Superposed Laser Output Light

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By

Tsedeke Abate

Addis Ababa, Ethiopia

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FACULTY OF CHEMICAL AND PHYSICAL SCIENCE
DEPARTMENT OF PHYSICS

The undersigned hereby certify that they have read and recommend to the School of Graduate Studies for acceptance a thesis entitled "**Superposed Laser Output Light**" by **Tsedeke Abate** in partial fulfillment of the requirements for the degree of **Master of Science in Physics**.

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Approved by the Examination Committee

Dr. Fesseha Kassahun, Advisor ____________________

Dr. Gizaw Mengistu, Examiner ____________________

Dr. Tadelech Atomssa, Examiner ____________________

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Author: **Tsedeke Abate**

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Abstract

We study the squeezing and statistical properties of the superposed output light from two identical three-level lasers. Applying the Q function for the superposed output light, we calculate the mean photon number, the variance of photon number, and the quadrature variance. It is found that the degree of squeezing increases with the linear gain coefficient. Moreover, the degree of squeezing of the superposed light is found to be twice that of the light from one three-level laser.

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Chapter 1

Introduction

Over the years, there has been a considerable interest in the analysis of the squeezing and statistical properties of the light generated by three-level lasers [1-16]. A three-level laser is a quantum optical system in which three-level atoms in a cascade configuration, initially prepared in a coherent supperposition of the top and bottom levels, are injected into a cavity coupled to a vacuum reservoir via a single-port mirror. When a three-level atom in a cascade configuration makes a transition from the top to the bottom level via the intermidate level, two photons are are generated. If the two photons have the same frequency, the three-level atom is called degenerate otherwise it is called non degenerate.

Several authors have found that the light produced by a degenerate three-level laser is in a squeezed state under certain conditions [1, 5, 6, 7, 8, 10]. In particular, the maximum intracavity squeezing is found at threshold and at steady state to be 50%. Recently, Beyene [17] has studied the squeezing properties of superposed light from two identical three-level lasers. He found that for A=3 and κ=0.8, the squeezing is 95.8% below the coherent-state level.

Although previous studies deal mainly with the degree of squeezing and statistical properties of the cavity light, it is the output light that is accessible to the experimental observation. It is found that the squeezing of the cavity light produced by a degenerate three-level laser is greater than that of the corresponding output light [6].

In this thesis we study the squeezing and statistical properties of the output light generated from a degenerate three-level laser and two identical three-level lasers. To this end, we first derive c-number Langevin equations associated with the normal ordering using the master equation. Employing the solutions of the resulting c-number Langevin equations along with the properties of the noise forces, we obtain the antinormally ordered characterisic function defined in Hiesenberg picture, which is used to calculate the Q-function for the cavity mode light produced by the degenerate three-level laser. Applying the input-output relation, we find the Q-function for the output light from the three-level laser. We then determine the Q-function for superposed output light beams produced by two identical three-level lasers.

Applying the Q-function for the output light produced by a three-level laser, we calculate mean photon number, the variance of photon number, and the quadrature variance. Finally, the Q-function of the superposed output light beams is used to calculate the mean photon number, the variance of photon number and the quadrature variance.

Chapter 2

The Q function of laser light

In this chapter we seek to determine the Q-function for the output light generated from a degenerate three-level laser and two identical three-level lasers. To this end, we first derive c-number Langevin equations using the master equaion. Applying the solutions of the resulting c-number Langavin equations along with the properties of the noise force, we then calculate the antinormally-ordered characteristic function with the aid of which the Q function of the cavity laser light is determined. Finally, applying the input-output relation, we calculate the Q-function for superposed output light beams produced by two identical three-level lasers.

2.1 The Q function of the cavity laser light

The three level laser in which three level atoms initially prepared in a coherent superposition of the top and bottom levels are injected at a constant rate in to a cavity coupled to a vacuum reservoir and removed from the cavity. We denote the top, middle, and bottom levels of these atoms by $|a\rangle$, $|b\rangle$, and $|c\rangle$ respectively. We assume the cavity mode to be at resonance with the two transitions $|a\rangle \to |b\rangle$ and $|b\rangle \to |c\rangle$, with direct transition between levels $|a\rangle$ and $|c\rangle$ to be electric dipole forbidden. The interaction of a three-level atom with the cavity mode can be described by the Hamiltonian [1]

$$\hat{H} = ig\big[\big(|a\rangle\langle b| + |b\rangle\langle c|\big)\hat{a} - \hat{a}^{\dagger}\big(|b\rangle\langle a| + |c\rangle\langle b|\big)\big], \tag{2.1.1}$$

where g is the coupling constant between the atom and the cavity mode and $\hat{a}$ is the annihilation operator for the cavity mode. We take the initial state of a three-level atoms to be

$$|\psi_A(0)\rangle = c_a(0)|a\rangle + c_c(0)|c\rangle \tag{2.1.2}$$

and hence the initial density operator for a single atom has the form

$$\hat{\rho}_A(0) = \rho_{aa}^{(0)}|a\rangle\langle a| + \rho_{ac}^{(0)}|a\rangle\langle c| + \rho_{ca}^{(0)}|c\rangle\langle a| + \rho_{cc}^{(0)}|c\rangle\langle c|, \tag{2.1.3}$$

where $\rho_{aa}^{(0)} = |c_a|^2$, $\rho_{ac}^{(0)} = c_a c_c^*$, $\rho_{ca}^{(0)} = c_c c_a^*$, and $\rho_{cc}^{(0)} = |c_c|^2$.

Suppose $\hat{\rho}_{AR}(t, t_j)$ is the density operator for single atom plus the cavity mode at time t, with the atom injected at time t_j such that $(t - \tau) \leq t_j \leq t$. The density operator for all atoms in the cavity plus the cavity mode at time t can then be written as

$$\hat{\rho}_{AR}(t) = r_a \sum \hat{\rho}_{AR}(t, t')\Delta t_j, \tag{2.1.4}$$

where r_a is the rate at which atoms are injected in to the cavity. Now converting the summation in to integration in the limit $\Delta t_j \rightarrow 0$, we have

$$\hat{\rho}_{AR}(t) = r_a \int_{t-\tau}^{t} \hat{\rho}_{AR}(t, t')dt' \tag{2.1.5}$$

and on differentiating with respect to t, there follows

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a(\hat{\rho}_{AR}(t, t) - \hat{\rho}_{AR}(t, t - \tau)) + r_a \int_{t-\tau}^{t} \frac{\partial}{\partial t}\hat{\rho}_{AR}(t, t')dt'. \tag{2.1.6}$$

We observe that $\hat{\rho}_{AR}(t, t)$ is the density operator for the cavity mode plus an atom injected at time t. This operator can thus be expressed as

$$\hat{\rho}_{AR}(t, t) = \hat{\rho}_A(t)\hat{\rho}(t), \tag{2.1.7}$$

with $\hat{\rho}(t)$ being the density operator for the cavity mode alone. We also note that $\hat{\rho}_{AR}(t, t-\tau)$ represents the density operator for an atom plus the cavity mode at time t, with the atom being removed from the cavity at this time. This operator can also be put in the form

$$\hat{\rho}_{AR}(t, t-\tau) = \hat{\rho}_A(t-\tau)\hat{\rho}(t). \tag{2.1.8}$$

Now in view of (2.1.7) and (2.1.8), one can write (2.1.6) as

$$\begin{aligned} \frac{d}{dt}\hat{\rho}_{AR}(t) &= r_a(\hat{\rho}_A(t) - \hat{\rho}(t-\tau))\hat{\rho}(t) \\ &+ r_a \int_{t-\tau}^{t} \frac{\partial}{\partial t}\hat{\rho}_{AR}(t, t')dt'. \end{aligned} \tag{2.1.9}$$

In the absence of damping of the cavity mode by a vacuum reservoir, the density operator $\hat{\rho}_{AR}(t, t')$ evolves in time according to

$$\frac{\partial}{\partial t}\hat{\rho}_{AR}(t, t') = -i[\hat{H}, \hat{\rho}_{AR}(t, t')], \tag{2.1.10}$$

so that using this relation along with (2.1.5), one can put Eq. (2.1.9) in the form

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a(\hat{\rho}_A(t) - \hat{\rho}(t-\tau))\hat{\rho}(t) - i[\hat{H}, \hat{\rho}_{AR}(t)]. \tag{2.1.11}$$

Furthermore, tracing over the atomic variables and taking in to account the damping of the cavity mode by a vacuum reservoir, we have

$$\frac{d\hat{\rho}}{dt} = -iTr_A[\hat{H}, \hat{\rho}_{AR}(t)] + \frac{1}{2}\kappa(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}), \tag{2.1.12}$$

where we have used the fact that

$$Tr\hat{\rho}_A(t) = Tr\hat{\rho}_A(t-\tau) = 1. \tag{2.1.13}$$

Employing (2.1.1) the master equation for the cavity mode can be put in the form

$$\begin{aligned} \frac{d\hat{\rho}}{dt} &= g(\hat{\rho}_{ab}\hat{a}^\dagger - \hat{a}^\dagger\hat{\rho}_{ab} + \hat{\rho}_{bc}\hat{a}^\dagger - \hat{a}^\dagger\hat{\rho}_{bc} + \hat{a}\hat{\rho}_{ba} - \hat{\rho}_{ba}\hat{a} + \hat{a}\hat{\rho}_{cb} - \hat{\rho}_{cb}\hat{a}) \\ &+ \frac{1}{2}\kappa(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}), \end{aligned} \tag{2.1.14}$$

in which the matrix element $\hat{\rho}_{\alpha\beta}$ is defined by

$$\hat{\rho}_{\alpha\beta} = \langle\alpha|\hat{\rho}_{AR}|\beta\rangle, \tag{2.1.15}$$

with α, β=a,b,c.

On the other hand, we see from (2.1.11) that

$$\begin{aligned} \frac{d\hat{\rho}_{\alpha\beta}}{dt} &= r_a(\langle\alpha|\hat{\rho}_A(0)|\beta\rangle - \langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle)\hat{\rho}(t) \\ &\quad -i(\langle\alpha|\hat{H}\hat{\rho}_{AR}|\beta\rangle - \langle\alpha|\hat{\rho}_{AR}\hat{H}|\beta\rangle) - \gamma\hat{\rho}_{\alpha\beta}, \end{aligned} \tag{2.1.16}$$

where the last term is included to account for the decay of the atoms due spontaneous photon emission. Here γ, considered to be the same for all the three levels, is the atomic decay constant. The atoms are removed from the cavity after they have decayed to a level other than the middle or bottom level. We then see that

$$\langle\alpha|\hat{\rho}_{AR}(t-\tau)|\beta\rangle = 0, \tag{2.1.17}$$

and hence Eq. (2.1.16) reduces to

$$\frac{d\hat{\rho}_{\alpha\beta}}{dt} = r_a\langle\alpha|\hat{\rho}_A(0)|\beta\rangle\hat{\rho}(t) - i(\langle\alpha|\hat{H}\hat{\rho}_{AR}|\beta\rangle - \langle\alpha|\hat{\rho}_{AR}\hat{H}|\beta\rangle) - \gamma\hat{\rho}_{\alpha\beta}. \tag{2.1.18}$$

Applying this equation and taking into account (2.1.1) and (2.1.3), one readily obtains

$$\frac{d\hat{\rho}_{ab}}{dt} = g(\hat{\rho}_{ac}\hat{a}^\dagger + \hat{a}\hat{\rho}_{bb} + \hat{\rho}_{aa}\hat{a}) - \gamma\hat{\rho}_{ab}, \tag{2.1.19}$$

$$\frac{d\hat{\rho}_{bc}}{dt} = g(\hat{a}\hat{\rho}_{cc} - \hat{\rho}_{bb}\hat{a} - \hat{a}^\dagger\hat{\rho}_{ac}) - \gamma\hat{\rho}_{bc}, \tag{2.1.20}$$

$$\frac{d\hat{\rho}_{aa}}{dt} = r_a\rho_{aa}^{(0)}\hat{\rho} + g(\hat{\rho}_{ab}\hat{a}^\dagger + \hat{a}\hat{\rho}_{ba}) - \gamma\hat{\rho}_{aa}, \tag{2.1.21}$$

$$\frac{d\hat{\rho}_{bb}}{dt} = g(\hat{\rho}_{bc}\hat{a}^\dagger + \hat{a}\hat{\rho}_{cb} - \hat{a}^\dagger\hat{\rho}_{ab} - \hat{\rho}_{ba}\hat{a}) - \gamma\hat{\rho}_{bb}, \tag{2.1.22}$$

$$\frac{d\hat{\rho}_{ac}}{dt} = r_a\rho_{ac}^{(0)}\hat{\rho} + g(\hat{a}\hat{\rho}_{bc} - \hat{\rho}_{ab}\hat{a}) - \gamma\hat{\rho}_{ac}, \tag{2.1.23}$$

$$\frac{d\hat{\rho}_{cc}}{dt} = r_a\rho_{cc}^{(0)}\hat{\rho} - g(\hat{a}^\dagger\hat{\rho}_{bc} + \hat{\rho}_{cb}\hat{a}) - \gamma\hat{\rho}_{cc}. \tag{2.1.24}$$

We confine ourselves to linear analysis and this can be achieved by dropping the g-terms in (2.1.21), (2.1.22), (2.1.23), and (2.1.24) and applying the large-time approximation. Thus upon dropping g-terms and applying the large-time approximation scheme, we get from Eqs. (2.1.21), (2.1.22), (2.1.23), and (2.1.24) and that

$$\hat{\rho}_{aa} = \frac{r_a \rho_{aa}^{(0)}}{\gamma}\hat{\rho}, \tag{2.1.25}$$

$$\hat{\rho}_{bb} = 0, \tag{2.1.26}$$

$$\hat{\rho}_{ac} = \frac{r_a \rho_{ac}^{(0)}}{\gamma}\hat{\rho}, \tag{2.1.27}$$

$$\hat{\rho}_{cc} = \frac{r_a \rho_{cc}^{(0)}}{\gamma}\hat{\rho}. \tag{2.1.28}$$

Now combination of (2.1.19), (2.1.25), (2.1.26), and (2.1.27) as well as (2.1.20), (2.1.26), (2.1.27), and (2.1.28) leads to

$$\frac{d\hat{\rho}_{ab}}{dt} = \frac{g r_a}{\gamma}(\rho_{ac}^{(0)}\hat{\rho}\hat{a}^\dagger - \rho_{aa}^{(0)}\hat{\rho}\hat{a}) - \gamma\hat{\rho}_{ab}, \tag{2.1.29}$$

$$\frac{d\hat{\rho}_{bc}}{dt} = \frac{g r_a}{\gamma}(\rho_{cc}^{(0)}\hat{a}\hat{\rho} - \rho_{ac}^{(0)}\hat{a}^\dagger\hat{\rho}) - \gamma\hat{\rho}_{bc}. \tag{2.1.30}$$

Using once more the large-time approximation scheme, we easily find

$$\hat{\rho}_{ab} = \frac{g r_a}{\gamma^2}(\rho_{ac}^{(0)}\hat{\rho}\hat{a}^\dagger - \rho_{aa}^{(0)}\hat{\rho}\hat{a}), \tag{2.1.31}$$

$$\hat{\rho}_{bc} = \frac{g r_a}{\gamma^2}(\rho_{cc}^{(0)}\hat{a}\hat{\rho} - \rho_{ac}^{(0)}\hat{a}^\dagger\hat{\rho}). \tag{2.1.32}$$

Finally, on account of (2.1.31), (2.1.32), the master equation for the cavity mode given by (2.1.14) takes the form

$$\begin{aligned}
\frac{d\hat{\rho}}{dt} &= \frac{1}{2}A\rho_{aa}^{(0)}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}) \\
&+\frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}) \\
&+\frac{1}{2}A\rho_{ac}^{(0)}(\hat{\rho}\hat{a}^{\dagger 2} + \hat{a}^{\dagger 2}\hat{\rho} - 2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger) \\
&+\frac{1}{2}A\rho_{ca}^{(0)}(\hat{\rho}\hat{a}^2 + \hat{a}^2\hat{\rho} - 2\hat{a}\hat{\rho}\hat{a}),
\end{aligned} \tag{2.1.33}$$

where

$$A = \frac{2r_a g^2}{\gamma^2} \tag{2.1.34}$$

is the linear gain coefficient. It is worth mentioning that the quantum properties of the light generated by the three-level laser are determined by the master equation (2.1.33). It is easy to observe that with $\rho_{aa}^{(0)} = 1$ and $\rho_{ac}^{(0)} = \rho_{cc}^{(0)} = 0$, this equation reduces to the master equation for a two-level laser operating below threshold.

Using the relation

$$\langle \hat{A} \rangle = Tr(\frac{d\rho}{dt} A)$$

along with Eq. (2.1.33), one readily obtains

$$\begin{aligned}
\frac{d}{dt}\langle \hat{a} \rangle \quad = \quad & \frac{1}{2} A\rho_{aa}^{(0)} Tr(2\hat{a}^\dagger \hat{\rho}\hat{a}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger \hat{a} - \hat{a}\hat{a}^\dagger \hat{\rho}\hat{a}) \\
& + \frac{1}{2}(A\rho_{cc}^{(0)} + \kappa) Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger \hat{a} - \hat{\rho}\hat{a}^\dagger \hat{a}\hat{a} - \hat{a}^\dagger \hat{a}\hat{\rho}\hat{a}) \\
& + \frac{1}{2} A\rho_{ac}^{(0)} Tr(\hat{\rho}\hat{a}^{\dagger 2}\hat{a} + \hat{a}^{\dagger 2}\hat{\rho}\hat{a} - 2\hat{a}^\dagger \hat{\rho}\hat{a}^\dagger \hat{a}) \\
& + \frac{1}{2} A\rho_{ca}^{(0)} Tr(\hat{\rho}\hat{a}^3 + \hat{a}^2 \hat{\rho}\hat{a} - 2\hat{a}\hat{\rho}\hat{a}\hat{a}).
\end{aligned}$$

Employing the cyclic property of trace operation and commutation relation

$$[\hat{a}, \hat{a}^\dagger] = 1,$$

we get

$$\begin{aligned}
\frac{d}{dt}\langle \hat{a}(t) \rangle \quad & = \frac{1}{2} A\rho_{aa}^{(0)} \langle \hat{a}(t) \rangle - \frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)\langle \hat{a}(t) \rangle, \\
& = -\frac{1}{2}\mu \langle \hat{a}(t) \rangle
\end{aligned} \tag{2.1.35}$$

In the same way, one can show that

$$\frac{d}{dt}\langle \hat{a}(t)\hat{a}(t) \rangle = -\mu \langle \hat{a}^2(t) \rangle + A\rho_{ac}^{(0)}, \tag{2.1.36}$$

$$\frac{d}{dt}\langle \hat{a}^\dagger(t)\hat{a}(t) \rangle = -\mu \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle + A\rho_{aa}^{(0)}, \tag{2.1.37}$$

in which

$$\mu = A(\rho_{cc}^{(0)} - \rho_{aa}^{(0)}) + \kappa. \tag{2.1.38}$$

We see that the c-number equations corresponding to Eqs. (2.1.35), (2.1.36), and (2.1.37) are

$$\frac{d}{dt}\langle\alpha(t)\rangle = -\frac{1}{2}\mu\langle\alpha(t)\rangle, \tag{2.1.39}$$

$$\frac{d}{dt}\langle\alpha(t)\alpha(t)\rangle = -\mu\langle\alpha^2(t)\rangle + A\rho_{ac}^{(0)}, \tag{2.1.40}$$

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = -\mu\langle\alpha^*(t)\alpha(t)\rangle + A\rho_{aa}^{(0)}. \tag{2.1.41}$$

On the basis of Eq. (2.1.39) one can write

$$\frac{d}{dt}\alpha(t) = -\frac{1}{2}\mu\alpha(t) + f(t), \tag{2.1.42}$$

where $f(t)$ is a noise force whose correlation properties remain to be determined. We note that Eq. (2.1.39) and the expectation value of Eq. (2.1.42) will have the same form if

$$\langle f(t)\rangle = 0. \tag{2.1.43}$$

The comlex conjugate of Eq. (2.1.42) can be written as

$$\frac{d}{dt}\alpha^*(t) = -\frac{1}{2}\mu\alpha^*(t) + f^*(t). \tag{2.1.44}$$

But we have the relation

$$\frac{d}{dt}\big\langle\alpha(t)\alpha(t)\big\rangle = \big\langle\frac{d\alpha(t)}{dt}\alpha(t)\big\rangle + \big\langle\alpha(t)\frac{d\alpha(t)}{dt}\big\rangle. \tag{2.1.45}$$

Inserting Eq. (2.1.42) in (2.1.45), one can easily get

$$\frac{d}{dt}\langle\alpha(t)\alpha(t)\rangle = -\mu\langle\alpha^2(t)\rangle + \langle f(t)\alpha(t)\rangle + \langle\alpha(t)f(t)\rangle. \tag{2.1.46}$$

Now comparison of Eqs. (2.1.40) and (2.1.46) shows that

$$\langle f(t)\alpha(t)\rangle + \langle\alpha(t)f(t)\rangle = A\rho_{ac}^{(0)}. \tag{2.1.47}$$

A formal solution of Eq.(2.1.42) can be written as

$$\alpha(t) = \alpha(0)e^{-\mu t/2} + \int_0^t e^{-\mu(t-t')/2} f(t')dt'. \tag{2.1.48}$$

Multiplying Eq. (2.1.48) by f(t) from the left side and taking the expectation value, we get

$$\langle f(t)\alpha(t)\rangle = \langle f(t)\alpha(0)\rangle e^{-\mu t/2} + \int_0^t e^{-\mu(t-t')/2} \langle f(t)f(t')\rangle dt'. \tag{2.1.49}$$

Since a noise force at a certain time should not affect a light mode variable at an earlier time, we have

$$\langle f(t)\alpha(0)\rangle = \langle f(t)\rangle\langle\alpha(0)\rangle = 0. \tag{2.1.50}$$

In view of this result Eq. (2.1.49) reduces to

$$\langle f(t)\alpha(t)\rangle = \int_0^t e^{-\mu(t-t')/2} \langle f(t)f(t')\rangle dt'. \tag{2.1.51}$$

Similarily multiplying Eq. (2.1.48) by f(t) from the right side and taking the expectation value together with the assertion that a noise force at a certain time should not affect a light mode variable at an earlier time, one can get

$$\langle \alpha(t)f(t)\rangle = \int_0^t e^{-\mu(t-t')/2} \langle f(t')f(t)\rangle dt'. \tag{2.1.52}$$

Thus taking into account (2.1.47), along with, (2.1.51), and (2.1.52), we get

$$\int_0^t e^{-\mu(t-t')/2} \langle f(t)f(t')\rangle dt' + \int_0^t e^{-\mu(t-t')/2} \langle f(t')f(t)\rangle dt' = A\rho_{ac}^{(0)}, \tag{2.1.53}$$

and assuming that

$$\langle f(t)f(t')\rangle = \langle f(t')f(t)\rangle, \tag{2.1.54}$$

we have

$$2\int_0^t e^{-\mu(t-t')/2} \langle f(t)f(t')\rangle dt' = A\rho_{ac}^{(0)}. \tag{2.1.55}$$

On the bais of this result, we assert that

$$\langle f(t)f(t')\rangle = A\rho_{ac}^{(0)}\delta(t-t'). \tag{2.1.56}$$

Furthermore, employing the relation

$$\frac{d}{dt}\big\langle\alpha^*(t)\alpha(t)\big\rangle = \big\langle\frac{d\alpha^*(t)}{dt}\alpha(t)\big\rangle + \big\langle\alpha^*(t)\frac{d\alpha(t)}{dt}\big\rangle, \tag{2.1.57}$$

along with (2.1.42) and (2.1.44), one can easily establish that

$$\frac{d}{dt}\big\langle\alpha^*(t)\alpha(t)\big\rangle = -\mu\langle\alpha^*(t)\alpha(t)\rangle + \langle f^*(t)\alpha(t)\rangle + \langle\alpha^*(t)f(t)\rangle. \tag{2.1.58}$$

Inspection of Eqs. (2.1.41) and (2.1.58) indicates that

$$\langle f^*(t)\alpha(t)\rangle + \langle\alpha^*(t)f(t)\rangle = A\rho_{aa}^{(0)}. \tag{2.1.59}$$

With the aid of Eq. (2.1.48), we get

$$\langle f^*(t)\alpha(t)\rangle = \langle f^*(t)\alpha(0)\rangle e^{-\mu t/2} + \int_0^t e^{-\mu(t-t')/2}\langle f^*(t)f(t')\rangle dt'. \tag{2.1.60}$$

Since a noise force at a certain time should not affect a light mode variable at an earlier time, we note that

$$\langle f^*(t)\alpha(t)\rangle = \int_0^t e^{-\mu(t-t')/2}\langle f^*(t)f(t')\rangle dt'. \tag{2.1.61}$$

Taking the complex conjugate this equation, we obtain

$$\langle\alpha^*(t)f(t)\rangle = \int_0^t e^{-\mu(t-t')/2}\langle f^*(t')f(t)\rangle dt'. \tag{2.1.62}$$

Hence on account of (2.1.59) along with (2.1.61) and (2.1.62), we have

$$\int_0^t e^{-\mu(t-t')/2}\langle f^*(t)f(t')\rangle dt' + \int_0^t e^{-\mu(t-t')/2}\langle f^*(t')f(t)\rangle dt' = A\rho_{aa}^{(0)}, \tag{2.1.63}$$

and on taking

$$\langle f^*(t)f(t')\rangle = \langle f^*(t')f(t)\rangle, \tag{2.1.64}$$

we see that

$$2\int_0^t e^{-\mu(t-t')/2}\langle f^*(t)f(t')\rangle dt' = A\rho_{aa}^{(0)}. \tag{2.1.65}$$

Now on the basis of this result, we assert that

$$\langle f^*(t)f(t')\rangle = A\rho_{aa}^{(0)}\delta(t-t'). \tag{2.1.66}$$

Eqs. (2.1.43), (2.1.56), and (2.1.66) describes the correlation properties of the noise force f(t) associated with the normal ordering.

We now proceed to obtain the Q-function of the cavity laser light. The Q function for a single-mode light is defined by

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi^2} \int d^2 z \phi_a(z, t) e^{(z^* \alpha(t) - z\alpha^*(t))}, \tag{2.1.67}$$

with antinormally ordered characteristic function defined in the Heisenberg picture by

$$\phi_a(z^*, z, t) = Tr(\hat{\rho}(0) e^{-z^* \hat{a}(t)} e^{z\hat{a}^\dagger(t)}). \tag{2.1.68}$$

With the aid of the identity

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{B}} e^{\hat{A}} e^{[\hat{A}, \hat{B}]}, \tag{2.1.69}$$

one can check that

$$e^{-z^* \hat{a}(t)} e^{z\hat{a}^\dagger(t)} = e^{-z^* z} e^{z\hat{a}^\dagger(t)} e^{-z^* \hat{a}(t)}. \tag{2.1.70}$$

In view of this result, expression (2.1.68) can be expressed in terms of c-number variables associated with normal ordering as

$$\phi_a(z^*, z, t) = e^{-z^* z} \langle e^{(z\alpha^*(t) - z^* \alpha(t))} \rangle. \tag{2.1.71}$$

Taking into account of the expectation value of Eq. (2.1.48) along with the assumption that the cavity mode to be initially in a vacuum state, we can write

$$\langle e^{(z\alpha^* - z^* \alpha)} \rangle = exp\Big[\frac{1}{2} \langle (z\alpha^* - z^* \alpha)^2 \rangle\Big], \tag{2.1.72}$$

Therefore, one can then express (2.1.71) in the form

$$\phi_a(z^*, z, t) = e^{-z^* z} e^{\frac{1}{2} \langle z^2 \alpha^{*2} + z^{*2} \alpha^2 - 2z^* z\alpha^* \alpha \rangle}. \tag{2.1.73}$$

In view of Eq. (2.1.48), one can easily get

$$\begin{aligned} \langle \alpha^2(t) \rangle &= \langle \alpha^2(0) \rangle e^{-\mu t} + \int_0^t e^{-\mu(2t - t')/2} \langle \alpha(0) f(t') \rangle dt' \\ &+ \int_0^t e^{-\mu(2t - t'')/2} \langle f(t') \alpha(0) \rangle dt'' \\ &+ \int_0^t e^{-\mu(2t - t' - t'')/2} \langle f(t') f(t'') \rangle dt' dt''. \end{aligned} \tag{2.1.74}$$

Since a noise force at a certain time should not affect a light mode variable at an earlier time, we have

$$\langle\alpha^2(t)\rangle = \langle\alpha^2(0)\rangle e^{-\mu t} + \int_0^t e^{-\mu(2t-t'-t'')/2}\langle f(t')f(t'')\rangle dt'dt''. \tag{2.1.75}$$

Assuming the cavity mode to be initially in a vacuum state, we have

$$\langle\alpha^2(0)\rangle = 0. \tag{2.1.76}$$

In view of this result, Eq. (2.1.75) reduces to

$$\langle\alpha^2(t)\rangle = \int_0^t e^{-\mu(2t-t'-t'')/2}\langle f(t')f(t'')\rangle dt'dt''. \tag{2.1.77}$$

With the aid of Eq. (2.1.56), we find

$$\langle\alpha^2(t)\rangle = \frac{A\rho_{ac}^{(0)}}{\mu}[1-e^{-\mu t}]. \tag{2.1.78}$$

similarily one can get

$$\langle\alpha^{*2}(t)\rangle = \frac{A\rho_{ac}^{(0)}}{\mu}[1-e^{-\mu t}]. \tag{2.1.79}$$

From Eq. (2.1.48), one can write

$$\alpha^*(t) = \alpha^*(0)e^{-\mu t/2} + \int_0^t e^{-\mu(t-t')/2} f^*(t'')dt''. \tag{2.1.80}$$

Now on account of (2.1.48) and (2.1.80), we have

$$\begin{aligned}\langle\alpha^*(t)\alpha(t)\rangle &= \langle\alpha^*(0)\alpha(0)\rangle e^{-\mu t} + \int_0^t e^{-\mu(2t-t')/2}\langle\alpha(0)f(t')\rangle dt' \\ &\quad + \int_0^t e^{-\mu(2t-t'')/2}\langle f^*(t'')\alpha(0)\rangle dt'' \\ &\quad + \int_0^t e^{-\mu(2t-t-t'')/2}\langle f^*(t'')f(t')\rangle dt'dt''.\end{aligned} \tag{2.1.81}$$

Assuming the cavity mode to be initially in a vacuum state and taking in to account of the assertion that a noise force at a certain time should not affect a light mode variable at an earlier time, we get

$$\langle\alpha^*(t)\alpha(t)\rangle = \int_0^t e^{-\mu(2t-t-t'')/2}\langle f^*(t'')f(t')\rangle dt'dt''. \tag{2.1.82}$$

Hence in view of (2.1.66) and Eq. (2.1.81) takes the form

$$\langle\alpha^*(t)\alpha(t)\rangle = \frac{A\rho_{aa}^{(0)}}{\mu}[1 - e^{-\mu t}]. \tag{2.1.83}$$

It proves to be more convenient to introduce a new parameter defined by

$$\rho_{aa}^{(0)} = \frac{1-\eta}{2}, \tag{2.1.84}$$

so that in view of the fact that

$$\rho_{aa}^{(0)} + \rho_{cc}^{(0)} = 1, \tag{2.1.85}$$

and

$$|\rho_{ac}^{(0)}|^2 = \rho_{aa}^{(0)}\rho_{cc}^{(0)}. \tag{2.1.86}$$

one easily finds

$$\rho_{cc}^{(0)} = \frac{1+\eta}{2}, \tag{2.1.87}$$

and

$$|\rho_{ac}^{(0)}| = \frac{1}{2}(1-\eta^2)^{1/2}. \tag{2.1.88}$$

Therefore, on the basis of these relations, Eqs. (2.1.78), (2.1.79), and (2.1.83) can be rewritten as

$$\langle\alpha^2(t)\rangle = \langle\alpha^{*2}(t)\rangle = \frac{A(1-\eta^2)^{1/2}}{2(A\eta+\kappa)}[1 - e^{-\mu t}], \tag{2.1.89}$$

and

$$\langle\alpha^*(t)\alpha(t)\rangle = \frac{A(1-\eta)}{2(A\eta+\kappa)}[1 - e^{-\mu t}]. \tag{2.1.90}$$

On substituting (2.1.89) and (2.1.90) in to Eq. (2.1.73), there follows

$$\phi_a(z^*, z, t) = exp[-az^*z + b(z^2 + z^{*2})/2], \tag{2.1.91}$$

in which

$$a = 1 + \frac{A(1-\eta)}{2(A\eta+\kappa)}[1 - e^{-\mu t}], \tag{2.1.92}$$

and

$$b = \frac{A(1-\eta^2)^{1/2}}{2(A\eta+\kappa)}[1 - e^{-\mu t}]. \tag{2.1.93}$$

Now introducing (2.1.91) into Eq. (2.1.67), we obtain

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi^2} \int d^2 z exp[-az^* z + z^* \alpha - z\alpha^* + b(z^2 + z^{*2})/2]. \tag{2.1.94}$$

Thus on performing the integration employing the relation

$$\begin{aligned} &\int \frac{d^2 z}{\pi} exp\big(- azz^* + bz + cz^* + Az^2 + Bz^{*2}\big) \\ &= \left[\frac{1}{a^2 - 4AB}\right]^{1/2} exp\left[\frac{abc + Ac^2 + Bb^2}{a^2 - 4AB}\right], a > 0 \end{aligned} \tag{2.1.95}$$

one readily obtains

$$Q(\alpha^*, \alpha, t) = \frac{(u^2 - v^2)^{1/2}}{\pi} exp[-u\alpha^*\alpha + v(\alpha^2 + \alpha^{*2})/2], \tag{2.1.96}$$

where

$$u = \frac{a}{a^2 - b^2}, \tag{2.1.97}$$

and

$$v = \frac{b}{a^2 - b^2}. \tag{2.1.98}$$

One can check that

$$\int d^2\alpha Q(\alpha^*, \alpha, t) = 1. \tag{2.1.99}$$

2.2 The Q function of the superposed laser output light

We next proceed to obtain Q-function of the superposed output light beams produced by a pair of three-level lasers. To this end, we first need to determine the Q-function of the output light from a three-level laser. According to the well known input-output relation, we have

$$\hat{a}_{out}(t) = \sqrt{\kappa}\hat{a}(t) - \hat{a}_{in}(t). \tag{2.2.1}$$

For a cavity mode coupled to a vacuum reservoir and with the cavity mode represented by c-number variables associated with the normal ordering, one can write

$$\alpha_{out}(t) = \sqrt{\kappa}\alpha(t). \tag{2.2.2}$$

In view of (2.1.97) and (2.2.2) the Q-function of the output light from a three level laser is expressed as

$$Q(\alpha^*_{out}, \alpha_{out}, t) = \frac{(u^2 - v^2)^{1/2}}{\pi} exp[-\frac{u}{\kappa}\alpha^*_{out}\alpha_{out} + \frac{v}{k}(\alpha^2_{out} + \alpha^{*2}_{out})/2]. \qquad (2.2.3)$$

One can easily check that

$$\int \frac{d^2\alpha_{out}}{\kappa} Q(\alpha^*_{out}, \alpha_{out}, t) = 1. \qquad (2.2.4)$$

This implies that the Q-function given by (2.2.3) is normalized.

Now the Q-function for the superposition of the two light beams can be given as [1]

$$\begin{aligned} Q(\alpha^*, \alpha, t) = \frac{1}{\pi}\int d^2\beta d^2\gamma Q(\beta^*, \alpha - \gamma) Q(\gamma^*, \alpha - \beta) \times \\ exp(-\alpha\alpha^* - \beta\beta^* - \gamma\gamma^* + \alpha^*\beta + \alpha\beta^* + \alpha^*\gamma + \alpha\gamma^* - \beta^*\gamma - \beta\gamma^*). \qquad (2.2.5) \end{aligned}$$

Hence applying (2.2.2), one can put Eq. (2.2.5) in the form

$$\begin{aligned} Q(\alpha^*_{out}, \alpha_{out}, t) = \frac{1}{\pi}\int \frac{d^2\beta_{out}}{\kappa}\frac{d^2\gamma_{out}}{\kappa} Q(\beta^*_{out}, \alpha_{out} - \gamma_{out}) Q(\gamma^*_{out}, \alpha_{out} - \beta_{out}) \times \\ exp(-\frac{\alpha_{out}\alpha^*_{out}}{\kappa} - \frac{\beta_{out}\beta^*_{out}}{\kappa} - \frac{\gamma_{out}\gamma^*_{out}}{\kappa} + \frac{\alpha^*_{out}\beta_{out}}{\kappa} + \frac{\alpha_{out}\beta^*_{out}}{\kappa} + \frac{\alpha^*_{out}\gamma_{out}}{\kappa} \\ + \frac{\alpha_{out}\gamma^*_{out}}{\kappa} - \frac{\beta^*_{out}\gamma_{out}}{\kappa} - \frac{\beta_{out}\gamma^*_{out}}{\kappa}), \qquad (2.2.6) \end{aligned}$$

is the Q-function for the superposition of the two output light beams.

Let $Q(\beta^*, \beta)$ and $Q(\gamma^*, \gamma)$ be Q-functions of the first and second light beams respectively. Therefore, on account of (2.2.3), the Q-function of one light beam can be written as

$$\begin{aligned} Q(\beta^*_{out}, \alpha_{out} - \gamma_{out}) &= \frac{(u^2 - v^2)^{1/2}}{\pi} exp[-\frac{u}{\kappa}\beta^*_{out}\alpha_{out} + \frac{u}{\kappa}\beta^*_{out}\gamma_{out} \\ &+ \frac{v}{k}(\alpha^2_{out} - 2\alpha\gamma + \gamma^2 + \beta^{*2}_{out})/2], \qquad (2.2.7) \end{aligned}$$

and the Q-function for another light beam can be expressed as

$$\begin{aligned} Q(\gamma^*_{out}, \alpha_{out} - \beta_{out}) &= \frac{(u^2 - v^2)^{1/2}}{\pi} exp[-\frac{u}{\kappa}\gamma^*_{out}\alpha_{out} + \frac{u}{\kappa}\gamma^*_{out}\beta_{out} \\ &+ \frac{v}{k}(\alpha^2_{out} - 2\alpha\beta + \beta^2 + \gamma^{*2}_{out})/2], \qquad (2.2.8) \end{aligned}$$

where u and v are given by Eqs. (2.1.98) and (2.1.99).

Thus, on account of (2.2.7), and (2.2.8) the Q-function for superposed identical laser light beams can be put in the form

$$\begin{aligned}
&Q(\alpha_{out}^*, \alpha_{out}, t) = \frac{(u^2 - v^2)}{\kappa^2} exp(-\frac{\alpha_{out}\alpha_{out}^*}{\kappa} + \frac{v\alpha^2}{\kappa}) \\
&\times \int \frac{d^2\beta_{out}}{\pi} exp[-\frac{\beta_{out}\beta_{out}^*}{\kappa} + (\frac{\alpha_{out}^*}{\kappa} - \frac{v\alpha_{out}}{\kappa})\beta \\
&+(\frac{\alpha_{out}}{\kappa} - \frac{u\alpha_{out}}{\kappa})\beta^* + \frac{v}{2\kappa}(\beta^2 + \beta^{*2})] \\
&\times \int \frac{d^2\gamma_{out}}{\pi} exp[-\frac{\gamma_{out}\gamma_{out}^*}{\kappa} + \frac{u\beta_{out}^*}{\kappa} - \frac{v\alpha_{out}}{\kappa} + \frac{\alpha_{out}^*}{\kappa} - \frac{\beta_{out}^*}{\kappa})\gamma \\
&+(-\frac{u\alpha_{out}}{\kappa} - \frac{u\beta_{out}}{\kappa} + \frac{\alpha_{out}}{\kappa} - \frac{\beta_{out}}{\kappa})\gamma^* + \frac{v}{2\kappa}(\gamma^2 + \gamma^{*2})].
\end{aligned} \tag{2.2.9}$$

Let the second integral be

$$\begin{aligned}
I_2 \;\; = \;\; &\int \frac{d^2\gamma_{out}}{\pi} exp[-\frac{\gamma_{out}\gamma_{out}^*}{\kappa} + \frac{u\beta_{out}^*}{\kappa} - \frac{v\alpha_{out}}{\kappa} + \frac{\alpha_{out}^*}{\kappa} - \frac{\beta_{out}^*}{\kappa})\gamma \\
&+(-\frac{u\alpha_{out}}{\kappa} - \frac{u\beta_{out}}{\kappa} + \frac{\alpha_{out}}{\kappa} - \frac{\beta_{out}}{\kappa})\gamma^* + \frac{v}{2\kappa}(\gamma^2 + \gamma^{*2})],
\end{aligned} \tag{2.2.10}$$

carrying out the integration applying (2.1.96), we get

$$\begin{aligned}
I_2 \;\; = \;\; &\frac{\kappa}{(1-v^2)^{1/2}} exp[\frac{-u\alpha_{out}\alpha_{out}^* + \alpha_{out}\alpha_{out}^* - v\alpha_{out}\alpha_{out}^* + v^3\alpha_{out}^2/2}{\kappa(1-v^2)} \\
&\frac{-v\alpha_{out}^2/2 + u^2v\alpha_{out}^2 + v\alpha_{out}^{*2}/2}{\kappa(1-v^2)}] \\
&\times exp[\frac{u^2 - 2u}{\kappa(1-v^2)}\beta_{out}\beta_{out}^* + \frac{vu\alpha_{out} + u\alpha_{out}^* - \alpha_{out}^* + u^2v\alpha_{out} - v\alpha_{out}}{\kappa(1-v^2)}\beta_{out} \\
&+\frac{-u^2\alpha_{out} - 2u\alpha_{out} - \alpha_{out} - uv^2\alpha_{out} + v^2\alpha_{out} - v\alpha_{out}^*}{\kappa(1-v^2)}\beta_{out}^* \\
&+\frac{u^2v + v}{\kappa(1-v^2)}(\frac{\beta_{out}^2}{2} + \frac{\beta_{out}^{*\,2}}{2})].
\end{aligned} \tag{2.2.11}$$

Therefore, upon substituting (2.2.15) into (2.2.13), there follows

$$
\begin{aligned}
Q(\alpha_{out}^*, \alpha_{out}, t) \;\; = \;\; & \frac{1}{\pi\kappa}\left[\frac{(u^2 - v^2)}{(1 - v^2)}\right]^{1/2} \\
& exp[\frac{-u\alpha_{out}\alpha_{out}^*}{\kappa(1 - v^2)} + \frac{v + vu^2 - v^3}{\kappa(1 - v^2)}\alpha_{out}^2 + \frac{v}{\kappa(1 - v^2)}\alpha_{out}^{*2} \\
& \times \int \frac{d^2\beta_{out}}{\pi} exp\Bigg[- \frac{2u - u^2 - v^2}{\kappa(1 - v^2)}\beta_{out}\beta_{out}^* \\
& + \frac{(vu - vu^2 + v^3 - v)\alpha_{out} + (u - v^2)\alpha_{out}^*}{\kappa(1 - v^2)}\beta_{out} \\
& + \frac{(u - u^2)\alpha_{out} + (vu - v)\alpha_{out}^*}{\kappa(1 - v^2)}\beta_{out}^* \\
& + \frac{vu^2 - 2vu + 2v - v^3}{\kappa(1 - v^2)}(\frac{\beta_{out}^2}{2} + \frac{\beta_{out}^{*2}}{2})\Bigg]. \qquad (2.2.12)
\end{aligned}
$$

Finally, after carrying out the integration applying Eq. (2.1.96), we get

$$Q(\alpha_{out}^*, \alpha_{out}, t) = \frac{C}{\pi} exp[-\frac{D}{\kappa}\alpha_{out}^*\alpha_{out} + \frac{E}{k}(\alpha_{out}^2 + \alpha_{out}^{*2})/2], \qquad (2.2.13)$$

where

$$C = \sqrt{\frac{u^2 - v^2}{4 - 4u + u^2 - v^2}}, \qquad (2.2.14)$$

$$D = \frac{2u - u^2 + v^2}{4 - 4u + u^2 - v^2}, \qquad (2.2.15)$$

and

$$E = \frac{2v}{4 - 4u + u^2 - v^2}. \qquad (2.2.16)$$

One can also easily check that

$$\int \frac{d^2\alpha_{out}}{\kappa} Q(\alpha_{out}^*, \alpha_{out}, t) = 1. \qquad (2.2.17)$$

This indicates that the Q function given by (2.2.17) is normalized.

Chapter 3

The output light from a three-level laser

In this chapter we seek to study the statistical properties of the output light generated from a three-level laser, which can be described by the mean photon number and variance of photon number. To this end, applying the Q-function for the output light from a three-level laser, we determine the mean photon number and the variance of the photon number. Using the same Q function, we also calculate the quadrature variances for the output mode.

3.1 The mean photon number

In this section, we want to calculate the mean photon number of the output light, resulted from a three level laser.

The expectation value of an operator function $\hat{A}(\hat{a}^\dagger, \hat{a})$ is expressible as

$$\langle \hat{A} \rangle = \frac{1}{\pi} \int d^2\alpha \; d^2\beta |\langle \alpha|\beta \rangle|^2 Q(\alpha^*, \beta) A_n(\beta^*, \alpha), \tag{3.1.1}$$

where $A_n(\beta^*, \alpha)$ is the c-number function correponding to operator function $\hat{A}(\hat{a}^\dagger, \hat{a})$ in the normal order.

Applying the input-out put relation (2.2.2), we have

$$\bar{n}_{out} = \frac{m^2}{\pi}\int d^2\alpha_{out} d^2\beta_{out} |\langle\alpha_{out}|\beta_{out}\rangle|^2 Q(\alpha^*_{out}, \beta_{out})\beta^*_{out}\alpha_{out}, \tag{3.1.2}$$

where

$$|\langle\alpha_{out}|\beta_{out}\rangle|^2 = exp(-\frac{\alpha^*_{out}\alpha_{out}}{\kappa} - \frac{\beta^*_{out}\beta_{out}}{\kappa} + \frac{\alpha_{out}\beta^*_{out}}{\kappa} + \frac{\alpha^*_{out}\beta_{out}}{\kappa}). \tag{3.1.3}$$

From (2.2.3), we see that

$$Q(\alpha^*_{out}, \beta_{out}, t) = \frac{(u^2 - v^2)^{1/2}}{\pi} exp[-\frac{u}{\kappa}\alpha^*_{out}\beta_{out} + \frac{v}{k}(\beta^2_{out} + \alpha^{*2}_{out})/2]. \tag{3.1.4}$$

On account of (3.1.3) and (3.1.4), one can write

$$\begin{aligned} \bar{n}_{out} &= m^2 N \int \frac{d^2\alpha_{out}}{\pi} exp(-m\alpha_{out}\alpha^*_{out} + mv\alpha^{*2}_{out}/2)\alpha_{out} \\ &\quad \times \int \frac{d^2\beta_{out}}{\pi} exp\Big[- m\beta_{out}\beta^*_{out} + (m\alpha^*_{out} - mu\alpha^*_{out})\beta_{out} \\ &\quad + m\alpha_{out}\beta^*_{out} + mv\beta^2_{out}/2\Big]\beta^*_{out}, \end{aligned} \tag{3.1.5}$$

where

$$N = (U^2 - V^2)^{1/2}, \tag{3.1.6}$$

and

$$m = \frac{1}{\kappa}. \tag{3.1.7}$$

Let

$$\begin{aligned} I_1 &= \int \frac{d^2\beta_{out}}{\pi} exp\Big[- m\beta_{out}\beta^*_{out} + (m\alpha^*_{out} - mu\alpha^*_{out})\beta_{out} \\ &\quad + m\alpha_{out}\beta^*_{out} + mv\beta^2_{out}/2\Big]\beta^*_{out}, \\ &= \frac{\partial}{\partial\gamma_{out}} \int \frac{d^2\beta_{out}}{\pi} exp\Big[- m\beta_{out}\beta^*_{out} + (m\alpha^*_{out} - mu\alpha^*_{out})\beta_{out} \\ &\quad + (m\alpha_{out} + \gamma_{out})\beta^*_{out} + mv\beta^2_{out}/2\Big]_{\gamma=0}. \end{aligned} \tag{3.1.8}$$

Carrying out the integration over β_{out} and differentiating with respect to γ_{out}, we get

$$I_1 = \frac{\alpha^*_{out} - u\alpha^*_{out} + v\alpha_{out}}{m} exp(m\alpha_{out}\alpha^*_{out} - mu\alpha^*_{out}\alpha_{out} + mv\alpha^2_{out}/2). \tag{3.1.9}$$

Now upon substituting (3.1.9) into (3.1.5), we have

$$\begin{aligned}\bar{n}_{out} \quad &= \quad Nm(1-u)\int\frac{d^2\alpha_{out}}{\pi}exp[-mu\alpha_{out}\alpha^*_{out}+mv(\alpha^2_{out}/2+\alpha^{*2}_{out}/2]\alpha_{out}\alpha^*_{out}\\ &\quad +Nmv\int\frac{d^2\alpha_{out}}{\pi}exp[-mu\alpha_{out}\alpha^*_{out}+mv(\alpha^2_{out}/2+\alpha^{*2}_{out}/2]\alpha^2_{out}. \qquad (3.1.10)\end{aligned}$$

Let the first integral be

$$\begin{aligned}I_2 \quad &= Nm(1-u)\int\frac{d^2\alpha_{out}}{\pi}exp[-mu\alpha_{out}\alpha^*_{out}+mv(\alpha^2_{out}/2+\alpha^{*2}_{out}/2]\alpha_{out}\alpha^*_{out},\\ &= \frac{\partial}{\partial\beta_{out}}\frac{\partial}{\partial\gamma_{out}}Nm(1-u)\int\frac{d^2\alpha_{out}}{\pi}exp[-mu\alpha_{out}\alpha^*_{out}+\beta_{out}\alpha_{out}\\ &+\gamma_{out}\alpha^*_{out}+mv(\alpha^2_{out}/2+\alpha^{*2}_{out}/2)]_{\gamma_{out}=\beta_{out}=0},\\ &= \frac{\partial}{\partial\beta_{out}}\frac{\partial}{\partial\gamma_{out}}\frac{N(1-u)}{(u^2-v^2)^{1/2}}exp[\frac{u\beta_{out}\gamma_{out}+v(\gamma^2_{out}+\gamma^2_{out})}{m(u^2-v^2)}]_{\gamma_{out}=\beta_{out}=0},\\ &= \frac{\kappa N(u-u^2)}{(u^2-v^2)^{3/2}}. \qquad (3.1.11)\end{aligned}$$

And let the second integral be

$$\begin{aligned}I_3 \quad &= Nmv\int\frac{^2\alpha_{out}}{\pi}exp[-mu\alpha_{out}\alpha^*_{out}+mv(\alpha^2_{out}/2+\alpha^{*2}_{out}/2]\alpha^2_{out},\\ &= Nmv\frac{\partial^2}{\partial\beta^2_{out}}\int\frac{d^2\alpha_{out}}{\pi}exp[-mu\alpha_{out}\alpha^*_{out}+\beta_{out}\alpha_{out}+mv(\alpha^2_{out}/2+\alpha^{*2}_{out}/2]_{\beta_{out}=0},\\ &= NV\frac{\partial^2}{\partial\beta^2_{out}}\frac{1}{(u^2-v^2)^{1/2}}exp[\frac{v\beta^2_{out}/2}{m(u^2-v^2)}]_{\beta_{out}=0},\\ &= \frac{\kappa Nv^2}{(u^2-v^2)^{3/2}}. \qquad (3.1.12)\end{aligned}$$

Hence on account of (3.1.10) along with (3.1.11) and (3.1.12), we get

$$\bar{n}_{out} = \kappa(\frac{u}{u^2-v^2}-1). \qquad (3.1.13)$$

Now in view of (2.1.92) and (2.1.93) the mean photon number takes the form

$$\begin{aligned}\bar{n}_{out} \quad &= \kappa(-1+a)\\ &= \frac{\kappa A(1-\eta)}{2(A\eta+\kappa)}[1-e^{-\mu t}]. \qquad (3.1.14)\end{aligned}$$

At steady state this tuns to be

$$\bar{n}_{out} = \frac{\kappa A(1-\eta)}{2(A\eta+\kappa)}. \qquad (3.1.15)$$

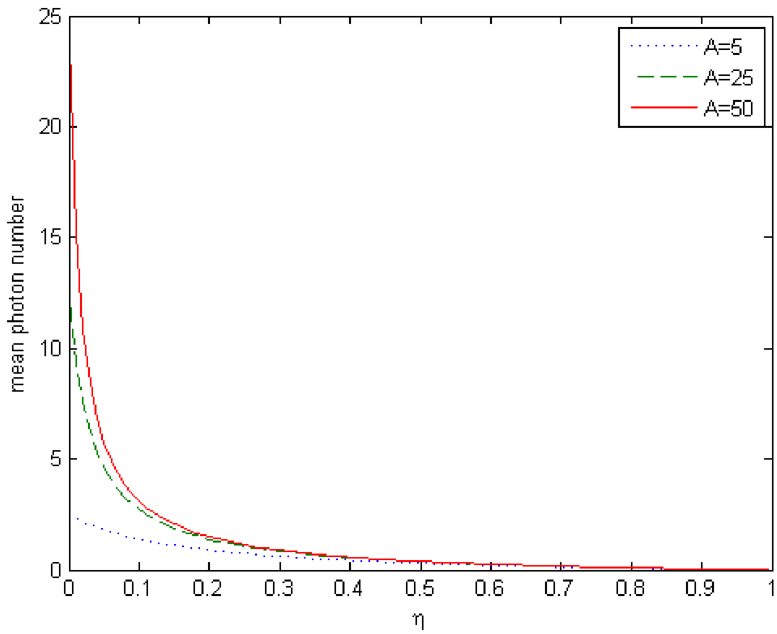

Figure 3.1: Plots of Eq. (3.1.15) versus η for $\kappa = 0.8$ and for different values of the linear gain coefficient.

Fig 3.1 and Eq.(3.1.15) shows that the mean photon number of the output light increases with the linear gain coefficient and decreases with η.

3.2 The photon-number variance

The variance of the photon number for the output mode is expressible as

$$(\Delta\hat{n}_{out})^2 = \langle\hat{n}^2_{out}\rangle - \bar{n}^2_{out}. \tag{3.2.1}$$

Using the commutaion relation $[\hat{a}_{out}, \hat{a}^\dagger_{out}] = \kappa$ along with Eq. (2.2.2), we can write

$$\begin{aligned}
\langle\hat{n}^2_{out}\rangle &= \langle\hat{a}^\dagger_{out}\hat{a}_{out}\hat{a}^\dagger_{out}\hat{a}_{out}\rangle,\\
&= \langle\hat{a}^\dagger_{out}(\hat{a}^\dagger_{out}\hat{a}_{out} + \kappa)\hat{a}_{out}\rangle,\\
&= \langle(\hat{a}^{\dagger 2}_{out}\hat{a}^2_{out} + \kappa\hat{a}^\dagger_{out}\hat{a}_{out})\rangle,\\
&= \langle\hat{a}^{\dagger 2}_{out}\hat{a}^2_{out}\rangle + \kappa\langle\hat{a}^\dagger_{out}\hat{a}_{out}\rangle,\\
&= \langle\hat{a}^{\dagger 2}_{out}\hat{a}^2_{out}\rangle + \kappa\bar{n}_{out}. \qquad (3.2.2)
\end{aligned}$$

We recall that the expectation value of $\hat{a}^{\dagger 2}\hat{a}^2$ is expressible as

$$\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle = \frac{1}{\pi}\int d^2\alpha\ d^2\beta |\langle\alpha|\beta\rangle|^2 Q(\alpha^*,\beta)\beta^{*2}\alpha^2. \tag{3.2.3}$$

Thus on account of (2.2.2), there follows

$$\langle \hat{a}^{\dagger 2}_{out}\hat{a}^2_{out}\rangle = \frac{m^2}{\pi}\int d^2\alpha_{out} d^2\beta_{out} |\langle\alpha_{out}|\beta_{out}\rangle|^2 Q(\alpha^*_{out},\beta_{out})\beta^{*2}_{out}\alpha^2_{out}, \tag{3.2.4}$$

where m is given by Eq. (3.1.7).

In view of (3.1.3), (3.1.4), and (3.2.4), we have

$$\begin{aligned}\langle \hat{a}^{\dagger 2}_{out}\hat{a}^2_{out}\rangle &= m^2 N\int \frac{d^2\alpha_{out}}{\pi} exp(-m\alpha_{out}\alpha^*_{out} + mv\alpha^{*2}_{out})\alpha^2_{out}\\ &\times \int\frac{d^2\beta_{out}}{\pi} exp\Big[-m\beta_{out}\beta^*_{out} + (m\alpha^*_{out} - mu\alpha^*_{out})\beta_{out}\\ &+m\alpha_{out}\beta^*_{out} + mv\beta^2_{out}/2\Big]\beta^{*2}_{out}. \end{aligned}\tag{3.2.5}$$

Let

$$\begin{aligned} I_1 &= \int\frac{d^2\beta_{out}}{\pi} exp[-m\beta_{out}\beta^*_{out} + (m\alpha^*_{out} - mu\alpha^*_{out})\beta_{out}\\ &+m\alpha_{out}\beta^*_{out} + mv\beta^2_{out}/2]\beta^{*2}_{out},\\ &= \frac{\partial^2}{\partial\gamma^2_{out}}\int\frac{d^2\beta_{out}}{\pi} exp[-m\beta_{out}\beta^*_{out} + (m\alpha^*_{out} - mu\alpha^*_{out})\beta_{out}\\ &+(m\alpha_{out}+\gamma_{out})\beta^*_{out} + mv\beta^2_{out}/2]_{\gamma\_out=0},\\ &= \frac{1}{m}\frac{\partial^2}{\partial\gamma^2_{out}} exp[-m\alpha_{out}\alpha^*_{out} + \alpha^*_{out}\gamma_{out} - mu\alpha^*_{out}\alpha_{out} - u\alpha^*_{out}\gamma_{out}\\ &+mv\alpha^2_{out}/2 + v\alpha_{out}\gamma_{out} + \frac{v}{m}\gamma^2_{out}/2]_{\gamma_{out}=0},\\ &= \frac{2v(1-u)\alpha_{out}\alpha^*_{out} + (1-2u+u^2)\alpha^{*2}_{out} + v^2\alpha^2_{out} + v/m}{m}\\ &\times exp(-m\alpha_{out}\alpha^*_{out} - mu\alpha^*_{out}\alpha_{out} + mv\alpha^2_{out}/2). \end{aligned}\tag{3.2.6}$$

Applying this result in (3.2.5), we get

$$\begin{aligned}\langle \hat{a}^{\dagger 2}_{out}\hat{a}^{2}_{out}\rangle &= 2mNv(1-u)\int \frac{d^2\alpha_{out}}{\pi} exp[-mu\alpha_{out}\alpha^*_{out} + mv(\alpha^2_{out}+\alpha^{*2}_{out})]\alpha^*_{out}\alpha^3_{out} \\ &+ mN(1-2u+u^2)\int \frac{d^2\alpha_{out}}{\pi} exp[-mu\alpha_{out}\alpha^*_{out} + mv(\alpha^2_{out}+\alpha^{*2}_{out})]\alpha^{*2}_{out}\alpha^2_{out} \\ &+ mNv^2\int \frac{d^2\alpha_{out}}{\pi} exp[-mu\alpha_{out}\alpha^*_{out} + mv(\alpha^2_{out}+\alpha^{*2}_{out})]\alpha^4_{out} \\ &+ Nv\int \frac{d^2\alpha_{out}}{\pi} exp[-mu\alpha_{out}\alpha^*_{out} + mv(\alpha^2_{out}+\alpha^{*2}_{out})]\alpha^2_{out}. \end{aligned} \tag{3.2.7}$$

Let I_1, I_2, I_3, and I_4 repersents the first, second, third, and the fourth integral respectively. Therefore, Eq.(3.2.7) can be re-written as

$$\langle \hat{a}^{\dagger 2}_{out}\hat{a}^{2}_{out}\rangle = I_1 + I_2 + I_3 + I_4. \tag{3.2.8}$$

Now the first integral in (3.2.7) can be written as

$$\begin{aligned} I_1 &= 2mNv(1-u)\frac{\partial}{\partial\gamma_{out}}\frac{\partial^3}{\partial\beta^3_{out}}\int \frac{d^2\alpha_{out}}{\pi} exp[-mu\alpha_{out}\alpha^*_{out} + \beta_{out}\alpha_{out} + \gamma_{out}\alpha^*_{out} \\ &+ mv(\alpha^2_{out}+\alpha^{*2}_{out})]_{\beta_{out}=\gamma_{out}} = 0, \\ &= 2mNv(1-u)\frac{\partial}{\partial\gamma_{out}}\frac{\partial^3}{\partial\beta^3_{out}} exp[\frac{u\beta_{out}\gamma_{out} + v(\beta^2_{out}/2+\gamma^2_{out}/2)}{m(u^2-v^2)}]_{\beta_{out}} = \gamma_{out} = 0, \\ &= \frac{6\kappa^2 v^2(u-u^2)}{(u^2-v^2)^2}. \end{aligned} \tag{3.2.9}$$

Following similar procedure, one can get

$$I_2 = \frac{\kappa^2(1-2u+u^2)(v^2+2u^2)}{(u^2-v^2)^2}, \tag{3.2.10}$$

$$I_3 = \frac{3\kappa^2 v^4}{(u^2-v^2)^2}, \tag{3.2.11}$$

and

$$I_4 = \frac{\kappa^2 v^2}{(u^2-v^2)}. \tag{3.2.12}$$

Now substitution of (3.2.9), (3.2.10), (3.2.11), and (3.2.12) into (3.2.8) yields

$$\langle \hat{a}^{\dagger 2}_{out}\hat{a}^{2}_{out}\rangle = \frac{\kappa^2[v^2+2u^2+2u^4-4u^3-4u^2v^2+4uv^2+2v^4]}{(u^2-v^2)^2}. \tag{3.2.13}$$

On account of (2.1.97), (2.1.98), and (3.2.13), we have

$$\langle \hat{a}^{\dagger 2}_{out}\hat{a}^{2}_{out}\rangle = \kappa^2[2-4a+2a^2+b^2] \tag{3.2.14}$$

In view of (2.1.92), and (2.1.93), Eq. (3.2.14) takes at steady the form

$$\langle \hat{a}^{\dagger 2}_{out}\hat{a}^{2}_{out}\rangle = \frac{\kappa^2 A^2(\eta-1)^2}{2(A\eta+\kappa)^2} + \frac{\kappa^2 A^2(1-\eta^2)}{4(A\eta+\kappa)^2} \tag{3.2.15}$$

Now combination of (3.2.2) and (3.2.15) yields

$$\begin{aligned}\langle \hat{n}^2_{out}\rangle \quad &= \frac{\kappa^2 A^2(\eta-1)^2}{2(A\eta+\kappa)^2} + \frac{\kappa^2 A^2(1-\eta^2)}{4(A\eta+\kappa)^2} + \kappa\bar{n}_{out}. \\ &= 2\bar{n}^2_{out} + \kappa\bar{n}_{out} + \frac{\kappa^2 A^2(1-\eta^2)}{4(A\eta+\kappa)^2}\end{aligned} \tag{3.2.16}$$

With the aid of Eqs. (3.2.1), and (3.2.16) the variance of the photon number at steady state is expressible as

$$\begin{aligned}(\Delta\hat{n}_{out})^2_{ss} \quad &= 2\bar{n}^2_{out} + \kappa\bar{n}_{out} + \frac{\kappa^2 A^2(1-\eta^2)}{4(A\eta+\kappa)^2} - \bar{n}^2_{out}, \\ &= \kappa\bar{n}_{out} + \frac{\kappa^2 A^2(1-\eta)^2}{4(A\eta+\kappa)^2} + \frac{\kappa^2 A^2(1-\eta^2)}{4(A\eta+\kappa)^2}, \\ &= \kappa\bar{n}_{out} + \frac{\kappa^2 A^2(1-\eta)}{2(A\eta+\kappa)^2}, \\ &= \kappa\bar{n}_{out}(1+\frac{A}{A\eta+\kappa}).\end{aligned} \tag{3.2.17}$$

Fig 3.2 represents the mean photon number versus η (solid curve) and the variance of the photon number versus η (dashed curve) for $\kappa = 0.8$ and $A = 2$. We observe from the figure that the variance of the photon number is greater than the mean photon number. This clearly tells us that the photon statistics is super-Poissonian.

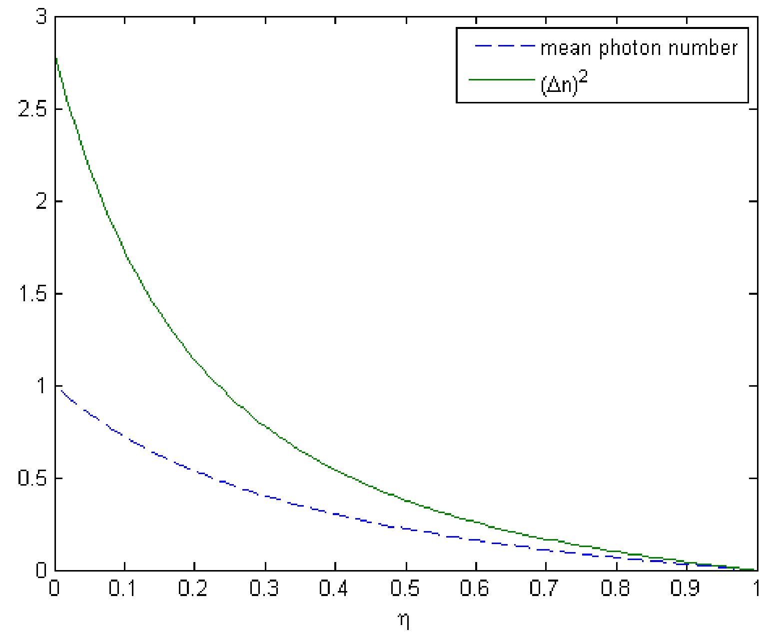

Figure 3.2: Plots of Eq. (3.1.15) and Eq. (3.2.17) versus η for $\kappa = 0.8$ and $A = 2$.

3.3 The Quadrature variance

In this section we seek to study the squeezing properties of the output light produced by a three-level laser. To this end, we first determine the quadraure variances.

The squeezing properties of a single light are described by two quadrature operators defined as [1]

$$\hat{a}_+ = \hat{a} + \hat{a}^\dagger, \tag{3.3.1}$$

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}). \tag{3.3.2}$$

These operators are Hermitian and satisfy the commutation relation

$$\left[\hat{a}_+, \hat{a}_-\right] = 2i. \tag{3.3.3}$$

The operators $\hat{a}_+$ and $\hat{a}_-$ represent physical quantities called the plus and minus quadratures. A single-mode light is said to be in a squeezed state if either $\Delta a_+ < 1$ or $\Delta a_- < 1$ such that $\Delta a_+ \Delta a_- < 1$. The quadrature variance for a light beam interms of quadrature operators can be expressed as

$$(\Delta a_{\pm})^2 = \langle \hat{a}_{\pm}^2 \rangle - \langle \hat{a}_{\pm} \rangle^2. \tag{3.3.4}$$

On account of (3.3.1), (3.3.2), and (3.3.4), this expression can be put in the form

$$\begin{aligned}(\Delta a_{\pm})^2 &= 1 \pm \langle \hat{a}^2 \rangle \pm \langle \hat{a}^{\dagger 2} \rangle + 2\langle \hat{a}^{\dagger} \hat{a} \rangle \mp \langle \hat{a} \rangle^2 \\ &\quad \mp \langle \hat{a}^{\dagger} \rangle^2 \mp 2\langle \hat{a} \rangle \langle \hat{a}^{\dagger} \rangle. \end{aligned} \tag{3.3.5}$$

The expectation value of an operator function $\hat{A}(\hat{a}^{\dagger}, \hat{a})$ is expressible as

$$\langle \hat{A} \rangle = \frac{1}{\pi} \int d^2\alpha \ d^2\beta |\langle \alpha|\beta \rangle|^2 Q(\alpha^*, \beta) A_n(\beta^*, \alpha), \tag{3.3.6}$$

where $A_n(\beta^*, \alpha)$ is the c-number function correponding to operator function $\hat{A}(\hat{a}^{\dagger}, \hat{a})$ in the normal order.

Hence on account of (3.1.4), we have

$$\begin{aligned}\langle \hat{a} \rangle &= N \int \frac{d^2\alpha}{\pi} exp(-\alpha\alpha^* + v\alpha^{*2}/2)\alpha \\ &\quad \times \int \frac{d^2\beta}{\pi} exp[-\beta\beta^* + (\alpha^* - u\alpha^*)\beta + \alpha\beta^* + v\beta^2/2], \end{aligned} \tag{3.3.7}$$

Let

$$\begin{aligned} I_1 &= \int \frac{d^2\beta}{\pi} exp[-\beta\beta^* + (\alpha^* - u\alpha^*)\beta + \alpha\beta^* + v\beta^2/2], \\ &= exp(\alpha\alpha^* - u\alpha^*\alpha + v\alpha^2/2). \end{aligned} \tag{3.3.8}$$

In view of this result, we see that

$$\begin{aligned} \langle \hat{a} \rangle &= N \int \frac{d^2\alpha}{\pi} exp[-u\alpha\alpha^* + v(\alpha^2/2 + \alpha^{*2}/2]\alpha, \\ &= N \frac{\partial}{\partial\beta} \int \frac{d^2\alpha}{\pi} exp[-u\alpha\alpha^* + \beta\alpha + v(\alpha^2/2 + \alpha^{*2}/2]_{\beta=0}, \\ &= N \frac{\partial}{\partial\beta} \frac{1}{(u^2 - v^2)^{1/2}} exp[\frac{v\beta^2/2}{(u^2 - v^2)}]_{\beta=0}, \\ &= 0. \end{aligned} \tag{3.3.9}$$

Similarily

$$\langle \hat{a}^\dagger \rangle = 0. \tag{3.3.10}$$

On the other hand,

$$\begin{aligned}\langle \hat{a}^2 \rangle &= N\int \frac{d^2\alpha}{\pi} exp(-\alpha\alpha^* + v\alpha^{*2}/2)\alpha^2 \\ &\times \int \frac{d^2\beta}{\pi} exp[-\beta\beta^* + (\alpha^* - u\alpha^*)\beta + \alpha\beta^* + v\beta^2/2],\end{aligned} \tag{3.3.11}$$

With the aid of (3.3.8), we have

$$\begin{aligned}\langle \hat{a}^2 \rangle \quad &= N\int \frac{d^2\alpha}{\pi} exp[-u\alpha\alpha^* + v(\alpha^2/2 + \alpha^{*2}/2]\alpha^2, \\ &= N\frac{\partial^2}{\partial\beta^2}\int \frac{d^2\alpha}{\pi} exp[-u\alpha\alpha^* + \beta\alpha + v(\alpha^2/2 + \alpha^{*2}/2]_{\beta=0}, \\ &= \frac{\partial^2}{\partial\beta^2} exp[\frac{v\beta^2/2}{(u^2 - v^2)}]_{\beta=0}, \\ &= \frac{v}{u^2 - v^2}.\end{aligned} \tag{3.3.12}$$

With the help of (2.1.97) and (2.1.98), Eq. (3.3.12) can be written as

$$\langle \hat{a}^2 \rangle = b. \tag{3.3.13}$$

In the same way, one can get

$$\langle \hat{a}^{\dagger 2} \rangle = b. \tag{3.3.14}$$

And,

$$\langle \hat{n} \rangle = \langle \hat{a}^\dagger \hat{a} \rangle = \int d^2\alpha Q(\alpha^*, \alpha)(\alpha\alpha^* - 1) \tag{3.3.15}$$

where $\alpha\alpha^* - 1$ is the c-number function corresponding to the number operator $\hat{n}$ in the antinormal order.

$$\langle \hat{a}^\dagger \hat{a} \rangle = \int d^2\alpha Q(\alpha^*, \alpha)\alpha\alpha^* - \int d^2\alpha Q(\alpha^*, \alpha). \tag{3.3.16}$$

In view of Eq. (2.1.96) and (2.1.99), we can write

$$\begin{aligned}\langle \hat{a}^\dagger \hat{a} \rangle \quad &= (u^2 - v^2)\left[\int \frac{d^2\alpha}{\pi} exp(-u\alpha\alpha^* + v(\alpha^2 + \alpha^{*2})/2)\alpha\alpha^*\right] - 1, \\ &= -(u^2 - v^2)\frac{\partial}{\partial u}\left[\int \frac{d^2\alpha}{\pi} exp(-u\alpha\alpha^* + v(\alpha^2 + \alpha^{*2})/2)\right] - 1.\end{aligned} \tag{3.3.17}$$

Carrying out the integration applying Eq. (2.1.95), we get

$$\langle \hat{a}^\dagger \hat{a} \rangle = \frac{u}{u^2 - v^2} - 1 \tag{3.3.18}$$

On account of Eqs. (2.1.92), (2.1.93), (2.1.97), and (2.1.98) the mean photon number takes the form

$$\langle \hat{a}^\dagger \hat{a} \rangle = \frac{A(1-\eta)}{2(A\eta + \kappa)} \left[1 - e^{-\mu t}\right]. \tag{3.3.19}$$

At steady state this turns to be

$$\langle \hat{a}^\dagger \hat{a} \rangle = \frac{A(1-\eta)}{2(A\eta + \kappa)} \tag{3.3.20}$$

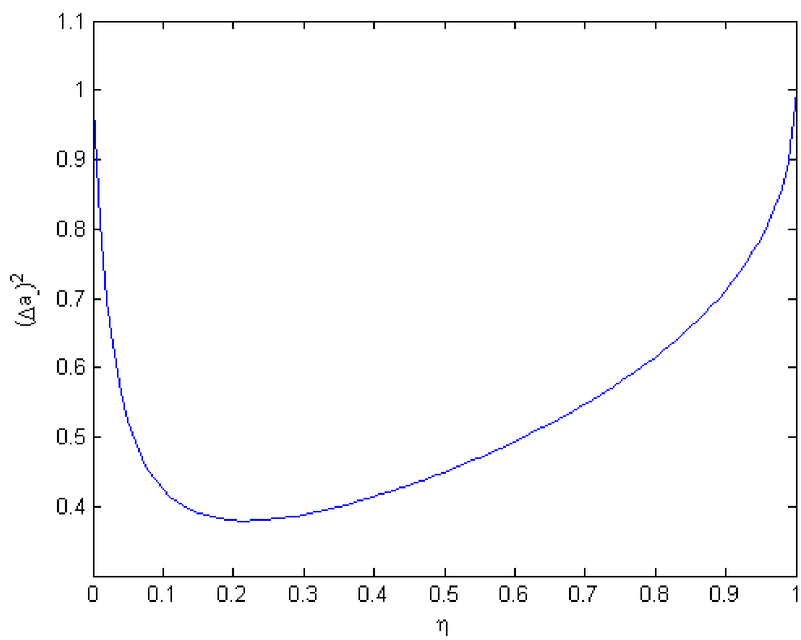

Figure 3.3: Plot of Eq. (3.3.27) versus η for $\kappa = 0.8$ and $A = 25$.

The plot in fig 3.3 clearly shows that the light mode is in the squeezed state and the squeezing occurs in the minus quadrature for values of η between zero and one.

Using Eqs. (3.3.5), (3.3.9), (3.3.10), (3.3.13), and (3.3.14), the quadrature variance for the plus quadrature can be expressed in the form

$$(\Delta a_+)^2 = 1 + 2b + 2\langle \hat{a}^\dagger \hat{a} \rangle. \tag{3.3.21}$$

In view of Eqs. (2.1.93), (3.1.20) and (3.3.21), the plus quadrature variance takes at steady state the form

$$(\Delta a_+)^2 = \frac{A(1+\sqrt{1-\eta^2})+\kappa}{A\eta+\kappa}. \tag{3.3.22}$$

Using Eqs. (3.3.5), (3.3.9), (3.3.10), (3.3.13), and (3.3.14), the quadrature variance for the minus quadrature can be expressed in the form

$$(\Delta a_-)^2 = 1 - 2b + 2\langle\hat{a}^\dagger\hat{a}\rangle. \tag{3.3.23}$$

On account of Eqs. (2.1.94), (3.3.20) and (3.3.23), the minus quadrature variance takes at steady state the form

$$(\Delta a_-)^2 = \frac{A(1-\sqrt{1-\eta^2})+\kappa}{A\eta+\kappa}. \tag{3.3.24}$$

The quadrature variance of the output light can be expressed in terms of the quadrature variance of the cavity light as [11]

$$(\Delta a_\pm)^2_{out} = \kappa(\Delta a_\pm)^2 + (1-\kappa)(\Delta a_\pm)^2_{in}, \tag{3.3.25}$$

where $(\Delta a_\pm)^2_{in}$ is the quadrature variance of the reservoir mode.

Applying (3.3.22) in Eq. (3.3.25), the plus quadrature variance of the output light for vacuum reservoir can be written as

$$(\Delta a_+)^2_{out} = \kappa\Big[\frac{A(1+\sqrt{1-\eta^2})+\kappa}{A\eta+\kappa}\Big] + 1 - \kappa, \tag{3.3.26}$$

and applying (3.3.24) in Eq. (3.3.25), the minus quadrature variance of the output light takes the form

$$(\Delta a_-)^2_{out} = \kappa\Big[\frac{A(1-\sqrt{1-\eta^2})+\kappa}{A\eta+\kappa}\Big] + 1 - \kappa. \tag{3.3.27}$$

Fig 3.4 represents the minus quadrature for a three-level laser at steady state versus η for $\kappa = 0.8$ and for different values of the linear gain coefficient. We see from the plots that the degree of squeezing increases with the linear gain coefficient. We observe from these plots that almost perfect squeezing can be obtained for large values of A and for

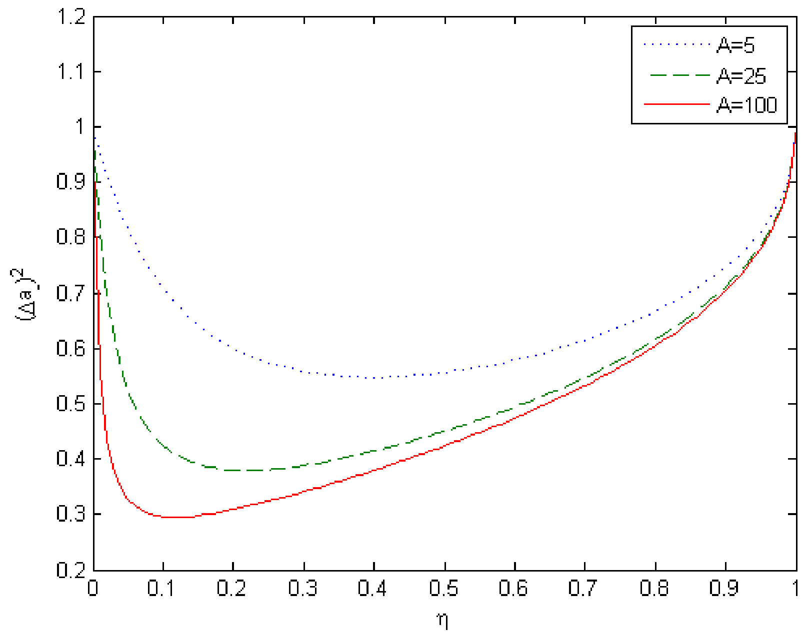

Figure 3.4: Plots of Eq. (3.3.27) versus η for $\kappa = 0.8$ and for different values of the linear gain coefficient.

small values of η.

Since squeezing occurs in the minus quadrature, the degree of squeezing (q) of the output light is defined as

$$q = 1 - (\Delta a_-)^2_{out}. \tag{3.3.28}$$

For $A = 100$, $\eta = 0.1111$, and $\kappa = 0.8$, we get $(\Delta a_-)^2_{out} = 0.2953$ and $q = 0.705$.

This indicates that the squeezing of the single output laser light is 70.5% below the coherent-state level. It is less than the intra-cavity squeezing, which is 88.09% below the coherent-state level [17].

Chapter 4

The Superposed output light from three-level laser

In this chapter we seek to study the statistical properties of superposed output light generated by two identical three-level lasers. To this end, applying the Q function for the superposed output light from two identical three-level lasers, we determine the mean photon number and the variance of the photon number. Using the same Q function, we also calculate the quadrature variances for the output mode.

4.1 The mean photon number

The expectation value of an operator function $\hat{A}(\hat{a}^\dagger, \hat{a})$ is expressible as

$$\langle \hat{A} \rangle = \frac{1}{\pi} \int d^2\alpha \; d^2\beta |\langle \alpha | \beta \rangle|^2 Q(\alpha^*, \beta) A_n(\beta^*, \alpha), \tag{4.1.1}$$

where $A_n(\beta^*, \alpha)$ is the c-number function correponding to operator function $\hat{A}(\hat{a}^\dagger, \hat{a})$ in the normal order. Applying the input-out put relation (2.2.2), we have

$$\bar{n}_{out} = \frac{m^2}{\pi} \int d^2\alpha_{out} d^2\beta_{out} |\langle \alpha_{out} | \beta_{out} \rangle|^2 Q(\alpha^*_{out}, \beta_{out}) \beta^*_{out} \alpha_{out}, \tag{4.1.2}$$

From Eq.(2.2.13), we have

$$Q(\alpha^*_{out}, \beta_{out}, t) = \frac{C}{\pi} exp[-\frac{D}{\kappa}\alpha^*_{out}\beta_{out} + \frac{E}{k}(\beta^2_{out} + \alpha^{*2}_{out})/2]. \tag{4.1.3}$$

On account of (4.1.2) and (4.1.3), one can write

$$\begin{aligned}\bar{n}_{out} &= m^2 C \int d^2\alpha_{out} exp(-m\alpha_{out}\alpha^*_{out} + mE\alpha^{*2}_{out}/2)\alpha_{out} \\ &\times \int \frac{d^2\beta_{out}}{\pi} exp\Big[- m\beta_{out}\beta^*_{out} + (m\alpha^*_{out} - mD\alpha^*_{out})\beta_{out} \\ &+ m\alpha_{out}\beta^*_{out} + mE\beta^2_{out}/2\Big]\beta^*_{out}. \end{aligned} \tag{4.1.4}$$

Let

$$\begin{aligned} I_1 &= \int \frac{d^2\beta_{out}}{\pi} exp[-m\beta_{out}\beta^*_{out} + (m\alpha^*_{out} - mD\alpha^*_{out})\beta_{out} \\ &+ m\alpha_{out}\beta^*_{out} + mE\beta^2_{out}/2\Big]\beta^*, \\ &= \frac{d}{d\gamma_{out}} \int \frac{d^2\beta_{out}}{\pi} exp\Big[- m\beta_{out}\beta^*_{out} + (m\alpha^*_{out} - mD\alpha^*_{out})\beta_{out} \\ &+ (m\alpha_{out} + \gamma_{out})\beta^*_{out} + mE\beta^2_{out}/2\Big]_{\gamma=0}. \end{aligned} \tag{4.1.5}$$

Carrying out the integration over β_{out} and differentiating with respect to γ_{out}, we get

$$I_1 = \frac{\alpha^*_{out} - D\alpha^*_{out} + E\alpha_{out}}{m} exp(m\alpha_{out}\alpha^*_{out} - mD\alpha^*_{out}\alpha_{out} + mE\alpha^2_{out}/2). \tag{4.1.6}$$

Now upon substituting (4.1.6) into (4.1.4), we have

$$\begin{aligned}\bar{n}_{out} &= Cm(1-D)\int \frac{d^2\alpha_{out}}{\pi} exp[-mD\alpha_{out}\alpha^*_{out} + mE(\alpha^2_{out}/2 + \alpha^{*2}_{out}/2]\alpha_{out}\alpha^*_{out} \\ &+ CmE \int \frac{d^2\alpha_{out}}{\pi} exp[-mD\alpha_{out}\alpha^*_{out} + mE(\alpha^2_{out}/2 + \alpha^{*2}_{out}/2]\alpha^2_{out}. \end{aligned} \tag{4.1.7}$$

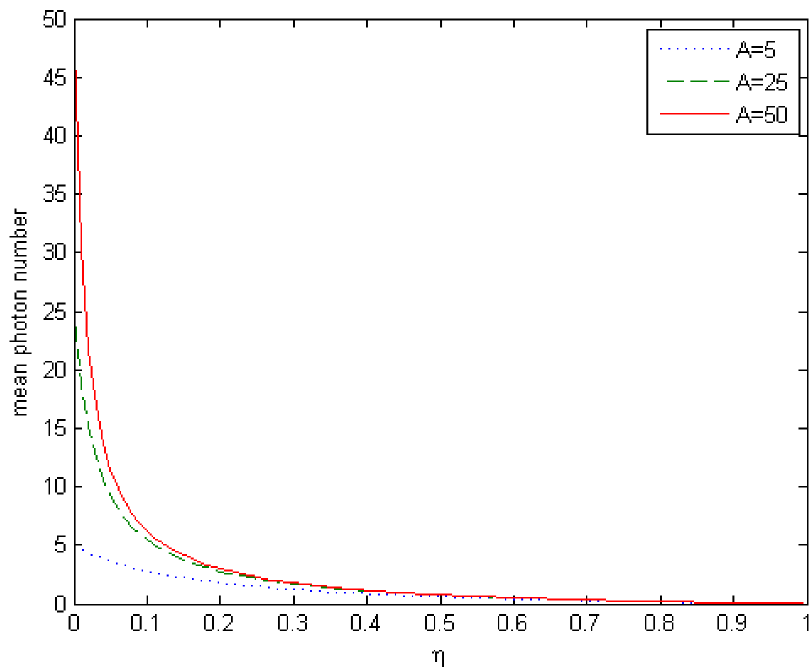

Figure 4.1: Plots of Eq. (4.1.12) versus η for $\kappa = 0.8$ and for different values of the linear gain coefficient.

Let the first integral be

$$\begin{aligned}
I_2 &= Cm(1-D)\int \frac{d^2\alpha_{out}}{\pi} exp[-mD\alpha_{out}\alpha^*_{out} + mE(\alpha^2_{out}/2 + \alpha^{*2}_{out}/2)]\alpha_{out}\alpha^*_{out}, \\
&= \frac{\partial}{\partial\beta_{out}}\frac{\partial}{\partial\gamma_{out}} Cm(1-D)\int \frac{d^2\alpha_{out}}{\pi} exp[-mD\alpha_{out}\alpha^*_{out} + \beta_{out}\alpha_{out} \\
&+\gamma_{out}\alpha^*_{out} + mE(\alpha^2_{out}/2 + \alpha^{*2}_{out}/2)]_{\gamma_{out}=\beta_{out}=0}, \\
&= \frac{\partial}{\partial\beta_{out}}\frac{\partial}{\partial\gamma_{out}} \frac{C(1-D)}{(D^2-E^2)^{1/2}} exp[\frac{D\beta_{out}\gamma_{out} + E(\gamma^2_{out} + \gamma^2_{out})}{m(D^2-E^2)}]_{\gamma_{out}=\beta_{out}=0}, \\
&= \frac{\kappa C(D-D^2)}{(D^2-E^2)^{3/2}}. \qquad (4.1.8)
\end{aligned}$$

And let the second integral be

$$
\begin{aligned}
I_3 \quad &= CmE\int \frac{{}^2\alpha_{out}}{\pi} exp[-mD\alpha_{out}\alpha^*_{out} + mE(\alpha^2_{out}/2 + \alpha^{*2}_{out}/2]\alpha^2_{out}, \\
&= CmE\frac{\partial^2}{\partial\beta^2_{out}}\int \frac{d^2\alpha_{out}}{\pi} exp[-mD\alpha_{out}\alpha^*_{out} + \beta_{out}\alpha_{out} + mE(\alpha^2_{out}/2 + \alpha^{*2}_{out}/2]_{\beta_{out}=0}, \\
&= CE\frac{\partial^2}{\partial\beta^2_{out}}\frac{1}{(D^2-E^2)^{1/2}} exp[\frac{E\beta^2_{out}/2}{m(D^2-E^2)}]_{\beta_{out}=0}, \\
&= \frac{\kappa CE^2}{(D^2-E^2)^{3/2}}.
\end{aligned} \tag{4.1.9}
$$

Now combination of (4.1.7), (4.1.8), and (4.1.9) leads to

$$
\begin{aligned}
\bar{n}_{out} \quad &= \frac{\kappa C[E^2 + D - D^2]}{(D^2-E^2)^{3/2}}, \\
&= \frac{\kappa CD}{(D^2-E^2)^{3/2}} - \frac{\kappa C}{(D^2-E^2)^{1/2}}, \\
&= \kappa[\frac{D}{D^2-E^2} - 1].
\end{aligned} \tag{4.1.10}
$$

Finally, in view of (2.1.92), (2.1.93), (2.2.15), (2.2.16) and, (4.1.10) the mean photon number takes the form

$$
\begin{aligned}
\bar{n}_{out} \quad &= 2\kappa(-1 + a) \\
&= \frac{\kappa A(1-\eta)}{A\eta + \kappa}[1 - e^{-\mu t}].
\end{aligned} \tag{4.1.11}
$$

At steady state this reduces to

$$
\bar{n}_{out} = \frac{\kappa A(1-\eta)}{A\eta + \kappa}. \tag{4.1.12}
$$

This represents the mean photon number of the superposed output light beams is twice that of a single laser light beam.

Fig 4.1 and Eq. (4.1.12) show that the mean photon number of the superposed output light increases with the linear gain coefficient and decreases with η.

4.2 The photon-number variance

The variance of photon number is expressible as

$$(\Delta\hat{n}_{out})^2 = \langle\hat{n}_{out}^2\rangle - \bar{n}_{out}^2. \tag{4.2.1}$$

But from chapter three, we have

$$\langle\hat{n}_{out}^2\rangle = \langle\hat{a}_{out}^{\dagger 2}\hat{a}_{out}^2\rangle + \kappa\bar{n}_{out}. \tag{4.2.2}$$

Hence the expectation value of $\hat{a}^{\dagger 2}\hat{a}^2$ can be expressed as

$$\langle\hat{a}^{\dagger 2}\hat{a}^2\rangle = \frac{1}{\pi}\int d^2\alpha\ d^2\beta|\langle\alpha|\beta\rangle|^2 Q(\alpha^*,\beta)\beta^{*2}\alpha^2. \tag{4.2.3}$$

Thus on account of (2.2.2), there follows

$$\langle\hat{a}_{out}^{\dagger 2}\hat{a}_{out}^2\rangle = \frac{m^2}{\pi}\int d^2\alpha_{out}d^2\beta_{out}|\langle\alpha_{out}|\beta_{out}\rangle|^2 Q(\alpha_{out}^*,\beta_{out})\beta_{out}^{*2}\alpha_{out}^2. \tag{4.2.4}$$

In view of (3.1.3), (4.1.3), and (4.2.3), we have

$$\begin{aligned}\langle\hat{a}_{out}^{\dagger 2}\hat{a}_{out}^2\rangle &= m^2C\int\frac{d^2\alpha_{out}}{\pi}exp(-m\alpha_{out}\alpha_{out}^* + mE\alpha_{out}^{*2})\alpha_{out}^2 \\ &\times\int\frac{d^2\beta_{out}}{\pi}exp\Big[-m\beta_{out}\beta_{out}^* + (m\alpha_{out}^* - mD\alpha_{out}^*)\beta_{out} \\ &+m\alpha_{out}\beta_{out}^* + mE\beta_{out}^2/2\Big]\beta_{out}^{*2}.\end{aligned} \tag{4.2.5}$$

Let

$$\begin{aligned}I_1 &= \int\frac{d^2\beta_{out}}{\pi}exp[-m\beta_{out}\beta_{out}^* + (m\alpha_{out}^* - mD\alpha_{out}^*)\beta_{out} \\ &+m\alpha_{out}\beta_{out}^* + mE\beta_{out}^2/2]\beta_{out}^{*2}, \\ &= \frac{\partial^2}{\partial\gamma_{out}^2}\int\frac{d^2\beta_{out}}{\pi}exp[-m\beta_{out}\beta_{out}^* + (m\alpha_{out}^* - mD\alpha_{out}^*)\beta_{out} \\ &+(m\alpha_{out} + \gamma_{out})\beta_{out}^* + mE\beta_{out}^2/2]_{\gamma\_out=0}, \\ &= \frac{1}{m}\frac{\partial^2}{\partial\gamma_{out}^2}exp[-m\alpha_{out}\alpha_{out}^* + \alpha_{out}^*\gamma_{out} - mD\alpha_{out}^*\alpha_{out} - D\alpha_{out}^*\gamma_{out} \\ &+mE\alpha_{out}^2/2 + E\alpha_{out}\gamma_{out} + \frac{E}{m}\gamma_{out}^2/2]_{\gamma\_out=0}, \\ &= \frac{2E(1-D)}{m}\alpha_{out}\alpha_{out}^* + \frac{1-2D+D^2}{m}\alpha_{out}^{*2} + \frac{E^2}{m}\alpha_{out}^2 + E/m^2 \\ &\times exp(-m\alpha_{out}\alpha_{out}^* - mD\alpha_{out}^*\alpha_{out} + mE\alpha_{out}^2/2).\end{aligned} \tag{4.2.6}$$

Applying this result in (4.2.5), we get

$$\begin{aligned}
\langle \hat{n}^2_{out} \rangle &= 2mCE(1-D)\int \frac{d^2\alpha_{out}}{\pi} exp[-mD\alpha_{out}\alpha^*_{out} + mE(\alpha^2_{out} + \alpha^{*2}_{out})]\alpha^*_{out}\alpha^3_{out} \\
&+ mC(1-2D+D^2)\int \frac{d^2\alpha_{out}}{\pi} exp[-mD\alpha_{out}\alpha^*_{out} + mE(\alpha^2_{out} + \alpha^{*2}_{out})]\alpha^{*2}_{out}\alpha^2_{out} \\
&+ mCE^2 \int \frac{d^2\alpha_{out}}{\pi} exp[-mD\alpha_{out}\alpha^*_{out} + mE(\alpha^2_{out} + \alpha^{*2}_{out})]\alpha^4_{out} \\
&+ CE \int \frac{d^2\alpha_{out}}{\pi} exp[-mD\alpha_{out}\alpha^*_{out} + mE(\alpha^2_{out} + \alpha^{*2}_{out})]\alpha^2_{out}. \qquad (4.2.7)
\end{aligned}$$

Let I_1, I_2, I_3, and I_4 repersents the first, second, third, and the fourth integral respectively. Therefore, Eq.(4.2.7) can be re-written as

$$\langle \hat{n}^2_{out} \rangle = I_1 + I_2 + I_3 + I_4. \qquad (4.2.8)$$

Now the first integral in (4.2.5) can be written as

$$\begin{aligned}
I_1 &= 2mCE(1-D)\frac{\partial}{\partial \gamma_{out}}\frac{\partial^3}{\partial \beta^3_{out}} \int \frac{d^2\alpha_{out}}{\pi} exp[-mD\alpha_{out}\alpha^*_{out} + \beta_{out}\alpha_{out} + \gamma_{out}\alpha^*_{out} \\
&+ mE(\alpha^2_{out} + \alpha^{*2}_{out})]_{\beta_{out}} = \gamma_{out} = 0, \\
&= 2mCE(1-D)\frac{\partial}{\partial \gamma_{out}}\frac{\partial^3}{\partial \beta^3_{out}} exp[\frac{D\beta_{out}\gamma_{out} + E(\beta^2_{out}/2 + \gamma^2_{out}/2)}{m(D^2-E^2)}]_{\beta_{out}} = \gamma_{out} = 0, \\
&= \frac{6\kappa^2 E^2(D-D^2)}{(D^2-E^2)^2}. \qquad (4.2.9)
\end{aligned}$$

Following similar procedure, one can get

$$I_2 = \frac{\kappa^2(1-2D+D^2)(E^2+2D^2)}{(D^2-E^2)^2}, \qquad (4.2.10)$$

$$I_3 = \frac{3\kappa^2 E^4}{(D^2-E^2)^2}, \qquad (4.2.11)$$

and

$$I_4 = \frac{\kappa^2 E^2}{D^2-E^2}. \qquad (4.2.12)$$

Upon substituting (4.2.9), (4.2.10), (4.2.11), and (4.2.12) into (4.2.8) gives

$$\langle \hat{a}^{\dagger 2}_{out}\hat{a}^2_{out} \rangle = \frac{\kappa^2[E^2 + 2D^2 + 2D^4 - 4D^3 - 4D^2E^2 + 4DE^2 + 2E^4]}{(D^2-E^2)^2}. (4.2.13)$$

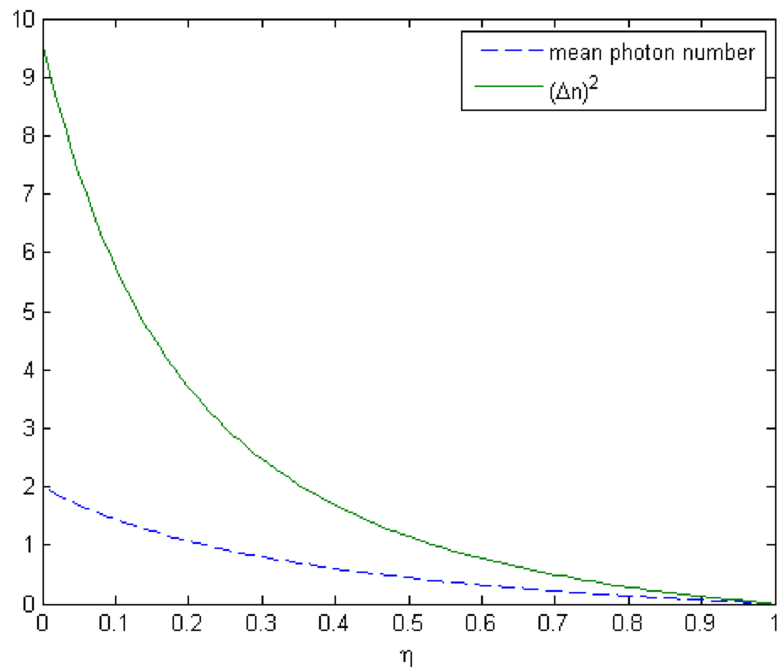

Figure 4.2: Plots of Eq. (4.1.12) and Eq. (4.2.17) versus η for $\kappa = 0.8$ and $A = 2$.

On account of (2.2.15), (2.2.16), and (4.2.13), we have

$$\langle \hat{a}^{\dagger 2}_{out}\hat{a}^{2}_{out}\rangle = 4\kappa^2[2 - 4a + 2a^2 + b^2] \tag{4.2.14}$$

In view of (2.1.92), and (2.1.93), Eq. (4.2.14) takes at steady the form

$$\langle \hat{a}^{\dagger 2}_{out}\hat{a}^{2}_{out}\rangle_{ss} = \frac{2\kappa^2 A^2(\eta - 1)^2}{(A\eta + \kappa)^2} + \frac{\kappa^2 A^2(1 - \eta^2)}{(A\eta + \kappa)^2} \tag{4.2.15}$$

Now combination of (4.2.2) and (4.2.15) yields

$$\begin{aligned}\langle \hat{n}^2_{out}\rangle_{ss} \quad &= \frac{2\kappa^2 A^2(\eta - 1)^2}{(A\eta + \kappa)^2} + \frac{\kappa^2 A^2(1 - \eta^2)}{(A\eta + \kappa)^2} + \kappa\bar{n}_{out}. \\ &= 2\bar{n}^2_{out} + \kappa\bar{n}_{out} + \frac{\kappa^2 A^2(1 - \eta^2)}{4(A\eta + \kappa)^2} \end{aligned} \tag{4.2.16}$$

With the aid of Eqs. (4.2.1), and (4.2.16) the variance of the photon number at steady state is expressible as

$$\begin{aligned}(\Delta\hat{n}_{out})^2_{ss} &= 2\bar{n}^2_{out} + \kappa\bar{n}_{out} + \frac{\kappa^2A^2(1-\eta^2)}{(A\eta+\kappa)^2} - \bar{n}^2_{out}, \\ &= \kappa\bar{n}_{out} + \bar{n}^2_{out} + \frac{\kappa^2A^2(1-\eta^2)}{(A\eta+\kappa)^2}, \\ &= \kappa\bar{n}_{out} + \frac{2\kappa^2A^2(1-\eta)}{(A\eta+\kappa)^2}, \\ &= \kappa\bar{n}_{out}(1+\frac{2A}{A\eta+\kappa}).\end{aligned} \tag{4.2.17}$$

We observe from the figure 4.2 and Eq. (4.2.17) that the variance of the photon number is greater than the mean photon number. This shows that the photon statistics is super-Poisonian.

4.3 The Quadrature variance

In this section we seek to study the squeezing properties of the output light produced by a pair of three-level lasers. To this end, we first determine the quadraure variances.

The squeezing properties of a single light are described by two quadrature operators defined as [1]

$$\hat{a}_+ = \hat{a} + \hat{a}^\dagger, \tag{4.3.1}$$

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}). \tag{4.3.2}$$

These operators are Hermitian and satisfy the commutation relation

$$\left[\hat{a}_+, \hat{a}_-\right] = 2i. \tag{4.3.3}$$

The operators $\hat{a}_+$ and $\hat{a}_-$ represent physical quantities called the plus and minus quadratures. A single-mode light is said to be in a squeezed state if either $\Delta a_+ < 1$ or $\Delta a_- < 1$ such that $\Delta a_+\Delta a_- \geq 1$. The quadrature variance for a light beam interms of quadrature operators can be expressed as

$$(\Delta a_\pm)^2 = \langle\hat{a}^2_\pm\rangle - \langle\hat{a}_\pm\rangle^2. \tag{4.3.4}$$

On account of (4.3.1), (4.3.2), and (4.3.4), this expression can be put in the form

$$\begin{aligned}(\Delta a_{\pm})^2 &= 1 \pm \langle \hat{a}^2 \rangle \pm \langle \hat{a}^{\dagger 2} \rangle + 2\langle \hat{a}^{\dagger}\hat{a} \rangle \mp \langle \hat{a} \rangle^2 \\ &\quad \mp \langle \hat{a}^{\dagger} \rangle^2 \mp 2\langle \hat{a} \rangle \langle \hat{a}^{\dagger} \rangle. \end{aligned} \tag{4.3.5}$$

We recall that the expectation value of an operator function $\hat{A}(\hat{a}^{\dagger}, \hat{a})$ is expressible as

$$\langle \hat{A} \rangle = \frac{1}{\pi} \int d^2\alpha \ d^2\beta |\langle \alpha | \beta \rangle|^2 Q(\alpha^*, \beta) A_n(\beta^*, \alpha), \tag{4.3.6}$$

where $A_n(\beta^*, \alpha)$ is the c-number function correponding to operator function $\hat{A}(\hat{a}^{\dagger}, \hat{a})$ in the normal order.

Hence on account of (4.1.3), we have

$$\begin{aligned}\langle \hat{a} \rangle &= C \int \frac{d^2\alpha}{\pi} exp(-\alpha\alpha^* + E\alpha^{*2}/2)\alpha \\ &\quad \times \int \frac{d^2\beta}{\pi} exp[-\beta\beta^* + (\alpha^* - D\alpha^*)\beta + \alpha\beta^* + E\beta^2/2], \end{aligned} \tag{4.3.7}$$

Let

$$\begin{aligned} I_1 &= \int \frac{d^2\beta}{\pi} exp[-\beta\beta^* + (\alpha^* - D\alpha^*)\beta + \alpha\beta^* + E\beta^2/2], \\ &= exp(\alpha\alpha^* - D\alpha^*\alpha + E\alpha^2/2). \end{aligned} \tag{4.3.8}$$

In view of this result, we see that

$$\begin{aligned}\langle \hat{a} \rangle &= C \int \frac{d^2\alpha}{\pi} exp[-D\alpha\alpha^* + E(\alpha^2/2 + \alpha^{*2}/2]\alpha, \\ &= C\frac{\partial}{\partial\beta} \int \frac{d^2\alpha}{\pi} exp[-D\alpha\alpha^* + \beta\alpha + E(\alpha^2/2 + \alpha^{*2}/2]_{\beta=0}, \\ &= E\frac{\partial}{\partial\beta} \frac{1}{(D^2 - E^2)^{1/2}} exp[\frac{E\beta^2/2}{(D^2 - E^2)}]_{\beta=0}, \\ &= 0. \end{aligned} \tag{4.3.9}$$

Similarily

$$\langle \hat{a}^{\dagger} \rangle = 0. \tag{4.3.10}$$

On the other hand,

$$\begin{aligned}\langle \hat{a}^2 \rangle \;\; &= \;\; C \int \frac{d^2\alpha}{\pi} exp(-\alpha\alpha^* + E\alpha^{*2}/2)\alpha^2 \\ &\times \int \frac{d^2\beta}{\pi} exp[-\beta\beta^* + (\alpha^* - D\alpha^*)\beta + \alpha\beta^* + E\beta^2/2], \end{aligned} \tag{4.3.11}$$

With the aid of (4.3.8), we have

$$\begin{aligned}\langle \hat{a}^2 \rangle \quad &= C \int \frac{d^2\alpha}{\pi} exp[-D\alpha\alpha^* + E(\alpha^2/2 + \alpha^{*2}/2]\alpha^2, \\ &= C\frac{\partial^2}{\partial\beta^2} \int \frac{d^2\alpha}{\pi} exp[-D\alpha\alpha^* + \beta\alpha + E(\alpha^2/2 + \alpha^{*2}/2]_{\beta=0}, \\ &= \frac{\partial^2}{\partial\beta^2} exp[\frac{E\beta^2/2}{(D^2 - E^2)}]_{\beta=0}, \\ &= \frac{E}{D^2 - E^2}. \end{aligned} \tag{4.3.12}$$

With the help of (2.1.97) and (2.1.98), Eq. (4.3.12) can be written as

$$\langle \hat{a}^2 \rangle = 2b. \tag{4.3.13}$$

In the same way, one can get

$$\langle \hat{a}^{\dagger 2} \rangle = 2b. \tag{4.3.14}$$

And,

$$\langle \hat{n} \rangle = \langle \hat{a}^\dagger \hat{a} \rangle = \int d^2\alpha Q(\alpha^*, \alpha)(\alpha\alpha^* - 1) \tag{4.3.15}$$

where $\alpha\alpha^* - 1$ is the c-number function corresponding to the number operator $\hat{n}$ in the antinormal order.

$$\langle \hat{a}^\dagger \hat{a} \rangle = \int d^2\alpha Q(\alpha^*, \alpha)\alpha\alpha^* - \int d^2\alpha Q(\alpha^*, \alpha). \tag{4.3.16}$$

In view of Eq. (2.1.96) and (2.1.99), we can write

$$\begin{aligned}\langle \hat{a}^\dagger \hat{a} \rangle \quad &= (D^2 - E^2)\left[\int \frac{d^2\alpha}{\pi} exp(-D\alpha\alpha^* + E(\alpha^2 + \alpha^{*2})/2)\alpha\alpha^*\right] - 1, \\ &= -(D^2 - E^2)\frac{\partial}{\partial D}\left[\int \frac{d^2\alpha}{\pi} exp(-D\alpha\alpha^* + E(\alpha^2 + \alpha^{*2})/2)\right] - 1. \end{aligned} \tag{4.3.17}$$

Carrying out the integration applying Eq. (2.1.95), we get

$$\langle \hat{a}^\dagger \hat{a} \rangle = \frac{D}{D^2 - E^2} - 1 \tag{4.3.18}$$

On account of Eqs. (2.1.93), (2.1.94), (2.1.98), (2.1.99), (2.2.15), and (2.2.16) the mean photon number takes the form

$$\langle \hat{a}^\dagger \hat{a} \rangle = \frac{A(1-\eta)}{A\eta + \kappa}[1 - e^{-\mu t}]. \tag{4.3.19}$$

At steady state this turns to

$$\langle \hat{a}^\dagger \hat{a} \rangle = \frac{A(1-\eta)}{A\eta + \kappa} \tag{4.3.20}$$

Using Eqs. (4.3.5), (4.3.9), (4.3.10), (4.3.13), and (4.3.14), the quadrature variance for the plus quadrature can be expressed in the form

$$(\Delta a_+)^2 = 1 + 4b + 2\langle \hat{a}^\dagger \hat{a} \rangle. \tag{4.3.21}$$

In view of Eqs. (2.1.93), (4.1.20) and (4.3.21), the plus quadrature variance takes at steady state the form

$$(\Delta a_+)^2 = \frac{2A(1 + \sqrt{1-\eta^2}) - A\eta + \kappa}{A\eta + \kappa}. \tag{4.3.22}$$

Using Eqs. (4.3.5), (4.3.9), (4.3.10), (4.3.13), and (4.3.14), the quadrature variance for the minus quadrature can be expressed in the form

$$(\Delta a_-)^2 = 1 - 4b + 2\langle \hat{a}^\dagger \hat{a} \rangle. \tag{4.3.23}$$

On account of Eqs. (2.1.93), (4.3.20) and (4.3.23), the minus quadrature variance takes at steady state the form

$$(\Delta a_-)^2 = \frac{2A(1 - \sqrt{1-\eta^2}) - A\eta + \kappa}{A\eta + \kappa}. \tag{4.3.24}$$

The quadrature variance of the output light can be expressed in terms of the quadrature variance of the cavity light as [11]

$$(\Delta a_\pm)^2_{out} = \kappa(\Delta a_\pm)^2 + (1-\kappa)(\Delta a_\pm)^2_{in}, \tag{4.3.25}$$

where $(\Delta a_{\pm})^2_{in}$ is the quadrature variance of the reservoir mode.

Applying (4.3.22) in Eq. (4.3.25), the plus quadrature variance of the output light for vacuum reservoir can be written as

$$(\Delta a_{+})^2_{out} = \kappa\Big[\frac{2A(1+\sqrt{1-\eta^2}) - A\eta + \kappa}{A\eta + \kappa}\Big] + 1 - \kappa, \tag{4.3.26}$$

and applying (4.3.24) in Eq. (4.3.25), the minus quadrature variance of the output light takes the form

$$(\Delta a_{-})^2_{out} = \kappa\Big[\frac{2A(1-\sqrt{1-\eta^2}) - A\eta + \kappa}{A\eta + \kappa}\Big] + 1 - \kappa. \tag{4.3.27}$$

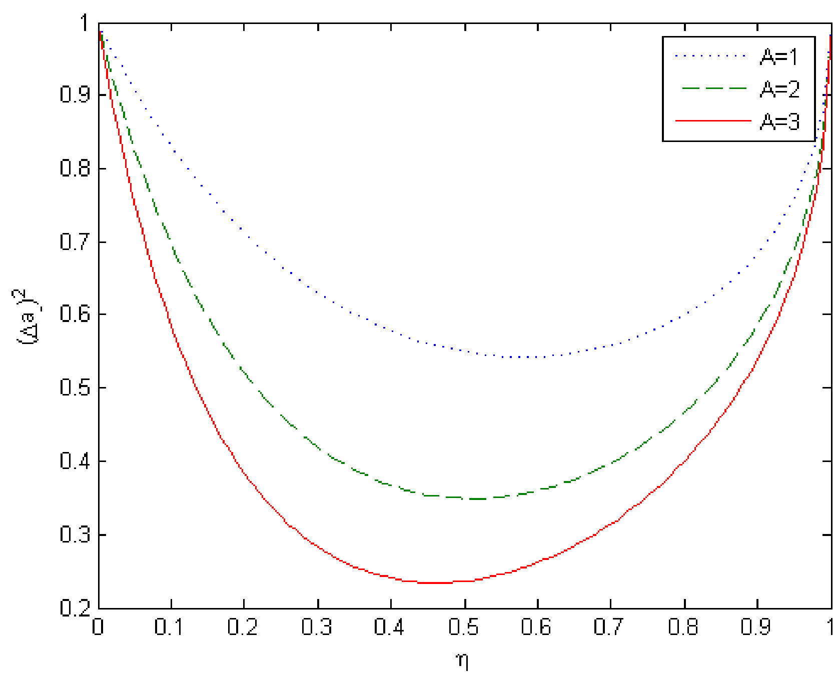

Figure 4.3: Plots of Eq. (4.3.27) versus η for $\kappa = 0.8$ and for different values of the linear gain coefficient.

This can also be expressed interms of a single three-level laser output light as

$$
\begin{aligned}
(\Delta a_-)^2_{out} \quad &= \kappa\big[\frac{2A(1-\sqrt{1-\eta^2})}{A\eta+\kappa}\big] + 1 - \kappa - \frac{\kappa(A\eta-\kappa)}{A\eta+\kappa}, \\
&= 2\kappa\big[\frac{A(1-\sqrt{1-\eta^2})+\kappa}{A\eta+\kappa}\big] + 1 - \kappa - \frac{\kappa(A\eta-\kappa)}{A\eta+\kappa} - \frac{2\kappa^2}{A\eta+\kappa}, \\
&= 2\kappa\big[\frac{A(1-\sqrt{1-\eta^2})+\kappa}{A\eta+\kappa}\big] + 1 - \kappa - \kappa \\
&= 2\Big[\kappa\big[\frac{A(1-\sqrt{1-\eta^2})+\kappa}{A\eta+\kappa}\big] + 1 - \kappa\Big] - \kappa - 1 + \kappa \\
&= 2\Big[\kappa\big[\frac{A(1-\sqrt{1-\eta^2})+\kappa}{A\eta+\kappa}\big] + 1 - \kappa\Big] - 1 \qquad (4.3.28)
\end{aligned}
$$

Taking into account (4.3.28) along with (3.3.27), we find

$$(\Delta a_-)^2_{out} = 2(\Delta a_-)^2_{sout} - 1, \qquad (4.3.29)$$

where $(\Delta a_-)^2_{sout}$ is the the quadrature variance of the output light from one three-level laser.

Fig 4.3 represents the minus quadrature for the superposed output light beams produced by a pair of three-level lasers at steady state versus η for $\kappa = 0.8$ and for different values of the linear gain coefficient. We see from the plots that the degree of squeezing increases with the linear gain coefficient. We observe from these plots that almost perfect squeezing can be obtained for large value of A and for small value of η.

The solid and dashed curves in Fig 4.4 represent the minus quadrature variance for the output light generated from a degenerate three-level laser and two identical three-level lasers for $A = 3$ and $\kappa = 0.8$. Using Eqs. (3.3.28) and (4.3.27) along with the values of $A = 3$, $\eta = 0.4545$, and $\kappa = 0.8$, we get for the superposed laser output light $(\Delta a_-)^2_{out} = 0.234$ and $q = 0.766$. Moreover, using Eqs. (3.3.27) and (3.3.28) along with the values of $A = 3$, $\eta = 0.4545$, and $\kappa = 0.8$, we get for the single laser light $(\Delta a_-)^2_{out} = 0.617$ and $q = 0.383$. From these results, we see that the squeezing of the superposed laser output light and that of the single laser output light are 76.6% and 38.3%

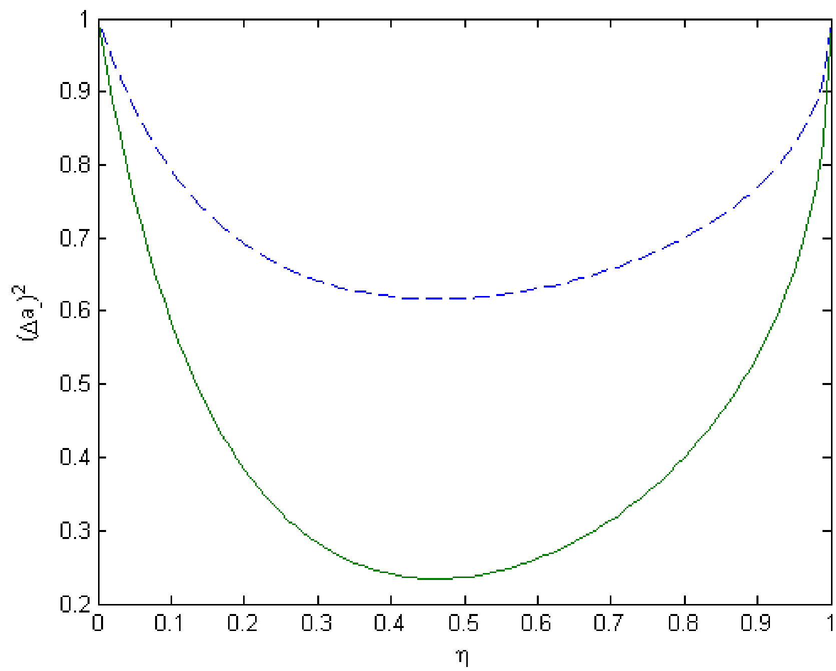

Figure 4.4: Plots of minus quadrature variance for single laser light (dashed curve) and a pair of laser light beams (solid curve) at steady state versus η for $\kappa = 0.8$, and $A = 25$.

below the coherent-state level, respectively. This shows that the degree of squeezing of the superposed light is twice that of the single laser light beam.

Chapter 5

Conclusion

In this thesis we have studied the squeezing and statistical properties of the output light generated from a degenerate three-level laser and two identical three-level lasers, could be analysed with the aid of c-number Langevin equations. Employing the solutions of these equations along with the properties of the noise forces, we have obtained the antinormally ordered characteristic function from which we have found the Q-function for the light generated by a three-level laser. Applying this Q function, we determine the Q function for superposed output light beams produced by a pair of three-level lasers.

Employing the Q-function for the output light produced by a three-level laser, we have calculated the mean photon number, variance of photon number, and the quadrature variance. It is found that the mean photon number increases with the linear gain coefficient and the photon statistics is super-Poissonian. We have also seen that the quadrature squeezing increases with the linear gain coefficient.

On the other hand, applying the Q-function of superposed output light beams we have calculated the mean photon number, variance of photon number, and the quadrature variance. Moreover, we found that the photon statistics is super-Poissonian. We have observed that the effect of superposition of two laser light beams is to increase the mean photn number and decrease the minus quadrature which helps in getting a better squeezing. We have seen that the degree of squeezing of the superposed laser output light is twice that of a single laser output light beam.

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Declaration

This thesis is my original work, has not been presented for a degree in any other University and that all the sources of material used for the thesis have been dully acknowledged.

Name: Tsedeke Abate

Signature:– – – – – – – – – – – –

This thesis has been submitted for examination with my approval as University advisor.

Name: Dr. Fesseha Kassahun

Signature:– – – – – – – – – – –

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