# Chromatic Polynomials

**By: Getachew Derso**

Addis Ababa University
Faculty of Computer and Mathematical Sciences
Department of Mathematics

**Advisers: Melkamu Zeleke (Professor)**
**&**
**Seyoum Getu (PhD)**

**A Project submitted to the Office of Graduate Programs of Addis Ababa University in Chromatic polynomials of the requirements for the Degree of Master of Science in Mathematics**

January, 2011

# Declaration Letter

**I, Getachew Derso, declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other similar title to me.**

**Getachew Derso**

________________________________

Addis Ababa University
January, 2011

# Permission Letter

This is to certify that this project is compiled by Mr. **Getachew Derso** in the department of Mathematics, College of Mathematics and Computational Sciences, Addis Ababa University, under my supervision.

Melkamu Zeleke (Prof.)

______________________________

Seyoum Getu (PhD)

____________________________

Addis Ababa University

January, 2011

# Addis Ababa University
# Office of Graduate Program
# Faculty of Science
# Department of Mathematics

Approved by the Board of Examiners:

______________________ ____________________

Department Head Signature

______________________ ____________________

Examiner Signature

______________________ ____________________

Examiner Signature

---

Tel. 251-911-014093 e-mail: gettderso@yahoo.com

Table of contents

# List of Figures

**Figures** **Pages**

# I. ACKNOWLEDGEMENT

In preparing this project, I would like to express my appreciation to my advisors **Professor Melkamu zeleke** and **Doctor Seyoum Getu** for their patience in repeatedly reading the draft manuscript of this project and for making constructive comments and suggestions from which I have immeasurably benefited.

Second, I would like to extend grateful acknowledgement to my family in general my father, my mother and my brothers in particular for their blessed support and much appreciated willingness to shoulder social responsibility on my behalf.

Last but not least, I am also indebted for many friends and to all those who helped me one way or another for the accomplishment of my study.

By : Getachew Derso

January 25,2011

## II. Summary of the project

The four-color theorem states that any planar graph $G = (V, E)$ can be properly colored using at most four colors. The attempt to prove this famous mathematical problem gave rise to the development of many tools for solving problems related to graph coloring.

In 1912, Birkhoff introduced a function $P(G,\lambda)$, define for all positive integer $\lambda$,to be the number of all proper colorings of a graph $G = (V,E)$.This function turns out to be a polynomial in $\lambda$ , and one could prove the four-color theorem by showing that $P(G,\lambda) > 0$ for all graphs $G = (V, E)$.

The polynomial $P(G,\lambda)$ is defined for all real and complex values of $\lambda$ ,and it is called the chromatic polynomial of $G = (V, E)$. In this project, we describe the properties of chromatic polynomials, discuss some practical methods for computing them, and look at the roots of chromatic polynomials . Classic results such as Whitney's Broken-cycle theorem and Read's Unimodal conjecture on the coefficients of $P(G,\lambda)$ will also be discussed.

# III Introduction

## 1.1. History and Concepts.

In 1912, George Birkhoff [4] introduced chromatic polynomials in an unsuccessful attempt to solve the four-color conjecture. The proof of the four-color theorem was obtained in 1976 [1, 2] by means of a computer analysis of almost 2000 cases. This proof did not use the theory of chromatic polynomials.

Meanwhile, new methods in chromatic polynomial theory have led to research in directions other than attempting to prove the four-color theorem. The study of chromatic polynomials is currently an active branch of graph theory, with many fundamental problems as yet unsolved. One such problem is to find a necessary and sufficient condition for a polynomial to be the chromatic polynomial of a graph. Another is the classification of all graphs which are uniquely determined, up to isomorphism, by their chromatic polynomial.

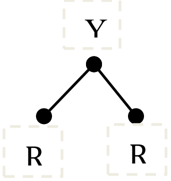
For the purposes of this report, a graph G is a pair of finite sets $(V, E)$ ,where V is a non-empty set of n vertices, and E is a set of e distinct unordered pairs $(a, b)$ , with $a, b \in V(G)$ and a ≠ b. These pairs are called edges. A graph is planar if it can be drawn on a plane such that its edges do not cross.

Given a graph G, we can label its vertices l,...,n. Now we introduce a set of $\lambda$ colors, and assign a color to each of the n vertices so that two vertices joined by an edge do not receive the same color. Such an assignment is a proper coloring of G ; by a coloring of G , we shall mean a proper coloring. Note that not all of the $\lambda$ colors need be used. If such a coloring is possible, then G is $\lambda$ -colorable. The smallest integer for which G is $\lambda$ -colorable is the chromatic number of G, and is denoted by $\chi(G)$ .

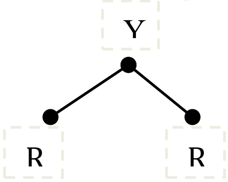
This coloring procedure is a direct abstraction of the cartographer's problem of coloring a map of political regions so that two regions with a common border are colored with distinct colors. Simply think of the regions as vertices; join two vertices by an edge if the corresponding regions have a border in common. Any map the cartographer draws will correspond to a planar graph. The four-color theorem states that all planar graphs are 4-colorable.

Chromatic polynomials were invented to count the number of different colorings of a graph. What does it mean for two colorings to be different? If we start with a coloring of G, and permute the colors, then we obtain a new coloring, provided the permutation is not the identity. These two colorings are considered to be different: this property is known as color difference . For example, given the set of colors {R, Y ,B} representing red, yellow and blue respectively,

a.

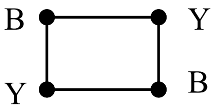


b.

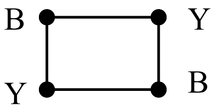


the coloring in (a) is different from the coloring in (b) . Here the permutation is the one that interchanges the two colors red and yellow. Two colorings are considered to be the same under color indifference if one coloring can be obtained from the other by a permutation of the colors. Under both color difference and color indifference, the vertices of the graph are fixed; i.e., permutations of the vertices are not allowed.

(1)



(2)



Thus (1) is different from (2) under both color difference and color indifference. From this point on, a coloring will mean a proper coloring with color difference.

In his 1912 paper, Birkhoff showed that the function which describes the number of ways of coloring G, with $\lambda$ or fewer colors, is in fact a polynomial in $\lambda$ .The proof is as follows:

Start with a graph G. Let $m_i$ be the number of ways of coloring G by using exactly i colors, with color indifference. There are $\lambda$ ways of selecting the first of the i colors, $\lambda$ - 1 ways of selecting the second of the i colors, $\lambda$ - 2 ways of selecting the third color ,..., and $\lambda - i + 1$ ways of selecting the $i^{th}$ color. Thus, the expression $m_i(\lambda - 1)(\lambda - 2) \ldots (\lambda - i + 1)$ represents the number of ways of coloring G by using exactly i colors with color difference.

If i is the number of colors actually used to color G, then i can range from 1 to n , the number of vertices. Summing over i gives the total number of ways G can be colored in $\lambda$ or fewer colors.

$$P(G,\lambda) = \sum_{i=1}^{n} m_i(\lambda-1)(\lambda-2)\dots(\lambda-i+1)$$

Therefore $P(G,\lambda)$ is a polynomial, since it is expressed as a linear combination of polynomials. $P(G,\lambda)$ is called the chromatic polynomial of G in the variable $\lambda$.

There are 3 chapters in this project:

In chapter one , we shall introduce the basic concepts of graph theory.

In chapter two, we have six sections. In section one we introduce the concepts of coloring and chromatic number, in section two and three ,we present chromatic polynomial for some special graphs and important theorems. In section four we introduce properties of chromatic polynomials. In section five, we shall see representation of chromatic polynomial in tree , null and complete graph form. To end the chapter, we introduce umbral product of polynomials.

In chapter three, we have four sections. In section one, we shall see chromatic polynomial identities for tree, cycle and wheel graph. In section two, suppose $P(G,\lambda)$ expressed in power form i.e $P(G,\lambda) = \sum_{i\geq 0} a_i\lambda^i$ we shall see some of the useful results relating the coefficients $a_i's$ and the number of certain subgraphs of G. To end the chapter, we disscuse the unimodial conjecture and strong logarithmic concavity conjecture on $a_i$ ‘s.

# Chapter-one

## 1. Basic Concepts in Graph Theory

---

Informally, a graph is a diagram consisting of points, called vertices, joined together by lines, called edges; each edge joins exactly two vertices. A graph G is a triple consisting of a vertex set of $V(G)$, an edge set E(G), and a relation that associates with each edge two vertices (not necessarily distinct) called its endpoints.

### ❖ Definition of Graph

A graph $G = (V, E)$ consists of a (finite) set denoted by V, or by V(G) if one wishes to make clear which graph is under consideration, and a collection E, or E(G), of unordered pairs {u, v} of distinct elements from V. Each element of V is called a vertex or a point or a node, and each element of E is called an edge or a line or a link. Formally , a graph G is an ordered pair of disjoint sets (V, E), where $E \subseteq V \times V$. Set V is called the vertex or node set, while set E is the edge set of graph G. Typically, it is assumed that self-loops (i.e. edges of the form (u, u), for some $u \in V$) are not contained in a graph.

### ❖ Directed and Undirected Graph

A graph $G = (V, E)$ is directed if the edge set is composed of ordered vertex (node) pairs. A graph is undirected if the edge set is composed of unordered vertex pair.

### ❖ Vertex Cardinality

The number of vertices, the cardinality of V, is called the order of graph and devoted by |V|. We usually use $n$ to denote the order of G. The number of edges, the cardinality of E, is called the size of graph and denoted by |E|. We usually use $m$ to denote the size of G.

### ❖ Neighbor Vertex and Neighborhood

We write $v_i v_j \in E(G)$ to mean $\{v_i, v_j\} \in$ E(G), and if e = $v_i v_j \in$ E(G), we say $v_i$ *and* $v_j$ are adjacent.

Formally, given a graph G = (V, E), two vertices $v_i$ , $v_j \in$ V are said to be neighbors, or adjacent nodes, if ( $v_i$ , $v_j$) ∈ E. If G is directed, we distinguish between incoming neighbors of $v_i$ (those vertices $v_j \in$ V such that ($v_j$, $v_i$) ∈ E) and outgoing neighbors of $v_i$ (those vertices $v_j \in$ V such that ($v_i$, $v_j$) ∈ E).

- **Vertex Degree**

The degree deg(v) of vertex v is the number of edges incident on v . The degree sequence of graph is (deg($v_1$), deg($v_2$), ..., deg($v_n$)), typically written in nondecreasing or nonincreasing order. The minimum and maximum degree of vertices in V(G) are denoted by $\delta$(G) and Δ(G), respectively. If $\delta$(G) = Δ(G) = r, then graph G is said to be regular of degree r, or simply r-regular.

Formally, given a graph G = (V, E), the degree of a vertex v ∈ V is the number of its neighbors in the graph. That is,

$$\deg(v) = |\ \{u \in V : (v, w) \in E\}|.$$

If G is directed, we distinguish between in-degree (number of incoming neighbors) and out-degree (number of outgoing neighbors) of a vertex.

- **Loop and Multiple Edges**

A loop is an edge whose endpoints are equal i.e., an edge joining a vertex to itself is called a loop. We say that the graph has multiple edges if in the graph two or more edges joining the same pair of vertices.

### 1.1 Simple Graph

A graph with no loops or multiple edges is called a simple graph. We specify a simple graph by its set of vertices and set of edges, treating the edge set as a set of unordered pairs of vertices and write $e = uv$ (or $e = vu$) for an edge $e$ with endpoints $u$ and $v$.

When $u$ and $v$ are endpoints of an edge, they are adjacent and are neighbors.

- **Connected Graph**

A graph that is in one piece is said to be connected, whereas one which splits into several pieces is disconnected.

A graph G is connected if there is a path in G between any given pair of vertices, otherwise it is disconnected. Every disconnected graph can be split up into a number of connected subgraphs, called components.

## 1.2 Subgraph

Let $G$ be a graph with vertex set $V(G)$ and edge set E(G). A subgraph of $G$ is a graph all of whose vertices belong to $V(G)$ and all of whose edges belong to $E(G)$.

- **Degree (or Valency )**

Let $G$ be a graph with loops, and let $v$ be a vertex of $G$. The degree of $v$ is the number of edges meeting at $v$, and is denoted by $deg(v)$.

- **Regular Graph**

A graph is regular if all the vertices of G have the same degree. In particular, if the degree of each vertex is r, the G is regular of degree r.

**The Handshaking Lemma:** In any graph, the sum of all the vertex-degree is equal to twice the number of edges.

***Proof*** Since each edge has two ends, it must contribute exactly 2 to the sum of the degrees. The result follows immediately.

The Following are the consequences of the Handshaking lemma.

1. In any graph, the sum of all the vertex-degree is an even number.
2. In any graph, the number of vertices of odd degree is even.

3. If G is a graph which has n vertices and is regular of degree r, then G has exactly 1/2 nr edges.

## 1.3 Isomorphic Graphs

Two graph G and H are isomorphic if H can be obtained from G by relabeling the vertices - that is, if there is a one-to-one correspondence between the vertices of G and those of H, such that the number of edges joining any pair of vertices in G is equal to the number of edges joining the corresponding pair of vertices in H. For example, the two graphs are not isomorphic.

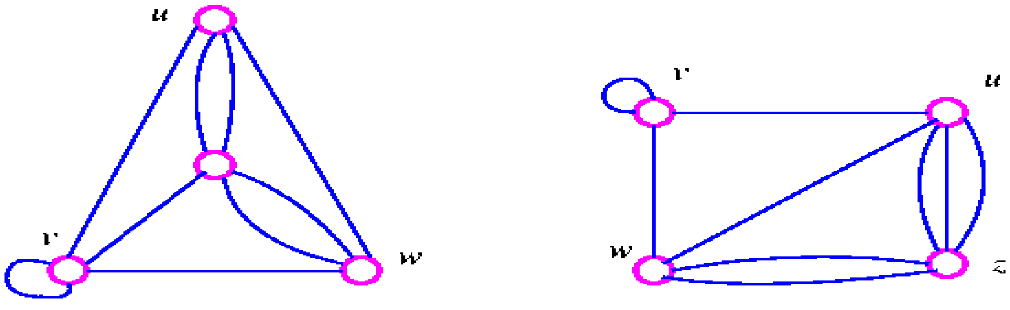


Figure 1.1

The word isomorphic derives from the Greek for same and form.

### Walk

A walk of length k in a graph $G$ is a succession of $k$ edges of G of the form $uv$, $vw$, $wx$, xy,$yz$.

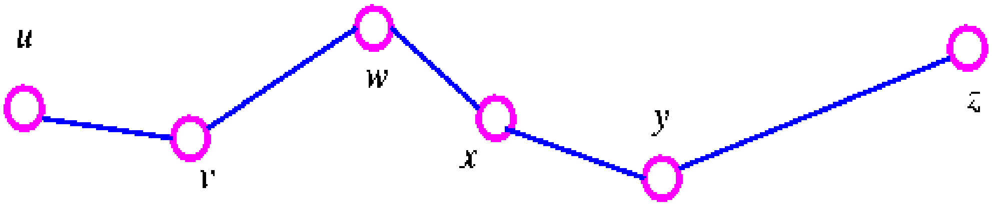


Figure 1.2

We denote this walk by $uvwxyz$ and refer to it as a walk between $u$ and $z$.

- **Trail and Path**

If all the edges (but no necessarily all the vertices) of a walk are different, then the walk is called a trail. If, in addition, all the vertices are different, then the trail is called path.

### 1.4 Some common graphs

- **Complete Graphs**

A complete graph is a graph in which every two distinct vertices are joined by exactly one edge. The complete graph with $n$ vertices is denoted by $K_n$.

The graph $K_n$ is regular of degree $n$-1, and therefore has $\frac{1}{2}n(n-1)$ edges, by consequence 3 of the handshaking lemma.

- **Null Graphs**

A null graph is a graph containing no edges. The null graph with n vertices is denoted by $N_n$.

Note that $N_n$ is regular of degree 0.

- **Cycle Graphs**

A cycle graph is a graph consisting of a single cycle. The cycle graph with n vertices is denoted by $C_n$.

Note that: $C_n$ is regular of degree 2, and has $n$ edges.

- **Path Graphs**

A path graph is a graph consisting of a single path. The path graph with n vertices is denoted by $P_n$.

Note that path graph, $P_n$, has $n$-1 edges, and can be obtained from cycle graph, $C_n$, by removing any edge.

- **Bipartite Graphs**

A bipartite graph is a graph whose vertex-set can be split into two sets in such a way that each edge of the graph joins a vertex in the first set to a vertex in the second set.

- **Complete Bipartite Graph**

A complete bipartite graph is a bipartite graph in which each vertex in the first set is joined to each vertex in the second set by exactly one edge.

The complete bipartite graph with $r$ vertices and s vertices is denoted by $K_{r,s}$.

Note that $K_{r,s}$ has r + s vertices (r vertices of degree s, and s vertices of degree r), and rs edges. Note also that $K_{r,s} = K_{s,r}$.

**An Important Note:** A complete bipartite graph of the form $K_{1,s}$ is called a star graph.

## 1.5: New graphs formed from the given ones

There are many ways of producing a new graph from one or more given graphs.

Let G be the a graph and x, y $\in$ V(G)

if xy $\in V(G)$,then we write G.xy to denote the graph obtained from G by contracting x and y . if xy $\notin$ E(G), then we denote by G+xy the graph obtained by adding a new edge to G.

The complement of a graph G is that graph whose vertex set is V(G) and where uv is an edge of $\bar{G}$ if and only if uv is not an edge of G. Observe that if G is a graph of order n and size m, then $\overline{G}$ is a graph of order n and size $\binom{n}{2} - m$. Furthermore, if G is isomorphic to G, then G is said to be self-complementary. Both $P_4$and $C_5$ are self-complementary graphs.

An edge e in **G, G** said to be subdivided if it is deleted and replaced by a chain connecting its ends, the internal vertices of this chain being new. A subdivision of G is a graph that can be obtained from G by a finite sequence of edge subdivisions .A subdivision of G is called G-homemorph.

For any k≥2, let $\Theta_k$ be the multi graph with 2 vertices and k edges. Any subdivision of $\Theta_k$ is called a multi-bridge graph.

For any $a_1, a_2, a_3, \ldots, a_k \in N$, we denote by $\Theta(a_1, a_2, a_3, \ldots, a_k)$ the graph obtained by replacing the edges of $\Theta_k$ with paths of lengths $a_1, a_2, a_3, \ldots, a_k$ respectively.

The graph $\Theta(a_1, a_2, a_3)$ is called a $\Theta$-graph and the graph $\Theta(a_1, a_2, a_3)$ is called the generalized $\Theta$-graph.

For two (vertex-disjoint) graphs G and H, the union G ∪ H of G and H is the (disconnected) graph with

$$V(G \cup H) = V(G) \cup V(H) \text{ and } E(G \cup H) = E(G) \cup E(H).$$

The join G + H of two vertex-disjoint graphs G and H has V(G + H) = V(G) ∪ V(H) and $E(G+H) = E(G) \cup E(H) \cup \{ uv : u \in V(G), v \in V(H)\}$.

Therefore, $K(p,q) = O_p + O_q$ for positive integers p and q ,where $O_p$ and $O_q$ are null graph with p and q vertices.

More generally

$$k(P_1, P_2, P_3, \dots, P_t) = \left(\dots + \left(O_{p1} + O_{p2}\right) + \cdots . O_{pn}\right)$$

Also, The Wheel of order n, denoted by $W_n$ is defiend as

$$W_n = C_{n-1} + K_1$$

### ❖ Tree Graph

A tree is a connected graph which has no cycles.

### ❖ Spanning Tree

If $G$ is a connected graph, the spanning tree in $G$ is a subgraph of $G$ which includes every vertex of $G$ and is also a tree.

## 1.6 Connectivity

### ❖ Bridge

A bridge is a single edge whose removal disconnects a graph.

### ❖ Edge Connectivity

The edge-connectivity $\lambda(G)$ of a connected graph $G$ is the smallest number of edges whose removal disconnects $G$. When $\lambda(G) \geq k$, the graph $G$ is said to be $k$-edge-connected.

### ❖ Cut Set

A cut set of a connected graph $G$ is a set S of edges with the following properties

- The removal of all edges in S disconnects $G$.
- The removal of some (but not all) of edges in S does not disconnects $G$.

### ❖ Vertex Connectivity

The connectivity (or vertex connectivity) $K(G)$ of a connected graph $G$ (other than a complete graph) is the minimum number of vertices whose removal disconnects $G$. When $K(G) \geq k$, the graph is said to be $k$-connected (or $k$-vertex connected). When we remove a vertex, we must also remove the edges incident to it.

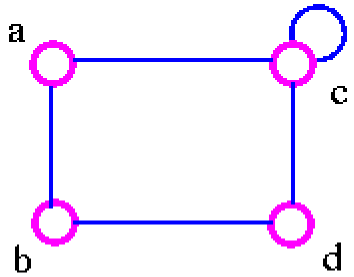


**Figure 1.3**

The above graph G cannot be disconnected by removing a single vertex, but the removal of two non-adjacent vertices (such as b and c) disconnects it. The graph G has connectivity 2.

### ❖ Cut-Vertex

A cut-vertex is a single vertex whose removal disconnects a graph.It is important to note that the above definition breaks down if G is a complete graph, since we cannot then disconnect G by removing vertices. Therefore, we make the following definition.

### ❖ Connectivity of Complete Graph

The connectivity $k(k_n)$ of the complete graph $k_n$ is $n$-1. When $n-1 \geq k$, the graph $k_n$ is said to be k-connected.

### ❖ Vertex-Cut set

A vertex-cut set of a connected graph G is a set S of vertices with the following properties.

a. The removal of all the vertices in S disconnects G.
b. The removal of some (but not all) of vertices in S does not disconnects G.

Consider the following graph

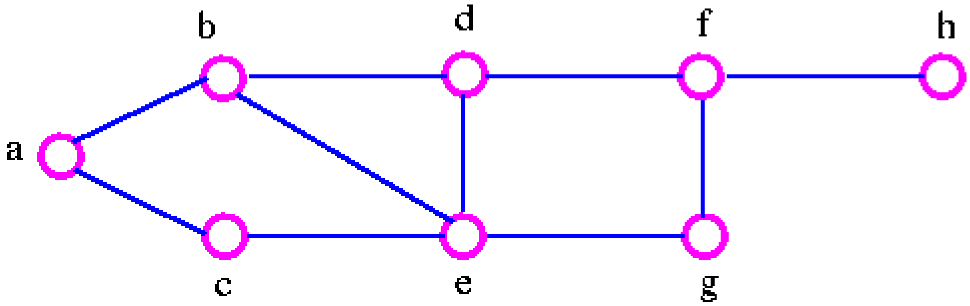
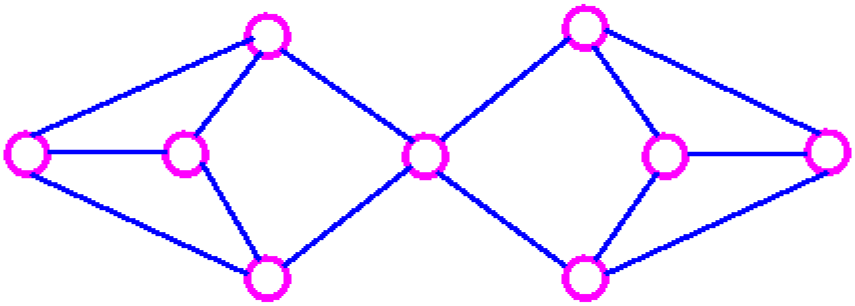


Figure 1.4

We can disconnects the graph by removing the two vertices b and e, but we cannot disconnect it by removing just one of these vertices. The vertex-cutset of G is {b, e}. Note that the connectivity k(G) does not exceed the edge-connectivity λ(G). This inequality holds for all connected graph.

Formally, for any connected graph G we have $K(G) \leq \lambda(G) \leq \delta(G)$ where δ(G) is the smallest vertex-degree in G. But it is certainly possible for both inequality in the above theorem to be strict inequalities (that is, $k(G) < \lambda(G) < \delta(G)$) For example, in the following graph, **Figure 1.5**



$K(G)=1$, $\lambda(G) = 2$, and $\delta(G) = 3$.

## *1.7 Trees*

*An acyclic graph is a graph with no cycles. A tree is a connected acyclic graph. Thus each component of a forest is tree, and any tree is a connected forest.*

**Theorem:** The following are equivalent in a graph G with n vertices.

i. G is a tree.
ii. There is a unique path between every pair of vertices in G.

iii. G is connected, and every edge in G is a bridge.
iv. G is connected, and it has (n - 1) edges.
v. G is acyclic, and it has (n - 1) edges.
vi. G is acyclic, and whenever any two arbitrary nonadjacent vertices in G are joined by and edge, the resulting enlarged graph G' has a unique cycle.
vii. G is connected, and whenever any two arbitrary nonadjacent vertices in G are joined by an edge, the resulting enlarged graph has a unique cycle.

**Theorem** Let T be a graph with n vertices. Then the following statements are equivalent.

a. T is connected and contains no cycles.
b. T is connected and has n-1 edges.
c. T has n-1 edges and contains no cycles.
d. T is connected and each edge is a bridge.
e. Any two vertices of T are connected by exactly one path.
f. T contains no cycles, but the addition of any new edge creates exactly one cycles.

### ❖ Spanning Trees

Let G be a connected graph. A spanning tree in G is a sub graph of G that includes all the vertices of G and is also a tree. The edges of the trees are called branches.

**Theorem** A graph is connected if and only if it has a spanning tree.

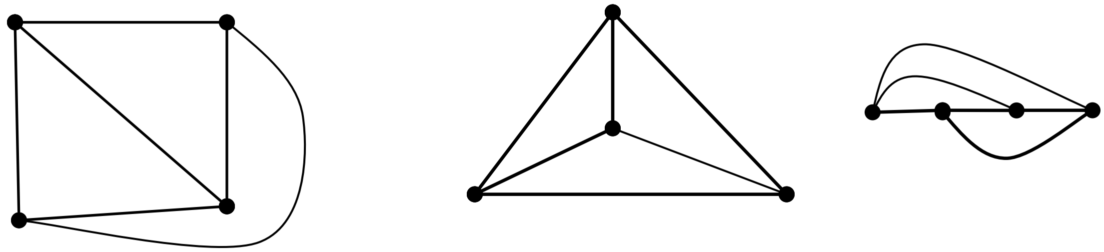
***Proof*** Let G be a connected graph. Delete edges from G that are not bridges until we get a connected sub graph H in which each edge is a bridge. Then H is a spanning tree. On the other hand, if there is a spanning tree in G, there is a path between any pair of vertices in G; thus G is connected.

## 1.8 Graph Planarity

A graph G is planar if it can be drawn in the plane in such a way that no two edges meet each other except at a vertex to which they are incident. Any such drawing is called a plane drawing of G.

For example, the graph $K_4$ is planar, since it can be drawn in the plane without edges crossing.

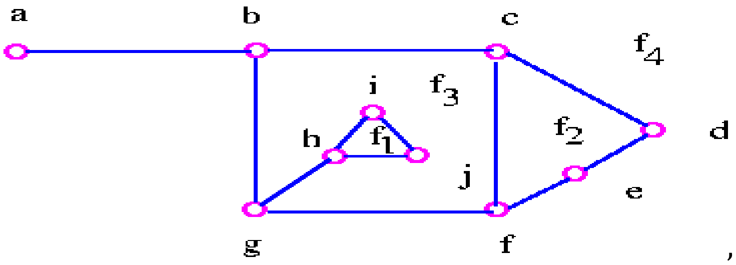
The three plane drawings of $K_4$ are: **Figure 1.6**



On the other hand, the complete bipartite graph $K_{3,3}$ is not planar.

**Euler's Formula**

If G is a planar graph, then any plane drawing of G divides the plane into regions, called faces. One of these faces is unbounded, and is called the infinite face. If *f* is any face, then the degree of *f* (denoted by deg *f*) is the number of edges encountered in a walk around the boundary of the face *f*. If all faces have the same degree (g, say), the G is face-regular of degree g. For example, the following graph G has four faces



, $f_4$ being the infinite face.

**Figure 1.7**

It is easy to see from above graph that deg $f_1$=3, deg $f_2$=4, deg $f_3$=9, deg $f_4$=8.

Note that the sum of all the degrees of the faces is equal to twice the number of edges in the the graph , since each edge either borders two different faces (such as bg, cd, and cf) or occurs twice when walk around a single face (such as ab and gh). The Euler's formula relates the number of vertices, edges and faces of a planar graph. If n, m, and f denote the number of vertices, edges, and faces respectively of a connected planar graph, then we get $n-m+f = 2$.

The Euler formula tells us that all plane drawings of a connected planar graph have the same number of faces namely, $2+m-n$.

**Theorem (Euler's Formula)** *Let G be a connected planar graph, and let* $\mathbf{n}$*,* $\mathbf{m}$ *and* $\mathbf{f}$ *denote, respectively, the numbers of vertices, edges, and faces in a plane drawing of G. Then* $\mathbf{n - m + f = 2}$*.*

***Proof*** We employ mathematical induction on edges, m. The induction is obvious for m=0 since in this case n=1 and f=1. Assume that the result is true for all connected plane graphs with fewer than m edges, where m is greater than or equal to 1, and suppose that G has m edges. If G is a tree, then n=m+1 and f=1 so the desired formula follows. On the other hand, if G is not a tree, let e be a cycle edge of G and consider G-e. The connected plane graph G-e has n vertices, m-1 edges, and f-1 faces so that by the inductive hypothesis,

$$n - (m - 1) + (f - 1) = 2$$

which implies that $$n - m + f = 2.$$

**Theorem** A graph is planar if and only if it does not contain a subdivision of $K_5$ and $K_{3,3}$ as a sub graph[7].

# Chapter-Two

# $\lambda$ – Coloring of Graphs and its enumeration

In this chapter, we introduce the concept of vertex coloring .we define the chromatic polynomial and chromatic number. We also calculate the chromatic polynomial for some special class of graphs as Examples and introduce some of its basic properties. To end this chapter, we will see umbral product of polynomials.

## 2.1: Coloring Graphs and Chromatic number

Graph coloring, or more specifically vertex coloring means the assignment of colors to the vertices of a graph in such a way that no two adjacent vertices share the same color.

While the colors used can be elements of any set, actual colors (such as red, blue, green, and yellow) are often chosen only when a small number of colors are being used; otherwise, positive integers (typically 1, 2 . . . λ for some positive integer λ) are commonly used for the colors. A reason for using positive integers as colors is that we are often interested in the number of colors being used. While all λ colors are typically used in a λ -coloring of a graph, there are occasions when only some of the λ colors are used. Suppose we have a λ -coloring of a graph G, where each color is one of the integers 1, 2. . . λ as mentioned above. If $V_i(1 \leq i \leq \lambda)$ is the set of vertices in G colored i (where one or more of these sets may be empty), then each nonempty set $V_i$ is called a color class and the nonempty elements of $\{V_1, V_2 . . . V_n\}$ produce a partition of V (G). Because no two adjacent vertices of G are assigned the same color by f, each nonempty color class $V_i(1 \leq i \leq \lambda)$ is an independent set of vertices of G.

**Definition 2.1.1:** Let G be a graph and $\lambda \in \mathbb{N}$. A mapping f: $V(G) \longrightarrow \{1,2,3,\ldots, \lambda\}$ is called a λ-coloring of G if $f(u) \neq f(v)$, whenever the vertices u and v are adjacent in G.

Two λ-coloring f and g of G are regarded as distinct if $f(x) \neq g(x)$ for some vertex x in G.

$P(G, \lambda)$ is the number of distinct λ-colorings of G.

P(G,0)=0……By convention

$P(G, \lambda) \geq 1$ if and only if G is λ-colorable

**Definition 2.1.2:** The chromatic number is the minimum λ for which the graph is λ-colorable.

Notation: Chromatic number $= \chi(G)$

i.e $\chi(G) = \min\{\lambda \in \mathbb{N} : P(G, \lambda) \geq 1\}$

There is no general formula for the chromatic number of a graph.

Observation: 1-The chromatic number of Null graph of length n is 1.

2- The chromatic number of path graph of length n is 2.

3- The chromatic number of Cycle graph of length n is $\begin{cases} 2 & if\ n\ is\ even \\ 3 & if\ n\ is\ odd \end{cases}$.

4 - The chromatic number of complete graph length n is n.

5 - The chromatic number of bipartite graph is 2.

6- The chromatic number of k-partite length n is 2

**Example 2.1.3 (Application): Chemical Storage**

A company manufactures n chemicals $C_1, C_2, C_3, \dots, C_n$. Certain pairs of these chemicals are incompatible and would cause explosions if brought into contact with each other. As a precautionary measure, the company wishes to divide its warehouse into compartments, and store incompatible chemicals in different compartments.

What is the least number of compartments into which the warehouse should be partitioned? We obtain a graph G on the vertex set $\{v_1, v_{2,} v_3, \dots, v_n\}$ by joining two vertices $v_i$ and $v_j$ if and only if the chemicals $C_i$ and $C_j$ are incompatible. It is easy to see that the least number of compartments into which the warehouse should be partitioned is equal to the chromatic number of G.

### 2.1.4 The Four Color Problem History

If the countries in a map of South America were to be colored in such a way that every two countries with a common boundary are colored differently, then this map could be colored using only four colors. Is this true of every map?

While it is not difficult to color a map of South America with four colors, it is not possible to color this map with less than four colors. In fact, every two of Brazil, Argentina, Bolivia, and Paraguay are neighboring countries and so four colors are required to color only these four countries.

It is probably clear why we might want two countries colored differently if they have a common boundary – so they can be easily distinguished as different countries in the map. It may not be clear, however, why we would think that four colors would be enough to color the countries of every map. After all, we can probably envision a complicated map having a large number of countries with some countries having several neighboring countries, so constructed that a great many colors might possibly be needed to color the entire map. Here we understand neighboring countries to mean two countries with a boundary line in common, not simply a single point in common. In 1852 Francis Guthrie (1831–1899), a former graduate of University College London, observed that the counties of England could be colored with four colors so that neighboring counties were colored differently. This led him to ask whether the counties of every map (real or imagined) can be colored with four or fewer colors so that every two neighboring counties are colored differently. Francis mentioned this problem to his younger brother Frederick, who at the time was taking a class from the well-known mathematician Augustus De Morgan. With the approval of his brother, Frederick mentioned this problem to De Morgan, who considered the problem to be new but was unable to solve it. Despite De Morgan's great interest in the problem, few other mathematicians who were aware of the problem seemed to share this interest. A quarter century passed with little activity on the problem. At a meeting of the London Mathematical Society in 1878, the great mathematician Arthur Cayley inquired about the status of this Four Color Problem. This revived interest in the problem and would lead to an 1879 article written by the

British lawyer Alfred Bray Kempe containing a proposed proof that every map can be colored with four or fewer colors so that neighboring counties are colored differently. For the next ten years, the Four Color Problem was considered to be solved. However, an 1890 article by the British mathematician Percy John Heawood presented a map and a partial coloring of the counties of the map, which Heawood showed was a ounterexample to the technique used by Kempe. Although this counterexample did not imply that there were maps requiring five or more colors, it did show that Kempe's method was unsuccessful. Nevertheless, Heawood was able to use Kempes technique to prove that every map could be colored with five or fewer colors.

The Four Color Problem can be stated strictly in terms of plane (or planar) graphs, rather than in terms of maps. Let G be a plane graph. Then G is k-region colorable if each region of G can be assigned one of k given colors so that neighboring (adjacent) regions are colored differently. Since it was believed by many that the question posed in the Four Color Problem had an affirmative answer, this led to the following.

### 2.1.5. The Four Color Theorem:

Every plane graph is 4-region colorable.

Note: The chromatic number of complete graph of length n is n.

It is not contradictory with the four color theorem because $K_n, for\ n > 5$ is not planar graph.

## 2.2. Chromatic polynomial for some graph G

In this subsection, we shall enumerate P(G, λ) for some special graph G ,For illustration.

**Theorem 2.2.1: Null graph ($N_n$)** The Chromatic polynomial of the empty graph on n vertices is P($N_n$, λ)= $\lambda^n$

**Proof:** let λ ϵ N and $N_n$ be empty graph.

Each of the n vertices can be independently colored using any of the λ colors, which gives a total of $\lambda^n$ possibilities.

P(G, λ)= $\lambda^n$ is called Power form of P(G, λ).

More generally, if G = $\cup_{i=1}^{k} G_i$, then

$$P(G,\lambda) = \prod_{i}^{k} P(G_i,\lambda)$$

**Theorem 2.2.2: Complete graph ( $K_n$)**

The chromatic polynomial of graph **G** is $\lambda(\lambda-1)(\lambda-2)(\lambda-3)\ldots(\lambda-n+1)$ if and only if **G** is a complete graph on n vertices .

**Proof:** With **λ** colors, the first vertex **G** can be colored in λ ways, a second vertex of **G** can be colored in λ-1 ways if and only if the first vertex is adjacent to the second vertex, the third vertex of **G** can be colored in λ-2 ways if and only if the third vertex is adjacent to first two vertices…, and the n<sup>th</sup> vertex can be colored in λ-n+1 ways, if and only if every vertex is adjacent to every other vertex, i.e. this happen if and only if **G** is Complete

$$\therefore \; P(K_n,\lambda)= \lambda(\lambda-1)(\lambda-2)(\lambda-3)\ldots(\lambda-n+1)$$

$P(K_n,\lambda)= \lambda(\lambda-1)(\lambda-2)(\lambda-3)\ldots(\lambda-n+1)$…is called factorial form denoted by

$P(K_n,\lambda) =(\lambda)_n$

Ingeneral , when $P(K_n,\lambda)$ is expressed in terms of the power of λ, we have

$$P(K_n,\lambda)= \lambda(\lambda-1)(\lambda-2)(\lambda-3)\ldots(\lambda-n+1)=\sum_{k=1}^{n} S(n,k)\lambda^{n},$$

where S(n, k) are the (signed) stirling numbers of the first kind ,which are defined recursively by

$$S(n,k) = S(n-1,k-1) - (n-1)\, S(n-1,k)$$

With the boundary conditions that

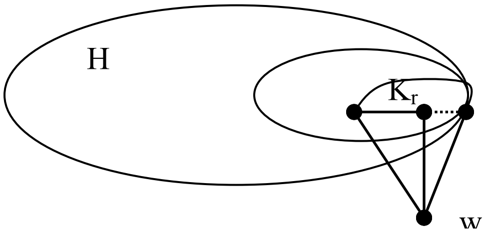
$S(r, 0) =0$ for all $r \in N$

$S(r, r)=1$ for all $r \in N_0$

**Example 2.2.3** Let H be a graph containing a $K_r$ as a sub graph, and let G be the graph obtained from H by adding a new vertex w which is linked With each vertex in $K_r$ (and no others) as shown in Figure 2.1. Then we have

$$P(G,\lambda) = (\lambda - r)\, P(H,\lambda)$$

*Figure 2.1*



***Theorem 2.2.4**: The chromatic polynomial of* $K_{2,n}$ *is*

$$(\lambda)(\lambda-1)((\lambda-1)^{n-1}+(\lambda-1)^{n})$$

***Proof:*** Let $v$ and $w$ be the vertices in the bipartite set of size 2.

We consider two cases.

**Case -1:** If $v$ and $w$ are colored the same, then we choose a single color of the available λ colors. As both $v$ and $w$ are adjacent to all of the remaining n vertices, we color using the remaining (λ -1)colors. As these n vertices are independent, we can color freely, and so arrive at $\lambda(\lambda-1)^n$ colorings for this case.

**Case -2:** If $v$ and $w$ are colored differently, then there are $\lambda(\lambda-1)$ ways to color them. As before, we use the remaining $\lambda-2$ colors to color the remaining n vertices freely, giving $\lambda(\lambda-1)(\lambda-1)^n$ colorings for this case.

Using the addition principle for combining mutually exclusive cases, we get

$$\begin{aligned} P(K_{2,n},\lambda) &= \lambda(\lambda-1)^{n}+\lambda(\lambda-1)(\lambda-2)^{n} \\ &= \lambda(\lambda-1)((\lambda-1)^{n-1}+(\lambda-2)^{n}) \end{aligned}$$ □

***Theorem 2.2.5**: The chromatic polynomial of* $K_{3,n}$ *is*

$$(\lambda)(\lambda-1)(\lambda-2)(\lambda-3)^{n}+\ 3(\lambda)(\lambda-2)^{n}+\lambda(\lambda-1)^{n}$$

***Proof:*** Let $v$, $u$ and $w$ be the vertices in the bipartite set of size 3.if we color v, u and w first ,then there are three cases to consider :

**Case -1:** All of $v$,$u$ and w have been given different colors, λ choice for v, $\lambda - 1$ for u and $\lambda - 2$ for w. In this case there are $\lambda - 3$ choices for each of $y_1, y_2, y_3, \dots, y_n$. There are $\lambda(\lambda-1)(\lambda-2)(\lambda-3)^n$ ways all together.

**Case -2:** Two of v ,u and w have been given the same colors and the other different color. Note that there are $\binom{3}{2} = 3$ choices for which two have the same color, and $\lambda - 1$ for different colors. in this case, there are $\lambda - 2$ choices for $y_1, y_2, y_3, \dots, y_n$

There are $\lambda(\lambda-1)(\lambda-2)^n$ ways all together.

**Case -3:** All of $v$, $u$ and w have been given the same color: λ choice for $v$, $u$ and w.

In this case there are $\lambda - 1$ choices for $y_1, y_2, y_3, \dots, y_n$ .

There are $\lambda(\lambda-2)^n$ ways all together.

Combining these by the addition principle

$$P(K_{3,n}, \lambda) = (\lambda)(\lambda-1)(\lambda-2)(\lambda-3)^n + 3(\lambda)(\lambda-1)(\lambda-2)^n + \lambda(\lambda-1)^n \qquad \square$$

**Question 1:** Explain why the chromatic polynomial $P(G, \lambda)$ of a planar graph G cannot contain a term $(\lambda - k)$ , for any $k \geq 4$

**Solution:** if $P(G, \lambda)$ contains the $(\lambda - k)$ , then $P(G, k) = 0$

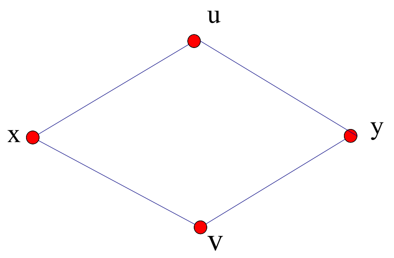
$\Longrightarrow$ The graph G is not 4- colorable

Since any planar graph is 4-colorable, and therefore k-colorable for $k \geq 4$

Its chromatic polynomial cannot contain a term $(\lambda - k)$ , for any $k \geq 4$

**Question 2:** Find the chromatic polynomial of $C_4$ ?

**Figure 2.2** $C_4$



Let f be a λ -coloring of C4.

**Case** 1: f (x) = f (y).

There are λ - 1 ways to color the vertices $u$ and $v$ independently, and thus the number of such λ -colorings f is $\lambda(\lambda - 1)^2$

**Case 2:** f (x) $\neq$ f (y). There are λ - 2 ways to color the vertices u and *v* independently, and thus the number of such λ -colorings f is $\lambda(\lambda-1)(\lambda-2)^2$.

We thus conclude that

$$\begin{aligned} P(C_4, \lambda) &= \lambda(\lambda-1) + \lambda(\lambda-1)(\lambda-2)^2 \\ &= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda \\ &= (\lambda-1)^4 + (\lambda-1) \end{aligned}$$

### 2.3 Basic results on enumeration of *P (G, λ )*

It is known that there is no fixed formula to calculate $\chi(G)$ .The problem of evaluating P(G, **λ** ) is at least as hard as that of determining $\chi(G)$. In spite of this, there are results which are useful for evaluating P(G, **λ** ) more efficiently for some classes of graphs.

Two of them will be introduced in this section. The first result that we shall introduce provides us with a recursive way to compute P(G ,**λ** ), and its proof is just an extension of the idea used in Counting P(C4, **λ**) as shown in Question 2

**Theorem 2.3.1(Fundamental addition deletion Theorem)**

Let **x** and **y** be two non-adjacent vertices in a graph G.

Then

$$P(G,\lambda) = P(G+xy,\lambda) + P(G\,.\,xy,\lambda) \dots\dots\dots\dots\dots\dots\dots (2.1)$$

***Proof:*** Let f be a **λ** -coloring of G.

The λ coloring G has two cases:

**Case -1:** λ coloring of G assigning different colors to the vertices x and y i.e. f (x) $\neq$ f (y) $\Longrightarrow$ A proper λ coloring of (case 1) is equivalent to a proper **λ** –coloring of P(G . xy, **λ**)

**Case -1:** λ coloring of G assigning same colors to the vertices x and y i.e. f (x) = f (y) $\Longrightarrow$ A proper λ coloring of (case 2) is equivalent to a proper **λ** –coloring of P(G + xy, **λ**)

Hence,

$$P(G,\lambda) = P(G+xy,\lambda) + P(G\,.\,xy,\lambda)$$

**Theorem 2.3.1.2 Fundamental Reduction theorem (FRT)**

Let **x** and **y** be two adjacent vertices in a graph H

Then

$$P(H,\lambda) = P(H-xy,\lambda) - P(H.xy,\lambda) \text{ .................... } (2.2)$$

***Proof :*** If we treat the graph G+xy in theorem 2.3.1 as a given graph H , then Theorem 2.3.2 can be restated .

**Theorem 2.3.1.3:** For every graph G of order $n$,

G is a tree if and only if $P(G,\lambda) = \lambda(\lambda-1)^{n-1}$ ........................(2.3)

*(⟹) **Proof:** Suppose G is a tree*

***WTS:*** $P(T_n,\lambda) = \lambda(\lambda-1)^{n-1}$

(i) Proof by induction on $n$:

When $n = 2$,

it is clear that $P(T_2\,,\lambda) = P(K_2\,,\lambda) = \lambda(\lambda-1)^1 = \lambda(\lambda-1)^{2-1}$.

(ii)Now, assume for every tree of order $n$-1 ,it is true, and let $T$ be a tree of order $n$. Let $v$ be a leaf in $T$ and $e$ the edge incident to $v$.

We observe that that $(T-e)$ has two components: the isolated vertex $v$ and a tree of order n-*1*.By the induction hypothesis and Theorem 2.3.1

$P(T_{n-1},\lambda) = \lambda^2(\lambda-1)^{n-2}$ and

On the other hand, $(T.e)$ is also a tree of order $(n-1)$ and

$$P(T.e,\lambda) = \lambda(\lambda-1)^{n-2}$$

Now, by using the Deletion-contraction Theorem,

We have,

$$\begin{aligned} P(T_n,\lambda) &= P(T_{n-1},\lambda) - P(T.e,\lambda) \\ &= \lambda^2(\lambda-1)^{n-2} - \lambda(\lambda-1)^{n-2} \\ &= \lambda(\lambda-1)^{n-2}(\lambda-1) \\ &= \lambda(\lambda-1)^{n-1} \end{aligned}$$

(⟸) **WTS: G is a tree**

Suppose $P(G,\lambda) = \lambda(\lambda-1)^{n-1}$ be the chromatic polynomial of G

⟹ The order of G is n.

$\Rightarrow$ The coefficient of $\lambda^{n-1}$ is –(n-1). So the size of G is n-1.

If G is not the connected graph, then its chromatic polynomial is product of the chromatic polynomials of its components. Therefore the coefficients of λ in it has to be zero. But the coefficients of λ in $\lambda(\lambda-1)^{n-1}$ is not zero

so G is connected graph with n vertices and $(\lambda-1)$ edges.

G is a Tree. ☐

**Corollary 3.2.1.4:** The chromatic polynomials of $K_{1,n}$ is $\lambda(\lambda-1)^n$

**Proof:**

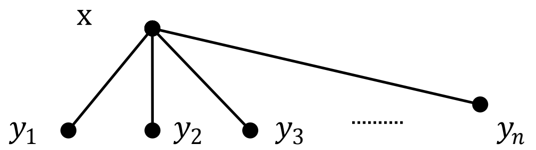


**Figure 2.3**

The graph is a tree on (n+1) vertices so

$$P(K_{1,n},\lambda) = \lambda(\lambda-1)^n \qquad \square$$

***Theorem 3.2.5:*** For any cycle graph length n,

$$P(C_n,\lambda) = (\lambda-1)^n + (-1)^n(\lambda-1) \qquad \text{.....................} \qquad (2.4)$$

***Proof:*** Proof by induction on n:

(i) For n = 3, we observe that $C_3$ is the same as $K_3$, so we have

$$P(C_3,\lambda) = P(K_3,\lambda) = (\lambda)(\lambda-1)(\lambda-2) = \lambda^3 - \lambda^2 + 2\lambda$$

Comparing this with theorem’s formula

$$\begin{aligned} P(C_3,\lambda) = P(K_3,\lambda) &= (\lambda-1)^3 + (-1)^3(\lambda-1) \\ &=(\lambda^3 - 3\lambda^2 + 3\lambda - 1) - (\lambda-1) \\ &= \lambda^3 - \lambda^2 + 2\lambda \end{aligned}$$

$\Rightarrow$ It is true for n=3

(ii)Assume it is true for $C_{n-1}$,

We want to show it is also true for $C_n$.

Let e be an edge in $C_n$. By deleting and contracting e, we will have $P_{n-1}$ path of length $(n-1)$, and $C_{n-1}$ respectively. By the induction hypothesis, the

Deletion-contraction Theorem, and the fact that every path is a tree, we have

$$\begin{aligned} P(C_n, \lambda) &= P(P_n, \lambda) + \mathrm{P}(C_{n-1}, \lambda) \\ &= \lambda(\lambda - 1)^{n-1} - ((\lambda - 1)^{n-1} + (-1)^{n-1}(\lambda - 1)\,) \\ &= (\lambda - 1)^{n} + (-1)^{n}(\lambda - 1) \end{aligned} \qquad \square$$

**Theorem 3.2.1.6:** The chromatic polynomial of the wheel graph *is*

$$P(W_n, \lambda) = \lambda((\lambda - 2)^{n-1} + (-1)^{n-1}(\lambda - 2)) \ldots\ldots\ldots\ldots\ldots\ldots \qquad (2.5)$$

***Proof:*** Let $v$ be the central vertex of the wheel graph $W_n$ . First, choose any of the available λ colors to color $v$. Since $v$ is adjacent to all other vertices, the chosen color cannot be used again. Hence, it remains to color $W_n$-v (i.e. $C_n$) using $\lambda - 1$ colors. Making use of the above theorem, we have

$$\begin{aligned} P(W_n, \lambda) &= \lambda P(C_n, \lambda - 1) \\ &= \lambda((\lambda - 2)^{n-1} + (-1)^{n-1}(\lambda - 2) \end{aligned} \qquad \square$$

*Let $G_1$ and $G_2$ be two graphs, and $r \in \mathbb{N}_0$ with $r \leq$ min {ω ($G_1$), ω ($G_1$)), where ω (H) is the clique number of a graph H. Choose a $K_r$, from each $G_i$, i = 1,2, and form a new graph G from the union of $G_1$ and $G_2$ by identifying the two chosen $K_r$ 's in an arbitrary manner as shown in Figure 2.4.*

*We call G a $K_r$ -gluing of $G_1$ and $G_2$ , and denote $\mathcal{G}$ [$G_1 U_r G_2$ ] the family of all $K_r$ -gluing of $G_1$ and $G_2$. When r = 0, G is just the disjoint union of of $G_1$ and $G_2$; when r = 1, G is also called a vertex-gluing of $G_1$ and $G_2$ , and when r = 2, G is also called an edge-gluing of $G_1$ and $G_2$.*

*The following result, due to Zykov (1949), provides a shortcut for evaluating P(G, λ) if G is a $K_r$ -gluing of some graphs.*

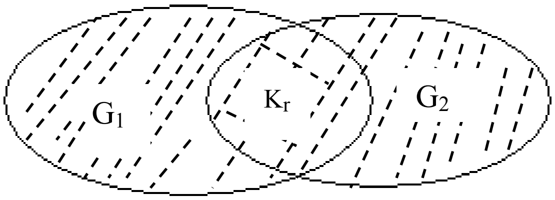


*Figure 2.4*

### 2.3.2: Zukav's Theorem

Let G be undirected graph with sub graphs $G_1, G_2$.

If $G \in \mathcal{G}\,[G_1 \cup G_2]$ and $G_1 \cap G_2 = K_r$ , then

$$P(G, \lambda) = [P(G_1, \lambda) * P(G_2, \lambda)] / P(K_r, \lambda)$$

**Proof:** Since $G_1 \cap G_2 = K_n$, it follows that $K_n$ sub graphs $G_1$ and $G_2$, and $\chi(G_1), \chi(G_2) \geq n$ . Given λ colors, there are $P(K_n, \lambda) = (\lambda)_n$ ways to color $K_n$ there are $P(G_1, \lambda)/(\lambda)_n$ ways to color $G_1 - Kn$, and there are $P(G_2, \lambda)/(\lambda)_n$ to color $G_2 - K_n$ .

By the product rule, we have

$$\begin{aligned} P(G, \lambda) &= P(G_1 - K_n, \lambda) * P(G_2 - K_n, \lambda) * P(K_n, \lambda) \\ &= [P(G_1, \lambda)/(\lambda)_n] * [P(G_2, \lambda)/(\lambda)_n] * (\lambda)_n \\ &= [P(G_1, \lambda) * P(G_2, \lambda)] / (\lambda)_n \qquad \square \end{aligned}$$

***Example 2.3.4*** *The graph G is an edge-gluing of $C_3$ and $C_4$.,*

**G:**

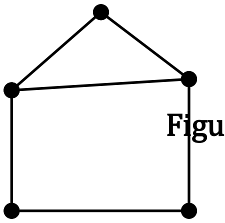


**Figure 2.5**

*Thus, by Theorem 2.3.2, we*

$$P(G,\lambda) = \frac{P(C_3,\lambda)\,P(C_4,\lambda)}{P(K_2,\lambda)} = \frac{(\lambda)(\lambda-1)(\lambda-2)\left((\lambda-1)^4+(-1)^4(\lambda-1)\right)}{\lambda(\lambda-1)}$$

$$=\lambda^5 - 6\lambda^4 + 14\lambda^3 - \lambda^2 + 6\lambda$$

**Example 2.3.5** *The graph G of Figure 2.5* $K_r$ *is a -gluing of H and* $K_{r+1}$*. Thus, by Theorem 2.3.2,*

$$P(G,\lambda) = \frac{P(H,\lambda)P(K_{r+1},\lambda)}{P(K_r,\lambda)} = P(H,\lambda)(\lambda - r)$$

**Theorem 2.3.2.1** *Let H be a connected graph, and u and v two nonadjacent distinct vertices of H. Let G be the graph obtained from H by adding a u - v path P of length r as shown in Figure 1.4. Then*

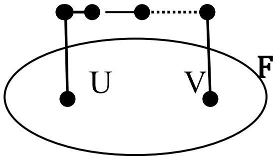


**Figure 2.6**

$$P(G,\lambda) = \frac{1}{(\lambda)(\lambda-1)} P(H,\lambda)P(C_{r+1},\lambda) + (-1)^r\, P(H.uv,\lambda)$$

**Proof:** Using Fundamental addition deletion theorem, we have

$$P(G,\lambda) = P(G+uv,\lambda) + P(G.uv,\lambda)$$

$$\Rightarrow \quad = \frac{1}{(\lambda)(\lambda-1)} P(C_{r+1},\lambda)P(H+uv,\lambda) + \frac{1}{\lambda} P(H.uv,\lambda)P(C_r,\lambda)$$

$$\Rightarrow \quad = \frac{1}{(\lambda)(\lambda-1)} P(C_{r+1},\lambda)[P(H-uv,\lambda) - P(H.uv,\lambda)] + \frac{1}{\lambda} P(H.uv,\lambda)P(C_r,\lambda)$$

Where

$$P(H+uv,\lambda) = P(H,\lambda) - P(H.uv,\lambda)$$

$$\Rightarrow \; = \frac{1}{(\lambda)(\lambda-1)} P(C_{r+1},\lambda)P(H,\lambda) - \frac{1}{(\lambda)(\lambda-1)} P(C_{r+1},\lambda)P(H.uv,\lambda) + \frac{1}{\lambda}P(H.uv,\lambda)P(C_r,\lambda)$$

$$\Rightarrow \; = \frac{1}{(\lambda)(\lambda-1)} P(C_{r+1},\lambda)P(H,\lambda) + P(H.uv,\lambda)\left[\frac{1}{\lambda}P(C_r,\lambda) - \frac{1}{(\lambda)(\lambda-1)} P(C_{r+1},\lambda)\right]$$

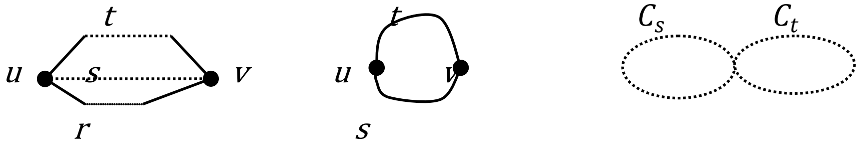
$$\Rightarrow \; = \frac{1}{(\lambda)(\lambda-1)} P(C_{r+1},\lambda)P(H,\lambda) + P(H.uv,\lambda)\left[\frac{1}{\lambda}((\lambda-1)^{r} + (-1)^{r}(\lambda-1) - \frac{1}{(\lambda)(\lambda-1)}((\lambda-1)^{r+1} + (-1)^{r+1}(\lambda-1))\right]$$

Where $C_r = (\lambda-1)^{r} + (-1)^{r}(\lambda-1)$

$$\Rightarrow \; = \frac{1}{(\lambda)(\lambda-1)} P(C_{r+1},\lambda)P(H,\lambda) + (-1)^{r} P(H.uv,\lambda)$$ □

**Example 2.3.7** *By applying Theorem 2.3.2, we can express P(G, λ), where* $G = \theta(r, s, t)$ *is the generalized* $\theta$ *-graph and* $r, s, t \geq 2$*, in terms of* $P(C_i, \lambda)$ *'s as shown below.*

*Figure 2.7*



$$P(G,\lambda) = \frac{1}{(\lambda)(\lambda-1)} P(C_{r+1},\lambda)P(C_{s+t},\lambda) + (-1)^{r} P(C_s,\lambda)P(C_t,\lambda)$$

## 2.4: Properties of Chromatic polynomial

**Theorem:** The necessary condition to be satisfied by the chromatic polynomial of a simple graph of order n and size m

Let G be the graphs with n vertices and m edge and

$P(G,\lambda) = a_n\lambda^{n} + a_{n-1}(\lambda)^{n-1} + a_{n-2}(\lambda)^{n-2} + \cdots + (-1)^{n-1}a_1 + a_0$

where $a_i \in \mathbb{Z}, for\ i = 0,1,2, \dots, n$

**a)** Degree of P(G, λ ) = n = |V| (the number of vertices of G) and Coefficient of $\lambda^n$ is 1. (a monic polynomial) .

**b**) The constant term of P(G, λ) is 0.

**c**) The sum of the coefficients is zero if G is connected.

**d**) The coefficients of P(G, λ ) alternate in sign

**e)** $|a_{n-1}|$ = |E| (the number of edges of G).

**Proof:** Suppose that G has n vertices.

We will proceed by induction on the number of edges.

(**i**)First, consider G has only one edge. Then the n-2 isolated vertices can be colored in any way, and the two vertices on the edge can be colored in λ(λ − 1)ways, so,

$$P(G, \lambda) = \lambda (\lambda - 1)\lambda^{n-2} = \lambda^{??} - \lambda^{??-2}$$

So, the degree of P (G, λ) in this case is n and the polynomial is monic.

**(ii)** Assume that the statement is true for all graphs with fewer than k edges. Assume that G has k edges and let ß be one of them. By the fundamental reduction theorem,

$$P(G, \lambda) = P(G - \beta, \lambda) + P(G.\beta, \lambda)$$

Now, since $G - \beta$ has fewer than k edges, by the induction hypothesis, $P(G - \beta, \lambda)$ is monic of degree n. Since $P(G.xy, \lambda)$ has fewer than k edges and only n-1 vertices, P($G''_ß$, λ ) is monic of degree n-1. Upon subtraction of these polynomials, we see that there is no term in P($G''_ß$, λ) that can remove the $x^n$ term in P($G'_ß$, x), so P(G, λ) will be monic of degree n. By induction, the statement is true for all graphs with n vertices and any number of edges.

**Proof (b):** We cannot color the graph with 0 color .i.e $P(G, 0) = 0\ and$

$$P(G, 0) = a_0$$

∴ The constant term is Zero. □

**Proof(C):** Suppose G is connected and , has order greater than one.

Since G is connected we can not color graph G with 1 and we have

$P(G ,\lambda) = \mathrm{a_n}\lambda^n + (\lambda)^{\mathrm{n-1}} + \mathrm{a_{n-2}}(\lambda)^{\mathrm{n-2}} + \cdots + (-1)^{\mathrm{n-1}}\mathrm{a_1} + a_0$

$\Rightarrow \quad P(G ,1) = \mathrm{a_n}1^n + (1)^{\mathrm{n-1}} + \mathrm{a_{n-2}}(1)^{\mathrm{n-2}} + \cdots + (1)^{\mathrm{n-1}}\mathrm{a_1} + a_0$

$\Rightarrow \sum_{i=0}^{n} a_i = 0 , where\ a_i \in \mathbb{Z}$

**Proof(d):** We will proceed by induction on the number of edges of G.

**(i)** For m=1

$$P(\mathrm{G},\lambda ) = \lambda\,(\lambda - 1)\lambda^{\mathrm{n-2}} = \lambda^n - \lambda^{n-2}$$

**(ii)** Now, assume true for all graphs on n vertices with fewer than k edges. Let G be a graph with k edges and consider the fundamental reduction theorem. Since G'ß and G"ß both have fewer than k edges, by the induction hypothesis we may assume that their chromatic polynomials have the form:

$P(G^{'} ,\lambda) = \lambda^n - \mathrm{a_{n-1}}(\lambda)^{\mathrm{n-1}} + \mathrm{a_{n-2}}(\lambda)^{\mathrm{n-2}} + \cdots + (-1)^{\mathrm{n-1}}\mathrm{a_1} \ , where\ a_i \geq 0$

$P(G^{''} ,\lambda) = \lambda^{n-1} - \mathrm{b_{n-2}}(\lambda)^{\mathrm{n-2}} + \mathrm{b_{n-3}}(\lambda)^{\mathrm{n-3}} + \cdots + (-1)^{\mathrm{n-1}}\mathrm{b_1} , where\ b_i \geq 0$

By the fundamental reduction theorem, we have

$$P(G,\lambda) \ = \ P(G - \beta,\lambda) + P(G.\beta,\lambda)$$

$$= \lambda^n - \mathrm{a_{n-1}}(\lambda)^{\mathrm{n-1}} + \mathrm{a_{n-2}}(\lambda)^{\mathrm{n-2}} + \cdots + (-1)^{\mathrm{n-1}}\mathrm{a_1} - (\lambda^n - \mathrm{b_{n-1}}(\lambda)^{\mathrm{n-1}} + \mathrm{b_{n-2}}(\lambda)^{\mathrm{n-2}} + \cdots + (-1)^{\mathrm{n-1}}\mathrm{b_1}) \ ,$$

$$= \lambda^n - (\mathrm{a_{n-1}} + 1)(\lambda)^{\mathrm{n-1}} + (\mathrm{b_{n-3}} + \mathrm{a_{n-2}})(\lambda)^{\mathrm{n-2}} + \cdots$$

By induction, the statement is true for all graphs with n vertices and any number of edges.

∴ The coefficients of $P(G, \lambda)$ alternate in sign.

**Proof (e)**: We will again proceed by induction on the number of edges of G.

**(i)**As in the proofs of the above theorem(d), the chromatic polynomial of a graph with n vertices and one edge is $\lambda^{n} - \lambda^{n-1}$, so our statement is true for such a graph, |-1| = 1.

**(ii)**Now, assume true for all graphs on n vertices with fewer than k edges. Let G be a graph with k edges and consider the fundamental reduction theorem. Since $G'_{ß}$ and $G''_{ß}$ both have fewer than k edges, by the induction hypothesis we may assume that their chromatic polynomials have the form:

$$P(G^{'}, \lambda) = \lambda^{n} - a_{n-1}(\lambda)^{n-1} + a_{n-2}(\lambda)^{n-2} + \cdots + (-1)^{n-1}a_1 \ , where\ a_i \geq 0$$

$$P(G^{''}, \lambda) = \lambda^{n-1} - b_{n-2}(\lambda)^{n-2} + b_{n-3}(\lambda)^{n-3} + \cdots + (-1)^{n-1}b_1 \ , where\ b_i \geq 0$$

where $a_{n-1}$ is the number of edges in $G'_{ß}$ and $b_{n-2}$ is the number of edges in $G''_{ß}$.
Now, by the fundamental reduction theorem,
By the fundamental reduction theorem, we have

$$P(G, \lambda) = P(G - \beta, \lambda) + P(G.\beta, \lambda)$$

$$= \lambda^{n} - a_{n-1}(\lambda)^{n-1} + a_{n-2}(\lambda)^{n-2} + \cdots + (-1)^{n-1}a_1 - (\lambda^{n} - b_{n-1}(\lambda)^{n-1} + b_{n-2}(\lambda)^{n-2} + \cdots + (-1)^{n-1}b_1) \ ,$$

$$= \lambda^{n} - (a_{n-1} + 1)(\lambda)^{n-1} + (b_{n-2} + a_{n-2})(\lambda)^{n-2} + \cdots$$

since $G'_{ß}$ is obtained from G by removal of just one edge, $a_{n-1} = k - 1$, so $a_{n-1} + 1 = k$, and the statement is true.
By induction, the statement is true for all graphs with n vertices and any number of edges.

□

These properties do *not* characterize the chromatic polynomials amongst all the polynomials with integer coefficients. For example,

$P(x) = x^4 - 4x^3 + 3x^2$ is not the chromatic polynomial of any graph.

### 2.5. Representation of a Chromatic Polynomial in the Null Graph, Tree, and Complete Graph Bases

The set of polynomial functions over the field of rational numbers is a vector space, with the usual operations of polynomial addition and multiplication. Chromatic polynomials are contained in the subspace whose vectors are of finite length, and where the components of the vector are integers. Chromatic polynomials are not a subspace by themselves because scalar multiplication of a chromatic polynomial by a number other than 1 produces a polynomial which is not a chromatic polynomial.

A set of polynomials ***S*** spans the set of chromatic polynomials if any chromatic polynomial can be written as a linear combination of elements of ***S.*** This set ***S*** is **linearly independent** if no element in **S** can be written as a linear combination of other elements in S.

A basis for the set of chromatic polynomials is a linearly independent set of polynomials spanning the set of chromatic polynomials. When we write a chromatic polynomial, we are actually listing the coefficients of this polynomial relative to a given basis.

The <u>**thre**</u>e most commonly used bases are given below:

**I.** A chromatic polynomial for graph G expressed in the null graph basis (also called the standard basis) is written as a unique, finite, linear combination of $\{1, \lambda, \lambda^2, \lambda^3, \lambda^4, \ldots, \lambda^n, \ldots\}$ i.e., for graph G with n vertices,

$$P(G\,,\lambda) = \lambda^n - a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + (-1)^{n-1}a_1$$

$$\text{Where } a_i \geq 0, i = 1, 2, \ldots. n-1$$

As explained in theorem 2.4.1 the coefficient of $\lambda^n$ is always 1, and $P(G\,,\lambda)$ has no constant term. By applying repeatedly FRT : the chromatic polynomial every graph can be written linear combination of the empty graph.

This basis is called the **null graph basis.**

**Example 2.5.1** Find the chromatic polynomial of $C_4$?

$C_4$ :

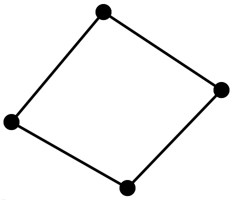
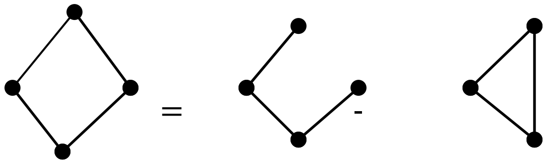
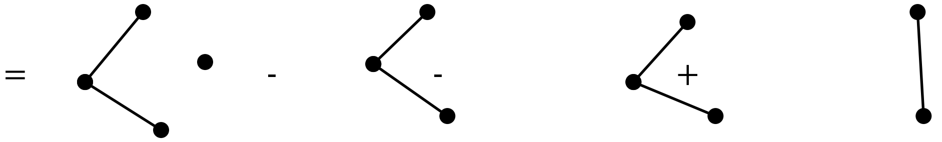
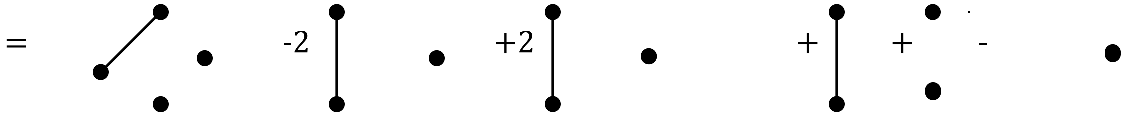


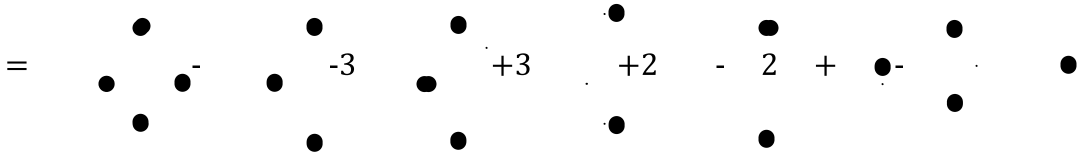
**Figure 2.8**

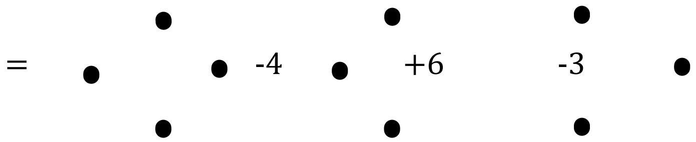
**Solution:**











$$P(C_4,\lambda) = P(N_4,\lambda) - 4P(N_3,\lambda) + 6P(N_2,\lambda) - 3P(N_1,\lambda)$$

$$= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda = (\lambda - 1)^4 + (-1)^4(\lambda - 1)$$

**Theorem 2.5.2.** For any graph G, the coefficients of its chromatic polynomial relative to the null graph basis alternate in sign.

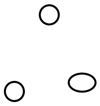
**Proof:** The proof proceeds by 2-way induction.

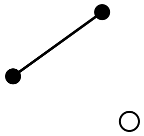
For n = 1, 2, and 3, the following graphs verify the theorem:

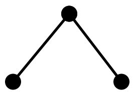
For n=1  …. $\lambda$

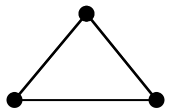
For n=2  …$\lambda^2$  ….$\lambda^2 - \lambda$

For n=3

 …… $\lambda^3$

 ……….. $\lambda^3 - \lambda^2$

 …… $\lambda^3 - 2\lambda^2 + \lambda$

 $\lambda^3 - 3\lambda^2 + 2\lambda$

Let the theorem be true for all graphs with n vertices or less. Consider the graphs with n + 1 vertices .For the null graph $N_{n+1}$ , $P(N_{n+1}, \lambda) = \lambda^{n+1}$ „ and the theorem is true. Now let the theorem be true for all graphs with n vertex and k edges or less. For any graph G' with n + 1 vertices and k + 1 edges. By fundamental reduction theorem, we have

$$P(G',\lambda) = P(G,\lambda) - P(G'',\lambda)$$

, where G has n+1 vertices and k edges, and G" has n vertices and $\leq$ k edges. Since the theorem is true for G and G",

Thus $P(G,\lambda) = \lambda^{n+1} - \mathrm{a}_{1,n}\lambda^{n} + \mathrm{a}_{2,n-1}\lambda^{n-1} + \cdots + (-1)^{n}\mathrm{a}_{1,1}\lambda$

Where $a_{1,i} \geq 0, i = 1, 2, \ldots . n$

$P(G'',\lambda) = \lambda^{n} - \mathrm{a}_{2,n-1}\lambda^{n-1} + \mathrm{a}_{2,n-2}\lambda^{n-2} + \cdots + (-1)^{n-1}\mathrm{a}_{1}\lambda$

Where $a_{2,i} \geq 0, i = 1, 2, \ldots . n-1$

$P(G',\lambda) = P(G,\lambda) - P(G'',\lambda)$

$$=\lambda^{n+1} - (\mathrm{a}_{1,n} + 1)\lambda^{n} + (\mathrm{a}_{1,n-1} + \mathrm{a}_{2,n-1})\lambda^{n-1} + \cdots + (\mathrm{a}_{1,1} + \mathrm{a}_{2,1})(-1)^{n}\lambda$$

$\Longrightarrow$ Alternate in sign.

**II.** A chromatic polynomial for graph G expressed in the tree basis is written as a unique, finite, linear combination of $\{1, \lambda, \lambda(\lambda-1)^{1}, \lambda(\lambda-1)^{2}, \ldots, \lambda(\lambda-)^{n} \ldots .\}$

i.e

$$P(G',\lambda) = \lambda(\lambda-)^{n-1} + C_{n-1}\lambda(\lambda-)^{n-2} + C_{n-2}\lambda(\lambda-)^{n-2} + \cdots + C_{1}\lambda$$

$= \sum_{i=1}^{n-1} C_i \lambda(\lambda-)^{i}$ where $C_i$ are integers, i=1, 2, 3... n-1

This is called the tree basis .

The method of computing $P(G',\lambda)$ in a tree form was suggested by Nijenhuis and Wilf(1987)(see also james and Riha 1975).

Given a connected graph G with a relatively small number of edges, comparing with the strategy by reducing G to tree takes fewer numbers of steps. Thus, from a computational point view,(2.2.9) has its own advantage. Theorem 2.1.3. For any connected graph G, the coefficients of its chromatic polynomial relative to the tree basis alternate in sign.

**Proof:** Nijenhuis and Wilf calculate chromatic polynomials by deleting edges with theorem FRT until we can represent the chromatic polynomial of G as the sums and differences of chromatic polynomials of trees. At each stage an edge is chosen so that its removal will not disconnect the graph. This can always be done because a connected graph contains a spanning tree as a subgraph.

A graph that is not connected does not have a spanning tree. At some point in the algorithm, we will have to add at least 1 edge to obtain a tree, rather than continue the process of deleting edges. Adding an edge destroys the alternating sign property.

**III.** A chromatic polynomial for graph G represented in the complete graph basis (also called the falling factorial basis) is written as a unique, finite, linear combination of $\{1, (\lambda)_1, (\lambda)_2, \ldots, (\lambda)_n, \ldots\}$ i.e if G has n vertices, then

$$P(G,\lambda) = (\lambda)_n + b_{n-1}(\lambda)_{n-1} + \cdots b_1(\lambda)_1 \qquad b_i \geq 0, i = 1,2,\ldots,n-1$$

The proof proceeds by noting that the end product of repeated applications of fundamental addition- deletion theorem yields a sum of complete graphs.

**Theorem 2.5.2** Let G be a graph of order n. Then

$$\mathrm{P}(\mathrm{G},\lambda) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \alpha(\mathrm{G},\mathrm{i})\,(\lambda)_{\mathrm{i}}$$

where $\alpha(\mathrm{G}, \mathrm{i})$ is the number of ways of partitioning V(G) into i independent sets.

**Proof:** There is a bijection between the family of colorings of G using exactly i colors' from {1,2, . . . , λ) and the family of partitions of V(G) into i independent sets; and further, for any such partition, there are $(\lambda)_{\mathrm{i}}$ λ -colorings' of G. Thus the number of λ -colorings of G is given $\sum_{\mathrm{i}=1}^{\mathrm{n}} \alpha(\mathrm{G},\mathrm{i})\,(\lambda)_{\mathrm{i}}$

**Example 2.5.3:** Find the chromatic polynomial of the graph shown in fig 2.9 ?

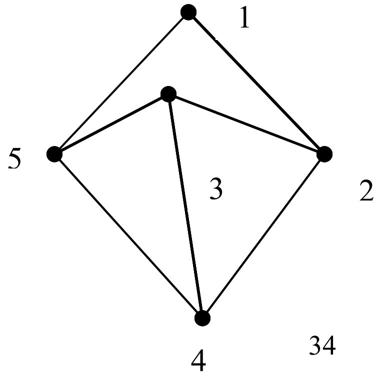


**Figure 2.9**

Let $\alpha$(G,i) be the number of ways of the partitioning vertex set V= {1,2,3,4,5} in to i independent subset. There is at least one edge in the graph, eso $\alpha$(G,1) = 0 ,

It is not possible to partition V in to two independent subset so $\alpha$(G,2) = 0

There are two ways of partitioning V in to three independent subset:

{{3},{1,4},{2,5}}

{{4},{1,3},{2,5}}, $\alpha$(G,3) = 2

There are three ways of partitioning V in to four independent subset:

{{1},{3},{4},{1,5}}

{{2},{3},{5},{1,4}}

{{2},{4},{5},{1,3}} , $\alpha$(G,4) = 3

Finally there are one ways of partitioning V in to five independent subset:

{{1},{2},{3},{4},{5}} , $\alpha$(G,5) = 1

Thus, $\mathrm{P}(\mathrm{G},\lambda) = \sum_{i=1}^{5} \alpha(\mathrm{G},\mathrm{i})\,(\lambda)_{\mathrm{i}}$

$= \alpha(\mathrm{G},5)(\lambda)_5 + \alpha(\mathrm{G},4)(\lambda)_4 + \alpha(\mathrm{G},3)(\lambda)_3 + \alpha(\mathrm{G},2)(\lambda)_2 + \alpha(\mathrm{G},1)(\lambda)_1$

$= (\lambda)_5 + 3(\lambda)_4 + 2(\lambda)_3$ ☐

If $\chi(G) = \mathrm{r}$ , then $\alpha$(G, i) =0 when i< r, $(\mathrm{r})_{\mathrm{I}}$ =0 when i > r.

Thus

$$\mathrm{P}(\mathrm{G},\mathrm{r}) = \sum_{i=1}^{n} \alpha(\mathrm{G},\mathrm{i})\,(\mathrm{r})_{\mathrm{i}} = \alpha(\mathrm{G},\mathrm{r})\mathrm{r}!$$

**Corollary 2.5.6** If $\chi(G) = \mathrm{r}$, then $\alpha(\mathrm{G,r}) = \mathrm{P}(\mathrm{G,r})/r!$

**Proof:** since

$$\mathrm{P}(\mathrm{G,r}) = \sum_{\mathrm{i=1}}^{\mathrm{n}} \alpha(\mathrm{G,i})\,(\mathrm{r})_{\mathrm{i}} = \alpha(\mathrm{G,r})\mathrm{r}!$$

$$\Longrightarrow \qquad \mathrm{P}(\mathrm{G,r}) = \alpha(\mathrm{G,r})\mathrm{r}!$$

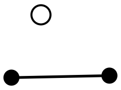
$$\Longrightarrow \qquad \alpha(\mathrm{G,r}) = \mathrm{P}(\mathrm{G,r})/r!$$

An independent set in G corresponds to a clique in $\bar{G}$, and vice versa.
Thus, as pointed out in Frucht (1985) and R.Y Liu (1987),
$b_i$ = (= a(G, i)) is the number of spanning subgraphs of consisting $\overline{G}$

of ***i*** components, each of which being a complete graph .........(*)

**Example 2.5.7:** Find $P(P_3,\lambda)$ *and* $P(C_4,\lambda)$, then

$\overline{P_3}$ :

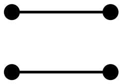


By (*), $b_2 = b_3 = 1, b_1 = 0$ *and so*, by theorem addition deletion theorem

$$\mathrm{P}(\mathrm{G},\lambda) = (\lambda)_3 + (\lambda)_2$$

If $\mathrm{G} = C_4$ ,then

$\overline{C_4}$:



Thus by (*) $b_4 = 1, b_3 = 2, b_1 = 0$ so, by theorem 2.5.2
$\mathrm{P}(\mathrm{G},\lambda) = (\lambda)_4 + 2(\lambda)_3 + (\lambda)_2$

### 2.6 The join of graphs and the umbral product

In this section ,we will introduce a way of expressing $P(G+H,\lambda)$,where G+H is the join of the graphs G and H ,in terms of $P(G,\lambda)$ and $P(H,\lambda)$.

Let $G_1 and\ G_2$ be two disjoint graphs. Suppose $|V(G_i)| = n_i$ for i= 1,2 .By the join graph of these two graphs, denoted by $G_1 + G_2$,we we mean the graph obtained by connecting every vertex in one graph to all vertices in the other .Now ,assume that $P(G_1,\lambda)$ and $P(G_2,\lambda)$ expressed in factorial form ,

$\sum_{r=1}^{n_1} \alpha(G_1,r)\,(\lambda)_r\ and \sum_{s=1}^{n_2} \alpha(G_1,s)\,(\lambda)_s$ respectively. The umbral product of these two form, denoted $P(G_1,\lambda)\ \otimes P(G_2,\lambda)$ is also a factorial form, obtained by applying the standard polynomial product, as if treating the factorials $(\lambda)_r \otimes (\lambda)_s = (\lambda)_{r+s}$

**Theorem 2.6.1(umbral product theorem)**

Let $G_1 and\ G_2$ be any two graphs with $P(G_i,\lambda)$ expressed in factorial form, I = 1, 2. Then

$$P(G_1+G_2,\lambda) = P(G_1,\lambda)\ \otimes P(G_2,\lambda)$$

**Proof :** Let $P(G_1,\lambda) = \sum_{r=1}^{n_1} \alpha(G_1,r)\,(\lambda)_r$ and $P(G_2,\lambda) = \sum_{s=1}^{n_2} \alpha(G_2,s)\,(\lambda)_s$

First we will prove that $(1 \le t \le n_1{+}n_2)$ the color – partitions of G, is equal to

$\sum_{r+s=t}^{n_1+n_2} \alpha(G_1,r)\alpha(G_2,s)$ ,for 1 ≤ r ≤ $n_{1,} and$ 1 ≤ s ≤ $n_2$

A color-class of G is either a color class of $G_1$or color-class of $G_2$ , Since every vertex in $G_1$ is adjacent to every vertex in $G_2$ , and vice versa.

Due to this fact the number of color partition of G in to t color classes is equal to the sum of color- partitions of $G_1$ and $G_2$ in to r and s color-classes, respectively proved that t =s +r.

Now we can write

$$P(G_1,\lambda)\ \otimes P(G_2,\lambda) = \sum_{r=1}^{n_1} \alpha(G_1,r)\,(\lambda)_r \otimes \sum_{s=1}^{n_2} \alpha(G_2,s)\,(\lambda)_s$$

$$=\sum_{r=1}^{n_{1,}} \sum_{s=1}^{n_{2,}} \alpha(G_1,r)\ \alpha(G_2,s)\,(\lambda)_{r+s}$$

$$= \sum_{t}^{n_{1,}+n_{2,}} \alpha(G,t)(\lambda)_t$$

$$= P(G_1+G_2,\lambda) \qquad \square$$

**Example 2.6.2 :** Let $G_1 = P_3$ and $G_2 = C_4$.

Find $P(G_1+G_2,\lambda)$ ?

**Solution :** By applying addition deletion theorem repeatedly, we have

$P(G_2,\lambda) = (\lambda)_3 + (\lambda)_2$

$P(G_1,\lambda) = (\lambda)_4 + 2(\lambda)_3 + (\lambda)_2$

$P(G_1 + G_2,\lambda) = P(G_1,\lambda) \otimes P(G_2,\lambda)$……by theorem 2.6.1

$= ((\lambda)_3 + (\lambda)_2) \otimes ((\lambda)_4 + 2(\lambda)_3 + (\lambda)_2)$

$= (\lambda)_7 + 3(\lambda)_6 + 3(\lambda)_5 + (\lambda)_4$ □

**Corollary 2.6.3:** For any graph H.

$$P(H + K_1,\lambda) = \lambda P(H,\lambda - 1)$$

**Proof:** Let $P(H,\lambda) = \sum_{i\geq 1} b_i(\lambda)_i$

By theorem 2.6.1, we have

$P(H + K_1,\lambda) = P(H,\lambda) \otimes P(K_1,\lambda)$

$= \sum_{i\geq 1} b_i(\lambda)_i \otimes \lambda$

$= \sum_{i\geq 1} b_i(\lambda)_{i+1}$

$= \lambda \sum_{i\geq 1} b_i(\lambda - 1)_i$

$= \lambda P(H,\lambda - 1)$ □

**Corollary 2.6.4:** For the wheel $W_n$ of order n ≥ 4

$$P(W_n,\lambda)\lambda\big((\lambda - 2)^{n-1} + (-1)^{n-1}(\lambda - 2)\big)$$

**Proof:** $P(W_n,\lambda) = P(C_{n-1} + K_1,\lambda) = \lambda P(C_{n-1},\lambda - 1)$….by the above corollary (2.6.4)

$$= \lambda\big((\lambda - 2)^{n-1} + (-1)^{n-1}(\lambda - 2)\big) \quad □$$

We shall next compute $P\big(K_{(p,q)},\lambda\big)$ .for this purpose, we first introduce a sequence of numbers S(n,k), called the stirling numbers of the second kind ,which are defined recursively as

$$S(n,k) = S(n - 1,k - 1) + kS(n - 1,k)$$

Where n, k ∈ ℕ , with the boundary condition that

$$\begin{cases} S(r,1) = 1, for\ all\ r \in \mathbb{N} \\ S(r,r) = 1, for\ all\ r \in \mathbb{N}_0 \\ S(r.0) = S(0,r) = 0, for\ all\ r \in \mathbb{N}, \end{cases}$$

Combinatorially, $S(n,k)$ counts the number of ways of distributing n distinct objects in to k identical boxes such that no box is empty. It is known that when the power $\lambda^n = \sum_{k=1}^{n} S(n,k)(\lambda)_k$

Thus by theorem 2.6.1, we have

$$\begin{aligned} P\left(K_{(p,q)},\lambda\right) &= P\left(O_p + O_q, \lambda\right) \\ &= P\left(O_p,\lambda\right) \otimes P\left(O_q,\lambda\right) \\ &= (\textstyle\sum_{r=1}^{p} S(n,r)(\lambda)_r) \otimes (\sum_{s=1}^{q} S(n,s)(\lambda)_s) \\ &= \textstyle\sum_{r=1}^{p} \sum_{s=1}^{q} S(n,r)\, S(n,s)(\lambda)_{r+s} \end{aligned}$$

**More generally**, for the complete m-partite graph $\mathrm{K}\left(p_1, p_{2,}p_3, \dots, p_m\right)$ , we have

$$P\left(K_{(p_1,p_{2,}p_3,\dots,p_m)},\lambda\right) = \textstyle\sum_{r_m=1}^{p_m} \dots \sum_{r_1=1}^{p_1} \prod_{i=1}^{m} S(p,r_i)\,(\lambda)_{r_1+\dots+r_m}$$

# Chapter-Three

# Chromatic polynomials

---

## 3.1. Chromatic polynomial identities

In this section, we will be looking at three polynomial identities. Similar to combinatorial proofs, we will calculate the chromatic polynomials for trees, cycles and wheels in two ways.

Proofs for these identities were found mainly by using the deletion-contraction algorithm.

**Identity 3.1.1**: For $n > 1$ and $\lambda \in \mathbb{R}$ ,

$$\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{n-1-i}\lambda^{n-1} = \lambda(\lambda-1)^{n-1}$$

which is $P(T_n, \lambda)$, the chromatic polynomial of a tree with n vertices.

**Proof :** One way to compute $P(T_n, \lambda)$ is to directly count the number of ways to color a tree with $\lambda$ colors. Let $T_n$ be a tree with n vertices, $n \geq 2$, and note that $T_n$ has at least two pendent vertices. Starting at one of the pendent vertices, we can color that vertex with any of the $\lambda$ colors. The vertex or vertices that are adjacent to the pendent vertex can then be colored in $\lambda$ -1 ways, since adjacent vertices can't have the same coloring. If we continue coloring the adjacent vertices until all n vertices are colored, we would have all the vertices, other than the first colored in $\lambda - 1$ ways. Hence

$$P(T_n, \lambda) = \lambda(\lambda-1)^{n-1}$$

Another way to calculate $P(T_n, \lambda)$ is by using the deletion-contraction algorithm and count the number of ways to delete edges from the graph G so it reduces to a null graph. Note that the chromatic polynomial of a null graph of order n is $\lambda^n$.

To get the null graph of order n from a tree of order n, we must delete all $n - 1$ edge, since there are no cycles, a contraction always removes exactly one edge. There is $\binom{n-1}{n-1}$ , only one, way to delete all the edges, which gives us the polynomial $\binom{n-1}{n-1}\lambda^n$ .

To get the null graph of order $n - 1$ from a tree of order n, we must delete $n - 2$ edges and contract one. This gives us $\binom{n-1}{n-2}$ ways to delete the edges, a polynomial of $-\binom{n-1}{n-2}\lambda^{n-1}$ . We want to contract and delete the graph's edges until we get to the null graph of one vertex.

In general, to get a null graph of order k from $T_n$ , there are $\binom{n-1}{k-1}$ ways to delete the edge, which has a polynomial of $(-1)^{n-k}\binom{n-1}{k-1}\lambda^k$ null graphs of order k. Thus,

$$P(T_n, \lambda) = \binom{n-1}{n-1}\lambda^n - \binom{n-1}{n-2}\lambda^{n-1} + \cdots + \binom{n-1}{0}(-1)^{n-1}\lambda$$
$$= \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{n-1-i}\lambda^{n-i} \qquad \square$$

Of course, we can also derive Identity 3.1.1 by using the binomial theorem, we have that

$$\lambda(\lambda-1)^{n-1} = \lambda[\binom{n-1}{n-1}\lambda^{n-1} - \binom{n-1}{n-2}\lambda^{n-2} + \cdots + \binom{n-1}{0}(-1)^{n-1}\lambda^0]$$
$$= \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{n-1-i}\lambda^{n-i} \qquad \square$$

**Identity 3.1.2.** For $n > 2$ and $\lambda \in \mathbb{R}$ ,

$$\left[\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{n-1-i}\lambda^{n-i}\right] + (-1)^n\lambda = \sum_{i=0}^{n-2}(-1)^i\lambda(\lambda-1)^{n-i-1}$$
$$=(\lambda-1)^n + (-1)^n(\lambda-1)$$

which is $P(C_n, \lambda)$ the chromatic polynomial of a cycle with n vertices.

**Proof :** One way to calculate $P(C_n, \lambda)$ is by counting the number of ways to delete edges, reducing the original graph to a null graph. To get a null graph of n-vertices, from an n-cycle, we need to delete all the edges, which we can do in $\binom{n}{n}$ ways and contributes the polynomial term of $\binom{n}{n}\lambda^n$ *to* $P(C_n, \lambda)$ .

To get a null graph of $(n - 1)$-vertices, we can delete all but one edge, which we can do in $\binom{n}{n-1}$ ways. Generalizing, if we want the null graph of k vertices, $2 \leq k \leq n$, we can delete in $\binom{n}{k}$ ways, and have the polynomial $(-1)^k\binom{n}{k}\lambda^k$ .

However, when we want the null graph of one vertex,

we can either delete 1 edge, which can be done in $\binom{n}{1}$ ways, or contract all the edges, which gives a term of

$$(-1)^{n-1}[\binom{n}{1}\lambda - \lambda]$$

Hence,

$$P(\mathrm{C_n}, \lambda) = \lambda^n - \binom{n}{n-1}\lambda^{n-1} + \cdots + \binom{n}{1}(-1)^{n-1}\lambda + (-1)^n\lambda$$
$$= \sum_{i=0}^{n-1}(-1)^i\binom{n}{n-i}\lambda^{n-i} + (-1)^n\lambda$$

On the other hand, we can derive $P(\mathrm{C_n}, \lambda)$ by contacting and deleting the cycle down to trees. Note that deleting and contracting an edge from an n-cycle gives us an n-tree and an $(\mathrm{n}-1)$-cycle. By deleting an edge from the n-cycle, we end up with a n-tree, which has the chromatic polynomial of $\lambda(\lambda-1)^{\mathrm{n}-1}$. Looking at the contraction step of the n-cycle, we get an $(\mathrm{n}-1)$-cycle. We then want to take the (n $-$ 1)-cycle and delete and contract it down to a 3-cycle, which gives us the polynomial terms

$(-1)^{n-k}\lambda(\lambda-1)^{\mathrm{n}-1}, 4 \leq \mathrm{k} \leq \mathrm{n}$. The 3-cycle deletes to a 3-tree, a chromatic polynomial of $\lambda(\lambda-1)^2$ and contracts to a 2-tree, a term of $\lambda(\lambda-1)^1$ .If we add up all the polynomials we got from the trees of sizes two to $\mathrm{n}-1$, we get that

$$P(\mathrm{C_n}, \lambda) = \lambda(\lambda-1)^{\mathrm{n}-1} - \lambda(\lambda-1)^{\mathrm{n}-2} + \lambda(\lambda-1)^{\mathrm{n}-3} - \cdots (-1)^{\mathrm{n}}\lambda(\lambda-1)^1$$
$$= \sum_{\mathrm{i}=0}^{\mathrm{n}-2}(-1)^{\mathrm{i}}\lambda(\lambda-1)^{\mathrm{n}-\mathrm{i}-1}$$

We have seen in section () that

$$P(\mathrm{C_n}, \lambda) = (\lambda-1)^n + (-1)^n(\lambda-1)$$

Using binomial theorem, we can show that the first part 3.1.1 is equal to 3.1.2

Since

$$(\lambda-1)^n + (-1)^n(\lambda-1) = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\lambda^i + (-1)^n(\lambda-1)$$
$$= [\,\lambda^n - \binom{n}{n-1}\lambda^{n-1} + \binom{n}{n-2}\lambda^{n-2} + \cdots + \binom{n}{0}(-1)^{n-1}\,] +$$
$$(-1)^n(\lambda-1)$$
$$= \lambda^n - \binom{n}{n-1}\lambda^{n-1} + \binom{n}{n-2}\lambda^{n-2} + \cdots + \binom{n}{1}(-1)^{n-1}\lambda + (-1)^n\,\lambda$$
$$= \sum_{i=0}^{n-1}(-1)^i\binom{n}{n-i}\lambda^{n-i} + (-1)^n\lambda \qquad \square$$

**Identity 3.1.3**. For $\mathrm{n} > 3$ and $\lambda \in \mathbb{R}$,

$$\lambda[(\lambda-2)^{\mathrm{n}-1}+(-1)^{\mathrm{n}-1}(\lambda-2)]$$

$$=(-1)^{\mathrm{n}-1}\lambda(\lambda-1)+\sum_{\mathrm{i}=1}^{\mathrm{n}-1}(-1)^{\mathrm{n}-\mathrm{i}-1}\binom{\mathrm{n}-1}{\mathrm{i}}\lambda(\lambda-1)^{\mathrm{i}}$$

which is $P(\mathrm{W_n}, \lambda)$ the chromatic polynomial of a wheel with n vertices.

**Proof:** One way to find $P(\mathrm{W_n}, \lambda)$ is by looking first at the vertex which is adjacent to all other vertices, call it v, and see how many different ways we can color it. We can color v in λ ways. After coloring v, we can then color the rest of the graph, which is now a cycle, with $\lambda - 1$ colors. From Identity 3.1.2, we get that the chromatic polynomial of the cycle is

$(\lambda-1)^n+(-1)^n(\lambda-1)$. Hence

$$P(\mathrm{W_n}, \lambda)=\lambda[\,(\lambda-1)^n+(-1)^n(\lambda-1)]$$

Another way to calculate $P(\mathrm{W_n}, \lambda)$ is by counting the number of ways to contract edges. Using the deletion-contraction algorithm, we will, one-by-one, contract the edges on the outside of the wheel, to produce trees. Given an n-wheel, there are $\binom{n-1}{0}$ ways to contract the edges to form an n-tree, because we would need to delete all n−1 of the outside edges. This gives us the term $\binom{n-1}{0}\lambda(\lambda)^{n-1}$ in $(\mathrm{W_n}, \lambda)$. To get a (n−1)-tree, we can contract one edge from the outside. There are $\binom{n-1}{1}$ ways to do this, giving us the term $-\binom{n-1}{1}\lambda(\lambda)^{n-2}$. Note that when doing a Contraction to an n-wheel, the result is an $(\mathrm{n}-1)$ wheel, if $\mathrm{n} > 4$. In general, if we want the tree of order k, $4 \leq k \leq n$ we can contract in $\binom{n-1}{n-k}$ ways. Looking at when we want a 3-tree, we can contract $\mathrm{n}-3$ edges. However, when we start with n – 3 contractions, we get a 3-cycle. So instead of two deletions, we just do one. This still gives us $\binom{n-1}{n-3}$ ways to contract to get a 3-tree. To get a 2-tree, we can contract n−2 edges.. Looking at the cases where we start with $\mathrm{n}-3$ contractions, we see that the case with $\mathrm{n}-3$ contractions, one deletion, and one contraction actually gives us a 3-tree, without doing the last contraction, so we can take out that case. The second case where we start with n−2 contractions and one deletion gives us a 2-tree without doing the last deletion. Counting all these combinations, we get $\binom{n-1}{n-2} - 1$ ways to contract to a 2-tree. Combining the terms, we get that

$$P(\mathrm{W_n}, \lambda) = \binom{n-1}{n-1}\lambda(\lambda-1)^{n-1} - (-1)^{n-(n-2)-1}\binom{n-1}{n-2}\lambda(\lambda-1)^{n-2} + \cdots +$$

$$= (-1)^{n-2-1}\binom{n-1}{2}\lambda(\lambda-1)^2 + (-1)^{n-1-1}\left[\binom{n-1}{1} - 1\right]\lambda(\lambda-1)^1$$

$$= \sum_{i=1}^{n-1}\left[(-1)^{n-i-1}\binom{n-1}{i}\lambda(\lambda-1)^i\right] - (-1)^{n-2}\lambda(\lambda-1)$$

### 3.2 An interpretation of Chromatic polynomial coefficients

Suppose that $P(G,\lambda)$ is expressed in power form $\sum a_i\lambda^i$ .In this section, we shall present some useful results relating the coefficients $a_i$'s and the number of certain spanning sub graphs of G. This result for general graph was given by Whitney (1932b).but its original idea was due to Birkhoff (1912) who applied it to evaluate the number of λ −colorings for planar graph.

**Theorem 3.2.1:** Let G be an (n, m)- graph. Then

$$P(G,\lambda) = \sum_{p=1}^{n}\left(\sum_{r=0}^{m}(-1)^r N(p,r)\right)\lambda^p$$

Where $N(p,r)$ *denotes* the number of spanning sub graphs of G with p components and r edges.

**Proof:** Let $\theta: V(G) \to \{1,2,\dots,\lambda\}$ a λ −mapping of G.

Let A be the set of all λ −mapping.Then$|A| = \lambda^n$.

Let $E(G) = \{e_1, e_2, \dots, e_m\}$.For each i=1, 2,...,m.

Let $q_i$denote the property that the two vertices incident with $e_i$ have the same image under $\theta$ , for any $\{i_1, i_2, \dots, i_r\} \subseteq \{1,2,\dots,m\}$.

Let $N(q_{i_1}, q_{i_2}, \dots, q_{i_r})$ denote the number of mapping in A that satisfy none of the $q_i$'s .By the principle of inclusion and exclusion. we have

$$P(G,\lambda) = \lambda^n - \sum_{i=1}^{m} N(q_i) + \sum_{1\le i<j\le m} N(q_i, q_j) + \cdots, (-1)^m(q_1, q_2, \dots, q_m)$$

Consider a typical term

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq m} N\left(q_{i_1}, q_{i_2}, \dots, q_{i_r}\right)$$

Let H be the spanning sub graph of G with r edges $e_{i_1}, e_{i_2} \dots e_{i_r}$ . Then λ -mapping of G that satisfies the properties $q_{i_1}, q_{i_2}, \dots, q_{i_r}$ is indeed a λ-mapping of H that satisfies properties $q_{i_1}, q_{i_2}, \dots, q_{i_r}$.Now with respect to such a λ -mapping of H ,all vertices in a component of H, all vertices in a component of H have the same image.

Suppose there are p (p = 1,2,..,n) components in H. Then the number of mapping in A satisfies properties $q_{i_1}, q_{i_2}, \dots, q_{i_r}$,

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq m} N\left(q_{i_1}, q_{i_2}, \dots, q_{i_r}\right) = \sum_{p=1}^{n} N(p, r)\, \lambda^p$$

Hence, note that $N(n, 0) = 1$ and $N(p, 0) = 1$ for $1 \leq p \leq n - 1$. We have

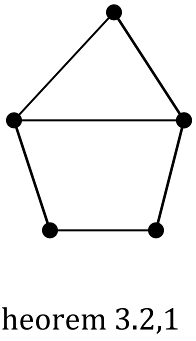
$$P(G, \lambda) = \lambda^n - \sum_{p=1}^{n} N(p, 1)\, \lambda^p + \sum_{p=1}^{n} N(p, 2)\, \lambda^p + \cdots + (-1)^m \sum_{p=1}^{n} N(p, m)\, \lambda^p$$

$$= \sum_{r=1}^{m} \sum_{p=1}^{n} (-1)^r N(p, m)\, \lambda^p$$

$$= \sum_{p=1}^{n} \left( \sum_{r=0}^{m} (-1)^r N(p, r) \right) \lambda^p$$

**Example 3.2.2:** Consider the (5,6)- graph in Figure 3.1 as shown below

**Figure 3.1**



Let us find the coefficients of $\lambda^2$ by Theorem 3.2,1

We have

$$P(G,\lambda)=\sum_{p=1}^{n}\left(\sum_{r=0}^{m}(-1)^{r}N(p,r)\right)\lambda^{p}$$

For p=2

Observe that

$$N(2,0)=N(2,1)=N(2,2)=N(2,5)=N(2,6)=0$$

And $N(2,3)=19\ and\ N(2,4)=4$

$$\Longrightarrow [\lambda^{2}]\,P(G,\lambda)=-19+4=-15$$

Whitney observed that there is cancellation between $N(2,3)$ and $N(2,4)$.This observation led Whitney to introduce the following notation of broken cycles to simplify the computation.

### 3.3 Broken cycle Theorem

**Definition**: Let G be an (n, m)-graph, and $\beta : E(G) \to \{1,2,\dots,m\}$ be a bijection. for any cycle C in G. let $e \in E(G)$ such that $\beta(e) > \beta(x)$ for any x in $E(C)\backslash\{e\}$ . we call the path C-e a Broken cycle in G with respect to $\beta$.

**Example:** Let G be a graph as shown in Figure 3.2 below

G:

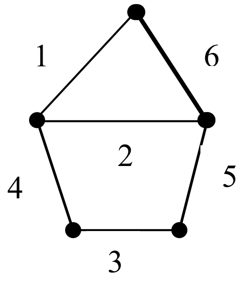
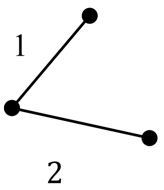
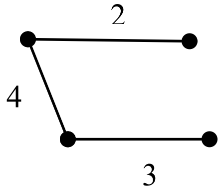
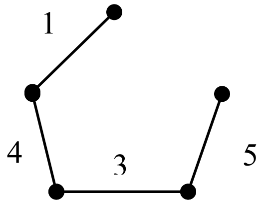


Figure 3.2

Broken cycles are







### 3.3.1 Whitney's Broken cycle Theorem

Let G be an (n, m)-graph, and $\beta : E(G) \to \{1,2,\dots,m\}$ be a bijection.Then

$$P(G,\lambda)=\sum_{i=1}^{n}(-1)^{n-i}h_{i}(G)\lambda^{i}$$

.Where $h_i(G)$ is the number of spanning sub graphs of G that have exactly n-i edges and that contain no broken cycles wrt $\beta$ .

Proof: Assume that there are q ≥ 0 broken cycles in G. For each broken cycle ,

let $\beta(B) = Max\{\beta(f): f \in E(B)\}$. Arrange the broken cycles as $B_1, B_2, \dots, B_q$ such that $\beta(B_1) \leq \beta(B_2) \leq \cdots \leq \beta(B_q)$ .Let $S_{q+1}$ be the family of spanning sub graphs of G that do not contain any $B_i$ as a sub graph .Then $\{s_1, s_2, \dots, s_{q+1}\}$ forms a partition of the family of spanning subgraph of G.

Consider $s_1$, assume that $B_1 = C - f_1$ ,for some cycle C and $f_1 \in E(C)$ .observe that each member H in $s_1$ not containing $f_1$ ,gives rise to a unique member H+$f_1$ in $s_1$ containing $f_1$, vise versa. Since $e(H + f_1) = e(H) + 1$ but C$c(H + f_1) = c(H)$ by Theorem 3.2,2, the contribution of H and H+$f_1$, to $P(G, \lambda)$ are cancelled off..

Consider $s_2$ , assume that $B_2 = C - f_2$ , for some cycle C and $f_2 \in E(C)$. It follows from the ordering of B's that $f_2 \notin E(B_1)(and\ so\ f_2 \notin E(B_1) \cup E(B_2))$ .Again, each member H in $s_2$ not containing $f_2$ gives rise to a unique member H+$f_2$ in $s_2$ containing $f_2$ ,and vise versa. Thus, their contribution to $P(G, \lambda)$ are cancelled off.

Likewise, the total contribution of the member in $s_i$ to $P(G, \lambda)$ is cancelled off, for each i=1, 2, 3... q

We are now left with the family $s_{q+1}$. let $H \in S_{q+1}$ and assume that e$(H) = s$.As H is a forest $0 \leq s \leq n - 1$ and $c(H) = n - s$. Let $N'(n - s, s)$ denotes the number of graphs in $S_{q+1}$ with n-s components and s edges. Thus, by Theorem 3.2.2

$$P(G, \lambda) = \sum_{p=1}^{n} \left( \sum_{r=0}^{m} (-1)^r N(p, r) \right) \lambda^p$$

$$= \sum_{s=0}^{n-1} (-1)^s N'(n - s, s) \lambda^{n - s}$$

$$= \sum_{s=0}^{n-1} (-1)^s h_{n-s}(G) \lambda^{n - s}$$

$$P(G,\lambda) = \sum_{i=0}^{n} (-1)^{n-i} h_i(G)\lambda^i$$

### 3.4 Unimodal Conjecture

Read (1968) observed that for any graph $G$ it appears that the coefficient of $P(G,\lambda)$ always increase in absolute value first and decrease eventually.

For example:

$$P(T_8,\lambda) = \lambda^8 - 7\lambda^7 + 21\lambda^6 - 35\lambda^5 - 21\lambda^3 + 7\lambda^2 - \lambda$$
$$P(c_8,\lambda) = \lambda^8 - 8\lambda^7 + 28\lambda^6 - 5\lambda^5 + 70\lambda^4 - 56\lambda^3 + 28\lambda^2 - 7\lambda$$

**Definition:** If $P(G,\lambda) = \sum_{i=1}^{n} (-1)^{n-i} h_i \lambda^i$ where $h_i \in \mathbb{N}$, then there is always seems to exist $k \in \mathbb{N}$ with $2 \le k \le n-1$ such that

$$h_1 \le h_2 \le \ \ldots \ \le h_{k-1} \le h_k \ge h_{k+1} \ge \ \ldots \ \ge h_n$$

This is called unimodal Conjecture(**UC**)

**Theorem 3.4.1** Let $G$ be a connected graph of order $n$. Then for any $i \in \mathbb{N}$ with $i \le n -$ 1 $\quad h_{i+1} \le \left(\frac{n}{i} - i\right) h_i$ where $h_i \in \mathbb{N}$

where equality holds if and ony if $G$ is tree.

**Proof:** By induction on $n$ and $m$.

i. For ? ?= 2,3

? ?(? ,?? ?) = ? ?$^2$ − ? ? $\Rightarrow 1 \le \left(\frac{2}{1} - 1\right) * 1 \Rightarrow 1 \le 1$

? ?(? ,?? ?) = ? ?$^3$−2? ?$^2$+ ? ?or ? ?$^3$−3? ?$^2$+ 2? ?

$1 * h_3 \le \left(\frac{3}{2} - 1\right) * h_2 \Rightarrow 1 \le 1$

∴ It is true for ? ?= 2 and ? ?= 3

ii. Assume it is true for all connected graph order ? ?≤ ? ?and ? ?is of size less than ? ?, where ? ,?? ?∈ ℕ with ? ?≥ 4 and ? ?− 1 < ? ?≤ (? ? over 2)

iii. If we take any connected graph ? ?,then either ? ?is a tree or it is not a tree.

**Case 1:** If $G$ is tree, then by (2.3)

$$P(G,\lambda) = \lambda(\lambda - 1)^{n-1}$$

$$h_i = \left[\lambda^i\right] P(G,\lambda) = \binom{n-1}{i-1} \quad \text{………} (1)$$

$$h_{i+1} = \left[\lambda^{i+1}\right] P(G,\lambda) = \binom{n-1}{i} \text{……} (2)$$

(2) From (1) and (2) , we get

$$h_{i+1} = \left(\frac{n}{i} - 1\right) h_i$$

**Case2:** If $G$ is not a tree, then $G$ has a cycle ( since G is connected).

Let $e$ be an edge in the cycle of $G$. Then $G.e$ is connected graph of order n-1 and $G - e$ is a connected $(n, m-1)$ graph. By induction hypothesis, the result holds for $G.e$ and $G - e$. Thus,

$$h_{i+1}(G-e) \leq \left(\frac{n}{i} - 1\right) h_i(G-e) \ , i = 1,2,\dots,n-1$$

and

$$h_{i+1}(G.e) \leq \left(\frac{n-1}{i} - 1\right) h_i(G.e) \ , i = 1,2,\dots,n-2$$

By (Theorem…) (FRT), we have

$$h_i = h_i(G-e) + h_i(G.e) \qquad\qquad i = 1,2,\dots,n-2$$

Let $i \in \mathbb{N}$ with $i \leq n-2$ we have

$$\begin{aligned} h_{i+1} &= h_{i+1}(G-e) + h_{i+1}(G.e) \\ &\leq \left(\frac{n}{i} - 1\right) h_i(G-e) + \left(\frac{n-1}{i} - 1\right) h_i(G.e) \end{aligned}$$

As $h_i(G-e) \geq 1$. It follows that

$$\begin{aligned} h_{i+1} &< \left(\frac{n}{i} - 1\right) (h_i(G-e) + h_i(G.e)) \\ &= \left(\frac{n}{i} - 1\right) h_i \end{aligned}$$

Consider now the case $i = n-1$. As $h_n(G.e) = 0$ and $h_{n-1}\,(G.e) = 1$ we have

$$\begin{aligned} h_n &= h_n(G-e) \\ \Rightarrow \quad &\leq \left(\frac{n}{n-1} - 1\right) h_{n-1}\,(G-e) \\ \Rightarrow \quad &= \left(\frac{n}{n-1} - 1\right)\,(h_{n-1} - 1) \ \blacksquare \end{aligned}$$

### Strong Logarithmic Concavity Conjecture (SLCC)

**Corollary 3.4.2** Let $G$ be a connected graph of order $n$.

(i) If ? ?is odd, then

$$h_{? ?} < h_{? ?-1} < \dots < h_{\frac{? ?+1}{2}}$$

(ii) If ? ?is even, then

$$h_{??} < h_{??-1} < \dots < h_{\frac{??}{2}+1} \le h_{\frac{??}{2}}$$

where $h_{\frac{??}{2}+1} = h_{\frac{??}{2}}$ if and only if ? ?is tree.

**Proof:** If $n$ is odd, then $\frac{n}{i} - 1 < 1$ for all i; with $\frac{n+1}{2} \le i < n$.

If $n$ is even, then $\frac{n}{i} - 1 < 1$ for all i; with $\frac{n}{i} + 1 \le i < n$ and $\frac{n}{i} - 1 = 1$ when $i = \frac{n}{2}$.

Thus by Theorem 3.4.1 when $n$ is odd, we have

$$h_n < h_{n-1} < \dots < h_{\frac{n+1}{2}}$$

and when $n$ is even, we have

$$h_n < h_{n-1} < \dots < h_{\frac{n}{2}+1} \le h_{\frac{n}{2}}$$

If $n$ is even and $G$ is a tree, then

$$h_{\frac{n}{2}+1} = h_{\frac{n}{2}} = \binom{n-1}{\frac{n}{2}-1}$$

If $n$ is even but $G$ is not a tree, then by Theorem 3.4.1 we have

$$h_{\frac{n}{2}+1} < h_{\frac{n}{2}} \qquad \square$$

## IV. Conclusion

There is little doubt that the best known and most studied area within graph theory is coloring. When the set is associated with a graph in some manner, then we are dealing with graph colorings. With its origins embedded in attempts to solve the famous Four Color Problem, graph colorings has become a subject of great interest, largely because of its diverse theoretical results, its unsolved problems, and it's numerous applications.

As further study of this topic, we could explore the calculation of Chromatic Polynomials given different restrictions on the colorings of a graph. For example, we could explore colorings for which particular pairs of adjacent vertices are the same color or colorings for which adjacent vertices are not similar colors. Another possible topic could be Chromatic Polynomials for edge colorings. Richard Stanley's text Enumerative combinatorics provides guidance for further exploration of these ideas.

We conclude this project with a listing of avenues for further research. Much remains to be done before the characterization of graphs and their chromatic polynomials is complete.

Two major problems are as yet unsolved, although some of their subproblems have been successfully analyzed. These two problems are:

**1.** What are the necessary and sufficient conditions for a polynomial to be a chromatic polynomial?

**2.** Is the unimodal conjecture true, and if so, relative to which bases is it true?

**1.** What are the necessary and sufficient conditions for a polynomial to be a chromatic polynomial?

Certainly the leading coefficient must be equal to one and the polynomial cannot contain a constant term. Relative to the null graph basis, coefficients must alternate in sign. Wilf [3] and Eisenberg [6] have sharpened the upper and lower bounds for the coefficients relative to this basis, but no sufficient conditions have yet been discovered.

**2.** Is the unimodal conjecture true, and if so, relative to which bases is it true?

R. C. Read [11] conjectured that the absolute value of the coefficients of the chromatic polynomial of any graph G, relative to the null graph basis, is unimodal. This means the coefficients first increase in absolute magnitude, and then decrease; two successive coefficients may be equal, but there is never one coefficient surrounded by larger coefficients. Read's conjecture allows for any number of equal coefficients.

This conjecture is true for all graphs with seven vertices or less, as verified by computer generation of these polynomials. Of the thousands of chromatic polynomials computed by many researchers in this area, no counterexample has been found. But the truth of the unimodal conjecture, relative to the null graph basis, has not yet been proven or disproven.

## Reference

[1] K. Appel and W. Haken , Every planar map is four colorable:
Part I— Discharging, Illinois J. Math., 21 (1977) , 429-490.

[2] K. Appel, W. Haken, and J. Koch, Every planar map is four-colorable:
Illinois J. Math., 21 (1977), 491-567.

[3] F M Dong and K L Teo Chromatic polynomials and chromaticity of Graphs
Math, December (2004), 1-54

[4] C. Y. Chao and E. G. Whitehead, Jr., Chromatically unique graphs.
Discrete Math., 27(1979), 171-177.

[5] V. Chvatal, A note on coefficients of chromatic polynomials, J. Combinatorial
Theory, 9 (1970), 95-96.

[6] B. Eisenberg, On the Coefficients of the Chromatic Polynomial of a Graph,
Adelphi University, Sept. 1970.

[7] F. Harary, Graph Theory, Addison-Wesley, 1969. -109-

[8] S. G. Hoggar, Chromatic Polynomials and Logarithmic Concavity,
J. Combinatorial Theory, Series B. 16 (1974) , 248-254,

[9] B. Loerinc, Chromatic uniqueness of the generalized.
Discrete Math., 2 (1978), 313-316.

[10] B. Loerinc, Computing Chromatic Polynomials for Special Families of Graphs, ,
New York University, Oct. 1979.

[11] R. C. Read, An introduction to chromatic polynomials,
J. Combinatorial Theory, 4(1968), 52-71. John Wiley and Sons, 1958.