SPECTRAL THEORY OF BOUNDED SELF-ADJOINT LINEAR OPERATORS

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Declaration

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship, or any other similar title to me.

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February 2013 **GEMECHU ADEM TUKE**

PERMISSION

This is to certify that this project is compiled by ***Mr. GEMECHU ADEM TUKE in*** *the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.*

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February 2013 **Mengistu Goa Sangago (PhD)**

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Abstract

Spectral theory is very broad but important aspect of applied functional analysis which extends to eigenvectors and eigenvalue theory of a square matrix. The name was introduced by David Hilbert in his original formulation of Hilbert space theory. It was later discovered that spectral theory could explain features of atomic spectral in quantum mechanics.

While solving the system of linear algebraic equations, differential equations or integral equations, we come across the problem related to inverse operation. Spectral theory is concerned with such inverse problems.

In this paper we consider the spectrum of a hermitian (or self-adjoint) operator. We also study the Hilbert-adjoint operator $\boldsymbol{T}^*$ of the bounded linear operator T and the Riesz representation of bounded linear functional by inner products

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CHAPTER ONE

PRELIMINARY CONCEPT

1.1 BASIC DEFINITION

<u>1.1.2 Definition</u>: Let X be a vector space over a field K. Then a mapping $\|.\|: X \to [0, \infty)$ is said to be a norm on X if for each x, y$\in X$, $\lambda \in K$

1.$\|x\| \geq 0$ and $\|x\| = 0$ if and $only$ if $x = 0$ $(positive\ definiteness)$

2.$\|\lambda x\| = |\lambda|\|x\|$ $(Homogeneity)$

3.$\|x + y\| \leq \|x\| + \|y\|$ $(triangle\ inequality)$

If norm is defined on a vector space X, then X is said to be a **normed space**

<u>1.1.3 Definition</u>: A linear functional on Hilbert space H is a linear map from H to $\mathbb{C}$. That is $\varphi: H \to \mathbb{C}$

<u>1.1.4 Definition</u>: A linear functional φ is bounded, or Continuous, if there exists a constant k such that

$$|\varphi(x)| \leq k\|x\| \quad (4)$$

The norm of a bounded linear functional is

$$\|\varphi\| = sup_{\|x\|=1}|\varphi(x)| \quad (5)$$

If $y \in H, then$ $\varphi_y(x) = \langle y, x\rangle$ is a bounded linear functional on H, with $\|\varphi_y\| = \|y\|$

<u>1.1.5 Definition</u>: Let X be a vector space over K and let $\langle ., . \rangle: X \times X \to K$ where for all $x, y, z \in X$

1. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ $(linearity)$

2. $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ $(Homogeneity)$

3. $\langle x, y\rangle = \overline{\langle y, x\rangle}$ $(conjugate\ symmetry)$

4. $\langle x, x\rangle \geq 0\ and\ \langle x, x\rangle = 0$ if and only if $x = 0$ $(positive\ definiteness)$

Then the pair $(X, \langle .,.\rangle)$ is said to be an **inner product space.**

1.1.6 Definition: A Hilbert space is a complete inner product space.

Examples of Hilbert spaces

1. The Euclidean space $\mathbb{R}^n$ is a Hilbert space with inner product defined by

$\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n$.Where $x = (x_1, x_2, \cdots x_n)$ and $y = (y_1 \cdots y_n)$. We obtain that

$\|x\| = \langle x, x\rangle^{\frac{1}{2}} = ({x_1}^2 + \cdots + {x_n}^2)$ and metric defined on it is given

$d(x, y) = \|x - y\| = \langle x - y, x - y\rangle^{\frac{1}{2}} = [(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2]^{\frac{1}{2}}$

2. The unitary space $\mathbb{C}^n : \langle x, y\rangle = x_1\overline{y_1} + \cdots x_n\overline{y_n}$.

3. The Hilbert space $l^2 = \{\sum_{j=1}^{\infty}|x_j|^2 < \infty\}$, with inner product defined by $\langle x, y\rangle = \sum_{j=1}^{\infty} x_j\overline{y_j}$ and the norm $\|x\| = \langle x, x\rangle^{\frac{1}{2}} = \left(\sum_{j=1}^{\infty}|x_j|^2\right)^{\frac{1}{2}}$

1.1.7 Definition: If $T: X \to X$ is a linear operator on linear space X, then $\lambda \in K$ is called **eigenvalue** of T if there exists a non- zero x$\in$ X then

$$Tx = \lambda x \qquad (1)$$

1.1.8 Definition (Resolvent set): Let H be a Hilbert space and let $T: D(T) \to H$ be a linear operator with domain $D(T) \subseteq H$. For any $\lambda \in \mathbb{C}$, we define the operator $T_\lambda = \lambda I - T$.Then R_λ is inverse operator. That is λ is regular value if provided,

a) R_λ exists
b) R_λ is a bounded linear operator;
c) R_λ is defined on a dense subspace of H

The resolvent set of T is the set of all regular values of T

$\rho(T)=\{\lambda \in \mathbb{C}: \lambda \text{ is a regular value of } T\}$

<u>1.1.9 Definition:</u> The set $\sigma(T)$ of all $\lambda \in \mathbb{C}$ that are not regular values of T is called spectrum of T

That is $\sigma(T) = \mathbb{C} - \rho(T)$

The spectrum of an operator T is usually divided into three disjoint sets.

1. Point spectrum, denoted $\sigma_p(T)$

2. Continuous spectrum, denoted $\sigma_c(T)$

3. Residual spectrum, denoted $\sigma_r(T)$

Where

$\boldsymbol{\sigma_p(T)} = \{\lambda \in \sigma(T): ker(\lambda I - T) \neq \{0\}\}$

$\boldsymbol{\sigma_c(T)} = \{\lambda \in \sigma(T): (\lambda I - T)^{-1} \text{ is densely defined but not bounded}\}$

$\boldsymbol{\sigma_r(T)} = \{\lambda \in \sigma(T): R(\lambda I - T) \text{ is not dense in } H\}$

Elements of $\sigma_p(T)$ are called eigenvalues

<u>1.1.10 Definition:</u> Two vectors x *and* y in inner product space are said to be orthogonal

if $\langle x, y\rangle = 0$

<u>1.1.11 Definition</u>: The set containing those vectors x , y *such that* $\langle x, y\rangle = 0$ *and* $\|x\|=1$ is said to be orthonormal set.

1.1.12 Definition: Orthogonal complement for non-empty subset S of Hilbert space we define the orthogonal complement to S, denoted by $S^{\perp}$.

$$S^{\perp} = \{y \in H : \langle x, y \rangle = 0 \; for \; all \; x \in S\}$$

1.1.13 Definition: Let $X \; and \; Y$ be normed space. A sequence $(T_n) \; of \; operators \; T_n \in B(X, Y)$ is said to be

a) **Uniformly operator convergent:** If (T_n) converges in the norm of B(X,Y) to T ∈B(X,Y)

i.e $$\|T_n - T\| \to \mathbf{0} \qquad (2)$$

b) **Strongly operator convergent:** If $(T_n x) \; converges \; strongly \; in \; Y \; for \; every \; x \in X$

$$\|T_n x - Tx\| \to \mathbf{0} \qquad (3)$$

1.1.14 Definition: A bounded linear operator $T: H \to H$ on Hilbert space H is said to be unitary if T is bijective and

$$T^* = T^{-1}$$

1.1.15 Definition: A bounded linear operator $T: H \to H$ on a Hilbert space H is said to be normal

if $TT^* = T^*T$

1.2 BASIC THEOREM

1.2.1 THEOREM (Cauchy-Schwarz inequality): If X is inner product space then

$$|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| \leq \|\boldsymbol{x}\| \|\boldsymbol{y}\| \qquad (7)$$

The equality holds in (7) if and only if $x \; and \; y \; are \; linearly \; depedent.$

Proof:

If $y = 0$, then (7) holds since$\langle x, 0 \rangle = 0$.

Let $y \neq 0$. For every scalar α we have

$0 \leq \|x - \alpha y\|^2$=$\langle x - \alpha y, x - \alpha y \rangle = \langle x, x \rangle - \overline{\alpha}\langle x, y \rangle - \alpha[\langle y, x \rangle - \overline{\alpha}\langle y, y \rangle]$

We see that expression in the brackets $[.,.,.]$ is zero if we choose

$\overline{\alpha} = \frac{\langle y,x\rangle}{\langle y,y\rangle}$. The remaining inequality is $0 \leq \langle x,x\rangle - \frac{\langle y,x\rangle}{\langle y,y\rangle}\langle x,y\rangle = \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2}$

Hence, $|\langle x,y\rangle| \leq \|x\|\|y\|$ ■

1.2.2 THEOREM (Continuity of inner product): If in an inner product $x_n \to x$ and

$y_n \to y$. Then $\langle x_n, y_n\rangle \to \langle x,y\rangle$

Proof:

Subtracting and adding a term, using the triangle inequality for numbers and finally, the Schwarz inequality, we obtain.

$$|\langle x_n, y_n\rangle - \langle x,y\rangle| = |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x,y\rangle|$$

$$\leq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle| \to 0$$

Since $x_n - x \to 0$ and $y_n - y \to 0$ $as\ n \to \infty$ ■

1.2.3 THEOREM (Riesz representation theorem): Let φ be a continuous linear functional on a Hilbert space H. Then there is a unique $z \in H$such that $\varphi(x) = \langle x,z\rangle\ for\ all\ x \in H$

Proof:

Let $N = \{x \in H: \varphi(x) = 0\}$ since φ is continuous, N is closed. If $N = H$, we take $Z = 0$

$H = N \oplus N^{\perp}$ (by theorem of projection)

Take $y_0 \in N^{\perp}$with $\|y_0\| = 1$ and consider for any $x \in H$ the vector

$y = \varphi(x)y_0 - \varphi(y_0)x$

Since $\varphi(y) = 0$ $for\ all\ x \in H$

$0 = \langle y, y_0\rangle = \langle(\varphi(x)y_0 - \varphi(y_0)x)x, y_0\rangle = \varphi(x)\langle y_0, y_0\rangle - \varphi(y_0)\langle x, y_0\rangle$

$$= \varphi(x) - \langle x,z\rangle$$

Hence, $\varphi(x) = \langle x,z\rangle$, where $z = \overline{\varphi(y_0)y_0}$

Show that the uniqueness,

$\varphi(x)$ =$\langle x, v\rangle = \langle x, w\rangle$. Then take $x = v - w$

$0 = \langle x, v\rangle - \langle x, w\rangle = \langle x, v - w\rangle = \|v - w\|^2$

$\Rightarrow v - w = 0$

Thus,$v = w$

1.2.4 THEOREM (Bounded inverse theorem): Let $T: V \to W$ be a bounded, linear and bijective operator from Banach space V into Banach space W then the inverse operator $T^{-1}: W \to V$ is also bounded.

The proof of these theorems so long if you want see on [1]

1.2.5 THEOREM(Principle of uniform boundness):Let(T_n) be sequence of bounded, linear operator $T_n: X \to Y$ from a Banach space$\boldsymbol{X}$ into a normed space $\boldsymbol{Y}$ such that is bounded for every $x \in X, say$

$$\|T_n x\| \leq C_N \quad n = 1,2 \ldots .. \qquad (8)$$

Where C_N is real number, then the sequence of the norms $\|T_n\|$is bounded, that is there is C a such that$\|T_n\| \leq C$

The proof of these theorems so long if you want see on [3]

CHAPTER TWO

Spectral Theory of Bounded Self-Adjoint Linear Operators

2.1 Spectral properties of bounded self- adjoint linear operators

2.1.1 Definition: Let $H_1 and\ H_2$ be complex Hilbert spaces and T: $H_1 \rightarrow H_2$ be a bounded linear operator. Then the **Hilbert-adjoint** operator $T^*: H_2 \rightarrow H_1$ is defined to be the operator satisfying.

$$< Tx, y >=< x, T^*y > \quad for\ all\ \ x \in H_1, y \in H_2$$

2.1.2 Theorem: The Hilbert-adjoint operator T^* of T exists, is unique and is a bounded linear operator with norm

$$\| T^* \| = \| T \|$$

Proof:

(1)$T^*\ \ is\ \ bounded.$

(2) T^* is Linear operator.

(3) $T^*\ is\ unique$

Take $y\epsilon\ H\ and\ \ T \in B(H)$.We define a bounded linear functional $X\ on\ H\ by\ x\ (x) =< Tx, y >$ $for\ all\ \ x, y \in H$

$\Rightarrow$ X Bounded since $|\ x\ (x)\ | = |< Tx, y >| \leq \| Tx \| \| y \|$. Now by **Riesz representation theorem** there is unique $z \in H\ such\ that\ x\ (x) =< x, z > \ for\ all\ x \in H$. Here we observe that z and y must have a certain relation so we can define an operator $T^*y = z$ satisfying

$< Tx, y >=< x, T^*y > \ for\ all\ x \in H$

Now let's see T^* Linear.

Let $y_1, y_2 \in H$ and $\alpha, \beta \in \mathbb{C}, so\ for\ all\ x \in H$

$< x, T^*(\alpha y_1 + \beta y_2) >=< Tx, \alpha y_1 + \beta y_2 >=< Tx, \alpha y_1 > +< Tx, \beta y_2 >$

$= \overline{\alpha} < Tx, y_1 > +\overline{\beta} < Tx, y_2 >=< x, \alpha T^* y_1 > +< x, \beta T^* y_2 >=< x, \alpha T^* y_1 + \beta T^* y_2 >$

$\Rightarrow T^*(\alpha y_1 + \beta y_2) = \alpha T^* y_1 + \beta T^* y_2$.

T^* $Linear\ operator.$

We want show that $\| \boldsymbol{T}^* \| = \| \boldsymbol{T} \|$

Let $y \in H$

$\| T^* y \|^2 =< T^* y, T^* y >=< y, TT^* y > \leq \|y\| \|TT^* y\| \leq \|y\| \|T\| \|T^* y\|$

$\Rightarrow \|T^* y\| \leq \|y\| \|T\|$, This shows that T^* is bounded

$\Rightarrow \|T^*\| \leq \|T\|$…………………….. (1)

Similarly, let $x \in H$

$\|Tx\|^2 =< Tx, Tx >=< x, T^* Tx > \leq \|x\| \|T^* Tx\| \leq \|x\| \|T^*\| \|Tx\|$

$\Rightarrow \|Tx\| \leq \|x\| \|T^*\|$

$\Rightarrow \| T \| \leq \| T^* \|$………. (2).

Combining (1) and (2) we get

$$\| T^* \| = \| T \| \qquad \blacksquare$$

To show the uniqueness of T^*

Suppose that $S \in B(H, H)$ also satisfied $\langle Tx, y \rangle = \langle x, Sy \rangle$ $\forall x, y \in H$. Then for each fixed y we would have that

$\langle x, Sy - T^* y \rangle = 0$ for every x, which implies $Sy - T^* y = 0$

Hence, $S = T^*$

Therefore, T^* is unique.

2.1.3 Definition: A bounded linear operator T: $H \to H$ on a complex Hilbert space H is said to be self-adjoint or hermitian if

$$T = T^*$$

Equivalently, a bounded linear operator T is said to be self-adjoint if

$$< Tx, y > = < x, Ty > \quad for\ all\ x, y \in H \qquad (1)$$

2.1.4 Theorem: Let T:$H \to H$ be a bounded linear operator on a Hilbert space H.Then:

(1) If T is self-adjoint, then $< Tx, x >$ is real for all $x \in H$

(2) If H is complex and $< Tx, x >$ is real for all $x \in H$, then the operator T is self-adjoint

Proof.

1. If T is self-adjoint, then for all x

$$\overline{\langle Tx, x\rangle} = < x, Tx > \qquad (1)$$

By definition $< Tx, y > = < x, T^*y >$ and since T is self-adjoint, we have

$$< Tx, x > = < x, Tx > \qquad (2)$$

Combining equation (1) and (2) gives

$$\overline{\langle Tx, x\rangle} = < Tx, x >$$

Hence $< Tx, x >$ is equal to its complex conjugate which implies that it is real.

2. If $< Tx, x >$ is real for all $x \in H, then$

$$< Tx, x > = \overline{\langle Tx, x\rangle} = \overline{\langle x, T^*x\rangle} = < T^*x, x >$$

Hence

$$0 =< Tx, x > -< T^{*}x, x >= \langle (T - T^{*})x, x\rangle$$

Thus, $T - T^{*} = 0$

Therefore $T = T^{*}$

Remark 1: If T Self-adjoint or unitary, then T is normal; in general the converse is not true.

Example 1: If $I: H \to H$ is identity operator, then $T = 2iI$ is normal, since $T^{*} = -2iI$, so that $TT^{*} = T^{*}T = 4I$ but $T \neq T^{*}$ and $T^{*} \neq T^{-1} = \frac{-1}{2}iI$

2.1.5 Theorem: Let $T: H \to H$ be a bounded self-adjoint linear operator on complex Hilbert space H.Then:

(a) All the eigenvalue of T are real.

(b) Eigenvectors corresponding to distinct eigenvalue of T are mutually orthogonal.

Proof

(a) Let λ be any eigenvalue of T and x a corresponding eigenvector.

Then $x \neq 0$ *and* $Tx = \lambda x$

$\Longrightarrow \lambda\langle x, x\rangle = \langle \lambda x, x\rangle = \langle Tx,x\rangle = \langle x, Tx\rangle = \langle x, \lambda x\rangle = \overline{\lambda}\langle x, x\rangle$, using the self-adjointness of T.

$\Rightarrow \lambda\langle x, x\rangle - \overline{\lambda}\langle x, x\rangle = 0$

$\Longrightarrow (\lambda - \overline{\lambda})\langle x, x\rangle = 0$. Here $\langle x, x\rangle = \|x\|^{2} \neq 0$

$\lambda = \overline{\lambda}$

(b) Let λ *and* μ be eigenvalues of T and let x *and* y be corresponding eigenvectors.

Then $Tx = \lambda x$ and $Ty = \mu y$ since T is self-adjoint and μ is real.

$\lambda\langle x, y\rangle = \langle \lambda x, y\rangle = \langle Tx,y\rangle = \langle x, Ty\rangle = \langle x, \mu y\rangle = \mu\langle x, y\rangle$

$\Rightarrow \lambda\langle x,y\rangle - \mu\langle x,y\rangle = 0$

$\Longrightarrow (\lambda-\mu)\langle x,y\rangle = 0$,$\lambda \neq \mu$

$\langle x,y\rangle = 0$ $\quad\quad$ $i.e\ \ x \perp y$

Example 2: $T = \begin{pmatrix} 2 & 1-i \\ 1+i & 1 \end{pmatrix}$

Spectrum $of\ (T) = [0,3]$ distinct real numbers with eigenvectors.

$$x_1 = \begin{bmatrix} -1+i \\ 2 \end{bmatrix}\ and\ x_2 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}\ respectively$$

Finally $\langle x_1, x_2\rangle = {x_1}^T\overline{x_2} = [-1+i \quad 2]\begin{bmatrix} 1+i \\ 1 \end{bmatrix} = 0$ $\ so\ the\ eigenvectors\ are\ orthogonal.$

2.1.6 Theorem: Let $T: H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Then a number λ belongs to the resolvent set $\rho(T)$ of T if and only if there exists a $c > 0$ such that for every $x\epsilon H$

$$(2) \qquad\qquad \|T_\lambda x\| \geq c\|x\| \qquad\qquad T_\lambda = T - \lambda I$$

Proof: $\Longrightarrow$) Suppose that $\lambda\epsilon\ \rho(T)$, $R_\lambda = {T^{-1}}_\lambda : H \to H$ exists and is bounded. Set$\|R_\lambda\| = k$ where $k > 0$ since $R_\lambda \neq 0$ now I=$R_\lambda T_\lambda$, so that for every $x \in H$

$\Longrightarrow T_\lambda x = y$,$\Rightarrow x = {T^{-1}}_\lambda$y ,since $R_\lambda = {T^{-1}}_\lambda$

$\Longrightarrow x = R_\lambda T_\lambda x$

$\Longrightarrow \|x\| = \|R_\lambda T_\lambda x\| \leq \|R_\lambda\|\|T_\lambda x\| = k\|T_\lambda x\|$

$\Longrightarrow \|x\| \leq k\|T_\lambda x\|$

$\Longrightarrow \|T_\lambda x\| \geq \frac{1}{k}\|x\|$, where $c = \frac{1}{k}$

$\|T_\lambda x\| \geq c\|x\|$

2.1.7 Theorem: The spectrum $\sigma(T)$ of a bounded self-adjoint linear operator $T: H \to H$ on a complex Hilbert space H is real.

Proof:

By theorem 2.1.6 we show that a $\lambda = \alpha + i\beta$ $(\alpha, \beta \ real)$ with $\beta \neq 0$ which belongs to $\rho(T)$, so that $\sigma(T) \subset \mathbb{R}$

$\langle T_\lambda x, x\rangle = \langle Tx, x\rangle - \lambda\langle x, x\rangle \ \ for\ all\ x \in H \ldots\ldots\ldots\ldots\ldots\ldots (1)$

Since $\langle Tx, x\rangle\ and\ \langle x, x\rangle\ are\ \ real$

$\overline{\langle T_\lambda x, x\rangle} = \langle Tx, x\rangle - \bar{\lambda}\langle x, x\rangle,\ here\ \bar{\lambda} = \ \alpha - i\beta \ldots\ldots\ldots\ldots\ldots. (2)$

Subtract equation (2) from (1)

$\overline{\langle T_\lambda x, x\rangle}$ -$\langle T_\lambda x, x\rangle = \left(\lambda - \bar{\lambda}\right)\langle x, x\rangle = 2i\beta\|x\|^2$

So that -2iIm$\langle T_\lambda x, x\rangle = 2i\beta\|x\|^2$

-Im$\langle T_\lambda x, x\rangle = \beta\|x\|^2$

Taking absolute value and Cauchy Schwarz inequality, we obtain

$|\beta|\|x\|^2 = |\mathrm{Im}\langle T_\lambda x, x\rangle| \leq \|T_\lambda x\|\|x\|$

$|\beta|\|x\| \leq \|T_\lambda x\|, for\ \|x\| \neq 0.$ If $\beta \neq 0, then\ \lambda \in\ \rho(T)$

Hence for $\lambda \in \sigma(T)$ we must have $\beta = 0 \ \ that\ is\ \lambda\ is\ real.$

2.2 Further Spectral Properties of Bounded Self-Adjoint Linear Operators.

<u>2.2.1 Theorem</u>: The spectrum $\sigma(T)$ of a bounded self-adjoint linear operator $T: H \to H$ on a complex Hilbert space H lies in the closed interval [m, M] on the real axis ,where

$$\boldsymbol{m = inf_{\|x\|=1}\langle Tx, x\rangle} \qquad\qquad \boldsymbol{M = sup_{\|x\|=1}\langle Tx, x\rangle}$$

Proof:

$\sigma(T)$ lies on the real axis by theorem of 2.1.7. We show that any real $\lambda = M + c \; with \; c > 0$ belongs to the resolvent set $\rho(T)$. For every $x \neq 0 \; and \; v = \|x\|^{-1}x \; we \; have \;\; x = \|x\|v \;\; and$

$$\langle Tx, x\rangle = \|x\|^2\langle Tv, v\rangle \leq \|x\|^2 sup_{\|v\|=1}\langle Tv, v\rangle = \langle x, x\rangle M$$

Hence $-\langle Tx, x\rangle \geq -\langle x, x\rangle M \; and \; by \; cauchy \; schwarz \; inequality. \; we \; obtain$

$$\|T_\lambda x\|\|x\| \geq -\langle T_\lambda x, x\rangle = -\langle Tx, x\rangle + \lambda\langle x, x\rangle = -\langle x, x\rangle M + \lambda\langle x, x\rangle$$

$$\geq (-M + \lambda)\langle x, x\rangle \;\; \geq c\|x\|^2$$

$$\|T_\lambda x\|\|x\| \geq c\|x\|^2, \;\; \|x\| \neq 0$$

$$\|T_\lambda x\| \geq c\|x\|$$

$$\lambda \in \rho(T)$$

Example 3: Consider the operator $T: l^2 \to l^2$ given by $Tx = (0, x_1, x_2 \cdots)$,

Where $X = (x_{1\,,}x_2, x_3, \cdots)$.Show that eigenvalue and spectrum of operator

Solution: we use to show two cases

Case I: If $\lambda = 0, then \; 0 \in \sigma_P(T)$

$Tx = \lambda x \;\Rightarrow Tx = 0x = 0$, for some x which is impossible.

Case II: If $\lambda \neq 0, then \; \lambda \in \sigma_P(T)$ there exists $x \neq 0$ such that $Tx = \lambda x$

$\Rightarrow \;(0, x_1, x_2, \ldots) = (\lambda x_1, \lambda x_2, ..)$

$\Rightarrow \;\lambda x_1 = 0, since \; \lambda \neq 0, x_1 = 0$

$x_1 = \lambda x_2, since \; \lambda \neq 0, x_2 = 0$

$\cdots$and so $x = 0$ a contradiction

$\sigma_P(T) = \emptyset$

Therefore $0 \in \sigma(T)$, because T has no inverse.

i.e $\quad 0 \notin \rho(T)$

2.2.2 Theorem: For any bounded self-adjoint linear operator T on a complex Hilbert space H we have

$$\|T\| = max(|m|, |M|) = sup_{\|x\|=1}|\langle Tx, x\rangle|$$

Proof: By the Cauchy Schwarz inequality

$sup_{\|x\|=1}|\langle Tx, x\rangle| \leq sup_{\|x\|=1}\|Tx\|\|x\| = \|T\|$. *where* K= $sup_{\|x\|=1}|\langle Tx, x\rangle|$

$K \leq \|T\|$…………………… (1)

We want show $\|T\| \leq K$

If Tz =0 then $0 \leq K. \|z\|^2$

$\|Tz\| \leq K \quad \forall z \in H, \|z\| = 1$

Otherwise $Tz \neq 0$ for any z of norm 1

$$v = \|Tz\|^{\frac{1}{2}}z \;\; and \;\; w = \|Tz\|^{-\frac{1}{2}}Tz$$

Now set y_1=v+w and $y_2 = v - w$.

$$\langle Ty_1, y_1\rangle - \langle Ty_2, y_2\rangle = 2(\langle Tv, w\rangle + \langle Tw, v\rangle)$$

$= 2(\langle Tz, Tz\rangle + \langle T^2z, z\rangle)$, since T is self-adjoint

$$= 2(\langle Tz, Tz\rangle + \langle Tz, Tz\rangle) = 2(2\langle Tz, Tz\rangle)$$

$$= 4\langle Tz, Tz\rangle = 4\|Tz\|^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now for every $y \neq 0$ *and* $x = \|y\|^{-1}y$ *we have* $y = \|y\|x$ *and*

$|\langle Ty, y\rangle| = \|y\|^2|\langle Tx, x\rangle| \leq \|y\|^2 sup_{\|x\|=1}|\langle Tx, x\rangle| = K\|y\|^2$, so that by the triangle inequality.

$$|\langle Ty_1, y_1\rangle - \langle Ty_2, y_2\rangle| \le |\langle Ty_1, y_1\rangle| + |\langle Ty_2, y_2\rangle|$$

$$\le K(\|y_1\|^2 + \|y_2\|^2)$$

$$\le 2K(\|v\|^2 + \|w\|^2)$$

$$= 4K\|Tz\| \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Combining (1) and (2) we get

$$4\|Tz\|^2 \le 4K\|Tz\|, \quad \|Tz\| \neq 0$$

$\|Tz\| \le K$ *by taking* supremum *over all z of norm* 1. $\|T\| \le K$ *and* $K \le \|T\|$

$$\|T\| = K = max(|m|, |M|) = sup_{\|x\|=1}|\langle Tx, x\rangle| \qquad ■$$

2.2.3 Theorem: The residual spectrum $\sigma_r(T)$ of a bounded self-adjoint linear operator $T: H \to H$ on a complex Hilbert space H is empty.

Proof.

Suppose not $\sigma_r(T) \neq \emptyset$. Let $\lambda \in \sigma_r(T)$. By definition of $\sigma_r(T)$, the inverse of T_λ exists but its domain $\mathfrak{D}(T^{-1}{}_\lambda)$ is not dense in H. Hence, by the projection theorem there is a $y \neq 0$ *in* H which is orthogonal to $\mathfrak{D}(T^{-1}{}_\lambda)$, but $\mathfrak{D}(T^{-1}{}_\lambda)$ is a range of T_λ

$\langle T_\lambda x, y\rangle = 0$ *for all* $x \in H$. Since λ is real and T self-adjoint, we obtain $\langle x, T_\lambda y\rangle = 0$ *for all* x

Taking $x = T_\lambda y$ *we get* $\|T_\lambda y\|^2 = 0$, *so that* $T_\lambda y = Ty - \lambda y = 0$ *since* $y \neq 0$. This shows that λ is an eigenvalue of T, but contradicts.

$$\lambda \in \sigma_r(T)$$

$$Hence, \ \sigma_r(T) = \emptyset \qquad ■$$

2.3 Positive Operators

In this section we can see the definition of **partial order** is a binary relation "≤" over a set p which is reflexive, antisymmetric, and transitive.

i.e. For all a, b and c in p

We have that

1. $a \leq a$ (Reflexive)
2. $if\ a \leq b\ \ and\ \ b \leq a\ then\ a = b$ $(antisymmetric)$
3. $if\ a \leq b\ and\ b \leq c\ then\ a \leq c$ $(transitivity)$

<u>2.3.1 Definition</u>: A bounded linear operator $T: H \to H$ is called a positive operator if and only if T self-adjoint and $\langle Tx, x\rangle \geq 0\ \ for\ all\ \ x \in H$

A bounded linear operator $T: H \to H$ is said to be **positive,** written

(1) $T \geq 0 \quad if\ and\ only\ if\ \ \langle Tx, x\rangle \geq 0\ \ for\ all\ \ x \in H$

<u>Remark 2</u>: (a) The sum of positive operators is positive.

(b) Every Positive operator on a complex Hilbert space is self-adjoint

Proof(b).

Assume that $T \in B(H)$ is positive.

$\Rightarrow \langle T^*x, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle} = \langle Tx, x\rangle \geq 0$

$\Rightarrow \langle (T^* - T)x, x\rangle = 0$ for all $x \in H$

$\Rightarrow T^* - T = 0$

Hence, $T^* = T$

Example 4: Let $T: L^2(0,1) \to L^2(0,1)$ be a linear operator defined by

$$Tf(x) = xf(x) \qquad for\ all\ \ f \in L^2(0,1) \quad for\ all\ x \in (0,1)$$

Show that T is a positive operator.

Solution. For all $f \in L^2(0,1)$we have that $T\ is\ self-adjoint\ \ since$

$$\langle Tf, g\rangle = \int_0^1 xf(x)\overline{g(x)}dx = \int_0^1 f(x)\overline{xg(x)}dx = \langle f, Tg\rangle\ for\ all\ \ f, g \in L^2(0,1)$$

$$\langle Tf, f\rangle = \int_0^1 xf(x)\overline{f(x)} = \int_0^1 x|f(x)|^2\, dx \geq 0$$

Therefore T is positive operator

The following theorem is like the familiar statement about real numbers which says that when you multiply two non-negative real numbers the result is a non-negative real numbers.

<u>**2.3.2 Theorem**</u>**:** Let S and T be a positive and self-adjoint such that $ST = TS$. Then their product ST is positive.

Proof.
Since $(ST)^* = T^*S^* = TS$ we see that ST is self-adjoint if and only if $ST = TS$.
We show that ST is positive.
This is true for $n = 0$. Assume it is true for n
We consider

$$(2) \qquad S_1 = \frac{1}{\|S\|}S \qquad\qquad S_{n+1} = S_n - {S_n}^2 \qquad (n = 1,2,3\ldots\ldots..)$$

and prove by induction that

$$(\mathbf{3}) \qquad 0 \leq S_n \leq I$$

(a) For $n = 1$ the inequality (3) holds. Indeed, the assumption $0 \leq S\ implies\ S_1 \leq I$ is obtained by application of the schawrz inequality $\|Sx\| \leq \|s\|\|x\|$

$$\langle S_1 x, x\rangle = \frac{1}{\|S\|}\langle Sx, x\rangle \le \frac{1}{\|S\|}\|sx\|\|x\| \le \|x\|^2 = \langle Ix, x\rangle$$

$$S_1 \le I$$

Suppose (**3**) holds for an $n = k \;\; that \;\; is$

$$0 \le S_k \le I \qquad thus \qquad 0 \le I - S_k \le I$$

Then, since S_k is self-adjoint, for every $x \in H \;\; and \quad y = S_k x$

We obtain

$$\langle {S_k}^2 (I - S_k)x, x\rangle = \langle (I - S_k)S_k x, S_k x\rangle = \langle (I - S_k)y, y\rangle \ge 0$$

By definition this proves

$${S_k}^2 (I - S_k) \ge 0$$

Similarly,

$$S_k (I - S_k)^2 \ge 0 \qquad \text{Since } S_k \text{ is self-adjiont}$$

It is clear that from remark (2) sum of positive operators is positive.

$$0 \le {S_k}^2 (I - S_k) + S_k (I - S_k)^2 = S_k - {S_k}^2 = S_{k+1}$$

Hence, $0 \le S_{k+1} \le I$ ■

(**b**) We now show that$\langle STx, x\rangle \ge 0 \; for \; all \; x \in H$. Form (3) we obtain successively

$S_1 = {S_1}^2 + S_2, \;\; S_2 = {S_2}^2 + S_3 \;\; , \;\; S_3 = {S_3}^2 + S_4$

$={S_1}^2 + {S_2}^2 + {S_3}^2 + S_4$

…

$={S_1}^2 + {S_2}^2 + {S_3}^2 + {S_4}^2 + \cdots + {S_n}^2 + S_{n+1}$

Since $S_{n+1} \ge 0 \;\; this \; implies$

(5) $${S_1}^2 + {S_2}^2 + {S_3}^2 + {S_4}^2 + \cdots + {S_n}^2 = S_1 - S_{n+1} \le S_1$$

By the definition of $\le$ and self-adjiontness of $S_j \; s$

$\sum_{j=1}^{n} \left\|S_j x\right\|^2 = \sum_{j=1}^{n} \langle S_j x, S_j x\rangle = \sum_{j=1}^{n} \langle {S_j}^2 x, x\rangle \le \langle S_1 x, x\rangle$

Since n is arbitrary, the infinite series $\|S_1x\|^2 + \|S_2x\|^2 + \cdots$ absolutely converges.

$\lim_{n\to} \sum_{j=1}^{n}\left\|S_jx\right\|^2 = \|S_1x\|^2$ $\quad hence \quad \lim_{n\to\infty} S_nx = Sx$

(6) $$\left(\sum_{j=1}^{n} {S_j}^2\right)x = (S_1 - S_{n+1})x \to S_1x$$

$\lim_{n\to} \left(\sum_{j=1}^{n} {S_j}^2\right)x = \lim_{n\to} (S_1 - S_{n+1})x \to S_1x$

All the $S_j\, s$ commute with T since they are sums and product of $S_1 = \|S\|^{-1}S$ $and\ S\ and\ T\ commute$.Using $S = \|S\|S_1, formula$ (6) $T \geq 0$ and the continutity of inner product, we thus obtain for every $x \in H\ and\ \ y_j = S_jx$

$$\langle STx, x\rangle = \|S\|\langle TS_1x, x\rangle$$

$$= \|S\| \lim_{n\to} \sum_{j=1}^{n}\langle T{S_j}^2x, x\rangle$$

$$= \|S\| \lim_{n\to} \sum_{j=1}^{n}\langle Ty_j, y_j\rangle \geq 0$$

$Therefore\ \ ST\ is\ \ positive$ ■

In this section we consider a condition which, together with boundeness, will ensure that a sequence is convergent.

2.3.3 Definition: A monotone sequence (T_n) of self-adjoint linear operators $T_n$on Hilbert space H is a sequence (T_n) which is either monotone increasing, that is,

$$T_1 \leq T_2 \leq T_3 \leq \cdots$$

Or monotone decreasing, that is,

$$T_1 \geq T_2 \geq T_3 \geq \cdots$$

The following theorem is a generalization of the familiar fact that if you have an increasing sequence of real numbers which is bounded above, then the sequence converges

2.3.4 Theorem: Let(T_n) be a sequence of bounded self-adjoint linear operators on a complex Hilbert space H such that

(7) $$T_1 \leq T_2 \leq T_3 \leq \cdots T_n \leq \cdots \leq K$$

Where K is bounded self-adjoint linear operator on H. Suppose that any T_j commutes with K and with every T_m. Then (T_n) is strongly operator convergent $(T_n x \to Tx\ for\ all\ x \in H)$ and limit operator T is linear, bounded and self-adjoint and satisfies $T \leq K$

Proof.

We consider $S_n = K - T_n$.

(a) The sequence $(\langle {S_n}^2 x, x\rangle)$ converges $for\ every\ x \in H$

(b) $T_n x \to Tx$, where T is linear, self-adjoint and bounded by uniform boundnees theorem

(a) $\langle {S_n}^2 x, x\rangle = \langle S_n x, S_n x\rangle$

Therefore S_n is self-adjoint

$${S_m}^2 - S_n S_m = (S_m - S_n)S_m = (T_n - T_m)(k - T_m)$$

Let $m < n$.Then $T_n - T_m$ and $K - T_m$ are positive by (7), since these operators commute, their product is positive. Hence on the left ${S_m}^2 - S_n S_m \geq 0,\ that\ is\ {S_m}^2 \geq S_n S_m\ for\ m < n.$

$Similary$

$$S_n S_m - {S_n}^2 = S_n(S_m - S_n) = (k - T_n)(T_n - T_m) \geq 0$$

So that $S_n S_m \geq {S_n}^2$, together

$${S_m}^2 \geq S_n S_m \geq {S_n}^2 \qquad m < n$$

By definition, using the self-adjointness of S_n

$$\langle {S_m}^2 x, x\rangle \geq \langle S_n S_m x, x\rangle \geq \langle {S_n}^2 x, x\rangle = \langle S_n x, S_n x\rangle = \|S_n x\|^2 \geq 0 \qquad (8)$$

This show that $(\langle {S_n}^2 x, x\rangle)$ with fixed x is a monotone decreasing sequence of non-negative numbers.

Hence, $(\langle {S_n}^2 x, x\rangle)\ converges.$ ■

(b) We show that $T_n x \to Tx$ is converges. By assumption, every T_n commute with every T_m and with K. Hence the $S_j s$ all commute.

These operators are self-adjoint, since $-2\langle S_n S_m x, x\rangle \leq -2\langle {S_n}^2 x, x\rangle \ \ by\ (8)\ where\ \ m < n$

$$\|S_m x - S_n x\|^2 = \langle (S_m - S_n)x, (S_m - S_n)x\rangle$$

$$= \langle (S_m - S_n)^2 x, x\rangle$$

$$= \langle {S_m}^2 x, x\rangle - 2\langle S_n S_m x, x\rangle + \langle {S_n}^2 x, x\rangle$$

$$\leq \langle {S_m}^2 x, x\rangle - \langle {S_n}^2 x, x\rangle$$

Since $(S_n x)$ is Cauchy sequence. Then H is completes.

We show that T is self-adjoint because T_n is self-adjoint and the inner product is continuous.

$$\langle Tx, x\rangle = \lim_{n\to} \langle T_n x, x\rangle = \lim_{n\to} \langle x, T_n x\rangle = \langle x, Tx\rangle$$

We show that T is bounded

Now we must show that T is bounded and $(T_n x)$ converges to it

$sup_n \|T_n x\|$ must be finite for every x, there is M such that $M > 0$

$\Rightarrow \|T_n\| \leq M \qquad\qquad \forall n$

$\Rightarrow \|T_n x\| \leq M\|x\|$

$\Rightarrow \lim_{n\to} \ \|T_n x\| \leq M\|x\|\ this\ \ implies\ \ \|Tx\| \leq M\|x\|$

Hence T is bounded, by uniform boundness theorem.

Finally $\langle Tx, x\rangle = \lim_{n\to\infty} \langle T_n x, x\rangle \leqq \langle Kx, x\rangle$

$T \leqq K$ ∎

2.4 Square Roots of a Positive Operators

2.4.1 Definition: Let $T: H \to H$ be positive bounded self-adjoint linear operator on complex Hilbert space H. Then a bounded self-adjoint operator A is called a square root of T if,

(1) $$A^2 = T$$

If, in addition, $A \geqq 0$, then A is called a positive square root of T and is denoted by

$$A = T^{\frac{1}{2}} \quad exists\ and\ is\ unique$$

The following result characterizes the uniqueness and existence of square root of positive operators.

2.4.2 Theorem: Every positive bounded self-adjoint linear operator $T: H \to H$ on a complex Hilbert space H has a positive square root A, which is unique. This operator A commutes with every bounded linear operator on H which commutes with T.

Proof.

(a) We show that if the theorem holds under the additional assumption $T \leq I$ it also holds without that assumption

If $T = 0, we\ can\ take\ A = T^{\frac{1}{2}} = 0$

Let $T \neq 0$ by cauchy schwarz inequality.

$\langle Tx, x\rangle \leq \|Tx\|\|x\| \leq \|T\|\|x\|^2 = \langle Ix, x\rangle$

$\langle \frac{T}{\|T\|}x, x\rangle = \langle Ix, x\rangle\ \ since\ \|T\| \neq 0 \qquad set\ \ Q = \left(\frac{1}{\|T\|}\right)T, we\ obtain$

$\langle Qx, x\rangle \leq \|x\|^2 = \langle Ix, x\rangle\ \ that\ is\ \ Q \leq I$

Assuming that Q has a unique positive square root.

$B = Q^{\frac{1}{2}} \quad we\ have \quad B^2 = Q\ and\ \ T = \|T\|Q \Rightarrow T^{\frac{1}{2}} = \|T\|^{\frac{1}{2}}Q^{\frac{1}{2}} = T^{\frac{1}{2}}B$

$\left(\|T\|^{\frac{1}{2}}B\right)^2 = \|T\|B^2 = \|T\|Q = T.$ Hence if we prove the theorem under the additional assumption $T \leq I$ ■

(b) We obtain the existence of the operator $A = T^{\frac{1}{2}}\ from\ \ A_n x \to Ax\ where\ \ A_0 = 0$

$$and \quad (2) \qquad A_{n+1} = A_n + \frac{1}{2}\left(T - {A_n}^2\right) \qquad n = 0,1,2, \dots$$

We consider (2), since $A_0 = 0$, *we have* $A_1 = \frac{1}{2}T$, $A_2 = T - \frac{1}{8}T^2$ ….,

$$A_{n+1} = A_n + \frac{1}{2}\left(T - {A_n}^2\right) \qquad n = 0,1,2,\dots$$

Each A_n is polynomial in T, and they also commutes with T. We now prove

(3) $A_n \leqq I \qquad n = 0,1,2,\dots$

(4) $A_n \leqq A_{n+1} \qquad n = 0,1,2,\dots$

(5) $A_n x \to Ax \quad , A = T^{\frac{1}{2}}$

(6) $ST = TS, \quad \Rightarrow AS = SA$

Where S is a bounded linear operator on H.

Proof (3)

This is true if $n = 0$. Suppose it is true for n. Since $\mathrm{I} - \mathrm{A}_{n-1}$ is self-adjoint

$(\mathrm{I} - \mathrm{A}_{n-1})^2 \geq 0$. Also $\mathrm{T} \leq \mathrm{I}$. This implies $\mathrm{I} - \mathrm{T} \geq 0$. *From this and* (2) *we obtain* (3)

$$\begin{aligned}\mathrm{I} - \mathrm{A}_n &= \mathrm{I} - \mathrm{A}_{n-1} + \tfrac{1}{2}\left(\mathrm{T} - {\mathrm{A}_{n-1}}^2\right)\\ &\geq \mathrm{I} - \mathrm{A}_{n-1} + \tfrac{1}{2}\left(\mathrm{I} - {\mathrm{A}_{n-1}}^2\right)\\ &= \mathrm{I} - \mathrm{A}_{n-1} - \tfrac{1}{2}(\mathrm{I} - \mathrm{A}_{n-1})(\mathrm{I} + \mathrm{A}_{n-1})\\ &= (\mathrm{I} - \mathrm{A}_{n-1})(\mathrm{I} - \tfrac{1}{2}(\mathrm{I} + \mathrm{A}_{n-1}))\\ &= (\mathrm{I} - \mathrm{A}_{n-1})(\mathrm{I} - \mathrm{A}_{n-1})\tfrac{1}{2} \geq 0\end{aligned}$$

$$\mathrm{I} - \mathrm{A}_n \geq 0 \qquad \blacksquare$$

Proof (4).

We use induction (2) gives 0=$\mathrm{A}_0 \leq \mathrm{A}_1 = \frac{1}{2}\mathrm{T}$

We show that $A_{n-1} \leq A_n$ for fixed n implies $A_n \leq A_{n+1}$ from (2) we calculate.

$$A_{n+1} - A_n = A_n + \frac{1}{2}\left(T - A_n{}^2\right) - A_{n-1} - \frac{1}{2}\left(T - A_{n-1}{}^2\right)$$

$$=A_n - \frac{1}{2}A_n{}^2 - A_{n-1} + \frac{1}{2}A_{n-1}{}^2$$

$$= A_n - A_{n-1} - \frac{1}{2}A_n{}^2 + \frac{1}{2}A_{n-1}{}^2$$

$$\geq (A_n - A_{n-1})[\mathrm{I} - \frac{1}{2}(A_n + A_{n-1})]$$

$$= (A_n - A_{n-1})[\ \mathrm{I} - \frac{1}{2}(2I)]=0$$

Hence $A_n - A_{n-1} \geq 0$ and $\left(\ I - \frac{1}{2}(A_n + A_{n-1})\right) \geq 0$

Therefore $A_n \leq A_{n+1} \qquad n = 0,1,2, \ldots$ ■

Proof (5)

(A_n) Monotone increasing by (4) and $A_n \leq I$ by (3). Hence (by the theorem monotone sequence) implies the existence of a bounded self-adjoint linear operator A.

Such that $A_n x \to Ax\ \ for\ all\ x \in H$ since $(A_n x)$ converges (2) gives

$$A_{n+1}x - A_n x = \frac{1}{2}\left(Tx - A_n{}^2 x\right) \to 0\ \ as \qquad n \to \infty$$

$$Tx - A^2 x = 0\ \ for\ all\ x$$

$$Tx = A^2 x\ , \quad x \neq 0 \qquad\qquad hence \quad T = A^2$$

Also $A \geq 0. becuase\ 0 = A_0 \leq A_n\ by\ (4)$

$\Rightarrow \lim_{n\to\infty}\langle A_n x, x\rangle = \lim_{n\to\infty}\langle x, A_n x\rangle = \langle x, Ax\rangle = \langle Ax, x\rangle \geq 0$ by continuity of inner product

Proof (6)

Let $S \in L(H, H)$ be any linear operator that commute with A

$$SA_n = A_n S \qquad \forall n$$

Since $A_n \to A, \qquad A_n Sx \to ASx$

Using continuity of inner product S

$\lim_{n\to\infty} SA_n x = S \lim_{n\to\infty} A_n x = SAx = \lim_{n\to\infty} A_n Sx = ASx$

Hence $ST = TS \Rightarrow AS = SA$

(c) Uniqueness**:** Let both A and B be positive square root of T. Then $A^2 = B^2 = T. Also\ BT = B^2B = BB^2 = TB,\ So\ that\ AB = BA\ \ by\ (6\ )$

Let $x \in H$ be arbitrary and $y = (A - B)x$

Then $\langle Ay, y\rangle \geq 0\ and\ \langle By, y\rangle \geq 0\ because\ A \geq 0\ \ and\ B \geq 0$

Using $AB = BA\ \ and\ A^2 = B^2\ we\ obtain$

$\langle Ay, y\rangle + \langle By, y\rangle = \langle (A + B)y, y\rangle = \langle (A^2 - B^2)x, y\rangle = 0$

Hence $\langle Ay, y\rangle = \langle By, y\rangle = 0\ , since\ A \geq 0$ and A is self-adjoint. It has itself a positive square root C, that is $C^2 = A\ and\ C\ is\ self - adjoint.$

We obtain

$0 = \langle Ay, y\rangle = \langle C^2y, y\rangle = \langle Cy, Cy\rangle = \|Cy\|^2\ and\ Cy = 0, also\ Ay = C^2y = C(Cy) = 0$
Similarly,

$B \geq 0\ and\ \ B\ is\ self - adjoint,$ it has itself a positive square root D.

That is $D^2 = B\ and\ D$ is self-adjoint.

$0 = \langle By, y\rangle = \langle D^2y, y\rangle = \langle Dy, Dy\rangle = \|Dy\|^2\ and\ Dy = 0\ , also\ By = D^2y = D(Dy) = 0$

Hence $By = 0,\ since\ B \geq 0$

Hence $(A - B)y = 0,\ using\ y = (A - B)x\quad for\ all\ x \in H$

$\|Ax - Bx\|^2 = \langle (A - B)^2x, x\rangle = \langle (A - B)y, x\rangle = 0$

Hence $A = B.\ Then\ A\ is\ unique.$

CHAPTER THREE

PROJECTIONS

3.1 BASIC DEFINITION AND PROPERTIES OF PROJECTION.

3.1.1 Definition: Let H be a Hilbert space over Complex number. A bounded linear operator P on H is called

1) a projection if P^2 =P
2) an orthogonal projection if P^2 = P and P* = P

Note: The Ran (P) = $P(H)$ of a projection on a Hilbert space H always is a closed linear subspace of H on which P acts like the identity. If in addition P is orthogonal, then P acts like zero operators on $\mathcal{N}(P)$.

If $x = y + z$ with $y \in R(P)$ and $z \in \mathcal{N}(P)$ is the decomposition guaranteed by the projection theorem, then $Px = y$. Thus the projection theorem sets up a one-to-one correspondence between orthogonal projection and closed linear subspaces.

$$H = Y \oplus Y^{\perp}$$

$$x = y + z = Px + (I - P)x$$

This show that the projection of H onto $Y^{\perp}\ is\ I - P$

Example 5: Let $P: C^3 \to C^3$ be the linear operator defined

by $P(x, y, z) = (x, y, 0)\ for\ all\ x, y, z \in C^3$

Then P is orthogonal projection

Solution: Since C^3 is finite dimensional $P \in B(C^3)\ and\ P^2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = P \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = P \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Therefore P is idempotent.

Follows from $\langle P(x,y,z),(u,v,w)\rangle = x\bar{u} + y\bar{v} = \langle (x,y,z), P(u,v,w)\rangle$

Then P is self-adjoint

Therefore P *is orthogonal projection.*

Orthogonal projection P has $Imp = \{(x,y,0): x,y \in \mathbb{C}\}$ orthogonal projection P matrix representation.

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In general a $n \times n$ diagonal matrix whose diagonal element either 0 *or* 1 is the matrix of an orthogonal projection $B(C^3)$

<u>3.1.2 Theorem</u>: A bounded linear operator $P: H \to H$ on a Hilbert space H is a projection if and only if P is self-adjoint and idempotent.

Proof.

($\Rightarrow$)Suppose that p is a projection on H and denote $P(H)$ *by* Y. Then $P^2 = P$ because for every $x \in H$ *and* $Px = y \in Y$ *we have*

$$P^2x = Py = y, hence \quad P^2x = Px \Rightarrow P^2 = P \ \ is\ idempotent.$$

Now consider any two vectors x_1 *and* x_2 *in* H, *f*rom decomposition we can write

$$x_1 = y_1 + z_1 \ and\ x_2 = y_2 + z_2\ where\ y_1, y_2 \in Y = R(P)\ and\ z_1, z_2 \in Y^{\perp} = N(P)$$

Then $\langle y_1, z_2\rangle = \langle y_2, z_1\rangle = 0$ *becuase* $Y \perp Y^{\perp}$

We show that P is self-adjoint

$$\langle Px_1, x_2\rangle = \langle y_1, y_2 + z_2\rangle = \langle y_1, y_2\rangle = \langle y_1 + z_1, y_2\rangle = \langle x_1, Px_2\rangle$$

Hence P is self-adjoint

$(\Leftarrow)$ $suppose\ that\ P\ self-adjoint\ and\ idempotent, denote\ P(H)\ by\ Y$.Then for every$x \in H$. Using projection theorem.

$$x = Px + (I - P)x$$

Orthogonality. $Y = P(H) \perp (I - P)(H)$. Follows from,

$$\langle Px, (I-P)v \rangle = \langle x, P(I-P)v \rangle = \langle x, Pv - P^2 v \rangle = \langle x, 0 \rangle = 0$$

Let $Y = P(H) = R(P)$, then Y is a linear subspace of H. If $y_n = Px_n$,with $y_n \longrightarrow z$, then

$$Py_n = P^2 x_n = Py_n = y_n$$

So,

$$z = \lim_{n \longrightarrow \infty} Py_n = Pz \in Y$$

Hence is Y closed subspace of H

Then P is projection on H.

<u>3.1.3 Theorem</u>. For any projection P on a Hilbert space H,

(3) $\langle Px, x \rangle = \|Px\|^2$

(4) $0 \leq P \leq I$

(5) $\|P\| \leqq 1$; $\|P\| = 1\ if\ P(H) \neq \{0\}$

Proof .

(3) $$\langle Px, x \rangle = \langle P^2 x, x \rangle = \langle Px, Px \rangle = \|Px\|^2$$

(4) $$0 \leq \|Px\|^2 = \langle Px, Px \rangle = \langle Px, x \rangle \leq \|x\|^2 = \langle x, x \rangle$$

$$0 \leq P \leq I$$

(5) $By\ the\ using\ schwarz\ inequality$

$$\langle Px, x \rangle \leqq \|x\|^2$$

$$\|Px\|^2 \leqq \|x\|^2, \qquad since\ \|x\| \neq 0$$

$$\|P\| \leqq 1 \ldots\ldots\ldots..(1) \qquad for\ all\ x \in H$$

$$\|Px\| = \|P(Px)\| \leqq \|P\|\|Px\|$$

$$\|Px\| \leqq \|P\|\|Px\|, since\ \|P\| \neq 0$$

$$1 \leqq \|P\| \ldots\ldots.... \quad (2)$$

By combining (1) and (2) we get ; $\|P\| = 1$ ■

3.1.4 Theorem: Products projections on Hilbert space H are satisfying the following two conditions.

a) $P = P_1P_2$ is projection on H if and only if the projections P_1 *and* P_2 commute. Then P projects H onto $Y = Y_1 \cap Y_2\ where\ Y_j = P_j(H)$

b) Two closed subspaces Y and V of H are orthogonal if and only if the corresponding projections satisfy $P_YP_V = 0$

Proof.

($\Leftarrow$) Suppose that P_1 *and* P_2 commute, then show that P is self-adjoint and idempotent

$$\langle Px_1, x_2\rangle = \langle P_1P_2x_1, x_2\rangle = \langle P_2x_1, P_1x_2\rangle = \langle x_1, P_2P_1x_2\rangle = \langle x_{1,} Px_2\rangle$$

Hence, $P\ is\ self-adjoint$

$$P^2 = (P_1P_2)^2 = (P_1P_2)(P_1P_2) = {P_1}^2{P_2}^2 = P_1P_2 = P$$

Then P is idempotent.

Hence, P is projection.

Then $Px = P_1(P_2x) = P_2(P_1x) = x\ \ for\ all\ x \in H$

Since P_1 projects H onto Y_1, we must have

$$Px = P_1(P_2x) \in Y_1$$

$$Px = P_2(P_1x) \in Y_2$$

$Px \in Y_1 \cap Y_2$ $since$ $x \in H$ was arbitrary. This show P projects H $into$ $Y = Y_1 \cap Y_2$.

Actually, P projects H onto Y. Indeed, if $y \in Y, then\ y \in Y_1$ and $y \in Y_2$

$$Py = P_1P_2y = P_1y = y$$

Then $y \in Y_1 \cap Y_2$, $y \in Y$ $Hence\ Y = Y_1 \cap Y_2$

(⇒)Suppose $P = P_1P_2$ is projection defined on H. It must be self-adjoint

$$P_1P_2 = (P_1P_2)^* = P_2{}^*P_1{}^* = P_2P_1$$

Then $P_1P_2 = P_2P_1$

b) (⇒) If $Y \perp V, then\ Y \cap V = \{0\}\ and\ P_YP_Vx = \{0\}\ for\ all\ x \in H$ by part (a), so that $P_YP_V = 0$

(⇐) $if\ P_YP_V = 0\ then\ for\ every\ y \in Y\ and\ v \in V$

We obtain

$$\langle y, v\rangle = \langle P_Yy, P_Vv\rangle = \langle y, P_YP_Vv\rangle = \langle y, 0\rangle = 0$$

Hence $Y \perp V$ ■

<u>3.1.5 Theorem</u>: Let $P_1\ and\ P_2$ be projections on Hilbert space H. Then:

a) The sum $P = P_1 + P_2$ is a projection on H if and only if $Y_1 = P_1(H)\ and\ Y_2 = P_2(H)$ are orthogonal.

b) If $P = P_1 + P_2$ is a projection, P projects H onto $Y = Y_1 \oplus Y_2$

Proof.

(⇒)Suppose $that\ \ P = P_1{+}P_2$ is a projection.

Let $x \in Y_1 \Rightarrow \|x\|^2 \geq \|(P_1{+}P_2)x\|^2 = \langle (P_1{+}P_2)x, (P_1{+}P_2)x\rangle$

$$= \langle (P_1{+}P_2)^2 x, x\rangle = \langle (P_1{+}P_2)x, x\rangle$$

$$= \langle P_1 x, x\rangle + \langle P_2 x, x\rangle$$

$$= \|x\|^2 + \langle {P_2}^2 x, x\rangle$$

$$= \|x\|^2 + \|P_2 x\|^2 \Rightarrow \|P_2 x\| = 0$$

For any $y \in Y_2$ and $x \in Y_1$

$$\langle x, y\rangle = \langle x, P_2 y\rangle = \langle P_2 x, y\rangle = \langle 0, y\rangle = 0$$

$$\langle x, y\rangle = 0$$

Therefore $\ Y_1 \perp Y_2$

(⇐) If $\ Y_1 \perp Y_2$ then $P_2 P_1 = P_1 P_2 = 0$ whichimplies $P^2 = P$

$$P^2 = (P_1 + P_2)^2 = {P_1}^2 + P_1 P_2 + P_2 P_1 = P_1 + P_2 = P$$

P is idempotent.

$$P^* = (P_1 + P_2)^* = P_1 + P_2 = P$$

P is self-adjoint.

Hence, P is projection.

b) We determine the closed subspace $Y \subset H$ onto which P projects,

Since $P_1 + P_2\ for\ every\ x \in H$

$$y = Px = P_1 x + P_2 x$$

Here $P_1 x\ \in Y_1\ and\ \ P_2 x \in Y_2, hence\ y \in Y_1 \bigoplus Y_2$

So that $Y \subset Y_1 \oplus Y_2 \ldots \ldots \ldots \ldots (1)$

We show that $Y \supset Y_1 \oplus Y_2$

Let $v \in Y_1 \oplus Y_2$ *be arbitary . then* $v = y_1 + y_2$ *, here* $y_1 \in Y_1$ *and* $y_2 \in Y_2$ *. applying* P

Using $Y_1 \perp Y_2$ *, we thus obtain,*

$Pv = P_1(y_1+y_2) + P_2(y_1+y_2) = P_1y_1+P_2y_2 = y_1+y_2 = v$

Hence $v \in Y$ *and* $Y \supset Y_1 \oplus Y_2 \ldots \ldots \ldots \ldots \ldots \ldots \ldots . (2)$

Combining (1) *and* (2) *we have*

$$Y = Y_1 \oplus Y_2$$

3.2 Further properties of projections

3.2.1 Theorem: Let P_1 and P_2 be projections defined on a Hilbert Space H. Denote by $Y_1 = P_1(H)$ and $Y_2 = P_2(H)$ the subspace onto which H is projected by P_1 and P_2, and let $\mathcal{N}(P_1)$ and $\mathcal{N}(P_2)$ be the Null space of these projections. Then the following conditions are equivalent.

1. $P_2P_1 = P_1P_2 = P_1$
2. $Y_1 \subset Y_2 = P_1(H) \subset P_2(H)$
3. $\mathcal{N}(P_1) \supset \mathcal{N}(P_2)$
4. $\|P_1x\| \le \|P_2x\|$
5. $P_1 \le P_2$

Proof

($2 \Rightarrow 1$)Suppose that $Y_1 \subset Y_2 = P_1(H) \subset P_2(H)$

We want show that $P_2P_1 = P_1P_2 = P_1$

For every $x \in H$, we have $P_1x \in Y_1$. Hence, $P_1x \in Y_2$ by (2)

$P_2(P_1x) = P_1x$

$\Rightarrow P_2P_1 = P_1$

$\Rightarrow (P_2P_1)^* = {P_1}^*{P_2}^* = P_1P_2 = P_2P_1 = P_1$. Since, P_1 is self-adjoint

$P_2P_1 = P_1P_2 = P_1$

$(1 \Rightarrow 4)$ Suppose $P_2P_1 = P_1P_2 = P_1$

We want show that $\|P_1x\| \leq \|P_2x\|$ for all $x \in H$

$\Rightarrow \langle P_1x, x\rangle \leq \|x\|^2$, since $P_1x = x$

$\Rightarrow \|P_1x\|^2 \leq \|x\|^2$, for all $x \in H$

$\Rightarrow \|P_1\|^2\|x\|^2 \leq \|x\|^2, \|x\| \neq 0$

We have $\|P_1\| \leq 1$ by (1) $\langle Px, x\rangle = \|Px\|^2$

$\|P_1x\| = \|P_1P_2x\| \leq \|P_1\|\|P_2x\| \leq \|P_2x\|$

$\|P_1x\| \leq \|P_2x\|$

$(4 \Rightarrow 5)$ Suppose that $\|P_1x\| \leq \|P_2x\|$

We want show that $P_1 \leq P_2$

From $\langle Px, x\rangle = \|Px\|^2$ and (4) in present theorem we have for all $x \in H$

$\langle P_1x, x\rangle = \|P_1x\|^2 \leq \|P_2x\|^2 = \langle P_2x, x\rangle$

$P_1 \leq P_2$. By the definition (2.3)

$(5 \Rightarrow 3)$Suppose that $P_1 \leq P_2$

We want show that $\mathcal{N}(P_1) \supset \mathcal{N}(P_2)$

Let $x \in \mathcal{N}(P_2) \Longrightarrow P_2x = 0$ by (3) section 3 and (5) in the present theorem

$\|P_1x\|^2 = \langle P_1x, x\rangle \leq \langle P_2x, x\rangle = \langle 0, x\rangle = 0$

Hence, $P_1 x = 0, x \in \mathcal{N}(P_1)$

$\mathcal{N}(P_1) \supset \mathcal{N}(P_2)$

3.2 .2 Theorem: Let P_1 *and* P_2 be Projections on a Hilbert space H. Then:

a) The difference $P = P_2 - P_1$ is a projection if and only if $Y_1 \subset Y_2$ where $Y_j = P_j(H)$
b) If $P = P_2 - P_1$ is a projection, P projects H onto Y, where Y is the orthogonal complement of Y_1 *in* Y_2

Proof

(⇒) Suppose that $P = P_2 - P_1$ is a projection

We want to show that $Y_1 \subset Y_2$

If $P = P_2 - P_1$ is a projection, $P = P^2$

$P_2 - P_1 = (P_2 - P_1)^2 = {P_2}^2 - P_2P_1 - P_1P_2 + {P_1}^2$

(6) $$P_1P_2 + P_2P_1 = 2P_1$$

Multiplication by P_2 from the left and from the right gives

$$P_2P_1P_2 + P_2P_1 = 2P_2P_1$$

$$P_1P_2 + P_2P_1P_2 = 2\,P_1P_2$$

Hence $P_2P_1P_2 = P_2P_1$, $P_2P_1P_2 = P_1P_2$ and by (6)

(7) $$P_2P_1 = P_1P_2 = P_1$$

$$Y_1 \subseteq Y_2$$

(⇐) Suppose that $Y_1 \subset Y_2$, where $Y_j = P_j(H)$

$$\Rightarrow P^2 = (P_2 - P_1)^2 = {P_2}^2 - P_2P_1 - P_1P_2 + {P_1}^2 = P_2 - P_1$$

P is idempotent

$\Rightarrow P^* = (P_2 - P_1)^* = {P_2}^* - {P_1}^* = P_2 - P_1$

P is self-adjoint

$Hence,\ P$ is projection.

(b) $Y = P(H)$consists of all vectors of the form

(8) $$y = Px = P_2x - P_1x \qquad for\ all\ x \in H$$

Since $P_2P_1 = P_1P_2 = P_1\ implies\ Y_1 \subset Y_2$

$$P_2y = {P_2}^2x - P_2P_1x = P_2x - P_1x = y$$

This shows that y$\in Y_2$, also from (8) and (1)

$$P_1y = P_1P_2x - {P_1}^2x = P_1x - P_1x = 0$$

$\Rightarrow P_1y = 0$

$y \in \mathcal{N}(P_1) = Y_1^{\perp}$

Together $y \in V\ \ where\ V = Y_2 \cap Y_1^{\perp}$, since the projection of H onto $Y_1^{\perp}$ is $I - P_1$, every $v \in V$ is the form

(9) $$v = (\,I - P_1)y_2 \quad , \quad (y_2 \in Y_2)$$

Using again, $P_2P_1 = P_1$, we obtain from (9), since $P_2y_2 = y_2$

$$Pv = (P_2 - P_1)(I - P_1)y_2$$

$$= \left(P_2 - P_2P_1 - P_1 + {P_1}^2\right)y_2$$

$$= (P_2 - P_2P_1 - P_1 + P_1)y_2 = (P_2 - P_1)y_2$$

$$= P_2y_2 - P_1y_2 = y_2 - P_1y_2 = v$$

$Pv = v$ so that $v \in Y$, since $v \in V$ was arbitrary

$$Y \supseteq V$$

$$Y = P(H) = V = Y_2 \cap Y_1^{\perp}$$

<u>3.2.3 Theorem</u>: Let P_n be a monotone increasing sequence of projection P_n defined on a Hilbert Space H. Then:

a) (P_n) is strongly operator convergent, say $P_n x \to Px$ for every $x \in H$, and the limit operator P is a projection defined on H.

b) P projects H onto

$$P(H) = \overline{\bigcup_{n=1} P_n(H)}$$

c) P has the null space

$$\mathcal{N}(P) = \bigcap_{n=1}^{\infty} \mathcal{N}(P_n)$$

Proof

a) Let $m < n$, by assumption, $P_m \leq P_n$, so that we have $P_m(H) \subset P_n(H)$ and $P_n - P_m$ is projection. Hence for every fixed $x \in H$,we obtain by 3.1.3

$$\|P_n x - P_m x\|^2 = \|(P_n - P_m)x\|^2$$
$$= \langle (P_n - P_m)x, x \rangle$$
$$= \|P_n x\|^2 - \|P_m x\|^2$$

Now $\|P_n\| \leq 1$ by 3.1.3, so that $\|P_n x\| \leq \|x\|$ for all n.

Hence, $(\|P_n x\|)$ is bounded a sequence of numbers.$(\|P_n x\|)$is also monotone by 3.2.1. Since$(P_n)$is monotone. Hence $\lim_{n\to\infty} P_n x \longrightarrow Px$ is converges. Since $(P_n x)$ is Cauchy.

$\Longrightarrow \lim_{n\to\infty} P_n x = Px$, since H is complete.

$\Longrightarrow \|P_n x\| \leq \|P_n\|\|x\|$

$\Longrightarrow \|P_n\| \leq \|x\|$

P_n is bounded

$\langle P_n x, x\rangle = \langle x, P_n^* x\rangle = \langle x, P_n x\rangle$

P_n is self-ad joint

$\langle P_n^2 x, x\rangle = \langle P_n x, P_n x\rangle = \langle x, P_n x\rangle$

P is projection

b) We determine $P(H)$. Let $m < n$.Then $P_m \leq P_n$

i.e $P_n - P_m \geq 0$ and $\langle (P_n - P_m)x, x\rangle \geq 0$

The continuity of inner product

$\lim_{n\to\infty}\langle (P_n - P_m)x, x\rangle = \lim_{n\to\infty}\langle x, (P_n - P_m)x\rangle$

$= \lim_{n\to\infty}\langle x, (P - P_m)x\rangle$

$\langle (P - P_m)x, x\rangle \geq 0$

That is $P_m \leq P$ and 3.2.1 yields $P_m(H) \subset P(H)$ for every m. Hence $\cup P_m(H) \subset P(H)$

Furthermore, for every m and for every $x \in H$, we have

$P_m x \in P_m(H) \subset \cup P_m(H)$

Since $P_m x \longrightarrow Px$, $Px \in \overline{\cup P_m(H)}$

Hence $P(H) \subset \overline{\cup p_m(H)}$

$\cup P_m(H) \subset P(H) \subset \overline{\cup P_m(H)}$

$\Rightarrow P(H) = \mathcal{N}(I - P)$

$P(H) = \overline{\cup_{m=1}^{\infty} P_m(H)}$

c) We determine $\mathcal{N}(P)$

$\mathcal{N}(P) = P(H)^{\perp} \subset P_n(H)^{\perp}$ for all n. Since $P(H) \supset P_n(H)$ by part (b).

Hence $\mathcal{N}(P) \subset \cap\, P_n(H)^{\perp} = \cap\, \mathcal{N}(P_n)$

$\Rightarrow \mathcal{N}(P) \subset \cap\, \mathcal{N}(P_n)$ ……………………………. (1)

On other hand, if $x \in \cap\, \mathcal{N}(P_n)$, then $x \in \mathcal{N}(P_n)$ for all n

So that $P_n x = 0$ and $\quad P_n x \longrightarrow Px \implies Px = 0$

That is $x \in \mathcal{N}(P)$.

Since $x \in \mathcal{N}(P_n)$ was arbitrary.

$\cap\, \mathcal{N}(P_n) \subset \mathcal{N}(P)$…………………………………… (2)

By combining (1) and (2) we gets

$\mathcal{N}(P) = \bigcap_{n=1}^{\infty} \mathcal{N}(P_n)$

4. CONCLUSION

The spectral theory mainly deals with a systemic study of inverse operators, their general properties and their relation to the original operators.

Operators on Hilbert space is a basis for a comprehensive study of the spectral theory. In particular, the hermitian or self-adjoint operators are very important in many applications. In this project work, we have generalized this idea by showing that the spectrum of a self-adjoint operator on a Hilbert space also consists entirely of real values.

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