Addis Ababa University

Iterative Approximation of Fixed Points of ρ-nonexpansive Multivalued Mappings in Modular Function Spaces

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by

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Declaration

I, Wondimu Woldie Kassu, with student ID number $GSR/2557/06$, hereby declare that this thesis is my own work and that it has not been previously submitted for assessment or completion of any post graduate qualification to another university or for another qualification.

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This is to certify that the thesis prepared by Wondimu Woldie Kassu, entitled: **Iterative Approximation of Fixed Points of ρ-nonexpansive Multivalued Mappings in Modular Function Spaces** and submitted in fulfillment of the requirements for the Degree of Doctor of Philosophy(in Mathematics) complies with the regulations of the University and meets the accepted standards with respect to originality and quality.

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Abstract

The existence and iterative approximation of fixed points of single-valued and multi-valued mappings in modular function spaces have been studied by many well known Mathematicians. Due to its applicability in real world problems such as Market Economy and Game theory and other applied mathematics such as Differential equations and Optimization theory, the study of fixed point theory has continued in modular function spaces.

In this thesis, we constructed a Mann-type iterative scheme and proved the ρ-convergence of the scheme to common fixed point of finite family of ρ-nonexpansive multi-valued mappings. We also proved the ρ-convergence of Ishikawa-type iterative scheme to common fixed point of two ρ-nonexpansive multivalued mappings under certain mild conditions on the mappings and the set on which the mappings are defined. Moreover, we introduced a new class of multi-valued mappings in modular function spaces called ρ-quasi-nonexpansive mapping and proved the ρ-convergence of Mann-type iterative scheme to common fixed point of finite family of this class of mappings in modular function spaces.

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Wondimu Woldie Kassu

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Contents

Chapter 1

Introduction

1.1 Background of the study

In this chapter, we give a brief introduction of fixed point theory, some fundamental concepts and notations relevant to the development of fixed point theory. A brief survey of the development of fixed point theory on metric spaces, Banach spaces and modular function spaces has been presented. The concept of function modular, basic definitions, notations, some examples which support our definitions and some lemmas and propositions which will be used in the sequel are also stated. We now proceed as follows:

Historically, the concept of fixed point was initiated by H. Poincare in 1886. The concept of metric spaces was introduced by M. Frech in 1906 which furnished the common idealization of mathematical, physical and other scientific constructs in which the notion of distance appears. The study of fixed point theory has continued for the last hundred years in metric and Banach spaces. Since 1990, the study of fixed point theory in modular spaces, spaces that generalize some classes of Banach spaces attracted the attention of well known mathematicians (see, for example, [1, 15, 16, 18, 24, 31, 33]).

Let $(X, \mathcal{A}, \mu)$ be a measure space. We define $L^p = \{f : X \to [-\infty, \infty]$ measurable $: \int_X |f|^p d\mu < \infty\}$.

A problem that many well knowm mathematicians dealt with for almost the last sixty years is how to generalize the classical L^p spaces. The first attempt began by Brinbaum and Orlicz [9]. Their approach was considering spaces of functions with some growth properties that is different from the power type growth control provided by the L^p-norm. This generalization found many applications in differential and integral equations with kernels of non-power types.

Another attempt of generalization of L^P spaces was due to Orlicz [51]. For an Orilcz function $\varphi : [0, \infty] \longrightarrow [0, \infty]$ which is non-decreasing and convex with $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$, the space L^φ is defined by $L^\varphi = \{f : \mathbb{R} \to \mathbb{R} : \int_{\mathbb{R}} \varphi(a|f(x)|)dx < \infty\}$ for some $a > 0$. The Luxemburg norm (see [44])of a measurable function f is defined by

$$\|f\|_\varphi := \inf\{c > 0 : \int \varphi\Big(\frac{|f|}{c}\Big) d\mu \leq 1\}.$$

In fact, the assumption of convexity for Orlicz functions φ can be omitted, two examples of such function are:

$$\varphi(t) = e^t - 1 \text{ and } \varphi(t) = ln(1 + t).$$

We see that the Orlicz space L^φ is a real generalization of L^p at least for $1 \leq p < \infty$. Indeed, if $\varphi(t) = t^p$, then $L^\varphi = L^p$ with equality of norms.

Afterwards, many well known mathematicians were involved in generalizing L^p spaces due to its applicability. Nakano [50] is the pioneer researcher who introduced modular spaces based on replacing the particular integral form by an abstarct functional called a modular. Another generalization of L^P spaces is due to Musielak and Orlicz [46].

The first fixed point theorem was, in 1912, due to famous Dutch Mathematician L.E.J. Brouwer [12]. An important generalization of Brouwer's theorem was discovered by J. Schauder [54] in 1930. A point is often called *fixed point* when it remains invariant, irrespective of the type of transformation it undergoes. For a mapping T that has a set X as both domain and range, a fixed point is a point $x \in X$ for which $T(x) = x$. And for a mapping that has X as domain and 2^X (where 2^X is the set of all nonempty subsets of X) as range set, a fixed point is a point $x \in X$ such that $x \in T(x)$.

The fixed point theory has played a crucial role in the problem of nonlinear analysis which is the mixture of analysis, algebra and topology. A fixed point theorem is the one which guarantees the existence of a fixed point of a mapping T under suitable assumptions both on the domain and mapping T. Apart from establishing the existence of fixed point, it often becomes necessary to prove the uniqueness of the fixed point.

Besides, from computational point of view, an algorithm for calculating the value of fixed point to a given degree of accuracy is also desirable. Often this algorithm involves the iterates of the given mapping. Basically, the question about the existence, uniqueness and approximation of fixed point provide three significant aspects of the general fixed point principle. In 1922, Polish Mathematician, S. Banach [10] introduced his revolutionary principle called *"Banach Contraction Mapping Principle,"* perhaps, his principle may be one of few most significant theorems that answers all the three questions, existence, uniqueness and constructive algorithm.

Iterative processes have been widely used as a successful tool in approximating fixed points of nonlinear mappings. The problem of finding fixed points of nonlinear mappings is strongly related to the zero of some nonlinear mappings. In many cases, solutions of such mapping equations happen to be fixed points of certain nonlinear mappings . Indeed, for a mapping T whose set of fixed points is nonempty, the mapping equation $x = Tx$ is clearly equivalent to $0 = (I - T)x$, I is an identity mapping. Since the problem of approximating solution of mapping equations can be reformulated as that of finding fixed points of certain mapping, one may turn to the theory of fixed points to search for tools that can serve as solution techniques for finding the zeros of such mapping.

Besides the existence of fixed point of mappings, many known mathematicians have shown their interests towards approximation techniques of fixed points of both single and multivaled mappings in metric and Banach spaces. As many analysts agree, the existence of fixed points of a certain class of mappings should be accompanied with the approximation technique with minimum error. For this reason, nowadays, the approximation technique of fixed points of nonlinear mappings has got the attention of well known researchers in the area of nonlinear analysis (see for example [21],[22], [45] and the references therein for single valued mappings and [52], [53], [55], [56],[57] for multivalued mappings). It is well known that iterative approximation for fixed points of nonlinear mappings has been successfully used to develop efficient and powerful numerical methods for solving various nonlinear equations and variational problems, often of great importance for applications in various areas of pure and applied science. Our purpose, in this thesis, is to construct an iterative approximation and prove its convergence to fixed points of nonlinear multi-valued nonexapnsive mappings in modular function spaces.

Next we present how the remaining sections and chapters of this thesis organized:

In section 1.2, we present the definition of function modular, modular function space; give some examples of modular function space; give some basic properties of modular function spaces. Chapter 2 is devoted to literature review on the iterative approximation of fixed point and the methodology we studied.

In Chapter 3, we study some existing results from literature and tools that we apply to obtain our main results in the consequent chapter.

Chapter 4 is totally devoted to main results of this thesis. In section 4.1, we prove ρ-convergence of the Ishikawa-type scheme to a common fixed point of finite family of ρ-nonexpansive multivalued mappings in modular function spaces. Section 4.2 deals with the ρ-convergence of an Ishikawa iterative scheme to a common fixed point of two multivalued ρ-nonexpansive mappings. In the last section of this chapter, section 4.3, we introduce a new class of multivalued mappings called ρ-quasi-nonexpansive mappings and establish convergence theorems for Mann-type scheme to common fixed point of finite family of this class of mappings in modular function spaces.

The last chapter of the thesis, Chapter 5, is devoted to the discussion of our main results and how and what these results extended existing results in literature and indicate the problems of our future work.

1.2 Preliminaries

Before giving the definition of modular function space, we recall some important features of multivalued mappings in metric spaces, which will be adapted to the modular function space settings. We give the definition of Hausdorff metric introduced by Nadler [49].

Let (M, d) be a metric space. We denote the set of all nonempty closed and bounded subsets of M by $CB(M)$, family of nonempty compact subsets of M by $\mathcal{C}(M)$. Let $x \in M$ and $B \subset M$, we define the distance from to B by

$$d(x, B) = \inf\{d(x, b) : b \in B\}.$$

Definition 1.2.1. Let (M, d) be a metric space. We define $H : CB(M) \times CB(M) \longrightarrow [0, \infty)$ by

$$H(A, B) = max\{\sup_{a \in A} d(a, B),\ \sup_{b \in B} d(b, A)\},$$

for all $A, B \in CB(M)$.

Remark 1.2.2.

1. The function H is a metric on $CB(M)$ and called a Hausdorff metric.
2. If (M, d) is a complete metric spaces, then $(H, CB(M))$ is also a complete metric space.

Definition 1.2.3. Let (M, d) be a metric space. A multivalued mapping $T : M \longrightarrow CB(M)$ is called Lipschitzian if there exists an $L \geq 0$ such that for all $x, y \in M$,

$$H(Tx, Ty) \leq Ld(x, y).$$

If $L \leq 1$, we say that T is nonexpansive and if $L \in [0, 1)$, we say that T is contraction mapping.

Definition 1.2.4. An element x in M is called a fixed point of a multivalued mapping T on M, if $x \in Tx$ and we denote the set of all fixed points of T by $F(T)$.

Lemma 1.2.5. [49] Let $A, B \in CB(M)$ and $a \in A$. Then for every $\epsilon > 0$, there exists $b \in B$ such that

$$d(a, b) \leq H(A, B) + \epsilon.$$

Theorem 1.2.6. [49] Let M be a complete metric space and $T : M \to CB(M)$ a contraction mapping. Then, T has a fixed point in M.

1.2.1 Basic definitions and examples of modular function spaces

Now, we present the notion of convex function modular. We also give the definition of modular function spaces and give some examples which suport our definition. For more details on the definitions given the reader may consult [23, 28, 29, 35].

Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a nontrivial δ-ring of subsets of Ω such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \cup_{n=1} K_n$. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}_\infty$ we denote the space of all extended measurable functions, that is, all functions $f : \Omega \longrightarrow [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \mathcal{E}$, $|g_n| \leq |f|$ and $g_n(w) \longrightarrow f(w)$ for all $w \in \Omega$. By 1_A we denote the characteristic function of the set A.

Example 1.2.7. Let (Ω, Σ, μ) be a σ-finite measure space. If we consider $\mathcal{P}$ to be the class of sets of finite measure in Σ, then it satisfies the above conditions on $\mathcal{P}$.

Definition 1.2.8. Let $\rho : \mathcal{M}_\infty \longrightarrow [0, \infty]$ be a nontrivial, convex and even function. We say that ρ is a regular convex function pseudo-modular if it satisfies the following:

a) $\rho(0) = 0$;

b) ρ is monotone; that is, $|f(w)| \leq |g(w)|$ for all $w \in \Omega$ implies $\rho(f) \leq \rho(g)$ where $f, g \in \mathcal{M}_\infty$;

c) ρ is orthogonally subadditive; that is, $\rho(f.1_{A \cup B}) \leq \rho(f.1_A) + \rho(f.1_B)$ for any $A, B \in \Sigma$ such that $A \cap B \neq \emptyset$, where $f \in \mathcal{M}_\infty$;

d) ρ has Fatou property; that is, $|f_n(w)| \uparrow |f(w)|$ for all $w \in \Omega$ implies that $\rho(f_n) \uparrow \rho(f)$ where $f \in \mathcal{M}_\infty$ and

e) ρ is order continuous in $\mathcal{E}$; that is, $g_n \in \mathcal{E}$ and $|g_n| \downarrow 0$ implies that $\rho(g_n) \downarrow 0$.

We say that a set $A \in \Sigma$ is ρ-null if $\rho(g.1_A) = 0$ for every $g \in \mathcal{E}$. We say that a property p holds ρ-almost everywhere if the exceptional set $\{w \in \Omega : p(w) \textit{ does not hold}\}$ is ρ-null. As usual we identify any pair of measurable functions f and g differing only on ρ-null set by $f = g$ ρ-a.e. With this in mind we define

$$\mathcal{M} = \{f \in \mathcal{M}_\infty : |f(w)| < \infty \ \rho - a.e\},$$

where $f \in \mathcal{M}$ is actually an equivalence class of functions equal ρ-a.e rather than an individual function.

Definition 1.2.9. Let ρ be a regular convex function pseudo-modular.

a) We say that ρ is a regular convex function semi-modular if $\rho(\alpha f) = 0$ for every $\alpha > 0$ implies that $f = 0$ ρ-a.e.

b) We say that ρ is a regular convex function modular if $\rho(f) = 0$ implies that $f = 0$ ρ-a.e.

The class of all nonzero regular convex function modulars defined on Ω is denoted by $\mathcal{R}$.

Example 1.2.10. Let $\Omega = [0,1]$. Let $\mathcal{M}_\infty$ be the set of all Lebesgue measurable functions on Ω. Define $\rho : \mathcal{M}_\infty \to [0,\infty]$ by

$$\rho(f) = \int_0^1 |f|.$$

Then clearly, ρ is convex function modular.

Remark 1.2.11. Let us denote $\rho(f,E) = \rho(f.1_E)$ for $f \in \mathcal{M}, E \in \Sigma$. Also by convention for $\alpha > 0$ we will write $\rho(\alpha, E)$ instead of $\rho(\alpha 1_E)$. We will use these notations when convenient. It is easy to prove that $\rho(f,E)$ is a convex function modular in the sense of Definition 1.2.8.

Remark 1.2.12. Note that if ρ is a regular convex function modular, then to verify that a set E is ρ-null it suffices to prove that there exists $\alpha > 0$ such that $\rho(\alpha, E) = 0$.

Definition 1.2.13. Let ρ be a convex function modular.

1. A modular function space is the vector space $L_\rho(\Omega,\Sigma)$ or briefly L_ρ, defined by

$$L_\rho = \{f \in \mathcal{M} : \rho(\lambda f) \to 0 \ as \ \lambda \to 0\}.$$

2. The following formula defines a norm in L_ρ frequently called the *Luxemburg norm*:

$$\|f\|_\rho = \inf\{\alpha > 0 : \rho\Big(\frac{f}{\alpha}\Big) \leq 1\}.$$

We now turn to some examples of modular function spaces. The following examples and more can be found in [23] and the references therein.

Example 1.2.14. If we define ρ by

$$\rho(f) = \int_{\mathbb{R}} |f(t)|^p dm(t)$$

for $f \in L^p$-space for $p \geq 1$, it generates modular function spaces. In this case dm denotes the Lebesgue measure on $\mathbb{R}$.

Example 1.2.15. Let l^p $(1 < p < \infty)$ denote the Banach space of real sequences such that $l^p = \{f = (f_1, f_2, ...) : \sum_{n=1}^{\infty} |f_n|^p < \infty\}$. Consider $\rho(f) = \sum_{n=1}^{\infty} |f_n|^p$, then it defines convex function modular and the space l^p defines modular function space.

Example 1.2.16. Let L^φ be an Orlicz space which is the space of Lebesgue measurable real valued functions f that satisfy

$$\int_{\mathbb{R}} \varphi(|f(x)|) d\lambda(x) < \infty,$$

where $\varphi : [0, \infty] \longrightarrow [0, \infty]$ is non-decreasing and convex function with $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$ is an Orilcz function, with the *Luxemburg* norm defined by

$$\|f\|_\varphi := \inf\{c > 0 : \int \varphi\Big(\frac{|f|}{c}\Big) d\lambda \leq 1\},$$

where λ is a Lebesgue measure on $\mathbb{R}$. If we define a function modular by

$$\rho(f) = \int_{\mathbb{R}} \varphi(|f(t)| d\lambda(t)),$$

then the space L^φ defines a modular function space.

In studying fixed point theory in modular function spaces, we repeatedly use the notion of ρ-convergence. In the following definition we describe this concept and other important terms in the modular function spaces.

Definition 1.2.17. [28, 29, 35] Let $\rho \in \mathcal{R}$.

a) We say that $\{f_n\}$ is ρ-convergent to f and write $f_n \to f(\rho)$ if $\rho(f_n - f) \to 0$.

b) A sequence $\{f_n\}$ in L_ρ is called a ρ-Cauchy sequence if $\rho(f_n - f_m) \to 0$ as $n, m \to \infty$.

c) A set $B \subset L_\rho$ is called ρ-closed if for any sequence $\{f_n\} \subset B$, the convergence $f_n \to f \ in \ (\rho)$ implies that f belongs to B.

d) A set $B \subset L_\rho$ is called ρ-bounded if its ρ-diameter is finite, where the ρ-diameter of B is defined as $\delta_\rho(B) = \sup\{\rho(f - g) : f \in B, g \in B\}$.

e) A set $B \subset L_\rho$ is called ρ-compact if for any $\{f_n\}$ in B, there exists a subsequence $\{f_{n_k}\}$ and an $f \in B$ such that $\rho(f_{n_k} - f) \to 0$.

f) A set $B \subset L_\rho$ is called ρ- a.e closed if for any sequence $\{f_n\}$ in B,which ρ-a.e converges to some f, we have $f \in B$.

g) A set $B \subset L_\rho$ is called ρ-a.e compact if for any sequence $\{f_n\}$ in B, there exists a subsequence $\{f_{n_k}\}$ which ρ-a.e converges to some $f \in B$.

h) Let $f \in L_\rho \ and \ B \subset L_\rho$. The ρ-distance between $f \ and \ B$ is defined as $d_\rho(f, B) = \inf\{\rho(f-g) : g \in B\}$.

i) A set $B \subset L_\rho$ is called strongly ρ-bounded if there exits $\beta > 1$ such that $\mathcal{M}_\beta(B) = \sup\{\rho(\beta(f-g)) : f, g \in B\} < \infty$.

Definition 1.2.18. [28, 29, 35] We say that a function modular $\rho \in \mathcal{R}$ has the Δ_2 - property if $\rho(2f_n) \to 0$ whenever $\rho(f_n) \to 0$.

Definition 1.2.19. [28, 29, 35] We say that a function modular $\rho \in \mathcal{R}$ has the Δ_2 - type condition if there exists a constant $0 < k < \infty$ such that for every $f \in L_\rho$, we have $\rho(2f) \leq k\rho(f)$.

Remark 1.2.20. [28, 29, 35] If ρ satisfies the Δ_2 - type condition, then it satisfies Δ_2 - property, but the converse is not true.

Definition 1.2.21. [1] We say that a function modular $\rho \in \mathcal{R}$ is uniformly continuous if for every $\epsilon >\ 0$ and $L\ >\ 0$, there exists $\delta >\ 0$ such that for every $g, h \in L\rho$ with $\rho(h) \leq \delta$ and $\rho(g) \leq L$, we have $|\rho(g) - \rho(h+g)| \leq \epsilon$.

Definition 1.2.22. [1] We say that L_ρ has property (R) if and only if every non-increasing sequence $\{C_n\}$ of nonempty, ρ-bounded, ρ-closed, convex subsets of L_ρ has nonempty intersection.

Definition 1.2.23. [25, 37] Let $\rho \in \mathcal{R}$ and $r > 0$, $\epsilon > 0$. Define,

$$D_1(r, \epsilon) = \{(f, g) : f, g \in L_\rho,\ \rho(f) \leq r,\ \rho(g) \leq r,\ \rho(f-g) \geq \epsilon r\}.$$

Let

$$\delta_1(r, \epsilon) = \inf\{1 - \frac{1}{r}\rho(\frac{f+g}{2}) : (f, g) \in D_1(r, \epsilon)\},\ if\ D_1(r, \epsilon) \neq \emptyset$$

and $\delta_1(r, \epsilon) = 1$, if $D_1(r, \epsilon) = \emptyset$. Then, we say that

1. ρ satisfies (UC1) if for every $r > 0$, $\epsilon > 0$, we have $\delta_1(r, \epsilon) > 0$.

2. ρ satisfies (UUC1) if for every $s \geq 0$, $\epsilon > 0$, there exists $\eta_1(s,\epsilon) > 0$ depending only on s and ϵ such that
$$\delta_1(r,\epsilon) > \eta_1(s,\epsilon) > 0,$$
for any $r > s$.

Definition 1.2.24. [25, 37] Let $\rho \in \mathcal{R}$ and $r > 0$, $\epsilon > 0$. Define,

$$D_2(r,\epsilon) = \{(f,g) : f,g \in L_\rho,\ \rho(f) \leq r,\ \rho(g) \leq r,\ \rho(\frac{f-g}{2}) \geq \epsilon r\}.$$

Let $\delta_2(r,\epsilon) = \inf\{1 - \frac{1}{r}\rho(\frac{f+g}{2}) : (f,g) \in D_2(r,\epsilon)\},\ if\ D_2(r,\epsilon) \neq \emptyset$ and $\delta_2(r,\epsilon) = 1,\ if\ D_2(r,\epsilon) = \emptyset$.Then, we that

1. ρ satisfies (UC2) if for every $r > 0$, $\epsilon > 0$, we have that $\delta_2(r,\epsilon) > 0$.
2. ρ satisfies (UUC2),if for every $s \geq 0,\ \epsilon > 0$, there exists $\eta_2(s,\epsilon) > 0$ depending only on s and ϵ such that
$$\delta_2(r,\epsilon) > \eta_2(s,\epsilon) > 0,$$
for any $r > s$.
3. We say that a function modular ρ is strictly convex (SC), if for every $f,g \in L_\rho$ such that $\rho(f) = \rho(g)$ and $\rho(\alpha f + (1-\alpha)g) = \alpha\rho(f) + (1-\alpha)\rho(g)$ for some $\alpha \in (0,1)$, we have $f = g$.

Proposition 1.2.25. [25, 37] From Definition 1.2.23 and Definition 1.2.24 the following properties hold:

(a) (UUCi) implies (UCi) for $i = 1, 2$;

(b) $\delta_1(r,\epsilon) \leq \delta_2(r,\epsilon)$;

(c) (UC1) implies (UC2);

(d) (UC2) implies (SC);

(e) (UUC1) imlies (UUC2).

The following definition of Opial's properties are used in modular function spaces:

Definition 1.2.26. Let $\rho \in \mathcal{R}$. We say that

1. L_ρ satisfies the ρ-a.e. Opial's property if for every $\{f_n\} \subset L_\rho$ ρ-a.e converges to 0, such that there exists $\beta < 1$ for which
$$\sup_n \rho(\beta f_n) = M < \infty,$$
we have
$$\liminf_{n\to\infty} \rho(f_n) < \liminf_{n\to\infty} \rho(f_n + f)$$
for every $f \in E_\rho$ not equal to 0.

2. L_ρ satisfies the ρ-a.e- strongly Opial's property if for every $\{f_n\} \subset L_\rho$ which is ρ-a.e convergent to 0, such that there exists $\beta < 1$ for which
$$\sup_n \rho(\beta f_n) = M < \infty,$$
the following equality holds for every $g \in E_\rho$
$$\liminf_{n\to\infty} \rho(f_n + g) = \liminf_{n\to\infty} \rho(f_n) + \rho(g).$$

1.2.2 Some basic properties of modular function spaces

In this section, we give some important properties that function modular ρ possesses . It is very important to note that the ρ-limit is actually unique, in the sense, if we assume that it has two limits, then they are equal ρ-a.e . However, that ρ-convergence does not necessarily imply ρ-Cauchy condition unlike the metric counterpart. This is because the function modular ρ fails to satisfy triangle inequality in general. Also, $f_n \to f$ in (ρ) does not imply in general $\lambda f_n \to \lambda f$, for $\lambda > 1$. The following proposition states the uniqueness of ρ-limit.

Proposition 1.2.27. Assume that we have a sequence $f_n \in L_\rho$ such that $\rho(f_n - f) \to 0$ and $\rho(f_n - g) \to 0$. Then, $f = g$ ρ-a.e.

Proof. Since ρ is convex, we have

$$\begin{aligned} \rho(\frac{f-g}{2}) &= \rho(\frac{f - f_n + f_n - g}{2}) \\ &\leq \frac{1}{2}\rho(f - f_n) + \frac{1}{2}\rho(f_n - g) \to 0, \end{aligned}$$

which gives, $\rho(\frac{f-g}{2}) = 0$. This implies that $f = g$ ρ-a.e. $\square$

Lemma 1.2.28. [14] Let ρ be a function modular satisfying the Δ_2-condition and let $\{f_n\}$ be a sequence in L_ρ such that $f_n \to f \in L_\rho$ ρ-a.e, and there exists $k > 1$ such that

$$\sup_{n\geq 1} \rho(k(f_n - f)) < \infty.$$

Then,

$$\liminf_{n\to\infty} \rho(f_n - g) = \liminf_{n\to\infty} \rho(f_n - f) + \rho(f - g), \ \forall\ g \in L_\rho.$$

Moreover, one has

$$\rho(f) \leq \liminf_{n\to\infty} \rho(f_n).$$

Proposition 1.2.29. [23] Let $\rho \in \mathcal{R}$. Then, the following hold.

1. If $\|f\|_\rho < 1$, then $\rho(f) \leq \|f\|_\rho$
2. $\|f\|_\rho \leq 1$ if and only if $\rho(f) \leq 1$

Proposition 1.2.30. [23] Let $\rho \in \mathcal{R}$ $f_n, \ f \ \in L_\rho$. Then $\|f_n - f\|_\rho \ \to \ 0$ implies $\rho(f_n - f) \to 0$.

Proposition 1.2.31. [23] Let $\rho \in \mathcal{R}$ and $f \in L_\rho$. Then $\|f\|_\rho > 1$ implies $\rho(f) \geq \|f\|_\rho$

Proposition 1.2.32. [23] *Let*$\rho \in \mathcal{R}, \ f_n, \ f \ \in \ L_\rho$. Then $\|f_n - f\|_\rho \ \to \ 0$ if and only if $\rho(\lambda(f_n - f)) \to 0$ for all $\lambda > 0$.

Due to the following proposition, it is very important to note that for any sequence in modular function space L_ρ, the ρ- limit, if it exists, belongs to L_ρ.

Proposition 1.2.33. [35] Assume a sequence $f_n \in L_\rho$ ρ-converges to an $f \in \mathcal{M}$, that is, $\rho(f_n - f) \to 0$. Then, $f \in L_\rho$.

Proposition 1.2.34. [35] If $\{f_n\}$ converges uniformly to f on a set $E \in \mathcal{P}$, then for every $\alpha > 0$, $\lim\limits_{n\to\infty} \rho(\alpha(f_n - f), E) = 0$.

Proposition 1.2.35. Let $\rho \in \mathcal{R}$ and $f, f_n \in \mathcal{M}_\infty$. If $f_n \to f$ ρ-a.e, then

$$\rho(f) \leq \liminf_{n\to\infty} \rho(f_n).$$

Theorem 1.2.36. [35] Let $\rho \in \mathcal{R}$. Assume that $\{f_n\}$ satisfies ρ-Cauchy condition. Then, there exists a sub-sequence $\{f_{n_k}\}$ of $\{f_n\}$ and function $f \in \mathcal{M}$ such that $f_{n_k} \to f$ ρ-a.e.

Proof. Denoting $E_{n,m}(\epsilon) = \{w \in \Omega : |f_n(w) - f_m(w)| \geq \epsilon\}$, then we get

$$\rho(\epsilon, E_{n,m}(\epsilon)) \leq \rho(f_n - f_m, E_{n,m}(\epsilon)) \leq \rho(f_n - f_m) \to 0.$$

Hence, for any $k \in \mathbb{N}$, there exists $n_0(k) \in \mathbb{N}$ such that if $n \geq n_0(k)$ and $m \geq n_0(k)$, then

$$\rho(2^{-k}, E_{n,m}(2^{-k})) < 2^{-k}.$$

Taking $n_1 = n_0(1),\ n_k \ = \ max\{n_{k-1}, n_0(k)\}$, we get the sequence $g_k = f_{n_k}$. Define

$$E_k = \{w \in \Omega : |g_k(w) - g_{k+1}(w)| \geq 2^{-k}\},$$

and observe that

$$\rho(2^{-k}, E_k) < 2^{-k}.$$

Let

$$G_j = \bigcup_{m=j}^{\infty} E_m$$

and observe that for every $j \geq i \geq k$ and $w \in \Omega \setminus G_k$, the following inequality holds:

$$|g_i(w) - g_j(w)| \leq \sum_{m=i}^{\infty} |g_m(w) - g_{m+1}(w)| < 2^{1-i}.$$

Define a measurable function,

$$h = \sum_{j=1}^{\infty} 2^{-j} \chi_{G_j \setminus G_{j+1}}$$

and calculate

$$\rho(h, G_k) = \rho(h, \bigcup_{n=k}^{\infty}(G_n \setminus G_{n+1})) \leq \sum_{n=k}^{\infty} \rho(2^{-n}, E_n) \leq 2^{1-k}.$$

Hence,

$$\rho(h, G) = 0,$$

where

$$G = \bigcap_{j=1}^{\infty} G_j.$$

Observe that if w does not belong to G, then there exists k such that for any $j \geq i \geq k$,

$$|g_i(w) - g_j(w)| < 2^{1-i}.$$

Therefore, there exists a real number $f(w)$ such that

$$f_{n_i}(w) = g_i(w) \to f(w).$$

Clearly, the function f is measurable. It remains to show that G is ρ-null.

Note that $G = \bigcup_{n=1}^{\infty} D_n$, where $D_n = G \cap \{w \in \Omega : h(w) \geq n^{-1}\}$. Therefore,

$$0 = \rho(h, G) \geq \rho(h, D_n) \geq \rho(n^{-1}, D_n).$$

Hence, D_n is ρ-null for every $n \in \mathbb{N}$, which implies that G is ρ-null. □

The completeness of L_ρ is very important in the sequel. Therefore, we state the following theorem that guarantees the completeness of L_ρ with respect to ρ-convergence and sketch the proof.

Theorem 1.2.37. [35] Let $\rho \in \mathcal{R}$. Then, L_ρ is complete with respect to ρ-convergence.

Proof. Let a sequence $\{f_n\} \subset L_\rho$ be ρ-Cauchy. By Theorem 1.2.36 there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$f_{n_k} \to f \in \mathcal{M} \ \rho - a.e.$$

Fix $\epsilon > 0$ and note that there exists $n_0 \in \mathbb{N}$ such that

$$\rho(f_m - f_n) < \epsilon,$$

provided $m, n \geq n_0$. Using Proposition 1.2.35 we get

$$\rho(f - f_{n)} \leq \liminf_{k\to\infty} \rho(f_{n_k} - f_n) \leq \epsilon \ for \ n \geq n_0.$$

It remains to show that $f \in L_\rho$. Let $\lambda_n \to 0$. Fix $\epsilon > 0$ and take $k \in \mathbb{N}$ such that

$$\rho(f - f_k) < \epsilon.$$

For n sufficiently large, we obtain

$$\rho(\lambda_n(f - f_k)) \leq \rho(f - f_k) < \epsilon,$$

which implies that

$$\rho(\lambda_n(f - f_k)) \to 0 \ as \ n \to \infty.$$

Thus, $f_k - f$ belongs to L_ρ. Since $f_k \in L_\rho$ and L_ρ is a linear space, then f belongs to L_ρ as desired. □

1.2.3 Some single valued mappings in modular spaces

This section presents, the definition of ρ-Lipschitz single valued mappings and some existence theorems of fixed points of ρ-Lipschitz single valued mappings in modular function space settings. Let us begin with the definition of ρ-Lipschitz single valued mappings.

Definition 1.2.38. [34] Let $\rho \in \mathcal{R}$ and $C \subset L_\rho$ be nonempty, and ρ-closed set. A mapping $T : C \to C$ is called a ρ-Lipschitzian or shortly Lipschitzian if there exists a number $L \geq 0$ such that

$$\rho(T(f) - T(g)) \leq L\rho(f - g),$$

for all $f, g \in C$. If $L < 1$, T is called ρ-contraction or shortly contraction mapping. Moreover, if $L \leq 1$, then T is called ρ-nonexpansive mapping.

A point $f \in C$ is called fixed point of T if $T(f) = f$. The set of all fixed points of T is denoted by $F(T)$.

Definition 1.2.39. [34] Let $\rho \in \mathcal{R}$ and $C \subset L_\rho$ be nonempty, and ρ-closed set. A mapping $T : C \to C$ is called a pointwise ρ-contraction or shortly pointwise contraction if there exists $\alpha : C \to [0, 1)$ such that

$$\rho(T(f) - T(g)) \leq \alpha(f)\rho(f - g),$$

for all $f, g \in C$.

Definition 1.2.40. [34] Let $\rho \in \mathcal{R}$ and $C \subset L_\rho$ be a nonempty ρ-closed set. A mapping $T : C \to C$ is called an asymptotic pointwise mapping if there exists a sequence of mappings $\alpha_n : C \to [0, \infty)$ such that

$$\rho(T^n(f) - T^n(g)) \leq \alpha_n(f)\rho(f - g),$$

for all $f, g \in C$. The following are the mappings that follow immediately.

(a) If $\alpha_n(f) = 1$ for every $f \in C$ and $n \in \mathbb{N}$, then T is called ρ-nonexpansive or shortly nonexpansive.

(b) If $\{\alpha_n\}$ converges poinywise to $\alpha : C \to [0,1)$, then T is called asymptotic pointwise ρ-contraction or shortly asymptotic ponitwise contraction.

(c) If $\limsup_n \alpha_n(f) \leq 1$ for any $f \in C$, then T is called asymptotic pointwise ρ-nonexpansive or shortly asymptotic pointwise nonexpansive.

(d) If α_n is a constant function for every n, and $\limsup_n \alpha_n \leq 1$ for any $f \in C$, then T is called asymptotically ρ-nonexpansive.

Theorem 1.2.41. [35]**(Banach Contraction Mapping Principle)** Let $\rho \in \mathcal{R}$ and $C \subset L_\rho$ be nonempty, ρ-closed and ρ-bounded set. Let $T : C \to C$ be a ρ-contraction. Then, T has a unique fixed point $\overline{f} \in C$. Moreover, for any $f \in C$,

$$\rho(T^n(f) - \overline{f}) \to 0,$$

as $n \to \infty$, where T^n is the n^{th} iterate of T.

Proof. Since T is ρ-contraction, there exists $\alpha \in [0,1)$ such that

$$\rho(T(f) - T(g)) \ \leq \ \alpha\rho(f-g), \ for\ all\ f, g \in C.$$

Now, let us fix $f_0 \in C$. Since C is ρ-bounded, we observe that

$$\begin{aligned}\rho(T^{n+k}(f_0) - T^n(f_0)) \ &\leq \ \alpha\rho(T^{n+k-1}(f_0) - T^{n-1}(f_0)) \\ &\leq \ \alpha^n \rho(T^k(f_0) - f_0) \\ &\leq \ \alpha^n \delta_\rho(C) \to 0, \ as\ n \to \infty.\end{aligned}$$

Hence, $\{T^n(f_0)\}$ is ρ-Cauchy. Now, the ρ-completeness of L_ρ, Theorem 1.2.37, implies that there exists $\overline{f} \in L_\rho$ such that $\lim_{n\to\infty} \rho(T^n(f_0)-$

$\overline{f}) = 0$. Because C is ρ-closed, we get that $\overline{f} \in C$. Since,

$$\begin{aligned} \rho(\frac{\overline{f} - T(\overline{f})}{2}) &\leq \rho(\overline{f} - T^n(f_0)) + \rho(T^n(f_0) - T(\overline{f})) \\ &\leq \rho(\overline{f} - T^n(f_0) + \alpha\rho(T^{n-1}(f_0) - \overline{f}) \to 0, \end{aligned}$$

as $n \to \infty$, we get $T(\overline{f}) = \overline{f}$, which means that $\overline{f}$ is a fixed point of T. To prove the uniqueness part, if we consider $T(f_1) = f_1$ and $T(f_2) = f_2$, then we have

$$\rho(f_1 - f_2) = \rho(T(f_1) - T(f_2)) \leq \alpha\rho(f_1 - f_2). \qquad (1.1)$$

Since $\alpha < 1$ and right-hand side of inequality (1.1) is finite, equality can hold only if $f_1 = f_2$. ☐

1.2.4 Some multivalued mappings in modular spaces

This section deals with the definition of Hausdorff distance in the modular function space framework. We also give the definition of multivalued ρ-Lipschitzian mappings and state some existence theorems for fixed point of ρ-noexpansive mappings, the class of mappings we focus on, in modular function spaces.

Let $\rho \in \mathcal{R}$ and C be a nonempty subset of the modular space L_ρ. We denote a collection of all nonempty ρ-closed and ρ- bounded subsets of C by $\mathcal{C}_\rho(C)$ and a collection of all nonempty ρ-compact subsets of C by $\mathcal{K}_\rho(C)$.

Definition 1.2.42. [24] A set $C \subset L_\rho$ is called ρ-proximinal if for each $f \in L_\rho$, there exists an element $g \in C$ such that

$$\rho(f - g) = d_\rho(f, C) = \inf\{\rho(f - h) : h \in C\}.$$

We denote the family of nonempty ρ-bounded ρ-proximinal subsets of C by $P_\rho(C)$.

Definition 1.2.43. [24] We define a Hausdorff distance on $\mathcal{C}_\rho(C)$ by,

$$H_\rho(A, B) = max\{\sup_{f\in A} d_\rho(f, B),\ \sup_{g\in B} d_\rho(g, A)\},$$

$A, B \in \mathcal{C}_\rho(C)$.

Definition 1.2.44. [24] A multivalued mapping $T : C \to \mathcal{C}_\rho(C)$ is called ρ-Lipschitzian if there exists a number $k \geq 0$ such that

$$H_\rho(T(f), T(g)) \leq k\rho(f - g)\ for\ all\ f, g \in C.$$

If $k \leq 1$ then, T is called ρ-nonexpansive and if $k < 1$, T is called ρ-contractive.

Definition 1.2.45. [24] A multivalued mapping $T : C \to \mathcal{C}_\rho(C)$ is said to satisfy Condition (I) if there exists a nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$, $\varphi(r) > 0$ for all $r \in (0, \infty)$ such that

$$d_\rho(f, Tf) \geq \varphi(d_\rho(f, F_\rho(T))),$$

for all $f \in C$.

Definition 1.2.46. A family of mappings $T_i : C \to C_\rho(C)$, $i = 1, 2, ..., m$, is said to satisfy Condition (II) if there exists a nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$, $\varphi(r) > 0$ for $r \in (0, \infty)$ such that

$$d_\rho(f, T_i(f)) \geq \varphi(d_\rho(f, \cap_{i=1}^m F_\rho(T_i)))$$

for some $i = 1, 2, ..., m$.

Chapter 2

Literature Review

2.1 Review on iterative approximation of fixed point

The most revolutionary fixed point result is due to Banach [10] (Banach Contraction Mapping Principle) which provides existence, uniqueness and approximation technique for contraction self mapping on complete metric space. Proving existence theorem for fixed point of nonexpansive mappings in Banach spaces was not easy as it was done for the contraction mappings. However, in 1965, the Kirk's breakthrough theorem appeared for nonexpansive self mappings defined on the weakly compact subsets of Banach spaces with normal structure.

Theorem 2.1.1. [41] Let X be a Banach space and suppose that C is a nonempty weakly compact convex subset of X which has the normal structure property. Then any nonexpansive mapping $T : C \longrightarrow C$ has a fixed point.

We note that, Kirk's theorem, unlike Banach contraction theorem, does not give any iterative technique of fixed point approximation.

When a fixed point exists, it is very important to study the technique of its approximation. Very often, the solution to a problem shall be translated into a fixed point. In some cases its existence can be ensured by inspection and in such cases it is better to think about the approximation technique than existence. Because of this, mathematicians started to think of approximation techniques. Indeed, unlike the Banach contraction principle, Picard iteration does not converge to a fixed point of nonexpansive self mappings on a closed and convex subsets of Banach spaces. Thus, mathematicians developed another iteration schemes to comeover this problem. Here we focuse on frequently used iteration schemes called Mann [45] and Ishikawa [21] iteration schemes.

Let X be a Banach space. Let C be a nonempty closed convex subset of X and $T : C \longrightarrow C$ be a mapping. The first iteration scheme due to Mann [45] is defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T(x_n),\ n \geq 0, \tag{2.1}$$

where the initial guess $x_0 \in C$ is taken arbitrarily and the sequence $\{\alpha_n\}$ is in the interval $[0, 1]$ such that $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$. The second iteration scheme due to Ishikawa [21] is defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_{n+})T(\beta_n x_n + (1 - \beta_n)T(x_n)),\ n \geq 0, \tag{2.2}$$

where the initial guess $x_0 \in C$ is taken arbitrarily and the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in the interval $[0, 1]$ such that $0 < \alpha_n \leq \beta_n < 1$, $\lim\limits_{n\to\infty} \beta_n = 0$, $\sum\limits_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Using Mann-type and Ishikawa-type iteration schemes, approximation of fixed points of both single and multivalued mappings were carried out by well known mathematicians in Banach spaces (see eg. [52, 53, 55–57] and the references therein).

As it was described previously, modular function spaces are generalizations of some classes of Banach spaces. Therefore, analysts are interested in working on the fixed point theory in these spaces. The work which paved the way to this direction is conducted by Khamsi et al. [31] in 1990. Following their results, many works had evolved. For instance in 2006, Dhompongsa et al. [18], proved the existence theorem for fixed points of multivalued nonexpansive mappings in modular function spaces. Similarily, in 2009, Latif and Kutib [33] proved existence theorem for fixed points of multi-valued ρ- nonexpansive-type mappings in the setting of modular function space.

The existence for fixed points of asymptotic nonexpansive, single valued mappings and multi-valued ρ-nonexpansive mappings in modular function spaces are proved for the following theorem in the literature.

Theorem 2.1.2. [23] Let $\rho \in \mathcal{R}$. Let $C \subset L_\rho$ be nonempty, ρ-closed, and ρ-bounded. Let $T : C \longrightarrow C$ be a pointwise ρ-contraction or asymptotic pointwise ρ-contraction. Then, T has at most one fixed point $f_0 \in C$.

Theorem 2.1.3. [23] Assume $\rho \in \mathcal{R}$ is (UUC1). Let C be a ρ-closed ρ-bounded convex nonempty subset of L_ρ. Then, any $T : C \longrightarrow C$ pointwise asymptotically nonexpansive mapping has a fixed point.

Moreover, the set of all fixed points $F(T)$ is ρ-closed and convex.

The existence for fixed points of multi-valued ρ-contractive and ρ-nonexpansive mappings were obtained due to Kutib and Latif [33]. They proved the following theorems in the setting of modular function spaces.

Theorem 2.1.4. [33] Let C be a nonempty ρ-closed subset of the modular function space L_ρ. Then any $T : C \to \mathcal{C}_\rho(C)$ ρ- contractive mapping has a fixed point, that is, there exists $f \in C$ such that $f \in T(f)$.

Proposition 2.1.5. [33] Let C be a ρ -closed, convex and ρ- bounded nonempty subset of the modular space L_ρ. Let $T : C \longrightarrow \mathcal{C}_\rho(C)$ be a ρ-nonexpansive mapping. Then, there exists an approximate fixed point sequence $\{f_n\}$ in C, that is, for any $n \geq 1$, there exists $F_n \in T(f_n)$ such that

$$\lim_{n\to\infty} \rho(f_n - F_n) \; = \; 0.$$

In particular, one has $\lim\limits_{n\to\infty} d_\rho(f_n, T(f_n)) = 0$, where

$$d_\rho(f_n, T(f_n)) = \inf\{\rho(f_n - g) : g \in T(f_n)\}.$$

Theorem 2.1.6. [33] Let C be a nonempty ρ-closed convex subset of the modular function space L_ρ. Assume that C is ρ-a.e compact. Then each ρ-nonexpansive mapping $T : C \longrightarrow \mathcal{K}_\rho(C)$ has a fixed point.

The first fixed point iteration process studied in the setting of modular function spaces appeared in 2012 for the class of asymptotic pointwise nonexapnsive mappings was due to Dehaish and Kozlowski [14]. It appeared as the pioneering work on the convergence of Mann-type and Ishikawa-type schemes in modular function spaces. Dehaish and Kozlowski introduced the generalized Mann-type and Ishikawa-type iteration schemes as follows:

Let $C \subset L_\rho$ be nonempty ρ-closed convex set. Let $T \in \mathcal{T}_r(C)$ and $\{n_k\}$ be an increasing sequence of natural numbers. Let $\{t_k\}$ and $\{s_k\}$ be sequences bounded away from 0 and 1. The generalized Mann-type (respectively, Ishikawa-type) iteration schemes generated by the mapping T, $\{t_k\}$, $\{s_k\}$ and sequence $\{n_k\}$, denoted by $gM(T, \{t_k\}, \{n_k\})$ (respectively, $gI(T, \{t_k\}, \{s_k\}, \{n_k\})$) is defined by

$$x_{k+1} = t_k T^{n_k}(x_k) + (1 - t_k)x_k, \tag{2.3}$$

and

$$x_{k+1} = t_k T^{n_k}(s_k T^{n_k}(x_k) + (1 - s_k)x_k) + (1 - t_k)x_k, \tag{2.4}$$

where $x_1 \in C$ is chosen arbitrarily and $\mathcal{T}_r(C)$ is a class of all asymptotic pointwise ρ-nonexpansive mappings on C such that a_n is bounded for every $n \geq 1$ and $\sum_{n=1}^{\infty} b_n(x) < \infty$, where $a_n(x) = max\{\alpha_n(x), 1\}$, $b_n(x) = a_n(x) - 1$ such that $T : C \to C$ is given by

$$\rho(T^n(x) - T^n(y)) \leq \alpha_n(x)\rho(x - y),$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

Using Mann-type given by (2.3) and Ishikawa-type by (2.4), Dehaish and Kozlowski proved the following convergence theorem:

Theorem 2.1.7. [14] Let $\rho \in \mathcal{R}$ satisfy conditions $(UUC1)$ and Δ_2. Let $C \subset L_\rho$ be a ρ-compact, ρ-bounded and convex set, and let $T \in \mathcal{T}_r(C)$. Let $\{t_k\} \subset (0,1)$ and $\{s_k\} \subset (0,1)$ be bounded away from 0 and 1. Assume that the generalized Mann-type scheme (respectively Ishikawa-type scheme) given by (2.3) and Ishikawa-type by (2.4) respectively are well defined, that is, $\limsup_{k\to\infty} a_{n_k}(x_k) = 1$. Then there exists a fixed point $x \in F(T)$ such that the sequence $\{x_k\}$ generated by the schemes converges strongly to a fixed point of T, that is, $\lim\limits_{k\to\infty} \rho(x_k - x) = 0$.

In 2014, Alsulam and Kozlowski [6] proved convergence theorems to fixed points of ρ-nonexpansive, singlevalued, mappings using both Mann and Ishikawa iteration schemes. Moreover, it was proved that the set of fixed points is nonempty. In fact, they proved the following theorems.

Theorem 2.1.8. [6] Let $\rho \in \mathcal{R}$. Assume that

(1) ρ satisfies (UUC1);

(2) ρ has strong Opial property;

(3) ρ has Δ_2 -property and uniformly continuous.

Let $C \subset L_\rho$ be nonempty, ρ-a.e compact, convex, strongly ρ- bounded and ρ-closed set. Let $T : C \longrightarrow C$ be ρ-nonexpansive mapping and $\alpha \in (0,1)$. Let $\{x_k\}$ be a sequence of elements of C generated by Mann process

$$x_{k+1} = \alpha T(x_k) + (1-\alpha)x_k,$$

where $x_1 \in C$ is chosen arbitrarily. Then, there exists $x \in F_\rho(T)$ such that $x_n \to x$ ρ-a.e.

Theorem 2.1.9. [6] Let $\rho \in \mathcal{R}$. Assume that

(1) ρ satisfies (UUC1);

(2) ρ has strong Opial property;

(3) ρ has Δ_2- property and uniformly continuous.

Let $C \subset L_\rho$ be nonempty, ρ-a.e compact, convex, strongly ρ- bounded and ρ-closed set. Let $T : C \longrightarrow C$ be ρ-nonexpansive mapping and $\alpha \in (0,1)$, $\beta \in (0,1)$. Let $\{x_k\}$ be a sequence of elements of C generated by Ishikawa scheme

$$x_{k+1} = \alpha T((\beta x_k) + (1-\beta)x_k) + (1-\alpha)x_k,$$

where $x_1 \in C$ is chosen arbitrarily. Then, there exists $x \in F_\rho(T)$ such that $x_n \to x$ ρ-a.e.

On the other hand, in 2014 Abdou et.al [1], proved ρ-convergence of sequence generated by Ishikawa-type iteration scheme to common fixed point of two ρ-nonexpansive, single valued, mappings in the setting of modular function space. While proving convergence theorems, Abdou et.al dropped condition $(UUC1)$ and used strict convexity, which is weaker than $UUC1$ condition. Moreover, they dropped Δ_2-property and used only uniform continuity of modular function ρ. Abdou et.al defined the following Ishikawa-type iteration scheme for two nonexpansive mappings .

Let $C \subset L_\rho$ be nonempty ρ-closed ρ-bounded and convex set. Let

$S, T : C \to C$ be ρ-nonexpansive mappings. Define Ishikawa iteration scheme by,

$$\begin{cases} f_1 \in C, \\ f_{n+1} = \alpha_n S(\beta_n T(f_n) + (1-\beta_n)f_n) + (1-\alpha_n)f_n, \end{cases} \tag{2.5}$$

where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$.

If we put $S = T$, in the above iterative scheme (2.5), then it automatically reduces to the original Ishikawa iterative scheme (2.2) for a mapping T.

Abdou et.al proved the following theorem that guarantees the convergence of sequence generated by Ishikawa-type scheme (2.5) to common fixed point of two ρ-nonexpansive mappings in the setting of modular function spaces:

Theorem 2.1.10. [1] Let $\rho \in \mathcal{R}$ be strictly convex and uniformly continuous. Let $C \subset L_\rho$ be a nonempty, ρ-bounded, ρ-compact and convex set. Let $S, T : C \to C$ be two ρ-nonexpansive mappings. Assume $F = F_\rho(T) \cap F_\rho(S) \neq \emptyset$. Let $\{f_n\} \subset C$ be as defined in (2.5), where $\alpha_n, \beta_n \in [a, b]$ with $0 < a \leq b < 1$. Then, $\{f_n\}$ ρ-converges to a common fixed point of S and T.

In the study of fixed point theory, it is importantnt to note that the problem of the extension of known fixed point results obtained for singlevalued mappings to the case of multivalued mappings is of area of interest, becuase of its great importance in the areas of applied science. Having the above convergence theorems proved for fixed points of single valued mappings, perhaps, the first attempt on the convergence theorems of iterative algorithm for fixed points of multivalued mappings in the setting of modular function spaces was introduced in 2014 by Khan and Abbas [24]. Khan and Abbas studied the following Mann-type iterative scheme to approximate fixed point of multi-valued ρ-nonexpansive mappings in the setting of modular function spaces. Let $C \subset L_\rho$ be nonempty ρ-closed ρ-bounded and convex set. Let $T : C \longrightarrow P_\rho(C)$ be a multivalued mapping.

$$\begin{cases} f_1 \in C, \\ f_{n+1} = (1-\alpha_n) f_n + \alpha_n u_n, \end{cases} \tag{2.6}$$

where $u_n \in P_\rho^T(f_n)$ and $\{\alpha_n\} \subset (0,1)$ is bounded away from both 0 and 1. Using the Mann-type iteration scheme in (2.6), Khan and Abbas proved the following convergence theorems:

Theorem 2.1.11. [24] Let ρ satisfy (UUC1) and let C be a nonempty ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T : C \to P_\rho(C)$ be a multivalued mapping such that P_ρ^T is a ρ-nonexpansive mapping. Suppose $F_\rho(T) \neq \emptyset$. Let $\{f_n\} \subset C$ be defined by (2.6). Then,

$$\lim_{n\to\infty} \rho(f_n - c) \; exists \; for \; all \; c \in F_\rho(T)$$

and

$$\lim_{n\to\infty} d_\rho(f_n, P_\rho^T(f_n)) = 0.$$

Theorem 2.1.12. [24] Let ρ satisfy (UUC1) and let C be a nonempty ρ-compact, ρ-bounded and convex subset of L_ρ. Let $T : C \to P_\rho(C)$ be a multivalued mapping such that P_ρ^T is ρ-nonexpansive mapping.

Suppose that $F_\rho(T) \neq \emptyset$. Let $\{f_n\}$ be defined as in (2.6). Then $\{f_n\}$ ρ-converges to a fixed point of T.

Theorem 2.1.13. [24] Let ρ satisfy (UUC1) and let C be a nonempty ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T : C \to P_\rho(C)$ be a multivalued mapping with $F_\rho(T) \neq \emptyset$ and satisfying Condition (I) such that P_ρ^T is ρ-nonexpansive mapping. Let $\{f_n\}$ be as defined in (2.6). Then, $\{f_n\}$ ρ-converges to a fixed point of T.

Cosequently, many authors have studied approximation of fixed points of nonlinear multivalued mappings in modular function spaces, for instance, [2, 6, 15, 16] and many others in the references therein. Following the results extensively seen in the literature, we are motivated and inspired by the research going in this area. Furthermore, the study of fixed point theory in the setting of modular function spaces is not well developed compared to its applicability in areas of applied science and applied mathematics such as differential equations and Optimization theory. This in turn, motivated us to set our area of research in this direction. Now, we are in a position to state our main research problems.

2.2 Statement of the Problems

Based on the review made above, we identify the following problems on the convergence theorems of iterative approximation of fixed points in modular function spaces.

We note that, Khan and Abbas [24] proved the ρ-convergence theorems for fixed points of multivalued mappings in modular function spaces. The first problem that originated from this result is as follows:

1. *Can we introduce an iterative scheme which approximates a common fixed point of a finite family of ρ-nonexpansive multi-valued mappings?*

Next, we focus on the results of Abdou et al. [1]. They have initiated the approximation of common fixed point of two single-valued ρ-nonexpansive mappings in modular function spaces using Ishikawa-type iterative scheme. Based on their results, we pose our second problem as follows:

2. *Can we state and prove a convergence theorem for common fixed point of two multi-valued ρ-nonexpansive mappings in modular function spaces using Ishikawa iteration process?*

The next problem is based on the extension of the class of mappings that contains the class of ρ-nonexpansive multivalued mappings in the setting of modular function space. We define a new class of multi-valued mappings called ρ-quasi-nonexpansive and pose the following problem:

3. *Can we establish an iterative scheme and study the convergence of the scheme to common fixed point of finite family of multivalued ρ-quasi-nonexpansive mappings using Mann-type iterative scheme?*

2.3 Aim and Objectives

The aim of this thesis is to extend some existing results in the literature. To achieve this aim, the specific objectives are to

- establish convergence theorem which approximates fixed point of multivalued ρ-nonexpansive mappings in modular function spaces.
- introduce an iterative scheme and study its convergence to a common fixed point of a finite family of ρ- nonexpansive multivalued mappings in modular function spaces.
- extend the convergence theorems of Abdou et al. [1] which are proved for the ρ-nonexpansive single valued mappings to the case of ρ- nonexpansive multi-valued mappings in modular function spaces.
- study Mann-type iteration scheme and prove strong converegence of the scheme to a common fixed point of the class of ρ-quasi-nonexpansive mappings which is more general than the class of ρ-nonexpansive mappings.

2.4 Scope of the Study

This study is confined to approximation of fixed points of ρ-nonexpansive multi-valued mappings in Modular function spaces. Moreover, our iteration schemes are restricted to Mann-type and Ishikawa-type iterative schemes.

2.5 Methodology

In this section, we study the methodology we use to obtain our main results.

2.5.1 Mann-type iterative scheme in Modular function spaces

We first introduce a Mann-type iterative scheme which could be used to approximate a common fixed of finite family of multivalued ρ-nonexpansive mappings. We define it as follows:

Let $C \subset L_\rho$ be nonempty convex set and $T_i : C \longrightarrow P_\rho(C)$, $i = 1, 2, ..., m$, be a finite family of multi-valued mappings. Fix $f_1 \in C$ and define a sequence $\{f_n\} \subset C$ as follows:

$$f_{n+1} = \alpha_{0,n} f_n + \alpha_{1,n} g_{1,n} + \alpha_{2,n} g_{2,n} + ... + \alpha_{m,n} g_{m,n}, \tag{2.7}$$

where $g_{i,n} \in P_\rho^{T_i}(f_n)$ and $\{\alpha_{i,n}\} \subset (0,1)$ is bounded away from 0 and 1 such that $\sum\limits_{i=0}^{m} \alpha_{i,n} = 1$. We call the scheme (2.7) Mann-type iteration scheme for multi-valued mappings T_i, $i = 1, 2, 2...m$.

Note that, if $m = 1$ in the above iterative algorithm (2.7), then it automatically reduces to the Mann-type iterative scheme defined by (2.6).

2.5.2 Ishikawa-type iterative scheme in modular function spaces

Secondly, we study an Ishikawa-type iterative scheme which could be used to approximate a common fixed point of two multi-valued ρ-nonexpansive mappings in modular function space. We define the scheme for two multi-valued mappings as follows:
Let $C \subset L_\rho$ be nonempty and convex and $S, T : C \longrightarrow P_\rho(C)$ be multi-valued mappings. Define the sequence $\{f_n\}_{n=1}^{\infty}$ by

$$\begin{cases} f_1 \in C, \\ g_n = \beta_n u_n + (1-\beta_n) f_n, \\ f_{n+1} = \alpha_n v_n + (1-\alpha_n) f_n \end{cases} \tag{2.8}$$

where $u_n \in P_\rho^S(f_n)$, $v_n \in P_\rho^T(g_n)$, and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0, 1)$ such that both are bounded away from 0 and 1. This algorithm coincides with the Ishikawa-type scheme (2.5) used by Abdou et al.[1] if our mappings are to be single valued.

2.5.3 Mann-type iterativescheme and ρ-quasi-nonexpansive multivalued mappings

To obtain our third research question, the following Mann-type iterative scheme is also studied for a finite family of mappings in modular function spaces. Let $C \subset L_\rho$ be nonempty convex set and $T_i : C \to \mathcal{C}_\rho(C)$, $i = 1, 2, ..., m$, be a finite family of multi-valued mappings. Fix $f_1 \in C$ and define a sequence $\{f_n\} \subset C$ as follows:

$$f_{n+1} = \alpha_{0,n} f_n + \alpha_{1,n} u_{1,n} + \alpha_{2,n} u_{2,n} + \ldots + \alpha_{m,n} u_{m,n}, \tag{2.9}$$

where $u_{i,n} \in T_i(f_n)$ and $\{\alpha_{i,n}\} \subset (0,1)$ is bounded away from 0 and 1 such that $\sum_{i=0}^{m} \alpha_{i,n} = 1$.

Chapter 3

Theory of Methods

3.1 Some Auxiliary Mathematical Tools

We state the following results which shall be needed in the sequel. For the clarity sake, we shall also include the proofs of the known ones, though the proofs are given in the cited articles.

Lemma 3.1.1. [24] Let $T : C \to P_\rho(C)$ be a multivalued mapping and

$$P_\rho^T(f) = \{g \in Tf : \rho(f - g) = d_\rho(f, T(f))\}.$$

Then, the following are equivalent:

(i) $f \in F_\rho(T)$, that is $f \in Tf$;

(ii) $P_\rho^T(f) = \{f\}$, that is, $f = g$ for each $g \in P_\rho^T(f)$;

(iii) $f \in F_\rho(P_\rho^T)$, that is, $f \in P_\rho^T(f)$. Further, $F_\rho(T) = F(P_\rho^T)$ where $F(P_\rho^T)$ denotes the set of fixed points of P_ρ^T.

Proof. Suppose (i) holds. Since $f \in F_\rho(T)$ implies that $f \in Tf$, so $d_\rho(f, Tf) = 0$. By definition of $P_\rho^T(f)$ for any $g \in P_\rho^T(f)$, we have $\rho(f - g) = d_\rho(f, Tf) = 0$. Thus, $f = g$. That is, $P_\rho^T(f) = \{f\}$. Hence (ii) holds. Condition (ii) implies (iii) is obvious. Now, to prove (iii) implies (i). Since $f \in F_\rho(P_\rho^T)$, by definition of P_ρ^T, we have

$$d_\rho(f, Tf) = \rho(f - g) = 0.$$

Thus, $f \in Tf$ by the ρ-closedness of Tf. Hence, (i) holds. $\square$

Lemma 3.1.2. [14] Let ρ satisfies (UUC1)and $\{f_n\}$ and $\{g_n\}$ be sequence in L_ρ. Let $\{t_n\} \subset (0, 1)$ be bounded away from both 0 and 1. If there exists $R > 0$ such that $\limsup_{n\to\infty} \rho(f_n) \leq R$, $\limsup_{n\to\infty} \rho(g_n) \leq R$ and $\lim_{n\to\infty} \rho(t_n f_n + (1 - t_n)g_n) = R$, then

$$\lim_{n\to\infty} \rho(f_n - g_n) = 0.$$

Lemma 3.1.3. [28] The following statements are equivalent:

(i) ρ satisfies the Δ_2-condition.

(ii) $\rho(f_n - f) \to 0$ if and only if $\rho(\lambda(f_n - f)) \to 0$, for all $\lambda >\ 0$ if and only if $\|f_n - f\|_\rho \to 0$.

Proof. (i) $\implies$ (ii). By Proposition 1.2.30, $\|f_n - f\|_\rho \to 0$ implies that $\rho(f_n - f) \to 0$. But, if $\rho(f_n - f) \to 0$ then by Δ_2 property, we have $\rho(2(f_n - f)) \to 0$. Now, Proposition 1.2.31 $\rho(\lambda(f_n - f)) \to 0$ implies $\|f_n - f\|_\rho \to 0$. In particular, $\rho(2(f_n - f)) \to 0$ implies $\|f_n - f\|_\rho \to 0$ as desired.
(ii) $\implies$ (i). Suppose $\rho(f_n - f) \to 0$ if and only if $\|f_n - f\|_\rho \to 0$. Then $\|\lambda(f_n - f)\|_\rho \to 0$ for every $\lambda > 0$. By Proposition 1.2.30, $\rho(\lambda(f_n - f)) \to 0$ for all $\lambda > 0$. In particular, $\rho(2(f_n - f)) \to 0$. Then, it follows that ρ has Δ_2 property. $\square$

3.2 Tools from Our Results

The following lemmas are obtained in our work. We use them to prove our convergence theorems in the sequel.

Lemma 3.2.1. *Let $\rho \in \mathcal{R}$. Let $A, B \in P_\rho(L_\rho)$. For every $f \in A$, there exists $g \in B$ such that $\rho(f-g) \leq H_\rho(A,B)$.*

Proof. Let $f \in A$. Then, by proximinality of B there exists $b \in B$ such that $\rho(f-b) = d_\rho(f,B)$. By the definition of Hausdorff distance, we have

$$\begin{aligned} H_\rho(A,B) &= \max\{\sup_{h\in A} d_\rho(h,B), \sup_{k\in B} d_\rho(k,A)\} \\ &\geq \sup_{h\in A} d_\rho(h,B) \\ &\geq d_\rho(f,B) \\ &= \rho(f-h). \end{aligned}$$

□

Lemma 3.2.2. Let $\rho \in \mathcal{R}$ satisfy (UUC1) and Δ_2-property. Let $C \subset L_\rho$ be ρ-closed, ρ-bounded and convex set. Suppose $T_i : C \to P_\rho(C)$, $i = 1, 2, ..., m$, is a finite family of multi-valued mappings such that $P_\rho^{T_i}$ is ρ-nonexpansive mapping for $i = 1, 2, ..., m$. Assume that $F := \bigcap_{i=1}^{m} F_\rho(T_i) \neq \emptyset$. Let $\{f_n\}$ be as defined in (2.7). Then,

1. $\lim_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F$;
2. $\lim_{n\to\infty} d_\rho(f_n, T_i(f_n)) = 0$, for all $i = 1, 2, ..., m$.

Proof. Let $p \in F$ be arbitrary. Then, by Lemma 3.1.1 $P_\rho^{T_i}(p) = \{p\}$, $\forall\ i \in \{1, 2, ..., m\}$. From equation (2.7), we have

$$\begin{aligned} \rho(f_{n+1} - p) &= \rho(\alpha_{0,n} f_n + \alpha_{1,n} g_{1,n} + \alpha_{2,n} g_{2,n} + ... + \alpha_{m,n} g_{m,n} - p) \\ &= \rho(\alpha_{0,n}(f_n - p) + \alpha_{1,n}(g_{1,n} - p) + \alpha_{2,n}(g_{2,n} - p) + ... \\ &\quad + \alpha_{n,m}(g_{m,n} - p)) \end{aligned}$$

$$\begin{aligned} &\leq \alpha_{0,n}\rho(f_n - p) + \alpha_{1,n}\rho(g_{1,n} - p) + \alpha_{2,n}\rho(g_{2,n} - p) + ... \\ &\quad +\alpha_{m,n}\rho(g_{m,n} - p). \end{aligned} \tag{3.1}$$

Since $g_{i,n} \in P_\rho^{T_i}(f_n)$ and $P_\rho^{T_i}$ is ρ-nonexpansive for all $i = 1, 2, ..., m$, we have

$$\begin{aligned} \rho(g_{i,n} - p) &\leq H_\rho(P_\rho^{T_i}(f_n), P_\rho^{T_i}(p)) \\ &\leq \rho(f_n - p) \end{aligned} \tag{3.2}$$

for all $i = 1, 2, ..., m$.

Now, from (3.1), (3.2) and the assumption $\sum_{i=0}^{m} \alpha_{i,n} = 1$, we obtain that

$$\rho(f_{n+1} - p) \leq \rho(f_n - p), \text{ for all } p \in F. \tag{3.3}$$

Therefore, $\lim_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F$.

Let

$$\lim_{n\to\infty} \rho(f_n - p) = r \tag{3.4}$$

for some $r \geq 0$. So from (3.2) and (3.4), we have

$$\limsup_{n\to\infty} \rho(g_{m,n} - p) \leq r. \tag{3.5}$$

Consider

$$\begin{aligned}
& \rho\Big(\frac{\alpha_{0,n}}{1-\alpha_{m,n}}(f_n-p)+\frac{1}{1-\alpha_{m,n}}\sum_{i=1}^{m-1}\alpha_{i,n}(g_{i,n}-p)\Big) \\
\leq\ & \frac{\alpha_{0,n}}{1-\alpha_{m,n}}\rho(f_n-p)+\frac{1}{1-\alpha_{m,n}}\sum_{i=1}^{m-1}\alpha_{i,n}\rho(g_{i,n}-p) \\
\leq\ & \frac{\sum_{i=0}^{m-1}\alpha_{i,n}}{1-\alpha_{m,n}}\rho(f_n-p) \\
=\ & \rho(f_n-p).
\end{aligned}$$

Therefore, from (3.4) we obtain

$$\limsup_{n\to\infty}\rho\Big(\frac{\alpha_{0,n}}{1-\alpha_{m,n}}(f_n-p)+\frac{\sum_{i=1}^{m-1}\alpha_{i,n}(g_{i,n}-p)}{1-\alpha_{m,n}}\Big)\ \leq\ r. \quad (3.6)$$

Thus (3.4), (3.5), (3.6) and Lemma 3.1.2, give that

$$\lim_{n\to\infty}\rho\Big(\frac{\alpha_{0,n}}{1-\alpha_{m,n}}f_n+\frac{\sum_{i=1}^{m-1}\alpha_{i,n}g_{i,n}}{1-\alpha_{m,n}}-g_{m,n}\Big)\ =\ 0. \quad (3.7)$$

Now,

$$\begin{aligned}
\rho(f_{n+1}-g_{m,n}) &= \rho\Big(\alpha_{0,n}f_n+\sum_{i=1}^{m}\alpha_{i,n}g_{i,n}-g_{m,n}\Big) \\
&= \rho\Big(\alpha_{0,n}f_n+\sum_{i=1}^{m-1}\alpha_{i,n}g_{i,n}-(1-\alpha_{m,n})g_{m,n}\Big) \\
&= \rho\Big((1-\alpha_{m,n})[\frac{\alpha_{0,n}}{1-\alpha_{m,n}}f_n+\frac{\sum_{i=1}^{m-1}\alpha_{i,n}g_{i,n}}{1-\alpha_{m,n}}-g_{m,n}]\Big).
\end{aligned}$$

From (3.7) by the Δ_2- property of ρ and Lemma 3.1.3, we get that

$$\lim_{n\to\infty}\rho(f_{n+1}-g_{m,n})=0.$$

In the same way, we can show that

$$\lim_{n\to\infty} \rho(f_{n+1} - f_n) = 0$$

and

$$\lim_{n\to\infty} \rho(f_{n+1} - g_{i,n}) = 0,$$

for all $i = 1, 2, ..., m - 1$.

Now by the convexity of ρ, we get

$$\begin{aligned} \rho(\frac{f_n - g_{i,n}}{2}) &= \rho(\frac{f_n - f_{n+1}}{2} + \frac{f_{n+1} - g_{i,n}}{2}) \\ &\leq \frac{\rho(f_n - f_{n+1})}{2} + \frac{\rho(f_{n+1} - g_{i,n})}{2}. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} \rho(\frac{f_n - g_{i,n}}{2}) = 0.$$

Since ρ satisfies Δ_2- property by Lemma 3.1.3 we obtain

$$\lim_{n\to\infty} \rho(f_n - g_{i,n}) = 0, \tag{3.8}$$

for all $i = 1, 2, ..., m$.

Since $g_{i,n} \in P_\rho^{T_i}(f_n)$, we have

$$d_\rho(f_n, T_i(f_n)) = \rho(f_n - g_{i,n})$$

for all $i = 1, 2, ..., m$.

Therefore,

$$\lim_{n\to\infty} d_\rho(f_n, T_i(f_n)) = 0$$

follows immediately from (3.8) for all $i = 1, 2, ..., m$. □

Lemma 3.2.3. Let $\rho \in \mathcal{R}$ and $C \subset L_\rho$ be nonempty ρ-bounded and convex set. Suppose $S, T : C \to P_\rho(C)$ be multi-valued mappings such that P_ρ^T and P_ρ^S are ρ-nonexpansive mappings with $F = F_\rho(T) \cap F_\rho(S) \neq \emptyset$. Let $\{f_n\}$ be defined as in (2.8). Then, $\lim\limits_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F$.

Proof. Let $p \in F$ be arbitrary. Then by Lemma 3.1.1, we have $P_\rho^T(p) = \{p\}$ and $P_\rho^S(p) = \{p\}$. Furthermore, from (2.8), we get

$$\begin{aligned} \rho(f_{n+1} - p) &= \rho((1-\alpha_n)f_n + \alpha_n v_n - p) \\ &= \rho((1-\alpha_n)(f_n - p) + \alpha_n(v_n - p) \\ &\leq (1-\alpha_n)\rho(f_n - p) + \alpha_n \rho(v_n - p) \\ &\leq (1-\alpha_n)\rho(f_n - p) + \alpha_n H_\rho(P^T(g_n), P^T(p)) \\ &\leq (1-\alpha_n)\rho(f_n - p) + \alpha_n \rho(g_n - p). \end{aligned} \tag{3.9}$$

Agaim from (2.8), we obtain that

$$\begin{aligned} \rho(g_n - p) &= \rho((1-\beta_n)f_n + \beta_n u_n - p) \\ &\leq (1-\beta_n)\rho(f_n - p) + \beta_n \rho(u_n - p) \\ &\leq (1-\beta_n)\rho(f_n - p) + \beta_n H_\rho(P^S(f_n), P^S(p)) \\ &\leq (1-\beta_n)\rho(f_n - p) + \beta_n \rho(f_n - p) \\ &= \rho(f_n - p). \end{aligned} \tag{3.10}$$

Now, from (3.9) and (3.10), we get that

$$\rho(f_{n+1} - p) \leq \rho(f_n - p).$$

Therefore, the sequence $\{\rho(f_n - p)\}$ is decreasing. Hence, $\lim\limits_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F$. □

Chapter 4

Main Results

4.1 Convergence of Mann-type scheme for finite family of mappings in modular function spaces

In this section, we state and prove convergence theorems for Mann-type iteration scheme defined by (2.7) to a common fixed point of finite family of multivalued ρ-nonexpansive mappings in modular function spaces.

Theorem 4.1.1. Let $\rho \in \mathcal{R}$ satisfy (UUC1) and Δ_2-property. Let $C \subset L_\rho$ be ρ-closed, ρ-bounded and convex set. Suppose $T_i : C \to P_\rho(C)$, $i = 1, 2, ..., m$, be a finite family of multi-valued mappings such that $P_\rho^{T_i}$ is ρ-nonexpansive mapping for each $i = 1, 2, ..., m$. Let $F := \bigcap_{i=1}^{m} F_\rho(T_i) \neq \emptyset$ and $\{f_n\}$ be as defined in (2.7). Then, $\{f_n\}$ ρ-converges to a point in F if and only if $\liminf_{n\to\infty} d_\rho(f_n, F) = 0$.

Proof. The necessity is straight forward. Now, we prove the other way round. Suppose that $\liminf_{n\to\infty} d_\rho(f_n, F) = 0$. From the proof of Lemma 3.2.2, equation (3.3), we have

$$\rho(f_{n+1} - p) \leq \rho(f_n - p), \text{ for all p } \in F.$$

This implies that

$$d_\rho(f_{n+1}, F) \leq d_\rho(f_n, F).$$

So that $\lim\limits_{n\to\infty} d_\rho(f_n, F)$ exists. But by hypothesis, $\liminf\limits_{n\to\infty} d_\rho(f_n, F) = 0$. Therefore, it must be the case that

$$\lim_{n\to\infty} d_\rho(f_n, F) = 0.$$

Next, we show that $\{f_n\}$ is a ρ-Cauchy sequence in C. Let $\epsilon > 0$ be arbitrary. Then there exists an integer $n_0 \in \mathbb{N}$ such that

$$d_\rho(f_n, F) < \frac{\epsilon}{2}, \ \forall\ n \geq n_0.$$

In particular, $inf\{\rho(f_{n_0} - p) : p \in F\} < \frac{\epsilon}{2}$. Thus, there must exist a $p_0 \in F$ such that

$$\rho(f_{n_0} - p_0) \ < \ \epsilon. \tag{4.1}$$

Now, for $m, n \geq n_0$, we have

$$\begin{aligned} \rho(\frac{f_m - f_n}{2}) &\leq \frac{1}{2}\rho(f_m - p_0) + \frac{1}{2}\rho(f_n - p_0) \\ &\leq \frac{\rho(f_{n_0} - p_0)}{2} + \frac{\rho(f_{n_0} - p_0}{2}) \\ &< \epsilon. \end{aligned}$$

Since ρ satisfies Δ_2 condition, by Lemma 3.1.3, we get that $\{f_n\}$ is a ρ-Cauchy sequence in C. Since L_ρ is complete with respect to ρ-convergence and C is ρ-closed , there exists an $f \in C$ such that $\rho(f_n - f) \to 0$.

Next, we show that f is a common fixed point of $\{T_i\}_{i=1}^m$. Let $g_i \in P_\rho^{T_i}(f)$ be arbitrary. Then by Lemma 3.2.1 there exits $g_{n,i} \in P_\rho^{T_i}(f_n)$

such that $\rho(g_{n,i}-g_i) \leq H_\rho(P_\rho^{T_i}(f_n), P_\rho^{T_i}(f))$ for all $i=1,2,...,m$. Then by convexity of ρ and Lemma 3.2.2, we have

$$\begin{aligned}
\rho(\frac{f-g_i}{3}) &= \rho(\frac{f-f_n}{3}+\frac{f_n-g_{n,i}}{3}+\frac{g_{n,i}-g_i}{3}) \\
&\leq \frac{1}{3}\rho(f-f_n)+\frac{1}{3}\rho(f_n-g_{n,i})+\frac{1}{3}\rho(g_{n,i}-g_i) \\
&\leq \rho(f-f_n)+d_\rho(f_n,P_\rho^{T_i}(f_n))+H_\rho(P_\rho^{T_i}(f_n),P_\rho^{T_i}(f)) \\
&\leq \rho(f-f_n)+d_\rho(f_n,T_i(f_n))+\rho(f-f_n) \to 0 \text{ as n } \to \infty.
\end{aligned}$$

Thus, $f=g_i$. Since $g_i \in P_\rho^{T_i}(f)$ by Lemma 3.1.1, we have $f \in T_i(f)$ for all i= 1,2,...,m . Therefore, $f \in F$, i.e., $\{f_n\}$ ρ- converges to a common fixed point of the family $\{T_i\}_{i=1}^m$. □

Theorem 4.1.2. Let $\rho \in \mathcal{R}$ satisfy (UUC1) and Δ_2-property. Let $C \subset L_\rho$ be ρ-closed, ρ-bounded and convex set. Suppose $T_i : C \to P_\rho(C)$, $i=1,2,...,m$, be a finite family of multi-valued mappings satisfying Condition (II) such that $P_\rho^{T_i}$ is ρ-nonexpansive mapping for each $i=1,2,...,m$. Assume that $F := \bigcap_{i=1}^{m} F_\rho(T_i) \neq \emptyset$. Let $\{f_n\}$ be as defined in (2.7). Then $\{f_n\}$ ρ-converges to a point in F.

Proof. By Lemma 3.2.2, $\lim_{n\to\infty}\rho(f_n-p)$ exists for all $p \in F$. If $\lim_{n\to\infty}\rho(f_n-p)=0$, there is nothing to prove. Suppose $\lim_{n\to\infty}\rho(f_n-p)=R>0$. Again from the proof of Lemma 3.2.2, we have that

$$\rho(f_{n+1}-p) \leq \rho(f_n-p) \text{ for all p in } F.$$

This implies ,

$$d_\rho(f_{n+1},F) \leq d_\rho(f_n,F).$$

Hence, $\lim_{n\to\infty} d_\rho(f_n, F)$ exists. By Condition (II) and Lemma 3.2.2,

$$0 = \lim_{n\to\infty} d_\rho(f_n, T_i(f_n)) \geq \lim_{n\to\infty} \varphi(d_\rho(f_n, F))$$

for some $i = 1, 2, ..., m$. Thus, $\lim_{n\to\infty} \varphi(d_\rho(f_n, F)) = 0$. Since φ is nondecreasing and $\varphi(0) = 0$, $\lim_{n\to\infty} d_\rho(f_n, F) = 0$. By Theorem 4.1.1 the desired result follows. □

4.2 Convergence of Ishikawa-type scheme for two multi-valued mappings in modular function spaces

In this section, we state and prove the convergence theorems for Ishikawa-type iteration scheme defined by (2.8) to a common fixed point of two multivalued mappings in modular function spaces. We thus have the following theorems.

Theorem 4.2.1. Let $\rho \in \mathcal{R}$ satisfy (UUC1) and $C \subset L_\rho$ be ρ-closed, ρ-bounded and convex set. Suppose $S, T : C \to P_\rho(C)$ be multi-valued mappings such that P_ρ^T and P_ρ^S are ρ-nonexpansive mappings with $F = F_\rho(T) \cap F_\rho(S) \neq \emptyset$. Let $\{f_n\}$ be defined as in (2.8). Then, $\{f_n\}$ is a common ρ-approximate fixed point sequence for S and T.

Proof. By Lemma 3.2.3, $\lim_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F$. Let

$$\lim_{n\to\infty} \rho(f_n - p) = r. \tag{4.2}$$

By (3.10) and (4.2), we have that

$$\limsup_{n\to\infty} \rho(g_n - p) \leq r. \tag{4.3}$$

Since $u_n \in P^S(f_n)$, $\rho(u_n - p) \leq H_\rho(P^S(f_n), P^S(p)) \leq \rho(f_n - p)$. Therefore,

$$\limsup_{n\to\infty} \rho(u_n - p) \leq r. \tag{4.4}$$

Similarly,

$$\limsup_{n\to\infty} \rho(v_n - p) \leq r. \tag{4.5}$$

Since $\alpha_n \in (0,1)$ is bounded away from 0 and 1, there exists $\alpha \in (0,1)$ such that $\lim\limits_{n\to\infty} \alpha_n = \alpha$ (sub-sequentially if necessary).

Now,

$$\begin{aligned} \rho(f_{n+1} - p) &= \rho(\alpha_n v_n + (1-\alpha_n) f_n - p) \\ &\leq \alpha_n \rho(v_n - p) + (1-\alpha_n)\rho(f_n - p). \end{aligned} \tag{4.6}$$

Applying *liminf* to both sides of (4.6) and using (4.2), we get that

$$\begin{aligned} \liminf_{n\to\infty} \rho(f_{n+1} - p) &\leq \alpha \liminf_{n\to\infty} \rho(v_n - p) + (1-\alpha) \liminf_{n\to\infty} \rho(f_n - p) \\ r &\leq \alpha \liminf_{n\to\infty} \rho(v_n - p) + (1-\alpha) r \\ r &\leq \liminf_{n\to\infty} \rho(v_n - p). \end{aligned} \tag{4.7}$$

Combining (4.5) and (4.7), we obtain

$$\lim_{n\to\infty} \rho(v_n - p) = r. \tag{4.8}$$

Since $v_n \in P^T(g_n)$, we have $\rho(v_n - p) \leq \rho(g_n - p)$. Thus,

$$\liminf_{n\to\infty} \rho(g_n - p) \geq r. \tag{4.9}$$

From (4.3) and (4.9), we get

$$\lim_{n\to\infty} \rho(g_n - p) = r. \tag{4.10}$$

Thus, using (4.10) we obtain

$$\lim_{n\to\infty} \rho((1-\beta_n)(f_n - p) + \beta_n(u_n - p)) = r. \tag{4.11}$$

Therefore, Lemma 3.1.2 gives that

$$\lim_{n\to\infty} \rho(f_n - u_n) = 0. \tag{4.12}$$

Since $u_n \in P^S(f_n)$, $\lim\limits_{n\to\infty} d_\rho(f_n, S(f_n)) = 0$ follows from (4.12); that is, $\{f_n\}$ is ρ-approximate fixed point sequence for S. From (4.2), we have also

$$\lim_{n\to\infty} \rho((1-\alpha_n)(f_n - p) + \alpha_n(v_n - p)) = r. \tag{4.13}$$

From (4.2), (4.5), (4.13) and Lemma 3.1.2, we get

$$\lim_{n\to\infty} \rho(f_n - v_n) = 0. \tag{4.14}$$

From (2.8), we have

$$\rho(g_n - f_n) \le \rho(u_n - f_n).$$

Applying (4.12), we obtain that

$$\lim_{n\to\infty} \rho(g_n - f_n) = 0.$$

Since P^T is ρ-nonexpansive, we get that

$$H_\rho(P^T(f_n), P^T(g_n)) \le \rho(g_n - f_n).$$

Thus,

$$\lim_{n\to\infty} H_\rho(P^T(f_n), P^T(g_n)) = 0. \tag{4.15}$$

Since $v_n \in P^T(g_n)$, $d_\rho(f_n, P^T(g_n)) \leq \rho(f_n - v_n)$.

Thus, from (4.14), we have

$$\lim_{n\to\infty} d_\rho(f_n, P^T(g_n)) = 0. \tag{4.16}$$

Then,

$$d_\rho(f_n, P^T(f_n)) \;\leq\; d_\rho(f_n, P^T(g_n)) + H_\rho(P^T(f_n), P^T(g_n)).$$

Applying (4.15) and (4.16), we obtain that

$$\lim_{n\to\infty} d_\rho(f_n, P^T(f_n)) = 0 \tag{4.17}$$

But,

$$d_\rho(f_n, T(f_n)) \leq d_\rho(f_n, P^T(f_n)).$$

Therefore, the desired result follows from (4.17); that is, $\{f_n\}$ is ρ-approximate fixed point sequence for T. Hence, $\{f_n\}$ is a common ρ-approximate fixed point sequence for S and T. $\square$

Theorem 4.2.2. Let $\rho \in \mathcal{R}$ satisfy (UUC1) and $C \subset L_\rho$ be nonempty ρ-compact, ρ-bounded and convex set. Suppose $S, T : C \to P_\rho(C)$ are multivalued mappings such that P_ρ^T and P_ρ^S are ρ-nonexpansive mappings with $F = F_\rho(T) \cap F_\rho(S) \neq \emptyset$. Let $\{f_n\}$ be defined as in (2.8). Then, $\{f_n\}$ ρ-converges to a common fixed point of S and T.

Proof. By the ρ-compactness of C, there exists a sub-sequence $\{f_{n_k}\}$ of $\{f_n\}$ and $f \in C$ such that $\rho(f_{n_k} - f) \to 0$ as $k \to \infty$.

We show that f is a common fixed point of S and T; that is, $f \in S(f)$ and $f \in T(f)$. We first show that $f \in T(f)$. Let $g \in P_\rho^T(f)$ and

$h \in P_\rho^S(f)$ be arbitrary. Then by Lemma 3.2.1, there exists $g_k \in P_\rho^T(f_{n_k})$ and $h_k \in P_\rho^S(f_{n_k})$ such that $\rho(g_k - g) \leq H_\rho(P_\rho^T(f_{n_k}), P_\rho^T(f))$ and $\rho(h_k - h) \leq H_\rho(P_\rho^S(f_{n_k}), P_\rho^S(f))$. Now, by convexity of ρ, we have

$$\begin{aligned}
\rho(\frac{f-g}{3}) &= \rho(\frac{f-f_{n_k}}{3} + \frac{f_{n_k}-g_k}{3} + \frac{g_k-g}{3}) \\
&\leq \frac{1}{3}\rho(f-f_{n_k}) + \frac{1}{3}\rho(f_{n_k}-g_k) + \frac{1}{3}\rho(g_k-g) \\
&\leq \rho(f-f_{n_k}) + d_\rho(f_{n_k}, P_\rho^T(f_{n_k})) + \rho(g_k-g) \\
&\leq \rho(f-f_{n_k}) + d_\rho(f_{n_k}, P_\rho^T(f_{n_k})) + H_\rho(P_\rho^T(f_{n_k}), P_\rho^T(f)) \\
&\leq \rho(f-f_{n_k}) + d_\rho(f_{n_k}, P_\rho^T(f_{n_k})) + \rho(f-f_{n_k}) \to 0, \ as\ k \ \to \infty.
\end{aligned}$$

Hence, $f = g$ ρ-a.e. Since $g \in P_\rho^T(f)$ was arbitrary, we have $P_\rho^T(f) = \{f\}$. Thus, by Lemma 3.1.1, $f \in T(f)$. By the similar argument, we have $f \in S(f)$. This completes the proof. □

Theorem 4.2.3. Let $\rho \in \mathcal{R}$ satisfy (UUC1) and Δ_2-property. Let $C \subset L_\rho$ be ρ-closed, ρ-bounded and convex set. Suppose $S, T : C \to P_\rho(C)$ be multi-valued mappings such that P_ρ^S and P_ρ^T are ρ-nonexpansive mappings with $F = F_\rho(T) \cap F_\rho(S) \neq \emptyset$ and $\{f_n\}$ be defined as in (2.8). Suppose S and T satisfy Condition (II). Then, $\{f_n\}$ ρ-converges to a point in F.

Proof. By Lemma 3.2.3 we know that $\lim_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F$. If $\lim_{n\to\infty} \rho(f_n - p) = 0$, then we are done. Assume that $\lim_{n\to\infty} \rho(f_n - p) = r > 0$. By the same lemma, we have

$$\rho(f_{n+1} - p) \leq \rho(f_n - p), \text{ for all p } \in F.$$

This implies that

$$d_\rho(f_{n+1}, F) \leq d_\rho(f_n, F).$$

So that $\lim\limits_{n\to\infty} d_\rho(f_n, F)$ exists. By Theorem 4.2.1 and Condition (II) either

$$0 = \lim_{n\to\infty} d_\rho(f_n, T(f_n)) \geq \lim_{n\to\infty} \varphi(d_\rho(f_n, F))$$

or

$$0 = \lim_{n\to\infty} d_\rho(f_n, S(f_n)) \geq \lim_{n\to\infty} \varphi(d_\rho(f_n, F)).$$

In both cases, $\lim\limits_{n\to\infty} \varphi(d_\rho(f_n, F)) = 0$. Since φ is increasing and $\varphi(0) = 0$, it must be the case that $\lim\limits_{n\to\infty} d_\rho(f_n, F) = 0$.

Let $\epsilon > 0$ be arbitrary. Then there exists an integer $n_0 \in \mathbb{N}$ such that

$$d_\rho(f_n, F) < \frac{\epsilon}{2}, \ \forall n \geq n_0.$$

In particular, $inf\{\rho(f_{n_0} - p) : p \in F\} < \frac{\epsilon}{2}$. Thus, there must exist a $p_0 \in F$ such that

$$\rho(f_{n_0} - p_0) \ < \ \epsilon. \tag{4.18}$$

Now for $m, n \geq n_0$, we have

$$\begin{aligned} \rho(\frac{f_m - f_n}{2}) &\leq \frac{1}{2}\rho(f_m - p_0) + \frac{1}{2}\rho(f_n - p_0) \\ &\leq \frac{\rho(f_{n_0} - p_0)}{2} + \frac{\rho(f_{n_0} - p_0)}{2} \\ &< \epsilon. \end{aligned}$$

Since ρ satisfies Δ_2 condition, by Lemma 3.1.3 we get $\{f_n\}$ is a ρ-Cauchy sequence in C. Since L_ρ is complete and C is ρ-closed, there exists an $f \in C$ such that $\rho(f_n - f) \to 0$.

Now by Theorem 4.2.2 the desired result follows. □

Remark 4.2.4. *If, in Theorem 4.2.2 and Theorem 4.2.3, we consider* $S = I$*, the identity mapping on* C*, the Scheme (2.8) reduces to the following known Mann iterative scheme*

$$f_{n+1} = \alpha_n v_n + (1 - \alpha_n) f_n, \tag{4.19}$$

where $v_n \in P_\rho^T(f_n)$*, and* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *that is bounded away from* 0 *and* 1*.*

4.3 Approximating fixed points of a multivalued ρ-quasi-nonexpansive mappings

Now, we introduce a new class of mappings called ρ-quasi-nonexpansive. Indeed, the definition of this type of mappings is given in Banach spaces. We just adapt the definition in Banach spaces to the modular function spaces framework.

Definition 4.3.1. A mapping $T : C \to \mathcal{C}_\rho(C)$ is said to be ρ-quasi-nonexpansive if $F_\rho(T) \neq \emptyset$ and $H_\rho(T(f), T(h)) \leq \rho(f - h)$ for all $f \in C$ and $h \in F_\rho(T)$.

The following lemma shows that the class of ρ-nonexpansive multivalued mappings with $F_\rho \neq \emptyset$ in the modular function spaces is also included in the class of ρ-quasi-nonexpansive multivalued mappings.

Lemma 4.3.2. If a mapping $T : C \longrightarrow \mathcal{C}_\rho(C)$ is ρ-nonexpansive with $F_\rho(T) \neq \emptyset$, then T is ρ-quasi-nonexpansive.

Proof. Suppose T is ρ-nonexpansive and $F_\rho(T) \neq \emptyset$. The by ρ-nonexpansiveness of T, we have $H_\rho(Tf, Tp) \leq \rho(f - p)$ for all $p \in$

$F_\rho(T)$ and $f \in C$. This implies that T is ρ-quasi-nonexpansive. $\square$

The following example shows that, the class of ρ-quasi-nonexpansive multivalued mappings properly contains the class of ρ-nonexpansive multi-valued mappings in modular function spaces.

Example 4.3.3. Consider $L_\rho = \mathbb{R}$ and $\rho(x) = |x|$, the usual absolute value function on the set of real numbers. Then clearly, ρ is convex function modular on $\mathbb{R}$. Let $C \subset L_\rho$ be given by $C = [0, 5]$. Then define $T : C \to \mathcal{C}_\rho(C)$ by

$$T(x) = \begin{cases} [0, \frac{x}{5}], if x \neq 5 \\ \\ \{1\}, if x = 5 \end{cases}$$

then T is ρ-quasi-nonexpansive ; but it is not ρ-nonexpansive. In fact, $F_\rho(T) = \{0\}$ and hence $H_\rho(Tx, T0) \leq \frac{|x|}{5} \leq |x - 0| = \rho(x - 0)$ for all $x \in C$. Thus, T is ρ-quasi-nonexpansive multivalued mapping, but T is not ρ-nonexpansive. To see this, take $x_0 = \frac{25}{6}$ and $y_0 = 5$, then $Tx_0 = [0, \frac{5}{6}]$ and $Ty_0 = \{1\}$ and $\rho(x_0, y_0) = |x_0 - y_0| = |\frac{25}{6} - 5| = \frac{5}{6}$. But the Hausdorff distance of Tx_0 and Ty_0 is given by,

$$\begin{aligned} H_\rho(Tx_0, Ty_0) &= max\Big\{\sup_{a \in \{1\}} d_\rho\Big(a, \Big[0, \frac{5}{6}\Big]\Big), \sup_{b \in [0, \frac{5}{6}]} d_\rho\Big(b, \{1\}\Big)\Big\} \\ &= 1 > \frac{5}{6} = |x_0 - y_0| = \rho(x_0, y_0), \end{aligned}$$

which proves that T is not ρ-nonexpansive.

Lemma 4.3.4. Let $\rho \in \mathcal{R}$ and C be a nonempty ρ-closed, ρ-bounded and convex subset of L_ρ. Suppose $T_i : C \to \mathcal{C}_\rho(C)$, $i = 1, 2, ..., m$, is a finite family of ρ-quasi-nonexpansive multi-valued mappings. Assume that $F := \bigcap_{i=1}^{m} F_\rho(T_i) \neq \emptyset$ and $T_i(p) = \{p\}$ for all $p \in F$. Then, the common fixed point set F is ρ-closed.

Proof. Let $\{g_n\}$ be a sequence in F such that $\lim_{n\to\infty} \rho(g_n - f) = 0$ for some $f \in C$. We must show that $f \in F$. Since $g_n \in F_\rho(T_i)$ and T_i is ρ-quasi-nonexpansive for

$i = 1, 2, ..., m$,

$$H_\rho(T_i g_n, T_i f) \leq \rho(g_n - f).$$

Now, let $h_i \in T_i f$ for $i = 1, 2, ..., m$. Then,

$$\rho(h_i - g_n) \leq H_\rho(T_i f, T_i g_n) \leq \rho(g_n - f).$$

Now, by convexity of ρ, we get

$$\begin{aligned} \rho(\frac{h_i - f}{2}) &= \rho(\frac{h_i - g_n}{2} + \frac{g_n - f}{2}) \\ &\leq \frac{1}{2}(\rho(h_i - g_n) + \rho(g_n - f)) \\ &\leq \rho(g_n - f) \to \ 0 \ , as \ n \to \infty. \end{aligned}$$

This implies that, $h_i = f$, that is, $f \in T_i f$ for all $i = 1, 2, ..., m$. Since $T_i(f)$ is ρ-closed, $f \in F_\rho(T_i)$ for all $i = 1, 2, ..., m$. Thus, the common fixed point set is ρ-closed. □

Theorem 4.3.5. Let $\rho \in \mathcal{R}$ satisfy (UUC1) and Δ_2-property. Let $C \subset L_\rho$ be ρ-closed, ρ-bounded and convex set. Suppose $T_i : C \to \mathcal{C}_\rho(C)$, $i = 1, 2, ..., m$, is a finite family of ρ-quasi-nonexpansive multi-valued mappings. Assume that $F := \bigcap_{i=1}^{m} F_\rho(T_i) \neq \emptyset$ and $T_i(p) = \{p\}$ for all $p \in F$. Let $\{f_n\}$ be as defined in (2.9).Then,

1. $\lim_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F$;
2. $\lim_{n\to\infty} d_\rho(f_n, T_i(f_n)) = 0$, for all $i = 1, 2, ..., m$.

Proof. Let $p \in F$ be arbitrary. From equation (2.9), we have

$$\begin{aligned} \rho(f_{n+1} - p) &= \rho(\alpha_{0,n} f_n + \alpha_{1,n} u_{1,n} + \alpha_{2,n} u_{2,n} + ... + \alpha_{m,n} u_{m,n} - p) \\ &= \rho(\alpha_{0,n}(f_n - p) + \alpha_{1,n}(u_{1,n} - p) + \alpha_{2,n}(u_{2,n} - p) + ... \\ &\quad + \alpha_{m,n}(u_{m,n} - p)) \\ &\leq \alpha_{0,n}\rho(f_n - p) + \alpha_{1,n}\rho(u_{1,n} - p) + \alpha_{2,n}\rho(u_{2,n} - p) + ... \\ &\quad + \alpha_{m,n}\rho(u_{m,n} - p). \end{aligned} \tag{4.20}$$

Since $u_{i,n} \in T_i(f_n)$, $T_i(p) = \{p\}$ and T_i is ρ-quasi-nonexpansive for $i = 1, 2, ..., m$, we have

$$\begin{aligned} \rho(u_{i,n} - p) &\leq H_\rho(T_i(f_n), T_i(p)) \\ &\leq \rho(f_n - p) \end{aligned} \tag{4.21}$$

for all $i = 1, 2, ..., m$.

Now, from (4.20), (4.21) and the assumption $\sum_{i=0}^{m} \alpha_{i,n} = 1$, we obtain that

$$\rho(f_{n+1} - p) \leq \rho(f_n - p), \text{ for all } p \in F.$$

Therefore, $\lim_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F$.

Let

$$\lim_{n\to\infty} \rho(f_n - p) = r \tag{4.22}$$

for some $r \geq 0$. So from (4.21) and (4.22), we have

$$\limsup_{n\to\infty} \rho(u_{m,n} - p) \leq r. \tag{4.23}$$

Next, observe that

$$\begin{aligned}
& \rho\Big(\frac{\alpha_{0,n}}{1-\alpha_{m,n}}(f_n - p) + \frac{1}{1-\alpha_{m,n}}\sum_{i=1}^{m-1}\alpha_{i,n}(u_{i,n}-p)\Big) \\
\leq\ & \frac{\alpha_{0,n}}{1-\alpha_{m,n}}\rho(f_n - p) + \frac{1}{1-\alpha_{m,n}}\sum_{i=1}^{m-1}\alpha_{i,n}\rho(u_{i,n}-p) \\
\leq\ & \frac{1}{1-\alpha_{m,n}}\sum_{i=0}^{m-1}\alpha_{i,n}\rho(f_n - p) \\
=\ & \rho(f_n - p).
\end{aligned}$$

Therefore, from (4.22) we obtain

$$\begin{aligned}
& \limsup_{n\to\infty}\rho\Big(\frac{\alpha_{0,n}}{1-\alpha_{m,n}}(f_n - p) + \frac{1}{1-\alpha_{m,n}}\sum_{i=1}^{m-1}\alpha_{i,n}(u_{i,n}-p)\Big) \\
\leq\ & r. \qquad (4.24)
\end{aligned}$$

Thus (4.22), (4.23), (4.24) and Lemma 3.1.2, give that

$$\lim_{n\to\infty}\rho\Big(\frac{\alpha_{0,n}}{1-\alpha_{m,n}}f_n + \frac{1}{1-\alpha_{m,n}}\sum_{i=1}^{m-1}\alpha_{i,n}u_{i,n} - u_{m,n}\Big) = 0.(4.25)$$

Now,

$$\begin{aligned}
\rho(f_{n+1} - u_{m,n}) &= \rho\Big(\alpha_{0,n}f_n + \sum_{i=1}^{m}\alpha_{i,n}u_{i,n} - u_{m,n}\Big) \\
&= \rho\Big(\alpha_{0,n}f_n + \sum_{i=1}^{m-1}\alpha_{i,n}u_{i,n} - (1-\alpha_{n,m})u_{m,n}\Big) \\
&= \rho\Big((1-\alpha_{m,n})\Big[\frac{\alpha_{0,n}}{1-\alpha_{m,n}}f_n + \frac{1}{1-\alpha_{m,n}}\sum_{i=1}^{m-1}\alpha_{i,n}u_{i,n} - u_{m,n}\Big]\Big).
\end{aligned}$$

From (4.25), the Δ_2- property of ρ and Lemma 3.1.3, we get that

$$\lim_{n\to\infty}\rho(f_{n+1} - u_{m,n}) = 0.$$

In the same way, we can show that

$$\lim_{n\to\infty}\rho(f_{n+1}-f_n)=0$$

and

$$\lim_{n\to\infty}\rho(f_{n+1}-u_{i,n})=0,$$

for all $i=1,2,...,m-1$.

Now, by the convexity of ρ, we get

$$\begin{aligned}\rho(\frac{f_n-u_{i,n}}{2}) &= \rho(\frac{f_n-f_{n+1}}{2}+\frac{f_{n+1}-u_{i,n}}{2}) \\ &\leq \frac{\rho(f_n-f_{n+1})}{2}+\frac{\rho(f_{n+1}-u_{i,n})}{2}.\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\rho(\frac{f_n-u_{i,n}}{2})=0.$$

Since ρ satisfies Δ_2- property by Lemma 3.1.3, we obtain

$$\lim_{n\to\infty}\rho(f_n-u_{i,n}) = 0, \tag{4.26}$$

for all $i=1,2,...,m$.

Since $u_{i,n}\in T_i(f_n)$, we have

$$d_\rho(f_n,T_i(f_n))\leq\rho(f_n-u_{n,i})$$

for all $i=1,2,...,m$.

Therefore,

$$\lim_{n\to\infty}d_\rho(f_n,T_i(f_n))=0$$

follows immediately from (4.26) for all $i=1,2,...,m$.

Theorem 4.3.6. Let $\rho \in \mathcal{R}$ satisfy (UUC1) and Δ_2-property. Let $C \subset L_\rho$ be ρ-closed, ρ-bounded and convex set. Suppose $T_i : C \to \mathcal{C}_\rho(C)$, $i = 1, 2, ..., m$, is a finite family of ρ-quasi-nonexpansive multi-valued mappings. Let $F := \bigcap_{i=1}^{m} F_\rho(T_i) \neq \emptyset$ with $T_i(p) = \{p\}$ for all $p \in F$ and $\{f_n\}$ be as defined in (2.9). Then, $\{f_n\}$ ρ-converges to a point in F if and only if $\liminf_{n\to\infty} d_\rho(f_n, F) = 0$.

Proof. The necessity is straight forward. Now, we prove the other way round. Suppose that $\liminf_{n\to\infty} d_\rho(f_n, F) = 0$. From the proof of Theorem 4.3.5, we have

$$\rho(f_{n+1} - p) \leq \rho(f_n - p), \text{ for all p } \in F.$$

This implies that

$$d_\rho(f_{n+1}, F) \leq d_\rho(f_n, F).$$

So that $\lim_{n\to\infty} d_\rho(f_n, F)$ exists. But by hypothesis, $\liminf_{n\to\infty} d_\rho(f_n, F) = 0$. Therefore, it must be the case that

$$\lim_{n\to\infty} d_\rho(f_n, F) = 0.$$

Consider a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a sequence $\{p_k\}$ in F such that

$$\rho(f_{n_k} - p_k) < \frac{1}{3^k} \text{ and } \rho(f_{n_{k+j}} - f_{n_k}) < \frac{1}{3^{k+j}} \text{ for all } k,\ j \geq 1.$$

We show that $\{p_k\}$ is a ρ-Cauchy sequence in F.

Observe that for $j \geq 1$,

$$\begin{aligned}
\rho(\frac{p_{k+j}-p_k}{3}) &= \rho(\frac{p_{k+j}-f_{n_{k+j}}+f_{n_{k+j}}-f_{n_k}+f_{n_k}-p_k}{3}) \\
&\leq \frac{1}{3}\rho(p_{k+j}-f_{n_{k+j}})+\frac{1}{3}\rho(f_{n_{k+j}}-f_{n_k})+\frac{1}{3}\rho(f_{n_k}-p_k) \\
&< \frac{1}{3^{k+j+1}}+\frac{1}{3^{k+1}}+\frac{1}{3^{k+j+1}} \longrightarrow 0 \ as \ k,\ j \to \infty.
\end{aligned}$$

Since ρ-satisfies Δ_2-condition, we obtain by Lemma 3.1.3 that $\{p_k\}$ is a ρ-Cauchy sequence in F. But, we know that L_ρ is complete and F is ρ-closed, by Lemma 4.3.4, $\{p_k\}$ ρ-converges to a point in F, say p. Next we show that $\{f_n\}$ ρ-converges to p. In fact, by convexity of ρ, we have

$$\begin{aligned}
\rho(\frac{f_{n_k}-p}{2}) &= \rho(\frac{f_{n_k}-p_k+p_k-p}{2}) \\
&\leq \rho(f_{n_k}-p_k)+\rho(p_k-p) \to 0 \ as \ k \to \infty.
\end{aligned}$$

By the Δ_2-condition of ρ, we have $\rho(f_{n_k}-p) \to 0$, as k $\to \infty$. Since $\lim_{n\to\infty} \rho(f_n-p)$ exists, the sequence $\{f_n\}$ ρ-converges to p. Which completes the proof.

Theorem 4.3.7. *Let $\rho \in \mathcal{R}$ satisfy (UUC1) and Δ_2-property. Let $C \subset L_\rho$ be ρ-closed, ρ-bounded and convex set. Suppose $T_i : C \to \mathcal{C}_\rho(C)$, $i = 1, 2, ..., m$, is a finite family of ρ-quasi-nonexpansive multi-valued mappings satisfying Condition (II). Assume that $F := \bigcap_{i=1}^{m} F_\rho(T_i) \neq \emptyset$ and $T_i(p) = \{p\}$ for all $p \in F$. Let $\{f_n\}$ be as defined in (2.9). Then $\{f_n\}$ ρ-converges to a point in F.*

Proof. From Theorem 4.3.5, we have $\lim_{n\to\infty} d_\rho(f_n,\ T_i(f_n)) = 0$ and $d_\rho(f_{n+1}, F) \leq d_\rho(f_n, F)$. Hence, $\lim_{n\to\infty} d_\rho(f_n, F)$ exists. It then follows from the definition of Condition (II) that,

$$0 = \lim_{n\to\infty} d_\rho(f_n, T_i(f_n)) \geq \lim_{n\to\infty} \varphi(d_\rho(f_n, F))$$

for some $i = 1, 2, ..., m$.

Thus, $\lim\limits_{n\to\infty} \varphi(d_\rho(f_n, F)) = 0$. Since φ is nondecreasing, $\varphi(0) = 0$, and $\varphi(t) > 0$ for all $t \in (0, \infty)$, we have

$$\lim_{n\to\infty} d_\rho(f_n, F) = 0.$$

The rest of the proof follows from the proof of Theorem 4.3.6, as desired.

Chapter 5

Discussion, Conclusion and Recommendation

5.1 Discussion

The results obtained in this thesis improve several results that appeared recently in modular function spaces. In particular, our Theorem 4.1.1 (through Lemma 3.2.2) closed a gap observed in Theorem 2.1.12 of Khan and Abbas [24] which was given without clear reasons on how the final result was obtained. Moreover, the first result obtained in this thesis is published in the journal of *"Communications in Optimization Theory, 2016"* . On the other hand, Theorem 4.3.5 extends Theorem 2.1.11 of Khan and Abbas [24] from the class of ρ-nonexpansive mappings to more general class of ρ-quasi-nonexpansive mappings. Furthermore, the results obtained in this thesis extend the corresponding results of Abdou et al. [1] from single valued mappings to multivalued mappings. Finally, Theorems 4.3.6 and 4.3.7 improve and extend the corresponding results obtained in Theorems 4.1.1 and 4.1.2.

5.2 Conclusion

In conclusion, we gladly state that the main aim and specific objectives of research are achieved. Our method of proof is independent of interest.

5.3 Recommendation

Our recommendations for the future work and for any researcher interested in this area are as follows:

- The Mann and Ishikawa iterative schemes we studied are for the class of mappings defined on the modular function spaces satisfying (UUC1) and Δ_2-conditions. We have the following researchable problems
    1. Can we obtain the same results by dropping the strong condition Δ_2-property in the modular function spaces?
    2. Can we extend the results to the modular spaces satisfying uniform continuity and strictly convex conditions?
- In all results obtained so far in this direction, nothing has been said about convergence theorems for mappings defined on the modular function space that satisfies property(R), the property that behaves like the reflexive property in Banach spaces, the question now is, can we extend the results in the literature as well as our results to the modular function spaces which have

property (R)?

- Since our convergence theorems are for the class of ρ-quasi-nonexpansive mappings, it is possible to pose the following question. Can we obtain the same convergence theorems if the class of mappings is more general than the class of ρ-quasi-nonexpansive?

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