ell$ interpolates f at

$$t_1 = \frac{\alpha+\beta}{2} - \frac{1}{2\sqrt{2}}(\beta - \alpha),\ t_2 = \frac{\alpha+\beta}{2} + \frac{1}{2\sqrt{2}}(\beta - \alpha).$$

**Solution**: $f''(t) = 2 > 0$, hence we have

$$2|\ell(t) - t^2| = \left(\beta - \frac{\alpha+\beta}{2} + \frac{1}{2\sqrt{2}}(\beta - 2)\right).\left(\beta - \frac{\alpha+\beta}{2} - \frac{1}{2\sqrt{2}}(\beta - \alpha)\right).2$$

implies

$$|\ell(t) - t^2| = \left|\frac{1}{8}(\beta - \alpha)^2\right|$$

i.e. $\quad \ell(t) - \frac{1}{8}(\beta - \alpha)^2 = t^2$.

Therefore the approximation error is $\frac{1}{8}(\beta - \alpha)^2$

## 3. DUALITY THEORY

A major topic of this chapter is the derivation of "strong" duality results, i.e to give theorems which specify when V(D) = V(P). Another important topic is the proof of existence of solutions to the primal and the dual problems.

At first we will try to discuss the "geometric" representation of the dual problem (D) which helpful to understand the other sections.

## 3.1 Geometric Interperitation of the dual problem

To give a geometric representation of the dual problem, first we introduce the concepts of a convex set and the special case of convex cone.

**Definition 3.1.1** A set $k \subseteq \Re^n$ is said to be convex if and only if $a_1$, $a_2 \in K$ implies $\lambda a_1 + (1 - \lambda) a_2 \in K$, $\lambda \in (0, 1)$.

Moreover if K is convex and $a_1, \ldots, a_q \in k$, then

$$\sum_{i=1}^{q} \lambda_i a_i \in k \text{ if } \sum_{i=1}^{q} \lambda_i = 1 \text{ and } \lambda_i \geq 0, i \in \{1, \ldots, q\}.$$

**Definition 3.1.2.** Let $A \subseteq \Re^n$. Then convex hull of A denoted by conv A is the set of all convex combinations of elements of A,

i.e Conv A = C,

where

$$c: = \{x \in \Re^n \mid \exists q \in N, \exists \lambda_i, \exists a_i \in A: x = \sum_{i=1}^{q} \lambda_i a_i,\ \sum_{i=1}^{q} \lambda_i = 1,$$

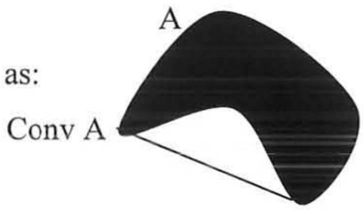
$$\lambda_i \geq 0, \text{i.}\in. \{1, \ldots, q\}\}.$$

Some of the properties of conv A are:

**Property:** Let $A \subseteq \Re^n$ be arbitrary set. Then

1. Conv A is convex for any set A,
2. A convex set which contains A must contain all convex combination given by c,
3. Conv A is the smallest convex set having A as a subset.

Conv A using a figure can be illustrated as:



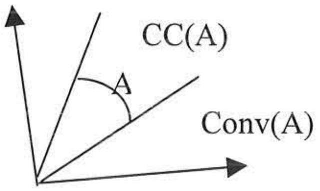
**Definition 3.1.3** A convex cone is a convex set with the property if $x \in C$, then $\lambda x \in C$ for all $\lambda x \in C$ for all $\lambda \geq 0$.

Let $A \neq \phi$, $A \subseteq \Re^n$. Then the $\{y \in \Re^n \quad | y = \lambda x, x \in A, x \geq 0\}$ is a cone generated by A denoted by cone A and is called the conichull of set A.

We denote the convex conic hull of A by cone (Conv A) or CC(A), where

$$CC(A) := \{z | z = \sum_{i=1}^{q} x_i a_i \text{ , } x_i \geq 0, a_i \in \text{conv}A, i. \in \{1, \ldots, q\}, q \geq 1\}.$$

Geometrically we have



Thus CC(A) consists of all non negative linear combination of elements of Conv A. Now we apply the above concepts to the set of vectors, which occur in the formulation the dual problem and the primal problem given as follows:

Let the primal linear optimization be given by

(P) $\quad f(y) : C^T y \to \min, c, y \in \Re^n$

s.t

$$\sum_{r=1}^{n} a_i(s)^T y_r \geq b(s), \forall s \in S',$$

where S is an indexed Set (finite or possibly infinite).

Then its dual is given by:

(D) $$\sum_{r=1}^{n} b_i(s_i) x_i \to \max$$

s.t

$$\sum_{i=1}^{q} x_i a_r (s_i) = c_r, r \in \{1, \ldots, n\}$$

$$\{s_1, \ldots, s_q\} \subseteq S$$

$$x_i \geq 0, i \in \{1, \ldots, q\}.$$

Now, we define the constraints of the primal problem interms of the set of vectors

$A_s\!: = \{a_s \mid s_sS\} \subseteq \Re^{n}$.

From the constraints of the dual problem and definition of CC(A), we find that $\{s_1, \ldots, s_q, x_1, \ldots, x_q\}$ is feasible for the dual problem if and only if $C = (c_1, \ldots, c_n)$ lies in $CC(A_s)$. We denote $CC(A_s)$ by "$M_n$" which is sometimes called the "moment cone".

**Lemma 3.1.1** The Dual problem (D) is feasible if and only if $C \in M_n$.

**Proof:** (1) Let (D) be feasible.

Then

$C = \sum_{i=1}^{q} x_i a_i$ , $x_i \geq 0$ by duality lemma. by definition of $M_n$, we have

$C \in M_n$.

(2) Let $C \in M_n$.

then $C = \sum_{i=1}^{q} x_i a_i$ , $x_i \geq 0$ by definition of $M_n$.

By duality lemma, we have the dual problem (D) is feasible.

**Example 3.1.** Let the primal problem of a linear optimization be:

$$\text{(P)} \qquad y_1 + \frac{1}{2} y_2 \rightarrow \min,$$

s.t

$$s y_1 + s^2 y^2 \geq e^s - 1, \; s \in [\,0, 1] = S.$$

**Solution**: For this primal Problem, we have

$$C = \left(1, \frac{1}{2}\right)^T, \; a_s \,, \begin{pmatrix} s \\ s^2 \end{pmatrix}, \; s \in [0, 1] \text{ and } A_{s\,:\,\{a}s \mid s \in S' \; [0, 1]\} \subseteq \Re^2.$$

From the constraints of the dual of (P) there are numbers $s_1, s_2 \in [0, 1]$ such that

$$c = x_1 a_{s_1} + x_2 a_{s_2} .$$

$$\text{i.e} \quad \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} = x_1 \begin{pmatrix} a_{s_2} \\ (a_{s_1})^2 \end{pmatrix} + x_2 \begin{pmatrix} a_{s_2} \\ (a_{s_2})^2 \end{pmatrix}$$

which imply

$$x_1 a_{s_1} + x_2 a_{s_2} = 1$$

$$x_1 (a_{s_1})^2 + (a_{s_1})^2 x_2 = \frac{1}{2} .$$

We get $x_1 = \dfrac{a_{s_2} - \frac{1}{2}}{a_{s_1}(a_{s_2} - a_{s_1})} \geq 0.$, $x_2 = \dfrac{a_{s_1} - \frac{1}{2}}{a_{s_2}(a_{s_1} - a_{s_2})}$

The inequality holds true [for both $x_1$ and $x_2$].

If $a_{s_2} \geq \frac{1}{2}$ and $a_{s_1} < \frac{1}{2}$ or $a_{s_2} \leq \frac{1}{2}$ and $a_{s_1} > \frac{1}{2}$

Hence if we choose $a_{s_1} = \frac{1}{4} \in [0, 1]$ and $a_{s_2} = \frac{3}{4} \in [0, 1]$ we have $x_1 = 2$ and $x_2 = \frac{2}{3}$, $x_1, x_2 \geq 0$.

$$\text{i.e.} \quad C = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} = 2 \begin{pmatrix} \frac{1}{4} \\ \frac{1}{10} \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 3/4 \\ 9/16 \end{pmatrix} .$$

By definition we have

$C \in M_n = CC(A_s)$. Which implies

The dual problem (D) of (P) is feasible by Lemma .1.1.

**Theorem 3.1.2 (Reduction theorem).**

Let $z \in R^p$ $(P \geq 1)$ be a nonnegative linear combination of $z_1, \ldots, z_q$ in $R^p$ and $q \geq 1$. i.e

$$Z = \sum_{i=1}^{q} x_i z_i , \; x_i \geq 0, \; i \in \{ 1, \ldots, q\} \qquad (1)$$

Then Z admits a representation

$$Z = \sum_{i=1}^{q} \bar{x}_i z_i \,,\ \bar{x}_i \geq 0, \text{ i. } \in \{1, \ldots, q\} \qquad (2)$$

such that there is { $\bar{x}_{i_1}$, …, $\bar{x}_{i_r}$ } $\subseteq$ {$x_1$, …, $x_q$} where r $\leq$ p, $\bar{x}_{i_j} > 0$

and $\bar{z}_{i_j}$ are linearly independent.

**Proof**: If $z_1$, …, $z_q$ are linearly independent, then q -, p and $x_i$ = $\bar{x}_i$ are uniquely determined by (1) and (2) and we are done.

Suppose $z_1$, …, $z_q$ are linearly dependent.
Then there are $\alpha_1$, …, $\alpha_q$ not all zero in R such that

$$\sum_{i=1}^{q} \alpha_i z_i = 0 \qquad (3)$$

Hence for each r with $\alpha_r \neq 0$,

$$z_r = - \sum_{i \neq r} \frac{\alpha_i}{\alpha_r} z_i \,.$$

Putting this in (1), we have

$$Z = \sum_{\substack{i=1 \\ i \neq r}}^{q} \left( x_i - x_r \frac{\alpha_i}{\alpha_r} \right) z_i \qquad (4)$$

Hence we get a representation of z as a linear combination of q – 1 of the vectors $z_1$, …, $z_q$. Now we can choose r so that (4) becomes a nonnegative linear combination.

i.e $\quad x_i - x_r\frac{\alpha_i}{\alpha_r} \geq 0$, i.e {1, …, r – 1, r + 1, …, q} (5)

we select r such that $\alpha_r > 0$.

(if all $\alpha_i$ in (3) are non positive we multiply (3) by –1).

Since $x_i \geq 0$ and $\alpha_r > 0$, we have

$$x_i - x_r\frac{\alpha_i}{\alpha_r} \geq 0 \text{ if } \alpha_i \leq 0$$

Hence (5) holds true if $\alpha_i \leq 0$.

For the case $\alpha_i > 0$, (5) implies

$$\frac{x_i}{\alpha_i} \geq \frac{x_r}{\alpha_r}$$

This holds true and hence (5), if we determine r such that

$$\frac{x_r}{\alpha_r} = \min \left\{ \frac{x_i}{\alpha_i} \mid \alpha_i > 0 \right\}$$

Then (4) expresses Z as a nonnegative liner combination of the vectors $z_1, \ldots, z_{r+1}, \ldots, z_q$. We may repeat this procedure until we have the representation (2). //

**Example 2** Let P = 2 and q = 4. And let

$$z_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, z_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, z_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, z_4 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, z = \begin{pmatrix} 1/4 \\ 2 \end{pmatrix}.$$

Then z admits the representation

$$Z = \frac{1}{4} z_1 + \frac{1}{2} z_2 + \frac{1}{4} z_3 + \frac{1}{4} z_4.$$

Since the vectors $z_1, \ldots, z_4$ are linearly dependent, we can find the representation of z with vectors less than 4.

We can express $z_1$ as

$$z_1 = \frac{1}{3} z_2 + \frac{1}{3} z_3$$

or $$-3z_1 + z_2 + z_3 = 0$$

Hence we have

$$\sum_{i=1}^{4} \alpha_1 z_i = 0, \text{ where } \alpha_1 = -3, \alpha_2 = 1, \alpha_3 = 1, \alpha_4 = 0$$

Since we must have $\alpha_r > 0$, r must be equal to 2 or 3. Now to determine the smaller of the coefficients

$$\frac{x_2}{\alpha_2} = \frac{1}{2} \text{ and } \frac{x_3}{\alpha_4} = \frac{1}{4}$$

$$\min \left\{ \frac{x_2}{\alpha_2}, \frac{x_3}{\alpha_3} \right\} = \frac{1}{4} \text{ imply } r = 3$$

Therefore we have the form of representation (2) in the Reduction theorem using the representation of z in (4).

i.e $Z = z_1 + \frac{1}{4} z_2 + \frac{1}{4} z_4.$ (*) //

We can carry out another reduction step on (*) to obtain z as a non-negative linear combination of the vectors $z_1$, $z_2$, $z_4$. As a consequence of the Reduction theorem we have the following Theorems with out proof.

**Theorem of Caratheodory:**

Let $A \subseteq \Re^n$. Then for each $z \in$ conv A, there are n + 1 reals $x_1, \ldots, x_{n+1} \geq 0$ and n + 1 points $a_+, \ldots, a_{n+1} \in A$, such that $x_1 + \ldots = x_{n+1} = 1$ and $z = \sum_{i=1}^{n+1} x_i a_i$ see the proof on [1]

**Theorem 3.1.3** Let $\{s_1, \ldots, s_q, x_1, \ldots, x_q\}, q \geq 1$ be feasible for (D). i.e

$$\sum_{i=1}^{q} a_r(s_i)x_i = c_r,\ r \in \{1, \ldots, n\},$$

$$x_i \geq 0,\ i. \in \{1, \ldots, q\}.$$

Then there is a subset $\{ s_{i_1}, \ldots, s_{i_n} \}$ is also feasible to (D).,

i. e

$$\sum_{j=1}^{n} a_r(s_{i_j})\bar{x}_{i_j} = c_r,\ r = 1, \ldots, n,$$

$$\bar{x}_{i_j} \geq 0,\ j = 1, \ldots, n.$$

The vectors $a\left(s_{i_j}\right)$ which belong to positive numbers $\bar{x}_{i_j}$ are linearly independent

The proof follows directly from the reduction theorem.

Now consider the n + 1 equations

$$\sum_{i=1}^{q} a_r(s_i)x_i = c_0,$$

$$\sum_{i=1}^{q} a_r(s_i)x_i = c_r,\ r \in \{1, \ldots, n\} \tag{5}$$

Then we obtain an important result that n + 1 points $s_{i_j}$ "are enough" to determine V(D). hence we formulate (D) as follows:

$$\text{(D)} \qquad \sum_{i=1}^{n+1} b(s_i)x_i \to \max,$$

$$\text{s.t} \qquad \sum_{i=1}^{n+1} a_r(s_i)x_i = c_r,\quad r \in \{1, \ldots, n\},$$

$$s_1, \ldots, s_{n+1} \in S,$$

$$x_1, \ldots, x_{n+1} \geq 0.$$

If we adjoin the real number b(s) to the vectors a(s) and consider the vectors

$$\widetilde{a}(s) = (b(s), a_t(s), \ldots, a_n(s))^T \in \Re^{n+1}$$

Then we have another moment cone $M_{n+1} \subseteq \Re^{n+1}$.

We write (5) in the form of

$$\sum_{i=1}^{q} \widetilde{a}(s_i)x_i = (c_0, c_1, \ldots, c_n)^T \tag{6}$$

If we let $\widetilde{A}_s := \{\widetilde{a}(s) \mid s \in S\} \subseteq \Re^{n+1}$, Then

$$M_{n+1} = CC\left(\widetilde{A}_s\right).$$

By the definition of convex conic hull every $\widetilde{z} \in M_{n+1}$ admits the representation

$$\widetilde{z} = \sum_{i=1}^{q} \widetilde{a}(s_i)x_i,\ x_i \geq 0.$$

Comparison with (6) gives

$\{s_1, \ldots, s_q, x_1, \ldots, x_q\}$ is feasible for (D) with the corresponding value $c_0$ of the dual objective function if any only if

$$(c_0, c_1, \ldots, c_n)^T \in M_{n+1} \text{ by Lemma 3.1.1} \tag{7}$$

A " geometric" formulation of the dual problem is

$$c_0 \rightarrow \max \qquad (8)$$

s.t

$$(c_0, c_1, \ldots, c_n)^T \in M_{n+1}$$

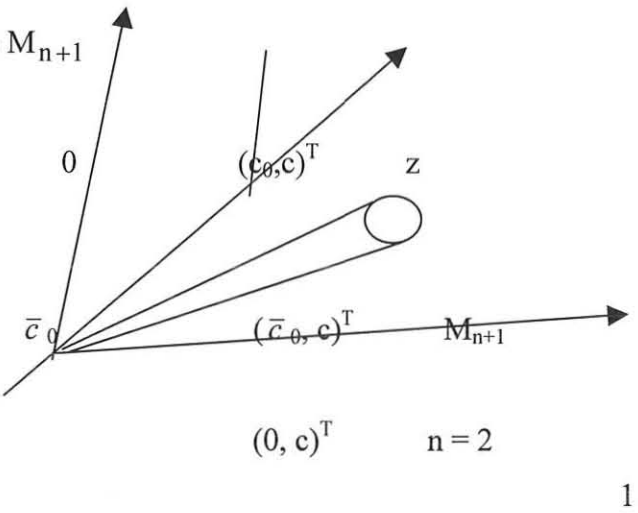


Fig 3.1

Fig 3.1 gives a geometric illustration of the dual problem (8). We have to find the point $(\bar{c}_0, c)^T$ of the straight line

$$\{ (c_0, c_1, \ldots, c_n), c_0 \in R\}$$

which belongs to $M_{n+1}$ and whose first component is maximum.

From a solution $(\bar{c}_0, c_1, \ldots, c_n)^T \in M_{n+1}$, we have

$$\bar{c}_0 = \sum_{i=1}^{n+1} b(\bar{s}_i)\bar{x}_i ,$$

and $$c = \sum_{i=1}^{n+1} a(\bar{s}_i)\bar{x}_i ,$$

where $\bar{x}_i$ are nonnegative number and $\bar{s}_i \in S$, i.$\in$ {1, …, n+1}.

Hence $\{ \bar{s}_1, \ldots, \bar{s}_{n+1}, \bar{x}_1, \ldots, \bar{x}_{n+1} \}$ is a solution to (D).

## 3.2 SOLUBILITY OF THE DUAL PROBLEM

The following important theorem on the solvability of the dual problem (D) is an immediate consequences of the formulation (8) above.

**Theorem 3.2.1.** Let for a given linear optimization problem the dual problem (D) be bounded and $M_{n+1}$ is closed. Then the problem (D) has a solution.

**Proof**: by a state diagram (10 given in chapter 1.3, (D) is bounded implies V(D) is finite, where V(D) is the maximum of the continuous function f given by

$f(z_0, z_1, \ldots, z_n) = z_0$ defined on the set

$$M_{n+1} \cap \{(z_0, z)^T \mid V(D) - 1 \leq , z_0 \leq V(D), Z = C\} \qquad (*)$$

where C is a vector in $\Re^n$, exist in the formulation of the primal problem given by (*) is compact as it is closed and bounded follows the maximum exist.
Since $M_{n+1}$ is closed,

$(z_0, z_1, \ldots, z_n) \in M_{n+1}$ implies

$$z_0 = \sum_{i=1}^{n+1} b(s_i)x_i ,$$

and $z = \sum_{i=1}^{n+1} b(s_i)x_i$ , $x_i \geq 0$, $s_i \in S$, i.e $\{1, \ldots, n+1\}$

Since (D) has feasible solution as it is bounded, we have

$\{s_1, \ldots, s_{n+1}, x_1, \ldots, x_{n+1}\}$ is a solution to (D). //

Quite often we shall encounter a special class of problems where the index set S and the functions $a_1, \ldots, a_n$, b which appears in the constraints of (P)

$$\sum_{r=1}^{n} a_r(s)y_r \geq b(s), s \in S$$

satisfying the following assumptions:

(*)

General assumptions on (P)
S is a compact subset of $R^k$ and the real valued functions $a_1, \ldots, a_n$, b which are defined on S are continuous there.

If S is finite, then trivially every real valued function on S is continuous. hence we can assume that

$$S = \{1, \ldots, m\} \subset R.$$

**Definition 3.2.1** If there is a vector $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_n)^T \in \Re^n$, such that

$$\sum_{r=1}^{n} a_r(s)\tilde{y}_r > b(s),\ s \in S, \qquad (1)$$

Then (P) is said to meet the slater condition.

If (P) satisfies (1), we also call (P) super consistent since (1) is a sharpening of the statement that $\tilde{y}$ is feasible for (P).

Suppose now the general assumption on (P) is satisfied. Then the slater condition (1) met, if one of the functions $a_1, \ldots, a_n$ is constant. For example:

Let $a(s1) = 1$, $s \in S$

then (1) is met if we take

$$\tilde{y} = (\tilde{y}_1, 0, \ldots, 0)^T,$$

where $\tilde{y}_1 > \max_{s \in S} b(s)$.

This is possible since b is continuous on a compact set.

**Remark**: The slater condition is an example of the so called regularity conditions which are introduced in the theory of optimization and which play a major role in the derivation of theorems on duality and existence of solutions.

Lemma 3.2.2. Let $A \subseteq R^p$ be a compact set.

Then con V(A) is also compact.

**Proof**: Let $a_1, \ldots, a_{p+1} \in A$ and

$(x_1, \ldots, x_{p+1}) \in D$, where

$D \subseteq R^{p+1}$ is defined by

$$D = \{x \in R^{p+1} \mid x_i \geq 0, \text{ i.e. } \{1, \ldots, p+1\} \text{ and } \sum_{i=1}^{p+1} x_i = 1\}$$

Then by definition

$$\text{Conv (A)} = \sum_{i=1}^{p+1} a_i x_i$$

Hence conv (A) is the image of the compact set $\underbrace{A \times A \times \cdots \times A \times D}_{p+1 \text{ times}}$

under the continuous mapping

$$(a_1, \ldots, a_{p+1}, x_1, \ldots, x_{p+1}) \rightarrow \sum_{i=1}^{p+1} a_i x_i$$

Since A is compact and the mapping is continuous we have

conv (A) is compact

**Theorem 3.2.3** Suppose that the general assumption of (P) is satisfied and (P) meets the slater condition. Then $M_{n+1}$ is closed.

**Proof**: Let z be an arbitrary vector in $\overline{M}_{n+1}$. We will show that z must be also in $M_{n+1}$.

Since $M_{n+1} = CC(\hat{A}_s)$ by definition, for $z \in \overline{M}_{n+1}$ we may associate a sequence $\{h_i\}_{i>1}$ in conv $(\widetilde{A}_s)$ and a sequence of nonnegative numbers $\{\lambda_i\}_{i>1}$ such that

$$z = \lim_{i \to \infty} \lambda_i h_i \qquad (1)$$

Since S is compact, and $a_1, \ldots, a_n$, b are continuous, the set $\widetilde{A}_s$ is compact. by Lemma 3.2.2, conv $(\widetilde{A}_s)$ is compact. Therefore we may pick a subsequence of $\{\lambda_i\}_{i>1}$ which converges to a vector $h \in$ conv $(\widetilde{A}_s)$. from (1) we may assume by

$$\lim_{i \to \infty} h_i = h, \; h \in \text{conv } (\widetilde{A}_s).$$

Case 1. If now the sequence $\{\lambda_i)_{i>1}$ is bounded, in similar way we can assume that it converges to $\lambda > 0$. Then we obtain

$$z = \lim_{i \to \infty} \lambda_i h_i = \lim_{i \to \infty} h_i \lim_{i \to \infty} h_i = \lambda h$$

and from $h \in$ conv $(\widetilde{A}_s)$, $\lambda \geq 0$, it follows that

$$z = \lambda h \in cc(\widetilde{A}_s) = M_{n+1}$$

Case 2.if $\{\lambda_i\}_{i > 1}$ is un bounded, then we may assume by using a suitable subsequence, that $\lambda_i > 0$, i = 1, 2, …, and $\lim \frac{1}{\lambda_i} = 0$. Hence we get

$$h = \lim_{i\to\infty} h_i = \lim_{i\to\infty} \frac{1}{\lambda_i}\lambda_i h_i = \lim_{i\to\infty} \frac{1}{\lambda_i} \lim_{i\to\infty} \lambda_i h_i = 0z = 0.$$

i.e the nuall vector of $R^{n+1}$ lies in conv($\widetilde{A}_s$). Hence there are $q \geq 1$ non negative numbers $\alpha_1, \ldots, \alpha_q$ and q points $s_1, \ldots, s_q$ in S such that

$$\sum_{i=1}^{q} \widetilde{a}(s_i)\alpha_i$$

$$\text{and} \quad \sum_{i=1}^{q} \alpha_i = 1 \qquad (2)$$

which implies $\sum_{i=1}^{q} b(s_i)\alpha_i = 0$

and $\sum_{\varepsilon=1}^{q} a_r(s_i)\alpha_i = 0$, r = 1, …, n.

Let $y \in \Re^n$ be an arbitrary vector. The last two equation give

$$\sum_{i=1}^{q} \alpha_i \left( \sum_{r=1}^{n} y_r a_r(s_i) - b(s_i) \right) = 0 \qquad (3)$$

Since (P) meets the slater condition there is $\widetilde{y} \in \Re^n$ such that

$$\sum_{r=1}^{n} \widetilde{y}_r a_r(s_i) - b(s_i) > 0, \text{ i. } \in \{1, \ldots, q\}$$

Now if we put $y = \widetilde{y}$ in (3) we get, since $\alpha_i \geq 0$, that $\alpha_1 = \ldots \alpha_q = 0$ must hold. But this contradicts (2). This rules out the possibility that $\{\lambda_i\}_{i > 1}$ is unbounded. Hence we have established the theorem. //

**Example.** Consider the linear optimization problem

$$C^T Y \rightarrow \min$$

$$\text{(P)} \qquad \text{s.t}$$

$$s^2 y_1 \geq s, \qquad s \in [0, 1]$$

Here we have $n = 1$, $S = [0, 1]$, $a_1(s) = s^2$, $b(s) = S$.

Since $a_1(0) = b(c) = 0$, the slater condition is not met.

$M_{n+1}$ is not closed since the vectors $(s_1, 0)^T$, $x_1 > 0$ are in $\overline{M}_{n+1}$ but not in $M_{n+1}$.

## 3.3 Separation Theorem and Duality

We start this section by developing a fundamental tool to be used in the proof of strong duality theorem, namely the statement that a point outside a closed convex set in $R^n$ may be "separated" from this set by a hyper plane i the sense of the following definition.

**Definition 4.3.1** Let M be a non empty, closed and convex subset of $R^p$ and $z \notin M$ a fixed point, the hyperplane

$$H(y; \eta) = \{x \in R^P \mid y^T x = \eta\}$$

is said to separate z from M if

$$y^T x \leq \eta < y^T z, \; x \in M$$

Geometrically

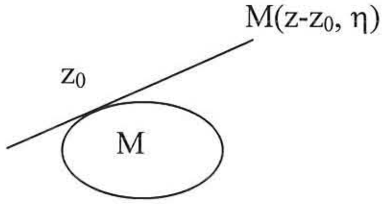


Fig. 4.3

From the geometric consideration, a vector y which defines a separating hyperplane is obtained by determining the projection $z_0$ of z on M and putting

$y = z - z_0$. We will therefore show that existence of projection theorem.

First let us see the role of the concept of separating hyperplane in the theory of the dual pair (P) – (D). Assume that the hyperplane

$$H(y; 0) \mid = \{ z \in \Re^{n+1} \mid \sum_{r=0}^{n} z_r y_r = 0\}$$

separates the moment cone $M_{n+1}$ from the point $V \notin M_{n+1}$.

Thus all of $M_{n+1}$ lies on one side of the hyperplane. hence

$$0 \geq \sum_{r=0}^{n} z_r y_r \text{ , for all } (z_0, m_1, z_n) \in M_{n+1} \qquad (1)$$

In particular since $M_{n+1} = CC(\widetilde{A}_s)$ we have

$z = \widetilde{a}(s) = (b(s), a_1(s), \ldots, a_n(s))^T \in M_{n+1}$ for all $s \in S$. Thus we find from (1) that

$$0 \geq b(s)y_0 + \sum_{r=1}^{n} a_r(s) y_r \text{ , } s \in S$$

If $y_0 > 0$ holds, then the last relation takes the form

$$\sum_{r=1}^{n} a_r(s) \text{ - } \frac{y_r}{y_0} \geq b(s), \ s \in S$$

Hence the vector $Y = \left( \frac{-y_1}{y_0}, \ldots, \frac{-y_n}{y_0} \right)$ is feasible for (P).

**Theorem 3.3.1** (Projection theorem)

Let $M \subseteq \Re^p$ be a nonempty, closed, convex set and let z be a fixed point out side of M. Then there is exactly one vector $z_0 \in M$ which lies "closest to z" i.e, $z_0$ is such that

$$0 < | z - z_0| \leq |z - x|. \ \forall x \in M.$$

**Proof**: 1. Existence

Since M is closed and $z \notin M$ we have

$$\rho = \inf_{x \in M} |z - x| > 0.$$

Obviously, it is sufficient to search for a vector $z_0$ in the set

$$\widetilde{M} = M \cap \{x \in R^p \mid |z - x| \le 2\rho\}.$$

Now the continuous real-valued function $x \to |z - x|$ assumes its minimum value on $\widetilde{M}$ as it is bounded and closed set. hence there is a $z_0 \in M$ such that

$$|z - z_0| \le |z - x|, \; x \in \widetilde{M}. \qquad (1)$$

From the construction of $\widetilde{M}$, (1) holds for all $x \in M$.

2. Uniqueness

Assume $z_1 \ne z_0$ such that

$$|z - z_1| \le |z - x|, \text{ for all } x \in M.$$

Put $z_2 = \dfrac{(z_1 + z_0)}{2}$

From parallelogram law we have

$$\begin{aligned} |z - z_2|^2 = \frac{1}{4} |\,(z - z_0) + (z - z_1)|^2 < \frac{1}{4} |\,(z - z_0) + (z - z_1)\,|^2 + \frac{1}{4} |z_0 - z_2|^2 \\ = \frac{1}{4} |(z - z_0) + (z - z_1)|^2 + \frac{1}{4} |(z - z_0) - (z - z_1)|^2 \\ = \frac{1}{2} (|z - z_0|^2 + |z - z_1|^2) = \\ = |z - z_0|^2. \end{aligned}$$

which implies

$$|z - z_2| < |z - z_0|.$$

But this contradicts the construction of $z_0$.

follows $z_0$ is unique. //

**Theorem 3.3.2** (Separation theorem).

Let $M \subseteq R^p$ be non empty, closed, and convex set. Let $z \notin M$ be a fixed point whose projection on M is $z_0$. If we put $y = z - z_0$ and $\eta = (z - z_0)^T z_0$ we get

$$y^T x \le \eta < y^T z, \; x \in M;$$

i.e the hyperplane $H(y; \eta)$ separates z from M.

**Proof:** Let $x \in M$ be an arbitrary vector and $0 < \eta \le 1$ be a fixed number. Then

$$(1-\mu)z_0 + \mu x = z_0 + \mu(x - z_0) \in M.$$

we also find that

$$\begin{aligned} |z - z_0|^2 &\le |\, z - (z_0 + \mu(x - x_0))\,|^2 \\ &= |z - z_0|^2 - \alpha\mu(z - z_0)^T(x - z_0) + \mu^2\,|x - z_0|^2) \end{aligned}$$

giving

$$(z\ - z_0)^T\,(x - z_0) \le \frac{1}{2}\ \mu|x - z_0|^2.$$

Letting $\mu \to 0$, we have

$$(z - z_0)^T\,(x - z_0) \le 0,$$

which follows

$$y^T x \le y \qquad\qquad (1)$$

For the other inequality we have

$$0 < |z - z_0|^2 = (z - z_0)^T\,(z - z_0) = y^T z - y^T z_0 = y^T z - \eta.$$

follows

$$\eta < y^T z \qquad\qquad (2)$$

From (1) and (2) we have

$$y^T x \le\ \eta < y^T z.\ //$$

Suppose now that the assumption of separation theorem hold, but specialize M to be a convex cone. Then $x \in M$ implies that $\lambda x \in M$ for all $\lambda > 0$. form (1) and (2) we have

$$y^T(\lambda x) \le \eta,\ \ \lambda > 0,$$

$$\text{or} \qquad y^T x\ \le \frac{\eta}{\lambda},\ \lambda > 0$$

Letting $\lambda \to \infty$ we have

$$y^T x \le 0,\ x \in M.$$

Thus if M is a convex cone we may put $\eta = 0$, we have

$$y^T x \le 0 < y^T z, \quad x \in M.$$

Now we can use the separation theorem to establish the strong duality theorem.

**Theorem 3.3.3** (First duality theorem)

Consider the dual pair (P), (D) with the following assumptions:

(i) The dual problem is consistent and has a finite value V(D);

(ii) The Moment cone $M_{n+1}$ is closed.

Then (P) is consistent as well and

$$V(P) = V(D);$$

i.e there is no duality gap. More over (D) is solvable.

**Proof.** Since (D) is bounded and $M_{n+1}$ is closed by theorem 3.2.1 (D) is solvable and we have

$$(c_0, c_1, \ldots, c_n)^T \in M_{n+1},$$

but $(c_0 + \varepsilon, c_1, \ldots, c_n) \notin M_{n+1}$ for any $\varepsilon > 0$.

Since $M_{n+1}$ is closed by separation theorem there is a hyperplane in $\Re^{n+1}$ which separates $(c_0 + \varepsilon, c)^T$ from the convex cone $M_{n+1}$. Hence there is a vector $(y_0, y_1, \ldots, y_n)^T \in \Re^{n+1}$ different form zero, such that

$$\sum_{r=0}^{n} x_r y_r \leq 0 < y_0(c_0 + \varepsilon) + \sum_{r=1}^{n} c_r y_r,$$

$$(x_0, x_1, \ldots, x_n)^T \in M_{n+1} \qquad (1)$$

in (1) we now put

$$(x_0, x_1, \ldots, x_n)^T = (c_0, c_1, \ldots, c_n)^T \in M_{n+1}$$

and we obtain

$$y_0 \varepsilon > 0.$$

Since $\varepsilon > 0$, we must have $y_0 > 0$. If we set

$$(x_0, x_1, \ldots, x_n)^T = (b(s), a_1(s), \ldots, a_n(s))^T \in \widetilde{A}_s \subseteq M_{n+1}$$

where $s \in S$ is arbitrary, we find from the left most inequality in (1)

$$\sum_{r=1}^{n} a_r(s) \left( \frac{-y_r}{y_0} \right) \geq b(s), \ s \in S.$$

Hence the vector

$$\widetilde{y} = \left( \frac{-y_1}{y_0}, \frac{-y_2}{y_0}, \ldots, \frac{-y_n}{y_0} \right) \in \Re^n$$

is feasible for (P).

The right side inequality implies

$$\sum_{r=1}^{n} c_r \left( {-y_r}/{y_0} \right) < c_0 + \varepsilon.$$

Hence we have

$$V(P) \leq \sum_{r=1}^{n} c_r \ \tilde{y}_r < c_0 + \varepsilon = V(D) + \varepsilon \leq V(P) + \varepsilon$$

Thus $V(P) - \varepsilon \leq V(D) \leq V(P)$

**Theorem 3.3.4.** Let the dual pair (P), (D) be given with the assumptions

i) General assumption on (P) holds;

ii) (D) is consistent;

iii) (P) meets the slater condition

Then (D) is solvable and the values of (P) and (D) coincides.

**Proof**: Since (D) is consistent, it is bounded.

i.e $V(D) < \infty$.

Moreover (i) and (iii) holds true implies

$M_{n+1}$ is closed, by Theorem 3.2.3

Hence by the first duality theorem, we get the desired result. //

Now for the special case of the linear optimization problem, where $s = \{1, \ldots, m\}$ we have the following theorem,

**Theorem 3.3.5** Let the primal problem $(P_\ell)$ and the dual problem $(D_\ell)$ be given, such that both are consistent. The both are solvable and the values of their respective objective function coincides.

**Proof**: Since S is finite and compact the general assumption on (P) is fulfilled. Moreover, since $(D_\ell)$ is consistent, $V(D_\ell) < \infty$ and $(P_\ell)$ is consistent implies $(P_\ell)$ meets the slatter condition.

Therefore by theorem 3.2.3 and first duality theorem $(P_\ell)$ and $(D_\ell)$ are solvable and their corresponding values coincide.//

## 3.4 Supporting Hyperplanes and Duality

**Definition 3.4.1** Let M be a non-empty convex subset of $\Re^p$, and let $z \in M$ be a fixed point. The Hyperplane

$$H(y; \eta) := \{x \in \Re^p \mid y^T x = \eta\}$$

is said to be a supporting hyperplane to M at z if

$$y^T x \leq \eta = y^T z, \quad x \in M$$

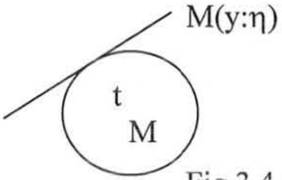


Fig 3.4.1

Supporting Hyperplane.

**Lemma 3.4.1** Let $z \in \dot{M}$ (interior of M). Then there are no supporting hyperplanes to M at Z.

**Proof**: Suppose M has supporiting hyperplane $H(y; \eta)$ at Z.

Since $z \in \dot{M}$, there is a $\lambda > 0$, such that

$$z_\lambda = z + \lambda y \in M.$$

Then we have

$$y^T z \lambda \leq 0.$$

But this contradicts the fact that $\lambda > 0$ and $y^T y > 0$. hence we have the desired result. //

**Theorem 3.4.2** Let M be a nonempty convex subset of $\Re^p$ and let z be on the boundary of M. (i.e. $z \in \mathrm{bd}M = \overline{M} \setminus \dot{M}$). Then there is a supporting hyperplane to M at z.

**Proof**: Let $z \in \mathrm{bd}M$ be a fixed point. Then there is a sequence $\{z_i\}$ of points, such that $z_i \notin \overline{M}$ and $\lim_{i} z_i = z$.

Let $z_{i_0}$ be the projection of $z_i$ on $\overline{M}$ such that

$$y_i = z_i - z_{i_0} .$$

then $y_i^T x < y_i^T z_i$, $x \in \overline{M}$ , $i \in (1, 2\ldots,)$ (by separation theorem).

Since $z_i \notin \overline{M}$, $y_i \neq 0$, $i \in \{1, 2, \ldots\}$

Setting $\tilde{y}_i = \frac{y_i}{|y_i|}$, $i \in N$

we get $|\tilde{y}_i| = 1$, and

$$y_i^T x < y_i^T z_i, \; x \in M, \; i \in M \qquad (1)$$

Consider the set $B = \{y \in \Re^P \mid |y| = 1\}$.

Clearly B is closed and bounded, hence compact.

Therefore there is a subsequence of $\{\tilde{y}_i\}_{i=1}$ which converges to a point $\tilde{y} \in B$.

By (1) and taking the limit, we have

$$y^T x \leq \tilde{y}^T z, \; x \in M, \; \tilde{y} \neq 0.$$

hence we have the desired result. //

**Definition 3.4.2** The dual problem (D) is called super consistent,

if $c \in \mathring{M}_n$, where $c = (c_1, \ldots, c_n)^T$.

**Theorem 3.4.3** Second Duality theorem.

Let the dual pair (P), (D) be given with the following assumptions:

i) V(D) is finite

ii) (D) is super consistent.

Then (P) is solvable and V(P) = V(D).

**Proof**: Since (D) is super consistent, we have

$$\sum_{i=1}^{q} a_r(s_i)x_i < c_r,\ r \in \{1, \ldots, n\}.$$

from the proof of Duality Lemma 1.1.1, we get

$$\sum_{i=1}^{q} a_r(s_i)\widetilde{y}_r \geq b(s),\ s \in S$$

which implies by definition $\widetilde{y}$ is feasible for (P).

Hence by weak duality lemma, V(P) is finite.

Let $\hat{c}_0 = V(P)$. (1)

Then $(\hat{c}_0, c_1, \ldots, c_n)^T \in bdM_{n+1}$. Other wise there is $c_0 > \hat{c}_0$ feasible to (D), such that

$$(c_0, c_1, \ldots, c_n)^T \in bdM_{n+1}.$$

But this is not possible, since it contradicts (1).

define the convex cone $\widetilde{M}_{n+1} = \{(\widetilde{z}_0, \widetilde{z}_1, \ldots, \widetilde{z}_n)^T \mid$ there is $(z_0, z_1, \ldots, z_n)^T \in M_{n+1}$ such that $\widetilde{z}_0 \leq z_0, \widetilde{z}_1, \ldots, \widetilde{z}_n = z_n\}$

Clearly $(\hat{c}_0, c_1, \ldots, c_n)^T \in bdM_{n+1}$.

By theorem 3.4.2, there is a non trivial supporting hyperplane to $\widetilde{M}_{n+1}$ at $(\hat{c}_0, c_1, \ldots, c_n)^T$; i.e there is $\widetilde{y} = (y_0, y)^T = (y_0, y_1, \ldots, y_n)^T \neq 0$ such that

$$\widetilde{y}^T z \leq 0 = y_0 \hat{c}_0 + y^T c,\ z \in \widetilde{M}_{n+1} \quad (2)$$

by convexity of $M_{n+1}$.

Since $\widetilde{A}_s \subseteq cc(A_s) = M_{n+1} \subseteq \widetilde{M}_{n+1}$, (2) implies

$$y_0 b(s) + \sum_{r=1}^{n} a_r(s)y_r \leq 0,\ s \in S \quad (3)$$

Now we want to show $y_0 > 0$.

From the definition of $\widetilde{M}_{n+1}$ it follows that

$$(\hat{c}_0 - \lambda, c)^T \in \widetilde{M}_{n+1},\ \lambda > 0.$$

therefore from (2) we get

$y_0 \hat{c}_0 + y^T c = 0$, we find the

$-y_0 \lambda \leq 0,\ \lambda \geq 0.$

hence we have $y_0 \geq 0$.

Now to remove the possibility $y_0 = 0$, Let us assume $y_0 = 0$ from(2) we have

$$\sum_{r=1}^{n} y_r z_r \leq \sum_{r=1}^{n} c_r y_r \,, \; z \in M_n, \qquad (4)$$

where $M_n$ is the projection of $M_{n+1}$ on the subspace of $\Re^{n+1}$ defined through the condition $z_0 = 0$. Therefore (4) means that there is a nontrivial supporting hyperplane to $M_n$ at c. (Since $\widetilde{y} \neq 0$. and $y_0 = 0$ we have $(y_1, \ldots, y_n) \neq 0$). But this contradicts the fact that $c \in \overset{\circ}{M}_n$ . (Lemma 3.4.1).

hence we have $y_0 > 0$.

Now, Let $\widetilde{y}_r = \dfrac{y_r}{y_0}$, $r \in \{1, \ldots, n\}$,

from (3), we obtain

$$\sum_{r=1}^{n} a_r(s) \widetilde{y}_r \geq b(s), \; s \in S.$$

Thus $(\widetilde{y}_1, \ldots, \widetilde{y}_n)^T$ is feasible for (P)

hence $V(D) \leq V(P) \leq \sum_{r=1}^{n} c_r \widetilde{y}_r$ .

By (2) we conclude that

$$\sum_{r=1}^{n} c_r \widetilde{y}_r = \hat{c}_0 = V(D).$$

Hence we have established the theorem.

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