



Superposed One-Mode Subharmonic Generators

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Abstract

We study the squeezing and statistical properties of the light produced by one-mode subharmonic generators. We first obtain c-number Langevin equations with the aid of the master equation for a one-mode subharmonic generator. The solutions of the resulting differential equations are then used to calculate the quadrature variance and squeezing spectrum. Furthermore, employing the same solutions, we also obtain the antinormally ordered characteristic function defined in the Heisenberg picture. With the help of the resulting characteristic function, we determine the Q function of the light produced by a one-mode subharmonic generator. The Q function is then used to calculate the mean and variance of the photon number and the photon number distribution.

On the other hand, we determine the Q function of the light beams produced by a pair of one-mode subharmonic generators. Then employing the resulting Q function, we calculate the mean and variance of the photon number, the photon number distribution, and the quadrature variance. The mean photon number of the light produced by the two subharmonic generators turns out to be twice that of the light produced by one-mode subharmonic generator. In addition, the squeezing of the superposed light beams turns out to be close to 100% below the coherent-state level.

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Chapter 1

Introduction

A considerable attention has been paid to squeezed states of light. The squeezed properties of a single-light mode are described by two Hermitian operators satisfying the uncertain relation $\Delta a_+ \Delta a_- \geq 1$. A light mode is said to be in a squeezed state if either $\Delta a_+ < 1$ and $\Delta a_- > 1$ or vice versa. Squeezed light has potential application in the detection of weak signals and in low-noise communications [1, 2].

Various quantum optical processes such as subharmonic and second harmonic generation produce squeezed light. A one-mode subharmonic generator, consisting of a nonlinear crystal pumped by coherent light and placed in a cavity coupled to a vacuum reservoir, is a prototype source of a single-mode squeezed light. In this system a pump photon of frequency 2ω is down converted into a pair of highly correlated signal photons each of frequency ω .

Subharmonic generator is one of the most reliable sources of squeezed light. A theoretical analysis of the quantum fluctuations and photon statistics of the signal mode produced by a subharmonic generator has been made by a number of authors [3-15]. A maximum of 50% squeezing of the intracavity signal mode produced by the subharmonic generator has been predicted by a number of authors [3, 4, 5, 6].

The main objective of this thesis is to study the squeezing and statistical properties of the light produced by a pair of one-mode subharmonic generators. We carry out our analysis of a one-mode subharmonic generator employing the solutions of c-number Langevin

equations, obtained with the aid of the master equation. We calculate, employing the solutions of the resulting equations, the quadrature variances and the squeezing spectrum. Applying the same solutions, we also obtain the antinormally ordered characteristic function with the aid of which the Q function is determined. In addition, with the aid of Q function, we obtain the mean and variance of the photon number and the photon number distribution.

On the other hand, we determine the Q function of the light beams produced by a pair of one-mode subharmonic generators. Then employing the resulting Q function, we calculate the mean and variance of the photon number, the photon number distribution, and the quadrature variance. The mean photon number of the light produced by the two subharmonic generators turns out to be twice that of the light produced by one-mode subharmonic generator. In addition, the squeezing of the superposed light beams turns out to be close to 100% below the coherent-state level.

Chapter 2

One-Mode Subharmonic Generator

2.1 c-number Langevin equations

We first obtain c-number Langevin equations associated with the normal ordering for the signal mode produced in one-mode subharmonic generation. It proves to be convenient to call a one-mode subharmonic generating system, in a cavity coupled to a vacuum reservoir via a single port-mirror, a one-mode subharmonic generator.

In a one-mode subharmonic generator, a pump photon of frequency 2ω is down converted into a pair of signal photons each of frequency ω . The process of one-mode subharmonic generation is described by the Hamiltonian

$$\hat{H} = \frac{i\lambda}{2}(\hat{b}^\dagger \hat{a}^2 - \hat{b} \hat{a}^{\dagger 2}). \quad (2.1.1)$$

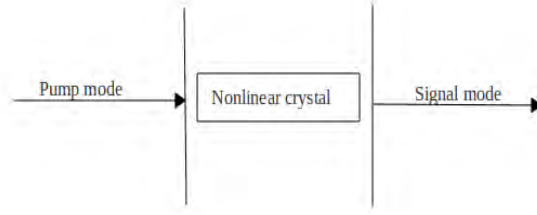


Figure 2.1: A one-mode subharmonic generator

With the pump mode represented by a real and constant c-number β , the process of one-mode subharmonic generation can be described by the Hamiltonian

$$\hat{H} = \frac{i\varepsilon}{2}(\hat{a}^2 - \hat{a}^{\dagger 2}), \quad (2.1.2)$$

where $\hat{a}$ is the annihilation operator for the signal mode and $\varepsilon = \lambda\beta$, with λ being the coupling constant. We recall that the master equation for a cavity mode coupled to a vacuum reservoir has the form

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}_s, \hat{\rho}] + \frac{\kappa}{2} \left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a} \right), \quad (2.1.3)$$

so that on taking into account Eq. (2.1.2), the master equation for the signal mode takes the form

$$\begin{aligned} \frac{d\hat{\rho}}{dt} = & \frac{\varepsilon}{2} \left(\hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^2 + \hat{\rho}\hat{a}^{\dagger 2} - \hat{a}^{\dagger 2}\hat{\rho} \right) \\ & + \frac{\kappa}{2} \left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a} \right), \end{aligned} \quad (2.1.4)$$

in which κ is the cavity damping constant. Now employing the relation

$$\frac{d}{dt}\langle\hat{a}\rangle = Tr\left(\frac{d\hat{\rho}}{dt}\hat{a}\right) \quad (2.1.5)$$

along with Eq.(2.1.4), we have

$$\begin{aligned} \frac{d}{dt}\langle\hat{a}(t)\rangle = & Tr\left[\frac{\varepsilon}{2}\left(\hat{a}^2\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^3 + \hat{\rho}\hat{a}^{\dagger 2}\hat{a} - \hat{a}^{\dagger 2}\hat{\rho}\hat{a}\right)\right. \\ & \left.+ \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2)\right] \end{aligned} \quad (2.1.6)$$

Applying the cyclic property of the trace operation, we readily find

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = \text{Tr}\left(\frac{\varepsilon}{2}(\hat{\rho}\hat{a}^\dagger\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}^\dagger) + \frac{\kappa}{2}(\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a})\right). \quad (2.1.7)$$

Now using the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = 1, \quad (2.1.8)$$

one can easily get

$$\hat{\rho}\hat{a}^\dagger\hat{a}^\dagger\hat{a} = \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}^\dagger - 2\hat{\rho}\hat{a}^\dagger. \quad (2.1.9)$$

Similarly, one can check that

$$\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a} = \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}. \quad (2.1.10)$$

Substituting Eqs. (2.1.9) and (2.1.10) into (2.1.7), we get

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = \text{Tr}\left(\frac{\varepsilon}{2}(-2\hat{\rho}\hat{a}^\dagger) + \frac{\kappa}{2}(-\hat{\rho}\hat{a})\right), \quad (2.1.11)$$

from which follows

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -\frac{\kappa}{2}\langle\hat{a}(t)\rangle - \varepsilon\langle\hat{a}^\dagger(t)\rangle. \quad (2.1.12)$$

Following a similar procedure, we readily find

$$\frac{d}{dt}\left\langle\hat{a}(t)\hat{a}(t)\right\rangle = -\kappa\langle\hat{a}^2(t)\rangle - 2\varepsilon\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle - \varepsilon, \quad (2.1.13)$$

$$\frac{d}{dt}\left\langle\hat{a}^\dagger(t)\hat{a}(t)\right\rangle = -\kappa\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle - \varepsilon\langle\hat{a}^{\dagger 2}(t)\rangle - \varepsilon\langle\hat{a}^2(t)\rangle. \quad (2.1.14)$$

We see that the c-number Langevin equations corresponding to Eqs. (2.1.12), (2.1.13) and (2.1.14) are

$$\frac{d}{dt}\langle\alpha(t)\rangle = -\frac{\kappa}{2}\langle\alpha(t)\rangle - \varepsilon\langle\alpha^*(t)\rangle, \quad (2.1.15)$$

$$\frac{d}{dt}\left\langle\alpha(t)\alpha(t)\right\rangle = -\kappa\langle\alpha^2(t)\rangle - 2\varepsilon\langle\alpha^*(t)\alpha(t)\rangle - \varepsilon, \quad (2.1.16)$$

$$\frac{d}{dt}\left\langle\alpha^*(t)\alpha(t)\right\rangle = -\kappa\langle\alpha^*(t)\alpha(t)\rangle - \varepsilon\langle\alpha^{*2}(t)\rangle - \varepsilon\langle\alpha^2(t)\rangle. \quad (2.1.17)$$

On the basis of Eq. (2.1.15), one can write

$$\frac{d}{dt}\alpha(t) = -\frac{\kappa}{2}\alpha(t) - \varepsilon\alpha^*(t) + f(t), \quad (2.1.18)$$

where $f(t)$ is a noise force whose correlation properties remain to be determined. We note that Eq. (2.1.15) and the expectation value of Eq. (2.1.18) will have the same form if

$$\langle f(t) \rangle = 0. \quad (2.1.19)$$

Moreover, using the relation

$$\frac{d}{dt} \left\langle \alpha(t)\alpha(t) \right\rangle = \left\langle \frac{d\alpha(t)}{dt} \alpha(t) \right\rangle + \left\langle \alpha(t) \frac{d\alpha(t)}{dt} \right\rangle, \quad (2.1.20)$$

along with Eq. (2.1.18), one can readily verify that

$$\frac{d}{dt} \left\langle \alpha(t)\alpha(t) \right\rangle = -\kappa \langle \alpha^2(t) \rangle - 2\varepsilon \langle \alpha^*(t)\alpha(t) \rangle + 2\langle \alpha(t)f(t) \rangle. \quad (2.1.21)$$

Now comparison of Eqs. (2.1.16) and (2.1.21) shows that

$$\langle \alpha(t)f(t) \rangle = -\frac{\varepsilon}{2}. \quad (2.1.22)$$

A formal solution of Eq. (2.1.18) can be written as

$$\alpha(t) = \alpha(0)e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2} (-\varepsilon\alpha^*(t') + f(t')) dt'. \quad (2.1.23)$$

We then note that

$$\langle \alpha(t)f(t) \rangle = \langle \alpha(0)f(t) \rangle e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2} [\langle f(t)f(t') \rangle - \varepsilon \langle \alpha^*(t')f(t) \rangle] dt'. \quad (2.1.24)$$

Since a noise force at a certain time should not affect a system variable at an earlier time, one can write

$$\langle \alpha(0)f(t) \rangle = \langle \alpha(0) \rangle \langle f(t) \rangle = 0 \quad (2.1.25)$$

and

$$\langle \alpha^*(t')f(t) \rangle = \langle \alpha^*(t') \rangle \langle f(t) \rangle = 0. \quad (2.1.26)$$

In view of these results, Eq. (2.1.24) reduces to

$$\langle \alpha(t)f(t) \rangle = \int_0^t e^{-\kappa(t-t')/2} \langle f(t)f(t') \rangle dt', \quad (2.1.27)$$

so that in view of Eq. (2.1.22), there follows

$$\int_0^t e^{\frac{-\kappa(t-t')}{2}} \langle f(t)f(t') \rangle dt' = \frac{-\varepsilon}{2}. \quad (2.1.28)$$

Now on the basis of the relation

$$\int_0^t e^{-a(t-t')/2} \langle f(t)g(t') \rangle dt' = D, \quad (2.1.29)$$

we assert that

$$\langle f(t)g(t') \rangle = 2D\delta(t-t'), \quad (2.1.30)$$

where D is a constant or some function of the time t. We then see that

$$\langle f(t)f(t') \rangle = -\varepsilon\delta(t-t'). \quad (2.1.31)$$

Furthermore, employing the relation

$$\frac{d}{dt} \left\langle \alpha^*(t)\alpha(t) \right\rangle = \left\langle \frac{d\alpha^*(t)}{dt} \alpha(t) \right\rangle + \left\langle \alpha^*(t) \frac{d\alpha(t)}{dt} \right\rangle \quad (2.1.32)$$

along with Eq. (2.1.18) and its complex conjugate, one can easily establish that

$$\frac{d}{dt} \left\langle \alpha^*(t)\alpha(t) \right\rangle = -\kappa \langle \alpha^*(t)\alpha(t) \rangle - \varepsilon \langle \alpha^{*2}(t) \rangle - \varepsilon \langle \alpha^2(t) \rangle + \langle \alpha(t)f^*(t) \rangle + \langle \alpha^*(t)f(t) \rangle. \quad (2.1.33)$$

On comparing this with Eq. (2.1.17), we have

$$\langle \alpha(t)f^*(t) \rangle + \langle \alpha^*(t)f(t) \rangle = 0. \quad (2.1.34)$$

In addition, using Eq. (2.1.18) and its complex conjugate, we find

$$\langle \alpha(t)f^*(t) \rangle = \langle \alpha(0)f^*(t) \rangle e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2} [\langle f^*(t)f(t') \rangle - \varepsilon \langle \alpha^*(t')f^*(t) \rangle] dt' \quad (2.1.35)$$

and

$$\langle \alpha^*(t)f(t) \rangle = \langle \alpha^*(0)f(t) \rangle e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2} [\langle f^*(t')f(t) \rangle - \varepsilon \langle \alpha^*(t')f(t) \rangle] dt'. \quad (2.1.36)$$

On account of the assertion that a noise force at time t should not affect system variables at earlier times, we have

$$\langle \alpha(t) f^*(t) \rangle = \int_0^t e^{-\kappa(t-t')/2} \langle f^*(t) f(t') \rangle dt', \quad (2.1.37)$$

and

$$\langle \alpha^*(t) f(t) \rangle = \int_0^t e^{-\kappa(t-t')/2} \langle f(t) f^*(t') \rangle dt'. \quad (2.1.38)$$

Now taking into account Eqs. (2.1.34), (2.1.37), (2.1.38) and assuming that

$$\langle f^*(t) f(t') \rangle = \langle f(t) f^*(t') \rangle, \quad (2.1.39)$$

we arrive at

$$2 \int_0^t e^{-\kappa(t-t')/2} \langle f^*(t') f(t) \rangle dt' = 0. \quad (2.1.40)$$

Therefore, in view of Eqs. (2.1.29) and (2.1.30), we get

$$\langle f^*(t) f(t') \rangle = \langle f(t) f^*(t') \rangle = 0. \quad (2.1.41)$$

It is worth mentioning that Eqs. (2.1.19), (2.1.31) and (2.1.41) describe the correlation properties of the noise force $f(t)$ associated with the normal ordering.

In order to obtain the solution of Eq. (2.1.18), we introduce a new variable defined by

$$\alpha_{\pm}(t) = \alpha^*(t) \pm \alpha(t). \quad (2.1.42)$$

It can then be shown using Eq. (2.1.18) and its complex conjugate that

$$\frac{d\alpha_{\pm}(t)}{dt} = -\frac{1}{2}\lambda_{\pm}\alpha_{\pm}(t) + f^*(t) \pm f(t), \quad (2.1.43)$$

where

$$\lambda_{\pm} = \kappa \pm 2\varepsilon. \quad (2.1.44)$$

According to Eq. (2.1.43) together with (2.1.44), the equation of evolution of α_{-} does not have a well-behaved solution for $\kappa < 2\varepsilon$. We then identify $\kappa = 2\varepsilon$ as the threshold condition. For $2\varepsilon < \kappa$, the solution of Eq. (2.1.43) can be written as

$$\alpha_{\pm}(t) = \alpha_{\pm}(0)e^{-\lambda_{\pm}t/2} + \int_0^t e^{-\frac{1}{2}\lambda_{\pm}(t-t')} f^*(t') \pm f(t') dt'. \quad (2.1.45)$$

Now with the aid of Eq. (2.1.42), one can write

$$\alpha(t) = \frac{1}{2}(\alpha_+(t) - \alpha_-(t)), \quad (2.1.46)$$

so that in view of Eq. (2.1.45), we get

$$\alpha(t) = A_+(t)\alpha(0) + A_-(t)\alpha^*(0) + B_+(t) - B_-(t), \quad (2.1.47)$$

in which

$$A_{\pm} = \frac{1}{2} \left(e^{-\frac{1}{2}\lambda_+ t} \pm e^{-\frac{1}{2}\lambda_- t} \right) \quad (2.1.48)$$

and

$$B_{\pm} = \frac{1}{2} \int_0^t e^{-\frac{1}{2}\lambda_{\pm}(t-t')/2} [f^*(t') \pm f(t')] dt'. \quad (2.1.49)$$

2.2 The Q function

We now proceed to calculate the Q function for the signal mode applying the anti-normally ordered characteristic function defined in the Heisenberg picture by

$$\Phi_a(z, t) = Tr(\hat{\rho}(0) e^{-z^* \hat{a}(t)} e^{z \hat{a}^\dagger(t)}). \quad (2.2.1)$$

Using the identity

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{B}} e^{\hat{A}} e^{[\hat{A}, \hat{B}]}, \quad (2.2.2)$$

Eq. (2.2.1) can be rewritten as

$$\Phi_a(z, t) = e^{-zz^*} Tr(\hat{\rho}(0) e^{z \hat{a}^\dagger(t)} e^{-z^* \hat{a}(t)}). \quad (2.2.3)$$

The characteristic function can be expressed in terms of c-number variables associated with the normal ordering as

$$\Phi_a(z, t) = e^{-zz^*} \langle \exp(z\alpha^* - z^*\alpha) \rangle. \quad (2.2.4)$$

If we assume the signal mode to be initially in a vacuum state, then we see from (2.1.47) along with (2.1.49) that

$$\langle \alpha(t) \rangle = 0. \quad (2.2.5)$$

Thus we observe that $\alpha(t)$ is a Gaussian variable with zero mean. On account of this, Eq. (2.2.4) can be put in the form [3]

$$\Phi_a(z, t) = e^{-zz^*} \exp\left[\frac{1}{2} \langle (z\alpha^* - z^*\alpha)^2 \rangle\right]. \quad (2.2.6)$$

It then follows that

$$\Phi_a(z, t) = e^{-zz^*} \exp\left[\frac{1}{2} \langle (z^2\alpha^{*2} + z^{*2}\alpha^2 - 2z^*z\alpha^*\alpha) \rangle\right]. \quad (2.2.7)$$

Using Eqs. (2.1.47) and (2.1.49) it can be readily verified that

$$\langle \alpha^2 \rangle = \langle \beta^2 \rangle, \quad (2.2.8)$$

$$\langle \alpha^{*2} \rangle = \langle \beta^{*2} \rangle, \quad (2.2.9)$$

$$\langle \alpha^*\alpha \rangle = \langle \beta^*\beta \rangle, \quad (2.2.10)$$

where

$$\beta = \beta_+ - \beta_-. \quad (2.2.11)$$

Furthermore, one easily finds with the aid of Eqs. (2.1.49) and (2.2.11) that

$$\langle \beta^2 \rangle = \langle \beta_+^2 \rangle + \langle \beta_-^2 \rangle - 2\langle \beta_+\beta_- \rangle, \quad (2.2.12)$$

$$\langle \beta^{*2} \rangle = \langle \beta_+^2 \rangle + \langle \beta_-^2 \rangle + 2\langle \beta_+\beta_- \rangle, \quad (2.2.13)$$

and

$$\langle \beta^*\beta \rangle = \langle \beta_+^2 \rangle - \langle \beta_-^2 \rangle. \quad (2.2.14)$$

Applying Eq. (2.1.49) along with (2.1.31) and (2.1.41), it can be established that

$$\langle \beta_+^2 \rangle = \frac{1}{4} \int_0^t e^{-\frac{1}{2}\lambda_+(t-t')/2} dt' dt'' \left[\langle f^*(t') f^*(t'') \rangle + \langle f^*(t') f(t'') \rangle + \langle f(t') f^*(t'') \rangle + \langle f(t') f(t'') \rangle \right]. \quad (2.2.15)$$

Taking into account Eqs. (2.1.31) and (2.1.41), we get

$$\langle \beta_+^2 \rangle = \frac{1}{4} \int_0^t e^{-\frac{1}{2}\lambda_+(t-t')/2} dt' dt'' \left[-\varepsilon \sigma(t-t') + (-\varepsilon \sigma(t-t')) \right], \quad (2.2.16)$$

so that on carrying out the integration, there follows

$$\langle \beta_+^2 \rangle = -\frac{\varepsilon}{2\lambda_+} (1 - e^{\lambda_+ t}). \quad (2.2.17)$$

Similarly, one can check that

$$\langle \beta_-^2 \rangle = -\frac{\varepsilon}{2\lambda_-} (1 - e^{\lambda_- t}) \quad (2.2.18)$$

and

$$\langle \beta_+ \beta_- \rangle = 0. \quad (2.2.19)$$

Now on account of this results, we have

$$\langle \beta^2 \rangle = \langle \beta^{*2} \rangle = -\frac{\varepsilon}{2\lambda_+} (1 - e^{\lambda_+ t}) - \frac{\varepsilon}{2\lambda_-} (1 - e^{\lambda_- t}) \quad (2.2.20)$$

$$\langle \beta^* \beta \rangle = -\frac{\varepsilon}{2\lambda_+} (1 - e^{\lambda_+ t}) + \frac{\varepsilon}{2\lambda_-} (1 - e^{\lambda_- t}). \quad (2.2.21)$$

Thus in view of Eqs. (2.2.20) and (2.2.21), Eq. (2.2.4) takes the form

$$\Phi_a(z, t) = \exp \left[-a z z^* + b(z^2 + z^{*2})/2 \right], \quad (2.2.22)$$

in which

$$a = 1 - \frac{\varepsilon}{2\lambda_+} (1 - e^{-\lambda_+ t}) + \frac{\varepsilon}{2\lambda_-} (1 - e^{-\lambda_- t}), \quad (2.2.23)$$

and

$$b = -\left(\frac{\varepsilon}{2\lambda_+} (1 - e^{-\lambda_+ t}) + \frac{\varepsilon}{2\lambda_-} (1 - e^{-\lambda_- t}) \right). \quad (2.2.24)$$

We now proceed to determine the Q function for the one-mode subharmonic generator. To this end, the Q function for the one-mode subharmonic generator is expressible as

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi^2} \int d^2 z \phi_a(z, t) e^{(z^* \alpha - z \alpha^*)}, \quad (2.2.25)$$

so that on combining Eq. (2.2.22) with (2.2.25), there follows

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi^2} \int d^2 z e^{(-a z z^* + z^* \alpha - z \alpha^* + b(z^2 + z^{*2})/2)}. \quad (2.2.26)$$

Thus on performing the integration employing the relation

$$\int \frac{d^2 z}{\pi} \exp \left[-a z z^* + b z + c z^* + A z^2 + B z^{*2} \right] = \left[\frac{1}{a^2 - 4AB} \right]^{\frac{1}{2}} \exp \left(\frac{abc + Ac^2 + Bb^2}{a^2 - 4AB} \right), a > 0 \quad (2.2.27)$$

we get

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi} (u^2 - v^2)^{\frac{1}{2}} \exp \left[-u \alpha \alpha^* + v(\alpha^2 + \alpha^{*2})/2 \right]. \quad (2.2.28)$$

where

$$u = \frac{a}{(a^2 - b^2)}, \quad (2.2.29)$$

and

$$v = \frac{b}{(a^2 - b^2)}. \quad (2.2.30)$$

This represents the Q function for the light produced by a one-mode subharmonic generator.

2.3 Quadrature squeezing

We wish here to study the squeezing properties of the signal mode produced by the one-mode subharmonic generator.

2.3.1 Quadrature variance

We now proceed to calculate the quadrature variance employing the c-number Langevin equations. To this end, we note that the variance of the plus and minus quadratures is expressible as

$$(\Delta a_{\pm})^2 = 1 + \langle : (\hat{a}_{\pm}(t), \hat{a}_{\pm}(t)) : \rangle, \quad (2.3.1)$$

where $::$ stands for normal ordering

$$\hat{a}_+ = (\hat{a}^\dagger + \hat{a}), \quad (2.3.2)$$

and

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}). \quad (2.3.3)$$

This quadrature variance can be written in terms of c-number variables associated with the normal ordering as

$$(\Delta a_{\pm})^2 = 1 \pm \langle \alpha_{\pm}(t), \alpha_{\pm}(t) \rangle, \quad (2.3.4)$$

in which

$$\alpha_{\pm}(t) = \alpha^* \pm \alpha. \quad (2.3.5)$$

Expression (2.3.4) can be rewritten as

$$(\Delta a_{\pm})^2 = 1 \pm \langle \alpha_{\pm}^2 \rangle \mp \langle \alpha_{\pm} \rangle^2. \quad (2.3.6)$$

We note that

$$\langle \alpha_{\pm}(t) \rangle = 0. \quad (2.3.7)$$

Hence expression (2.3.6) can be put in the form

$$(\Delta a_{\pm})^2 = 1 \pm \langle \alpha_{\pm}^2 \rangle. \quad (2.3.8)$$

In addition, using (2.1.43), we easily get

$$\frac{d}{dt} \langle \alpha_{\pm}^2(t) \rangle = -\lambda_{\pm} \langle \alpha_{\pm}^2(t) \rangle + 2 \langle \alpha_{\pm}(t) f^*(t) \rangle \pm 2 \langle \alpha_{\pm}(t) f(t) \rangle. \quad (2.3.9)$$

From Eqs. (2.1.19), (2.1.22) and (2.1.41), we note that

$$\langle \alpha(t) f^*(t) \rangle = \langle \alpha^*(t) f(t) \rangle = 0. \quad (2.3.10)$$

Now with the aid of (2.3.7) along with (2.1.22) and (2.3.10), we readily find

$$\langle \alpha_{\pm}(t) f^*(t) \rangle = -\frac{\varepsilon}{2} \quad (2.3.11)$$

$$\langle \alpha_{\pm}(t) f(t) \rangle = \mp \frac{\varepsilon}{2}. \quad (2.3.12)$$

Therefore, in view of these results Eq. (2.3.9) can be rewritten as

$$\frac{d}{dt} \langle \alpha_{\pm}^2(t) \rangle = -\lambda_{\pm} \langle \alpha_{\pm}^2(t) \rangle - 2\varepsilon. \quad (2.3.13)$$

A formal solution of Eq. (2.3.13) can be written as

$$\langle \alpha_{\pm}^2(t) \rangle = \langle \alpha_{\pm}^2(0) \rangle e^{-\lambda_{\pm} t} + \int_0^t e^{-\lambda_{\pm}(t-t')} [-2\varepsilon] dt'. \quad (2.3.14)$$

With the cavity mode initially in a vacuum state, the solution of this equation turns out to be

$$\langle \alpha_{\pm}^2(t) \rangle = -\frac{2\varepsilon}{\lambda_{\pm}} \left[1 - e^{-\lambda_{\pm} t} \right]. \quad (2.3.15)$$

Thus combination of (2.3.8) and (2.3.15) yields

$$(\Delta a_{\pm})^2 = 1 \mp \frac{2\varepsilon}{\lambda_{\pm}} \left[1 - e^{-\lambda_{\pm} t} \right]. \quad (2.3.16)$$

We observe that the signal mode is in a squeezed state and the squeezing occurs in the plus quadrature. The squeezing increases with time and reaches its maximum value at steady state

$$(\Delta a_{\pm})^2 = 1 \mp \frac{2\varepsilon}{\lambda_{\pm}}. \quad (2.3.17)$$

Moreover, on taking into account (2.1.44), we see that at steady state and at threshold

$$(\Delta a_{+})^2 = \frac{1}{2}. \quad (2.3.18)$$

And the variance of the minus quadrature goes to infinity. We note that at steady state and at threshold there is a 50% squeezing of the cavity signal mode.

2.3.2 Squeezing spectrum

We now proceed to obtain the spectrum of quadrature fluctuations, of the output signal mode produced by the one-mode subharmonic generator. We define the squeezing spectrum of a given output light with central frequency ω_0 by

$$S_{\pm}^{out}(\omega) = \frac{1}{\pi} \text{Re} \int_0^{\infty} \langle : \hat{a}_{\pm}^{out}(t), \hat{a}_{\pm}^{out}(t + \tau) : \rangle_{ss} e^{i(\omega - \omega_0)\tau} d\tau, \quad (2.3.19)$$

where Re denotes the real part and $::$ stands for normal ordering

$$\hat{a}_{+}^{out}(t) = \hat{a}^{\dagger out}(t) + \hat{a}^{out}(t), \quad (2.3.20)$$

and

$$\hat{a}_{-}^{out}(t) = i(\hat{a}^{\dagger out}(t) - \hat{a}^{out}(t)). \quad (2.3.21)$$

Upon integrating both sides of Eq. (2.3.19) over ω , we get

$$\int_{-\infty}^{\infty} S_{\pm}^{out}(\omega) d\omega = \int_{-\infty}^{\infty} \langle : \hat{a}_{\pm}^{out}(t), \hat{a}_{\pm}^{out}(t + \tau) : \rangle_{ss} e^{-i\omega_0\tau} \delta(\tau) d\tau. \quad (2.3.22)$$

Therefore steady state quadrature variance turns out to be

$$\int_{-\infty}^{\infty} S_{\pm}^{out}(\omega) d\omega = (: \Delta a_{\pm}^{out} :)^2, \quad (2.3.23)$$

Where $(: \Delta a_{\pm}^{out} :)^2$ is the normally-ordered quadrature variance of the output light. On the basis of this relation, we observe that $S_{\pm}^{out}(\omega) d\omega$ is the quadrature variance of the output light in the interval between ω and $\omega + d\omega$. The normally-ordered quadrature variance can then be written in the interval between $\omega' = -\lambda$ and $\omega = \lambda$ as [3]

$$(\Delta a_{\pm}^{out})_{\pm\lambda}^2 = \int_{-\lambda}^{\lambda} S_{\pm}^{out}(\omega') d\omega', \quad (2.3.24)$$

in which $\omega' = \omega - \omega_0$.

The squeezing spectrum is expressible in terms of c-number variables associated with the normal-ordering as

$$S_{\pm}^{out}(\omega) = \pm \frac{1}{\pi} \text{Re} \int_0^{\infty} \langle : \alpha_{\pm}^{out}(t), \alpha_{\pm}^{out}(t + \tau) : \rangle_{ss} e^{i(\omega - \omega_0)\tau} d\tau, \quad (2.3.25)$$

in which

$$\alpha_{\pm}^{out}(t) = \alpha^{*out}(t) \pm \alpha^{out}(t). \quad (2.3.26)$$

According to the input-output relation, we have

$$\hat{a}^{out}(t) = \sqrt{\kappa}\hat{a}(t) - \hat{a}_{in}(t). \quad (2.3.27)$$

For a cavity mode coupled to a vacuum reservoir and with the cavity mode represented by c-number variables associated with the normal ordering, one can write

$$\alpha_{\pm}^{out}(t) = \sqrt{\kappa}\alpha_{\pm}(t), \quad (2.3.28)$$

and

$$\alpha_{\pm}^{out}(t + \tau) = \sqrt{\kappa}\alpha_{\pm}(t + \tau). \quad (2.3.29)$$

Therefore, in view of (2.3.28) and (2.3.29), the squeezing spectrum of the output light can be put in the form

$$S_{\pm}^{out}(\omega) = \pm \frac{\kappa}{\pi} Re \int_0^{\infty} \langle : \alpha_{\pm}(t), \alpha_{\pm}(t + \tau) : \rangle_{ss} e^{i(\omega - \omega_0)\tau} d\tau. \quad (2.3.30)$$

We now proceed to obtain the squeezing spectrum of the output signal mode produced by the one-mode subharmonic generator. To this end, we note that the solution of Eq. (2.1.43) can be written as

$$\alpha_{\pm}(t + \tau) = \alpha_{\pm}(t) e^{-\lambda_{\pm}\tau/2} + \int_0^{\tau} e^{-\lambda_{\pm}(\tau - \tau')/2} [f^*(t + \tau') \pm f(t + \tau')] d\tau', \quad (2.3.31)$$

so that on multiplying by $\alpha_{\pm}(t)$ and taking the expectation value of the resulting expression, we get

$$\begin{aligned} \langle \alpha_{\pm}(t) \alpha_{\pm}(t + \tau) \rangle = & \langle \alpha_{\pm}^2(t) \rangle e^{-\lambda_{\pm}\tau/2} + \int_0^{\tau} e^{-\lambda_{\pm}(\tau - \tau')/2} [\langle \alpha_{\pm}(t) f^*(t + \tau') \rangle \\ & \pm \langle \alpha_{\pm}(t) f(t + \tau') \rangle] d\tau'. \end{aligned} \quad (2.3.32)$$

On account of the assertion that a noise force at time t should not affect system variable at earlier times, we have

$$\langle \alpha_{\pm}(t) \alpha_{\pm}(t + \tau) \rangle = \langle \alpha_{\pm}^2(t) \rangle e^{-\lambda_{\pm} \tau / 2}. \quad (2.3.33)$$

Upon substituting Eq. (2.3.33) along with (2.3.15) and (2.3.7) into (2.3.30), we have

$$S_{\pm}^{out}(\omega) = \mp \frac{2\varepsilon\kappa}{\pi\lambda_{\pm}} \text{Re} \int_0^{\infty} e^{-[\frac{\lambda_{\pm}}{2} - i(\omega - \omega_0)]\tau} d\tau. \quad (2.3.34)$$

Now on the basis of the relation

$$\int_0^{\infty} e^{-ax} dx = \frac{1}{a}, a > 0, \quad (2.3.35)$$

we then see that

$$S_{\pm}^{out}(\omega) = \mp \frac{2\varepsilon\kappa}{\pi\lambda_{\pm}} \text{Re} \left(\frac{1}{\frac{\lambda_{\pm}}{2} - i(\omega - \omega_0)} \right) \quad (2.3.36)$$

by rationalizing the denominator of the above expression, the squeezing spectrum of the output signal mode is found to be

$$S_{\pm}^{out}(\omega) = \mp \frac{\varepsilon\kappa/\pi}{(\omega - \omega_0)^2 + [\frac{\kappa}{2} \pm \varepsilon]^2}. \quad (2.3.37)$$

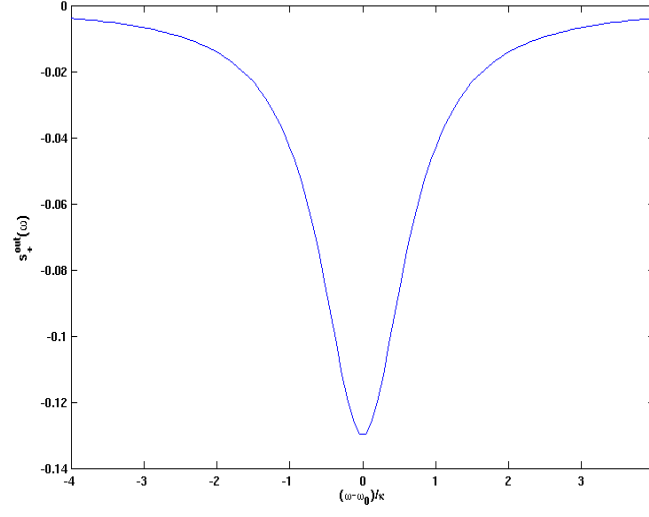


Figure 2.2: a Plot of the squeezing spectrum $S_+^{out}(\omega)$ versus $(\frac{\omega - \omega_0}{\kappa})$ for $\frac{\varepsilon}{\kappa} = 0.2$.

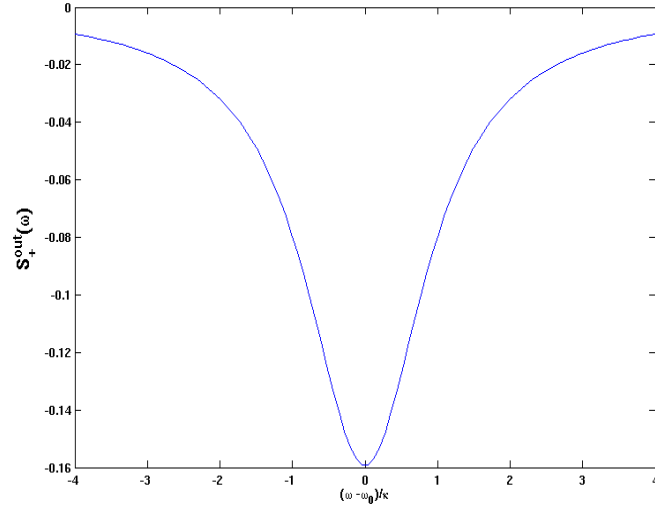


Figure 2.3: a Plot of the squeezing spectrum $S_+^{out}(\omega)$ versus $(\frac{\omega - \omega_0}{\kappa})$ for $\frac{\varepsilon}{\kappa} = 0.5$.

From Figs. 2.2 and 2.3 we see that the minimum value of the squeezing spectrum occurs at $\omega = \omega_0$ and the value of this spectrum increases with increasing or decreasing ω .

We note that at threshold ($\varepsilon = \frac{\kappa}{2}$) Eq. (2.3.37) becomes

$$S_{\pm}^{out}(\omega) = -\frac{\kappa}{2} \left(\frac{\kappa/\pi}{(\omega - \omega_0)^2 + \kappa^2} \right). \quad (2.3.38)$$

Where $(-\frac{\kappa}{2})$ is the normally-ordered quadrature variance of the output signal mode. Finally, Upon integrating Eq. (2.3.38) over the interval between $\omega' = -\lambda$ and $\omega' = \lambda$

$$(\Delta a_+^{out})_{\pm\lambda}^2 = -\frac{\kappa^2}{2\pi} \int_{-\lambda}^{\lambda} \frac{d\omega'}{\omega'^2 + \kappa^2}, \quad (2.3.39)$$

we readily get

$$(\Delta a_+^{out})_{\pm\lambda}^2 = -\frac{\kappa}{\pi} \tan^{-1} \left(\frac{\lambda}{\kappa} \right). \quad (2.3.40)$$

We have calculated the quadrature variance for $\kappa = 0.8$ and different frequency intervals. the quadrature variance of the output signal mode turns out to be $(\Delta a_+^{out})_{\pm 0.4}^2 = -0.1180$, $(\Delta a_+^{out})_{\pm 0.8}^2 = -0.2000$, $(\Delta a_+^{out})_{\pm 1.6}^2 = -0.2820$. These results show that the quadrature variance decreases as the frequency interval increases, with the minimum quadrature variance being -0.4 . We immediately note that the squeezing of the output signal mode increases as the frequency interval increases.

2.4 Photon statistics

It would be helpful to classify the photon statistics of light modes based on the relation between the variance and the mean of the photon number. Hence the photon statistics of a light mode for which $(\Delta n)^2 = \bar{n}$ is referred to as poissonian and the photon statistics of a light mode for which $(\Delta n)^2 > \bar{n}$ is called super-poissonian. Otherwise the photon statistics is said to be sub-poissonian.

2.4.1 The mean and variance of the photon number

The mean number of photons of the signal mode is expressible as

$$\langle \hat{n} \rangle = \int d^2\alpha Q(\alpha^*, \alpha) n_a(\alpha^*, \alpha), \quad (2.4.1)$$

where

$$n_a(\alpha^*, \alpha) = \alpha^* \alpha - 1, \quad (2.4.2)$$

is the c-number function corresponding to the operator function $\hat{n}(\hat{a}^\dagger, \hat{a})$ in the anti-normal order.

On account of Eqs. (2.2.28) and (2.4.2), expression (2.4.1) can be rewritten as

$$\bar{n} = (u^2 - v^2)^{\frac{1}{2}} \int \frac{d^2\alpha}{\pi} \exp[-u\alpha^*\alpha + v(\alpha^2 + \alpha^{*2})/2] \alpha^*\alpha - 1, \quad (2.4.3)$$

this can be put in the form

$$\bar{n} = (u^2 - v^2)^{\frac{1}{2}} \frac{d^2}{dn dm} \int \frac{d^2\alpha}{\pi} \exp[-u\alpha^*\alpha + n\alpha + m\alpha^* + v(\alpha^2 + \alpha^{*2})/2]_{n=m=0} - 1, \quad (2.4.4)$$

so that on carrying out the integration using (2.2.27) and applying the condition $n = m = 0$, we get

$$\bar{n} = \frac{u}{u^2 - v^2} - 1. \quad (2.4.5)$$

Now on account of Eqs. (2.2.29) and (2.2.30), we easily find

$$\bar{n} = a - 1. \quad (2.4.6)$$

And in view of (2.2.23), the mean photon number takes the form

$$\bar{n} = -\frac{\varepsilon}{2\lambda_+}(1 - e^{\lambda_+ t}) + \frac{\varepsilon}{2\lambda_-}(1 - e^{\lambda_- t}). \quad (2.4.7)$$

Thus at steady state, we see that

$$\bar{n} = \frac{2\varepsilon^2}{\kappa^2 - 4\varepsilon^2}. \quad (2.4.8)$$

We next proceed to obtain the variance of the photon number of the signal mode. The photon-number variance, defined by

$$(\Delta n)^2 = \langle (\hat{a}^\dagger \hat{a})^2 \rangle - \bar{n}^2. \quad (2.4.9)$$

Using Eq. (2.1.8), one can check that

$$\langle (\hat{a}^\dagger \hat{a})^2 \rangle = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - 3\langle \hat{a}^\dagger \hat{a} \rangle - 2, \quad (2.4.10)$$

so that expression (2.4.9), can be written as

$$(\Delta n)^2 = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - \bar{n}^2 - 3\bar{n} - 2. \quad (2.4.11)$$

Thus employing the Q function

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \int d^2\alpha Q(\alpha, t) \alpha^{*2} \alpha^2, \quad (2.4.12)$$

This can be put in the form

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = (u^2 - v^2)^{\frac{1}{2}} \frac{d^4}{d\eta^2 dz^2} \int \frac{d^2\alpha}{\pi} \exp[-u\alpha^* \alpha + \eta\alpha + z\alpha^* + \frac{v}{2}(\alpha^2 + \alpha^{*2})]_{\eta=z=0}. \quad (2.4.13)$$

On carrying out the integration using (2.2.27), we get

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{d^4}{d\eta^2 dz^2} \exp \left[\frac{uz\eta + \frac{v}{2}z^2 + \frac{v}{2}\eta^2}{(u^2 - v^2)} \right]_{\eta=z=0}, \quad (2.4.14)$$

so that carrying out the differentiation and applying the condition $z = \eta = 0$, one easily finds

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{2u^2 + v^2}{(u^2 - v^2)^2}. \quad (2.4.15)$$

On account of (2.2.29) and (2.2.30), we get

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = 2a^2 + b^2. \quad (2.4.16)$$

Now with the aid of Eqs. (2.4.6) and (2.4.16), Eq. (2.4.11) can be put in the form

$$(\Delta n)^2 = a^2 + b^2 - a. \quad (2.4.17)$$

Finally, applying Eqs. (2.2.23) and (2.2.24), the variance of the photon number is expressible as

$$(\Delta n)^2 = \frac{\varepsilon^2}{2\lambda_+^2} \left[1 - e^{-\lambda_+ t} \right]^2 + \frac{\varepsilon^2}{2\lambda_-^2} \left[1 - e^{-\lambda_- t} \right]^2 + \frac{\varepsilon}{2\lambda_-} \left(1 - e^{-\lambda_- t} \right) - \frac{\varepsilon}{2\lambda_+ t} \left(1 - e^{-\lambda_+ t} \right). \quad (2.4.18)$$

The photon number variance takes at steady state the form

$$(\Delta n)^2 = \left(\frac{\varepsilon \kappa}{\kappa^2 - 4\varepsilon^2} \right)^2 + \bar{n}^2 + \bar{n}. \quad (2.4.19)$$

This implies that the photon statistics of the light produced by one-mode subharmonic generator is super-Poissonian.

2.4.2 The photon number distribution

We next proceed to determine, using the Q function, the photon number distribution for the signal mode. The photon number distribution for the signal mode in one-mode subharmonic generator can be put in the form

$$P(n, t) = \frac{\pi}{n!} \frac{\partial^{2n}}{\partial \alpha^{*n} \partial \alpha^n} \left[Q(\alpha, t) e^{\alpha^* \alpha} \right]_{\alpha=\alpha^*=0}. \quad (2.4.20)$$

Thus employing (2.2.28), the photon number distribution can be written as

$$P(n, t) = \frac{1}{n!} (u^2 - v^2)^{\frac{1}{2}} \frac{\partial^{2n}}{\partial \alpha^{*n} \partial \alpha^n} \exp[(1-u)\alpha^* \alpha + \frac{v}{2}(\alpha^2 + \alpha^{*2})]_{\alpha^*=\alpha=0}, \quad (2.4.21)$$

upon expanding the exponential function in power series, we have

$$\exp\left((1-u)\alpha^* \alpha\right) = \sum_{k=0}^{\infty} \frac{(1-u)^k (\alpha^* \alpha)^k}{k!}, \quad (2.4.22)$$

$$\exp\left(\frac{v}{2}\alpha^{*2}\right) = \sum_{\ell=0}^{\infty} \frac{(v)^{\ell} (\alpha^{*2})^{\ell}}{2^{\ell} \ell!}, \quad (2.4.23)$$

and

$$\exp\left(\frac{v}{2}\alpha^2\right) = \sum_{m=0}^{\infty} \frac{(v)^m \alpha^{2m}}{2^m m!}. \quad (2.4.24)$$

On account of the power series expansion, expression (2.4.21) can be rewritten as

$$P(n, t) = \frac{1}{n!} (u^2 - v^2)^{\frac{1}{2}} \times \sum_{\ell m k} \frac{(1-u)^k (v)^{\ell+m}}{2^{\ell+m} \ell! m! k!} \frac{\partial^{2n}}{\partial \alpha^{*n} \partial \alpha^n} \left(\alpha^{*k+2\ell} \alpha^{k+2m} \right)_{\alpha^*=\alpha=0}, \quad (2.4.25)$$

moreover, using the identity

$$\frac{\partial^m}{\partial x^m} x^n = \frac{n!}{(n-m)!} x^{n-m}, \quad (2.4.26)$$

and applying the condition $\alpha = \alpha^* = 0$, one easily finds

$$P(n, t) = \frac{1}{n!} (u^2 - v^2)^{\frac{1}{2}} \times \sum_{\ell m k} \frac{(1-u)^k (v)^{\ell+m}}{2^{\ell+m} \ell! m! k!} \cdot \frac{(2\ell+k)!(2m+k)!}{(2\ell+k-n)!(2m+k-n)!} \delta_{2\ell+k,n} \delta_{2m+k,n}. \quad (2.4.27)$$

Hence in view of the property of the Kronecker delta, we see that $m = \ell$ and $k = n - 2\ell$ and on taking into account that a factorial is defined for nonnegative integers, we have

$$P(n, t) = (u^2 - v^2)^{\frac{1}{2}} \sum_{\ell=0}^{[n]} n! \frac{(1-u)^{n-2\ell} v^{2\ell}}{2^{2\ell} \ell!^2 (n-2\ell)!}. \quad (2.4.28)$$

where $[n] = \frac{n}{2}$ for even n and $[n] = \frac{(n-1)}{2}$ for odd n . From this result, we note that there is a finite probability to find odd number of signal photons. Although the signal photons are generated in pairs, it is possible for an odd number of signal photons to leave the cavity via the port-mirror. This must be then the reason for the possibility to observe an odd number of signal photons inside the cavity.

Chapter 3

Superposed One-Mode Subharmonic Generators

In chapter 2 we discussed, the statistical and squeezing properties of the light produced by one-mode subharmonic generator. Here we seek to study the statistical and squeezing properties of the light produced by a pair of one-mode subharmonic generators. To this end, first we determine the Q function for this light mode. With the aid of the resulting Q function, we calculate the mean and the variance of the photon number, the photon number distribution, and the quadrature variance.

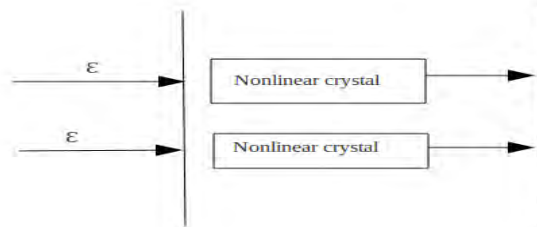


Figure 3.1: Superposed one-mode subharmonic generators

3.1 The Q function

We now proceed to determine the Q function for the light produced by superposed sub-harmonic generators. The Q function for this light mode is expressible as

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi} \int d^2\beta d^2\gamma Q(\beta^*, \alpha - \gamma, t) Q(\gamma^*, \alpha - \beta, t) \\ \times \exp[-\alpha^*\alpha - \beta^*\beta - \gamma^*\gamma + \alpha^*\beta + \alpha\beta^* + \alpha^*\gamma + \alpha\gamma^* \\ - \beta^*\gamma - \beta\gamma^*]. \quad (3.1.1)$$

Using Eq. (2.2.28), one can write

$$Q(\beta^*, \alpha - \gamma, t) = \frac{1}{\pi} \pi (u^2 - v^2)^{\frac{1}{2}} \exp[-u\beta^*\alpha + u\beta^*\gamma + v(\alpha^2 + \gamma^2 + \beta^{*2} - 2\alpha\gamma)/2]. \quad (3.1.2)$$

Using Eq. (2.2.28), we can also write

$$Q(\gamma^*, \alpha - \beta, t) = \frac{1}{\pi} \pi (u^2 - v^2)^{\frac{1}{2}} \exp[-u\gamma^*\alpha + u\gamma^*\beta + v(\alpha^2 + \beta^2 + \gamma^{*2} - 2\alpha\beta)/2]. \quad (3.1.3)$$

Introducing (3.1.2) and (3.1.3) into Eq. (3.1.1), we get

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi} \left((u^2 - v^2)(u^2 - v^2) \right)^{\frac{1}{2}} \exp\left[\frac{-2\alpha^*\alpha - 2v\alpha^2}{2}\right] \\ \times \int \frac{d^2\beta}{\pi} \exp\left[-\beta^*\beta + (\alpha^* + v\alpha)\beta + (\alpha - u\alpha)\beta^* + \frac{v}{2}(\beta^2 + \beta^{*2})\right] \\ \times \int \frac{d^2\gamma}{\pi} \exp\left[-\gamma\gamma^* + (\alpha^* + v\alpha + u\beta^* - \beta^*)\gamma + (\alpha - u\alpha - \beta + u\beta)\gamma^* \right. \\ \left. + \frac{v}{2}(\gamma^{*2} + \gamma^2)\right], \quad (3.1.4)$$

thus on performing the integration over γ , employing the relation given by (2.2.27) we find

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi} \left(\frac{(u^2 - v^2)^2}{1 - v^2} \right)^{\frac{1}{2}} \exp\left[\frac{-2\alpha^*\alpha + (u^2v - v^3 + v)\alpha^2 + v\alpha^{*2}}{2}\right] \int \frac{d^2\beta}{\pi} \\ \times \exp\left(\left[\frac{-(2u - u^2 - v^2)\beta^*\beta + [(uv + v^3 - u^2v - v)\alpha + (u - v^2)\alpha^*]\beta}{(1 - v^2)}\right] \right. \\ \left. + \frac{2[(u - u^2)\alpha + (uv - v)\alpha^*]\beta^* + (u^2v - v^3 + 2v - 2uv)(\beta^2 + \beta^{*2})}{2(1 - v^2)}\right). \quad (3.1.5)$$

Carrying out the integration over β , once more employing the relation given by (2.2.27), the Q function for the light mode produced by a pair of subharmonic generators finally takes the form

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi} \sqrt{S(t)} \exp \left[-T(t) \alpha^* \alpha + M(t) (\alpha^2 + \alpha^{*2}) \right], \quad (3.1.6)$$

where

$$S(t) = \frac{u^2 - v^2}{4 - 4u + u^2 - v^2}, \quad (3.1.7)$$

$$T(t) = \frac{-u^2 + 2u + v^2}{4 - 4u + u^2 - v^2} \quad (3.1.8)$$

and

$$M(t) = \frac{v}{4 - 4u + u^2 - v^2}. \quad (3.1.9)$$

Now let us integrate this function over the variable α , we have

$$\int d^2 \alpha Q(\alpha^*, \alpha, t) = \frac{1}{\pi} \sqrt{S(t)} \int d^2 \alpha \exp \left[-T(t) \alpha^* \alpha + M(t) (\alpha^2 + \alpha^{*2}) \right]. \quad (3.1.10)$$

Carrying out the integration using the relation given by (2.2.27), we readily find

$$\int d^2 \alpha Q(\alpha^*, \alpha, t) = 1, \quad (3.1.11)$$

which shows that the Q function is normalized.

3.2 Mean photon number

We next seek to calculate the mean photon number of the light mode produced by a pair of subharmonic generators. To this end, the mean photon number is expressible in terms of the Q function as

$$\bar{n} = \int d^2 \alpha Q(\alpha^*, \alpha) \alpha^* \alpha - 1. \quad (3.2.1)$$

On account of Eq. (3.1.6), expression (3.2.1) can be written as

$$\bar{n} = \frac{1}{\pi} \sqrt{S(t)} \int d^2\alpha \exp[-T(t)\alpha^*\alpha + M(t)(\alpha^2 + \alpha^{*2})] \alpha^*\alpha - 1. \quad (3.2.2)$$

This is equivalently rewritten in the form

$$\bar{n} = \sqrt{S(t)} \frac{d}{dz} \frac{d}{d\eta} \int \frac{d^2\alpha}{\pi} \exp[-T(t)\alpha^*\alpha + z\alpha + \eta\alpha^* + M(t)(\alpha^2 + \alpha^{*2})]_{z=\eta=0} - 1, \quad (3.2.3)$$

so that performing the integration employing the relation given by (2.2.27), we obtain

$$\bar{n} = \sqrt{S(t)} \frac{d^2}{dzd\eta} \left[\frac{1}{T(t)^2 - 4M^2(t)} \right]^{\frac{1}{2}} \exp \left[\frac{T(t)z\eta + M(t)(\eta^2 + z^2)}{T(t)^2 - 4M^2(t)} \right]_{z=\eta=0} - 1. \quad (3.2.4)$$

Hence, carrying out the differentiation and applying the condition $z = \eta = 0$, one easily finds

$$\bar{n} = \frac{2u - u^2 + v^2}{u^2 - v^2} - 1. \quad (3.2.5)$$

Now on account of Eqs. (2.2.29) and (2.2.30), we easily find

$$\bar{n} = 2a - 2. \quad (3.2.6)$$

And in view of (2.2.23), the mean photon number takes the form

$$\bar{n} = -\frac{\varepsilon}{\lambda_+} (1 - e^{\lambda_+ t}) + \frac{\varepsilon}{\lambda_-} (1 - e^{\lambda_- t}). \quad (3.2.7)$$

Thus at steady state, we see that

$$\bar{n} = \frac{4\varepsilon^2}{\kappa^2 - 4\varepsilon^2}. \quad (3.2.8)$$

Which is twice the mean photon number of the light produced by a one-mode subharmonic generator.

3.3 Variance of the photon number

We next proceed to obtain the variance of the photon number for the light mode produced by a pair of one-mode subharmonic generators. The variance of the photon-number is given by

$$(\Delta n)^2 = \langle \hat{n}^2 \rangle - \bar{n}^2. \quad (3.3.1)$$

Using Eq .(2.1.8), one can check that

$$\langle (\hat{a}^\dagger \hat{a})^2 \rangle = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - 3\langle \hat{a}^\dagger \hat{a} \rangle - 2, \quad (3.3.2)$$

so that expression (3.3.1) can be written as

$$(\Delta n)^2 = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - \bar{n}^2 - 3\bar{n} - 2. \quad (3.3.3)$$

Thus employing the Q function, we have

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \int d^2\alpha Q(\alpha, t) \alpha^{*2} \alpha^2. \quad (3.3.4)$$

This can be put in the form

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \sqrt{S(t)} \frac{d^4}{da^2 db^2} \int \frac{d^2\alpha}{\pi} \exp \left[-T(t) \alpha^* \alpha + a\alpha + b\alpha^* + M(t)(\alpha^2 + \alpha^{*2}) \right]_{a=b=0}, \quad (3.3.5)$$

on carrying out the integration using (2.2.27) , we get

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \left(\frac{S(t)}{T^2(t) - 4M^2(t)} \right)^{\frac{1}{2}} \frac{d^4}{da^2 db^2} \exp \left[\frac{abT(t) + M(t)(b^2 + a^2)}{(T^2(t) - 4M^2(t))} \right]_{a=b=0}, \quad (3.3.6)$$

so that carrying out the differentiation and applying the condition $a = b = 0$, one easily finds

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{2(2u - u^2 + v^2)}{(u^2 - v^2)^2}, \quad (3.3.7)$$

on account of (2.2.29) and (2.2.30) , we get

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = 8a^2 + 4b^2 - 8a + 2. \quad (3.3.8)$$

Now with the aid of Eqs. (3.2.6) and (3.3.8), Eq. (3.3.3) can be put in the form

$$(\Delta n)^2 = 4a^2 + 4b^2 - 6a + 2. \quad (3.3.9)$$

Finally, applying Eqs. (2.2.29) and (2.2.30), the variance of the photon number is expressible as

$$\begin{aligned} (\Delta n)^2 = & \frac{2\varepsilon^2}{\lambda_+^2} \left[1 - e^{-\lambda_+ t} \right]^2 + \frac{2\varepsilon^2}{\lambda_-^2} \left[1 - e^{-\lambda_- t} \right]^2 \\ & + \frac{\varepsilon}{\lambda_-} (1 - e^{-\lambda_- t}) - \frac{\varepsilon}{\lambda_+} (1 - e^{-\lambda_+ t}). \end{aligned} \quad (3.3.10)$$

At steady state, we see that

$$(\Delta n)^2 = \left(\frac{2\varepsilon\kappa}{\kappa^2 - 4\varepsilon^2} \right)^2 + \bar{n}^2 + \bar{n}. \quad (3.3.11)$$

We observe that variance of the photon number of the light mode produced by a pair of one-mode subharmonic generators is greater than that of the light produced by a one-mode subharmonic generator. We can also say that the photon statistics of this light is super-poissonian.

3.4 Photon number distribution

We next proceed to determine the photon number distribution for the light mode produced by a pair of one-mode subharmonic generators. The photon number distribution can be written as

$$P(n, t) = \frac{\pi}{n!} \frac{\partial^{2n}}{\partial \alpha^{*n} \partial \alpha^n} \left[Q(\alpha^*, \alpha, t) e^{\alpha^* \alpha} \right]_{\alpha=\alpha^*=0}, \quad (3.4.1)$$

thus employing (3.1.6) along with (3.4.1) the photon number distribution can be put in the form

$$P(n, t) = \frac{1}{n!} \sqrt{S(t)} \frac{\partial^{2n}}{\partial \alpha^{*n} \partial \alpha^n} \exp \left[(1 - T(t)) \alpha^* \alpha + M(t) (\alpha^2 + \alpha^{*2}) \right]_{\alpha^*=\alpha=0}, \quad (3.4.2)$$

on account of the power series expansions

$$e^{1-T(t)\alpha^*\alpha} = \sum_{i=0}^{\infty} \frac{(1-T(t))^i}{i!} (\alpha^* \alpha)^i, \quad (3.4.3)$$

$$e^{M(t)\alpha^2} = \sum_{j=0}^{\infty} \frac{(M(t))^j}{j!} \alpha^{2j}, \quad (3.4.4)$$

and

$$e^{P(t)\alpha^{*2}} = \sum_{m=0}^{\infty} \frac{(M(t))^m}{m!} \alpha^{*2m}. \quad (3.4.5)$$

Expression (3.4.2) can be put in the form

$$P(n) = \frac{1}{n!} \sqrt{S(t)} \sum_{ijm} \frac{(1-T(t))^i M(t)^{j+m}}{i! j! m!} \frac{\partial^{2n}}{\partial \alpha^{*n} \partial \alpha^n} [\alpha^{*2m+i} \alpha^{2j+i}]_{\alpha=\alpha^*=0}, \quad (3.4.6)$$

so that carrying out the differentiation using the identity given by (2.4.26) and applying the condition $\alpha = \alpha^* = 0$, one easily finds

$$P(n) = \frac{1}{n!} \sqrt{S(t)} \sum_{ijm} \frac{(1 - T(t))^i M(t)^{j+m}}{i!j!m!} \frac{(2m+i)!(2j+i)!}{(2m+i-n)!(2j+i-n)!} \delta_{2m+i,n} \delta_{2j+i,n}. \quad (3.4.7)$$

Hence in view of the property of the Kronecker delta, we see that

$$m = j, \quad (3.4.8)$$

and

$$i = n - 2j. \quad (3.4.9)$$

and on taking into account that a factorial is defined for non-negative integers, we have

$$P(n) = \sqrt{S(t)} \sum_{j=0}^{[n]} n! \frac{(1 - T(t))^{n-2j} M(t)^{2j}}{j!^2 (n-2j)!}. \quad (3.4.10)$$

where $[n] = \frac{n}{2}$ for even n and $[n] = \frac{(n-1)}{2}$ for odd n . From this result, we note that there is a finite probability to find odd number of signal photons. Although the signal photons are generated in pairs, it is possible for an odd number of signal photons to leave the cavity via the port-mirror. This must be then the reason for the possibility to observe an odd number of signal photons inside the cavity

3.5 Quadrature variance

In this section we seek to calculate, employing the Q function, the quadrature variances of the light mode produced by a pair of one-mode subharmonic generators. The squeezing properties of a single-mode light are described by two hermitian operators

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a} \quad (3.5.1)$$

and

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}). \quad (3.5.2)$$

These operators satisfy the commutation relation

$$[\hat{a}_+, \hat{a}_-] = 2i. \quad (3.5.3)$$

We define the quadrature variance by

$$(\Delta a_+)^2 = \langle \hat{a}_+^2 \rangle - \langle \hat{a}_+ \rangle^2 \quad (3.5.4)$$

and

$$(\Delta a_-)^2 = \langle \hat{a}_-^2 \rangle - \langle \hat{a}_- \rangle^2. \quad (3.5.5)$$

The expressions for the variance of the plus and minus quadratures, can be written as

$$(\Delta a_+)^2 = 1 + 2\langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}^{\dagger 2} \rangle + \langle \hat{a}^2 \rangle - \langle \hat{a}^\dagger \rangle^2 - \langle \hat{a} \rangle^2 - 2\langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle \quad (3.5.6)$$

and

$$(\Delta a_-)^2 = 1 + 2\langle \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^{\dagger 2} \rangle - \langle \hat{a}^2 \rangle + \langle \hat{a}^\dagger \rangle^2 + \langle \hat{a} \rangle^2 - 2\langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle \quad (3.5.7)$$

Now with the aid of Eq. (3.1.6) one can write

$$\langle \hat{a}^{\dagger 2} \rangle = \sqrt{S(t)} \frac{d^2}{dz^2} \int \frac{d^2 \alpha}{\pi} \exp[-T(t)\alpha^* \alpha + z\alpha^* + M(t)(\alpha^2 + \alpha^{*2})]_{z=0}, \quad (3.5.8)$$

on carrying out the integration using (2.2.27), we get

$$\langle \hat{a}^{\dagger 2} \rangle = \left(\frac{S(t)}{T^2(t) - 4M^2(t)} \right)^{\frac{1}{2}} \frac{d^2}{dz^2} \exp \left[\frac{M(t)z^2}{(T^2(t) - 4M^2(t))} \right]_{z=0}. \quad (3.5.9)$$

Carrying out the differentiation and applying the condition $z = 0$, one easily finds

$$\langle \hat{a}^{\dagger 2} \rangle = \frac{2v}{u^2 - v^2}. \quad (3.5.10)$$

It can also be shown in a similar manner that

$$\langle \hat{a}^2 \rangle = \frac{2v}{u^2 - v^2}, \quad (3.5.11)$$

and

$$\langle \hat{a} \rangle = \langle \hat{a}^\dagger \rangle = 0. \quad (3.5.12)$$

Substituting Eqs. (3.5.10), (3.5.11), (3.5.12) and (3.2.6) into Eqs. (3.5.6) and (3.5.7) , the quadrature variance turns out to be

$$(\Delta a_{\pm})^2 = 1 \pm 4a \pm 4b - 4. \quad (3.5.13)$$

Now on account of Eqs. (2.2.29) and (2.2.30), we get

$$(\Delta a_{\pm})^2 = 1 \mp \frac{4\varepsilon}{\lambda_{\pm}}(1 - e^{-\lambda_{\pm}t}). \quad (3.5.14)$$

This shows that the signal mode is in a squeezed state and the squeezing occurs in the plus quadrature. The squeezing increases with time and reaches its maximum value at steady state.

$$(\Delta a_{+})^2 = 1 - \frac{4\varepsilon}{\kappa + 2\varepsilon}. \quad (3.5.15)$$

Moreover, we see that the variance of the minus quadrature goes to infinity. We note that at steady state and for $\kappa = 0.8$ and $\varepsilon = 0.35$ there is a 93% squeezing of the superposed light mode.

Chapter 4

Conclusion

We have obtained, using the master equation, c-number Langevin equations for a one-mode subharmonic generator. Applying the solutions of the resulting differential equations, we have calculated the quadrature variance and the squeezing spectrum. We have found that at threshold and at steady state there is a 50% squeezing of the intracavity signal mode. We have also seen that the quadrature variance of the output signal mode decreases as the frequency interval increases, with the minimum quadrature variance being -0.4. This implies that the squeezing of the output signal mode increases as the frequency interval increases.

On the other hand, applying the same solutions, we have obtained the antinormally ordered characteristic function. By using this characteristic function, we have determined the Q function of the light produced by a one-mode subharmonic generator, which is then used to calculate the mean and variance of the photon number and the photon number distribution.

Finally, we have determined the Q function of the light beams produced by a pair of one-mode subharmonic generators. Using this Q function we have obtained the mean and the variance of the photon number. Our result shows that the mean of the photon number for this case is twice that of the light produced by a one-mode subharmonic generator . In

addition, the variance of the photon number is greater than that of the light produced by a one mode subharmonic generator. Moreover, employing the same Q function we have calculated the quadrature variance and the photon number distribution. We have found that at steady state for values of ε less than and close to 0.5κ , the degree of squeezing of the light produced by a pair of one-mode subharmonic generators is close to 100% below the coherent-state level.

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Declaration

This thesis is my original work, has not been presented for a degree in any other University and that all the sources of material used for the thesis have been dully acknowledged.

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