A SINGULAR BOUNDARY VALUE PROBLEM FOR A DEGENERATE ELLIPTIC PARTIAL DIFFERENTIAL EQUATION

By

Tesfa Biset

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The undersigned hereby certify that they have read and recommend to the College of Natural Sciences for acceptance a PhD dissertation entitled "**A Singular Boundary Value Problem for a Degenerate Elliptic Partial Differential Equation**" by **Tesfa Biset** in partial fulfillment of the requirements for PhD degree.

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External Examiner: ______________________

Research Supervisor: ______________________
Ahmed Mohammed (Prof.)

Examing Committee: ______________________

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Author: **Tesfa Biset**

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Signature of Author

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Abstract

A singular boundary value problem associated with the ∞-Laplace equation having, in general a non-separable inhomogeneous term is considered. The existence of ground state solution in viscosity sense to such a problem in bounded domains as well as in the whole Euclidean space $\mathbb{R}^N$, for $N \geqslant 2$ is established. The investigation of the problem is mainly based on sub-solution and super-solution methods, and existence of principal eigenfunctions to the eigenvalue of Dirichlet type problem $-\Delta_\infty u = \lambda f(x,u), u > 0$ in an open set Ω and $u(x) = 0$ on $\partial\Omega$, corresponding to the ∞-Laplace equation.

Keywords: ∞-Laplacian, Viscosity solution, Principal eigenvalue, Principal eigenfunction

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Chapter 1

Introduction

This dissertation is concerned with the investigation of singular boundary value problems related to the highly degenerate elliptic operator

$$\Delta_\infty u := \langle D^2 u Du, Du\rangle.$$

In the literature, Δ_∞ is referred to as the Infinity-Laplacian and this operator can be understood as the limit of the p-Laplacian as p tends to infinity. Interested reader may refer to the paper by Michael Crandall for the precise statement concerning this convergence. Historically, Δ_∞ appeared in a series of papers [1, 2] and [3] by Gunnar Aronsson in his work related to finding absolute minimizing functions in a given domain in $\mathbb{R}^N$. Aronsson used a heuristic argument to relate C^2 solutions of the infinity Laplacian equation $\Delta_\infty u = 0$ with the problem of seeking absolutely minimizing functions on domains. However, as observed by Aronsson himself, there are absolutely minimizing functions that are not twice differentiable and therefore not classical solutions of the above equation. A breakthrough was achieved in 1983 when Robert Jensen was able to show that a function u is absolutely minimizing if and only if $\Delta_\infty u = 0$ in a domain in the viscosity sense. Again, the reader may refer to the seminal paper of 1993 [19] for details. Given a bounded domain $\Omega \subseteq \mathbb{R}^N$, Jensen showed that u is a solution of the Dirichlet Problem

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

if and only if u is an absolutely minimizing Lipschitz function on Ω such that $u = g$ on the boundary of Ω. The paper and the results therein opened up a new interest in problems related to the operator Δ_∞. Another landmark paper related to the infinity-Laplacian was published in 2009 (see [29]) and the authors were able to provide a tug-of-war game interpretation to the solution of the Dirichlet Problem

$$\begin{cases} \Delta_\infty u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

for a given continuous function f on Ω and a Lipschitz function g on the boundary $\partial\Omega$.

In a recent paper [8], the authors studied the following singular boundary value problem.

$$\begin{cases} -\Delta_\infty u = b(x)h(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, $b \in C(\Omega)$ is a positive function, and

$h : (0,\infty) \to (0,\infty)$ is a continuous function that may be unbounded near 0. An interesting special case is $h(t) = t^{-\gamma}$ for $\gamma > 0$. The paper establishes that if $b \in C(\Omega) \cap L^\infty(\Omega)$ is positive and h is a non-increasing continuous function, then Problem (1.1) admits a unique solution $u \in C(\overline{\Omega})$.

The aim of this dissertation is to explore the solvability of the following parameter dependent singular boundary value problem on bounded as well as on the entire Euclidean spaces.

$$\begin{cases} -\Delta_\infty u = \lambda f(x,u),\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

where the boundary condition is understood in the sense that $u(x) \to 0$ as $|x| \to \infty$ when $\Omega = \mathbb{R}^N$.

The non-linearity is subjected to some natural, but general conditions and includes all known results as special cases.

Chapter 2

Preliminaries and Background

We next state some basic notations which will be used in the next sections.

Notations:

The set Ω is an open subset of $\mathbb{R}^N$, $N \geqslant 2$ and N a natural number, in most cases Ω is bounded. $\overline{\Omega}$ is the closure of Ω and $\partial\Omega$ is its boundary.

Given sets $\mathcal{O}$ and Ω, we write $\mathcal{O} \subset\subset \Omega$ to mean $\mathcal{O}$ is compactly contained in Ω.

The set of all continuous real valued functions in Ω is denoted by $C(\Omega)$ and $C(\overline{\Omega})$ if on $\overline{\Omega}$. The notion $u \in C^k$ indicates that u is a real valued function on a subset of $\mathbb{R}^N$ and it is k-times continuously differentiable. $C^2(\Omega)$ stands for the set of twice continuously differentiable functions in Ω. A smooth function usually means a C^2 function.

Given a set $E \subseteq \mathbb{R}$, $C(\Omega, E)$ denotes the class of continuous functions on Ω that have values in E.

If $f \in C(\Omega)$, $L^\infty(\Omega)$ is the standard space of bounded Lebesgue measurable functions with the usual norm,

$$\|f\|_{L^\infty(\Omega)} = \inf\{M \in \mathbb{R} : |f(x)| \leqslant M, x \text{ in } \Omega\}.$$

Balls are expressed as: $B_r(x) := \{y \in \mathbb{R}^N\colon\ |y - x| < r\}$ and
$\overline{B}_r(x) := \{y \in \mathbb{R}^N\colon\ |y - x| \leqslant r\}$.

$\circ(\varepsilon)$ denotes quantities whose quotients by ε approaches 0 as $\varepsilon \to 0$. We write
$f(x) \leqslant g(x) + \circ(|x - x_0|^2)$ as $x \to x_0$ to mean that there is a function $h(x) \geqslant 0$ such that $h(x) = \circ(|x - x_0|^2)$ as $x \to x_0$ and $f(x) = g(x) + h(x)$ in a neighbourhood of x_0.

Definition of viscosity solution:

Let $\Omega \subseteq \mathbb{R}^N$ be a non-empty open subset, and $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be continuous. A function $u \in C(\Omega)$ is said to be a **viscosity sub-solution** of the PDE

$$-\Delta_\infty u = H(x, u, Du) \tag{2.1}$$

at $x_0 \in \Omega$ if and only if for each $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a maximum at $x_0 \in \Omega$ the following inequality holds.

$$-\Delta_\infty \varphi(x_0) \leqslant H(x_0, u(x_0), D\varphi(x_0)). \tag{2.2}$$

A function $u \in C(\Omega)$ is said to be a **viscosity super-solution** of the PDE (2.1) at $x_0 \in \Omega$ if and only if for each $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a minimum at $x_0 \in \Omega$ the reverse inequality in (2.2) holds.

Finally, we say $u \in C(\Omega)$ is a **viscosity solution** of (2.1) at $x_0 \in \Omega$ if u is a viscosity sub-solution as well as a viscosity super-solution of (2.1) at x_0.

Viscosity sub-solution, super-solution and solution of (2.1) in a set are defined in an obvious way. If u is a viscosity sub-solution (resp., super-solution or solution) to (2.1) in Ω, we will indicate this by writing

$$-\Delta_\infty u \leqslant H(x, u, Du), (\text{resp., } \geqslant \text{ or } =) \qquad x \in \Omega.$$

All sub-solutions or super-solutions that will be considered in this dissertation are in the viscosity sense. Therefore, in the sequel we will refrain from prefixing them with the word viscosity.

The following comparison principle will be useful (see [25, 27, 28]).

Proposition 2.1 Let $c \in C(\Omega) \cap L^\infty(\Omega)$ such that $c > 0$, $c < 0$ or $c \equiv 0$ in Ω. If $u, v \in C(\overline{\Omega})$ such that $-\Delta_\infty u \leqslant c$ in Ω and $-\Delta_\infty v \geqslant c$ in Ω, then

$$\max_{\overline{\Omega}}(u - v) = \max_{\partial\Omega}(u - v).$$

Consider an open set $\Omega \subseteq \mathbb{R}^N$, and let $\mathbb{R}^+ := (0, \infty)$. Let $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ be a continuous function such that the following two conditions hold:

(F-a) There is $h \in C((0, t_0), \mathbb{R}^+)$ for some $t_0 > 0$ and $a \in C(\overline{\Omega}, [0, \infty))$ with $a(x) \not\equiv 0$ on Ω such that

$$f(x,t) \geqslant a(x)h(t), \quad \forall\ (x,t) \in \Omega \times (0, t_0).$$

(F-b) There are functions $g \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in C(\Omega, \mathbb{R}^+)$ such that

$$f(x,t) \leqslant b(x)g(t) \quad \forall\ (x,t) \in \Omega \times \mathbb{R}^+.$$

The following notations will be needed in relation to the functions h and g that appear in **(F-a)** and **(F-b)**, respectively.

$$g_\infty := \limsup_{t\to\infty} \frac{g(t)}{t^3} \qquad \text{and} \qquad h_0 := \liminf_{t\to 0^+} \frac{h(t)}{t^3}. \tag{2.3}$$

Moreover, we will always suppose that $0 < h_0 \leqslant \infty$.

Finally, the function b in Condition **(F-b)** above will be assumed to satisfy the following condition.

(B-w) The following problem admits a solution $w_\Omega \in C(\overline{\Omega})$.

$$\begin{cases} -\Delta_\infty u &= b(x), \quad u > 0 \quad \text{in } \Omega \\ u(x) &= 0 \qquad\qquad\quad\ \ \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

If Ω is unbounded, then the boundary condition in (2.4) is understood in the sense that $u(x) \to 0$ as $|x| \to \infty$ within Ω. We remark that if $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{R}^N$ are two open sets with Ω_1 bounded, then $0 < w_{\Omega_1} \leqslant w_{\Omega_2}$ in Ω_1. This is a consequence of the comparison principle. If Ω is bounded and $b \in C(\Omega) \cap L^\infty(\Omega)$, then it is known that (2.4) admits a solution (see [24]).

The existence of a principal eigenvalue and a principal eigenfunction for the following eigenvalue problem in bounded domains Ω will be an important ingredient in investigating the existence of solutions to Problem (1.2).

$$\begin{cases} -\Delta_\infty u &= \lambda a(x) u^3 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{2.5}$$

When $a(x) > 0$ in Ω, and Ω is bounded it is known (see [7]) that there is a positive principal eigenvalue $\Lambda_1(\Omega, a)$ and a corresponding principal eigenfunction $\phi > 0$ in Ω that satisfy (2.5).

In our work in this dissertation, we will establish a similar result when $a(x) \geqslant 0$ in Ω, but $a(x) \not\equiv 0$ in Ω. The proof of this will be provided in the appendix.
A property of the principal eigenvalue that will be useful here is the fact that
$\Lambda_1(\mathcal{O}, a) \geqslant \Lambda_1(\mathcal{Q}, a)$ whenever $\mathcal{O} \subseteq \mathcal{Q}$. Since a will be fixed, but we will consider different domains Ω, we will denote $\Lambda_1(\Omega, a)$ simply as $\Lambda_1(\Omega)$.

Chapter 3

A singular boundary value problem in bounded domains

Let $\Omega \subseteq \mathbb{R}^N$ be an open set. Let $h : \Omega \times (0, \infty) \to [0, \infty)$ be continuous and consider the problem

$$\begin{cases} -\Delta_\infty u = h(x, u) \text{ in } \Omega \\ \quad u(x) = \varpi \;\; \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\varpi \in C(\partial\Omega, [0, \infty))$

Note that $\varpi \equiv 0$ on $\partial\Omega$ is a possibility and $h(x, t)$ may exhibit singularity at $t = 0$, for instance $h(x, t) = b(x)t^{-\gamma}$ for $\gamma > 0$.

We also note that, by a sub-solution $v \in C(\overline{\Omega})$ (respectively a super-solution $w \in C(\overline{\Omega})$) of the Dirichlet Problem (3.1), we mean a sub-solution (respectively a super-solution) of $-\Delta_\infty u = h(x, u)$ as defined earlier such that $v \leqslant \varpi$ (resp.,$w \geqslant \varpi$) on $\partial\Omega$.

For bounded open set $\Omega \subseteq \mathbb{R}^N$ we have the following theorem.

Theorem 3.1 Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set. Suppose $v \in C(\overline{\Omega})$ is a sub-solution and $w \in C(\overline{\Omega})$ is a super-solution of (3.1) in Ω such that $0 < v \leqslant w$ in Ω and $v = \varpi = w$ on $\partial\Omega$. Then Problem (3.1) admits a solution $u \in C(\overline{\Omega})$ such that $v \leqslant u \leqslant w$ in $\overline{\Omega}$.

Proof: Let $\{\Omega_j\}$ be a sequence of sub-domains of Ω such that

$$\Omega_j \subset\subset \Omega_{j+1} \text{ and } \quad \Omega := \bigcup_{k=1}^{\infty} \Omega_k, \quad (j = 1, 2, \cdots)$$

For instance we can take

$$\Omega_j := \{x \in \Omega : dist(x, \partial\Omega) > \tfrac{1}{j}\}, \quad (j = 1, 2, \cdots).$$

Recalling that $v(x) > 0$ for all $x \in \Omega$, define $\widehat{h} : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\widehat{h}(x,t) := \begin{cases} h(x,t) & \text{for } t \geqslant v(x) \\ h(x, v(x)) & \text{for } t < v(x). \end{cases}$$

We observe that $\widehat{h} \in C(\Omega \times \mathbb{R})$ and satisfies $\sup_{\Omega_j \times I} \left|\widehat{h}(x,t)\right| < \infty$ for any compact interval $I \subseteq \mathbb{R}$, and any integer $j \geqslant 1$. Furthermore, for each positive integer j, v is a sub-solution and w is a super-solution of the following Dirichlet Problem.

$$\begin{cases} -\Delta_\infty u = \widehat{h}(x,u) & \text{in } \Omega_j \\ u = v & \text{on } \partial\Omega_j \end{cases} \tag{3.2}$$

For each j, let $u_j \in C(\overline{\Omega}_j)$ be the minimal solution of the Dirichlet Problem (3.2) with respect to the sub-solution v such that $v \leqslant u_j \leqslant w$ in Ω_j (see [[27], Theorem 2.5]). In particular, $u_{j+1} \geqslant u_j$ on $\partial\Omega_j$, and u_j is a minimal solution. We conclude that $u_j \leqslant u_{j+1}$ in Ω_j. We extend u_j to $\overline{\Omega}$ by defining it to be v on $\overline{\Omega}\backslash\overline{\Omega}_j$. This way, we obtain a sequence $\{u_j\}$ of functions in C($\overline{\Omega}$) such that $v \leqslant u_j \leqslant u_{j+1} \leqslant w$ on $\overline{\Omega}$ for all j.

Next, we define u by $\quad$ u(x):= $\lim\limits_{j\to\infty} u_j(x) \quad (x \in \overline{\Omega})$.

The sequence $\{u_j\}$ is uniformly bounded on $\overline{\Omega}$. We note that $\{u_j\}$ is equicontinuous (see [[8], Theorem 2.4]) on compact subsets of Ω. Therefore, given $\mathcal{O} \subset\subset \Omega$, the sequence $\{u_j\}$ has a subsequence that converges to u uniformly on $\mathcal{O}$. Consequently, we see that $u \in C(\Omega)$. Since $v \leqslant u \leqslant w$ on $\overline{\Omega}$, and $v = u = w$ on $\partial\Omega$, we conclude that $u \in (C\overline{\Omega})$.

It remains to show that $\Delta_\infty u = -\hat{h}(x,u)$ in the viscosity sense. For this we first show that $\Delta_\infty u \geqslant -\hat{h}(x,u)$.

Suppose that $\varphi \in C^2(\Omega)$ and $u - \varphi$ has a local maximum at some $x_0 \in \Omega$. Let j be large enough such that $x_0 \in \Omega_j$. Then

$$u(x) - \varphi(x) \leqslant u(x_0) - \varphi(x_0),\ x \in B_r(x_0) \subset\subset \Omega_j$$

for some $r > 0$. Let $\varepsilon > 0$ be small. Without loss of generality, we suppose that $\{u_k\}_{k \geqslant j}$ converges to u uniformly on $\overline{B}_r(x_0)$.

For $k \geqslant j$, let x_k be a point of maximum of

$$u_k(x) - \left(\varphi(x) + \tfrac{\varepsilon}{2}\left|x - x_0\right|^2\right)$$

on $\overline{B}_r(x_0)$. In particular,

$$u_k(x_k) - (\varphi(x_k) + \tfrac{\varepsilon}{2}\left|x_k - x_0\right|^2) \geqslant u_k(x_0) - \varphi(x_0) \tag{3.3}$$

Since $x_k \in \overline{B}_r(x_0)$, by passing to a sub-sequence if necessary, we assume that $x_k \to \hat{x}$ for some $\hat{x} \in \overline{B}_r(x_0)$. Taking limit in (3.3) as $k \to \infty$ we obtain

$$u(\hat{x}) - \left(\varphi(\hat{x}) + \tfrac{\varepsilon}{2}\left|\hat{x} - x_0\right|^2\right) \geqslant u(x_0) - \varphi(x_0).$$

which is

$$\frac{\varepsilon}{2}\left|\hat{x} - x_0\right|^2) \leqslant u(\hat{x}) - \varphi(\hat{x}) - (u(x_0) - \varphi(x_0)) \leqslant 0,$$

showing that $\hat{x} = x_0$. Thus, $x_k \in B_{r\backslash 2}(x_0)$ for sufficiently large k. Since u_k is a sub-solution and x_k is a point of local maximum of $u_k - (\varphi + \varepsilon\left|x - x_0\right|^2)$ on $B_r(x_0)$, we have

$$\Delta_\infty \varphi(x_k) + \circ(\varepsilon) \geqslant -\widehat{h}(x_k, u_k(x_k)). \tag{3.4}$$

Taking the limit in (3.4) and recalling that $u_k \to u$ locally uniformly on Ω_j we find that

$$\Delta_\infty \varphi(x_0) + \circ(\varepsilon) \geqslant -\widehat{h}(x_0, u(x_0)).$$

Next, we take limit $\varepsilon \to 0$ and conclude that $\Delta_\infty u \geqslant -\widehat{h}(x, u)$ in the viscosity sense.

A similar argument shows that u is a super-solution of $\Delta_\infty u = -\widehat{h}(x, u)$.

Since $u \geqslant v$ in Ω, by the definition of $\widehat{h}$ we see that $\widehat{h}(x,u) = h(x,u)$ in Ω, and thus u is the desired solution of Problem (3.1).

The next two results establish a super-solution and sub-solution to problem (3.1).

Theorem 3.2 Let $\Omega \subseteq \mathbb{R}^N$ be an open set, and $f\colon\ \Omega \times (0,\infty) \to \mathbb{R}$ be a continuous function that satisfies Condition **(F-b)** with b satisfying **(B-w)**. There exists a positive constant λ^*_Ω, depending on g_∞ and $\|w_\Omega\|_\infty$ such that corresponding to any $0 < \lambda < \lambda^*_\Omega$ there is $v_{\Omega,\lambda} \in C(\overline{\Omega})$ which is a super-solution of Problem (1.2).

Proof: We define a continuous function G: $(0,\infty) \times (0,\infty) \to (0,\infty)$ as follows.

$$G(t,\tau) := \begin{cases} t^3 \sup\limits_{t \leqslant s \leqslant \tau} g(s)s^{-3} & 0 < t \leqslant \tau \\ t^3 g(\tau)\tau^{-3} & 0 < \tau \leqslant t. \end{cases}$$

The function $G(t,\tau)t^{-3}$ is non-increasing in t. Let

$$\Gamma(\gamma) = \frac{2}{\gamma}\int_{\gamma/2}^{\gamma} \frac{G(s,\gamma)}{s^3} ds. \tag{3.5}$$

We note that Γ is a C^1 decreasing function. To check that Γ is decreasing, we differentiate (3.5),

$$\begin{aligned} \Gamma'(t) &= -\frac{2}{t^2}\int_{t/2}^{t} \frac{G(s,\gamma)}{s^3} ds + \frac{2}{t}\Big(\frac{G(t,\gamma)}{t^3} - \frac{1}{2}\frac{G(t/2,\gamma)}{(t/2)^3}\Big) \\ &\leqslant -\frac{1}{t}\frac{G(t,\gamma)}{t^3} + \frac{2}{t}\frac{G(t,\gamma)}{t^3} - \frac{1}{t}\frac{G(t/2,\gamma)}{(t/2)^3} \\ &= \frac{1}{t}\frac{G(t,\gamma)}{t^3} - \frac{1}{t}\frac{G(t/2,\gamma)}{(t/2)^3} \ \leqslant\ 0 \end{aligned} .$$

Moreover, we have

$$\frac{g(\gamma)}{\gamma^3} = \frac{G(\gamma,\gamma)}{\gamma^3} \leqslant \Gamma(\gamma) \leqslant \frac{G(\gamma/2,\gamma)}{(\gamma/2)^3} = \sup_{\gamma/2 \leqslant t \leqslant \tau} \frac{g(t)}{t^3} \leqslant \sup_{\gamma/2 \leqslant t < \infty} \frac{g(t)}{t^3} \tag{3.6}$$

Consequently, we have

$$\lim_{\gamma\to\infty}\Gamma(\gamma)=g_\infty.$$

Now, suppose that $g_\infty<\infty$. Set

$$\lambda_\Omega^*:=\frac{1}{g_\infty\left\|w_\Omega\right\|_\infty^3} \tag{3.7}$$

For $0<\lambda<\lambda_\Omega^*$ we have

$$\lim_{\gamma\to\infty}\Gamma(\gamma)=\frac{1}{\lambda_\Omega^*\left\|w_\Omega\right\|_\infty^3}<\frac{1}{\lambda\left\|w_\Omega\right\|_\infty^3}. \tag{3.8}$$

Consider the function

$$\Psi(t):=\int_0^t\frac{1}{\sqrt[3]{\Gamma(s)}}ds,\quad t>0. \tag{3.9}$$

Note that

$$\lim_{t\to\infty}\frac{\Psi(t)}{t}=\lim_{t\to\infty}\Psi'(t)=\lim_{t\to\infty}\frac{1}{\sqrt[3]{\Gamma(t)}}>\sqrt[3]{\lambda}\left\|w_\Omega\right\|_\infty.$$

Therefore, we can find $\theta\geqslant 1$ large enough such that

$$\Psi(\theta)\geqslant\theta\sqrt[3]{\lambda}\left\|w_\Omega\right\|_\infty. \tag{3.10}$$

On the other hand, suppose $g_\infty=\infty$. In this case, the function Ψ defined as in (3.9) satisfies

$$\alpha_*:=\sup_{t\geqslant 1}\frac{\Psi(t)}{t}<\infty. \tag{3.11}$$

We define

$$\lambda_{\Omega}^{*} := \left(\frac{\alpha_{*}}{\|w_{\Omega}\|_{\infty}} \right)^{3}, \tag{3.12}$$

so that for $0 < \lambda < \lambda_{\Omega}^{*}$. We have

$$\sqrt[3]{\lambda}\, \|w_{\Omega}\|_{\infty} < \sqrt[3]{\lambda_{\Omega}^{*}}\, \|w_{\Omega}\|_{\infty} = \alpha_{*}.$$

Therefore, there is $\gamma_{\infty} \geqslant 1$, depending on λ, such that

$$\sqrt[3]{\lambda}\, \|w_{\Omega}\|_{\infty} < \frac{\Psi(\gamma_{\infty})}{\gamma_{\infty}}. \tag{3.13}$$

Therefore, either (3.10) or (3.13) shows that there is $\theta \geqslant 1$ such that

$$\Psi(\theta) > \theta \sqrt[3]{\lambda}\, \|w_{\Omega}\|_{\infty}. \tag{3.14}$$

Let Φ be the inverse of Ψ, i.e. Φ satisfies

$$\int_{0}^{\Phi(t)} \frac{1}{\sqrt[3]{\Gamma(s)}} ds = t, \quad t \geqslant 0.$$

We now define,

$$v_{\Omega,\lambda}(x) := \Phi(\theta \sqrt[3]{\lambda} w_{\Omega}(x)), \quad (x \in \Omega), \tag{3.15}$$

where θ is as in (3.14), and w_{Ω} is a positive solution of (2.4) in bounded domain Ω, whose existence is guaranteed by Condition **(B-w)**. By the comparison principle, we note that $0 < w_{\Omega} \leqslant w_{b}$ in Ω.

We observe that

$$v_{\Omega,\lambda}(x) := \Phi(\theta\sqrt[3]{\lambda}w_\Omega(x)) \leqslant \Phi(\theta\sqrt[3]{\lambda}\,\|w_\Omega\|_\infty) \leqslant \theta.$$

We have $v_{\Omega,\lambda} > 0$ in Ω, and $v_{\Omega,\lambda} = 0$ on $\partial\Omega$. We now proceed to show that

$$-\Delta_\infty v_{\Omega,\lambda} \geqslant \lambda f(x, v_{\Omega,\lambda}), \quad (x \in \Omega). \tag{3.16}$$

For convenience we drop the subscript from $v_{\Omega,\lambda}$. So, let $(x_0, \varphi) \in \Omega \times C^2(\Omega)$ such that $v - \varphi$ has a local minimum at x_0. Without loss of generality, we can assume that x_0 is a global minimum of $v - \varphi$ in Ω, and that $v(x_0) = \varphi(x_0)$.

Denote $\eta(x) := \dfrac{1}{\theta\sqrt[3]{\lambda}}\Psi(\varphi) \in C^2(\Omega)$.

Then, we have $w_\Omega(x_0) = \eta(x_0)$, and $w_\Omega - \eta$ has a minimum at x_0. Since w_Ω is a viscosity super-solution of Problem (3.4), we see that

$$\Delta_\infty \eta(x_0) \leqslant -b(x_0).$$

We next compute

$$\begin{aligned}\Delta_\infty \eta &= \left(\frac{1}{\theta\sqrt[3]{\lambda}}\right)^3 \Psi''(\varphi)[\Psi'(\varphi)]^2 \,|D\varphi|^4 + \left(\frac{1}{\theta\sqrt[3]{\lambda}}\right)^3 [\Psi'(\varphi)]^3 \Delta_\infty \varphi \\ &\geqslant \left(\frac{1}{\theta\sqrt[3]{\lambda}}\right)^3 \frac{1}{\Gamma(\varphi)} \Delta_\infty \varphi.\end{aligned}$$

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The following chain of inequalities hold at x_0.

$$\begin{aligned}
\Delta_\infty \varphi &\leqslant \theta^3 \lambda \Gamma(\varphi) \Delta_\infty \eta \\
&\leqslant -\theta^3 \lambda \Gamma(\varphi) b \\
&\leqslant -\Big(\frac{\theta}{\varphi}\Big)^3 \lambda b g(\varphi) \quad \text{by} \quad (3.6) \\
&\leqslant -\lambda b g(\varphi), \quad \text{since} \quad \varphi(x_0) = v_\lambda(x_0) \leqslant \theta \\
&\leqslant -\lambda f(x_0, \varphi) \ \text{ by Condition } \ \text{(F-b)}. \qquad (3.17)
\end{aligned}$$

This shows that (3.16) holds.

For later use: Since $\theta \geqslant 1$, it follows from (3.17) that $-\Delta_\infty \varphi \leqslant -\lambda b(x) \Gamma(\varphi)$. Therefore,

$$\Delta_\infty v \leqslant -\lambda b(x) \Gamma(v). \qquad (3.18)$$

Remark 3.3 If $g_\infty = 0$ in (2.3), then Problem (1.2) admits a positive super-solution $v_{\Omega,\lambda} \in C(\overline{\Omega})$ for any $\lambda > 0$, provided that the hypotheses in Theorem 3.2 holds.

The following theorem provides a positive sub-solution to Problem (1.2) for an appropriate range of the parameter.

Theorem 3.4 Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain, and f: $\Omega \times (0, \infty) \to \mathbb{R}$ be a continuous function that satisfies Condition **(F-a)**. Given $\lambda > \Lambda_1(\Omega) h_0^{-1}$, there is $\phi_{\Omega,\lambda} \in C(\overline{\Omega}, (0, \infty))$ which is a sub-solution of Problem (1.2) in Ω.

Proof: Fix λ such that $\lambda > \Lambda_1(\Omega) h_0^{-1}$. If the constant h_0 in (2.3) is finite, there is a sufficiently small $\delta := \delta(\lambda)$ with $0 < \delta < t_0$ such that

$$h(t) \geqslant \Big(\frac{\Lambda_1(\Omega) h_0^{-1}}{\lambda}\Big) h_0 t^3 = \frac{\Lambda_1(\Omega)}{\lambda} t^3 \qquad \forall\, 0 < t \leqslant \delta.$$

On the other hand, if $h_0 = \infty$, then given any $\lambda > 0$, there is $0 < \delta := \delta(\lambda) < t_0$ such that

$$h(t) \geqslant \frac{\Lambda_1(\Omega)}{\lambda} t^3 \quad \forall\, 0 < t \leqslant \delta.$$

In any case, let $\phi_{\Omega,\lambda}$ be a positive eigenfunction of Problem (2.5) corresponding to $\Lambda_1(\Omega)$ such that $0 < \Phi_{\Omega,\lambda} \leqslant \min\{1, \delta(\lambda)\}$ in Ω. The above choice of δ, Condition **(F-a)**, together with the assumption on λ show that

$$\lambda f(x, \phi_{\Omega,\lambda}) \geqslant \lambda a(x) h(\phi_{\Omega,\lambda}) \geqslant \Lambda_1(\Omega) a(x) \phi_{\Omega,\lambda}^3 = -\Delta_\infty \phi_{\Omega,\lambda} \tag{3.19}$$

holds in Ω. This completes the proof of the theorem.

Remark 3.5: If $h_0 = \infty$ in (2.3), then under the hypotheses of Theorem 3.4, Problem (1.2) admits a positive sub-solution $\phi_{\Omega,\lambda} \in C(\overline{\Omega})$ for any $\lambda > 0$.

We next state the main result of this section.

Theorem 3.6 Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain, and $f\colon \Omega \times (0, \infty) \to \mathbb{R}$ be a continuous function that satisfies Conditions **(F-a)**, **(F-b)** and that b satisfies **(B-w)**. There exists a positive constant λ_Ω^*, depending on g_∞ and $\|w_b\|_{\Omega,\infty}$, such that for any λ with $\Lambda_1(\Omega) h_0^{-1} < \lambda < \lambda_\Omega^*$, Problem (1.2) admits a solution $u = u_\lambda \in C(\overline{\Omega})$.

Proof: Since $\lambda > \Lambda_1(\Omega)h_0^{-1}$, let $\phi_{\Omega,\lambda}$ be the sub-solution of (1.2) given in Theorem 3.4. Then, we have the following chain of inequalities:

$$\begin{aligned}\Delta_\infty \phi_{\Omega,\lambda} &\geqslant -\lambda f(x,\phi_{\Omega,\lambda}) \ \text{ by } \ (3.19)\\ &\geqslant -\lambda b(x) g(\phi_{\Omega,\lambda}) \text{ by Condition (F-b)}\\ &\geqslant -\lambda b(x)\phi^3\Gamma(\phi_{\Omega,\lambda}) \ \text{ by } \ (3.6)\\ &\geqslant -\lambda b(x)\Gamma(\phi_{\Omega,\lambda}), \quad \text{since } \phi_{\Omega,\lambda} \leqslant 1 \ \text{ in } \Omega,\end{aligned}$$

where Γ is the function introduced in (3.5).

Since $0 < \lambda < \lambda^*_\Omega$, let $v_{\Omega,\lambda}$ be a super-solution of Problem (1.2) as given by Theorem 3.2.

We claim that $\phi_{\Omega,\lambda} \leqslant v_{\Omega,\lambda}$ in Ω. Suppose not. Let

$$\mathcal{O} := \{x \in \Omega : \phi_{\Omega,\lambda}(x) > v_{\Omega,\lambda}(x)\}.$$

Then in $\mathcal{O}$, we have

$$\Delta_\infty \phi_{\Omega,\lambda} \geqslant -\lambda b(x)\Gamma(v_{\Omega,\lambda}) \text{ and } \Delta_\infty v_{\Omega,\lambda} \leqslant -\lambda b(x)\Gamma(v_{\Omega,\lambda}).$$

Recall that $b > 0$ in Ω. By comparison principle, we have $\phi_{\Omega,\lambda} \leqslant v_{\Omega,\lambda}$ in $\mathcal{O}$, which is a contradiction, and hence the claim holds.

As a summary, we see that $0 < \phi_{\Omega,\lambda} \leqslant v_{\Omega,\lambda}$ in Ω with $\phi_{\Omega,\lambda} = v_{\Omega,\lambda} = 0$ on $\partial\Omega$. We now invoke Theorem 3.1 to conclude that Problem (1.2) admits a solution $u \in C(\overline{\Omega})$ such that $\phi_{\Omega,\lambda} \leqslant u \leqslant v_{\Omega,\lambda}$ in $\overline{\Omega}$.

Remark 3.7: (1) If $g_\infty = 0$ and $h_\infty = \infty$ in (2.3), then under the assumption of Theorem 3.6, Problem (1.2) admits a positive solution $u_{\Omega,\lambda} \in C(\overline{\Omega})$ for any $\lambda > 0$.

(2) Given an open set $\Omega \subseteq \mathbb{R}^N$. Since λ^*_Ω, as defined in (3.7) or (3.12), depends on g_∞ and $\|w_\Omega\|_\infty$, where we recall that w_Ω is a solution of Problem (2.4) in Ω given in Condition **(B-w)**. By the standard Comparison Principle, we see that $w_\Omega \leqslant w_{\mathbb{R}^N}$ in Ω, and as a consequence we have $\lambda^*_{\mathbb{R}^N} \leqslant \lambda^*_\Omega$ for any bounded open set $\Omega \subseteq \mathbb{R}^N$. Therefore, for any $0 < \lambda < \lambda^*_{\mathbb{R}^N}$, and any open bounded subset $\Omega \subseteq \mathbb{R}^N$, let $v_{\Omega,\lambda}$ and $v_\lambda := v_{\mathbb{R}^N,\lambda}$ be the super-solution of Problem (1.2) corresponding to Ω and $\mathbb{R}^N$, respectively. The existence of these super-solutions is a consequence of Theorem 3.2. By definition (see(3.15)), we see that $v_{\Omega,\lambda} \leqslant v_\lambda$ on Ω. Therefore, given a bounded open set $\Omega \subseteq \mathbb{R}^N$, we note that $0 < u_{\Omega,\lambda} \leqslant v_\lambda$ on Ω, where $u_{\Omega,\lambda}$ is the solution of Problem (1.2) given in Theorem 3.6.

Chapter 4

A singular boundary value problem in $\mathbb{R}^N$

An alternative and useful reformulation of the definition of viscosity sub-solution and viscosity super-solution can be given in terms of sub-jets and super-jets as described below (see [13]).

The second order **super-jet** of u at $x_0 \in \Omega$, denoted by $J^{2,+}u(x_0)$, is defined as the set of all pairs $(p, X) \in \mathbb{R}^N \times S(N)$ such that

$$u(x) \leqslant u(x_0) + \langle p, x - x_0 \rangle + \tfrac{1}{2}\langle X(x - x_0), x - x_0 \rangle + \circ(|x - x_0|^2) \text{ as } x \to x_0.$$

The closure of $J^{2,+}u(x_0)$, denoted by $\overline{J}^{2,+}u(x_0)$, is defined as the collection of all pairs $(p, X) \in \mathbb{R}^N \times S(N)$ for which there exists a sequence $(x_k, p_k, X_k) \in \Omega \times J^{2,+}u(x_k)$ such that

$$(x_k, p_k, X_k) \to (x_0, p, X).$$

We note that $S(N)$ is the set of all $N \times N$ real symmetric matrices.

Similarly, the second order sub-jet of u at $x_0 \in \Omega$ denoted by $J^{2,-}u(x_0)$, is defined as follows:

The second order **sub-jet** of u at $x_0 \in \Omega$ is the set of all pairs $(p, X) \in \mathbb{R}^N \times S(N)$ such that

$$u(x) \geqslant u(x_0) + \langle p, x - x_0 \rangle + \tfrac{1}{2}\langle X(x - x_0), x - x_0 \rangle + \circ(|x - x_0|^2) \text{ as } x \to x_0.$$

The closure of $J^{2,-}u(x_0)$, denoted by $\overline{J}^{2,-}u(x_0)$, is defined as the collection of all pairs $(p, X) \in \mathbb{R}^N \times S(N)$ for which there exists a sequence $(x_k, p_k, X_k) \in \Omega \times J^{2,-}u(x_k)$ such that

$$(x_k, p_k, X_k) \to (x_0, p, X).$$

Note: From the above definitions, for u: $\Omega \to \mathbb{R}$, it follows that, if
$J^{2,+}u(x) \cap J^{2,-}u(x) \neq \varnothing$, then

$$Du(x) \text{ and } D^2u(x) \text{ exist and } J^{2,+}u(x) \cap J^{2,-}u(x) = \{(Du(x), D^2u(x))\}.$$

We now restate the definition of viscosity sub- and super-solutions of the PDE (partial differential equation) (2.1) in terms of $\overline{J}^{2,\pm}u(x_0)$.

A function $u \in C(\Omega)$ is a sub-solution of the PDE (2.1) in Ω if

$$\langle Xp, p\rangle \geqslant -H(x_0, u(x_0), p), \quad \forall\, x_0 \in \Omega \text{ and } \forall\,(p, X) \in \overline{j}^{2,+}u(x_0). \tag{4.1}$$

Similarly, $u \in C(\Omega)$ is a super-solution of the PDE (2.1) in Ω if

$$\langle Xp, p\rangle \leqslant -H(x_0, u(x_0), p), \quad \forall\, x_0 \in \Omega \text{ and } \forall\,(p, X) \in \overline{j}^{2,-}u(x_0). \tag{4.2}$$

The above definition is equivalent to the one given earlier in chapter 2. This equivalence is presented below. First we state the notion of "small order $\circ$". For $k \geqslant 1$,

$f(x) \leqslant \circ(|x|^k)$ as $x \to 0$ if and only if there is $\omega \in C([0,\infty),[0,\infty))$ such that $\omega(0) = 0$ and $\sup\limits_{x\in B_r\setminus\{0\}} \dfrac{f(x)}{|x|^k} \leqslant \omega(r)$.

Consider the general second order degenerate elliptic PDE

$$F(x, u(x), Du(x), D^2u(x)) = 0 \text{ in } \Omega\ .$$

Here, we may write the PDE (2.1) which is $-\Delta_\infty u = H(x, u, Du)$ as $F(x, u, Du, D^2u) = -\Delta_\infty u - H(x, u, Du) = 0.$

Equivalence of definitions for a sub-solution: Let u: $\Omega \to \mathbb{R}$. The following are equivalent:

(i) For $x_0 \in \Omega$ and $(p, X) \in \overline{J}^{2,+}u(x_0)$, $F(x_0, u(x_0), p, X) \leqslant 0$.

(ii) For any $\varphi \in C^2(\Omega)$, if u - φ attains its maximum at $x_0 \in \Omega$, then $F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leqslant 0$.

To show: (i) $\Rightarrow$ (ii) For $x_0 \in \Omega$ and any $\varphi \in C^2(\Omega)$, suppose $(u-\varphi)(x_0) = \max(u-\varphi)(x)$.

Taylor's expansion of φ at x_0 gives

$$\varphi(x) = \varphi(x_0) + \langle D\varphi(x_0), x-x_0\rangle + \tfrac{1}{2}\langle D^2\varphi(x_0)(x-x_0), x-x_0\rangle + \circ(|x-x_0|^2) \text{ as } x \to x_0.$$

$\Rightarrow \quad u(x) \leqslant u(x_0) + \langle D\varphi(x_0), x-x_0\rangle + \frac{1}{2}\langle D^2\varphi(x_0)(x-x_0), x-x_0\rangle + \circ(|x-x_0|^2)$ as $x \to x_0$, since $(u-\varphi)(x) \leqslant (u-\varphi)(x_0) = 0$.

Thus we get, $(D\varphi(x_0), D^2\varphi(x_0)) \in J^{2,+}u(x_0) \subseteq \overline{J}^{2,+}u(x_0)$.

(ii) $\Rightarrow$ (i) For $x_0 \in \Omega$ and $(p, X) \in J^{2,+}u(x_0)$, we can find non-decreasing continuous function

ω: $[0, \infty) \to [0, \infty)$ such that $\omega(0) = 0$ and

$$u(x) \leqslant u(x_0) + \langle p, x-x_0\rangle + \tfrac{1}{2}\langle X(x-x_0), x-x_0\rangle + |x-x_0|^2\, \omega(|x-x_0|) \text{ as } x \to x_0. \quad (*)$$

By the definition of small order $\circ$, we can find $\omega_0 \in C([0,\infty),[0,\infty))$ such that $\omega_0(0) = 0$, and

$$\omega_0(r) \geqslant \sup_{x\in B_r(x_0)\setminus\{x_0\}} \frac{1}{|x-x_0|^2}\Big\{u(x) - u(x_0) - \langle p, x-x_0\rangle - \frac{1}{2}\langle X(x-x_0), x-x_0\rangle\Big\}.$$

We have that $\omega(r) := \sup_{0\leqslant t\leqslant r} \omega_0(t)$ satisfies $(*)$.

We next define

$$\varphi(x) := \langle p, x-x_0\rangle + \tfrac{1}{2}\langle X(x-x_0), x-x_0\rangle + \psi(|x-x_0|),$$

$$\text{where} \quad \psi(t) := \int_t^{\sqrt{3}t} \Big(\int_s^{2s} \omega(r)dr\Big)ds \geqslant t^2\omega(t)$$

Computing, we get

$$(D\varphi(x_0), D^2\varphi(x_0)) = (p, X) \text{ and } (u-\varphi)(x_0) \geqslant (u-\varphi)(x) \text{ for } x \in \Omega.$$

Hence, for $x_0 \in \Omega$ and $(p, X) \in J^{2,+}u(x_0)$, we have $F(x_0, u(x_0), p, X) \leqslant 0$.

Next for $x_0 \in \Omega$, let $(p, X) \in \overline{J}^{2,+}u(x_0)$. Then we can find $(p_k, X_k) \in J^{2,+}u(x_k)$ with $x_k \in \Omega$ such that

$$\lim_{k\to\infty}(x_k, p_k, X_k) = (x_0, p, X) \text{ and } \quad F(x_k, u(x_k), p_k, X_k) \leqslant 0,$$

Sending $k \to \infty$ we obtain $F(x_0, u(x_0), p, X) \leqslant 0$

This implies that (i) holds.

Note: In view of the above equivalence, we see that for each $x_0 \in \Omega$,

$$J^{2,+}u(x_0) = \Big\{(D\varphi(x_0), D^2(x_0)) \in \mathbb{R}^N \times S(N) : \exists \varphi \in C^2(\Omega) \text{ such that } u - \varphi \\ \text{attains its maximum at} x_0\Big\}.$$

Similar holds for a super-solution.

Let us next consider the case $\Omega = \mathbb{R}^N$, and then we look at the following problem in $\mathbb{R}^N$.

$$\begin{cases} -\Delta_\infty u = \lambda f(x, u), \ u > 0 \ \text{ in } \mathbb{R}^N \\ u \to 0 \qquad \text{as } |x| \to \infty \end{cases} \tag{4.3}$$

Here, it is assumed that f satisfies **(F-a)** and **(F-b)** with $\Omega = \mathbb{R}^N$. We will also assume that condition **(B-w)** holds for $\Omega = \mathbb{R}^N$, so that Problem (2.4) admits a solution $w_{\mathbb{R}^N}$. In this case, the boundary condition in problem (2.4) is understood in the sense that $w(x) \to 0$ as $|x| \to \infty$.

We recall that, by Remark 3.7, $\|w_\Omega\|_\infty \leqslant \|w\|_\infty$ for any bounded domain $\Omega \subseteq \mathbb{R}^N$.

The following lemma (Source: Shigeaki Koike, A Beginner's Guide to the Theory of Viscosity Solutions, June 28, 2012, Saitama University, Japan; see also [13]) will be useful for the completion of the proof of the main result of this section.

Ishii's Lemma: Let u and w be upper semi-continuous functions defined on $\overline{\Omega}$. For $\phi \in C^2(\overline{\Omega} \times \overline{\Omega})$, let $(x_0, y_0) \in \overline{\Omega} \times \overline{\Omega}$ be a point such that

$$\max_{x,y\in\overline{\Omega}} \big(u(x) + w(y) - \phi(x,y)\big) = u(x_0) + w(x_0) - \phi(x_0, y_0).$$

Then for each $\mu > 1$, there are $X = X(\mu),\ Y = Y(\mu)) \in S(N)$ such that

$$\big(D_x\phi(x_0,y_0), X\big) \in \overline{J}_{\overline{\Omega}}^{2,+}u(x_0),\ \ \big(D_y\phi(x_0,y_0), Y\big) \in \overline{J}_{\overline{\Omega}}^{2,+}w(y_0)$$

and

$$-\Big(\mu + \|A\|\Big)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leqslant \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leqslant A + \frac{1}{\mu}A^2,$$

where $A = D^2\phi(x_0,y_0) \in S(2N)$.

Note: If we suppose that $u, w \in C^2(\overline{\Omega})$ and $(x_0, y_0) \in \Omega \times \Omega$, in the hypothesis of Ishii's Lemma, then we have,

$$X = D^2u(x_0),\ Y = D^2w(y_0), \text{ and } \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leqslant A$$

The last matrix inequality in the Lemma indicates that when u and w are only continuous, we get an error terms $\mu^{-1}A^2$, where in such a case $\mu > 1$ will be large. It is also to be noted that for $\phi(x,y) = \dfrac{1}{2\varepsilon}|x-y|^2$, we have

$$A = D^2\phi(x_0,y_0) = \frac{1}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix},\ A^2 = \tfrac{1}{\varepsilon}A \text{ and } \|A\| = \frac{1}{\varepsilon}$$

where $\quad \|A\|^2 := \sup\Big\{\langle A\begin{pmatrix} x \\ y \end{pmatrix}, A\begin{pmatrix} x \\ y \end{pmatrix}\rangle : |x|^2 + |y|^2 = 1\Big\}.$

In general, we may also use for $A \in S(N)$, $\|A\| = \max\big\{\,|\lambda_k| : \lambda_k \text{ is an eigenvalue of } A\big\}$.

Let μ be chosen to be $1/\varepsilon$. We have $0 < \varepsilon = \frac{1}{\mu} < 1$. Then, we have a simplified version of Ishii's Lemma,

$$-\frac{3}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leqslant \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leqslant \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Next, let $w(y) = -v(y)$. Then, with $\phi(x,y)$ as considered above, we get,

$$-\frac{3}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leqslant \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

For the remainder of this section, we will assume that $a(x)$ in Condition **(F-a)** is positive in Ω.

We now state the main result of this section.

Theorem 4.1 Let f satisfy conditions **(F-a)** and **(F-b)** in $\Omega = \mathbb{R}^N$, with b satisfying condition **(B-w)**. There exists a positive constant λ^* such that for any λ with $\Lambda_1(B(O,1))h_0^{-1} < \lambda < \lambda^*$, Problem (4.3) admits a solution $u = u_\lambda \in C(\mathbb{R}^N)$.

To prove this theorem we first state and prove the following lemma.

Lemma 4.2 Suppose f satisfies condition **(F-a)** with $a \in C(\Omega, (0,\infty))$, and let $\lambda > \Lambda_1(B(O,1))h_0^{-1}$. Given a positive integer ℓ, there is an eigenvalue $\varphi_\ell := \varphi_{B_\ell}$ of (2.5), depending on λ such that for any solution $u_k := u_{B_k,\lambda}$ of (4.3) in $B_k := B(O,k)$ we have

$$u_k \geqslant \varphi_\ell \quad \text{in} \;\; B(O,\ell) \;\forall k > \ell.$$

Proof: For each $(x,t) \in \mathbb{R}^N \times (0,\infty)$, let

$$\widehat{h}(x,t) = \inf\left\{ \frac{f(x,s)}{a(x)s^3} \;:\; 0 < s \leqslant t \right\}.$$

By assumption

$$\lambda > \Lambda_1(B(O,1))h_0^{-1}$$

where h_0 is as defined in (2.3) and h_0^{-1} is interpreted to be zero if $h_0 = \infty$.

Let us fix τ such that

$$\Lambda_1(B(0,1)) < \tau < h_0\lambda. \tag{4.4}$$

Then by definition of h_0, there is $0 < t^* := t^*(\lambda) \leqslant t_0$ such that $h(t) \geqslant (\tau/\lambda)t^3$ for $0 < t < t^*$. Therefore,

$$f(x,t) \geqslant a(x)h(t) \geqslant (\tau/\lambda)a(x)t^3,\ \ (x,t) \in \Omega \times (0,t^*). \tag{4.5}$$

$$\widehat{h}(x,t) \geqslant \frac{\tau}{\lambda},\quad (x,t) \in \mathbb{R}^N \times (0,t^*). \tag{4.6}$$

Next, let φ_ℓ be an eigenfunction of Problem (2.5) corresponding to the principal eigenvalue $\Lambda_\ell(B(O,\ell))$ on the ball $B_\ell := B(O,\ell)$. We assume that φ_ℓ has been normalized so that $0 < \varphi_\ell \leqslant t^*$ in $B(O,\ell)$.

For notational convenience, let

$v\colon = \varphi_\ell$, u:$= u_k$, $\Lambda_1 := \Lambda_1(B(O,1))$ and $\Lambda_\ell := \Lambda_1(B(O,\ell))$.

Suppose the conclusion of Lemma 4.2 in not true. Since $v \leqslant u$ on the boundary $\partial B(O,\ell)$, it follows that v/u attains its maximum in $B(O,\ell)$, and therefore

$$\log\left(\frac{v}{u}\right) = \log v - \log u$$

attains its positive maximum at some point $x_0 \in B(O, \ell)$. Put,

$\alpha(x) = \log v(x)$, and $\beta(x) = \log u(x)$.

Then, computation shows that

$$\Delta_\infty \alpha = -\frac{1}{v^4} |Dv|^4 + \frac{1}{v^3} \Delta_\infty v = -|D\alpha|^4 - \Lambda_\ell \alpha \quad \text{and} \tag{4.7}$$

$$\Delta_\infty \beta = -\frac{1}{u^4} |Du|^4 + \frac{1}{u^3} \Delta_\infty u = -|D\beta|^4 \text{ - } \frac{\lambda f(x,u)}{u^3}. \tag{4.8}$$

Thus we have,

$a\Lambda_\ell = -\Delta_\infty \alpha - |D\alpha|^4$ and $\dfrac{\lambda f(x,u)}{u^3} = -\Lambda_\infty \beta - |D\beta|^4$.

Note that the function $\alpha - \beta = \log\left(\frac{v}{u}\right)$ has a positive maximum in $\overline{B}(O, \ell)$, that is M:= $\max_{\overline{B}(O,\ell)} (\alpha - \beta) > 0$.

Next put

$$\Phi_\varepsilon(x, y) = \alpha(x) - \beta(y) - \frac{1}{2\varepsilon} |x - y|^2, \ \ (x, y) \in \overline{B}(O, \ell) \times \overline{B}(O, \ell). \tag{4.9}$$

Let us define,

$$M_\varepsilon := \Phi_\varepsilon(x_\varepsilon, y_\varepsilon) = \max_{\overline{B}(O,\ell)\times\overline{B}(O,\ell)} \Phi_\varepsilon(x, y), \text{ for some } (x_\varepsilon, y_\varepsilon) \in \overline{B}(O, \ell) \times \overline{B}(O, \ell)$$

Without loss of generality, we can assume that the sequence $\{(x_\varepsilon, y_\varepsilon)\}$, as $\varepsilon \to 0$, converges to $(z, z) \in \overline{B}(O, \ell) \times \overline{B}(O, \ell)$ where M $= (\alpha - \beta)(z)$. Since $M > 0$, we must have $z \in \Omega$. Consequently, we can suppose $(x_\varepsilon, y_\varepsilon) \in B(O, \ell) \times B(O, \ell)$ for sufficiently small $\varepsilon > 0$. By the Maximum Principle of Ishii (see ([18], Lemma 1) or ([13], Theorem 3.2)), there are $X_\varepsilon, Y_\varepsilon \in S(N)$ such that

$$(\eta_\varepsilon, X_\varepsilon) \in \overline{J}^{2,+}\alpha(x_\varepsilon), \qquad (-\eta_\varepsilon, Y_\varepsilon) \in \overline{J}^{2,-}\beta(y_\varepsilon) \quad \text{and}$$

$$-\frac{3}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leqslant \begin{pmatrix} X_\varepsilon & 0 \\ 0 & -Y_\varepsilon \end{pmatrix} \leqslant \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

where $\eta_\varepsilon := \frac{1}{\varepsilon}(x_\varepsilon, y_\varepsilon)$.

It follows from the last inequality that $X_\varepsilon \leqslant Y_\varepsilon$. From the definition of sub-solution and super-solution applied to equation (4.7) and (4.8) respectively, we get

$$-\Lambda_\ell a(x_\varepsilon) \leqslant \langle X_\varepsilon \eta_\varepsilon, \eta_\varepsilon \rangle + |\eta_\varepsilon|^4 \text{ and } \langle Y_\varepsilon \eta_\varepsilon, \eta_\varepsilon \rangle + |\eta_\varepsilon|^4 \leqslant -\frac{\lambda f(y_\varepsilon, u(y_\varepsilon))}{u^3(y_\varepsilon)}.$$

Recalling that $X_\varepsilon \leqslant Y_\varepsilon$, we conclude

$$\frac{\lambda f(y_\varepsilon, u(y_\varepsilon))}{u^3(y_\varepsilon)}) \leqslant \Lambda_\ell a(x_\varepsilon).$$

Taking the limit as $\varepsilon \to 0$, we obtain the inequality

$$\frac{\lambda f(z, u(z))}{u^3(z)}) \leqslant \Lambda_\ell a(z).$$

Therefore,

$$\lambda \hat{h}(z, u(z)) \leqslant \Lambda_\ell \leqslant \Lambda_1.$$

Since $v(z) > u(z)$ by assumption, and since $\hat{h}(x,t)$ is non-increasing in t for each x, we conclude that

$$\lambda \hat{h}(z, v(z)) \leqslant \Lambda_1.$$

Putting this last inequality together with the inequality in (4.6) we conclude that

$$\tau \leqslant \Lambda_1. \tag{4.10}$$

The inequality (4.10) is a contradiction to the original choice in (4.4) that $\tau > \Lambda_1(B(O,1))$. Therefore, the claim that $u_k \geqslant \varphi_\ell$ in $B(O,\ell)$ for all $k > \ell$ holds.

Proof of Theorem 4.1 Let $\lambda^* := \lambda^*_{\mathbb{R}^N}$ be the positive constant in Theorem 3.6 corresponding to $\Omega = \mathbb{R}^N$. Let $\Lambda_1(B(O,1))h_0^{-1} < \lambda < \lambda^*$. For each positive integer k, let $B_k := B(O,k)$ be the ball of radius k centered at the origin O. Since $\Lambda_1(B(O,k)) \leqslant \Lambda_1(B(O,1))$, we see that $\Lambda_1(B(O,k))h_0^{-1} < \lambda < \lambda^*_{B(O,k)}$. Since f satisfies conditions **(F-a)**, **(F-b)** (and b satisfies condition **(B-w)**) on $\Omega = B(O,k)$, we invoke Theorem 3.6 to pick a solution $u_k := u_{B_k,\lambda} \in C(\overline{B_k})$, for $k = 1, 2, \cdots$, of the following boundary value problem.

$$\begin{cases} -\Delta_\infty u = \lambda f(x,u) \text{ in } B_k \\ u(x) = 0 \text{ on } \partial B_k \end{cases} \tag{4.11}$$

According to Remark 3.7 we have $0 < u_k \leqslant v_\lambda$ in B_k. (4.12)

where v_λ is the super-solution of (1.2) given by Theorem 3.2 in $\Omega = \mathbb{R}^N$.

The inequality (4.12) shows that $\{u_k\}$ is uniformly bounded in $\mathbb{R}^N$. Therefore, the sequence $\{u_k\}$ is locally uniformly Lipschitz (see [[9], Theorem 2.4]). A standard Cantor diagonalization type argument produces a sub-sequence, which we still denote by $\{u_k\}$, that converges locally uniformly to some $u_\lambda \in C(\mathbb{R}^N)$ and that $0 \leqslant u_\lambda \leqslant v_\lambda$ in $\mathbb{R}^N$.

Next, fix an arbitrary positive integer ℓ, and take an eigenfunction φ_ℓ of (2.5) as in Lemma 4.2, so that

$$0 < \varphi_\ell \leqslant u_k \quad \text{in } B(O,\ell)\ \forall k > \ell.$$

As a consequence of this inequality, we have

$$0 < u_\lambda \leqslant v_\lambda \quad \text{in } B(O,\ell).$$

Since ℓ is arbitrary, we conclude that $u_\lambda > 0$ in $\mathbb{R}^N$. An argument similar to the one used in the proof of Theorem 3.1 shows that u_λ is actually a viscosity solution of Problem (4.3). Finally, it follows from $0 < u_\lambda \leqslant v_\lambda$ in $\mathbb{R}^N$ that $u(x) \to 0$ as $|x| \to \infty$. This completes the proof of Theorem 4.1.

Remark 4.3: Suppose that $g_\infty = 0$ and $h_0 = \infty$ in (2.3). If the conditions of Theorem 4.1 are met, then problem (1.2) admits a positive solution $u_\lambda \in C(\mathbb{R}^N)$ for any $\lambda > 0$.

Chapter 5

Appendix: An eigenvalue problem for the ∞-Laplacian

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain, and $a \in C(\Omega, [0, \infty))$ such that $a(x) \not\equiv 0$ in Ω. Here we are interested in the eigenvalue problem

$$\begin{cases} -\Delta_\infty u = \lambda a(x) u^3 \ \text{ in } \Omega \\ \qquad u = 0 \quad \text{ on } \partial\Omega. \end{cases} \tag{A.1}$$

To study Problem (A.1) we first study the following related perturbation problem.

$$\begin{cases} -\Delta_\infty u = \lambda a(x) u^3 + a(x) \ \text{ in } \Omega \\ \qquad u = 0 \quad \text{ on } \partial\Omega. \end{cases} \tag{A.2}$$

Consider the set,

$$E_a := \{\lambda \in \mathbb{R} : \text{Problem (A.2) admits a solution } v > 0 \ in \ \Omega\}.$$

I. We first show that E_a is bounded above. We adapt an argument used in [20] (see also [7]).

Note that if $\lambda < 0$, then $v \equiv -\lambda^{-1/3}$ solves $-\Delta_\infty v = \lambda a(x) v^3 + a(x)$. Therefore, by Theorem 3.1 of [9], Problem (A.2) has a positive solution in Ω.

If $\lambda = 0$, then a solution $w \in C(\overline{\Omega})$ of Problem (A.2) exists with $w > 0$ in Ω (see[8, 9]). Therefore, we see that $(-\infty, 0] \subseteq E_a$.

Next we show that E_a is bounded above. Since a is non-trivial, there is $x_0 \in \Omega$ such that $a(x_0) > 0$.

By continuity, there is $r > 0$ such that $B_r(x_0) \subseteq \Omega$, and $0 < \vartheta := \min_{x \in \overline{B}_r(x_0)} a(x)$.

Now, suppose $\lambda > 0$ and $-\Delta_\infty u = \lambda a(x)u^3 + a(x)$ in Ω for some $u \in C(\overline{\Omega})$ with $u > 0$ in Ω.

Let $\ell := \min_{\overline{B}_r(x_0)} u$ and $m(s) := \min_{\overline{B}_s(x_0)} u$ for $0 < s < dist(x_0, \partial\Omega)$.

We note that m is non-increasing on the interval $(0, dist(x_0, \partial\Omega))$. Moreover,

$$u(x) \geqslant m(|x - x_0|) \geqslant m(t) \quad \forall x \in B_t(x_0) \quad \text{and} \quad 0 < t < dist(x_0, \partial\Omega).$$

Let us consider the following initial value problem where $\eta > 0$ is a constant to be chosen.

$$(w'(s))^2 w''(s) = -\lambda\vartheta m(s)^3 - \vartheta, \quad w(0) = \eta, \quad w'(0) = 0.$$

This has a solution defined on (0, r) and we choose η so that $w(r) = \ell$. We now define

v in $B_r(x_0)$ as

$$v(x) := w(|x - x_0|), \quad x \in B_r(x_0).$$

One can show that v is a viscosity solution of

$$\Delta_\infty v = -\lambda\vartheta m(|x - x_0|)^3 - \vartheta; \quad \text{in } B_r(x_0) \text{ and } v = \ell \text{ on } \partial B_r(x_0).$$

Note that u satisfies

$$\Delta_\infty u = -\lambda a(x)u^3 - a(x) \leqslant -\lambda\vartheta m(|x - x_0|)^3 - \vartheta \quad \text{in } B_r(x_0) \text{ and } u \geqslant \ell \text{ on } \partial B_r(x_0).$$

Since $\lambda\vartheta m(|x - x_0|)^3 + \vartheta \geqslant 0$, by the comparison principle we find that $v \leqslant u$ in $B_r(x_0)$. Thus, given $0 < s \leqslant r$ and $x \in \partial B_s(x_0)$, for any $y \in \overline{B}_s(x_0)$, we see that

$$w(s) = w(|x - x_0|) = v(x) = v(x_s) \leqslant u(x_s) := \min_{\partial B_s(x_0)} u \leqslant u(y) .$$

It follows that $v(x) \leqslant m(s) = m(|x - x_0|)$ for all $x \in B_r(x_0)$.

Therefore,

$$\Delta_\infty v = -\lambda\vartheta m(|x - x_0|)^3 - \vartheta \leqslant -\lambda\vartheta v^3 - \vartheta \;\text{ in } B_r(x_0).$$

That is

$$(w'(s))^2 w''(s) \leqslant -\lambda\vartheta w(s)^3 \;\text{ in } B_r(x_0).$$

Note that $w' < 0$ in (0, r). Multiplying both sides of the above inequality by w' and integrating, we have

$$w'(s) \;\leqslant\; -\sqrt{2}\Big(\int_{w(s)}^{w(0)} \lambda\vartheta\xi^3\, d\xi\Big)^{1/4} = -\sqrt[4]{\lambda\vartheta(w(0)^4 - w(s)^4)},$$

from which it follows that

$$\int_{w(s)}^{\eta} \frac{1}{\sqrt[4]{\eta^4 - t^4}}\, dt \geqslant \sqrt[4]{\lambda\vartheta}\, s.$$

Therefore, we get the estimate

$$\begin{aligned}
\sqrt[4]{\lambda} \;&\leqslant\; \frac{1}{r\sqrt[4]{\vartheta}} \int_{\ell}^{\eta} \frac{1}{\sqrt[4]{\eta^4 - t^4}}\, dt \\
&\leqslant\; \frac{1}{r\sqrt[4]{\vartheta\eta^3}} \int_{\ell}^{\eta} (\eta - t)^{-1/4}\, dt \\
&=\; \frac{4}{3r\sqrt[4]{\vartheta}} \Big(\frac{\eta - \ell}{\eta}\Big)^{3/4} \leqslant\; \frac{4}{3r\sqrt[4]{\vartheta}}
\end{aligned}$$

This completes the proof of the boundedness of E_a.

II. Denote

$$\Lambda_1(\Omega, a) := \sup E_a \tag{A.3}$$

We next show that $\Lambda_1 := \Lambda_1(\Omega, a)$ (This is for notational convenience), as defined in (A.3), is an eigenvalue of Problem (A.1) that admits a positive eigenfunction. That is, we wish to show that there is $u \in C(\overline{\Omega})$ with $u > 0$ in Ω such that

$$\begin{cases} -\Delta_\infty u = \Lambda_1 a(x) u^3 & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{A.4}$$

Theorem A.1 Let $a \in C(\Omega, [0, \infty)) \cap L^\infty(\Omega)$ be non-trivial. Then Problem (A.4) has a solution $u \in C(\overline{\Omega})$ with $u > 0$ in Ω.

Proof: The idea of the proof is taken from [20]. Let λ_k be an increasing sequence of real numbers such that $\lambda_k \to \Lambda_1$. Let $w_k \in C(\overline{\Omega})$ be a positive solution of Problem (A.2) corresponding to λ_k.

We claim that $\{\sup_\Omega w_k\}$ is an unbounded sequence. Otherwise, suppose $\|\lambda a(x) w_k^3 + a(x)\|_{L^\infty(\Omega)} \leqslant M$. Then, if $\tilde{w} \in C(\overline{\Omega})$ such that $\Delta_\infty \tilde{w} = -M$ in Ω, and $\tilde{w} = 0$ on $\partial\Omega$, since obviously $\Delta_\infty w_k \geqslant -M$, by the Comparison Principle we conclude that

$$0 \leqslant w_k(x) \leqslant \tilde{w}(x), \quad x \in \overline{\Omega}\ .$$

Therefore, the sequence $\{w_k\}$ is equicontinuous and hence has a subsequence that converges locally uniformly to a solution $w \in C(\overline{\Omega})$ of

$$\begin{cases} -\Delta_\infty w = \Lambda_1 a(x) w^3 + a(x) & \text{in } \Omega \\ \quad w = 0 & \text{on } \partial\Omega. \end{cases} \tag{A.5}$$

By the Harnack inequality (see [6]), we see that $w > 0$ in Ω.

Let $w_\varepsilon := w + \varepsilon$, where $0 < \varepsilon < 1$ is to be chosen. Then,

$$
\begin{aligned}
-\Delta_\infty w_\varepsilon &= \Delta_\infty w \\
&= \Lambda_1 a(x) w^3 + a(x) \\
&= \Lambda_1 a(x)[(w+\varepsilon)^3 - 3(w+\varepsilon)^2\varepsilon + 3(w+\varepsilon)\varepsilon^2 - \varepsilon^3] + a(x) \\
&= \Lambda_1 a(x) w_\varepsilon^3 + a(x)\Big[\frac{1}{2} - 3\Lambda_1(w+\varepsilon)^2\varepsilon + 3\Lambda_1(w+\varepsilon)\varepsilon^2 - \Lambda_1\varepsilon^3\Big] + \frac{1}{2}a(x) \\
&\geqslant a(x)\Big\{\Lambda_1 w_\varepsilon^3 + \Big[\frac{1}{2} - 3\varepsilon\Lambda_1(1+\sup_\Omega w)^2\Big]\Big\} + \frac{1}{2}a(x),
\end{aligned}
$$

this is because $3(w+\varepsilon)\varepsilon^2 - \varepsilon^3 \geqslant 0$.

Therefore, if we let $z_\varepsilon := \sqrt[3]{2}\, w_\varepsilon$ we then obtain

$$
-\Delta_\infty z_\varepsilon \geqslant a(x)\Big\{\Lambda_1 z_\varepsilon^3 + \Big[1 - 6\varepsilon\Lambda_1(1+\sup_\Omega w)^2\Big]\Big\} + a(x).
$$

Now, choose $0 < \varepsilon < 1$ such that $6\varepsilon\Lambda_1(1+\sup_\Omega w)^2 < 1$. Then we note that,

$$
\begin{aligned}
\Lambda_1 z_\varepsilon^3 + \big[1 - 6\varepsilon\Lambda_1(1+\sup_\Omega w)^2\big] &= \Lambda_1 z_\varepsilon^3 + \Big[1 - 6\varepsilon\Lambda_1(1+\sup_\Omega w)^2\Big]\frac{z_\varepsilon^3}{2(w+\varepsilon)^3} \\
&\geqslant \Big[\Lambda_1 + \frac{1 - 6\varepsilon\Lambda_1(1+\sup_\Omega w)^2}{2(1+\sup_\Omega w)^3}\Big] z_\varepsilon^3,
\end{aligned}
$$

since $w + \varepsilon \leqslant 1 + \sup_\Omega w$.

Thus, we find that $\quad -\Delta_\infty z_\varepsilon \geqslant \mu a(x) z_\varepsilon^3 + a(x)$,

where $\quad \mu := \Lambda_1 + \dfrac{1 - 6\varepsilon\Lambda_1(1+\sup_\Omega w)^2}{2(1+\sup_\Omega w)^3}$.

Therefore, we observe that $v \equiv 0$ is a sub-solution of (A.5) and z_ε is a positive super-solution of Problem (A.5). Hence by Theorem 3.1 of [9], Problem (A.5) with $\lambda := \mu > \Lambda_1$ has a solution u with $0 \leqslant u \leqslant z_\varepsilon$.

Note that u is a non-trivial solution. Therefore, by the Harnack inequality (see[6, 22]), we note that u is a positive solution of (A.5). This is a contradiction to the definition of Λ_1 given in (A.3). Thus, $\{\sup_\Omega w_k\}$ is unbounded, as claimed.

Next, let

$$v_k := \frac{w_k}{\sup\limits_\Omega w_k}, \quad k = 1, 2, \cdots.$$

Then

$$-\Delta_\infty v_k = \frac{1}{(\sup\limits_\Omega w_k)^3}\big(\lambda_k a(x) w_k^3 \ + \ a(x)\big) = \lambda_k a(x) v_k^3 \ + \ \frac{a(x)}{(\sup\limits_\Omega w_k)^3}, \tag{A.7}$$

Let $N > 0$ such that

$$\left\| \lambda_k a(x) v_k^3 \ + \ \frac{a(x)}{(\sup\limits_\Omega w_k)^3} \right\|_{L^\infty(\Omega)} \leqslant N \qquad \text{for all } k \geqslant 1.$$

Let $\underline{w}$, $\overline{w} \in C(\overline{\Omega})$ such that

$$\Delta_\infty \underline{w} = N, \qquad \Delta_\infty \overline{w} = -N, \quad \text{in } \ \Omega \quad \text{ and } \ \underline{w} = 0 = \overline{w} \quad \text{on } \partial\Omega.$$

By the comparison principle, we see that

$$\underline{w}(x) \leqslant v_k(x) \leqslant \overline{w}(x) \qquad x \in C(\overline{\Omega}), \quad k = 1, 2, \cdots \tag{A.8}$$

Since $\{v_k\}$ is bounded, again, we can extract a subsequence that converges locally uniformly to $v \in C(\overline{\Omega})$ and hence letting $k \to \infty$ in the last equation (A.7), we find that

$$-\Delta_\infty v = \Lambda_1 a(x) v^3.$$

Note that $\underline{w} \leqslant v_k \leqslant \overline{w} \quad \text{in } C(\overline{\Omega})$, and therefore $v = 0$ on $\partial\Omega$.

We also note that $\sup_\Omega v = 1$. To see this, suppose that $\sup_\Omega v_k = v_k(x_k) = 1$ for $x_k \in \Omega$. Then a subsequence $\{x_k\}$ converges to some $x_0 \in C(\overline{\Omega})$. Suppose $x_0 \in \partial\Omega$. Then according to (A.8) we have $1 = v_k(x_k) \leqslant \overline{w}(x_k)$ and this implies that $\overline{w}(x_0) \geqslant 1$ which is not possible. Therefore $x_0 \in \Omega$. Thus $v_k(x_k) \to v(x_0) = 1$. Since $v \geqslant 0$ and $v(x_0) = 1$, we conclude, by Harnack inequality, that $v > 0$ in Ω. Thus, we have shown that $\Lambda_1(\Omega, a)$ is an eigenvalue, as claimed.

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Declaration

This dissertation is my original work and has not been presented for a degree in any other university, and that all sources of the materials used for the dissertation have been duly acknowledged.

Dr. Seid Mohammed (Advisor)

Signature ____________________

Prof. Ahmed Mohammed (Advisor)

Signature ______________________

Tesfa Biset

Signature ______________________