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Probabilistic Dynamic Programming for Inventory Problems

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Abstract

Stockout cost is the cost that happen when the demand is greater than the inventory. It may be either back_order cost or lost sale cost. Generally inventory problems has many features and cases. I formulate a probabilistic dynamic programming model to study problems over the finite planning horizon. This paper presents with the scope of stochastic dynamic programming.Mainly two models are discussed the first model is single period model and the second model is multi-stage model. Unfortunately, dynamic programming designed for multi-period problems not for single period but I discussed Christmas inventory problem or newsboy problem within single-period model.

In chapter 1 we will discuss definition of probabilistic or stochastic dynamic programming with illustrative example, historical background in short. In chapter 2 some preliminary concepts and definitions are included. Since this concepts are used for chapter four specially. In chapter 3 and 4 the main work of the paper with to models single-period model on chapter 3 and multi-stage or multi-period model in chapter 4. Finally conclusion and recommendation are in the last chapter. At the end a numerical example is included to illustrate how the model applied.

Keywords

Expected value	Inventory	Items	Poisson
Probabilistic	Stochastic	Stockout	Uncertainty

Permission

The undersigned here by certify that they have read and recommend to the College of Natural and Computational Science for acceptance of a thesis entitled ***Probabilistic Dynamic Programming for Inventory Problems*** by **Desta Zeleke** in partial fulfillment of the requirement for degree in masters of science in mathematics(Optimization).

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Chapter 1

Probabilistic or Stochastic Dynamic Programming

1.1 Introduction to Probabilistic or Stochastic Dynamic Programming

Most of multistage optimization problems are solving by dynamic programming which is a general method to solve such problems. Probabilistic or Stochastic Dynamic Programming(SDP) may be consider similarly as dynamic programming, but purposely to solve probabilistic multistage optimization problems.
A stochastic/probabilistic multistage optimization problem is a problem where one or several of the parameters in the problem are modeled as stochastic variables or processes. As many of the problems in the field of Operations Research deals with future planning and many future events are hard to predict with certainty, it is not hard to imagine the importance of SDP and related techniques. stochastic case - is always the actual situation.

One of the most studied area in economics is inventory problems because of the uncertainty of demands. The essential characteristic of all economic systems is that they are continually changing with time. For inventory systems, the processes generating demands and lead times change with time, as do the various costs of interest, and even the items carried by the system. In many cases, however, the changes occur slowly enough so that for considerable lengths of time the system can be treated as if it were in a steady state mode of operation. In other instances, however, the changes occur with such rapidity that they must be explicitly accounted.

The main purpose of this paper to study inventory problems that are modeled as multi-period models in which the mean rate of demand changes with time.

As might be expected, the difficulty of formulating and obtaining numerical solutions to realistic dynamic inventory models is considerably greater than for the case where it was permissible to assume that the system was in steady state. In fact, when demand is treated as a stochastic variable whose mean is time dependent. The natural formalism for setting up dynamic models in a form for numerical computations is dynamic programming.

This paper presents with the scope of stochastic dynamic programming.Mainly two models are discussed the first model is single period model and the second model is multi-stage model. Unfortunately, dynamic programming designed for multi-period problems not for single period but I discussed Christmas inventory problem or newsboy problem within single-period model.

In chapter 1 we will discuss definition of probabilistic or stochastic dynamic programming with illustrative example, historical background in short. In chapter 2 some preliminary concepts and definitions are included. Since this concepts are used for chapter four specially. In chapter 3 and 4 the main work of the paper with to models single-period model on chapter 3 and multi-stage or multi-period model in chapter 4. Finally conclusion and recommendation are in the last chapter.

1.2 Definition of probabilistic dynamic programming

Probabilistic Dynamic programming (PDP) is a dynamic programming that the state and the immediate returns are probabilistic rather than deterministic. Let us consider by illustrate example.

1.3 Illustrate Example

A wholesale distributor of Motorcycles is having trouble with shortages of a popular model and is currently reviewing the inventory policy for this model. The distributor purchases this model motorcycle from the manufacturer monthly and then supplies it to various motorcycle shops in the Easter Canada in response to purchase orders. What the total demand from motorcycle shops will be in any given month is quite uncertain/probabilistic. Therefore, the question is, How many motorcycles should be ordered from the manufacturer for any given month, given the stock level leading into that month?

The distributor has analyzed her costs and has determined that the following are important:

1. The ordering cost, i.e., the cost of placing an order plus the cost of the motorcycles being purchased, has two components: The administrative cost involved in placing an order is estimated as $200, and the actual cost of each bicycle is $35 for this wholesaler.

2. The holding cost, i.e., the cost of maintaining an inventory, is $1 per motorcycle remaining at the end of the month. This cost represents the costs of capital tied up, warehouse space, insurance, taxes, and so on.

3. The shortage cost is the cost of not having a motorcycle on hand when needed. This particular model is easily reordered from the manufacturer, and stores usually accept a delay in delivery. Still, although shortages are permissible, the distributor feels that she incurs a loss, which she estimates to be $15 per motorcycle per month of shortage. This estimated cost takes into account the possible loss of future sales because of the loss of customer goodwill. Other components of this cost include lost interest on delayed sales revenue, and additional administrative costs associated with shortages.

1.4 Historical Background and Previous Works

1.4.1 Historical Background

The control and maintenance of inventories of physical goods is a problem common to all enterprises in any sector of a given economy. Inventories must be maintained in agriculture, industry, retail establishments, and the military. For example in the United States the total dollar investment in inventories at any one time more than 50 billion dollars for defense projects alone and more than 95 billion for private enterprise sectors of the economy [1]. As researches indicated the total value of all inventory—including finished goods, partially finished goods, and raw materials—in the United States is more than a trillion dollars. This is more than $4,000 each for every man, woman, and child in the country. There are many reasons why organizations should maintain inventories of goods. The fundamental reason for doing so is that it is either physically impossible or economically unsound to have goods arrive in a given system precisely when demands for them occur.

Without inventories customers would have to wait until their orders were filled from a source or were manufactured.

In general, however, customers will not or cannot be allowed to wait for long periods of time. For this reason alone the carrying of inventories is necessary to almost all organizations that supply physical goods to "customers".

There are,a lot of reasons for holding inventories, among them are:-

1. The price of some raw material used by a manufacturer may exhibit considerable seasonal fluctuation.

2. The second reason for maintaining inventories, a reason particularly important to retail establishments,is that sales and profits can be increased if one has an inventory of goods to display to customers.

The intial causes for the use of mathematical methods in inventory analysis seems to have been supplied by the simultaneous growth of the manufacturing industries and various of engineering.

Now a day there are many operational books discussed about inventory problems most of their assumptions are demand is deterministic and for one period and few books are discussed about inventory problem with probabilistic demand and multi-period problem with very limited way. The full length book to deal with inventory problem was that of G.Hadley and T.M.Whitin[1]. It is relatively best explanation about mathematics models of inventory problems both deterministic and probabilistic cases.

1.4.2 Previous Works

Research on inventory problems has pursued by many operations researchers and authors. One of the earliest works is by Hadley and Whitin [1] which is titled by *"Analysis of Inventory System"* focused on the techniques of constructing and analyzing mathematical models of inventory system. Qin & Kar[3] were researchers to discussed about single period inventory problem, it known as the newsboy problem is a very significant consideration in terms of both theoretical and practical.

Hiller and Lieberman[4] are researchers they broadly discussed with different models and assumptions. they focused on similarly as the above with deterministic models and stochastic models according to the *predictability of demands* involved. In deterministic model review under continuous situations and series of periods and also under stochastic model illustrated by case studies.

Teha[6] the most known author he discussed on both deterministic and probabilistic models on his book titled on[6], with different perspectives. Which is demands are:-

1. Deterministic and constant (static) with time.
2. Deterministic and variable (dynamic) with time.
3. Probabilistic and stationary over time.
4. Probabilistic and non-stationary over time.

Beyes, P.Sethi and R.Sridhas[2], discussed about multi-product inventory models with stochastic demands and warehousing constraint and also they had conclude that if the products are separable and are linked only by the storage limitation, then he shown that the myopic ordering policy is optimal.

Li[7] studied about remanufacturing system under the assumption of uncertain production planning using stochastic dynamic programming.

Berman, Lichard and Larson[8] studied inventory problems using stochastic dynamic programming. Consider an industrial gases tanker vehicle visits n customers on a tour with possible $(n+1)^{th}$ customer added at the end. The over all objective is to adjust dynamically the amount of product provided on scene to each customer. So, as to minimize the total costs.

Gallego and Ryzin[5] discussed about optimal dynamic pricing of inventories with stochastic demand over finite horizon. The main objective of this work is by investigated the problems around dynamically pricing such inventories when demand is price sensitive and stochastic.

My work is mostly closed with Hadley and Whitin[1] works. They discussed different mathematical models with inventory problem. It also the most convenient guide for my research and I choose the models discusses in this book.

Chapter 2

Preliminary and Definitions

2.1 Definition of Poisson Distribution

A random variables which are defined over all the non-negative integers, so that a $p(x)$ can be defined for every non-negative integer x.we assume that the outcome of the experiment will yield one and only one non-negative integer x. Thus the various x values are mutually exclusive and we must have

$$\sum_{x=0}^{\infty} p(x) = 1,\ p(x) \geq 0 \tag{2.1}$$

Definition

The Poisson density will always be denoted by $p(x;\mu)$. The random variable x whose probabilities are given by

$$p(x) = p(x;\mu) = \frac{\mu^x}{x!} e^{-\mu}, \quad x = 0, 1, 2, ... \tag{2.2}$$

where p is a constant whose physical interpretation we will see chapter four. The cumulative and complementary cumulative Poisson functions will be denoted by $\hat{P}(x;\mu)$ and $P(x;\mu)$ respectively, where

$$\hat{P}(x) = \sum_{i=0}^{x} \frac{\mu^i}{i!} e^{-\mu} \quad P(x) = \sum_{i=x}^{\infty} \frac{\mu^i}{i!} e^{-\mu} \tag{2.3}$$

Observe that

$$\sum_{x=0}^{\infty} p(x,\mu) = e^{-\mu} \sum_{i=0}^{\infty} \frac{\mu^i}{i!} = e^{-\mu} e^{\mu} = 1$$

Since

$$e^{\mu} = \sum_{i=0}^{\infty} \frac{\mu^i}{i!}$$

Some properties of poisson distribution hold for all $i > 0$ and all non-negative integers x.

1. Property 1:

$$\sum_{i=x}^{\infty} ip(x, \mu) = \mu P(x-1; \mu)$$

2. Property 2:

$$\begin{aligned}\sum_{i=x}^{\infty}(i-x)p(i, \mu) &= \sum_{i=0}^{\infty} ip(x+i; \mu) \\ &= \sum_{i=x+1}^{\infty} P(i; \mu) \quad \textit{by property one} \\ &= \mu P(x-1; \mu) - xP(x; \mu)\end{aligned}$$

2.2 Some characteristics of Dynamic Programming

The basic feature in dynamic programming problems are as follow.

1. The problem can be divided in to stages with policy decision required at each stage.

2. Each stage has a number of states associated with the beginning of the stage. The states are the various possible conditions in which the system might be at that stage

3. The effect of the policy decision at each stage is transform the next states.

4. The solution procedure is designed to find an optimal policy for the overall problem. i.e., a prescription of the optimal policy decision at each stage for each of the possible states.

5. Given the current state, an optimal policy for the remaining stages is independent of the policy decisions adopted in previous stages. This is the principles of optimality for dynamic programming.

6. The solution procedure begins by finding the optimal policy for the last stage.

7. A recursive relationship that identifies the optimal policy for stage n, given the optimal policy for stage $n + 1$ is available.

8. When we use this recursive relationship, the solution procedure starts at the end most the time and move backward stage by stage. Each time we find optimal policy for that stage until initial stage, but some time might be used forward recursive relationship.

Chapter 3

Single-period Model of Inventory Problems

3.1 Definition Newsboy Problem

For a newsboy who sells newspapers on a street corner, the demand is uncertain, and the news boy must decide how many paper to buy from his supplier. If he buys too many papers he is left with unsold papers that has no value at the end of the day; if he buys too few papers he has lost the opportunity of making a higher profit. This types of problem known as ***a newsboy problem***

3.2 Newsboy Problem Model

Most single period problems are known as Christmas tree problem or Newsboy problem, since it can be phrased as a problem of deciding how many trees a dealer in Christmas trees should purchase for the season, or how many newspapers a boy should buy on a given day for his corner newsstand. Single-item inventory models occur when an item is ordered only once to satisfy the demand for the period. For example, fashion and food items become obsolete at the end of the season.
The symbols used in the development of the models include
K = Setup cost per order
h = Holding cost per held unit during the period
s = Penalty cost per shortage unit during the period
d = Random variable representing demand during the period
$p(d)$ = probability distribution function of demand during the period
x = Order quantity

y = Inventory on hand before an order is placed.
$E\{C(x)\}$ = Expected cost The model determines the optimal value of x that minimizes the sum of the expected holding and shortage costs. Given optimal $x(= x^*)$, the inventory policy calls for ordering $x^* - y$ if $y < x$; otherwise, no order is placed. If d is discrete, then $p(d)$ is defined only at discrete points and the associated cost function is

$$E\{C(x)\} = h\sum_{d=0}^{x}(x-d)p(d) + s\sum_{d=x+1}^{\infty}(d-x)p(d) \tag{3.1}$$

The necessary conditions for optimality are

$E\{C(x-1)\} \geq E\{C(x)\}$ and $E\{C(x+1)\} \geq E\{C(x)\}$

These conditions are sufficient because $E\{C(x)\}$ is a convex function. Then

$$E\{C(x-1)\} = h\sum_{d=0}^{x-1}(x-1-d)p(d) + s\sum_{d=x}^{\infty}(d-x+1)p(d)$$

$E\{C(x)\}$ has unique solution since $E\{C(x)\}$ is convex and its derivative becomes;

$$h\sum_{d=0}^{x-1}(x)p(d) - s\sum_{d=x}^{\infty}xp(d) = 0 \tag{3.2}$$

$$hp(d \leq x^* - 1) - s(1 - p(d \leq x^* - 1)) \leq 0 \qquad \textit{since lower sum} \tag{3.3}$$

$$p(d \leq x^* - 1) \leq \frac{p}{p+s} \tag{3.4}$$

$$h\sum_{d=0}^{x}(x)p(d) - s\sum_{d=x+1}^{\infty}xp(d) = 0 \tag{3.5}$$

$$hp(d \leq x^*) - s(1 - p(d \leq x^*)) \geq 0 \qquad \textit{since upper sum} \tag{3.6}$$

$$p(d \leq x^*) \geq \frac{p}{p+s} \tag{3.7}$$

From (3.2) and (3.5) we get

$$p(d \leq x^* - 1) \leq \frac{p}{p+s} \leq p(d \leq x^*) \tag{3.8}$$

3.3 Numerical Example

Example 1. *The owner of a newsstand wants to determine the number of Addis Zemen newspapers that must be stocked at the start of each day. The owner pays* 60 *cents for a copy and sells it for* 1.5 *birr. The sale of the newspaper typically occurs for* 1 *day. Newspapers left at the end of the day are recycled for an income of* 30 *cents per a copy. How many copies should the owner stock every morning, assuming that the demand for the day can be described as a discrete probability density function,* $p(d)$*, defined as*

d	200	220	300	320	340
$p(d)$	0.1	0.2	0.4	0.2	0.1

Solution 1. *The holding and penalty costs are not defined directly in this situation. The data of the problem indicate that each unsold copy will cost the owner* $60 - 30 = 30$ *cents and that the penalty for running out of stock is* $150 - 60 = 90$ *cents per copy. Thus, in terms of the parameters of the inventory problem, we have* $h = 30$ *cents per copy per day and* $s = 90$ *cents per copy per day.*
First, we determine the critical ratio as

$$\frac{s}{s+h} = \frac{90}{90+30} = 0.75$$

First, we determine the CDF $P\{d \leq x\}$ *as*

d	*200*	*220*	*300*	*320*	*340*
p(d)	*0.1*	*0.3*	*0.7*	*0.9*	*1.0*

For the computed critical ratio of 0.75, *we have*

$P(d \leq 300) \leq 0.75 \leq P(d \leq 320)$
It only follows that $x^* = 320$ *copies.*

Chapter 4

Multi-period Dynamic Inventory Model

4.1 Dynamic programming Model

Dynamic programming is a mathematical technique for making a sequence of interrelated decisions. It provides a systematic procedure for determining the optimal combination of decisions. Consider the problem of determining non-negative integers x_i which minimize the function $g(x_1, ..., x_n)$ defined by;

$$g(x_1, ..., x_n) = \sum_{i=1}^{n}(f_i(x_i)) = f_1(x_1) + f_2(x_2) + ... + f_n(x_n) \tag{4.1}$$

subject to the constraint

$$\sum_{i=1}^{n} a_i x_i \leq b \tag{4.2}$$

where the a_i, b are specified constants, and $f_i(x_i)$ is a function of x_i only.

We shall introduce dynamic programming by studying how one might attempt to solve numerically a problem such as that formulated above.
we select a value of x_n and we minimize g over $x_1, ..., x_{n-1}$ for this given value of x_n. By (4.2), we see that the variables $x_1, ..., x_{n-1}$ must satisfy

$$\sum_{i=1}^{n-1}(a_i(x_i)) \leq b - a_n x_n \tag{4.3}$$

so that the allowable range of variation for $x_1, ..., x_{n-1}$ will depend on the value of x_n can only have the integral values $0, 1, ..., \lfloor b/a_n \rfloor$, where $\lfloor b/a_n \rfloor$ denotes the largest integer less than or equal to b/a_n.

$$\min_{x_1,...,x_{n-1}} g = f(x_n) + \min_{x_1,...,x_{n-1}} \sum_{i=1}^{n-1} (f_i(x_i)) \tag{4.4}$$

and in computing

$$\min_{x_1,...,x_{n-1}} \sum_{i=1}^{n-1} (f_i(x_i)) \tag{4.5}$$

$x_1, ..., x_{n-1}$ must satisfy (4.3), and the minimum value at (4.5) will depend on $b - a_n x_n$ because of (4.3).
Let us denotes (4.5) by $Z_{n-1}(b - a_n x_n)$ for non negative integer.

Then $Z_{n-1}(b - a_n x_n)$ is the minimum of $\sum_{i=1}^{n-1} f_i(x_i)$. Equation (4.4) can be written

$$\min_{x_1,...,x_n} g = f(x_n) + Z_{n-1}(b - a_n x_n) \tag{4.6}$$

g^* is the optimal of g, we see that

$$g^* = \min_{x_n} [f(x_n) + Z_{n-1}(b - a_n x_n)] \tag{4.7}$$

Hence, if we knew the function $Z_{n-1}(\xi)$ for all integral arguments ξ from 0 to b, we could determine g^* simply by computing $f_n(0) + Z_{n-1}(b)$, $f_n(1) + Z_{n-1}(b - a_n)$, $f_n(2) + Z_{n-1}(b - 2a_n)$, etc., up to $x_n = \lfloor b/a_n \rfloor$, and picking the smallest of these. We would simultaneously determine $\hat{x}$, the optimal value of x_n, in doing so. The question then arises as to how we determine $Z_{n-1}(\xi)$ for any argument ξ. By definition

$$Z_{n-1}(\xi) = \min_{x_1,...,x_{n-1}} \sum_{i=1}^{n-1} (f_i(x_i)) \tag{4.8}$$

for non-negative integers $x_i, i = 1, ..., n-1$, satisfying

$$\sum_{i=1}^{n-1} a_i x_i \leq \xi$$

We can now resort to the same trick as above. Suppose that we pick a value of x_{n-1}, and minimize $\sum_{i=1}^{n-1} f_i(x_i)$ over $x_1, ..., x_{n-2}$ for non-negative integers $x_i, i = 1, ..., n-2$, satisfying

$$\sum_{i=1}^{n-2} a_i x_i \leq \xi - a_{n-1} x_{n-1}$$

For any argument ξ, let us now define the function $Z_{n-1}(\xi)$ by

$$Z_{n-1}(\xi) = \min_{x_1,\ldots,x_{n-2}} \sum_{i=1}^{n-2} (f_i(x_i)) \tag{4.9}$$

for non-negative integers $x_i, i = 1, ..., n-2$, satisfying

$$\sum_{i=1}^{n-2} a_i x_i \leq \xi$$

Then

$$Z_{n-1}(\xi) = \min_{x_{n-1}} [f_{n-1}(x_{n-1}) + Z_{n-2}(\xi - a_n x_n)] \tag{4.10}$$

We can continue working down in this way until we are reduced to evaluating

$$Z_1(\xi) = \min_{x_1} f_1(x_1) \tag{4.11}$$

for non-negative integers x_1 satisfying $x_1 \leq \lfloor \xi / a_1 \rfloor$. It is a straightforward task to evaluate $Z_1(\xi)$ for any given ξ. We need the value of $Z_1(\xi)$ in general for all integers ξ between 0 and b. Once we have computed $Z_1(\xi)$ for each ξ we can then compute $Z_2(\xi)$ by using

$$Z_2(\xi) = \min_{x_2} [f_2(x_2) + Z_1(\xi - a_2 x_2)] \tag{4.12}$$

Then we can also compute $Z_3(\xi)$, etc. The same manner until finally for $Z_{n-1}(\xi)$ we obtain g^* using (4.7). Then

$$g^* = Z_n(b) \tag{4.13}$$

and the $Z_k(\xi)$ can be computed recursively using the recurrence relation

$$Z_k(\xi) = \min_{0 \leq x_k \leq \lfloor \frac{\xi}{a_k} \rfloor} [f_k(x_k) + Z_{k-1}(\xi - a_k x_k)], k = 2, ..., n \tag{4.14}$$

In computing $Z_k(\xi)$ we also obtain for each ξ the value of x_k which yields the minimum. Denote by $\hat{x}_k$ the value of x_k which yields $Z_k(\xi)$ in. Suppose now that as we record the table of $Z_k(\xi)$ we also record a table of $\hat{x}_k$. At the last step, when $Z_n(b)$ is computed, we automatically determine x_n^*. By working backwards from x_n^*. To find x_{n-1}^*, we only need to determine $\hat{x}_{k-1}(\xi)$ for $\xi = b - a_n x_n^*$, i.e.,$x_{n-1}^* = \hat{x}_{n-1}(b - a_n x_n^*)$. Having determined x_{n-1}^* we are now able to compute x_{n-2}^* since $x_{n-2}^* = \hat{x}_{n-2}(b - a_n x_n^* - a_{n-1} x_{n-1}^*)$. We continue this procedure until finally we find that

$$x_1^* = \hat{x}_1(b - \sum_{i=2}^{n} a_i x_i^*)$$

Thus in general

$$x_k^* = \hat{x}_k(b - \sum_{i=k+1}^{n} a_i x_i^*), k = 1, ..., n-1 \tag{4.15}$$

In numerical computations, if one finds at stage k that more than one value of x_k yields $Z_k(\xi)$, one can tabulate each x_k which yields the minimum, thus making it possible at the end to find all alternative optimal solutions, or simply tabulate one of these x_k thus yielding finally only one of the alternative optimal solutions.

An interesting and helpful physical interpretation can be given to the $Z_k(\xi)$. $Z_k(\xi)$ is the minimum cost if only stages $1, 2, ..., k$ existed, and the quantity of the limiting resource which could be devoted to these stages was ξ. The physical interpretation of (4.14) is that an optimal policy for k stages must have the property that whatever the choice of the decision variable for the k^{th} stage, the policy must be optimal for the remaining $k-1$ stages for this choice of x_k.

4.2 Numerical Example

Example 2. *An isolated air base is supplied with spare parts only once every month when a plane from the main depot arrives to replenish the inventory. Let* $y_1 = 1$; $y_2 = 0$ *and* $y_3 = 2$; *are inventories on hand for item* $1, 2$ *and* 3 *respectively. There are three which are stocked at the air base. The plane which brings the spare parts can carry a volume* $b = 15cu.ft$ *and the unit volume of each items 1,2 and 3 are* $a_1 = 5cu.ft$, $a_2 = 2cu.ft$ *and* $a_3 = 4cu.ft$ *respectively.* $\pi_1 = \$5000$, $\pi_2 = \$2000$ *and* $\pi_3 = \$10,000$ *is incurred each time a demand occurs when the base is out of stock. Monthly demand for each of the items is essentially Poisson distributed with mean* $\mu_1 = 1$, $\mu_2 = 5$, $\mu_3 = 3$. *Determine the optimal values of* x_1, x_2 *and* x_3 *in order to minimize the expected stock-out cost?*

Solution 2. *If* x_i *units of item* i *are stocked, we desire minimize*

$$\begin{aligned} g &= \sum_{i=1}^{3} \pi_i \left[\sum_{y=x_i}^{\infty} (y - x_i) p(y; \mu_i) \right] \\ &= \sum_{i=1}^{3} \pi_i \left[\mu_i p(x_i; \mu_i) - x_i p(x_i; \mu_i) \right] \end{aligned}$$

by the characteristics of Poisson distribution for non negative integers x_i *which satisfy*

$$5x_1 + 2x_2 + 4x_3 \leq 15$$

we then define the functions

$$Z_k(\xi) = \min_{x_1,\ldots,x_k} \sum_{i=1}^{k} \pi_i \left[\mu_i p(x_i - 1; \mu_i) - x_i p(x_i; \mu_i)\right], k = 1, 2, 3$$

for non-negative integers x_i *satisfying*

$$\sum_{i=1}^{k} a_i x_i \leq \xi$$

Then, if x^* *is the optimal value of* g, $g^* = Z_3(15)$. *For* $k \geq 2, Z_k(\xi)$ *can be computed sequentially from*

$$Z_k(\xi) = \min_{0 \leq x_k \leq \lfloor \frac{\xi}{a_k} \rfloor} \{\pi_k[\mu_k P(x_k - 1; \mu_k) - x_k P(x_k; \mu_k)] + Z_{k-1}(\xi - a_k x_k)\}$$

$Z_1(\xi)$- *is given by*

$$Z_1(\xi) = \min_{0 \leq x_1 \leq \lfloor \frac{\xi}{5} \rfloor} \{\pi_1[\mu_1 P(x_1 - 1; \mu_1) - x_1 P(x_1; \mu_1)]\}$$

since $a_1 = 5$. *It is clear that the minimum in the above expression will be taken on when* $x_1 = \frac{\xi}{a_1}$, *since the expression in braces is the expected stock-out cost for item* 1 *when* $x_1 = 0, 1, ..., \frac{\xi}{a_1}$ *units are stocked. Hence*

$$Z_1(\xi) = \min_{0 \leq x_1 \leq \frac{\xi}{5}} \{\$5000[P(x_1 - 1; 1) - x_1 P(x_1; 1)]\}$$

For this example, however, it is easy to work backwards to determine which arguments ξ *will be needed. It saves some work here to do this at the outset. To calculate* $Z_3(15)$ *we need* $Z_2(15 - 4x_3)$ *where* x_3 *can take on every integral value from* 0 *to* $\lfloor 15/4 \rfloor = 3$. *Thus* $Z_2(\xi)$ *is needed for* $\xi = 15, 11, 7, 3$. *To find* $Z_2(\xi)$ *for any one of these arguments we need* $Z_1(\xi - 2x_2)$ *for all integral values of* x_2 *from* 0 *to* $\lfloor \xi/2 \rfloor$. *Thus* $Z_1(\xi)$ *will be needed only for the arguments* $\xi = 15, 13, 11, 9, 7, 5, 3, 1$.

x	$P(x;\mu_1=1)$	$P(x;\mu_2=5)$	$P(x;\mu_3=3)$
0	1.00000	1.00000	1.00000
1	0.63212	0.99326	0.95021
2	0.26424	0.95957	0.80085
3	0.08030	0.87534	0.57681
4	0.01899	0.73497	0.35277
5	0.00366	0.55951	0.18474
6	0.00059	0.38404	0.08392
7	0.00001	0.23781	0.03351

Table 4.1: Values of Poisson Distribution

The table below is the value of poisson distribution of random variable x *(i.e* $P(x;\mu_i)$ *where* $i=1,2,3$*).*

For $\xi=1$ *and* $\xi=3$ *, we have only* $x_1(\xi)=0$ *minimizer and* $Z_1(\xi=1)=5000[P(0,1)-0P(0,1)]=5000$ *is minimum value*

For $\xi=5$ *and* $\xi=7$*, we have the values* $x_1(\xi)=0,1$*. then the value of* $Z_1(\xi)$ *at* $x_1=0$ *already calculated on the above, but* $Z_1(\xi)$ *at* $x_1=1$ *is* $1,839.4$*. Then* $x_1=1$ *is the minimizer and the minimum values of* $Z_1(\xi=3)=1,839.4$*. We can continue with the same manner for For* $\xi=9$, $\xi=11$, $\xi=13$ *and* $\xi=15$*. Let us summarize by the following table.*

STAGE-1:

states(ξ)	$x_1=\frac{\xi}{5}$				$Z_1(\xi)$	$\hat{x}_1$
	0	1	2	3		
1	5000	-	-	-	5000	0
3	5000	-	-	-	5000	0
5	5000	1839.4	-	-	1839.4	1
7	5000	1839.4	-	-	1839.4	1
9	5000	1839.4	-	-	1839.4	1
11	5000	1839.4	518.2	-	518.2	2
13	5000	1839.4	518.2	-	518.2	2
15	5000	1839.4	518.2	116.7	116.7	3

Table 4.2: Values of $Z_1(\xi)$ and $\hat{x}_1$

To compute $Z_2(\xi)$ we use the recurrence relation
Given $\mu_2 = 5$ and $\pi_2 = 2,000$

$$Z_2(\xi) = \min_{0 \leq x_2 \leq \frac{\xi}{2}} \{\$2000[5P(x_2 - 1; 5) - x_2 P(x_2; 5)] + Z_1(\xi - 2x_2)\} \quad (4.16)$$

For $\xi = 3$, x_2 can be 0 or 1, and to determine which x_2 value is optimal, we calculate the value in (4.16) for $x_2 = 0$ and $x_2 = 1$, and take the smallest value. for $x_2 = 0$, $Z_2(\xi = 3) = 10,000 + Z_1(3) = 10,000 + 5,000 = 15,000$ for $x_2 = 1$, $Z_2(\xi = 3) = 13,013.48$.

For $\xi = 7$, x_2 can be 0 , 1 , 2 or 3. Therefore for $x_2 = 0$, $Z_2(\xi = 3)$=10000+$Z_1(7)$=10000+1839.4=11839.4 and for $x_2 = 1$, $Z_2(\xi = 3) = 8013.48 + Z_1(5) = 9853.18$, with the same procedure we can compute for $x_2 = 2$ and $x_2 = 3$. Let us summarize with the table below.

STAGE-2:

states(ξ)	The value of g at $x_2 = \frac{\xi}{2}$								$Z_2(\xi)$	$\hat{x}_2$
	0	1	2	3	4	5	6	7		
3	15,000	13,013.48	-	-	-	-	-	-	13,013.48	1
7	11,839.4	9,852.88	11,094.32	9343.66	-	-	-	-	9343.66	3
11	10,518.2	9,852.88	7,933.72	6,183.06	7,873.64	6,754.6	-	-	6,183.06	3
15	10,116.7	8.531.68	6,612.52	6183.06	4,713	3594	5986.61	5511.06	3,594	5

Table 4.3: Values of $Z_2(\xi)$ and $\hat{x}_2$

Let us compute the third stage by the following relation as follow

Given $\mu_3 = 3$ and $\pi_3 = 10,000$

$$Z_3(15) = \min_{0 \leq x_3 \leq \frac{\xi}{4}} \{\$10,000[3P(x_3 - 1; 3) - x_3 P(x_3; 3)] + Z_2(15 - 4x_3)\} \quad (4.17)$$

STAGE-3:

states(ξ)	The value of g at $x_3 = \frac{\xi}{4}$				$Z_3^*(\xi)$	x_3^*
	0	1	2	3		
15	33,594	7676.76	13,824.46	19,734.68	7676.76	1

Table 4.4: Values of $Z_3^*(\xi)$ and x_3^*

Since the last stage $\hat{x}_3(15) = x_3^* = 1$, *and* $g^* = Z_3(9) = 7676.76$. *then we only remains to determine* x_2^* *and* x_1^*.
We have from (4.15) *that* $x_2^* = \hat{x}_2(15 - 4x_3^*) = \hat{x}_2(11) = 3$. *Finally* $x_1^* = \hat{x}_1(15 - 2x_2^* - 4x_3^*) = \hat{x}_1(5) = 1$. *Thus 3 units of item 2 and 1 unit of items 1 and 3 should be stocked. The expected stockout cost is* 7676.76.

Chapter 5

Conclusion

In this paper I tried to point out some concepts of inventory problems with dynamic programming specifically probabilistic dynamic programming model. which means the state at the next stage is not completely determined by the state and policy decision at the current stage. Rather, there is a probability distribution for what the next state will be.

There are two cases I discussed to show the given problem. The first case is for single period and the second case is for multi-periods. For single period case I used a discrete probability distribution and for the second case I used Poisson distribution function with their means.

In this work continuous demand case not included some one who interested in this are can be develop this work and can be also consider different cases.

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