

ADDIS ABABA UNIVERSITY
COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES
DEPARTMENTS OF MATHEMATICS



“SOME APPLICATIONS OF ELLIPTIC FUNCTIONS AND ELLIPTIC INTEGRALS”

*A **thesis** submitted to the department of Mathematics of Addis Ababa University in partial fulfillment of the Masters of Science Degree in Mathematics.*

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This certifies that Ayana Mossies thesis, some applications of elliptic functions and elliptic integrals, which he prepared and submitted impartial fulfillment of the requirements for the master's degree in mathematics science, was compile in accordance with university regulations and satisfies recognized standards for originality and quality.

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Declaration

I, Ayana Mossie, hereby declare that this thesis titled “SOME APPLICATIONS OF ELLIPTIC FUNCTIONS AND ELLIPTIC INTEGRALS” is my own original work and has not been submitted for any other degree qualification at any other institution.

Name: Ayana Mossie

Signature: _____

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Abstract

One of the most important applications of elliptic integrals of the form $\int R(x, \sqrt{p_n(x)}) dx$ where R is rational algebraic function and $p_n(x)$ is polynomials of degree 3 or 4 with no repeated roots. Nowadays, incomplete and complete elliptic integrals of the first kind are estimated with high accuracy using advanced calculators. In this paper, several techniques are discussed to show how definite integrals of the form $\int R(x, \sqrt{p_n(x)}) dx$ can be converted to elliptic integrals of the first kind, and hence be estimated for optimal values. Indeed examples are provided in each step to help clarifications. Finally I have used eq.6 and eq.23 which is an important equations to show my analysis of ascending and descending transformation by taking initial values of $F(\emptyset, k_0)$ to provide a 30 decimal place values of numerical evaluations.

KEYWORDS: *Elliptic Functions, Elliptic Integral, Landen's Transformation, First kind.*

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CHAPTER ONE

1. Introduction and Preliminaries

The integrals of the form $\int R(x, \sqrt{p_n(x)}) dx$, where R is a rational function and $p(x)$ is a polynomial of third or fourth degree without repeat roots, are known as elliptic integrals. Elliptic functions are those that can be produced by inverting elliptic integrals. Elliptic curves are curves that need elliptic functions in order to parameterize. In fact, the first approximation of a planet's orbital length around the sun is represented by the length of an ellipse's arc. This thesis will focus on first-kind of elliptic integrals and Landen's transformation of elliptic functions.

Elliptic function and elliptic integrals have captivated mathematicians for centuries due to their rich structure and wide-ranging applications. At the heart of this fascinating field lie significant contributions from historical figures such as John Landen, Giovanni Fagnano, Adrien-Marie Legendre and Carl Gustav Jacob Jacobi. Each of mathematicians expanded boundaries of mathematical knowledge, laying the ground work for modern applications elliptic functions and integrals.

1.1. Historical context and contribution

Giovanni Fagnano (1698-1776)

Giovanni Fagnano, an Italian mathematician, made pivotal contribution to the theory of elliptic integral in the 19th century. His work primarily involved in the developments of the theory of elliptic integrals, which are integrals of functions involving square roots of polynomials of the third or the fourth degrees without repeat roots.

Fagnano's investigation led to the discovery of elliptic integrals, Fundamental properties and their connection with elliptic functions. His work provided a foundation for subsequent advancements in the field, influencing the mathematical understandings of elliptic integrals and their applications.

Adrien-Marie Legendre (1752-1833)

Adrien-Marie Legendre made a crucial contribution to the theory of elliptic integrals and functions. His work on elliptic integral was instrumental in developing the theory of elliptic functions. Legendre introduced the concept of the complete elliptic integral and provided comprehensive tables of these integrals. He also developed the theory of elliptic integrals in terms the modulus and amplitude, which are fundamental to the modern theory of elliptic function. His work laid the ground work for the subsequent development of elliptic functions and their applications.

Carl Gustav Jacob Jacobi (1804-1851)

Carl Gustav Jacob Jacobi, a German, mathematician, made ground breaking advancements in the theory of elliptic functions in the 19th century. Jacobi's work led to the developments of the theory of elliptic functions complex variables his formulation of the Jacobi theta functions and elliptic modular functions revolutionized the study of this mathematical object. Jacobi's contribution provided a comprehensive and systematic approach to elliptic functions, which became a cornerstone of modern mathematical analysis and profound implications for various branches of mathematics.

John Landen (1719-1790)

John Landen, an English mathematician, is renowned for his work on elliptic integrals and for introducing the Landen transformation also known as Land en's first kind. This transformation is a technique for simplifying the evaluations of elliptic integrals, particularly the complete elliptic integrals of the first and the second kind. Landen's transformation is particularly notable for its role in reducing the computations of elliptic integrals and transformations provided essential tools for subsequent mathematicians and scientists, aiding in the developments of the field.

The historical developments by Fagnano, Legendre, Jacobi and Landen have profound implications for modern mathematics and its applications. Elliptic integrals and functions have found applications in various fields, including number theory, algebraic geometry and mathematical physics. For example, Landen's transformation particularly useful in simplifying problems related to the calculations of elliptic integrals, which arise in the analysis of periodic phenomena, quantum, mechanics and general relativity.

In general, the historical contributions of Fagnano, Legendre, Jacobi and Landen have shaped the developments of elliptic functions and integral leading to their profound applications contemporary mathematics and science .The evaluations of these mathematical tools reflects a rich tradition of inquiry and innovation, providing a robust framework for solving complex problems and advancing our understanding of the mathematical universe.

Definition: Elliptic integrals are integrals of the form

$$F(\emptyset, k) = \int_0^{\emptyset} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}}, \quad 0 < k < 1 \quad (1)$$

where \emptyset is the upper limits (amplitude) of integration and k is a parameter known as the modulus.

1.2. Standard forms of elliptic integrals.

a) **Elliptic integrals of the first kind (F)**

$$F(\emptyset, k) = \int_0^{\emptyset} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}}, \quad 0 \leq k^2 \leq 1 \text{ and } 0 \leq \emptyset \leq \frac{\pi}{2}$$

b) **Elliptic integrals of the second kind (E)**

$$E(\emptyset, k) = \int_0^{\emptyset} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta, \quad 0 \leq k^2 \leq 1 \text{ and } 0 \leq \emptyset \leq \frac{\pi}{2}$$

c) **Elliptic integrals of the third kind(Π)**

$$\Pi(\emptyset, n, k) = \int_0^{\emptyset} \frac{d\theta}{(1+n \sin^2 \theta) \sqrt{(1-k^2 \sin^2 \theta)}}, \quad 0 \leq k^2 \leq 1 \text{ and } 0 \leq \emptyset \leq \frac{\pi}{2}.$$

where n is an additional parameter (characteristic) in the third kind.

Modulus and complimentary modulus:

The parameter k is known as modulus and its complimentary modulus is defined $k' = \sqrt{1 - k^2}$.

$$0 < k^2 < 1$$

Definition: An **elliptic function** is a complex function that is doubly periodic, meaning it has two independent periods. It is a generalization of trigonometric functions, which are singly periodic.

Formally, an elliptic function is a meromorphic function $f(z)$ of a complex variable z that satisfies the following conditions:

1. **Doubly Periodic:** There exist two distinct complex numbers ω_1 and ω_2 (called the fundamental periods) such that: $f(z+\omega_1)=f(z)f(z+\omega_2) = f(z)$ for all z in the complex plane.
2. **Meromorphic:** The function $f(z)$ has only isolated poles and no essential singularities. This means that around any point where $f(z)$ has a pole, it behaves like a rational function.

Elliptic functions are important in many areas of mathematics and physics, such as in the study of complex analysis, algebraic geometry, and theoretical physics. They play a significant role in describing the behavior of certain integrals and in the study of curves and surfaces.

1.3. Landen's Transformations

Landen's Transformations are identities that relate elliptic Integrals with different modulo or parameters. They simplify the computation of elliptic Integrals and are useful in various applications.

Landen's transformations provide a way to express elliptic integrals in terms of other elliptic integrals with different moduli or parameters. Here are the primary Landen transformations:

Transformation for the Elliptic Integral of the First Kind

One of Landen's transformations of the first kind involves the following relation:

$$F(\phi, k) = \frac{2}{1+\sqrt{1-k^2}} \int_0^{\phi'} \frac{d\theta}{\sqrt{(1-\sin^2\theta)\left(1-\frac{k^2}{(1+\sqrt{1-k^2})^2}\sin^2\theta\right)}} \quad \text{where } \phi' \text{ is related } \phi \text{ and through}$$

$$\phi' = 2\arctan\left(\sqrt{\frac{1-k^2}{1+\sqrt{1-k^2}}}\tan\frac{\phi}{2}\right).$$

Purpose and application:

Elliptic functions are important in many areas of mathematics and physics, such as in the study of complex analysis, algebraic geometry, and theoretical physics. They play a significant role in describing the behavior of certain integrals and in the study of curves and surfaces.

Landen's transformations are used to simplify elliptic integrals by converting them in to integrals with different parameters, often reducing complex expressions to more manageable forms. They are specifically useful in numerical computations and in simplifying the expressions for elliptic integrals in various mathematical and physical applications. And with $F(\psi, q)$ gave the connecting equations between the parameters ϕ, ψ and k, q

$$\sin(2\Psi - \phi) = k \sin \phi, \quad q = \frac{2\sqrt{k}}{1+k} \quad (2)$$

and stated the general relationship

$$F(\phi, k) = \frac{2}{1+k} F(\psi, q) \quad (3)$$

Equations (eq.2) are now known as Land en's ascending transformation and repeated applications of (eq.2) and (eq.3) allow iterative schemes for evaluation of such integrals. We explain these schemes in the following sections.

Legendre (1825) continued his development with another similar function that he called the elliptic integral of the Second Kind; in modern notation

$$E(\phi, k) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \theta} \, d\theta \quad (4)$$

and with $E(\psi, q)$ developed the relationship

$$E(\phi, k) + k \sin \phi = (1+k) E(\psi, q) + (1-k) F(\psi, q) \quad (5)$$

and concluded that this verified Land en's remarkable result for an arc of an hyperbola in terms of two elliptic arcs (Legendre 1825). Legendre did much work on elliptic functions and the classification of elliptic integrals and we adopt his notation in the following sections.

CHAPTER TWO

2. Jacobi elliptic Function

In terms of singularities, the simplest elliptic functions are those of order 2. In the previous section, we have seen that the order of an elliptic function is never less than 2. so a single irreducible double pole in each cell at which the residue is zero, and those which have two simple poles in each cell at which the two residues are equal in absolute value, but of opposite sign. The Jacobi elliptic functions are examples of the latter class.

2.1 MOTIVATION

Suppose we have the two integrals

$$u = \int_0^x \frac{dt}{\sqrt{1-t^2}} \quad (A)$$

$$\frac{\pi}{2} = \int_0^1 \frac{dt}{\sqrt{1-t^2}} \quad \text{where } -1 < x < 1 \text{ is real.}$$

$$u = \sin^{-1}x.$$

We can define a second function $\cos u$ by

$$\cos u = \sqrt{1 - \sin^2 u}.$$

By taking the square root positive for u between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, we have u . If we take the square root to be positive for u between zero and π , then this defines u as an odd function of x . By inversion of the integral, we have defined z as an odd function of u . If denote this function by $\sin u$, then (eq. (A)) reduces to the form as an even function of x . It follows that we have the identity

$$\sin^2 u + \cos^2 u = 1 \quad (B)$$

We can also note that $\sin 0 = 0$ and $\cos 0 = 1$.

Suppose the derivative of (A) with respect to x , which is clearly $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$.

It follows that: $\frac{d}{du} [\sin u] = \sqrt{1 - \sin^2 u} = \cos u$, as $x = \sin u$,

Moreover, by differentiation of (B), we obtain

$$\frac{d}{du} [\cos u] = -\sin u.$$

Now, we consider the equation

$$\omega = \sin u_1 \cos u_2 + \cos u_1 \sin u_2.$$

The partial derivatives of ω with respect to u_1 and u_2 are equal, so it follows that $\omega = f(u_1 + u_2)$, where $f(u_1 + u_2)$ is a function of $u_1 + u_2$. Therefore, $f(u_1 + u_2) = \sin u_1 \cos u_2 + \cos u_1 \sin u_2$.

If we set $u_2 = 0$, then $f(u_1) = \sin u_1$, and similarly $f(u_2) = \sin u_2$. Hence, $f(u_1 + u_2) = \sin(u_1 + u_2)$ and we obtain an addition formula $\sin(u_1 + u_2) = \sin u_1 \cos u_2 + \cos u_1 \sin u_2$.

By (eq. (B)), we also have $\cos(u_1 + u_2) = \cos u_1 \cos u_2 - \sin u_1 \sin u_2$. We can also use these two addition formula to see that both $\sin u$ and $\cos u$ are simply periodic functions with period 2π .

Definitions of the Jacobi elliptic functions

The Jacobi elliptic function $\text{sn}(u, k)$ is defined by means of the integral,

$$u = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad 0 < k < 1 \tag{C}$$

for some constant k . Therefore, by inversion of the integral,

we have $x = \text{sn}(u, k)$. It is clear that $\text{sn}0 = 0$. The functions $\text{cn}(u, k)$ and $\text{dn}(u, k)$ are defined by the identities:

$$\text{sn}^2 u + \text{cn}^2 u = 1 \tag{D}$$

$$k^2 \text{sn}^2 u + \text{dn}^2 u = 1. \tag{E}$$

It follows that $\text{cn}0 = 1 = \text{dn}0$.

Each of the Jacobi elliptic functions depend on a parameter k , called the modulus. We also have the complementary modulus k' defined by $k^2 + k'^2 = 1$.

When emphasizing a particular modulus, we write the three functions as $\text{sn}(u, k)$, $\text{cn}(u, k)$ and $\text{dn}(u, k)$. An alternative notation uses a parameter $m = k^2$.

2.2. Properties of Jacobi elliptic Function

Definitions and property: The Jacobi elliptic function are defined through the elliptic integral of the first kind: $u = \text{am}(u, k)$ where $\text{am}(u, k)$ is the amplitude function.

The most commonly, used elliptic functions are the Jacobi elliptic functions. They are denoted as $\text{sn}(u, k), \text{cn}(u, k),$ and $\text{dn}(u, k)$ where k is the modulus.

1. $\text{sn}(u, k)$ = the sign amplitude function .
2. $\text{cn}(u, k)$ = the cosine amplitude function.
3. $\text{dn}(u, k)$ = the delta amplitude function.

These functions satisfy the following relation:

$$\text{sn}^2(u, k) + \text{cn}^2(u, k) = 1 \quad \text{and} \quad \text{dn}^2(u, k) + k^2 \text{sn}^2(u, k) = 1$$

2.3. Properties

- Periodicity: Elliptic functions repeat their values in the complex plane with respect to their two fundamental periods.
- Meromorphic Nature: They have a Laurent series expansion around their poles, and their poles are isolated.

THEOREM 2.1 The function $\text{sn}(u, k)$ is an odd function of u , while $\text{cn}(u, k)$ and $\text{dn}(u, k)$ are even functions of u .

Proof: For odd and even properties

1. $\text{sn}(u, k)$ as an odd function : To show that $\text{sn}(u, k)$ is an odd function we need to prove that

$$\text{sn}(u, k) = -\text{sn}(-u, k).$$

Consider the Jacobi elliptic function definition and the corresponding relations. The function $\text{sn}(u, k)$ can be divided from the elliptic integral and is related to the sin of the amplitude function. The amplitude function $\text{am}(u, k)$ satisfy the property :

$$\text{am}(-u, k) = -\text{am}(u, k) . \text{Thus , } \text{sn}(-u, k) = \text{sn}(-\text{am}(u, k)) = -\text{sn}(\text{am}(u, k)) = -\text{sn}(u, k).$$

Therefore, $\text{sn}(u, k)$ is indeed an odd function.

2. $\text{cn}(u, k)$ as an even function: To show that $\text{cn}(u, k)$ is an even function. we

Need to prove that $\text{cn}(-u, k) = \text{cn}(u, k)$: Using the definition and the relationship between the elliptic integral and the amplitude function: $\text{am}(-u, k) = -\text{am}(u, k)$.

$$\text{cn}(-u, k) = \cos(\text{am}(-u, k)) = \cos(-\text{am}(u, k)) = \cos(\text{am}(u, k)) = \text{cn}(u, k).$$

Thus, $\text{cn}(u, k)$ is an even function .

3. $\text{Dn}(u, k)$ as even function: To show that $\text{dn}(u, k)$ is an even function.

We need to prove that: $\text{dn}(-u, k) = \text{dn}(u, k)$.

Similarly, using the amplitude function $\text{am}(-u, k) = -\text{am}(u, k)$

$\text{dn}(-u, k) = \delta(\text{am}(-u, k)) = \delta(-\text{am}(u, k))$: Since the delta amplitude function is even in its argument:

$$\delta(-\text{am}(u, k)) = \delta(\text{am}(u, k)) = \text{dn}(u, k).$$

Thus, $\text{dn}(u, k)$ is an even function.

THEOREM 2.2. The derivatives of the Jacobi elliptic functions are

$$\frac{d}{du} \{ \text{sn}u \} = \text{cn}u \text{dnu}, \quad \frac{d}{du} \{ \text{cn}u \} = -\text{sn}u \text{dnu}, \quad \text{and} \quad \frac{d}{du} \{ \text{dn}u \} = -k^2 \text{sn}u \text{cnu}.$$

Proof:

Deriving the differential equations

1. Derivatives of $\text{sn}(u, k)$. By definition, the Jacobi elliptic function $\text{sn}(u, k)$ and $\text{cn}(u, k)$ are related in terms of u by $\text{sn}^2(u, k) + \text{cn}^2(u, k) = 1$. Differentiating both sides with respect to u ,

$$\frac{d}{du} (\text{sn}^2(u, k) + \text{cn}^2(u, k)) = \frac{d}{du} (1).$$

Applying the chain rule:

$$2\text{sn}(u, k) \frac{d}{du} \text{sn}(u, k) + 2\text{cn}(u, k) \frac{d}{du} \text{cn}(u, k) = 0 .$$

Simplify to:

$$\operatorname{sn}(u, k) \frac{d}{du} \operatorname{sn}(u, k) + \operatorname{cn}(u, k) \frac{d}{du} \operatorname{cn}(u, k) = 0$$

form the known differential equation for $\operatorname{cn}(u, k)$: $\frac{d}{du} \operatorname{cn}(u, k) = -\operatorname{sn}(u, k) \operatorname{dn}(u, k)$.

Substitute the equation $\operatorname{sn}(u, k) \frac{d}{du} \operatorname{sn}(u, k) - \operatorname{cn}(u, k) \operatorname{sn}(u, k) \operatorname{dn}(u, k) = 0$.

Rearranging we get the required result: $\frac{d}{du} \operatorname{sn}(u, k) = \operatorname{cn}(u, k) \operatorname{dn}(u, k)$.

2. Derivatives of $\operatorname{cn}(u, k)$. From the definition of $\operatorname{cn}(u, k)$ and $\operatorname{dn}(u, k)$,

$$\operatorname{sn}^2(u, k) + \operatorname{cn}^2(u, k) = 1.$$

Differentiate both sides with respect to u , $\frac{d}{du} (\operatorname{sn}^2(u, k) + \operatorname{cn}^2(u, k)) = \frac{d}{du} (1)$.

$2\operatorname{sn}(u, k) \frac{d}{du} \operatorname{sn}(u, k) + 2\operatorname{cn}(u, k) \frac{d}{du} \operatorname{cn}(u, k) = 0$: Using $\frac{d}{du} \operatorname{sn}(u, k) = \operatorname{cn}(u, k) \operatorname{dn}(u, k)$ substitution in to the equation: We get $\operatorname{cn}(u, k) \frac{d}{du} \operatorname{cn}(u, k) + \operatorname{sn}(u, k) \operatorname{dn}(u, k) \operatorname{cn}(u, k) = 0$.

Rearranging we get the required result, $\frac{d}{du} \operatorname{cn}(u, k) = -\operatorname{sn}(u, k) \operatorname{dn}(u, k)$.

3. Derivatives of $\operatorname{dn}(u, k)$. The known relationship between $\operatorname{dn}(u, k)$ and $\operatorname{sn}(u, k)$ is

$\operatorname{dn}^2(u, k) = 1 - k^2 \operatorname{sn}^2(u, k)$ and differentiate with respect to u . We have

$2\operatorname{dn}(u, k) \frac{d}{du} \operatorname{dn}(u, k) = -2k^2 \operatorname{sn}(u, k) \frac{d}{du} \operatorname{sn}(u, k)$. Substitute $\frac{d}{du} \operatorname{sn}(u, k) = \operatorname{cn}(u, k) \operatorname{dn}(u, k)$ in to the derivative, we get $2\operatorname{dn}(u, k) \frac{d}{du} \operatorname{dn}(u, k) = -2k^2 \operatorname{sn}(u, k) \operatorname{cn}(u, k) \operatorname{dn}(u, k)$.

This simplified as, $\frac{d}{du} \operatorname{dn}(u, k) = -k^2 \operatorname{sn}(u, k) \operatorname{cn}(u, k)$.

CHAPTER THREE

3. ELLIPTIC INTEGRAL

3.1. Definition of Elliptic Integral

Definition 3.1.1. Any integral of a rational function x and $\sqrt{p(x)}$ where $p(x)$ is a cubic or quadratic polynomial with distinct roots is known as an elliptic integral. Such an integral can be written in terms of Legendre's (third kind) of elliptic integrals. These can be more conveniently expressed in terms of Jacobi elliptic functions. Elliptic integrals first arose from the attempts to find the perimeter of an ellipse:

3.2. LENGTHS OF CURVES

The ancient Greeks estimated the circumference of a circle by inscribing a polygon of n sides and then computing the perimeter of the polygon. They surmised that the larger n was, the better the perimeter of the polygon approximated the actual circumference of the circle. We will use this basic idea to define and compute the length of many curves. Let f have a continuous derivative on $[a, b]$. If f is linear, that is, if the graph of f is a line segment then the length L of the graph is the distance between $(a, f(a))$ and $(b, f(b))$, so that

$$L = \sqrt{(b - a)^2 + [f(b) - f(a)]^2} \quad (\text{see fig a below})$$

when f is not necessarily linear, we let $L = \{x_0, x_1, \dots, x_n\}$ be any partition of $[a, b]$, and approximate the graph of f by a polygonal line L whose vertices are $(x_0, f(x_0))$, $(x_1, f(x_1))$, \dots , $(x_n, f(x_n))$ (see fig b below). Let ΔL_k be the length of the portion of the graph of f joining $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$. If Δx_k is small, ΔL_k is approximately equal to the length of the line segment joining $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$. In other words

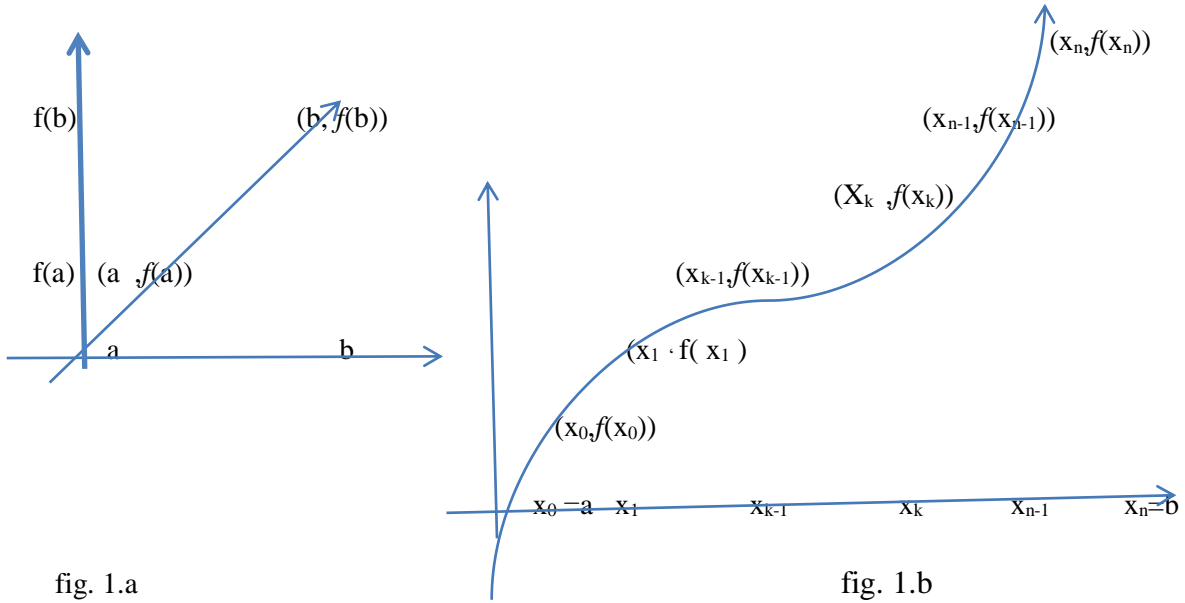
$$\Delta L_k \approx \sqrt{(x_k - x_{k-1})^2 + [f(x_k) - f(x_{k-1})]^2} \quad (\text{F})$$

The Mean Value Theorem, applied to f on the interval $[x_{k-1}, x_k]$, implies that

$$(f(x_k) - f(x_{k-1})) = f'(t_k) (x_k - x_{k-1}) \quad \text{for some } t_k \text{ in } (x_{k-1}, x_k).$$

Therefore (F) can be written as follows: $\Delta L_k \approx \sqrt{(x_k - x_{k-1})^2 + [f'(t_k)(x_k - x_{k-1})]^2}$

$$= \sqrt{(1 + [f'(t_k)]^2)(x_k - x_{k-1})^2} = \sqrt{(1 + [f'(t_k)]^2)} \Delta x_k$$



Therefore, the total length L of the graph of f , which is the sum of the lengths $\Delta L_1, \Delta L_2, \dots, \Delta L_n$ should be approximately

$$\sum_{k=1}^n \sqrt{1 + [f'(t_k)]^2} \Delta x_k$$

itself a Riemann sum for $\sqrt{1 + (f')^2}$ on $[a, b]$. Therefore it seems plausible that

$$L = \lim_{\|\phi\| \rightarrow 0} \sum_{k=1}^n \sqrt{1 + [f'(t_k)]^2} \Delta x_k \text{ Thus we get,}$$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

This leads us to make the following definition.

Definition 3.1.2: Let f have a continuous derivative on $[a, b]$. Then the length L of the graph of f on $[a, b]$ is given by $L = \int_a^b \sqrt{1 + (f'(x))^2} dx$.

Example: Let $f(x) = x^{\frac{3}{2}} + 2$ for $0 \leq x \leq \frac{4}{3}$. Find the length L of the graph of f in the given interval.

Solution: since f has a continuous derivative on $[0, \frac{4}{3}]$ which implies $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$. Thus from the definition:

$$L = \int_0^{\frac{4}{3}} \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2} dx = \int_0^{\frac{4}{3}} \sqrt{1 + \frac{9}{4}x} dx$$

To evaluate the integral, we let $u = 1 + \frac{9}{4}x$ so that $du = \frac{9}{4}dx$ if $x=0$, then $u=1$ and if $x=\frac{4}{3}$, then $u=4$.

$$\text{Therefore, } L = \int_0^{\frac{4}{3}} \sqrt{1 + \frac{9}{4}x} dx = \int_1^4 \sqrt{u} \frac{9}{4} du = \frac{4}{9} \left(\frac{2}{3}u^{\frac{3}{2}}\right) \Big|_1^4 = \frac{56}{27} \text{ units.}$$

Consider the equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Let $x = a \cos \theta$, $y = b \sin \theta$

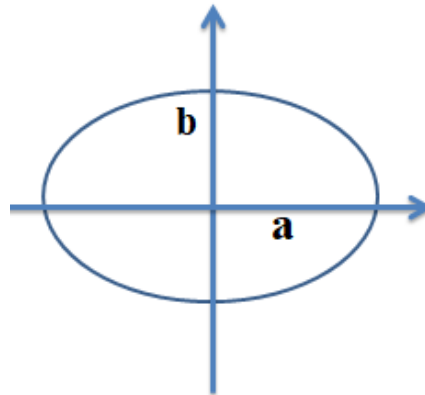


Figure 2. Graphical Illustration Ellipse

The equation of length of the curve in terms of polar coordinate is given by:

$$L = \int ds = 2 \int_{-a}^a \sqrt{dx^2 + dy^2} = 2 \int_{-a}^a \sqrt{1 + y'^2} dx = 2 \int_0^{2\pi} \sqrt{1 + b^2 \cos^2 \theta} (-a \sin \theta) d\theta .$$

$$L = 2 \int_0^{2\pi} \sqrt{1 + b^2 \cos^2 \theta} (-a \sin \theta) d\theta .$$

3.3. LANDEN TRANSFORMATION

Landen's transformation, also known as Landen's theorem or Landen's identity, is a useful mathematical tool in the study of elliptic integrals. It provides a way to transform elliptic integrals into simple forms and is particularly valuable in the contexts of numerical computation and analysis of elliptic integrals.

Landen's transformation is a technique to simplify the evaluation of elliptic integrals by relating them to integrals with different parameters. Specifically it allows us to express an elliptic integral of one modulus in terms of an elliptic integral of another modulus.

Recall that elliptic integrals are integrals of the form:

$F(\varnothing, k) = \int_0^{\varnothing} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}}$ where \varnothing is the amplitude, k is the elliptic modulus and $F(\varnothing, k)$ the incomplete elliptic integral of the first kind. The complete elliptic integral of the first kind obtained by setting \varnothing to $\frac{\pi}{2}$.

The three types of elliptic integrals are: first kind $F(\varnothing, k)$, second kind $E(\varnothing, k)$ and third kind $\Pi(n, \varnothing, k)$.

Landen's Transformation provides a formula that relates elliptic integrals with different modulus. For elliptic integrals of the first kind, the transformation can be expressed as:

$F(\varnothing, k') = \frac{2}{\sqrt{1+k^2}} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k'^2 \sin^2 \theta}}$ where k' the transformed modulus and is related to k by:

$K' = \sqrt{\frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}}$. This formula simplifies the evaluations of elliptic integrals by transforming them in to integrals with different modulus, which can sometimes be easier to compute or analyses.

Applications:

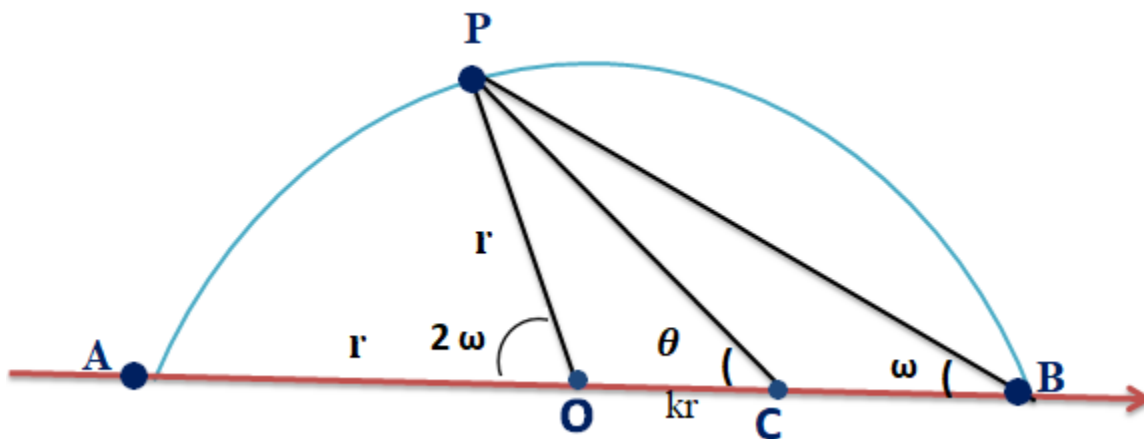
Numerical computations: Landen's transformation is particularly useful in numerical computation where it helps to reduce the computational complexity by transforming elliptic integrals in to forms that are more manageable.

Analytical work: it allows mathematicians to work with simpler expression or to drive asymptotic expressions and approximations for elliptic integrals.

Special cases and series expansions: The transformation can also be used to drive series expansions and special case of elliptic integrals by transforming to simpler integrals. In general Landen’s transformation is a powerful technique in the theory of elliptic integrals .It simplifies the evaluation and manipulation of these integrals by transforming them in to forms with different parameters. The transformation formula is particularly valuable for reducing the complexity of elliptic integrals in practical computations and theoretical analyses.

3.4. A Method of Evaluating Elliptic Integrals of the First Kind.

1. Let’s discuss about the technique



Landen’s transformation may be developed by considering the geometric relationships between two angles θ and ω and a parameter k ($0 \leq k \leq 1$). These relationships are shown in **Figure 3**.

Figure 3: Geometrical representation of half circle

- (i) AB is a diameter of a circle of radius r and center O , $AO= OB= OP= r$;
- (ii) C is a point on OB such that $OC = kr$ with $0 \leq k \leq 1$;

(iii) $\angle PBC = \omega$; $\angle PCO = \theta$; $\angle POA = 2\omega$ (property of chord AP and circumferential angle ω and central angle 2ω); $\angle OPC = 2\omega - \theta$ and $\omega \leq \theta$ because $\theta = \omega + \angle CPB \geq \omega$. Using the sine rule in triangle OPC, we obtain :

$$\sin (2\omega - \theta) = k \sin \theta \quad (6)$$

We now proceed with a sequence of manipulations to firstly obtain a useful differential relationship linking $d\theta$ and $d\omega$ [see (eq.7)] and then simplifying this equation into one where the functions on the left-hand and the right-hand-sides are similar in form and their variables linked by simple relationships [see (eq.18) and (eq.22)].

Let $y = 2\omega - \theta$ in (eq.6) then, using the chain rule, differentiate with respect to θ ;

giving $\cos y \, dy = k \cos \theta \, d\theta$. With $dy = 2d\omega - d\theta$, we write:

$\cos(2\omega - \theta) (2d\omega - d\theta) = k \cos \theta \, d\theta$ And this can be re-arranged as:

$$\frac{d\theta}{\cos(2\omega - \theta)} = \frac{2d\omega}{\cos(2\omega - \theta) + k \cos\theta} \quad (7)$$

Squaring both sides of (eq.6), then subtracting the result from unity and using

$$\cos^2(2\omega - \theta) = 1 - \sin^2(2\omega - \theta)$$

we may write as $\cos(2\omega - \theta) = \sqrt{1 - k^2 \sin^2\theta}$ (8)

Expanding (eq.6) with aid of $\sin(A-B) = \sin A \cos B - \cos A \sin B$ gives

$$\sin 2\omega \cos \theta - \cos 2\omega \sin \theta = k \sin \theta \quad (9)$$

and dividing both sides of (eq.9) by $\cos \theta$ and re-arranging gives

$$\tan \theta = \frac{\sin 2\omega}{k + \cos 2\omega} \quad (10)$$

From (eq.10), one can have a right-angled triangle whose length of hypotenuse is

$$\sqrt{\sin^2 2\omega + (k + \cos 2\omega)^2} = \sqrt{1 + 2k \cos 2\omega + k^2} \text{ and}$$

$$\sin \theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{\sin 2\omega}{\sqrt{1+2k\cos 2\omega+k^2}} \quad (11)$$

$$\cos \theta = \frac{\text{adjacent}}{\text{Hypotenues}} = \frac{k+\cos 2\omega}{\sqrt{1+2k\cos 2\omega+k^2}} \quad (12)$$

and multiplying (eq.11) and (eq.12) by $\sin \theta$ and $\cos \theta$ respectively and re-arranging gives

$$\sin^2 \theta \sqrt{1+2k\cos 2\omega+k^2} = \sin \theta \sin 2\omega \quad (13)$$

$$\cos^2 \theta \sqrt{1+2k\cos 2\omega+k^2} = \cos \theta (k+\cos 2\omega). \quad (14)$$

Adding (eq.13) and (eq.14) we may write as

$$\sin^2 \theta + \cos^2 \theta (\sqrt{1+2k\cos 2\omega+k^2}) = \sin \theta \sin 2\omega + \cos \theta (k+\cos 2\omega)$$

$$\begin{aligned} \sqrt{1+2k\cos 2\omega+k^2} &= \sin \theta \sin 2\omega + \cos \theta (k+\cos 2\omega) \quad \text{since } (\sin^2 \theta + \cos^2 \theta = 1) \\ &= \cos 2\omega \cos \theta + \sin 2\omega \sin \theta + k \cos \theta = \cos (2\omega - \theta) + k \cos \theta \end{aligned}$$

$$\cos (2\omega - \theta) + k \cos \theta = \sqrt{1+2k\cos 2\omega+k^2} \quad (15)$$

Note: taking $1+2k\cos 2\omega+k^2 = 1+2k(1-2\sin^2 \omega) + k^2$ since $(\cos 2\omega = 1-2\sin^2 \omega)$

$$\begin{aligned} &= 1+2k+k^2-4k\sin^2 \omega \\ &= (1+k)^2 - 4k\sin^2 \omega \\ &= (1+k)^2 \left\{ 1 - \frac{4k}{(1+k)^2} \sin^2 \omega \right\} \end{aligned} \quad (16)$$

So using (eq.15) and (eq.16), we have

$$\cos (2\omega - \theta) + k \cos \theta = (1+k) \sqrt{1 - \frac{4k}{(1+k)^2} \sin^2 \omega} \quad (17)$$

and substituting (eq.8) and (eq.17) into (eq.7) gives (Röch 2011, eq. (9))

$$\frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}} = \frac{2}{1+k} \frac{d\omega}{\sqrt{1 - \frac{4k}{(1+k)^2} \sin^2 \omega}}. \quad (18)$$

Now, since $0 \leq k \leq 1$ (see Figure 1 and the explanation), we use the rules for inequalities in the following development:

$$1+k \leq 2, (1+k)^2 \leq 4, 1 \leq \frac{4}{(1+k)^2}, k \leq \frac{4k}{(1+k)^2}. \quad (19)$$

$$\text{Defining: } q = \frac{2\sqrt{k}}{1+k} \quad (20)$$

and so $k^2 \leq k \leq \frac{4k}{(1+k)^2} = q^2$, then taking positive square roots,

$$k \leq q. \quad (21)$$

Using (eq.20) and (eq.1), we may write (eq.18) as:

$$F(\emptyset, k) = \frac{2}{1+k} F(\psi, q) \quad (22)$$

where q and k are related by (eq. 20) with $q \geq k$ and from (eq.6) (in Figure 1 with $\psi \equiv \omega$ and $(\emptyset \equiv \theta)$), we have $\sin(2\psi - \emptyset) = k \sin \emptyset$, with $\psi \leq \emptyset$ since $w < \theta$. (23)

The left-hand-side of (eq.22) has been replaced by a similar function (with a multiplier $\frac{2}{1+k}$ with smaller amplitude ψ and a larger modulus q). We summaries Legendre's result as:

If $F(\emptyset, k) = \int_0^\emptyset \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}}$ and $\sin(2\psi - \emptyset) = k \sin \emptyset$, $q = \frac{2\sqrt{k}}{1+k}$ where $\psi < \emptyset$, $q \leq k$, then $F(\emptyset, k) = \frac{2}{1+k} F(\psi, q)$.

These results are now used to develop an iterative scheme to evaluate an elliptic integral of the first kind .

Evaluating Elliptic Integral of the First Kind: Defining $K = k_0$, $q = k_1$, $\emptyset = \emptyset_0$ and $\psi = \emptyset_1$ we may write (eq.22) as $F(\emptyset_0, k_0) = \frac{2}{1+k_0} F(\emptyset_1, k_1)$ (24)

And the right hand side of (eq.24) can be iterated as

$$\frac{2}{1+k_0} \cdot \frac{2}{1+k_1} F(\emptyset_2, k_2) = \frac{2}{1+k_0} \cdot \frac{2}{1+k_1} \cdot \frac{2}{1+k_2} F(\emptyset_3, k_3).$$

Giving the sequence (Röch 2011, eq. (14))

$$F(\emptyset_0, k_0) = \frac{2}{1+k_0} \cdot \frac{2}{1+k_1} \cdot \frac{2}{1+k_2} \cdots \frac{2}{1+k_{n-1}} \cdot F(\emptyset_n, k_n) \quad (25)$$

where the moduli $\{k_n\}$ and $\{\emptyset_n\}$ amplitude are obtained from the recurrence relationships

$$k_{n+1} = \frac{2\sqrt{kn}}{1+kn} \text{ with starting value } k_0 \text{ and } k_{n+1} \geq k_n \quad (26)$$

$$\sin(2\emptyset_{n+1} - \emptyset_n) = k_n \sin \emptyset_n \quad (27)$$

with starting values \emptyset_0, k_0 and $\emptyset_{n+1} < \emptyset_n$. Now equation (eq.26) and (eq.27) are Landen's Ascending Transformation.

Note that as $n \rightarrow \infty$ the sequence of moduli $\{k_n\}$ converges to unity and $0 \leq k_n \leq 1$. For $0 \leq \emptyset_n \leq \frac{\pi}{2}$, $\sin(2\emptyset_{n+1} - \emptyset_n) = k_n \sin \emptyset_n \leq \sin \emptyset_n$ and $2\emptyset_{n+1} \leq 2\emptyset_n$ so the sequence of amplitudes $\{\emptyset_n\}$ is monotonically decreasing, bounded below by zero, and hence convergent.

Let $\lim_{n \rightarrow \infty} \emptyset_n = \emptyset'$, then with the aid of the standard result : $\int \sec \theta d\theta = \ln \tan \left(\frac{\pi}{4} + \frac{\theta}{2} \right)$

we may write as follows:

$$\lim_{n \rightarrow \infty} F(\emptyset_n, k_n) = F(\emptyset, 1) = \int_0^{\emptyset'} \frac{d\theta}{\sqrt{1-\sin^2\theta}} = \int_0^{\emptyset'} \sec \theta d\theta = \ln \tan \left(\frac{\pi}{4} + \frac{\emptyset'}{2} \right). \quad (28)$$

Using this result in (eq.25) we may write a formula for the evaluation of the elliptic integral of the first kind as

$$F(\emptyset_0, k_0) = \frac{2}{1+k_0} \cdot \frac{2}{1+k_1} \cdot \frac{2}{1+k_2} \cdots \ln \tan \left(\frac{\pi}{4} + \frac{\emptyset'}{2} \right) \quad (29)$$

A more efficient numerical evaluation of $F(\emptyset_0, k_0)$ in (eq.29) can be made by first re-arranging the recurrence (eq.26) as $\frac{2}{1+k_n} = \frac{kn+1}{\sqrt{kn}} = \frac{\sqrt{kn+1}\sqrt{kn+1}}{\sqrt{kn}}$ and in the right-hand-side of (eq.29) we may write

$$\frac{2}{1+k_0} \cdot \frac{2}{1+k_1} \cdot \frac{2}{1+k_2} \cdots \frac{2}{1+k_{n-1}} = \frac{\sqrt{k_1}\sqrt{k_1}}{\sqrt{k_0}} \frac{\sqrt{k_2}\sqrt{k_2}}{\sqrt{k_1}} \frac{\sqrt{k_3}\sqrt{k_3}}{\sqrt{k_2}} \cdots \frac{\sqrt{k_n}\sqrt{k_n}}{\sqrt{k_{n-1}}} = \sqrt{kn} \sqrt{\frac{k_1 k_2 k_3 \dots k_n}{k_0}}$$

And with the approximations $\emptyset' \approx \emptyset_n$ and $kn \approx 1$ (eq.29) can be approximated as (King 1924, eq. (8))

$$F(\emptyset_0, k_0) \approx \sqrt{\frac{k_1 k_2 k_3 \dots k_n}{k_0}} \ln \tan \left(\frac{\pi}{4} + \frac{\emptyset_n}{2} \right). \quad (30)$$

Using (eq.30) with the recurrence relationships (eq.26) and (eq.27) elliptic integrals of the first kind can be easily evaluated.

But from (eq.28) we have seen that $\int_0^{\phi} \sec \theta d\theta = \ln|\tan(\frac{\pi}{4} + \frac{\phi}{2})|$. We need to justify this is true.

To show this, we will evaluate the integral and simplify the expression on the right- hand side. Here's the step by step process.

1. Evaluating the integral

Consider the integral $I = \int_0^{\phi} \sec \theta d\theta$ to solve this we use standard result for the integrals of $\sec \theta$.

The anti-derivatives of $\sec \theta$ is $\ln|\sec \theta + \tan \theta|$. Hence: $\int_0^{\phi} \sec \theta d\theta = \ln|\sec \theta + \tan \theta| + C$.

Applying the limiting of integration from 0 to ϕ : $\int_0^{\phi} \sec \theta d\theta = [\ln|\sec \theta + \tan \theta|]_0^{\phi}$

$I = \ln|\sec \phi + \tan \phi| - \ln|\sec 0 + \tan 0| = \ln|\sec \phi + \tan \phi| - \ln 1 = \ln|\sec \phi + \tan \phi|$.

Thus, $\int_0^{\phi} \sec \theta d\theta = \ln|\sec \phi + \tan \phi|$.

2. Simplifying the expression

Now, we need to show that this is equal to $\ln|\sec \frac{\pi}{4} + \tan \frac{\phi}{2}|$.

Using the tangent addition formula:

$$\tan(\sec \frac{\pi}{4} + \tan x) = \frac{\tan \frac{\pi}{4} + \tan x}{1 - \tan \frac{\pi}{4} \tan x} = \frac{1 + \tan x}{1 - \tan x}.$$

Now letting $x = \frac{\phi}{2}$ this becomes

$$\tan(\frac{\pi}{4} + \frac{\phi}{2}) = \frac{\tan \frac{\pi}{4} + \tan \frac{\phi}{2}}{1 - \tan \frac{\pi}{4} \tan \frac{\phi}{2}}.$$

We need to express $\sec \phi + \tan \phi$ in terms of this tangent function .Using the double angle

formulas, we have $\tan \phi = \frac{2 \tan \frac{\phi}{2}}{1 - \tan^2 \frac{\phi}{2}}$ and $\sec \phi = \frac{1}{\cos \phi} = \frac{1}{\frac{1 - \tan^2 \frac{\phi}{2}}{1 + \tan^2 \frac{\phi}{2}}} = \frac{1 + \tan^2 \frac{\phi}{2}}{1 - \tan^2 \frac{\phi}{2}}$

Thus $\sec \phi + \tan \phi = \frac{1 + \tan^2 \frac{\phi}{2}}{1 - \tan^2 \frac{\phi}{2}} + \frac{2 \tan \frac{\phi}{2}}{1 - \tan^2 \frac{\phi}{2}} = \frac{1 + \tan^2 \frac{\phi}{2} + 2 \tan \frac{\phi}{2}}{1 - \tan^2 \frac{\phi}{2}} = \frac{(1 + \tan \frac{\phi}{2})^2}{1 - \tan^2 \frac{\phi}{2}}$.

Recognizing that $\tan(\frac{\pi}{4} + \frac{\phi}{2}) = \frac{1 + \tan \frac{\phi}{2}}{1 - \tan \frac{\phi}{2}}$.

We have $\sec \phi + \tan \phi = \frac{(1 + \tan^2 \frac{\phi}{2})^2}{1 - \tan^2 \frac{\phi}{2}} = \frac{1 + \tan \frac{\phi}{2}}{1 - \tan \frac{\phi}{2}} \tan(\frac{\pi}{4} + \frac{\phi}{2})$.

Thus, $\int_0^{\phi} \sec \theta d\theta = \ln|\sec \phi + \tan \phi| = \ln|\tan(\sec \frac{\pi}{4} + \tan \frac{\phi}{2})|$.

This complete the proof.

Using (eq.30) with recurrence relationship of (eq.26) and (eq.27), elliptic integrals of the first kind can be easily evaluated.

For example, the following table shows the sequence{ k_n } and { \emptyset_n } for $n=0,1,2,3,4,5,6,7,\dots$

For the evaluation of $F(\emptyset_0, k_0) = (\frac{\pi}{4}, 0.06)$ using Landen's ascending transformation.

| n | k_n | \emptyset_n |
|-----|------------------------|-------------------------|
| 0 | 0.06000000000000000000 | 45.00000000000000000000 |
| 1 | 0.46216787599682605626 | 23.71579195058573760503 |
| 2 | 0.92989226520499230066 | 17.21423343023637555399 |
| 3 | 0.99933994656426327554 | 16.59400265959260161458 |
| 4 | 0.99999994550521772983 | 16.58836792092492579036 |
| 5 | 0.99999999999999962878 | 16.58836745586785435456 |
| 6 | 0.99999999999999999999 | 16.58836745586785118657 |
| 7 | 0.99999999999999999999 | 16.58836745586785118648 |
| 8 | 0.99999999999999999999 | 16.58836745586785118639 |

The above table shows Landen's ascending transformation Sequences { k_n } and { \emptyset_n } for $F(\frac{1}{4}\pi, 0.06)$. For $n = 8$ the numerical values (20 decimal places) for (eq.30) are

$$\sqrt{\frac{K1 K2 K3 K4 K5 K6 k7 K8}{k0}} = 2.67545211433621425550517698478813902497$$

$$\emptyset_8 = 16.58836745586785118639$$

$$\ln \tan(\frac{1}{4}\pi + \frac{1}{2}\emptyset_8) = 0.29365326112932410501900766299187882821$$

$$F(\frac{1}{4}\pi, 0.06) = 0.785655238370174616731525093786$$

A Method of Evaluating Elliptic Integrals of the First Kind

Consider a re-arrangement of (eq.22) as

$$F(\psi, q) = \frac{1}{2}(1+k) F(\emptyset, k) \tag{31}$$

where an elliptic integral on the right-hand side with a higher amplitude \emptyset and smaller modulus k replaces an elliptic integral on the left-hand side with a specified amplitude ψ and modulus q . The following can be used to obtain the relationships between ψ , q (known) and \emptyset , k (unknown):

(A) Expression for k given q

From (eq.20) write

$$1-q^2 = 1 - \frac{4k}{(1+k)^2} = \frac{(1+k)^2 - 4k}{(1+k)^2} = \frac{1-2k+k^2}{(1+k)^2} = \frac{(1-k)^2}{(1+k)^2}. \quad (32)$$

The modulus q and complementary modulus q' are linked by

$$q^2 + q'^2 = 1 \quad (33)$$

And using this relationship and (eq.22) we write

$$q' = \sqrt{1 - q^2} = \frac{1-k}{1+k}. \quad (34)$$

Re-arranging (eq.34) and solving for k gives

$$k = \frac{1-q'}{1+q'} = \frac{1-\sqrt{1-q^2}}{1+q}. \quad (35)$$

(B) Expression for \emptyset given ψ

Using (eq6) with $\emptyset = \theta$, $\psi = \omega$, and $\sin(2\omega - \theta) = -\sin(\theta - 2\omega) = -\sin((\emptyset - \psi) - \psi)$, we write

$$-\sin((\emptyset - \psi) - \psi) = k \sin((\emptyset - \psi) + \psi).$$

And using $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ and re-arranging gives

$$\begin{aligned} \sin \psi \cos(\emptyset - \psi) - k \cos \psi \sin(\emptyset - \psi) &= k \cos \psi \sin(\emptyset - \psi) + \cos \psi \sin(\emptyset - \psi) \\ (1-k) \sin \psi \cos(\emptyset - \psi) &= (1+k) \sin(\emptyset - \psi) \cos \psi \end{aligned}$$

And

$$\tan(\emptyset - \psi) = \frac{1-k}{1+k} \tan \psi. \quad (36)$$

Substituting (eq.34) into (eq.36) gives

$$\tan(\emptyset - \psi) = \sqrt{1 - q^2} \tan \psi \quad (37)$$

Defining $\psi = \emptyset_0$, $\emptyset = \emptyset_1$ and $q = k_0$, $k = k_1$ we write (31) as

$$F(\emptyset_0, k_0) = \frac{1}{2}(1+k_1)F(\emptyset_1, k_1) \quad (38)$$

where $K_1 < k_0$ and $\emptyset_1 > \emptyset_0$.

The right-hand-side of (eq.38) can be iterated as

$$\frac{1}{2}(1+k_1) \frac{1}{2}(1+k_2) F(\emptyset_2, k_2) = \frac{1}{2}(1+k_1) \frac{1}{2}(1+k_2) \frac{1}{2}(1+k_3) F(\emptyset_3, k_3)$$

giving the sequence (RÖch 2011, eq. (22))

$$F(\emptyset_0, k_0) = \frac{1}{2} (1+k_1) \frac{1}{2} (1+k_2) \frac{1}{2} (1+k_3) \dots \frac{1}{2} (1+k_n) \quad (39)$$

Where the module $\{k_n\}$ and amplitudes $\{\emptyset_n\}$ are obtained from the recurrence relationships with starting value k_0

$$k_{n+1} = \frac{1 - \sqrt{1 - kn^2}}{1 + \sqrt{1 - kn^2}} \quad (40)$$

$$\tan(\emptyset_{n+1} - \emptyset_n) = \sqrt{1 - kn^2} \tan \emptyset_n \quad (41)$$

with starting value k_0 and \emptyset_0 . Equations (eq.40) and (eq.41) are Landen's descending transformation.

Now

$$k_n = \frac{1 - \sqrt{1 - K^2 n_1}}{1 + \sqrt{1 - K^2 n_1}} \cdot \frac{1 + \sqrt{1 - K^2 n_1}}{1 + \sqrt{1 - K^2 n_1}} \cdot \frac{K^2 n_1}{(1 + \sqrt{1 - K^2 n_1})^2}$$

and by re-arrangement, we may write

$$k_{n-1} = \sqrt{kn} (1 + \sqrt{1 - k^2 n_1})$$

and using (eq.40) we have

$$\begin{aligned} k_n - k_{n-1} &= \frac{1 - \sqrt{1 - k^2 n_1}}{1 + \sqrt{1 - k^2 n_1}} - \sqrt{kn} (1 + \sqrt{1 - k^2 n_1}) = \frac{1 - \sqrt{1 - k^2 n_1} - \sqrt{kn} (1 + \sqrt{1 - k^2 n_1})^2}{1 + \sqrt{1 - k^2 n_1}} \\ &= \frac{1 - \sqrt{1 - k^2 n_1} - k_{n-1} (1 + \sqrt{1 - k^2 n_1})}{1 + \sqrt{1 - k^2 n_1}} \end{aligned} \quad (42)$$

The numerator of (eq.42) can be written as

$$1 - k_{n-1} - (1 + k_{n-1}) \sqrt{1 - k^2 n_1} = \sqrt{1 - k_{n-1}} \{ \sqrt{1 - k_{n-1}} - (1 + k_{n-1}) \sqrt{1 + k_{n-1}} \} < 0.$$

Since the term in braces will be less than zero and the denominator greater than zero; and using this result in (eq.42) allows us to write $k_n < k_{n-1}$ for all n . And $0 < k_n < 1$ for all n . Thus the sequence $\{k_n\}$ is monotonically decreasing, bounded below by zero, and hence convergent, and

$$\lim_{n \rightarrow \infty} k_n = 0.$$

Suppose $\lim_{n \rightarrow \infty} \emptyset_n = \acute{\emptyset}$, then $\lim_{n \rightarrow \infty} F(\emptyset_n, k_n) = F(\acute{\emptyset}, 0) = \int_0^{\acute{\emptyset}} \frac{d\theta}{1-0} = \int_0^{\acute{\emptyset}} d\theta = \acute{\emptyset}$, therefore Landen's descending transformation (39) becomes

$$F(\emptyset_0, k_0) = \acute{\emptyset} \prod_{n=1}^{\infty} \frac{1}{2} (1+k_n) \quad (43)$$

The sequences $\{k_n\}$ and $\{\emptyset_n\}$ are obtained from the recurrence relationships (eq.40) and (eq.41).

For the purposes of evaluation we might write (eq.43) as

Using Landen's ascending transformation

$$\tan \phi = \frac{\sin 2\phi_1}{k + \cos 2\phi_1} \text{ or } k \sin \phi = \sin(2\phi_1 - \phi)$$

$F(k, \phi) = \frac{2}{1+k} \int_0^{\phi} \frac{d\phi_1}{\sqrt{1-k^2 \sin^2 \phi_1}}$, where $k_1 = \frac{2\sqrt{k}}{1+k}$, $k < k_1 < 1$. By successive application of the

transformation, we obtain a sequence of moduli k_n , $n=1,2,3, \dots$, $k_i = \frac{2\sqrt{k_{i-1}}}{1+k_{i-1}}$,

$$\lim_{n \rightarrow \infty} k_n = 1 \text{ and } \lim_{n \rightarrow \infty} \phi_n = \phi$$

from which we can write

$$F(k_0, \phi_0) = \sqrt{\frac{k_1 k_2 k_3 \dots}{k_0}} \int_0^{\phi} \frac{d\theta}{\sqrt{1-\sin^2 \theta}} = \sqrt{\frac{k_1 k_2 k_3 \dots}{k_0}} \ln \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right).$$

If the transformation $r = \sin \theta$ is made in the Legendre forms we obtain the following integrals

with $x = \sin \phi$. $F_1(k, x) = \int_0^x \frac{dv}{\sqrt{(1-v^2)(1-k^2 v^2)}}$

$$E_1(k, x) = \int_0^x \sqrt{\frac{1-k^2 v^2}{1-v^2}} dv.$$

$$H_1(k, n, x) = \int_0^x \frac{dv}{\sqrt{(1+nv^2)(1-v^2)(1-k^2 v^2)}}.$$

are called Jacob's forms for the elliptic integrals of the first, the second and third kinds respectively.

Note:- These are complete integrals If $x=1$, the incomplete elliptic integrals of the first kind is

defined as: $U = F(k, \phi) = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}}$, $0 < k < 1$,

where ϕ is the amplitude of $F(k, \phi)$ or u , written $\phi = \text{am } u$, and k is the modulus, $k = \text{mod } u$. The integral is also Legendre's form for the elliptic integral of the first kind. If $\phi = \frac{\pi}{2}$, the integral is called the complete elliptic integral of the first kind, denoted by $K(k)$, or simply K .

The incomplete elliptic integral of the second kind is defined by $E(k, \phi) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \theta} d\theta$. $0 < k < 1$,

also called Legendre's form for the elliptic integral of the second kind. If $\emptyset = \frac{\pi}{2}$, the integral is called the complete elliptic integral of the second kind, denoted by $E(k)$, or simply E . This is the form that arises in the determination of the length of arc of an ellipse.

The incomplete elliptic integral of the third kind is defined by:

$$H(k, n, \emptyset) = \int_0^{\emptyset} \frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}, \quad 0 < k < 1, n \neq 0.$$

also called Legendre's form for the elliptic integral of the third kind.

The last integral cannot be evaluated by elementary function viz. trigonometric, exponential, or logarithmic functions.

3.5. Kinds of Elliptic integrals

Note: If $R(x, y)$ is a rational algebraic function of x and y i.e., the quotient of two polynomials in x and y then $\int R(x, y)dx$ can be evaluated in terms of the usual elementary function (algebraic, trigonometry, inverse trigonometry, exponential and logarithmic).

If $y = \sqrt{ax + b}$ or $y = \sqrt{ax^2 + bx + c}$ with a, b, c constant. If $y = p(x)$ with $p(x)$ a cubic or quadratic polynomial, the integral R can be evaluated in terms of elliptic integral of first second or third kinds, or for special cases in terms of elementary function. If p is a polynomial of degree greater than 4, then R may be evaluated with the aid of hyper-elliptic functions.

3.6. Elliptic integrals of first kind and examples

Now for this thesis we only consider elliptic integrals of first kind in their Legendre's forms.

Definition: 3.2.1. The incomplete elliptic integral of the first kind is defined as

$$F(k, \emptyset) = \int_0^{\emptyset} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \quad 0 < k^2 < 1$$

Note. If $k=0$, then obviously $F(k, \emptyset) = \emptyset$. The complete form is mostly denoted as:

$$K(k) = F\left(\frac{\pi}{2}, k\right) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

In the following tables we represent the amounts of some elliptic integrals of first kind for some real values of k and \emptyset . These amounts will be used to evaluate the given integral in the next Examples. Table 1 refers to incomplete elliptic integrals of first kind and Table 2 shows some amount of the complete ones.

| Entry | k | \emptyset | $F(k, \emptyset)$ |
|-------|----------------------|-----------------|-------------------|
| 1 | $\frac{2}{3}$ | $\frac{\pi}{6}$ | 0.5342136 |
| 2 | $\sqrt{\frac{5}{6}}$ | $\frac{\pi}{4}$ | 0.8606394 |
| 3 | $\frac{1}{\sqrt{2}}$ | $\frac{\pi}{4}$ | 0.8260038 |
| 4 | $\sqrt{\frac{5}{6}}$ | $\frac{\pi}{3}$ | 1.2421478 |
| 5 | $\frac{2}{3}$ | $\frac{\pi}{3}$ | 1.1295143 |

Table 1: Some amounts of incomplete elliptic integral of first kind.

Note: All values given in Table 1 are calculated using [1]

| Entry | K | $F(\frac{\pi}{2}, k) = K$ |
|-------|----------------------|---------------------------|
| 1 | $\sqrt{\frac{1}{2}}$ | 1.8540666 |
| 2 | $\frac{2}{3}$ | 1.809701 |
| 3 | $\sqrt{\frac{5}{6}}$ | 2.3406433 |

Table 2: Some amounts complex elliptic integrals of First kind.

Note: All values given in Table 2 are calculated using [2]

Example 1:- $\int_0^2 \frac{dx}{\sqrt{(4-x^2)(9-x^2)}}$. Let $x=2\sin\theta$, then integral will be :

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{9-4\sin^2\theta}} = \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{4}{9}\sin^2\theta}}.$$

By using Table 1 the result will be:

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{9-4\sin^2\theta}} = \frac{1}{3} F\left(\frac{2}{3}, \frac{\pi}{2}\right) = 0.6032.$$

Example 2:- $\int_0^1 \frac{dx}{\sqrt{(1+x^2)(1+2x^2)}}$. Let $x=\tan\theta$, then the integral becomes

$$\int_0^1 \frac{dx}{\sqrt{(1+x^2)(1+2x^2)}} = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta d\theta}{\sqrt{(1+\tan^2\theta)(1+2\tan^2\theta)}} = \int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\cos^2\theta+2\sin^2\theta}} = \int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{2-\cos^2\theta}} = \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{1-\frac{1}{2}\cos^2\theta}}.$$

Let $\phi = \frac{\pi}{2} - \theta$ the integral becomes

$$= \frac{1}{\sqrt{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}}. \text{ By using Table 1 and Table 2 the result will be:}$$

$$= \frac{1}{\sqrt{2}} [F(\frac{1}{\sqrt{2}}, \frac{\pi}{2}) - F(\frac{1}{\sqrt{2}}, \frac{\pi}{4})] = \frac{1}{\sqrt{2}} [1.8540666 - .8260038] = 0.905540742977.$$

CHAPTER FOUR

4. Application of elliptic function for solution of certain nonlinear elliptic differential equation

Let $p_n(x)$ be a polynomial of degree 3 or 4 with no repeated real roots. Then, infinite integrals of the form $\int (x, \sqrt{p_n(x)}) dx$, cannot be solved in ease. However, definite form of these integrals can be converted to elliptic integrals of the first kind.

In all given transformations, n and m are supposed positive real numbers.

Theorem 4.1. Let $P_4(x) = (x^2 + m^2)(x^2 + n^2)$ be a polynomial with no real roots. If $m \leq n$, then

$$I = \int_{\theta_0}^{\theta_1} \frac{dx}{\sqrt{P_4(x)}} = \frac{1}{n} \left[F\left(\theta_1, \frac{\sqrt{n^2 - m^2}}{n}\right) - F\left(\theta_0, \frac{\sqrt{n^2 - m^2}}{n}\right) \right].$$

where, $\theta_i = \tan^{-1}\left(\frac{x_i}{m}\right)$ for $i = 0, 1$.

Proof:- The transformation $x = m \tan\theta$ helps us to convert I to an algebraic summation of elliptic integrals of first kind. Thus, we have $dx = m \sec^2\theta d\theta$, and $\theta = \tan^{-1}\left(\frac{x}{m}\right)$. By substituting the new variable, θ , we find:

$$\begin{aligned} I &= \int_{x_0}^{x_1} \frac{dx}{\sqrt{(x^2 + m^2)(x^2 + n^2)}} = \int_{\theta_0 = \tan^{-1}\left(\frac{x_0}{m}\right)}^{\theta_1 = \tan^{-1}\left(\frac{x_1}{m}\right)} \frac{m \sec^2\theta d\theta}{\sqrt{m^2(1 + \tan^2\theta)[n^2 - (n^2 - m^2)\sin^2\theta]}} \\ &= \int_{\theta_0}^{\theta_1} \frac{m \sec^2\theta d\theta}{\sqrt{m^2 n^2 \sec^4\theta \left(1 - \frac{n^2 - m^2}{n^2} \sin^2\theta\right)}} = \int_{\theta_0}^{\theta_1} \frac{m \sec^2\theta d\theta}{\sqrt{m^2 n^2 \sec^4\theta \left(1 - \frac{n^2 - m^2}{n^2}\right)}} \\ &= \int_{\theta_0}^{\theta_1} \frac{d\theta}{\sqrt{\left(1 - \frac{n^2 - m^2}{n^2} \sin^2\theta\right)}} = \frac{1}{n} \left[F\left(\theta_1, \frac{n^2 - m^2}{n}\right) - F\left(\theta_0, \frac{n^2 - m^2}{n}\right) \right]. \end{aligned}$$

The following solved examples illustrate the application of above theorem.

Example 1:- Evaluate the following definite integral:

$$I = \int_{\sqrt{\frac{5}{3}}}^{\sqrt{\frac{15}{3}}} \frac{dx}{\sqrt{(x^2 + 5)(x^2 + 9)}}.$$

Solution:- According to theorem 3, we have $m = \sqrt{5}$, $n = 3$,

$$\theta_0 = \tan^{-1}\left(\sqrt{\frac{5}{3}}\right) = \frac{\pi}{6} \quad \theta_1 = \tan^{-1}\left(\sqrt{\frac{15}{3}}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}.$$

$$\text{Now by the theorem } I = \int_{x_0}^{x_1} \frac{dx}{\sqrt{x^2 + 5(x^2 + 9)}} = \int_{\theta_0}^{\theta_1} \frac{\sqrt{5} \sec^2\theta d\theta}{\sqrt{(5 \tan^2\theta + 5)(\tan^2\theta + 9)}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{5} \sec^2\theta d\theta}{\sqrt{(5 \tan^2\theta + 5)(\tan^2\theta + 9)}}.$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{d\theta}{\sqrt{9-4\sin^2\theta}} = \frac{1}{3} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{d\theta}{\sqrt{1-\frac{4}{9}\sin^2\theta}}.$$

Applying entries 1 and 5 in table 1 implies

$$I = \frac{1}{3} [F(\frac{\pi}{3}, \frac{2}{3}) - F(\frac{\pi}{6}, \frac{2}{3})] = 0.1984336.$$

Example 2:- Evaluate the following definite integral:

$$I = \int_{\frac{1}{\sqrt{2}}}^{\infty} \frac{dx}{\sqrt{(2x^2+1)(x^2+1)}}.$$

Solution:- Since $2x^2+1=2(x^2+\frac{1}{2})$ we have $m=\frac{1}{\sqrt{2}}$, $n=1$

Let $x=\frac{1}{\sqrt{2}} \tan\theta$,

$$\theta_0 = \tan^{-1}(1) = \frac{\pi}{4}, \quad \theta_1 = \tan^{-1}(k) = \frac{\pi}{2} \text{ when } k \rightarrow \infty.$$

Thus according to theorem 3 and Table 1 and 2, we find

$$I = \frac{1}{\sqrt{2}} [F(\frac{\pi}{2}, \frac{1}{\sqrt{2}}) - F(\frac{\pi}{4}, \frac{1}{\sqrt{2}})] = 0.7269501.$$

Theorem 4.2. Let $p_3(x)$ be a cubic polynomial that decompose to linear factors.

If $p_3(x) = (x - a)(x - b)(x - c)$ where $a < b < c$ are real numbers,

$$I = \int_{x_0}^{x_1} \frac{dx}{\sqrt{p_3(x)}} = \frac{2}{\sqrt{c-a}} [F(\theta_1, \sqrt{\frac{b-a}{c-a}}) - F(\theta_0, \sqrt{\frac{b-a}{c-a}})]$$

where $\theta_i = \tan^{-1}(\frac{x_i-c}{c-b})$ for $i=0,1$.

Proof:- In this case ,we use two transformations. First, letting $x=u^2 + c$ implies $dx =2udu$ which

leads to $p_{3(x)} = u^2(u^2 + c - a)(u^2 + c - b)$.

Next using $u = \sqrt{c-b} \tan\theta$ and $du = \sqrt{c-b} \sec^2\theta d\theta$ according to the theorem 4.1, leads to the following procedure

$$\begin{aligned} \int \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} &= \int \frac{2udu}{\sqrt{u^2(u^2+c-a)(u^2+c-b)}} = \int \frac{2\sqrt{c-b}\sec^2\theta d\theta}{\sqrt{[(c-b)\tan^2\theta+(c-a)][(c-b)(\tan^2\theta)+(c-b)]} \\ &= \int \frac{2d\theta}{\sqrt{[(c-b)\sin^2\theta+(c-a)\cos^2\theta]}} = \int \frac{2d\theta}{\sqrt{[(c-a)-(b-a)\sin^2\theta]}} = \frac{2}{\sqrt{c-a}} \int \frac{d\theta}{\sqrt{1-\frac{b-a}{c-a}\sin^2\theta}}. \end{aligned}$$

Hence, $k = \sqrt{\frac{b-a}{c-a}}$ and furthermore $I = \frac{2}{\sqrt{c-a}} [F(\theta_1, \sqrt{\frac{b-a}{c-a}}) - F(\theta_0, \sqrt{\frac{b-a}{c-a}})]$.

Note that $\theta_0 = \tan^{-1}\left(\frac{u_0}{\sqrt{c-b}}\right) = \tan^{-1}\left(\frac{x_0-c}{\sqrt{c-b}}\right)$ and $\theta_1 = \tan^{-1}\left(\frac{u_1}{\sqrt{c-b}}\right) = \tan^{-1}\left(\frac{x_1-c}{\sqrt{c-b}}\right)$.

Example 3:- Evaluate the following definite integral: $I = \int_9^\infty \frac{dx}{\sqrt{(x-2)(x-7)(x-8)}}$.

Solution:- From the problem we have $a=2$, $b=7$, $c=8$. Thus, according to Theorem 4.2. Let $x = u^2 + 8$ and $u = \tan\theta$, then from the theorem, we have: $\theta_i = \tan^{-1}\left(\frac{x_i-c}{\sqrt{c-b}}\right)$ for $i=0,1$.

Thus $\theta_0 = \tan^{-1}\left(\frac{9-8}{\sqrt{8-7}}\right) = \theta_0 = \tan^{-1}(1) = \frac{\pi}{4}$ and $\theta_1 = \tan^{-1}(k)$ as $k \rightarrow \infty$ $\theta_1 = \frac{\pi}{2}$.

Moreover $K = \sqrt{\frac{b-a}{c-a}} = \sqrt{\frac{7-2}{9-8}} = \sqrt{\frac{5}{6}}$. Hence by using Table 1 we find

$$I = \frac{2}{\sqrt{6}} [F\left(\frac{\pi}{2}, \sqrt{\frac{5}{6}}\right) - F\left(\frac{\pi}{4}, \sqrt{\frac{5}{6}}\right)] = \frac{2}{\sqrt{6}} [2.3406433 - 0.8606394] = 1.208418124.$$

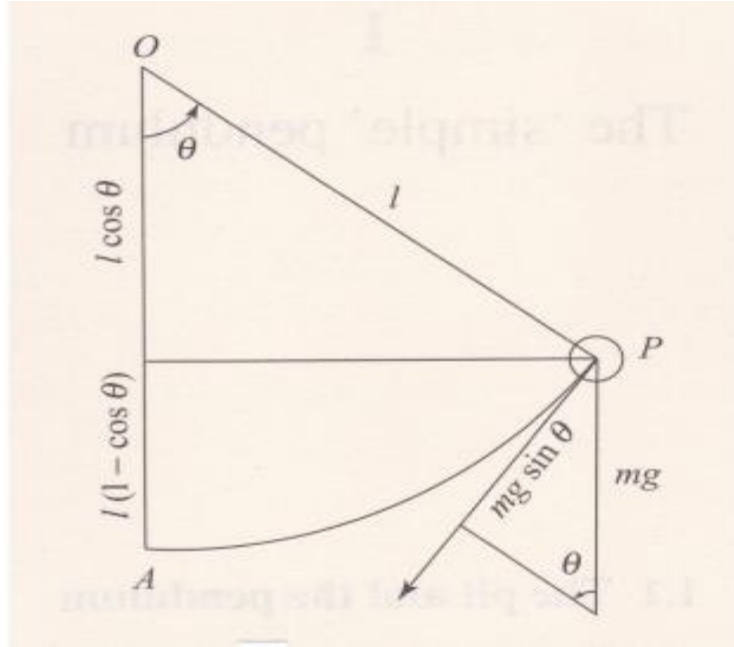


Figure.4. Simple pendulum representation adopted from (<https://doi.org/10.11113/jt.v65.1711>)

Applications example 4: Finite amplitude pendulum.

The equations of motions are:

$$ml\ddot{\theta} = -mg\sin\theta. \text{ Let } p = \dot{\theta} \Rightarrow p \frac{dp}{d\theta} = -gl\sin\theta \Rightarrow \frac{p^2}{2} = \frac{g}{l}\cos\theta + c$$

$$\text{I.C: At } t=0: \theta = \theta_0, \dot{\theta} = 0 \Rightarrow \frac{d\theta}{dt} = -\sqrt{\frac{2g}{l}} \sqrt{\cos\theta - \cos\theta_0},$$

The period, T, is given by

$$\begin{aligned} \frac{T}{4} &= -\sqrt{\frac{l}{2g}} \int_{\theta_0}^0 \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}, \text{ or, } T = 4 \sqrt{\frac{l}{g}} \int_{\theta_0}^0 \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = 2 \sqrt{\frac{l}{2g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2(\frac{\theta_0}{2}) - \sin^2(\frac{\theta}{2})}} \\ &= 4 \sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{1 - k^2 \sin^2 u}}, \sin \frac{\theta}{2} = \sin \frac{\theta_0}{2}, \text{ snu, } k = \sin \frac{\theta_0}{2}. \text{ Therefore } T = 4 \sqrt{\frac{l}{g}} F(k), \text{ an elliptic Integral.} \end{aligned}$$

For the special case of small oscillation $\theta_0 = 0$, we get the classical result: $T = 2\pi \sqrt{\frac{l}{g}}$.

CONCLUSION

In this thesis, we have explored the application of elliptic functions and elliptic integrals, with a particular focus on Landen's transformations of the first kind. This transformation, which is a powerful tool in the theory of elliptic functions, has proven to be instrumental in simplifying complex integrals and equations involving elliptic functions.

The analysis of Landen's transformations of the first kind reveals several key insights:

1. **Simplifications of elliptic integrals:** Landen's transformation provides a means to express elliptic integrals of one modulus in terms of elliptic integrals of another modulus. This ability to relate integrals with different modules facilitates the computation and evaluations of elliptic integrals of high complexity.

2. **Enhanced computational efficiency:** By leveraging Landen's transformation, we can significantly enhance computational efficiency. The transformation often simplifies the numerical evaluations of elliptic integrals, making it easier to solve practical problems that involve these functions.

3. **Insight modular properties:** Landen's transformation sheds light on the modular properties of elliptic integrals and functions. It highlights the deep connections between different elliptic integrals and provides a systematic approach to transforming and simplifying these integrals based on their modulus.

4. **Application in mathematical physics and engineering:** The transformation has notable applications in various fields, including mathematical physics and engineering. For instance, in the analysis of oscillatory systems and wave forms, where elliptic integrals frequently arise, Landen's transformation can simplify the process of solving differential equations and modeling physical phenomena.

5. **Theoretical implications:** The theoretical framework provided by Landen's transformation enriches our understanding of elliptic functions and integrals. It serves as a foundation for further research and exploration into more advanced topics in the theory of elliptic functions and their applications.

In summary, Landen's transformations of the first kind represent a crucial advancement in the study and application of elliptic functions and elliptic integrals. Its ability to simplify and relate integrals with different moduli enhances both the theoretical and practical aspects of working with elliptic functions. Further research may build up these findings to explore new applications and further refine the techniques associated with elliptic functions.

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