ADDIS ABABA UNIVERSITY

SCHOOL OF GRADUATE STUDIES

DEPARTMENT OF MATHEMATICS

GRADUATE PROJECT REPORT ON

Solution of some Retarded Functional Differential Equation by Schauder's Fixed point Theorem

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

Compiled By: *Tariku Birabasa*

Advisor: *Tadesse Abdi (PhD)*

January, 2011

Addis Ababa, Ethiopia

TABLE OF CONTENTS

Content **Page**

Declaration

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other similar title to me.

The Author

Permission

This is to certify that this project is a record of the research work by Mr. Tariku Birabasa in the department of Mathematics, Addis Ababa University, under my supervision. I thereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

Tadesse Abdi (PhD)

ACKNOWLEDGEMENT

I am forever grateful to my project advisor Dr. Tadesse Abdi for his constant encouragement and valuable suggestions, cooperation in providing necessary references, supervising the whole work and constructive comments.

I would like to thank my uncle Bogale Gelana for many faithful interactions and providing me a crucial writing and reading material, his laptop, which help me perform my work as per the time table.

I should also mention here that it was a great pleasure, to work and study with the 11 members of students in our field of study, Differential Equation, in Mathematics Department of Addis Ababa University.

Tariku Birabasa

ABSTRACT

So far, many things were said about differential equations without time delay, the so called ordinary differential equations or partial differential equations, and their solutions. The fixed point theorems have been used to show the existence and uniqueness of solution of initial value problem of these equations. Since time delay occurs naturally in just about every interaction of the real world, here in this project we see some differential equations with time delay, the so called functional differential equations or delay differential equations in general and use Schauder's fixed point theorem to show that a solution of a retarded functional differential equation is a fixed point of an integral operator obtained by variation of parameter method in particular.

NOTATIONS

I used the following notations in this project that should be clear for the reader.

- $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$ is the standard inner product in $\mathbb{R}^n$.
- $\| x \| = (\langle x, x\rangle)^{1/2}$ is a norm.
- $\mathbb{R}^+ = (0, \infty)$ is the set of positive real numbers.
- $\overline{\mathbb{R}^+} = [0, \infty)$ is the set of nonnegative real numbers.
- $\overline{\mathbb{R}_{t_o}} = [t_o - r_o, \infty)$, t_o , $r_o \in \overline{\mathbb{R}^+}$
- $C_{t_o} = C^o\left(\overline{\mathbb{R}_{t_o}}\right)$ is the space of all continuous functions on $\overline{\mathbb{R}_{t_o}}$.
- Co(A) = $\bigcap\{K \subseteq X : A \subseteq K$ is convex$\}$ is convex hull of A (a smallest convex set containing A) in the space X.
- $C([a, b], \mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[a, b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. i.e. for $\phi \in C([a, b], \mathbb{R}^n)$, then the norm of ϕ is defined by $\| \phi \| = \sup_{a \le \theta \le b} | \phi |$,where $| \phi |$ is a norm in $\mathbb{R}^n$.
- $C(X) = C^0(X)$ is the space of all continuous functions on X .
- $C^k(X)$ is the space of all functions whose derivative of order $\le k$ are continuous. Or the space of k times continuously differentiable functions.

INTRODUCTION

The structure of the paper is organized sequentially in such a way that it will be clear for the reader. The preliminary part contains some known definitions such as Metric Spaces, Normed Linear Spaces, Banach Spaces and Lipschitz Continuity which motivates the reader achieve the goal of this paper.

Chapter 1 is about fixed point theorems especially Banach-Cacciopoli and Schauder's theorems are stated and proved. Chapter 2 is devoted to some concepts about Functional Differential Equations. It tries to generalize equations with rate of changes of the present state (ordinary differential equations or partial differential equations) to equations with rate of changes of the past state (delay differential equations or functional differential equations). A substantial portion of the paper, Chapter 3, is devoted to show the solution of some Retarded Functional Differential Equations is the fixed point of an integral operator obtained by variation of constants formula.

PRELIMINARIES

Banach Spaces

Fixed point theorems can be considered in metric spaces where distance is used. Here we will use it in Banach spaces where norm is used since it is applied to many areas of current interest in analysis.

Definition 0.1: Let X be a non empty set and $d: X \times X \to \overline{\mathbb{R}^+}$ a function. Then *d* is called ***metric*** on X if the following properties hold.

i. $d(x, y) = 0$ if and only if $x = y$ for some $x, y \in X$;
ii. $d(x, y) = d(y, x)$ for all $x, y \in X$;
iii. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The value of metric *d* at (x, y) is distance between *x* and *y*; and the ordered pair (X, d) is called ***metric space***.

Definition 0.2: A norm on a linear space X is a functional whose value at x is denoted by $|| x ||$ with the following properties:

1. $|| x_1 + x_2 || \leq || x_1 || + || x_2 ||$, for all $x_1, x_2 \in X$;
2. $|| \alpha x || = | \alpha | || x ||$, for all $x \in X$ and scalar α ;
3. $|| x || \geq 0$ and $|| x || = 0$ if $x = 0$, for all $x \in X$.

$(X, ||.||)$ is called ***normed linear space***. Then every normed space $(X, ||.||)$ is a metric space (X, d) with induced metric $d(x, y) = || x - y ||$.

Definition 0.3: A sequence $\{x_n\}$ is said to be a ***Cauchy Sequence*** if for each $\varepsilon > 0$ there exists a positive integer *N* such that $| x_n - x_m | < \varepsilon$ for all $n \geq m \geq N$.

Definition 0.4: A normed linear space X is ***complete*** if every Cauchy sequence in X converges in X.

Definition 0.5: A complete normed linear space is called ***Banach space.***

Definition 0.6: Let $U \subseteq \mathbb{R}^n$ be open. A function $f: U \to \mathbb{R}$ is said to be:

i. Uniformly Lipschitz continuous if and only if there exists $L \in \mathbb{R}$ such that $|| f(x) - f(y) || \leq L || x - y ||$, for all $x, y \in U$. Here L is called Lipschitz constant.

ii. Lipschitz continuous if and only if for all $x \in U$ there exists a neighborhood V of x such that the restriction of f to $U \cap V$, $f|_{(U \cap V)}$ is uniformly Lipschitz continuous.

Definition 0.7: Let S and B be Banach spaces. A transformation T: V ⊆ $S \to B$ such that

$|| Tx - Ty || \leq L || x - y ||$ for some $L \in \mathbb{R}^+$ is said to be:

a) a ***contraction map*** on V if $0 < L < 1$

b) ***non expansive map*** on V if $L = 1$

Definition 0.8: A point $x \in U$ is said to be a ***fixed point*** of a transformation $T: U \to U$ if

$$Tx = x$$

Definition 0.9: Let D is a subset of a Banach space X and $T: D \to X$ be a map. Given $\varepsilon > 0$, a point $x \in D$ with $|| x - Tx || < \varepsilon$ is called an **ε-fixed point**.

Remark 0.1 A convergent sequence is bounded. A Cauchy sequence is convergent and hence it is bounded. So it has a bounded subsequence.

CHAPTER 1

FIXED POINT THEOREM

1.1. Banach-Cacciopoli Theorem

Cauchy-Piano and Picard-Lindeleöf theorems used fixed point theorem to solve the problem of existence of solutions of system of certain first order initial value problems of ordinary differential equations. Uniqueness of the solution is guaranteed by Lipschitz continuity.

Consider a system of first order initial value problem with ordinary differential equation

$$\begin{aligned} y'(t) &= f(t, y(t)) \\ y(0) &= y_o \end{aligned} \tag{1.1}$$

, where $f: I \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $I = [0, b]$

and an integral equation

$$y(t) = y_o + \int_0^t f\big(s, y(s)\big)ds \tag{1.2}$$

Then $y \in C^1(I)$ solves equation (1.1) if and only if $y \in C^o(I)$ solves equation (1.2) where for $U \in C^1(I)$ a norm is $\mid U \mid_1 =$ $\max\left\{\sup_{t\in I} \mid U(t) \mid , \sup_{t\in I} \mid U'(t) \mid\right\}$ and for $U \in C(I)$ a norm is $\mid U \mid_0 = \sup_{t\in I} \mid u(t) \mid$. ([3] , p. 6).

The equivalence of the above statement is shown as follows. To show the forward implication we have

$$\begin{aligned} y(t) &= \quad y_o + \int_0^t f\big(s, y(s)\big)ds \\ y'(t) &= \quad 0 + f\big(t, y(t)\big) \end{aligned}$$

$$= \ f(t, y(t))$$

and

$$y(0) = \ y_o + \underbrace{\int_0^0 f\big(s, y(s)\big)ds}_{=0}$$

$$= y_o$$

Conversely suppose y solves (1.1), then we have

$$\begin{cases} y \ is \ C^1(I) \\ y' = f\big(t, y(t)\big) \text{ and } y(0) = \ y_o \end{cases}$$

$$\frac{dy}{dt} = f\big(t, y(t)\big)$$

$$\int_{y_o}^{y} d\eta = \ \int_0^t f\big(s, y(s)\big)ds$$

$$\eta \ |_{y_o}^{y} = \ \int_0^t f\big(s, y(s)\big)ds$$

$$y - y_o = \int_0^t f\big(s, y(s)\big)ds$$

$$y = \ y_o \ + \int_0^t f\big(s, y(s)\big)ds$$

Define an ***integral operator*** $T: C(I) \to C(I)$ by

$$(Ty)(t) = y_o + \int_0^t f\big(s, y(s)\big)ds$$

$$= y(t)$$

Then the above equivalence is expressed briefly by “ y solves equation (1.1) if and only if $(Ty)(t) = y(t)$.”

In other words, classical solutions to equation (1.1) are fixed points of the integral operator.

Next we will see the following fixed point theorems.

Theorem 1.1: (Banach-Cacciopoli)

Let $T: V \to V$ is a contraction, where V is a closed subset of a Banach space X, then T has a unique fixed point in V.

Proof: since $T: V \to V$ is a contraction, then

$$\| Tx - Ty \| \le \lambda \| x - y \| \quad ,0 < \lambda < 1$$

Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in V such that $x_{n+1} = Tx_n$

$$\begin{aligned} | x_{n+1} - x_n | &= | Tx_n - Tx_{n-1} | \\ &\le \lambda | x_n - x_{n-1} | \quad , \qquad T \text{ is a contraction} \\ &= \lambda | Tx_{n-1} - Tx_{n-2} | \, , \quad Tx_n = x_{n+1} \\ &\le \lambda(\lambda | x_{n-1} - x_{n-2} |) \, , \qquad T \text{ is a contraction} \\ &= \lambda^2 | x_{n-1} - x_{n-2} | \end{aligned}$$

$$| x_{n+1} - x_n | \le \lambda^2 | x_{n-1} - x_{n-2} | \qquad (*)$$

Continuing in this fashion we find

$$| x_{n+1} - x_n | \le \lambda^i | x_{n+1-i} - x_{n-i} | \, , 1 \le i \le n$$

That is, $| x_{n+1} - x_n | \le \lambda^n | x_1 - x_o |$ (**)

In particular for $m > n > 0$, we have $x_n, x_{n+1}, \dots, x_{m-1}, x_m$

$$\begin{aligned} | x_m - x_n | &= | (x_m - x_{m-1}) + (x_{m-1} - x_n) | \\ &\le | x_m - x_{m-1} | + | x_{m-1} - x_n | \end{aligned}$$

$$= \mid x_m - x_{m-1} \mid + \mid (x_{m-1} - x_{m-2}) + (x_{m-2} - x_n) \mid$$

$$\leq \mid x_m - x_{m-1} \mid + \mid x_{m-1} - x_{m-2} \mid + \mid x_{m-2} - x_n \mid$$

$$\vdots$$

$$\leq \mid x_m - x_{m-1} \mid + \mid x_{m-1} - x_{m-2} \mid + \cdots + \mid x_{n+2} - x_{n+1} \mid + \mid x_{n+1} - x_n \mid$$

$$\leq \lambda^{m-1} \mid x_1 - x_o \mid + \lambda^{m-2} \mid x_1 - x_o \mid + \cdots + \lambda^{n+1} \mid x_1 - x_o \mid$$

$$+ \lambda^n \mid x_1 - x_o \mid \quad \text{by } (**)$$

$$= (\lambda^{m-1} + \lambda^{m-2} + \cdots + \lambda^{n+1} + \lambda^n) \mid x_1 - x_o \mid$$

$$= \lambda^n \left(\lambda^{(m-n)-1} + \lambda^{(m-n)-2} + \cdots + \lambda + 1\right) \mid x_1 - x_o \mid$$

$$= \lambda^n \left(\frac{1-\lambda^{m-n}}{1-\lambda}\right) \mid x_1 - x_o \mid \quad , \quad \text{since } 0 < \lambda < 1$$

$$\leq \frac{\lambda^n}{1-\lambda} \mid x_1 - x_o \mid \quad , \quad \text{since } 0 < \lambda < 1 \text{ we have } 1 - \lambda^{m-n} \leq 1$$

Therefore $\mid x_m - x_n \mid \leq \frac{\lambda^n}{1-\lambda} \mid x_1 - x_o \mid$. Which implies $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. And hence it has a convergent subsequence, say $\left\{x_{n_k}\right\}_{k=0}^{\infty}$. Since the Banach space X is complete there exists $\bar{x} \in X$ with

$$\lim_{k\to\infty} x_{n_k} = \bar{x}$$

The existence of the fixed point will be shown as

$$\| Tx_{n_k} - Tx_{n_k} \| = 0$$

$$\lim_{k\to\infty} \| Tx_{n_k} - Tx_{n_k} \| = 0$$

$$0 = \lim_{k\to\infty} \| Tx_{n_k} - Tx_{n_k} \| = \lim_{k\to\infty} \| x_{n_{k+1}} - Tx_{n_k} \|$$

$$= \| \lim_{k\to\infty} \left(x_{n_{k+1}} - Tx_{n_k}\right) \| \text{ , by continuity of norm in Banach space.}$$

$= \ || \lim_{k\to\infty} x_{n_{k+1}} - \lim_{k\to\infty} Tx_{n_k} \ ||$, definition of limit.

$= \ || \ \bar{x} - \lim_{k\to\infty} Tx_{n_k} \ ||$, $\lim_{k\to\infty} x_{n_k} = \bar{x}$

$= \ || \ \bar{x} - T\bar{x} \ ||$, $\lim_{k\to\infty} Tx_{n_k} = T\bar{x}$ by the continuity of T .

Therefore $0 = || \ \bar{x} - T\bar{x} \ ||$ and since we are in Banach space $\bar{x} - T\bar{x} = 0$. This implies

$$T\bar{x} = \bar{x}$$

Hence T has a fixed point $\bar{x}$.

Since V is a closed subset of Banach space, then $\bar{x} \in V$. Hence T has a fixed point in V .

To show the uniqueness of this fixed point, assume to the contrary that there exists

$$\bar{y} \in V \setminus \{\bar{x}\} \text{ such that } T\bar{y} = \bar{y}$$

Now $|| \ \bar{x} - \bar{y} \ || = \ || \ T\bar{x} - T\bar{y} \ ||$, T has fixed points $\bar{x}$ and $\bar{y}$

$\leq \lambda \ || \ \bar{x} - \bar{y} \ ||$, T is a contraction and $0 < \lambda < 1$

$|| \ \bar{x} - \bar{y} \ || \leq \lambda \ || \ \bar{x} - \bar{y} \ ||$

$|| \ \bar{x} - \bar{y} \ || - \lambda \ || \ \bar{x} - \bar{y} \ || \leq 0$

$\underbrace{(1-\lambda)}_{>0} \ || \ \bar{x} - \bar{y} \ || \leq 0$, but since $0 < \lambda < 1$, then $1-\lambda > 0$. This indicates

$$|| \ \bar{x} - \bar{y} \ || \leq 0$$

But since norm cannot be negative, we have

$$|| \ \bar{x} - \bar{y} \ || = 0$$

Then by definition of norm this implies that $\bar{x} - \bar{y} = 0$. From which we can conclude

$$\bar{x} = \bar{y}$$

Therefore T has a unique fixed point. ■

To apply this theorem to the above initial value problem (1.1), we consider Lipschitz condition on f in $I = [0, b]$

Let $(Ty_1)(t) = y_o + \int_0^b f(s, y_1(s))ds$ and $(Ty_2)(t) = y_o + \int_0^b f(s, y_2(s))ds$

$$|(Ty_1)(t) - (Ty_2)(t)| = |\int_0^b f(s, y_1(s))ds - \int_0^b f(s, y_2(s))ds|$$

$$\leq \int_0^b | f(s, y_1(s))ds - f(s, y_2(s)) | \, ds$$

$$\leq \int_0^b L \, | \, y_1(s) - y_2(s) \, | \, ds \quad , f \text{ is a contraction}$$

$$= bL \, | \, y_1(s) - y_2(s) \, |$$

$|(Ty_1)(t) - (Ty_2)(t)| \leq bL \, | \, y_1(s) - y_2(s) \, |$, this is a contraction if $bL < 1$

Now L is fixed, we take b small enough so that $bL < 1$. Then T becomes a contraction map. This gives a unique fixed point which is a solution of (1.1) on [0, b].

1.2. Schauder's Fixed point Theorem

Schauder's fixed point theorem shows us that a continuous map on a compact convex subset of a Banach space has a fixed point. The main idea is to approximate compact maps with finite dimension ranges. Before proving this theorem let us see the following theorem.

Theorem 1.2: Let D be a closed subset of a normed linear space X and $F: D \to X$ is a compact continuous map. Then F has a fixed point if and only if it has an ε-fixed point.

Note that F is compact means F(D) is contained in some compact subset of X. ([3]).

<u>Proof</u>:

To show the backward implication, assume that F has an ε-fixed point for each $\varepsilon > 0$.

Now for each $n = \{1,2, \ldots\}$ let d_n is a $\frac{1}{n}$-fixed point for F.

i.e.

$$\|\, d_n - F(d_n) \,\| < \frac{1}{n}$$

which implies that $d_n \to F(d_n)$ as $n \to \infty$ in S.

Since F is compact, F(D) is contained in a compact subset K of X , and therefore there exists a subsequence S of integers and an $x \in K$ such that

$$F(d_n) \to x \in K \text{ (as } n \to \infty \text{) in S.}$$

Since D is closed we have $x \in D$.

The continuity of F implies that $F(d_n) \to F(x)$ as $n \to \infty$ in S.

$$\|\, F(d_n) - F(x) \,\| \to 0$$
$$\|\, x - F(x) \,\| = 0$$
$$F(x) = x$$

Therefore F has a fixed point.

Conversely assume that F has a fixed point. Then there exists

$$x \in D \quad \text{such that } F(x) = x$$

Let $\varepsilon > 0$ be given, then $\|\, F(x) - x \,\| = 0 < \varepsilon$

$$\|\, F(x) - x \,\| < \varepsilon$$

Then F has an ε-fixed point ■

Theorem 1.3: (Schauder's fixed point theorem)

Let C be a closed, convex subset of a normed linear space X. Then every compact continuous map $F\!: C \to C$ has at least one fixed point.

Proof: By Theorem1.2 with C = D, it suffices to show that F has an ε-fixed point for every $\varepsilon > 0$.

Fix $\varepsilon > 0$, then there exist a finite dimensional, continuous map $F_\varepsilon : C \to C$ with $|| F_\varepsilon(x) - F(x) || < \varepsilon$ for $x \in C$ and $F_\varepsilon(C) \subseteq$ Co(A) $\subseteq C$ for some finite set A $\subseteq$ C , where Co(A) is the smallest convex set containing A.

Since Co(A) is closed and bounded and F_ε(Co(A)) $\subseteq$ Co(A), we may deduce that there exist $x_\varepsilon \in$ Co(A) with $x_\varepsilon = F_\varepsilon(x_\varepsilon)$

But $|| F_\varepsilon(x) - F(x) || < \varepsilon$ for $x \in C$, then

$$|| x_\varepsilon - F_\varepsilon(x_\varepsilon) || = || F_\varepsilon(x_\varepsilon) - F(x_\varepsilon) ||$$

From $|| F_\varepsilon(x) - F(x) || < \varepsilon$ we can see that $|| F_\varepsilon(x_\varepsilon) - F(x_\varepsilon) || < \varepsilon$, which implies

$$|| x_\varepsilon - F(x_\varepsilon) || < \varepsilon$$

Hence F has an ε-fixed point. Therefore by Theorem1.2 we can conclude that F has a fixed point which completes the proof. ■

CHAPTER 2

FUNCTIONAL DIFFERENTIAL EQUATIONS

Definition 2.1: A functional equation (FE) is an equation involving an unknown function for different argument values.

Example 2.1: a) $x(2t) + 2x(3t) = 1$

b) $x(t) = t^2 x(t+1) - [x(t-2)]^2$

c) $x\big(x(t)\big) = x(t) + 1$, etc.

The difference between the argument values of the unknown function and t in a functional equation is called ***argument deviations (delay)***. If all argument deviations are constant (example (b) above), then the functional equation is called a ***difference equation (discrete delay).***

Definition 2.2: A functional differential equation is an equation that involves a functional equation together with derivatives. So combing the nations of differential and functional equation we obtain the notion of functional differential equation (FDE). The order of a FDE is the order of the highest derivative of the unknown function entering in the equation. So a FE may be regarded as a FDE of order zero.

Example 2.2: a) $\dot{x}(t) = F\big(x(t)\big)$, unknown function as an argument value

b) $\dot{x}(t) = f(t, x(t))$, ordinary differential equation (ODE)

c) $\dot{x}(t) = \int_a^b F\big(t, s, x(s)\big)ds$, integro-differential equation

d) $\dot{x}(t) = -a(t)x(t) - b(t)g\Big(x\big(t - r(t)\big)\Big)$, $t \in \mathbb{R}^+$

In general, we can consider situations under investigation in two ways:

1) When a situation (model) does not depend on its past history, it consists of the so called ordinary differential equation (ODE) or partial differential equation (PDE). These equations admit the ***Principle of Causality*** which says that "The future state of the system is independent of the past and solely determined by the present."
2) Models incorporating past history include functional differential equation (FDE) or delay differential equation (DDE).

Although so much is known about differential equations without time delays (ordinary and partial differential equations) and are easier, in real, many of the processes both natural and manmade involve time delays (depend on the past history) (see [5]). Ordinary differential equations and partial differential equations have long played important roles in the history of theoretical population dynamics; they will no doubt continue to serve as indispensable tools in future investigations. Indeed, the use of delay differential equations in the modeling of population dynamics is currently very active. A more realistic model must include some of the past history of the system.

When the past dependence is through the state variable and not the derivative of the state variable we call it ***retarded functional differential equation*** (RFDE). Therefore RFDE is a dynamical system where the rate of change $\dot{x}(t)$ of the "state" $x(\cdot)$ at time t depends not only on the present value $x(t)$, but also on past values of $x(\cdot)$ where there is no derivative of the state variable x as an argument value.

Example2.3:a. $\dot{x}(t) = F(t, x(t), x(t-r))$, the rate of change $\dot{x}(t)$ does not depend on the derivative of the state variable x. Here $r > 0$ is called the ***delay*** and $x(t-r)$ is called the ***delayed argument***.

b. $\dot{x}(t) = -\alpha x(t-1)[1 + x(t)]$

If the delayed argument occurs in the highest order derivative of the state we call it ***neutral functional differential equation*** (NFDE).

Example 2.4: $\dot{x}(t) = rx(t)\left[1 - \left(\frac{x(t-\tau)}{k} + \frac{\rho\dot{x}(t-\tau)}{k}\right)\right]$

CHAPTER 3

SOLUTION OF SOME RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

Fixed point theorem is used to prove existence of solutions of various kinds of differential equations. Here we will use it to show that the fixed point of the integral operator obtained by variation of parameters (integrating factor method) is the solution of an initial value problem of retarded functional differential equation.

As mentioned earlier in chapter 2, a time delay occurs naturally in just about every interaction of the real world. Initial data for delay differential equations are generally continuous functions on a finite interval. This is distinct from the initial value problem for ordinary differential equations where initial data are points of Euclidean space.

For ordinary differential equations, we can view solutions of initial value problems as maps in the Euclidean space. In order to establish a similar view for solutions of delay differential equations, we need some definitions.

We adopt the following notations:

$C([a,b],\mathbb{R}^n)$ is as defined in the notation part. But when $[a,b]=[-r,0]$, where r is a positive constant, we generally denote $C=C([-r,0],\mathbb{R}^n)$.

For $\sigma\in\mathbb{R}, A\geq 0$, $x\in C([\sigma-r,\sigma+A],\mathbb{R}^n)$, and $t\in[\sigma,\sigma+A]$ we define $x_t\in C$ as:

$$x_t(\theta)=x(t+\theta)\ , \qquad \theta\in[-r,0]$$

Assume Ω is a subset of $\mathbb{R}\times C$, $f:\Omega\to\mathbb{R}^n$ is a given function, then we call

$$\dot{x}(t)=f(t,x_t) \tag{*}$$

the general first order RDDE (or RFDE) on Ω.

A function x is called a solution of (*) on $[\sigma - r, \sigma + A)$ if $x \in C([\sigma - r, \sigma + A), \mathbb{R}^n)$, $(t, x_t) \in \Omega$ and x_t satisfies (*) for $t \in [\sigma, \sigma + A)$.

For given $\sigma \in \mathbb{R}$, $\phi \in C$, we say $x(\sigma, \phi)$ is a solution of (*) with initial value ϕ at σ (or simply a solution through (σ, ϕ)), if there is an A > 0 such that $x(\sigma, \phi)$ is a solution of (*) on $[\sigma - r, \sigma + A)$ and $x_\sigma(\sigma, \phi) = \phi$. If the solution is unique, then, for each $t \geq 0$, we may define

$$T(t): \phi \mapsto x_t(\sigma, \phi)$$

Then T maps C into C. We call $T(t)$ the solution map of

$$\begin{cases} \dot{x}(t) = f(t, x_t) \\ \\ x_\sigma = \phi \end{cases} , t \geq \sigma$$

As we have seen in chapter 1 for first order ordinary differential equation initial value problem, a similar procedure shows us that finding a solution of (*) through (σ, ϕ) , $\sigma \in \mathbb{R}$, $\phi \in C$ is equivalent to solving the integral equation

$$x(t) = x_\sigma + \int_\sigma^t f(s, x_s) ds \quad , t \geq \sigma$$

<u>Proof</u>: $\dot{x}(t) = 0 + f(t, x_t) = f(t, x_t)$ and $x(\sigma) = x_\sigma + \underbrace{\int_\sigma^\sigma f(s, x_s) ds}_{=0}$

$$= x_\sigma = \phi$$

To show the other way we have $x_\sigma = \phi$ and $\dot{x}(t) = f(t, x_t)$, then

$$\frac{dx}{dt} = f(t, x_t)$$

$$dx = f(t, x_t) dt$$

$$\int_{x_\sigma}^x dk = \int_\sigma^t f(s, x_s) ds$$

$$x(t) = x_\sigma + \int_\sigma^t f(s, x_s) ds$$

To achieve the main goal of this project with the retarded functional differential equation

$$\dot{x}(t) = -a(t)x(t) - b(t)g\left(x\big(t - r(t)\big)\right)$$

let us use the following definitions and notations in order to apply Schauder's fixed point theorem on continuous maps in compact convex subset of a normed linear space.

Let r_o be a fixed non negative constant i.e. $r_o \in [0, \infty) = \overline{\mathbb{R}^+}$ and let h: $[-r_o, \infty) \rightarrow [1, \infty)$ be any strictly increasing continuous function with $h(-r_o) = 1$ and $h(t) \rightarrow \infty$ as $t \rightarrow \infty$.

For any $t_o \in \overline{\mathbb{R}^+}$ let $C_{t_o} = C^o\big(\overline{\mathbb{R}}_{t_o}\big)$, where $\overline{\mathbb{R}}_{t_o} \coloneqq [t_o - r_o, \infty)$. For $\Phi \in C^o\big(\overline{\mathbb{R}}_{t_o}\big)$ define

$$|| \Phi ||_h \coloneqq sup\left\{\frac{| \Phi(t) |}{h(t - t_o)} : t \geq t_o - r_o \right\} < \infty$$

We note since $t_o \in \overline{\mathbb{R}^+}$, $0 - r_o \leq t_o - r_o < t_o$, then

$$-r_o \leq t_o - r_o < t_o$$

<u>Claim</u>: $||.||_h$ is a norm on C_{t_o} and $(C_{t_o}, ||.||_h)$ is a normed linear space.

<u>Proof</u>: $\boxed{i}$ let $\phi_1, \phi_2 \in C_{t_o}$

$$|| \phi_1 + \phi_2 ||_h = sup\left\{\frac{| (\phi_1 + \phi_2)(t) |}{h(t - t_o)} : t \geq t_o - r_o\right\}$$

$$\leq sup\left\{\frac{| \phi_1(t) |}{h(t - t_o)} + \frac{| \phi_2(t) |}{h(t - t_o)}\right\} , t \geq t_o - r_o$$

$$\leq \underbrace{sup\left\{\frac{| \phi_1(t) |}{h(t - t_o)}\right\}}_{< \infty} + \underbrace{sup\left\{\frac{| \phi_2(t) |}{h(t - t_o)}\right\}}_{< \infty} < \infty$$

$$= || \phi_1 ||_h + || \phi_2 ||_h$$

Therefore $\| \phi_1+\phi_2 \|_h \leq \| \phi_1 \|_h + \| \phi_2 \|_h$.

$\boxed{ii}$ Let $\phi \in C_{t_o}$ and λ be scalar

$$\| \lambda\phi \|_h = \ sup\left\{\frac{|\lambda\phi(t)|}{h(t-t_o)}\right\}$$

$$= sup\left\{| \lambda | \frac{| \phi(t) |}{h(t-t_{o)}}\right\}$$

$$= | \lambda | \ sup\left\{\frac{| \phi(t) |}{h(t-t_o)}\right\} = | \lambda | \| \phi \|_h$$

Therefore $\| \lambda\phi \|_h = | \lambda | \| \phi \|_h$.

$\boxed{iii}$ Let $\phi \in C_{t_o}$, then

$$\| \phi \|_h = \ sup\left\{\frac{| \phi(t) |}{h(t-t_{o)}}\right\}$$

But $| \phi(t) | \geq 0$ and $h(t-t_o) \geq 1$ since h(t) is strictly increasing and $t \geq t_o - r_o \geq -r_o$

Therefore $\| \phi \|_h \geq 0$ and if $\phi(t) = 0$, then $\left\{\frac{|\phi(t)|}{h(t-t_{o)}}\right\} = 0 = \| \phi \|_h$. Hence $\| . \|_h$ is a norm on C_{t_o}.

Therefore $(C_{t_o}, \| . \|_h)$ is a normed linear space. ■

Now consider the retarded functional differential equation

$$\boxed{\dot{x}(t) = -a(t)x(t) - b(t)g\Big(x\big(t-r(t)\big)\Big)} \ , \quad t \in \overline{\mathbb{R}^+} \qquad (3.1)$$

where $\begin{cases} a, b \ \in \ C^o\left(\overline{\mathbb{R}^+}\right) \\ \quad g \ \in \ C^o(\mathbb{R}) \\ \quad r \ \in \ C^o\left(\overline{\mathbb{R}^+}, \overline{\mathbb{R}^+}\right) \end{cases}$

Let α be any fixed positive number. We assume that there are constants $\beta > 0$, $\gamma > 0$ and $r_o \geq 0$ so that

- $|g(x)| \leq \beta|x|$ for $x \in \mathbb{R}$ with $|x| \leq \alpha$,then , $|g(x)| \leq \beta\alpha$ (3.2)
- $sup\left\{ e^{\int_{\tau}^{t}(a(s)-\beta\gamma|b(s)|)\, ds} \ : t \in \overline{\mathbb{R}^{+}} \right\} \leq \gamma$ (3.3)

  where $\tau = \max(0, t - r(t))$
- $(t - r(t)) \geq -r_o$ (3.4)
- $\sigma = \sigma(t_o) := sup\left\{ \int_{t_o}^{t} (\beta\gamma \mid b(s) \mid - a(s))ds : t \geq t_o \right\} < \infty$ (3.5)

  define a number $\eta = \eta(t_o)$ by $\eta := \alpha$ (3.6)

Corresponding to Equation (3.1) consider the scalar linear equation

$$\dot{q} = (\beta\gamma \mid b(t) \mid - a(t))q \ , \quad t \in \overline{\mathbb{R}^{+}} \tag{3.7}$$

Let $q: \overline{\mathbb{R}}_{t_o} \to \mathbb{R}^{+}$ be a continuous function such that

$$q(t) = \eta \ \ for \ \ t_o - r_o \leq \mathrm{t} \leq t_o$$

This shows $q(t) = \alpha$ for $t_o - r_o \leq \mathrm{t} \leq t_o$.

And let $q(t)$ be the unique solution of the initial value problem (IVP)

$$\begin{cases} q' = (\beta\gamma \mid b(t) \mid - a(t))q \\ q(t_o) = q_o = \eta \end{cases} \quad , t \geq t_o$$

Then $q(t)$ can be expressed in two ways as:

$$q'(t) + a(t)q(t) = \beta\gamma \mid b(t) \mid q(t)$$

$$e^{\int_{s}^{t} a(u)du} q'(t) + a(t)q(t)e^{\int_{s}^{t} a(u)du} = e^{\int_{s}^{t} a(u)du} \beta\gamma \mid b(t) \mid q(t)$$

$$\frac{d}{dt}\left(q(t)e^{\int_{s}^{t} a(u)du}\right) = \beta\gamma e^{\int_{s}^{t} a(u)du} \mid b(t) \mid q(t)$$

$$\int_{t_o}^{t}\frac{d}{ds}\left(q(s)e^{\int_s^t a(u)du}\right)ds = \beta\gamma\int_{t_o}^{t} e^{\int_s^t a(u)du} \mid b(s) \mid q(s)\, ds$$

$$q(t)e^{\int_s^t a(u)du} = q(t_0) + \beta\gamma\int_{t_o}^{t} e^{\int_s^t a(u)du} \mid b(s) \mid q(s)\, ds$$

$$q(t) = \underbrace{q(t_0)}_{=\eta}\, e^{-\int_s^t a(u)du} + \beta\gamma e^{-\int_s^t a(u)du}\int_{t_o}^{t} e^{\int_s^t a(u)du} \mid b(s) \mid q(s)\, ds$$

$$q(t) = \eta e^{-\int_s^t a(u)du} + \beta\gamma e^{-\int_s^t a(u)du}\int_{t_o}^{t} e^{\int_s^t a(u)du} \mid b(s) \mid q(s)\, ds \qquad (3.8)$$

and

$$\frac{dq}{dt} = \left(\beta\gamma \mid b(t) \mid -a(t)\right)q$$

$$\frac{dq}{q} = \left(\beta\gamma \mid b(t) \mid -a(t)\right)dt$$

$$\frac{d}{dq}(\ln(q)) = \left(\beta\gamma \mid b(t) \mid -a(t)\right)dt$$

$$\int_{q_o}^{q}\frac{d}{dp}(\ln(p))\, dp = \int_{t_o}^{t}\left(\beta\gamma \mid b(s) \mid -a(s)\right)ds$$

$$\ln(p)\big|_{q_o}^{q} = \int_{t_o}^{t}\left(\beta\gamma \mid b(s) \mid -a(s)\right)ds$$

$$\ln(q) - \ln(q_o) = \int_{t_o}^{t}\left(\beta\gamma \mid b(s) \mid -a(s)\right)ds$$

$$\ln {}^{q}/_{q_o} = \int_{t_o}^{t}\left(\beta\gamma \mid b(s) \mid -a(s)\right)ds$$

$$\frac{q}{q_o} = e^{\int_{t_o}^{t}\left(\beta\gamma|b(s)|-a(s)\right)ds}$$

$$q = \underbrace{q_o}_{=\eta} e^{\int_{t_o}^{t}(\beta\gamma|b(s)|-a(s))ds}$$

$$q = \eta e^{\int_{t_o}^{t}(\beta\gamma|b(s)|-a(s))ds} \quad , \quad t \geq t_o \tag{3.9}$$

But $e^{\int_{t_o}^{t}(a(s)-\beta\gamma|b(s)|)ds} \leq \gamma$, from (3.3) then,

$$\frac{1}{\gamma} \leq e^{-\int_{t_o}^{t}(a(s)-\beta\gamma|b(s)|)ds}$$

$$= e^{\int_{t_o}^{t}(\beta\gamma|b(s)|-a(s))ds}$$

$$\frac{\eta}{\gamma} \leq \eta e^{\int_{t_o}^{t}(\beta\gamma|b(s)|-a(s))ds}$$

From (3.9) we have $\eta \leq \gamma q(t)$ (3.10)

Now $q(t) = \underbrace{\eta}_{>0} \underbrace{e^{\int_{t_o}^{t}(\beta\gamma|b(s)|-a(s))ds}}_{>0}$, $t \geq t_o$.

This shows us that $q(t) > 0$. Also we can have the following relation

$$q = \eta e^{\int_{t_o}^{t}(\beta\gamma|b(s)|-a(s))ds}$$

$$\leq \eta e^{sup \int_{t_o}^{t}(\beta\gamma|b(s)|-a(s))ds}$$

$$= \eta e^{\sigma}$$

Therefore we can conclude $0 < q(t) \leq \eta e^{\sigma}$. (3.11)

Definition 3.1: (compact)

A subset A of a normed linear space B is said to be compact if and only if every sequence $\{x_n\}$ in A has a convergent subsequence with limit in A.

Definition 3.2: let $\{f_m\}$ be a sequence of real-valued functions in a subset D of $\mathbb{R}^n$. Let $x \in D$. The sequence is ***equi-continuous*** at x if for all $\varepsilon > 0$ there exists $\delta > 0$, independent of m such that

$$| f_m(y) - f_m(x) | < \varepsilon \text{ for } y \in D \text{ with } | y - x | < \delta$$

Example 3.1: i) A prime example of a family of equi-continuous function is the space of Lipschitz functions on a compact set.

ii) $f(x) = x^n$, $n \in \mathbb{N}$ is equi-continuous only on closed interval of the form [0 , q] , where 0 <q < 1

If the sequence $\{f_m\}$ is equi-continuous at each point of a compact set S, then it is called ***uniformly equi-continuous*** (i.e. δ in the above definition can be chosen independently of $x \in S$).

Theorem 3.1: (Arzela-Ascoli)

Every bounded and equi-continuous sequence $\{f_m\}$ of real-valued functions on a compact subset S of $\mathbb{R}^n$ has a subsequence which converges uniformly on S.

Proof: let $\{f_m\}$ be a sequence of real-valued functions defined on a compact subset S of $\mathbb{R}^n$.

Since $\{f_m\}$ is bounded there exists a constant M > 0 such that $|f_m(x)| \leq$ M for every $m \in \mathbb{N}$ and every $x \in S$. Moreover, assume that the sequence $\{f_m\}$ is equi-continuous at every point of S (i.e. uniformly equi-continuous).

We want to show that there exists a subsequence say $\{g_m\}$ of $\{f_m\}$ which converges uniformly on S.

Since the sequence $\{f_m(x_1)\}$ is bounded, then it has a convergent subsequence. That is , we can choose a sequence m_{1j} such that $\left\{f_{m_{1j}}(x_1)\right\}$ converges as $j \to \infty$. Similarly we can choose a subsequence m_{2j} of the sequence m_{1j} such that $\left\{f_{m_{2j}}(x_2)\right\}$ converges.

Since m_{2j} is a subsequence of m_{1j} , $\{f_{m_{2j}}(x_1)\}$ converges as well.

Next, we choose a subsequence m_{3j} of the sequence m_{2j} such that $\{f_{m_{3j}}\}$ converges also at x_3 . We proceed in this manner.

Finally, consider the "diagonal" sequence $\{f_{m_{jj}}\}$. Except for the first i – 1 terms, m_{jj} is a subsequence of m_{ij} .

Hence $\{f_{m_{jj}}(x_i)\}$ converges for every $i \in \mathbb{N}$.

To simplify notation, we shall set $g_j = f_{m_{jj}}$ in the following. To conclude the proof, we show that the sequence $\{g_m\}$ is uniformly Cauchy.

Let $\varepsilon > 0$ be given. The sequence $\{g_m\}$ being a subsequence of $\{f_m\}$, is uniformly equi-continuous on S. Hence there is a $\delta > 0$ such that $| g_m(y) - g_m(x) | < \varepsilon/3$ whenever $| y - x | < \delta$

Since S is compact, there is a K $\in \mathbb{N}$ such that for every $x \in S$ there exists $i \in \{1,2,\dots,K\}$ with $| x_i - x | < \delta$.

Now choose N large enough so that $| g_m(x_i) - g_k(x_i) | < \varepsilon/3$ for m, k > N and every $i \in \{1,2,\dots,K\}$. For m, k > N and arbitrary $x \in S$, we now have

$$
\begin{aligned}
| g_m(x) - g_k(x) | &= | g_m(x) - g_m(x_i) + g_m(x_i) - g_k(x_i) + g_k(x_i) - g_k(x) | \\
&= | g_m(x) - g_m(x_i) | + | g_m(x_i) - g_k(x_i) | + | g_k(x_i) - g_k(x) | \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\
&= \varepsilon
\end{aligned}
$$

Therefore $|\ g_m(x) - g_k(x)\ | < \varepsilon$. By Cauchy criterion for uniform convergence $\{g_m\}$ is a uniformly Cauchy sequence.

Hence the sequence $\{g_m\}$ converges uniformly on S. ■

Let $\psi : \overline{\mathbb{R}}_{t_o} \rightarrow \mathbb{R}^+$ be continuous such that

$$sup\{\psi(\theta): -r_o \le \theta \le 0\} \le \eta \tag{3.12}$$

Let S be the set of continuous functions $\phi_n : \overline{\mathbb{R}}_{t_o} \rightarrow \mathbb{R}^+$ such that

$$\phi_n(t) = \psi(t - t_o)\ ,\ t_o - r_o \le \mathrm{t} \le t_o$$

$$|\ \phi_n(t)\ | \le q(t)\ ,\quad t \ge t_o$$

and

$|\ \phi_n(t_1) - \phi_n(t_2)\ | \le L\ |\ t_1 - t_2\ |$, $t_1, t_2 \in \mathbb{R}^+$ with $t_1 \le \tau_1 \le t_2\ , t_2 \le \tau_2$ where $q(t)$ is as defined by (3.9) with and where $L: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is defined by

$$L = L(\tau_1, \tau_2) = \max\{(|\ a(t)\ | + \beta\gamma\ |\ b(t)\ |)\alpha\}\ ,\quad \tau_1 \le t \le \tau_2$$

<u>Claim</u>:

a) S is non empty

b) ϕ_n is bounded

c) $\{\phi_n\}$ is equi-continuous

d) S is convex

<u>Proof</u>: $\boxed{\text{a}}$ let $\xi(t)$ is a function defined by

$$\xi(t) = \begin{cases} \psi(t - t_o)\ , & t_o - r_o \le \mathrm{t} \le t_o \\ \frac{\psi(0)}{\eta} q(t)\ , & t > t_o \end{cases}$$

Since $|\ \xi(t)\ | = \underbrace{\frac{\psi(0)}{\eta}}_{\le 1} q(t) \le q(t)$, $t > t_o$ from (3.12)

$$\xi(t) = \psi(t-t_o)\ ,\ t_o - r_o \leq \text{t} \leq t_o$$
$$|\ \xi(t)\ | \leq q(t)\ \ ,t > t_o$$

This implies the function ξ is an element of S. Therefore $S \neq \emptyset$ and hence S is non empty.

$\boxed{\text{b}}$ $|\ \phi_n(t)\ | = |\ \psi(t-t_o)\ | \leq \eta\ \ ,\ \ t_o - r_o \leq \text{t} \leq t_o$ and

$$\leq q(t)\ \ ,\ \ t > t_o$$

$$\leq \eta\, e^{\sigma}$$

Therefore $|\ \phi_n(t)\ | \leq max\{\eta\ ,\eta\, e^{\sigma}\}\ ,\ t \in \overline{\mathbb{R}}_{t_o}$ and then this implies ϕ_n is bounded.

$\boxed{c}$ We want to show for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all ϕ_n $|\ x - \bar{x}\ | < \delta$ whenever $|\ \phi_n(x) - \phi_n(\bar{x})\ | < \varepsilon$

Let $\varepsilon > 0$ be given and $|t_1 - t_2| < \delta$. Since $|\phi_n(t_1) - \phi_n(t_2)\ | \leq L\ |\ t_1 - t_2\ |$, if we choose $\delta = {}^{\varepsilon}/_{L}$, then

$$|\phi_n(t_1) - \phi_n(t_2)| \leq L\delta = L\left({}^{\varepsilon}/_{L}\right) = \varepsilon$$
$$|\phi_n(t_1) - \phi_n(t_2)| \leq \varepsilon$$

Therefore $\{\phi_n\}$ is equi-continuous.

From (b) and (c) $\{\phi_n\}_{n=1}^{\infty}$ is bounded and equi-continuous. Then by the Arzela-Ascoli theorem $\{\phi_n\}_{n=1}^{\infty}$ has a convergent subsequence. By Definition 3.1 S is compact.

$\boxed{\text{d}}$ Let ϕ_1 and $\phi_2 \in S$. If λ is such that $0 \leq \lambda \leq 1$ and $\Phi = \lambda\phi_1 + (1-\lambda)\phi_2$ then,

$$\Phi(\text{t}) = \lambda\phi_1(t) + (1-\lambda)\phi_2(t)$$

$$= \lambda\psi(t-t_o) + (1-\lambda)\psi(t-t_o)\ \text{, for } t_o - r_o \leq \text{t} \leq t_o$$

$$= \psi(t - t_o) \quad , \; t_o - r_o \le \text{t} \le t_o$$

and

$$| \, \Phi \, (\text{t}) \, | \; = \; | \, \lambda\phi_1(t) + \; (1 - \lambda)\phi_2(t) \, |$$

$$\le \lambda \, | \underbrace{\phi_1(t)}_{\le q(t)} | \; + (1 - \lambda) \, | \underbrace{\phi_2(t)}_{\le q(t)} | \quad , \; t > \; t_o$$

$$\le \lambda q(t) + \; (1 - \lambda) q(t) \; , \quad t > \; t_o$$

$$= q(t) \quad , \; t > \; t_o$$

Therefore $\Phi(\text{t}) = \; \psi(t - t_o) \;\; , \; t_o - r_o \le \text{t} \le t_o$ and

$$| \, \Phi \, (\text{t}) \, | \;\; \le q(t) \quad , \; t > \; t_o$$

Then $\Phi(\text{t}) \in \text{S}$ which implies $\lambda\phi_1(t) + \; (1 - \lambda)\phi_2(t) \; \in S$.

Hence S is convex. And then S is a compact and convex subset of a normed linear space $(C_{t_o}, ||\,.\,||_h)$.

Now consider the retarded functional differential equation (3.1)

$$\boxed{\dot{x}(t) = -a(t)x(t) - b(t)g\Big(x\big(t - r(t)\big)\Big)} \quad , \; t \in \overline{\mathbb{R}^+}$$

and a mapping P on S defined by

$$(P\phi)(t) = \begin{array}{l} \psi(0)e^{-\int_s^t a(u)du} - e^{-\int_s^t a(u)du} \int_{t_o}^t e^{\int_s^t a(u)du}\, b(s)g\Big(\phi\big(s - r(s)\big)\Big)\, ds \,, t > \; t_o \\ \psi(t - t_o) \quad , \qquad t_o - r_o \le \text{t} \le t_o \end{array}$$

P is an integral operator.

Claim: i) $P : S \to S$

ii) P has a fixed point

iii) a fixed point of P is a solution of the RFDE (3.1)

proof: $\boxed{i}$

$$| P\phi(t) | = | \psi(0)e^{-\int_s^t a(u)du} - e^{-\int_s^t a(u)du} \int_{t_o}^t e^{\int_s^t a(u)du} b(s)g\left(\phi\big(s-r(s)\big)\right) ds |$$

$$\leq | \underbrace{\psi(0)}_{\geq 0} \underbrace{e^{-\int_s^t a(u)du}}_{\geq 0} | + | \underbrace{e^{-\int_s^t a(u)du}}_{\geq 0} \int_{t_o}^t \underbrace{e^{\int_s^t a(u)du}}_{\geq 0} b(s)g\left(\phi\big(s-r(s)\big)\right) ds |$$

Then by Equation (3.2) we have

$$| P\phi(t) | = \underbrace{\psi(0)}_{=\eta} e^{-\int_s^t a(u)du} + e^{-\int_s^t a(u)du} \int_{t_o}^t e^{\int_s^t a(u)du} | b(s) | \, \underbrace{| g\left(\phi\big(s-r(s)\big)\right) |}_{\leq \beta\alpha} ds$$

$$\leq \eta \, e^{-\int_s^t a(u)du} + e^{-\int_s^t a(u)du} \int_{t_o}^t e^{\int_s^t a(u)du} | b(s) | \underbrace{\beta\alpha}_{\leq \beta\gamma q(s)} ds, \text{ by (3.6) \& (3.10)}$$

$$= \eta \, e^{-\int_s^t a(u)du} + \beta\gamma e^{-\int_s^t a(u)du} \int_{t_o}^t e^{\int_s^t a(u)du} | b(s) | q(s)ds$$

$$= q(t) \text{ , by (3.8)}$$

Therefore $| P\phi(t) | \leq q(t)$, $t > t_o$ and

$$P\phi(t) = \psi(t-t_o) \text{ , } t_o - r_o \leq t \leq t_o$$

This implies that $P\phi(t) \in S$.

Hence $P : S \to S$

$\boxed{ii}$ $P\phi$ is continuous since $\psi(0)$ and the integrands are continuous.

Therefore $P : S \to S$ is continuous, where S is compact and convex nonempty subset of the normed linear space $(C_{t_o}, || \, . \, ||_h)$.

Then P has a fixed point by Schauder's fixed point theorem. Therefore there exists $\phi \in S$ such that $(P\phi)(t) = \phi(t)$.

$\boxed{iii}$

$$(P\phi)(t) = \begin{cases} \psi(0)e^{-\int_s^t a(u)du} - e^{-\int_s^t a(u)du}\int_{t_o}^t e^{\int_s^t a(u)du}\, b(s)g\left(\phi\big(s-r(s)\big)\right)ds\,, t > t_o \\ \psi(t-t_o) \quad , \qquad t_o - r_o \le \mathrm{t} \le t_o \end{cases}$$

$$(\dot{P\phi}) = -a(t)\psi(0)e^{-\int_s^t a(u)du} - [-a(t)e^{-\int_s^t a(u)du}\int_{t_o}^t e^{\int_s^t a(u)du}\, b(s)g\left(\phi\big(s - r(s)\big)\right)ds + e^{-\int_s^t a(u)du}e^{\int_s^t a(u)du}b(t)g\left(\phi\big(t-r(t)\big)\right)]$$

$$= -a(t)\psi(0)e^{-\int_s^t a(u)du} + a(t)e^{-\int_s^t a(u)du}\int_{t_o}^t e^{\int_s^t a(u)du}\, b(s)g\left(\phi\big(s- r(s)\big)\right)ds - b(t)g\left(\phi\big(t-r(t)\big)\right)$$

$$= -a(t)\left[\psi(0)e^{-\int_s^t a(u)du} - e^{-\int_s^t a(u)du}\int_{t_o}^t e^{\int_s^t a(u)du}\, b(s)g\left(\phi\big(s- r(s)\big)\right)ds\right] - b(t)g\left(\phi\big(t-r(t)\big)\right)$$

$$= -a(t)(P\phi)(t) - b(t)g\left(\phi\big(t-\tau(t)\big)\right) \ , \ t > t_o$$

This implies $(P\phi)$ solves the RFDE (3.1). Therefore ϕ is the solution of the FDE. Hence the fixed point of P is a solution of the RFDE (3.1).

REFERENCES

[1] J. K. Hale, **Theory of Functional Differential Equations**, Springer, New York, 1977.

[2] Michael Rinardy and Robert C. Rogers, **An Introduction to Partial Differential Equations.**

[3] Ravi P. Agrawal, Maria Meehan and Donal O'Regan, **Fixed Point Theory and Applications**, Cambridge Univ. Press, 2004.

[4] Tensuo Furumochi, **Asymptotic Behavior of Solutions of Some Functional Differential Equation by Schauder's Theorem**, Electronic Journal of QTDE, 7th Coll. 2004, 1-5.

[5] Yang Kuang, **Delay Differential Equations with Applications in Population Dynamics**, Vol. 191, Academic Press, 1993.

[6] V. Kolmanovskii and Myshkis, **Introduction to the Theory and Application of Functional Differential Equations**, Kluwer Academic Publisher, 1999.