

ANALYSIS OF CONSTRAINED OPTIMAL CONTROL PROBLEMS



COLLEGE OF COMPUTATIONAL AND NATURAL SCIENCE
DEPARTMENT OF MATHEMATICS

A project Submitted in partial fulfillment of the requirement of
the degree of master of science in mathematics

By: Mengistu Chalchisa
Stream: Optimization
Advisor: Semu Mitiku(PhD)

July 2, 2013

ADDIS ABABA UNIVERSITY
DEPARTMENT OF MATHEMATICS

The undersigned hereby certify that they have read and recommend to the department of mathematics for acceptance of this project entitled "**ANALYSIS OF CONSTRAINED OPTIMAL CONTROL PROBLEMS**" by **Mengistu Chalchisa** in partial fulfillment of the requirements for the degree of Master of Science in mathematics.

Mengistu Chalchisa:

Signature:-----

Date-----

Adviser: Dr. Semu Mitiku Kassa

Signature:-----

Contents

| | |
|---|------------|
| abstract | i |
| Acknowledgment | ii |
| Nomenclature | iii |
| 1 BASIC CONCEPTS AND DEFINITIONS | 1 |
| 1.1 Introduction | 1 |
| 1.2 The Hamiltonian and Multipliers | 2 |
| 1.3 Pontryagin's maximum (or minimum) principle | 7 |
| 1.3.1 Bounded control | 8 |
| 2 Analysis of constrained Optimal control problems | 10 |
| 2.1 Introduction | 10 |
| 2.2 Problems with Mixed Inequality Constraints | 10 |
| 2.2.1 Necessary conditions for optimality | 11 |
| 2.2.2 Extension to General State Terminal Constraints | 12 |
| 2.3 Problems with pure state inequality constraints | 14 |
| 2.4 Direct adjoint approach | 16 |
| 2.5 The indirect adjoining approach with complementary slackness: (first order constraints) | 19 |
| 2.6 The indirect adjoining approach for higher order constraints | 22 |
| 2.7 The indirect adjoining approach with continuous adjoint functions | 24 |
| 2.8 Existence result | 25 |
| 2.9 Sufficient conditions and Uniqueness | 28 |
| Reference | 38 |

Abstract

This paper gives a review for optimal control problems with state variable inequality constraints. To solve the problem with pure state inequality constraint, we use different approaches such that: direct adjoint approach, the indirect adjoining approach with complementary slackness (first order constraints), the indirect adjoining approach for higher order constraints and the indirect adjoining approach with continuous adjoint functions. Furthermore, the application of optimal control problems conditions is demonstrated by solving illustrative examples.

Keywords: Optimal control problems, pure state inequality constraint and mixed state inequality constraint

Acknowledgment

I would like to thank my advisor, **Dr. Semu Mitiku**, to whom I am very grateful for all his support throughout this entire process. He has shown great patience and always has been helpful, giving many hours of his time and hard work in assisting me in finishing this project. Without his support, expertise and guidance, this project's completion would not have been possible. I am also very thankful to my friend **Bedilu Tsige** for his comment and supporting different material. I would also like to express my appreciation to **my families** for their continuous encouragement. Lastly all **my friends** who comments and helping me in all activities.

Nomenclature

$\lambda(t)$ co-state vector(adjoint vector)

$F[t]$ objective function(cost function)

$h[t]$ mixed constraint

$k[t]$ pure state constraint

$L[t]$ Lagrangian

$u, u(t)$ control vector

$u^*, u^*(t)$ optimal control vector

$U(x(t), t)$ optimal region

(x^*, u^*) feasible or admissible pair

$x, x(t)$ state vector

\dot{x} state equation

$x^*, x^*(t)$ optimal state vector

$H[t]$ Hamiltonian

0 initial time

T final\ terminal time

Chapter 1

BASIC CONCEPTS AND DEFINITIONS

1.1 Introduction

Optimal control deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved. A control problem includes a cost functional that is a function of state and control variables. An optimal control is a set of differential equations describing the paths of the control variables that minimize(or maximize) the cost functional. The optimal control can be derived using Pontryagin's maximum principle or by solving the Hamilton jacobian equation. Optimal control is an important part of optimization with many applications in different areas, especially in engineering and economics. The usual format of an optimal control problem is described by a number of parameters: $x = (x_1, x_2 \dots x_n)$, which evolves according to a state equation, $\dot{x}(t) = g(t, x(t), u(t))$, where $u = (u_1, u_2 \dots u_n)$ represents the control exercised on the system. This control vector should typically satisfy various types of constraints depending on the nature of the problem. We will consider only the restriction $u(t) \in U_{ad} \subseteq \mathfrak{R}^m$ for all t . The state equation is also complemented with initial or final conditions: $x(0) = x_0$, and $x(T) = x_T$, where T is the time horizon we are considering. We must also have an objective functional measuring how good a given control u is. The form of such an objective function is: $\max F(x, u) = \int_0^T f(t, x(t), u(t))dt$ where $f : [0, T] \times \mathfrak{R}^n \times \mathfrak{R}^m \rightarrow \mathfrak{R}$, measures the rate how good a given control u is. A pair (x, u) is said to be feasible or admissible, if the following are satisfied[9]:

1. Constraints on the control: $u(t) \in U_{ad} \forall t \in [0, T]$.
2. State law $\dot{x}(t) = g(t, x(t), u(t)) \forall t \in [0, T]$.

3. End point conditions $x(0) = x_0, x(T) = x_T$.

The optimal control problem is:

$$\max F(x, u) = \int_0^T f(t, x(t), u(t)) dt$$

subject to : $u(t) \in U_{ad}$

$$\dot{x}(t) = g(t, x(t), u(t))$$

$$x(0) = x_0, x(T) = x_T$$

1.2 The Hamiltonian and Multipliers

Consider:

$$\max F(x, u) = \int_0^T f(t, x(t), u(t)) dt$$

Among all pairs (x, u) such that:

$$\dot{x}(t) = g(t, x(t), u(t))$$

together with appropriate conditions at end points; but the state equation may be considered as a pointwise constraint that can be treated by introducing a multiplier or costate $\lambda(t)$. Consider the costate function:

$$\lambda : [0, T] \rightarrow \mathfrak{R}^n$$

and equation above are give the Lagrangian problem of the following:

$$L(x, u, \lambda, \dot{x}) = \int_0^T [f(t, x(t), u(t)) + \lambda(t)(g(t, x(t), u(t)) - \dot{x}(t))] dt$$

If we put:

$$G(t, x, u, \lambda, \dot{x}) = f(t, x(t), u(t)) + \lambda(t)(g(t, x(t), u(t)) - \dot{x}(t))$$

Then from this equation Euler Lagrangian equations system can be derived as follows:

$$\begin{aligned} \frac{\partial}{\partial x} G(t, x, u, \lambda, \dot{x}) &= \frac{\partial}{\partial x} (f(t, x(t), u(t)) + \lambda(t)(g(t, x(t), u(t)) - \dot{x}(t))) \\ f_x(t, x(t), u(t)) + \lambda(t)g_x(t, x(t), u(t)) &= 0 \end{aligned} \quad (1.1)$$

$$\begin{aligned}\frac{\partial}{\partial u}G(t, x, u, \lambda, \dot{x}) &= \frac{\partial}{\partial u}(f(t, x(t), u(t)) + \lambda(t)(g(t, x(t), u(t)) - \dot{x}(t))) \\ f_u(t, x(t), u(t)) + \lambda(t)g_u(t, x(t), u(t)) &= 0\end{aligned}\quad (1.2)$$

$$\begin{aligned}\frac{\partial}{\partial \lambda}G(t, x, u, \lambda, \dot{x}) &= \frac{\partial}{\partial \lambda}(f(t, x(t), u(t)) + \lambda(t)(g(t, x(t), u(t)) - \dot{x}(t))) \\ g(t, x(t), u(t)) - \dot{x}(t) &= 0\end{aligned}\quad (1.3)$$

$$\begin{aligned}\frac{\partial}{\partial \dot{x}}G(t, x, u, \lambda, \dot{x}) &= \frac{\partial}{\partial \dot{x}}(f(t, x(t), u(t)) + \lambda(t)(g(t, x(t), u(t)) - \dot{x}(t))) \\ -\lambda(t) &= 0\end{aligned}\quad (1.4)$$

Now from four above equation we can get the following three equations:

$$\begin{aligned}\frac{d}{dt}(-\lambda(t)) &= f_x(t, x(t), u(t)) + \lambda(t)g_x(t, x(t), u(t)) \\ f_x(t, x(t), u(t)) + \lambda(t)g_x(t, x(t), u(t)) + \dot{\lambda}(t) &= 0\end{aligned}\quad (1.5)$$

$$f_u(t, x(t), u(t)) + \lambda(t)g_u(t, x(t), u(t)) = 0 \quad (1.6)$$

$$g(t, x(t), u(t)) - \dot{x}(t) = 0 \quad (1.7)$$

The purpose of this equations are to determined the conditions for the control to maximize(or minimize) the objective functions.

Definition 1.1. *The control Hamiltonian function H of the optimal control problem [OC] is defined as:*

$$H : [0, \infty) \times \mathfrak{R}^m \times \mathfrak{R}^n \times \mathfrak{R}^{n*} \longrightarrow \mathfrak{R} \quad \text{with} \quad H(t, x, u, \lambda) = f(t, x, u) + \lambda g(t, x, u)$$

Thus the above equations (i.e equation of 1.7, ?? and ??) can be written as:

1. Adjoint condition

$$\dot{\lambda}(t) = -\frac{\partial}{\partial x}H(t, x(t), u(t), \lambda(t)).$$

2. Optimality condition

$$\frac{\partial}{\partial u}H(t, x(t), u(t), \lambda(t)) = 0.$$

3. State equation

$$\dot{x}(t) = g(t, x(t), u(t)).$$

4. Transversalis condition If we set $x(T)$ to be free then we have the following conditions corresponding to the bounder (initial value) case.

$$\lambda(T) = 0 \text{ the transversalis conditions}$$

Hence this four conditions are the necessary conditions for an optimal control problems[9].

Example 1.1 (9).

$$\begin{aligned} \min F(x, u) &= \int_0^1 u(t)^2 dt \\ \text{subject to: } \ddot{x}(t) &= u(t) \end{aligned}$$

End point conditions

$$x(0) = \dot{x}(0) = 1, \quad x(1) = 0$$

Solution. *To solve the problem first reduce the second order equations to first order system with components $x_1 = x$, $x_2 = \dot{x}$ so that $\dot{x}_1 = x_2$, $\dot{x}_2 = u$ and end point conditions: $x_1(0) = 1$, $x_1(1) = 0$ and $x_2(0) = 1$. The Hamiltonian is:*

$$H(t, x(t), u(t), \lambda(t)) = u(t)^2 + \lambda_1(t)x_2(t) + \lambda_2(t)u(t)$$

1. *The optimality conditions*

$$\frac{\partial}{\partial u} H(t, x(t), u(t)) = 0$$

$$2u + \lambda_2 = 0$$

$$u = -\frac{1}{2}\lambda_2$$

2. *The adjoint conditions*

$$-\frac{\partial}{\partial x_1} H(t, x(t), u(t)) = \dot{\lambda}_1 = 0$$

$$-\frac{\partial}{\partial x_2} H(t, x(t), u(t)) = \dot{\lambda}_2$$

$$\dot{\lambda}_2 = -\dot{\lambda}_1$$

Therefore the optimality equations together with end points are:

$$u = -\frac{1}{2}\lambda_2$$

$$\begin{aligned}\dot{\lambda}_1 &= 0 \\ \dot{\lambda}_2 &= -\dot{\lambda}_1\end{aligned}$$

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad x_1(0) = 1, \quad x_1(1) = 0, \quad x_2(0) = 1 \text{ and } \lambda_2(1) = 0.$$

Now from above conditions: $\dot{\lambda}_1(t) = \lambda_1 = c_1$ and $\lambda_1(t) = c_1$. Since $\dot{\lambda}_2 = -\dot{\lambda}_1$ then

$$\begin{aligned}\dot{\lambda}_2 &= -c_1 \\ \lambda_2(t) &= -c_1 t + c_2.\end{aligned}$$

Therefore,

$$u(t) = -\frac{1}{2}(c_1 t + c_2)$$

but

$$\begin{aligned}\dot{x}_2(t) &= u(t) \\ \dot{x}_2(t) &= -\frac{1}{2}(c_1 t + c_2) \\ x_2(t) &= \frac{1}{4}c_1 t^2 - \frac{1}{2}c_2 t + c_3.\end{aligned}$$

again

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_1(t) &= \frac{1}{4}c_1 t^2 - \frac{1}{2}c_2 t + c_3 \\ x_1(t) &= \frac{1}{12}c_1 t^3 - \frac{1}{4}c_2 t^2 + c_3 t + c_4,\end{aligned}$$

where c_1, c_2, c_3 and c_4 are constant that will be determined. Now let solve all constant using given conditions $x_1(0) = 1, x_1(1) = 0$.

$$x_1(0) = \frac{1}{12}c_1(0)^3 - \frac{1}{4}c_2(0)^2 + c_3(0) + c_4 = 1, \quad c_4 = 1$$

$$x_1(1) = \frac{1}{12}c_1(1)^3 - \frac{1}{4}c_2(1)^2 + c_3(1) + 1 = 0$$

$$\frac{1}{12}c_1 - \frac{1}{4}c_2 + c_3 + 1 = 0, \text{ where } x_2(0) = 1 \quad (1.8)$$

$$x_2(0) = \frac{1}{4}c_1(0)^2 - \frac{1}{2}c_2(0) + c_3 = 1, \quad c_3 = 1$$

therefore,

$$\frac{1}{12}c_1 - \frac{1}{4}c_2 + 2 = 0 \quad (1.9)$$

but $\lambda_2(1) = 0$

$$\begin{aligned}\lambda_2(1) &= -c_1 + c_2 \\ -c_1 + c_2 &= 0.\end{aligned}\tag{1.10}$$

From equation of 1.9 and 1.10 we can solve c_1 and c_2 simultaneously.

$$\begin{aligned}\frac{1}{12}c_1 - \frac{1}{4}c_2 &= -2 \\ -c_1 + c_2 &= 0\end{aligned}$$

hence $c_1 = 12$ and $c_2 = 12$. Then the optimal solutions are:

$$\begin{aligned}u(t) &= 6(t - 1) \\ x_1(t) &= t^3 - 3t^2 + t + 1 \\ x_2(t) &= 3t^2 - 6t + 1.\end{aligned}$$

The optimality conditions are necessary for all optimal solutions but they might not be sufficient in the sense that there might be other types of solutions that are not optimal.

Theorem 1.1. *Let f and g be linear and convex in (x, u) for each fixed $t \in [0, T]$. Then every solution of the system of optimality with the appropriate end point conditions (including transversality) will be an optimal solution of the control problem [9].*

Proof. *Assume that the pair (x, u) satisfies all the optimality conditions, and let (x^*, u^*) be any other admissible pair. Then let measure the difference of $F(x, u) - F(x^*, u^*)$*

$$\int_0^T [f(t, x(t), u(t)) - f(t, x^*(t), u^*(t))] dt$$

Due to the linearity and convexity then

$$\begin{aligned}f(t, x(t), u(t)) - f(t, x^*(t), u^*(t)) &\geq \frac{\partial}{\partial x} f(t, x, u)(x - x^*) + \frac{\partial}{\partial u} f(t, x, u)(u - u^*) \\ \int_0^T f(t, x(t), u(t)) - f(t, x^*(t), u^*(t)) &\geq \int_0^T \frac{\partial}{\partial x} f(t, x, u)(x - x^*) + \frac{\partial}{\partial u} f(t, x, u)(u - u^*)\end{aligned}$$

Using necessary conditions

$$\geq \int_0^T [(-\dot{\lambda} - \lambda \frac{\partial}{\partial x} g(t, x, u))(x - x^*) + (-\lambda \frac{\partial}{\partial u} g(t, x, u))(u - u^*)] dt$$

$$\geq - \int_0^T (\lambda \frac{\partial}{\partial x} g(t, x, u))(x-x^*)dt - \int_0^T \dot{\lambda}(x-x^*)dt - \int_0^T \lambda \frac{\partial}{\partial u} g(t, x, u)(u-u^*)dt.$$

Now let integrate using integra by part

$$\begin{aligned} & - \int_0^T \dot{\lambda}(t)x(t)dt + \int_0^T \dot{\lambda}(t)x^*(t)dt \\ & - [\lambda(t)x(t)]_0^T + \int_0^T \lambda(t)\dot{x}(t)dt + \lambda(t)x(t)]_0^T - \int_0^T \lambda(t)\dot{x}^*(t)dt \\ & = \int_0^T \lambda(t)(\dot{x}(t) - \dot{x}^*(t))dt \\ & - \int_0^T \lambda(t)((\dot{x}^*(t) - \dot{x}(t)) + \lambda(t)\frac{\partial}{\partial x}g(t, x, u)(x-x^*) + \lambda(t)\frac{\partial}{\partial u}g(t, x, u)(u-u^*))dt \\ & - \int_0^T \lambda(t)(g(t, x^*, u^*) - g(t, x, u))dt - \int_0^T \lambda(t)(\frac{\partial}{\partial x}g(t, x, u)(x-x^*) + \lambda(t)\frac{\partial}{\partial u}g(t, x, u)(u-u^*))dt \\ & \int_0^T g(t, x, u) - g(t, x^*, u^*)dt \geq \int_0^T \frac{\partial}{\partial x}g(t, x, u)(x-x^*) + \frac{\partial}{\partial u}g(t, x, u)(u-u^*)dt \end{aligned}$$

then by convexity of g we have:

$$F(x, u) - F(x^*, u^*) \geq -(\int_0^T \lambda(t)(g(t, x^*, u^*) - g(t, x, u))dt + \int_0^T \lambda(t)(g(t, x, u) - g(t, x^*, u^*)dt)).$$

There fore $F(x, u) \geq F(x^*, u^*)$ for all feasible pair (x, u) . Hence the pair (x^*, u^*) is the minimizer of optimal control problem.

1.3 Pontryagin's maximum (or minimum) principle

Pontryagin's maximum (or minimum) principle is used in optimal control problem to find the best possible control for taking a system from one state to another, especially in the presence of constraints for the state or input controls. The principle states informally that the Hamiltonian must be minimized (or maximized) over U , the set of all permissible controls. If $u^* \in U$ is the optimal control for the problem, then the principle states that:

$$H(t, x^*(t), u^*(t), \lambda^*(t)) \leq H(t, x(t), u(t), \lambda(t)), \quad \forall u \in U, \quad t \in [0, T]$$

where $x^* \in C^{(1)}[0, T]$ is the optimal state trajectory and $\lambda^* \in [0, T]$ is the optimal costate trajectory. Consider the optimal control problem in maximum form:

$$\max F(x, u) = \int_0^T f(t, x(t), u(t))dt + S(x(T), T)$$

subject to: $\dot{x}(t) = g(t, x(t), u(t)), x(0) = x_0$,

where $S(x(T), T)$ is known as the salvage functions. Suppose $x(t)$ and $u(t)$ represent the state trajectory and optimal control respectively. Then there exists an adjoint $\lambda(t)$ such that together $x(t)$, $u(t)$ and $\lambda(t)$ satisfies the following conditions[9]:

1. Adjoint condition

$$\dot{\lambda}^*(t) = -\frac{\partial}{\partial x} H(t, x^*(t), u^*(t), \lambda^*(t))$$

2. State equation

$$\dot{x}^*(t) = g(t, x^*(t), u^*(t)), \text{ with } x(0) = x_0.$$

3. The transversality condition

$$\lambda^*(T) = S_x(x^*(T), T)$$

4. The maximum conditions

$$H(t, x^*(t), u^*(t), \lambda^*(t)) \leq H(t, x(t), u(t), \lambda(t))$$

1.3.1 Bounded control

Consider the optimal control problems[8]:

$$\max F(x, u) = \int_0^T f(t, x(t), u(t)) dt + S(x(T), T)$$

subject to: $\dot{x}(t) = g(t, x(t), u(t)), x(0) = x_0$

$$a \leq u(t) \leq b$$

Hence the conditions for optimality for bounded controls are given:

1. State equation

$$\dot{x}(t) = g(t, x(t), u(t)), \text{ with } x(0) = x_0$$

2. Adjoint condition

$$\dot{\lambda}(t) = -\frac{\partial}{\partial x} H(t, x(t), u(t), \lambda(t))$$

3. The transversality condition

$$\lambda(T) = S_x(x(T), T)$$

4. The optimality condition

$$\begin{cases} u^* = a & \text{if } \frac{\partial}{\partial u} H(t, x(t), u(t), \lambda(t)) < 0 \\ a < u^* < b & \text{if } \frac{\partial}{\partial u} H(t, x(t), u(t), \lambda(t)) = 0 \\ u^* = b & \text{if } \frac{\partial}{\partial u} H(t, x(t), u(t), \lambda(t)) > 0 \end{cases}$$

Definition 1.2. The point t'_i 's at which the control switches between the minimum and the maximum is called the switching time. If the Hamiltonian problem is [8]:

$$H(t, x(t), u(t), \lambda(t)) = f_1(t, x) + u f_2(t, x) + \lambda(t)(g_1(t, x) + u g_2(t, x))$$

$$H(t, x(t), u(t), \lambda(t)) = f_1(t, x) + \lambda(t)g_1(t, x) + u(t)(f_2(t, x) + \lambda(t)g_2(t, x))$$

$$\frac{\partial}{\partial u} H(t, x(t), u(t), \lambda(t)) = f_2(t, x) + \lambda(t)g_2(t, x) = \psi(t)$$

which contains no information about u then the function $\psi(t) = \frac{\partial}{\partial u} H(t, x(t), u(t), \lambda(t))$ can be zero at some finite number of t 's.

$$\begin{cases} u^* = a & \text{if } \psi(t) < 0 \\ a < u^* < b & \text{if } \psi(t) = 0 \\ u^* = b & \text{if } \psi(t) > 0 \end{cases}$$

Hence $\psi(t) = f_2(t, x) + \lambda(t)g_2(t, x)$ is called the switching functions.

Definition 1.3. A control $u(\cdot) \in U_{ad}$ is called Bang Bang if for each $t \in [0, T]$ and each index $i = 1, 2, \dots, m$ we have $|u_i(t)| = 1$ where $u(t) = (u_1(t), u_2(t), \dots, u_n(t))$ [8].

Chapter 2

Analysis of constrained Optimal control problems

2.1 Introduction

Optimal control problems with state variable inequality constraints are an important in different areas, specially in mechanics, aerospace, management science and economics. These problems are not easy to solve and even the theory is not unambiguous, since there exist various forms of the necessary and sufficient optimality condition. More specifically, we deal with problems with both pure and mixed state variable constraints. Pure constraints are inequality constraints expressed only in terms of the state variables and possibly time. Mixed constraints are constraints on control variables that may depend on the state variables and time [10].

2.2 Problems with Mixed Inequality Constraints

Optimal control problems with state inequality constraint arise frequently in practical applications. Consider the problem to find a piecewise continuous control $u^* \in C[0, T]$ with associated response $x^* \in C^{(1)}[0, T]$ and a terminal time $T^* \in [0, T]$ such that the following constraints are satisfied and the cost function takes on its maximum value:

$$\begin{aligned} \max F &= \int_0^T f(t, x, u) dt \\ \text{subject to: } \dot{x}(t) &= g(t, x(t), u(t)), \quad x(0) = x_0, \quad x(T) = x_T \\ h(t, x(t), u(t)) &\leq 0 \end{aligned}$$

Assume that the components of $h(t, x(t), u(t))$ depend explicitly on the control u and the following constraint qualification condition holds

$$\left(\frac{\partial}{\partial u} h \quad \text{diag}(h) \right) \quad (2.1)$$

is full rank. In other words, the gradients with respect to u of all the active constraint $h(t, x(t), u(t)) \leq 0$ must be linearly independent. A possible way of attempting to solve optimal control problems with mixed inequality constraints are to form a Lagrangian function L by adjoining $h(t, x(t), u(t))$ to the Hamiltonian function H with a Lagrange multiplier vector function μ [3].

$$\begin{aligned} L(t, x, u, \lambda, \mu) &= H(t, x, u, \lambda) + \mu h(t, x, u) \\ \text{where } H(t, x, u, \lambda) &= f(t, x(t), u(t)) + \lambda g(t, x(t), u(t)) \end{aligned}$$

2.2.1 Necessary conditions for optimality

Consider the optimal control problems[7] and [13]:

$$\begin{aligned} \max F &= \int_0^T f(t, x, u) dt \\ \text{subject to: } \dot{x}(t) &= g(t, x(t), u(t)), \quad x(0) = x_0, \quad x(T) = x_T \\ h(t, x(t), u(t)) &\leq 0 \end{aligned}$$

with fixed initial time and free terminal time and where f , g and h are continuously differentiable with respect to (t, x, u) on $[0, T] \times \mathfrak{R}^m \times \mathfrak{R}^n$. Suppose that $u^* \in C[0, T]$ is a maximizer for the problem and let x^* denote the optimal response. If the constraint qualification conditions are holds for every $t \in [0, T]$:

1. The function

$$H(t, x^*(t), u, \lambda^*(t))$$

attains its maximum on $U(x^*(t), t)$ at $u = u^*(t)$, for every $t \in [0; T]$,

$$H(t, x^*(t), u^*(t), \lambda(t)) \geq H(t, x^*(t), u(t), \lambda(t))$$

$\forall u \in U(x^*(t), t)$ where

$$U(x(t), t) := \{u(t) \in \mathfrak{R}^n : h(t, x(t), u(t)) \leq 0\}$$

2. The quadruple $(t, x^*, u^*, \lambda^*, \mu^*)$ satisfies the equations

$$\dot{x}^*(t) = L_\lambda(t, x, u, \lambda, \mu)$$

$$\dot{\lambda}(t) = -L_x(t, x, u, \lambda, \mu)$$

$$0 = L_u(t, x, u, \lambda, \mu)$$

at each instant t of continuity of u^* .

3. The vector function μ^* is continuous at each instant of continuity of u^* and satisfies:

$$\mu(t)h(t, x(t), u(t)) = 0 \quad \mu(t) \geq 0$$

2.2.2 Extension to General State Terminal Constraints

The maximum principle given in above conditions can be extended to the case where general terminal constraints are specified on the state variables as

$$a(x(T), T) \geq 0$$

$$b(x(T), T) = 0$$

and a terminal term is added to the cost functional as

$$\max F = \int_0^T f(t, x, u)dt + S(x(T), T)$$

where a , b and S are continuously differentiable with respect to (t, x) for all $(t, x) \in [0, T] \times \mathfrak{R}^m$. Suppose that the terminal constraints satisfy the constraint qualification conditions[13]

$$\begin{pmatrix} \frac{\partial}{\partial x}a & \text{diag}(a) \\ \frac{\partial}{\partial x}b & 0 \end{pmatrix}$$

is full rank. Then in addition to the necessary condition of optimality there exists Lagrangian multiplier vectors $\alpha \in \mathfrak{R}^l$ and $\beta \in \mathfrak{R}^{l'}$ such that:

$$\lambda(T) = S_x(x(T), T) + \alpha a_x(x(T), T) + \beta b_x(x(T), T)$$

where

$$\alpha \geq 0, \quad \alpha a(x(T), T) = 0$$

Example 2.1. Consider the problem [12]

$$\max F = \int_0^1 u dt$$

$$\text{subject to: } \dot{x} = u, x(0) = 1$$

$$u \geq 0, x - u \geq 0$$

Note The constraints $u \geq 0$, $x - u \geq 0$ are a mixed type and they can be rewritten as $0 \leq u \leq x$.

Solution. The Hamiltonian is

$$H = u + \lambda u = H = (1 + \lambda)u$$

so that the optimal control has the form

$$u^* = \text{bang}[0, x; 1 + \lambda].$$

To get adjoint equation and the multipliers associated with constraint $u \geq 0, x - u \geq 0$ we form the Lagrangian

$$L = H + \mu_1 u + \mu_2(x - u) = \mu_2 x + (1 + \lambda + \mu_1 - \mu_2)u.$$

From this we get the adjoint equation

$$\dot{\lambda} = -L_x(t, x, u, \lambda, \mu) = -\mu_2, \quad \lambda(1) = 0$$

and also note that the optimal control must satisfy

$$\frac{\partial}{\partial u} L = 1 + \lambda + \mu_1 - \mu_2 = 0$$

where μ_1 and μ_2 must satisfy the complimentary slackness conditions

$$\mu_1 \geq 0, \mu_1 u = 0$$

$$\mu_2 \geq 0, \mu_2(x - u) = 0.$$

The mixed state constraint $x(t) - u(t) \geq 0$ being active for each $0 \leq t \leq 1$ then we have $u^*(t) = x^*(t)$. Since $x(0) = 1$ the control $u^* = x$ gives $x(t) = e^t$ as the solutions $x(t) = e^t$ it follows that $u^* = x > 0$ thus $\mu_1 = 0$.

Since

$$\frac{\partial}{\partial u} L = 1 + \lambda + \mu_1 - \mu_2 = 0$$

then

$$1 + \lambda - \mu_2 = 0 \quad \text{since} \quad \mu_1 = 0$$

$$\mu_2 = 1 + \lambda$$

and

$$\dot{\lambda} = -\mu_2 \quad \text{and} \quad \lambda(1) \quad \text{then}$$

$$\dot{\lambda} = -1 - \lambda$$

$$\dot{\lambda} + \lambda = -1$$

$$\lambda(t) = c_1 e^{-t} - 1 = 0, \quad \lambda(1) = 0$$

$$c_1 e^{-1} - 1 = 0, \quad c_1 = e$$

$$\lambda(t) = e^{1-t} - 1$$

$$\mu_2(t) = e^{1-t} \geq 0 \quad \text{and} \quad x - u^* = 0.$$

2.3 Problems with pure state inequality constraints

Consider the function:

$$k(t, x) \quad \text{where} \quad k : [0, T] \times \mathfrak{R}^n$$

then the pure state constraints $k(t, x) \geq 0$ are generally, more difficult to deal with since $k(t, x)$ does not explicitly depend on u , and x can be controlled only indirectly. It is therefore, convenient to differentiate $k(t, x)$ with respect to time t as many times as required until it contains a control variable. Let us for the moment define $k^i(t, x)$, $i = 1, 2 \dots, p$ recursively as follows:

$$\begin{aligned}
k^0(t, x, u) &= k(t, x) \\
k^1(t, x, u) &= \frac{d}{dt}k = k_x(t, x)g(t, x, u) + k_t(t, x) \\
k^2(t, x, u) &= \frac{d}{dt}k^1 = k_x^1(t, x)g(t, x, u) + k_t^1(t, x) \\
&\vdots \\
k^p(t, x, u) &= \frac{d}{dt}k^{p-1} = k_x^{p-1}(t, x, u)g(t, x, u) + k_t^{p-1}(t, x),
\end{aligned} \tag{2.2}$$

where subscripts denote partial derivatives. Depending on the context we also use a subscript such as i to denote the i^{th} component of a vector.

If

$$k_u^i(t, x, u) = 0 \text{ for } 0 \leq i \leq p-1, \quad k_u^p(t, x, u) \neq 0 \tag{2.3}$$

then the state constraint $k(t, x) \geq 0$ is of order p . In more general case of $k(t, x)$, the corresponding order p_i for each component $k_i(t, x)$ of $k(t, x)$ is obtained from equation of 2.2 and 2.3. If the state constraints will be order of $p = 1$ then it is easier to treat than the higher order cases [10]. With respect to the i^{th} constraint $k_i(t, x) \geq 0$, a subinterval $(\tau_1, \tau_2) \subset [0, T]$ with $\tau_1 < \tau_2$ is called an interior interval of a trajectory if $k_i(x(t), t) > 0$ for all $t \in (\tau_1, \tau_2)$. An interval $[\tau_1, \tau_2]$ with $\tau_1 < \tau_2$ is called a boundary interval if $k_i(x(t), t) = 0$ for $t \in [\tau_1, \tau_2]$. An instant τ_1 is called an entry time if there is an interior interval ending at $t = \tau_1$ and a boundary interval starting at τ_1 ; correspondingly, τ_2 is called an exist time if a boundary interval ends at τ_2 and an interior interval starts at τ_2 . If the trajectory x just touches the boundary at time τ , i.e., $k(\tau, x(\tau_c)) = 0$, and if the trajectory x is in the interior just before and after τ , then τ is called a contact time. Taken together, entry, exit, and contact times are called junction times. Assume that the following full rank conditions on any boundary interval $[\tau_1, \tau_2]$:

$$\begin{pmatrix} \frac{\partial}{\partial u} k_1^{p_1} \\ \vdots \\ \frac{\partial}{\partial u} k_{s'}^{p_{s'}} \end{pmatrix}$$

is full rank for all $t \in (\tau_1, \tau_2)$. Where $k_i^*[t] = 0$ for, $i = 1, 2, \dots, s' \leq s$ and $k_i^*(t, x) > 0$ for $i = s' + 1 \dots, s$ for $t \in (\tau_1, \tau_2)$ and p_i is the order of constraint $k_i(t, x) \geq 0$ i.e the gradients of $k_i^{p_i}(t, x)$ with respect to u of the active constraints $k_i(t, x) = 0$, $i = 1, 2, \dots, s'$ must be linearly independent along an optimal trajectory [3].

2.4 Direct adjoint approach

In this approach, the Hamiltonian H and Lagrangian L are defined as follows:

$$H(t, x, u, \lambda) = f(t, x, u) + \lambda g(t, x, u)$$

$$L(t, x, u, \lambda, \mu, \nu) = H(t, x, u, \lambda) + \mu h(t, x, u) + \nu k(t, x),$$

where the vector $\lambda \in \mathbb{R}^n[t]$ is the adjoint function and $\mu \in \mathbb{R}^s[t]$ and $\nu \in \mathbb{R}^q[t]$ are multipliers. This method derives its name from the fact that the mixed constraints $h(t, x, u) \geq 0$ as well as the pure state constraints $k(t, x) \geq 0$ are directly adjoined to the Hamiltonian in order to form the Lagrangian.

Theorem 2.1. *Let $\{x^*(\cdot), u^*(\cdot)\}$ be an optimal pair for optimal control problem over a fixed interval $[0, T]$, such that $u^*(\cdot)$ is right continuous with left hand limits and the constraint qualification condition of equation 2.1 holds for every triple (t, x^*, u^*) , $t \in [0, T]$ with $u \in U(t, x^*(t))$. Assume that $x^*(t)$ has only finitely many junction times where $\lambda(\cdot)$ are discontinuous at junction time. Then there exist a constant $\lambda_0(t) > 0$, a piecewise absolutely continuous costate trajectory $\lambda(\cdot)$ mapping $[0, T]$ in to \mathbb{R}^n , piecewise continuous multiplier function $\mu(\cdot)$ and $\nu(\cdot)$ mapping $[0, T]$ in to \mathbb{R}^s and \mathbb{R}^q respectively, a vector $\eta(\tau_i) \in \mathbb{R}^q$ for each point τ_i of discontinuity of $\lambda(\cdot)$ and $\alpha \in \mathbb{R}^l$ $\beta \in \mathbb{R}^l$ and $\gamma \in \mathbb{R}^q$ such that $(\lambda_0, \lambda(t), \mu, \nu, \alpha, \beta, \eta(\tau_1), \dots, \eta(\tau_i)) \neq 0$ for every t and the following conditions hold almost everywhere [10]:*

$$u^*(t) = \arg \max_{u \in U(t, x(t))} H(t, x^*, u, \lambda_0, \lambda(\cdot))$$

$$L_u^*[t] = H_u^*[t] + \mu h_u[t] = 0$$

$$\dot{\lambda} = -L_x^*[t]$$

$$\mu(t) \geq 0, \quad \mu h^*(t, x, u) = 0$$

$$\nu \geq 0, \quad \nu k^*(t, x) = 0$$

At the terminal time T , the following transversality conditions hold:

$$\lambda(T^-) = \lambda_0 S_x^*[T] + \alpha a_x[T] + \beta b_x[T] + \gamma k_x^*[T]$$

$$\alpha \geq 0, \quad \gamma \geq 0 \quad \alpha a[T] = \gamma k^*[T] = 0$$

For any time τ in a boundary interval and for any contact time τ , the costate trajectory λ may have a discontinuity given by the the following jump conditions:

$$\lambda(\tau^-) = \lambda(\tau^+) + \eta(\tau) k_x^*[\tau]$$

$$H^*(\tau^-) = H^*(\tau^+) - \eta(\tau)k_t^*[\tau]$$

$$\eta(\tau) \geq 0 \quad \eta(\tau)k^*[\tau] = 0$$

Where τ^+ and τ^- denote the left hand and the right hand side limits respectively.

Proof. 1. Now to formulate the maximum principle for the problem defined by the state equation, objective function, mixed inequality constraint and pure state variable inequality constraints, we form the Lagrangian as follows:

$$L(t, x, u, \lambda, \mu, \nu) = H(t, x, u, \lambda) + \lambda g(t, x, u) + \mu h(t, x, u) + \nu k(t, x),$$

where the Hamiltonian is:

$$H(t, x, u, \lambda) = f(t, x, u) + \lambda g(t, x, u)$$

Now we can show that the maximum principle from Hamiltonian equation; the only difference is that the expression of

$$F(x^*, u^*) - F(x, u) \geq 0$$

$$\int_0^T [f(t, x^*(t), u^*(t)) - f(t, x(t), u(t))]dt \geq 0$$

This equations are proved in Theorem (1.1) then it follows that a necessary conditions for $u^* = u$ to be maximizing control is that $F(x^*, u^*) \geq 0$ for all admissible. This implies that $H(t, x^*(t), u^*(t), \lambda^*(t)) \geq H(t, x^*(t), u(t), \lambda^*(t))$ for all admissible u and all $t \in [0, T]$. This state that u^* is maximize the Hamiltonian.

Therefore,

$$u^*(t) = \arg \max_{u \in U(t, x(t))} H(t, x^*, u, \lambda_0, \lambda(\cdot))$$

2. From equation of Euler Lagrangian equation derivative in chapter one we can get:

$$L_u^*[t] = H_u^*[t] + \mu h_u[t] = 0$$

$$\dot{\lambda} = -L_x^*[t].$$

Since the vector function $\mu(t)$ and $\nu(t)$ is piecewise continuous at each time t of continuity of u , then it is satisfies the following condition:

$$\mu(t) \geq 0, \quad \mu h^*(t, x, u) = 0$$

$$\nu \geq 0, \quad \nu k^*(t, x) = 0$$

3. Suppose that the point τ_i at which the control switches between the maximum and minimum time. Therefore, at this time the jump conditions form for the adjoint variable and the Hamiltonian function. Then we would have that $\lambda(\cdot)$ and $H(\cdot)$ are discontinuous at this time. Therefore, at any entry|contact time τ_i , the adjoint function and Hamiltonian function may have discontinuities of the form:

$$\lambda(\tau^-) = \lambda(\tau^+) + \eta(\tau)k_x^*[\tau]$$

$$H^*(\tau^-) = H^*(\tau^+) - \eta(\tau)k_t^*[\tau]$$

where the Lagrange multiplier vector $\eta(\tau)$ satisfies the condition $\eta(\tau)k_x^*[\tau] = 0$, $\eta(\tau) \geq 0$.

Proposition 2.1. The adjoint function λ is continuous at a junction time τ , i.e $\eta(\tau) = 0$ if either (1) or (2) below holds:

1. The control u^* is continuous at τ and

$$\begin{pmatrix} \frac{\partial}{\partial u} h^*[\tau] & \text{diag}(h^*[\tau]) & 0 \\ \frac{\partial}{\partial u} k^{1*}[\tau] & 0 & \text{diag}(k^*[\tau]) \end{pmatrix} \quad (2.4)$$

is full rank where $k^1(t, x, u)$ is defined in equation 2.3.

2. The entry or exit is non tangential i.e $k^{1*}(\tau^-) < 0$ or $k^{1*}(\tau^+) > 0$ then $\lambda(t)$ is continuous at time $t = \tau$ [10].

Definition 2.1. The Hamiltonian is said to be regular if along a given, $x(t), \lambda(t), \eta(t)$ and $H(x(t), u, \lambda(t), \eta(t))$ has a unique maximum in u for all $t \in [0, T]$.

Proposition 2.2. If the Hamiltonian is regular, which in this context means that the maximization of H with respect to u is unique, then u^* is continuous every where including the points on the boundary[9].

Proof. Suppose Hamiltonian is regular then we want to show that $u(t^-) = u(t^+)$. Let (x, u) be an admissible pair for optimal control problem; therefore, Hamiltonian look as follows:

$$H(t, x(t), u(t)) = f(t, x(t), u(t)) + \lambda(t)g(t, x(t), u(t))$$

then on the subinterval $[0, t^-]$ the control functions are:

$$H_u(t, x(t), u(t)) = 0 \Rightarrow f_u(t, x(t), u(t)) + \lambda(t)g_u(t, x(t), u(t)) = 0 \text{ where } t \in [0, t^-]$$

and also on the subinterval $[t^+, T]$ the control functions are:

$$H_u(t, x(t), u(t)) = 0 \Rightarrow f_u(t, x(t), u(t)) + \lambda(t)g_u(t, x(t), u(t)) = 0 \text{ where } t \in [t^+, T].$$

Hence the control functions are continuous if the $\lim_{t \rightarrow t^-} u(t) = \lim_{t \rightarrow t^+} u(t)$ but,

$$\begin{aligned} H_u(t^-, x(t^-), u(t^-)) &= f_u(t^-, x(t^-), u(t^-)) + \lambda(t^-)g_u(t^-, x(t^-), u(t^-)) \\ &= H_u(t^+, x(t^+), u(t^+)) = f_u(t^+, x(t^+), u(t^+)) + \lambda(t^+)g_u(t^+, x(t^+), u(t^+)) = 0 \end{aligned}$$

Therefore,

$$\begin{aligned} f_u(t^-, x(t^-), u(t^-)) + \lambda(t^-)g_u(t^-, x(t^-), u(t^-)) &= 0 \\ = f_u(t^+, x(t^+), u(t^+)) + \lambda(t^+)g_u(t^+, x(t^+), u(t^+)) &= 0 \end{aligned}$$

make both side limit

$$\begin{aligned} \lim_{t \rightarrow t^-} (f_u(t^-, x(t^-), u(t^-)) + \lambda(t^-)g_u(t^-, x(t^-), u(t^-))) &= 0 \\ = \lim_{t \rightarrow t^+} (f_u(t^+, x(t^+), u(t^+)) + \lambda(t^+)g_u(t^+, x(t^+), u(t^+))) &= 0 \end{aligned}$$

Therefore, $u(t)$ is continuous everywhere including the points on the boundary.

2.5 The indirect adjoining approach with complementary slackness: (first order constraints)

The idea behind this approach is the following. If the trajectory hits the boundary at time τ_1 i.e $k(x(\tau_1), \tau_1) = 0$ then for it to remain on the boundary up to time τ_2 requires.

$$k^1(t, x^*(t), u^*(t)) = 0 \text{ for } t \in (\tau_1, \tau_2)$$

where $k^1(t, x, u)$ may or may not depend explicitly on the control variables. This asserts that the phase velocity of a point moving along the trajectory is tangential to the boundary at time t . At the exit point τ_2 we must have $k^{1*}(\tau_2^+) \geq 0$. Thus, one could formally impose the constraint $k^1(t, x, u) \geq 0$ whenever $k(t, x) = 0$ in order to prevent the trajectory from violating the

constraint $k(t, x) \geq 0$. Then the Hamiltonian and Lagrangian can be defined as follows:

$$H^1(t, x, u, \lambda_0, \lambda^1) = \lambda_0 f(t, x, u) + \lambda^1 g(t, x, u)$$

$$L^1(t, x, u, \lambda_0, \lambda^1, \mu, \nu^1) = H^1(t, x, u, \lambda_0, \lambda^1) + \mu h(t, x, u) + \nu^1 k^1(t, x, u)$$

Because the derivative $k^1(t, x, u)$ of $k(t, x)$ rather than $k(t, x)$ itself is adjoined to H in forming the Lagrangian, this approach is known as the indirect adjoining approach.

The control region:

$$U^1(t, x) = \{u \in \mathfrak{R} \mid h(t, x, u) \geq 0, k^1(t, x, u) \geq 0 \text{ if } k(t, x) = 0\}$$

The necessary conditions of optimality that are used as a procedure while applying the indirect adjoining approach are now stated as follows.

Theorem 2.2. *Let $\{x^*(\cdot), u^*(\cdot)\}$ be an optimal pair for optimal control problem such that $x^*(\cdot)$ has only finitely many junction times and the strong constraint qualification condition, of equation 2.4 holds. Then there exists a constant $\lambda_0 \geq 0$ a piecewise absolutely continuous costate trajectory $\lambda^1(\cdot)$ mapping $[0, T]$ in to \mathfrak{R}^n , piecewise continuous multiplier function $\mu(\cdot)$ and $\nu^1(\cdot)$ mapping $[0, T]$ in to \mathfrak{R}^s and \mathfrak{R}^q respectively, a vector $\eta^1(\tau_i) \in \mathfrak{R}^q$ for each point τ_i of discontinuity of $\lambda^1(\cdot)$, $\alpha \in \mathfrak{R}^l$ and $\beta \in \mathfrak{R}^l$, not all zero, such that the following conditions hold almost everywhere[10]:*

$$u^*(t) = \arg \max_{u \in U(t, x(t))} H^1(t, x^*, u, \lambda_0, \lambda^1(\cdot))$$

$$\dot{\lambda}^1 = -L_x^{1*}[t]$$

$$L_u^{1*}[t] = 0$$

$$\mu(t) \geq 0, \quad \mu h^*(t, x, u) = 0$$

ν_i^1 is non increasing on boundary intervals of $k_i(t, x)$, $i = 1, 2, \dots, q$, with

$$\nu^1(t) \geq 0, \quad \dot{\nu}^1 \leq 0, \quad \nu^1 k^{1*}(t, x, u) = 0$$

and

$$\frac{d}{dt} H^{*1}[t] = \frac{d}{dt} L^{*1}[t] = L_t^{*1}[t]$$

whenever these derivatives exist. At the terminal time T the transversality conditions

$$\lambda^1(T^-) = \lambda_0 S_x^*[T] + \alpha a_x[T] + \beta b_x[T] + \gamma k_x^*[T]$$

$$\alpha \geq 0, \quad \gamma \geq 0 \text{ then } \alpha a[T] = \gamma k^*[T] = 0$$

holds. At each entry or contact time, the costate trajectory λ^1 may have a discontinuity of the form:

$$\begin{aligned}\lambda^1(\tau^-) &= \lambda^1(\tau^+) + \eta^1(\tau)k_x^*[\tau] \\ H^{*1}(\tau^-) &= H^{*1}(\tau^+) - \eta^1(\tau)k_t^*[\tau] \\ \eta^1(\tau) &\geq 0, \quad \eta^1(\tau)k^*[\tau] = 0\end{aligned}$$

Proof. Suppose that the constraint $k(t, x) \geq 0$ is called a constraint of first order since first derivative of $k(t, x)$ the first time at a term in control u appears in the expression by putting $g(t, x, u)$ for \dot{x} . Then in the case of first order constraints, we need to define $k^1(t, x, u)$ as follows:

$$k^1(t, x, u) = \frac{d}{dt}k(t, x) = \frac{\partial}{\partial x}kg(t, x, u) + \frac{d}{dt}k(t, x)$$

Then using first order conditions we can form Lagrangian function as follows:

$$L^1(t, x, u, \lambda_0, \lambda^1, \mu, \nu^1) = H^1(t, x, u, \lambda_0, \lambda^1) + \mu h(t, x, u) + \nu^1 k^1(t, x, u)$$

where the Hamiltonian is

$$H^1(t, x, u, \lambda_0, \lambda^1) = \lambda_0 f(t, x, u) + \lambda^1 g(t, x, u).$$

Then from above Lagrangian equation we can derivative the maximum principle states that the necessary conditions for u^* with the state trajectory x^* to be an optimal control for the problems. Then there exist adjoint variable λ and Lagrangian multipliers μ, ν, α, β and the jump parameter η which satisfies the conditions, then to find adjoint equation and optimality condition using partial derivative interims of x and u and the proof of the Hamiltonian maximizing condition is similar to Theorem (2.1) as follows:

$$H^1(t, x^*(t), u^*(t), \lambda^1(t)) \geq H^1(t, x^*(t), u, \lambda^1(t))$$

at each $t \in [0, T]$ for all u satisfying the following conditions

$$g(x^*(t), u, t) \geq 0,$$

and

$$k^1(x^*(t), u, t) \geq 0 \text{ whenever } k(x^*(t), t) = 0.$$

Therefore,

$$\begin{aligned}u^*(t) &= \arg \max_{u \in U(t, x(t))} H^1(t, x^*, u, \lambda_0, \lambda^1(\cdot)) \\ \dot{\lambda}^1 &= -L_x^{1*}[t]\end{aligned}$$

$$L_u^{1*}[t] = 0$$

$$\mu(t) \geq 0, \quad \mu h^*(t, x, u) = 0$$

Since the derivative of ν less than zero then ν is non increasing on the boundary interval of $k_i(t, x)$. At any entry| contact τ the control is switches between maximum and minimum. Therefore, at this time the adjoint function and Hamiltonian function have discontinuities, then it is the form of:

$$\lambda^1(\tau^-) = \lambda^1(\tau^+) + \eta^1(\tau)k_x^*[\tau]$$

$$H^{*1}(\tau^-) = H^{*1}(\tau^+) - \eta^1(\tau)k_t^*[\tau]$$

$$\eta^1(\tau) \geq 0, \quad \eta^1(\tau)k^*[\tau] = 0$$

2.6 The indirect adjoining approach for higher order constraints

In this section, we shall consider constraints of higher order, i.e $p \geq 2$. This means if $p = 1$ and $k^1(t, x, u)$ does not depend on the control variable u , then we differentiate $k(t, x)$ with respect to time t as many time as required until it contains a control variable u . Then such type of conditions are said to be indirect adjoint approach for higher order constraints. The Hamiltonian and Lagrangian of the indirect adjoining approach for the state constraint of order p are now:

$$H^p(t, x, u, \lambda_0, \lambda^p) = \lambda_0 f(t, x, u) + \lambda^p g(t, x, u)$$

$$L^p(t, x, u, \lambda_0, \lambda^p, \mu, \nu^p) = H^p(t, x, u, \lambda_0, \lambda^p) + \mu h(t, x, u) + \nu^p k^p(t, x, u)$$

with k^p defined in equation of 2.2. Then the control region $U^p(t, x)$ is defined as follows:

$$U^p(t, x) = \{u \in \mathfrak{R}^n \mid h(t, x, u) \geq 0, \quad k^p(t, x, u) \geq 0 \text{ if } k(t, x) = 0\}.$$

Theorem 2.3. *Let $(x^*(\cdot), u^*(\cdot))$ be an optimal pair for optimal control problem with $x^*(\cdot)$ having only finitely many junctions times, and where constraint $k(t, x)$ is of order p , let the constraint qualification condition, of equation 2.1, holds. Then there exist a constant $\lambda_0 \geq 0$, a piecewise absolutely continuous costate trajectory $\lambda^p(\cdot)$ mapping $[0, T]$ in to \mathfrak{R} , piecewise continuous multiplier function $\mu(\cdot)$ and $\nu(\cdot)$ mapping $[0, T]$ in to \mathfrak{R}^s and \mathfrak{R}^q , respectively, vectors $\eta^1(\tau_i) \dots \eta^p(\tau_i) \in \mathfrak{R}^q$ for each point τ_i of discontinuity of $\lambda^p(\cdot)$, $\alpha \in \mathfrak{R}^l$*

and $\beta \in \mathfrak{R}^l$ not all zero, such that the following conditions hold almost everywhere[10].

$$u^*(t) = \arg \max_{u \in U(t, x(t))} H^P(t, x^*, u, \lambda_0, \lambda^p(\cdot))$$

$$\lambda^p = -L_x^{*p}[t]$$

$$L_u^{*p}[t] = 0$$

$$\mu(t) \geq 0, \quad \mu h^*(t, x, u) = 0$$

The multiplier function ν^p is $p - 1$ times differentiable and $(\nu^p)^{p-1}$ is of bounded variation

$$(-1)^r (\nu^p)^r(t) \geq 0, \quad r = 0, 1, \dots, p \quad \nu^p k^{p*}(t, x, u) = 0$$

$$\frac{d}{dt} H^{*p}[t] = \frac{d}{dt} L^{*p}[t] = L_t^{*p}[t]$$

At the terminal time T , the transversality conditions with $\lambda^1(\cdot)$ replaced by $\lambda^p(\cdot)$. At entry times, the costate trajectory λ^p may have a discontinuity of the form:

$$\lambda^p(\tau^-) = \lambda^p(\tau^+) + \sum_{r=1}^p \eta^r(\tau) (k^{r-1})_x^*[\tau] \quad (2.5)$$

$$H^p[\tau^-] = H^p[\tau^+] + \sum_{r=1}^p \eta^r(\tau) (k^{r-1})_t^*[\tau] \quad (2.6)$$

$$\eta^r(\tau) \geq 0, \quad \eta^r(\tau) k_t^*[\tau] = 0, \quad r = 1, 2, \dots, p \quad (2.7)$$

Proof. The condition of the maximum principle states that the necessary conditions for u^* with the state trajectory x^* to be an optimal control for the problem is the same approach as first order state constraints (indirect adjoining approach). Suppose that the constraint $k(t, x) \geq 0$ is constraint of p order since $k(t, x)$ is derivative p times until it contains a control variable u . In the case of p order constraints, we need to define $k^p(t, x, u)$ as defined in the equations of (2.2). Then using p order conditions we can form Lagrangian function as follows:

$$L^p(t, x, u, \lambda_0, \lambda^p, \mu, \nu^p) = H^p(t, x, u, \lambda_0, \lambda^p) + \mu h(t, x, u) + \nu^p k^p(t, x, u)$$

where Hamiltonian is

$$H^p(t, x, u, \lambda_0, \lambda^p) = \lambda_0 f(t, x, u) + \lambda^p g(t, x, u)$$

Note p is indicate order.

Assume that the function g and k are continuously differentiable with respect

to all their argument up to order $p-1$ and p respectively, then the necessary condition of optimality as follows:

$$\begin{aligned} u^*(t) &= \arg \max_{u \in U(t, x(t))} H^p(t, x^*, u, \lambda_0, \lambda^p(\cdot)) \\ \lambda^p &= -L_x^{*p}[t] \\ L_u^{*p}[t] &= 0 \\ \mu(t) &\geq 0, \quad \mu h^*(t, x, u) = 0. \end{aligned}$$

If the switching function of order p the jump condition at entry times, the costate trajectory λ^p and Hamiltonian function may have a discontinuity of the form:

$$\begin{aligned} \lambda^p(\tau^-) &= \lambda^p(\tau^+) + \sum_{r=1}^p \eta^r(\tau) (k^{r-1})_x^*[\tau] \\ H^p[\tau^-] &= H^p[\tau^+] + \sum_{r=1}^p \eta^r(\tau) (k^{r-1})_t^*[\tau] \\ \eta^r(\tau) &\geq 0, \quad \eta^r(\tau) k_t^*[\tau] = 0, \quad r = 1, 2, \dots, p \end{aligned}$$

2.7 The indirect adjoining approach with continuous adjoint functions

In this section, the adjoint function $\tilde{\lambda}$ is continuous. The Hamiltonian H and the control region U respectively:

$$\tilde{H}(t, x, u, \lambda_0, \tilde{\lambda}, \mu, \tilde{\nu}) = \lambda_0 f(t, x, u) + \tilde{\lambda} g(t, x, u) + \mu h(t, x, u) + \tilde{\nu} k^1(t, x, u)$$

$$\tilde{U}(t, x) = \{u \in \mathfrak{R}^n \mid h(t, x, u) \geq 0, \quad k^1(t, x, u) \geq 0 \text{ if } k(t, x) = 0\}$$

$$\bigcup \{u \in \mathfrak{R}^n \mid h(t, x, u) \geq 0, \quad k^1(t, x, u) \leq 0 \text{ if } k(t, x) = 0\}$$

Theorem 2.4. *Let $(x^*(\cdot), u^*(\cdot))$ be an optimal pair for optimal control problem such that the strong constraint qualification condition, of equation 2.4, holds. Then there exist a constant $\lambda_0 \geq 0$, a continuous and a piecewise continuously differentiable adjoint function $\tilde{\lambda}(\cdot) : [0, T] \rightarrow \mathfrak{R}^n$ and multiplier function $\mu(\cdot) : [0, T] \rightarrow \mathfrak{R}^s$ and $\tilde{\nu} : [0, T] \rightarrow \mathfrak{R}^q$, such that the following conditions are satisfied whenever u is continuous.*

$$u^*(t) = \arg \max_{u \in U(t, x(t))} \tilde{H}(t, x^*, u, \lambda_0, \tilde{\lambda}(t), \mu(t), \tilde{\nu}(t))$$

$$\tilde{\lambda} = -\tilde{H}_x[t]$$

$$\begin{aligned}\tilde{H}_u[t] &= 0 \\ \frac{d}{dt}\tilde{H}[t] &= \tilde{H}_t[t]\end{aligned}$$

The multipliers $\mu(\cdot)$ and $\tilde{\nu}(\cdot)$ are continuous on intervals of continuity of $u^*(\cdot)$. Furthermore, $\tilde{\nu}(\cdot)$ is non increasing on $[0, T]$, continuous when ever $k_i^*[\cdot]$ is discontinuous (i.e when entry to or exit from the corresponding state constraint is non tangential), and constant on intervals up on which $k_i^*[\cdot] \geq 0$. At the terminal time T , the following transversality conditions holds[10]:

$$\begin{aligned}\tilde{\lambda}(T) &= \lambda_0 S_x^*[T] + \alpha a_x[T] + \beta b_x[T] \\ \alpha &\geq 0, \quad \alpha a[T] = 0\end{aligned}$$

Proof. Suppose u^* is continuous then the Hamiltonian is regular along a given $x(t)$, $\lambda(t)$, $\eta(t)$ and $H(t, x(t), u, \lambda(t), \eta(t))$ has a unique maximum in u for all $t \in [0, T]$ including the points on the boundary by Proposition (2.2). Therefore, the necessary conditions are holds since maximum principle is unique and using partial derivative we can obtained optimality conditions and the adjoint equation as follows:

$$\begin{aligned}u^*(t) &= \arg \max_{u \in U(t, x(t))} \tilde{H}(t, x^*, u, \lambda_0, \tilde{\lambda}(t), \mu(t), \tilde{\nu}(t)) \\ \tilde{\lambda} &= -\tilde{H}_x[t] \\ \tilde{H}_u[t] &= 0 \\ \frac{d}{dt}\tilde{H}[t] &= \tilde{H}_t[t].\end{aligned}$$

Since adjoint function is continuous then at the terminal time T , the following transversality conditions holds as follows:

$$\begin{aligned}\tilde{\lambda}(T) &= \lambda_0 S_x^*[T] + \alpha a_x[T] + \beta b_x[T] \\ \alpha &\geq 0, \quad \alpha a[T] = 0\end{aligned}$$

2.8 Existence result

We can review several different sets of optimality conditions for optimal control problems. Since optimality conditions do not mean much in the absence of an optimal solution then we briefly provide some existence results for the problems. Our purpose here is not to make a review of existence result. We choose to mention two characteristic:

- The first result uses strong assumptions such as boundedness of all admissible state and control paths.
- The second result uses growth conditions on the state and control variables[10].

The growth conditions[3] If f and g satisfy the following conditions for every bounded subset X of \mathfrak{R}^n , then there exist a constant c and a summable function d such that, for almost every t , for every $(x, u) \in \text{dom}f(t, x, u)$ with $x \in X$, we have:

$$\|g_x(t, x, u)\| \leq c\{|g(t, x, u)| + f(t, x, u)\} + d(t)$$

and for all ξ, ψ

$$|\xi|(1 + \|g_u(t, x, u)\|) \leq c\{|g_u(t, x, u)| + f(t, x, u)\} + d(t)$$

We define the (state dependent) control region:

$$U(t, x) = \{u \in \mathfrak{R}^n \mid h(t, x, u) \geq 0\} \subset \mathfrak{R}^n$$

and the set

$$N(t, x) = \{(f(t, x, u) + \gamma, g(t, x, u))/\gamma \leq 0, u \in U(t, x) \subset \mathfrak{R}^{n+1}\}$$

Lemma 2.1. *Let $U(y)$ be an upper semicontinuous set-valued mapping $\mathfrak{R}^m \rightarrow \mathfrak{R}^n$ with compact values. Then, on any compact (and hence on any bounded) set of y ; the values $U(y)$ are uniformly bounded, i.e., for any compact set $K \in \mathfrak{R}^m$ there exist a constant δ such that the set $U(y)$ is contained in the ball $B(0, \delta)$ for any $y \in K$ [2].*

Proof. *Since $U(y)$ is an upper semicontinuous mapping, for any y there exists a neighborhood $O(y)$ of y such that the inclusion $U(y^*) \subset U(y) + B_1(0, \delta)$ holds for any $y^* \in O(y)$. The union of these neighborhoods $O(y)$ over all $y \in K$ covers the entire compactum K ; and, by the definition of a compact set, a finite subcovering can be chosen from this covering. Namely, there exist finitely many points $y_1, \dots, y_m \in K$ and their neighborhoods $O(y_i)$ such that the set $U(y^*)$ is contained in $U(y_i) + B_1(0, \delta)$ for any $y^* \in O(y_i)$; and these neighborhoods cover the entire compactum K . The union V of the bounded sets $U(y_i) + B_1(0, \delta)$ over all $i = 1, \dots, m$ is also bounded, i.e., it is entirely contained in the ball $B(0, \delta)$ for some δ . Since for any $y \in K$; there exists a number i such that $y \in O(y_i)$; we have $U(y) \subset U(y_i) + B_1(0, \delta) \subset V$; and hence $U(y) \subset B(0, \delta)$.*

Corollary 2.1. *Suppose that $U(t, x)$ is an upper semicontinuous mapping $\mathbb{R}^{m+1} \rightarrow \mathbb{R}^n$ with compact values. Then, for any $T > 0$ and any bounded set $Q \subset \mathbb{R}^m$; there is an $R = R(T, Q)$ such that the inclusion $U(t, x) \subset B(0, \delta)$ holds for any $t \in [0, T]$ and any $x \in Q$ [2].*

Proof. *One should apply Lemma above to the mapping $U(y)$; where $y = (t, x)$; and to the compact set $K = [0; T] \times Q$.*

Theorem 2.5. *Consider the optimal control problem where T is free to vary in the interval $[0, T]$. Assume that f, g, h, k, S, a and b are continuous in all their arguments at all points $(t, x, u) \in [0, T] \times \mathbb{R}^m \times \mathbb{R}^n$. Suppose that there exist an admissible solution pair and that the following conditions holds:*

1. $N(t, x)$ is convex for all $(t, x) \in \mathbb{R}^m \times [0, T]$.
Suppose further that
2. There exists $\delta > 0$ such that $\|x(t)\| < \delta$ for all admissible pair $(x(t), u(t))$ and t .
and
3. There exists $\delta_1 > 0$ such that $\|u\| < \delta_1$ for all $u \in U(t, x)$ with $\|x(t)\| < \delta$.

Then there exists an optimal triple (T^, x^*, u^*) with $u^*(\cdot)$ measurable [10].*

Proof. *Let $(f_1(t, x, u) + \gamma_1, g_1(t, x, u))$ and $(f_2(t, x, u) + \gamma_2, g_2(t, x, u))$ be two value of $N(t, x)$. Then for all any $0 \leq a \leq 1$:*

$$a((f_1(t, x, u) + \gamma_1, g_1(t, x, u)) + (1 - a)(f_2(t, x, u) + \gamma_2, g_2(t, x, u)))$$

$$af_1(t, x, u) + a\gamma_1, ag_1(t, x, u) + (f_2(t, x, u) + \gamma_2, g_2(t, x, u)) - a(f_2(t, x, u) + \gamma_2, g_2(t, x, u))$$

collect like term together

$$(a((f_1(t, x, u) + \gamma_1) + (1 - a)(f_2(t, x, u) + \gamma_2)), ag_1(t, x, u) + (1 - a)g_2(t, x, u))$$

$$(a((f_1(t, x, u) - f_2(t, x, u)) + f_2(t, x, u) + (a(\gamma_1 - \gamma_2) + \gamma_2), ag_1(t, x, u) + (1 - a)g_2(t, x, u)))$$

Therefore, $N(x, t)$ is convex

By Lemma and Corollary 2.1 above there exists $\delta > 0$ such that $\|x(t)\| < \delta$ for all admissible pair $(x(t), u(t))$ and t and also there exists $\delta_1 > 0$ such that $\|u\| < \delta_1$ for all $u \in U(t, x)$ with $\|x(t)\| < \delta$. Therefore, the triple (T^, x^*, u^*) always belongs to the compact set $[0, T] \times B(0, \delta) \times B(0, \delta_1)$. Thus, the set of solutions $x(t)$ of optimal control problem is uniformly bounded and continuous, and the set of controls $u(t)$ is uniformly bounded.*

2.9 Sufficient conditions and Uniqueness

Theorem 2.6. *Let $(x^*(\cdot), u^*(\cdot))$ be a feasible pair for the optimal control problem with a fixed horizon time $T < \infty$. If there exists a piecewise continuously differentiable function $\lambda : [0, T] \rightarrow \mathbb{R}^n$ such that for every other feasible pair $(x(\cdot), u(\cdot))$ the following conditions holds [10]:*

1. *The maximum Hamiltonian:*

$$H(t, x^*(t), u^*(t), \lambda(t)) - H(t, x(t), u(t), \lambda(t)) \geq \dot{\lambda}(t)(x(t) - x^*(t)) \quad \forall t \in [0, T]$$

2. *The jump conditions*

$$(\lambda(\tau^-) - \lambda(\tau^+))(x(\tau) - x^*(\tau)) \geq 0$$

$\forall \tau \in [0, T]$ where λ is discontinuous and the transversality condition.

3.

$$\lambda(T)(x(T) - x^*(T)) \geq S(x(T), T) - S(x^*(T), T)$$

then (x^*, u^*) is optimal [7].

Proof. 1. Let $u \in U(t, x)$ and x be an admissible trajectory generated by u then by definition of Hamiltonian we:

$$\begin{aligned} f(t, x^*(t), u^*(t)) + \lambda(t)g(t, x^*(t), u^*(t)) - (f(t, x(t), u(t)) + \lambda(t)g(t, x(t), u(t))) \\ - \dot{\lambda}(t)(x(t) - x^*(t)) \geq 0 \end{aligned}$$

since

$$\dot{x} = g(t, x(t), u(t))$$

then

$$f(t, x^*(t), u^*(t)) + \lambda(t)\dot{x}^*(t) - f(t, x(t), u(t)) - \lambda(t)\dot{x}(t) - \dot{\lambda}(t)(x(t) - x^*(t)) \geq 0$$

Hence

$$f(t, x^*(t), u^*(t)) - f(t, x(t), u(t)) + \lambda(t)\dot{x}^*(t) - \lambda(t)\dot{x}(t) - \dot{\lambda}(t)(x(t) - x^*(t)) \geq 0$$

Then we have by definition of derivative:

$$f(t, x^*(t), u^*(t)) - f(t, x(t), u(t)) - \left(\frac{d}{dt}\right)[\lambda(t)(x^*(t) - x(t))] \geq 0$$

$$f(t, x^*(t), u^*(t)) - f(t, x(t), u(t)) \geq \left(\frac{d}{dt}\right)[\lambda(t)(x^*(t) - x(t))]$$

so that make integration both side then

$$\int_0^T f(t, x^*(t), u^*(t)) - f(t, x(t), u(t)) dt \geq \int_0^T \left(\frac{d}{dt}\right)[\lambda(t)(x^*(t) - x(t))] dt$$

$$\int_0^T f(t, x^*(t), u^*(t)) dt - \int_0^T f(t, x(t), u(t)) dt \geq \lambda(t_0)(x^*(0) - x(0)) - \lambda(T)(x^*(T) - x(T))$$

but, by initial and transversalis conditions $x^*(t_0) = x_0, x(t_0) = x_0$ and $\lambda(T) = 0$ such that:

$$\lambda(0)(x^*(0) - x(0)) - \lambda(T)(x^*(T) - x(T)) = 0$$

Hence

$$\int_0^T f(t, x^*(t), u^*(t)) dt - \int_0^T f(t, x(t), u(t)) dt \geq 0$$

2. The criterion for (x^*, u^*) to be optimal is the difference:

$$\int_0^T f(t, x^*(t), u^*(t)) dt - \int_0^T f(t, x(t), u(t)) dt \geq 0$$

for all admissible pairs (x, u) . Let use of definition of Hamiltonian and the fact that $\dot{x} = g(t, x(t), u(t))$ is satisfied for all admissible pairs, we easily obtained from conditions one above:

$$H(t, x^*(t), u^*(t), \lambda(t)) - H(t, x(t), u(t), \lambda(t)) \geq \dot{\lambda}(t)(x(t) - x^*(t)) \quad \forall t \in [0, T]$$

it follows that:

$$\geq \int_0^T \dot{\lambda}(t)(x(t) - x^*(t)) + \lambda(t)(\dot{x}(t) - \dot{x}^*(t)) dt$$

$$= \int_0^T \left(\frac{d}{dt}\right)[\lambda(t)(x^*(t) - x(t))] dt$$

Now we can write this equations as follows:

$$= \int_0^\tau \left(\frac{d}{dt}\right)[\lambda(t)(x^*(t) - x(t))] dt + \int_\tau^T \left(\frac{d}{dt}\right)[\lambda(t)(x^*(t) - x(t))] dt$$

then we obtain:

$$\begin{aligned} & \lambda(\tau^-)(x(\tau^-) - x^*(\tau^-)) - \lambda(0)(x(0) - x^*(0)) \\ & + \lambda(T)(x(T) - x^*(T)) - \lambda(\tau^+)(x(\tau^+) - x^*(\tau^+)). \end{aligned}$$

Using initial condition and transversalis condition $x^*(0) = x_0, x(0) = x_0$ and $\lambda(T) = 0$ such that:

$$\lambda(\tau^-)(x(\tau^-) - x^*(\tau^-)) - \lambda(\tau^+)(x(\tau^+) - x^*(\tau^+)) \geq 0$$

since $x(\tau^-) = x(\tau^+) = x(\tau)$ and $x^*(\tau^-) = x^*(\tau^+) = x^*(\tau)$ then we can write as follows:

$$\lambda(\tau^-)(x(\tau) - x^*(\tau)) - \lambda(\tau^+)(x(\tau) - x^*(\tau)) \geq 0$$

Hence

$$(\lambda(\tau^-) - \lambda(\tau^+))(x(\tau) - x^*(\tau)) \geq 0$$

since $\lambda(t)(x(t) - x^*(t)) \geq 0 \quad \forall t$ so that (x^*, u^*) is optimal pair.

Remark This theorem does not use any concavity or convexity assumption [4],[6].

Theorem 2.7. (Arrow type) Let $(x^*(\cdot), u^*(\cdot))$ be a feasible pair for optimal control problem with a fixed horizon time $T < \infty$. Assume that there exist a piecewise continuously differentiable function $\lambda(\cdot) : [0, T] \rightarrow \mathbb{R}^n$, a piecewise continuous function $\mu(\cdot) : [0, T] \rightarrow \mathbb{R}^s$ and $\eta(\cdot) : [0, T] \rightarrow \mathbb{R}^q$ such that all necessary conditions holds. Assume further that there exist $\alpha \in \mathbb{R}^l$ and $\beta \in \mathbb{R}^r$ such that the transversality conditions hold and assume that at all points τ_i of discontinuity of λ , there exists a $\eta(\tau_i) \in \mathbb{R}^n$ such that jump conditions hold. If the maximized Hamiltonian

$$H^0(t, x, \lambda) = \max_{u \in U(t, x)} H(t, x, u, \lambda)$$

is concave in x for all $(t, \lambda(t))$ and $S(x, T)$ is concave in x and $h(t, x, u)$ is quasiconcave in (x, u) , $k(t, x)$ and $a(t, x)$ are quasiconcave in x and $b(t, x)$ is linear in x then (x^*, u^*) is an optimal pair [10].

Proof. The Theorem remains valid if the concavity of $H(t, x^*, u^*, \lambda(t))$ in (x^*, u^*) at each t is replaced by the concavity of the maximized Hamiltonian $H^0(t, x^*, \lambda(t))$ in x at each t . Since the Theorem is valid if $H^0(t, x^*, \lambda(t))$ is concave in x then we are asked to show that $H^0(t, x^*, \lambda(t)) \in C^{(1)}$ in x , completing this proof. Suppose that the maximizing value of the control variable by using the notation $\tilde{u}(t, x, \lambda(t)) = \operatorname{argmax} H(t, x^*, u^*, \lambda(t))$. Now substituting $u = \tilde{u}(t, x, \lambda(t))$ into the Hamiltonian $H(\cdot)$ yields the value of the maximized Hamiltonian $H^0(\cdot)$, that is to say:

$$H^0(t, x^*, \lambda) = H(t, x^*, \tilde{u}^*(t, x, \lambda(t)), \lambda(t)) = f(t, x^*, \tilde{u}^*(t, x, \lambda(t))) + \lambda(t)g(t, x^*, \tilde{u}^*(t, x, \lambda(t)))$$

Now let $(x^*(\cdot), u^*(\cdot))$ be an admissible pair where $u = \tilde{u}(t, x, \lambda(t))$. Because $H^0(\cdot)$ is concave in x for all $t \in [0, T]$ given $\lambda(t)$ by hypothesis, it follows that:

$$H(t, x^*(t), \lambda(t)) \leq H(t, x^*(t), \lambda(t)) + H_x(t, x^*(t), \lambda(t))[x^*(t) - x(t)]$$

since this inequality holds for all $t \in [0, T]$ we can integrate it over $[0, T]$ and the inequality is preserved, there by yielding

$$\int_0^T H(t, x^*(t), \lambda(t))dt \leq \int_0^T H(t, x^*(t), \lambda(t)) + H_x(t, x^*(t), \lambda(t))[x^*(t) - x(t)]dt$$

since $H^0(t, x^*(t), \lambda(t)) = H(t, x^*(t), \tilde{u}^*(t, x, \lambda(t)), \lambda(t))$ then

$$\begin{aligned} H(t, x^*(t), \tilde{u}^*(t, x, \lambda(t)), \lambda(t)) &\leq H(t, x(t), \tilde{u}(t, x, \lambda(t)), \lambda(t)) \\ &\quad + H_x(t, x(t), \tilde{u}(t, x, \lambda(t)), \lambda(t))[x^*(t) - x(t)] \\ &\quad + H_u(t, x(t), \tilde{u}(t, x, \lambda(t)), \lambda(t))[u^*(t) - u(t)] \end{aligned} \quad (2.8)$$

Now recall that $(x(t), u(t))$ satisfy the necessary conditions and $H_u(t, x(t), \tilde{u}(t, x, \lambda(t)), \lambda(t)) = 0 \forall t \in [0, T]$. This implies that the last term on the right hand side of equation 2.8 is identically zero. Moreover, the inequality of equation 2.8 holds for all $t \in [0, T]$ so we can integrate it over $[0, T]$ and the inequality is preserved.

$$\begin{aligned} \int_0^T H(t, x^*(t), \tilde{u}^*(t, x, \lambda(t)), \lambda(t))dt &\leq \int_0^T H(t, x(t), \tilde{u}(t, x, \lambda(t)), \lambda(t))dt \\ &\quad + \int_0^T H_x(t, x(t), \tilde{u}(t, x, \lambda(t)), \lambda(t))[x^*(t) - x(t)]dt \end{aligned} \quad (2.9)$$

Now use the costate equation

$$\dot{\lambda}(t) = -H_x(t, x(t), \tilde{u}(t, x, \lambda(t)), \lambda(t)).$$

The definition of the Hamiltonian function

$$H(t, x^*(t), \tilde{u}^*(t, x, \lambda(t)), \lambda(t)) = f(t, x, \tilde{u}^*(t, x, \lambda(t))) + \lambda(t)g(t, x, \tilde{u}^*(t, x, \lambda(t)))$$

evaluated along the curves (x, u) and (x^*, u^*) and the definition of functional

$$F(x(\cdot), u(\cdot)) = \int_0^T f(t, x, u)dt$$

evaluated along the curves (x, u) and (x^*, u^*) to write equation of 2.9 in the form of

$$F(x^*(\cdot), u^*(\cdot)) \leq F(x(\cdot), u(\cdot)) + \int_0^t \lambda(t)[g(t, x^*, u^*) - g(t, x, u)] - \dot{\lambda}(t)[x^*(t) - x(t)]dt. \quad (2.10)$$

Let $h(t) = x^*(t) - x(t)$ and integrate the term

$$\int_0^T \dot{\lambda}(t)h(t)dt$$

of equation 2.10 integrate by parts to get

$$\int_0^T \dot{\lambda}(t)h(t)dt = \lambda(T)h(T) - \lambda(0)h(0) - \int_0^T \lambda(t)\dot{h}(t)dt. \quad (2.11)$$

Since $x^*(t)$ satisfies the state equation and initial condition by virtue of it being admissible as does $x(t)$ by virtue of it being a solution of the necessary conditions it therefore follows that:

$$h(0) = x^*(0) - x(0) = x_0 - x_0 = 0$$

Moreover, $\lambda(T) = 0$ by the necessary transversality condition. Hence using equation of 2.11 and these two results in equation of 2.10 yields.

$$F(x^*(\cdot), u^*(\cdot)) \leq F(x(\cdot), u(\cdot)) + \int_0^T \lambda(t)[g(t, x^*, u^*) - \dot{x}^*(t)] - [\dot{x}(t) - g(t, x, u)]dt. \quad (2.12)$$

Because the curves $(x^*(t), u^*(t))$ are admissible they must satisfy the state equation identically, i.e., $\dot{x}^*(t) = g(t, x^*(t), u^*(t))$. Similarly, since the curves $(x(t), u(t))$ are a solution of the necessary conditions they must satisfy the state equation identically too, i.e., $\dot{x}(t) = g(t, x(t), u(t))$. Thus the integrand of equation 2.12 is identically zero, thereby implying that $F[x^*(\cdot), u^*(\cdot)] \leq F[x(\cdot), u(\cdot)]$. If $H(\cdot)$ is strictly concave in (x^*, u^*) for all $t \in [0, T]$, then the inequality in equation of 2.8 becomes strict if either $x^*(t) \neq x(t)$ or $u^*(t) \neq u(t)$ for some $t \in [0, T]$ held. Carrying the strict inequality through the proof leads to the conclusion that $F[x^*(\cdot), u^*(\cdot)] < F[x(\cdot), u(\cdot)]$. This argument thus shows that any admissible pair $(x^*(t), u^*(t))$ which is not identically equal to $(x(t), u(t))$ is non optimal. Now the question becomes: Under what conditions is the Hamiltonian $H(\cdot)$ a concave function of $(x^*, u^*) \forall t \in [0, T]$? Therefore to answer the question let use the following theorem:

Theorem 2.8. A nonnegative linear combination of concave functions is also a concave function. That is, if $f^i(\cdot) : x \rightarrow \mathfrak{R}, i = 1, 2, \dots, m$, are concave functions on a convex subset $x \subset \mathfrak{R}^n$, then

$$f(x) = \sum_{i=1}^m \alpha^i f^i(x) \quad \text{where } \alpha^i \in \mathfrak{R}_+, \quad i = 1, 2, \dots, m$$

is also a concave function on $x \subset \mathfrak{R}^n$.

To see what Theorem 2.8 implies for Theorem 2.7, first recall the definition of the Hamiltonian, namely:

$$H(t, x, u, \lambda) = f(t, x, u) + \lambda g(t, x, u)$$

Therefore, if $f(\cdot)$ and $g(\cdot)$ are concave functions of $(x^*(t), u^*(t))$ for all $t \in [0, T]$, and if $\lambda(t) \geq 0$ for all $t \in [0, T]$, then $H(\cdot)$ is a concave function of $(x^*(t), u^*(t))$ for all $t \in [0, T]$ by Theorem 2.8 since the Hamiltonian is a nonnegative linear combination of concave functions. Similarly, if $f(\cdot)$ is concave in $(x^*(t), u^*(t))$ for all $t \in [0, T]$, $g(\cdot)$ is convex in $(x^*(t), u^*(t))$ for all $t \in [0, T]$, and $\lambda(t) \leq 0$ for all $t \in [0, T]$, then $H(\cdot)$ is concave in $(x^*(t), u^*(t))$ for all $t \in [0, T]$. To see this, first define $\mu(t) = -\lambda(t) \geq 0$. This definition allows us to rewrite the Hamiltonian in the form:

$$\begin{aligned} H(t, x, u, \lambda) &= f(t, x, u) + \lambda g(t, x, u) \\ &= f(t, x, u) + \mu(t)[-1g(t, x, u)] \end{aligned}$$

Since $g(\cdot)$ is convex in $(x^*(t), u^*(t))$ for all $t \in [0, T]$, $-g(\cdot)$ is concave in $(x^*(t), u^*(t))$ for all $t \in [0, T]$ by definition. Moreover, because $\mu(t) \geq 0$ for all $t \in [0, T]$, $H(\cdot)$ is concave in $(x^*(t), u^*(t))$ for all $t \in [0, T]$ by Theorem 2.8 since it is a nonnegative linear combination of concave functions. Thus, in either case, $H(\cdot)$ is concave in $(x^*(t), u^*(t))$ for all $t \in [0, T]$. Finally, if $g(\cdot)$ is linear in $(x^*(t), u^*(t))$ for all $t \in [0, T]$, then $\lambda(t)$ may be of any sign and $H(\cdot)$ will be concave in $(x^*(t), u^*(t))$ for all $t \in [0, T]$ if $f(\cdot)$ is concave in $(x^*(t), u^*(t))$ for all $t \in [0, T]$. This should be clear since if $g(\cdot)$ is linear in $(x^*(t), u^*(t))$ for all $t \in [0, T]$, then it is both concave and convex in $(x^*(t), u^*(t))$ for all $t \in [0, T]$, and therefore, $\lambda(t)g(\cdot)$ is both concave and convex in $(x^*(t), u^*(t))$ for all $t \in [0, T]$ regardless of the sign of $\lambda(t)$. Hence, if $g(\cdot)$ is linear in $(x^*(t), u^*(t))$ for all $t \in [0, T]$ and $f(\cdot)$ is concave in $(x^*(t), u^*(t))$ for all $t \in [0, T]$, then $H(\cdot)$ is concave in $(x^*(t), u^*(t))$ for all $t \in [0, T]$ since it is a nonnegative linear combination of concave functions. In this instance we may also conclude that a solution of the necessary conditions of optimal control problem is a solution to the optimal control problem by Theorem 2.7 since H^0 is concave in $(x^*(t), u^*(t))$ for all $t \in [0, T]$.

Corollary 2.2. *If the assumptions of Theorem 2.6 are satisfied and if Theorem 2.7 holds with strict inequality for $x(t) \neq x^*(t)$, then the optimal state trajectory $x^*(t)$ is uniquely determined[9].*

Corollary 2.3. *If the assumptions of Theorem 2.7 hold and if $H^0(\cdot)$ is strictly concave in x then the optimal state trajectory $x^*(\cdot)$ is unique.*

Remark Corollary 2.2 and 2.3 do not guarantee the uniqueness of the the optimal control u^* in the case of an infinite horizon $T = \infty$ then the above Theorem must be modified as follows [9].

Theorem 2.9. *If $T = \infty$ then Theorem 2.6 and 2.7 remain valid if the transversality condition:*

$$\lambda(T^-) = S_x^*[T] + \alpha a_x[T] + \beta b_x[T] + \gamma k_x^*[T]$$

$$\alpha \geq 0, \quad \gamma \geq 0 \quad \alpha a[T] = \gamma k^*[T] = 0$$

$$\lambda(T)(x(T) - x^*(T)) \geq S(x(T), T) - S(x^*(T), T)$$

are replaced by the following limiting transversality conditions[9]

$$\lim_{t \rightarrow \infty} \lambda(t)(x(t) - x^*(t)) \geq 0$$

for every other feasible state trajectory $x(\cdot)$

Proof. Let $(x^*(t), u^*(t))$ be any admissible pair. By hypothesis, $L(\cdot) \in C^{(1)}$ concave function of (x, u) for all $t \in [0, \infty)$. Therefore,

$$L(t, x^*, u^*, \lambda, \mu) = L(t, x, u, \lambda, \mu) + L_x(t, x, u, \lambda, \mu)[x^* - x] + L_u(t, x, u, \lambda, \mu)[u^* - u]$$

for all $t \in [0, \infty)$. Using the fact that $L_u(t, x, u, \lambda, \mu) = 0$ and then integrating both sides of the resulting reduced inequality over the interval $[0, \infty)$. Using the definitions of $L(\cdot)$ and $F[\cdot]$, yields:

$$\begin{aligned} F[x^*(\cdot), u^*(\cdot)] &\geq F[x(\cdot), u(\cdot)] + \int_0^\infty \lambda(g(t, x, u) - g(t, x^*, u^*))dt \\ &\quad + \int_0^\infty \mu(h(t, x, u) - h(t, x^*, u^*))dt \\ &\quad + \int_0^\infty L_x(t, x, u, \lambda, \mu)[x^* - x]dt \end{aligned}$$

by admissibility

$$\dot{x}(t) = g(t, x, u) \quad \text{for all } t \in [0, \infty)$$

and

$$\dot{x}^*(t) = g(t, x^*, u^*) \quad \text{for all } t \in [0, \infty)$$

while

$$\dot{\lambda} = -L_x(t, x, u, \lambda, \mu) \quad \text{for all } t \in [0, \infty)$$

then substituting these three results

$$\begin{aligned}
F[x^*(\cdot), u^*(\cdot)] &\geq F[x(\cdot), u(\cdot)] + \int_0^\infty \lambda(\dot{x}(t) - \dot{x}^*(t)) + \dot{\lambda}(x(t) - x^*(t))dt \\
&\quad + \int_0^\infty \mu(h(t, x, u) - h(t, x^*, u^*))dt \tag{2.13}
\end{aligned}$$

Moreover, implies that:

$$\mu(t)h(t, x, u) = 0 \quad \text{for } \mu(t) \geq 0$$

$$\mu(t)h(t, x^*, u^*) = 0 \quad \text{for } \mu(t) \geq 0$$

then

$$\int_0^\infty \mu(t)(h(t, x, u) - h(t, x^*, u^*))dt \leq 0 \tag{2.14}$$

Now using the inequality of equation 2.13 and 2.14 to be rewritten in the reduce form of

$$F[x^*(\cdot), u^*(\cdot)] \geq F[x(\cdot), u(\cdot)] + \int_0^\infty \lambda(\dot{x}(t) - \dot{x}^*(t)) + \dot{\lambda}(x(t) - x^*(t))dt \tag{2.15}$$

$$\begin{aligned}
&\frac{d}{dt}[\lambda(t)[x(t) - x^*(t)]] \\
&= \lambda(\dot{x}(t) - \dot{x}^*(t)) + \dot{\lambda}(t)[x(t) - x^*(t)]
\end{aligned}$$

and substitute this result in to equation of 2.15

$$\begin{aligned}
F[x^*(\cdot), u^*(\cdot)] &\geq F[x(\cdot), u(\cdot)] + \int_0^\infty \frac{d}{dt}[\lambda(t)[x(t) - x^*(t)]] \\
F[x(\cdot), u(\cdot)] + \lim_{t \rightarrow \infty} \lambda(t)[x(t) - x^*(t)] - \lambda(0)[x(0) - x^*(0)] \\
&F[x(\cdot), u(\cdot)] + \lim_{t \rightarrow \infty} \lambda(t)[x(t) - x^*(t)]
\end{aligned}$$

since by admissibility we have $x(0) = x_0$ and $x^*(0) = x_0$. Now if for every admissible control path $u(t)$, $\lim_{t \rightarrow \infty} \lambda(t)[x(t) - x^*(t)] \geq 0$ where $x(t)$ is the time path of the state variable corresponding to $u(t)$, then it follows that $F[x^*(\cdot), u^*(\cdot)] \geq F[x(\cdot), u(\cdot)]$ for all admissible functions $(x^*(\cdot), u^*(\cdot))$. If $L(\cdot)$ is a strictly concave function of (x^*, u^*) , for all $t \in [0, \infty)$ then the inequality becomes strict if either $x^*(t) \neq x(t)$ or $u^*(t) \neq u(t)$ for some $t \in [0, \infty)$. In this instance $F[x^*(\cdot), u^*(\cdot)] > F[x(\cdot), u(\cdot)]$ follows. This shows that any admissible pair of functions $(x^*(\cdot), u^*(\cdot))$ which are not identically equal to $(x(\cdot), u(\cdot))$ are suboptimal [5]. \square

Example 2.2. In this example we illustrate the approaches in Theorems 2.1, 2.2 and 2.4 by applying them to some illustrative examples. Consider the following example:[10]

$$\begin{aligned} \max \quad & \int_0^3 -x dt \\ \text{subject to} \quad & \dot{x} = u, \quad x(0) = 1 \\ & u + 1 \geq 0 \\ & 1 - u \geq 0 \\ & x \geq 0 \\ & x(3) = 1 \end{aligned}$$

Solution. The Hamiltonian is

$$H = -x + \lambda u$$

which implies the optimal control to be

$$u^* = \text{bang}[-1, 1; \lambda] \quad \text{when } x \geq 0$$

and which optimal control on the state constraint boundary is

$$u^* = \text{bang}[0, 1; \lambda] \quad \text{when } x = 0.$$

The boundary conditions $x(0) = x(3) = 1$. Thus

$$u^*(t) = \begin{cases} -1 & \text{for } t \in [0, 1) \\ 0 & \text{for } t \in [1, 2] \\ 1 & \text{for } t \in (2, 3] \end{cases}$$

and

$$x^*(t) = \begin{cases} 1 - t & \text{for } t \in [0, 1) \\ 0 & \text{for } t \in [1, 2] \\ t - 2 & \text{for } t \in (2, 3] \end{cases}$$

Now let first, we apply the direct adjoint approach and let form the Lagrangian L as:

$$L = H + \mu_1(u + 1) + \mu_2(1 - u) + \nu x.$$

The necessary condition of Theorem 2.1 are

$$L_u = \lambda + \mu_1 - \mu_2 = 0$$

$$\dot{\lambda} = -L_x = 1 - \nu$$

but

$$\begin{aligned}\mu_1 &\geq 0 \quad \mu_1(u+1) = 0 \\ \mu_2 &\geq 0 \quad \mu_2(1-u) = 0 \\ \nu &\geq 0 \quad \nu x = 0, \quad \dot{\nu} \leq 0 \\ \lambda(3) &= \beta \quad \text{where } \beta \in \mathfrak{R}.\end{aligned}$$

The enters of the boundary of $x = 0$ in a nontangential way at time $\tau_1 = 1$, since $k^1(1^-) \leq 0$ and also at time $\tau_2 = 2$ it leaves this boulder nontangential since $k^1(2^+) \geq 0$. Therefore, according to Proposition (2.1), λ is continuous at time $t = 1$ and $t = 2$ as well as in $[0, 1)$ and $(2, 3]$ where the state constraint is not active. Now consider the boundary interval $[1, 2]$, here $u = 0$ and implies that $\mu_1 = \mu_2 = 0$ then from

$$l_u = \lambda + \mu_1 + \mu_2 = 0, \quad \lambda = 0$$

Thus λ is also continuous in $(1, 2)$. Furthermore, since $\lambda = 0$ from equation

$$\dot{\lambda} = -L_x = 1 - \nu \quad \nu = 1.$$

Thus all multipliers are uniquely determined in $[1, 2]$. In $[0, 1)$ we have $x > 0$ and $\nu = 0$ then $\lambda = t - 1$ because of $\dot{\lambda} = 1$ and $\lambda(1) = 0$. Similarly in $(2, 3]$ we have $x > 0$ and $\nu = 0$ then $\lambda = t - 2$ because of $\dot{\lambda} = 1$ and $\lambda(2) = 0$. Now we can detrmind μ_1 and μ_2 from equation

$$L_u = \lambda + \mu_1 - \mu_2 = 0$$

and

$$\begin{aligned}\mu_1 &\geq 0 \quad \mu_1(u+1) = 0 \\ \mu_2 &\geq 0 \quad \mu_2(1-u) = 0.\end{aligned}$$

In $[0, 1)$ we have $\lambda = t - 1$ and $u = -1$ then $\mu_2 = 0$ and $\mu_1 = 1 - t$. Similarly in $(2, 3]$, $\lambda = t - 2$ and $u = 1$ then $\mu_2 = t - 2$ and $\mu_1 = 0$. In the inderect adjoining approach the Hamiltonian H^1 and Lagrangian L^1 are:

$$H^1 = -x + \lambda u$$

$$L^1 = H^1 + \mu_1(u+1) + \mu_2(1-u) + \nu u.$$

The necessary conditions of Theorem 2.2 are:

$$L_u^1 = \lambda + \mu_1 - \mu_2 + \nu = 0$$

$$\dot{\lambda} = -L_x = 1$$

where μ_1 , μ_2 and ν^1 satisfy the complementary slackness conditions:

$$\begin{aligned}\mu_1 &\geq 0 & \mu_1(u+1) &= 0 \\ \mu_2 &\geq 0 & \mu_2(1-u) &= 0 \\ \nu^1 &\geq 0 & \nu^1 x &= 0, & \dot{\nu}^1 &\leq 0 \\ \lambda^1(1^-) &= \lambda^1(1^+) + \eta^1(1) & \eta^1(1) &= 1 \geq 0.\end{aligned}$$

Since $x^*(t)$ enters the boundary zero at $t = 1$ there are no jumps in interval $(1, 2]$ and the solutions for $\lambda^1(t)$ is

$$\lambda^1(t) = t - 2 \quad t \in (1, 2].$$

Hence $\lambda^1(t) \leq 0$ and $x^*(t) = 0$ on $(1, 2]$ we have $u^*(t) = 0$. Now let us see what must happen at $t = 1$. we now from equation

$$\lambda^1(t) = t - 2 \quad t \in (1, 2] \quad \lambda^1(1^+) = -1.$$

Then

$$\begin{aligned}H^1(1^+) &= -x^*(1^+) + \lambda^1(1^+)u^*(1^+) = 0 \\ H^1(1^-) &= -x^*(1^-) + \lambda^1(1^-)u^*(1^-) = -\lambda^1(1^-).\end{aligned}$$

By equating $H(1^-)$ to $H(1^+)$ we obtain $\lambda^1(1^-) = 0$ then the value of the jump condition

$$\eta^1(1) = \lambda^1(1^-) - \lambda^1(1^+) = 1 \geq 0.$$

In time interval $[0, 1)$, $\mu_2 = 0$ since $u^* = -1$ and $\nu^1 = 0$ because $x > 0$ for $t \in [0, 1)$. Therefore,

$$\frac{\partial}{\partial u} L = \lambda + \mu_1 - \mu_2 + \nu = 0 \quad \text{then}$$

$$\lambda^1 + \mu_1 = 0 \quad \text{since } \mu_2 \text{ and } \nu = 0 \text{ for } t \in [0, 1)$$

hence

$$\mu_1(t) = -\lambda^1(t) = 2 - t \quad \text{for } t \in [0, 1) \text{ with } u = -1.$$

At $t = 1$ we have $x(1) = 0$ so the optimal control $u^*(1) = 0$. Now assume that we continue to use the control $u^*(t) = 0$ in the interval $[1, 2]$ then $x(t) = 0$ for $t \in [1, 2]$. Since $\lambda^1(t) \leq 0$ for $t \in [1, 2]$ then $u^*(1) = 0$ on the same interval then μ_1 and $\mu_2 = 0$ for $t \in [1, 2]$ but we can obtain $\nu^1(t) = -\lambda^1(t)$ for $t \in [1, 2]$. Therefore, the adjoint function λ is continuous every where, ν is constant in $[0, 1)$ and $(2, 3]$ where the state constraint is not active and ν is continuous at $t = 1, 2$ where $k^1(t, x, u) = \dot{x} = u$ is discontinuous. The adjoint function λ is continuous, since the entry to and the exit from the state constraint is nontangential.

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