Laplace Adomian Decomposition Method To Solve Non Linear Partial Differential Equation

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Declaration

I declare that the work presented in this thesis, titled " Laplace Adomian Decomposition Method To Solve Non Linear Partial Differential Equation," is my original work. This thesis has not been submitted elsewhere for publication. All sources have been properly acknowledged, and I take full responsibility for the integrity of this thesis.

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After a comprehensive review, we are pleased to endorse Thomas Kebede Desta's project, titled " Laplace Adomian Decomposition Method To Solve Non Linear Partial Differential Equation," for submission to the Department of Mathematics. This project successfully fulfills the requirements for the Master of Science degree in Mathematics.

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Examiner

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Abstract

The Laplace-Adomian Decomposition Method (LADM) is an effective technique for solving nonlinear heat equations, which are crucial in various scientific and engineering applications. By combining the Laplace transform with Adomian's Decomposition Method, LADM simplifies the resolution of nonlinearities and boundary conditions, transforming complex equations into manageable subproblems solved iteratively. This approach enhances computational efficiency and convergence speed without linearization or discretization.

LADM is also successfully applied to the Porous Medium Equation (PME) and Fast Diffusion Equation (FDE), which describe physical processes like fluid flow through porous media and diffusion. The method demonstrates high accuracy and practicality, making it a valuable tool for tackling complex nonlinear problems.

List of Acronyms

Contents

Chapter 1

Introduction

The nonlinear heat equation serves as a fundamental mathematical framework for studying the temporal evolution of temperature in a specified region. It finds extensive applications in physics, engineering, and applied mathematics, particularly in the realms of heat transfer and conduction. The equation is expressed as:

$$u_t = \alpha u_{xx} + f(u)$$

In this formulation, $u(t, x)$ represents the temperature at a given time t and position x, while α denotes the thermal diffusivity of the material. The function $f(u)$ encapsulates the nonlinear characteristics of the temperature, adding complexity that distinguishes it from linear heat equations [25].

To tackle such complexities, the Laplace-Adomian Decomposition Method (LADM) emerges as a potent strategy for solving nonlinear differential equations, including the nonlinear heat equation. This method synergistically combines the Laplace transform—a renowned technique for converting differential equations from the time domain into algebraic expressions in the frequency domain—with the Adomian Decomposition Method (ADM), which adeptly manages the nonlinear elements of the equation. Although the Laplace transform is effective, the inverse transform of nonlinear components poses significant challenges [10].

The Adomian Decomposition Method addresses these challenges by decomposing nonlinear terms into a series of Adomian polynomials. These polynomials are constructed specifically to resolve the nonlinearities, facilitating an iterative approach to derive solutions. By integrating the Laplace transform with ADM, LADM provides a systematic framework for solving intricate nonlinear differential equations.

This thesis focuses on the application of LADM to the nonlinear heat equation. The methodology involves first applying the Laplace transform to the nonlinear heat equation, followed by the decomposition of the resulting expressions into Adomian polynomials. The iterative solutions derived from this approach yield the temperature distribution, highlighting LADM's simplicity and efficacy in addressing complex nonlinear problems with precision and efficiency [22].

Subsequent sections will elaborate on the application of LADM to the nonlinear heat equation, delve into the theoretical underpinnings of the method, and provide illustrative examples of its implementation. This discussion will emphasize the effectiveness and versatility of LADM, affirming its utility as a vital tool for solving nonlinear differential equations in various scientific and engineering domains [28].

The challenge of addressing nonlinear heat equations is a prominent concern in science and engineering, largely due to the complexities introduced by nonlinear terms. Traditional solution methods often struggle to achieve both accuracy and efficiency, especially when confronted with complex boundary conditions and nonlinear source terms. While numerical methods are powerful, they can be computationally demanding and occasionally unstable, rendering them impractical for large-scale or highly nonlinear problems.

By integrating the Adomian Decomposition Method with the Laplace transform, the Laplace-Adomian Decomposition Method (LADM) provides a viable alternative. This innovative method effectively manages nonlinear features and reduces the problem to a more tractable algebraic form.

1.1 Literature Review

Heat transfer plays a crucial role in numerous scientific and engineering challenges. Effectively understanding and managing nonlinear heat equations are key aspects of this field. Traditional analytical methods often struggle with the complexities posed by nonlinear terms, making it challenging to obtain accurate results. This difficulty has prompted researchers to explore alternative approaches, leading to the development of the Laplace-Adomian Decomposition Method as a significant advancement [13].

1.1.1 Early Approaches and Challenges

Historically, numerical methods such as finite difference and finite element methods have been the primary tools for solving nonlinear heat balance equations. However, these methods often exhibit instability and sensitivity, especially in dealing with highly nonlinear problems. Although potentially useful, they can become computationally demanding. Analytical techniques, like perturbation methods, have also been explored, but they are generally limited to weakly nonlinear problems or scenarios involving small perturbations [18].

1.1.2 Inception of the Adomian Decomposition Method

In the 1980s, George Adomian introduced the Adomian Decomposition Method (ADM), a revolutionary technique for addressing differential equations that are challenging to solve with conventional methods. ADM simplifies complex, highly nonlinear expressions by breaking them down into a series of Adomian polynomials, which are manageable and can be addressed through an iterative process. This makes ADM distinct, offering a user-friendly approach that can handle a broad spectrum of nonlinear problems without requiring the simplifications or modifications typical of other methods [2].

1.1.3 Laplace Transform Combined with the Adomian Decomposition Method

The primary achievement and innovation of coupling the Laplace transform with the Adomian Decomposition Method (ADM), resulting in the Laplace-Adomian Decomposition Method (LADM), has significantly advanced the field of solving nonlinear differential equations. The main advantage of the Laplace transform lies in its ability to convert differential equations into algebraic equations, thus simplifying the solution process [30]. However, challenges arise during the inversion process, particularly when dealing with nonlinear terms .

LADM addresses this challenge by first applying ADM to decompose and solve the nonlinear components of the equation. Once the nonlinearities are managed, the inverse Laplace transform is applied to obtain the solution in the time domain. This innovative approach has proven effective in solving complex nonlinear problems that were previously difficult to address using traditional methods [5]. Therefore, LADM provides a robust framework for tackling nonlinear differential equations, expanding the range of problems that can be solved analytically.

1.1.4 Applications in Nonlinear Heat Equations

The efficiency of the Laplace-Adomian Decomposition Method (LADM) in solving nonlinear heat equations has been extensively demonstrated in the literature.

- **1999:** Wazwaz [29] demonstrated the efficiency of LADM for certain heat equations with complex boundary conditions and nonlinear source terms. His work highlighted that LADM is both accurate and reliable compared to traditional schemes.

- The Porous Medium Equation (PME) and the Fast Diffusion Equation (FDE) are important nonlinear partial differential equations that describe phenomena in various scientific fields [27].

- **The Porous Medium Equation (PME)** describes the flow of fluid through a porous material, such as the movement of groundwater or oil through rock. This equation is known for its nonlinear diffusion term, which makes it challenging to solve analytically. The PME is given by:

$$u_t = \Delta(u^m) \quad \text{for} \quad m > 1,$$

where u represents the quantity of interest, t is time, and m is a parameter that influences the nonlinearity of the diffusion term [**?**].

- **Fast Diffusion Equation (FDE):** The Fractional Diffusion Equation (FDE) describes processes where the diffusion rate diminishes as the concentration of the diffusing substance rises. This equation is characterized by a nonlinear diffusion term with $0 < m < 1$, making it particularly useful for modeling phenomena such as population dynamics and heat conduction in certain contexts. The FDE is expressed as follows:

$$u_t = \Delta(u^m) \quad \text{for} \quad 0 < m < 1,$$

This equation presents challenges regarding the existence and uniqueness of solutions and often requires specialized analytical methods [26].

1.1.5 Comparative Studies

Comparative studies have confirmed that LADM is preferred for treating nonlinear heat equations.

- El-Sayed [9] tested LADM against the Homotopy Perturbation Method and the Variational Iteration Method. Numerical tests indicated that LADM often provided a more straightforward implementation and a faster convergence rate than these alternative approaches.
- In 2004, Khuri [14] compared LADM with various decomposition techniques and demonstrated that LADM performed better in handling initial and boundary conditions. Khuri noted that LADM could produce a series of solutions that converge rapidly to exact solutions.

1.1.6 Recent Advances

Recent research has focused on boosting the efficiency and versatility of LADM.

- Zhou et al. (2017) [34] made modifications to Adomian polynomials, aiming to enhance convergence rates for highly nonlinear equations.
- Arikoglu and Ozkol (2020) [8] adopted a different approach by combining LADM alongside other analytical techniques, such as the Differential Transform Method (DTM), to address more complex heat transfer problems.

1.2 Objectives

1. Introduction to Laplace-Adomian Decomposition Method (LADM): This section provides the theoretical background of LADM and demonstrates

how integrating the Adomian Decomposition Method with the Laplace transform simplifies obtaining solutions to nonlinear differential equations.

2. LADM Implementation on Nonlinear Heat Equations: We implement LADM on various forms of nonlinear heat equations, focusing on different types of nonlinear source terms to find iterative solutions.

3. Expanding the Scope of LADM: This section explores how the Laplace Adomian Decomposition Method (LADM) can be applied to more complex variations of the nonlinear heat equation. These variations include scenarios with variable coefficients, fractional derivatives, and multidimensional domains, thereby increasing the method's flexibility.

4. Application to Porous Medium and Fast Diffusion Equations: In this study, we aim to extend the application of the Laplace Adomian Decomposition Method to address porous medium and fast diffusion equations. This approach will enable us to derive analytical solutions for these equations and gain deeper insights into the associated diffusion processes.

1.3 Advantages of the Laplace Adomian Decomposition Method

- Accuracy: The method rapidly converges to an exact or approximate solution.
- Simplicity: No discretization, linearization, or assumptions are involved.
- Versatility: The method is adept at solving a wide range of nonlinear ordinary and partial differential equations with high precision. It quickly converges to either exact or approximate solutions.
- Reduced Computational Effort: The method reduces computational effort by solving the problem in a series form.

Chapter 2

Preliminaries

Definitions and Concepts

1. Differential Equations

A differential equation is a mathematical equation that involves one functions and their derivatives.

Types

- Ordinary Differential Equation (ODE): A differential equation that involves one functions of a single independent variable along with their derivatives.
- Partial Differential Equation (PDE): A differential equation that involves one or more functions with partial derivatives concerning multiple independent variables.

2. Metric space

Definition 1. *(metric space, metric). Let X be a nonempty set. A function $p : X \times X \to \mathbb{R}$ is called a* ***metric*** *if it satisfies the following properties for all $x, y, z \in X$:*

1. *$p(x, y) \geq 0$ (non-negativity), and $p(x, y) = 0$ if and only if $x = y$ (identity of indiscernibles).*
2. *$p(x, y) = p(y, x)$ (symmetry).*
3. *$p(x, y) \leq p(x, z) + p(z, y)$ (triangle inequality).*

A ***metric space*** *is defined as the pair (X, p), where X is a nonempty set and p is a metric on that set.*

The triangle inequality is explicitly represented by property (3). A classic example of a metric space is the set of real numbers $\mathbb{R}$ with the metric defined by $p(x, y) = |x - y|$.

3. Normed Linear Spaces

*A **norm** is a nonnegative real-valued function $\|\cdot\|$ defined on a linear space X for any vectors $u, v \in X$ and any real number a, satisfying the following properties:*

1. $\|u\| = 0$ *if and only if* $u = 0$ *(positivity).*
2. $\|u + v\| \leq \|u\| + \|v\|$ *(triangle inequality).*
3. $\|au\| = |a|\|u\|$ *(homogeneity).*

*A **normed linear space** is defined as a linear space equipped with a norm. A metric p on a linear space X is induced by a norm $\|\cdot\|$ on X through the relationship:*

$$p(x, y) = \|x - y\| \quad \text{for all} \quad x, y \in X.$$

The triangle inequality for the norm is established by property (2).

4. Banach Space

A normed vector space A is termed a Banach Space if it is complete with respect to the metric $p(x, y)$.

5. Fixed Point

Definition 2. *A fixed point of a mapping $T : X \to X$ is an element $x \in X$ such that*

$$T(x) = x,$$

indicating that it is mapped to itself.

6. Contraction

Definition 3. *Let (X, p) be a metric space. A mapping $T : X \to X$ is called a contraction if there exists a positive constant $K < 1$ such that*

$$p(T(x), T(y)) \leq Kp(x, y) \quad \text{for all } x, y \in X.$$

Theorem 1

Let (X, p) be a non-empty complete metric space with a contraction mapping $T : X \to X$. Then T has a unique fixed point $x^* \in X$, and for any initial point $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_{n+1} = T(x_n)$ converges to x^*.

Theorem 2 (Banach's Fixed Point Theorem)

Let (X, p) be a complete metric space, and let $T : X \to X$ be a contraction. Then T has a unique fixed point $x \in X$ such that $T(x) = x$.

7. Inner Product

*Let V be a vector space over the field of real or complex numbers. An **inner product** on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ (or $\mathbb{C}$) that fulfills the following properties for all vectors $u, v, w \in V$ and for all scalars a:*

1. ***Linearity in the first argument:*** *For any $u, v, w \in V$ and scalar a,*
$$\langle au + v, w \rangle = a\langle u, w \rangle + \langle v, w \rangle.$$

2. ***Conjugate symmetry:*** *For all $u, v \in V$,*
$$\langle u, v \rangle = \overline{\langle v, u \rangle},$$
where $\overline{x}$ denotes the complex conjugate of x. In the case where V is a real vector space, this reduces to
$$\langle u, v \rangle = \langle v, u \rangle.$$

3. ***Positive-definiteness:*** *For any $v \in V$,*
$$\langle v, v \rangle > 0 \quad \text{if } v \neq 0,$$
and
$$\langle v, v \rangle = 0 \quad \text{if and only if } v = 0.$$

The inner product induces a norm on V defined by
$$\|v\| = \sqrt{\langle v, v \rangle}.$$

8. Hilbert Space

Definition 4. *A Hilbert space is a vector space (H) with an inner product ($\langle f, g \rangle$) such that the norm defined by*
$$||f|| = \sqrt{\langle f, f \rangle}$$
turns H into a complete metric space. If the metric defined by the norm is not complete, then H is instead known as an inner product space.

9. Nonlinear Function

Definition 5. *A function that does not satisfy the superposition principle; i.e., the output is not directly proportional to the input.*

Example 1.
$$f(u) = \beta u^2,$$
where β is a constant.

10. Heat Equation

A partial differential equation (PDE) that describes how temperature distributes in a region over time .

Forms:

- *Linear Heat Equation:*
$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}.$$
- *Nonlinear Heat Equation:*
$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + f(u) \qquad where\, f(u)\, is\, a\, non\, linear\, function.$$

Types of Nonlinear Heat Equations

Nonlinear heat equations come in various forms, each representing different physical phenomena and incorporating different types of nonlinearities [26]. Some of the key types include:

- *Porous Medium Equation: The porous medium equation is a nonlinear PDE given by:*
$$u_t = \Delta(u^m),$$
where $u = u(x,t)$ represent the density of the diffusing material, t represents time, Δ is the Laplacian operator, and m is a constant greater than 1 . This equation models diffusion processes in a medium where diffusivity depends on density. It arises in contexts such as groundwater flow, gas flow through porous media, and population dynamics [26].
- *Fast Diffusion Equation: The fast diffusion equation is another form of nonlinear diffusion equation, given by:*
$$u_t = \Delta(u^m),$$
but with $0 < m < 1$ Unlike the porous medium equation (PME), the fast diffusion equation (FDE) describes diffusion processes where diffusivity decreases with density. This equation is relevant in contexts such as thin film growth, plasma physics, and certain biological processes [11].

11. Thermal Diffusivity (α)

Definition: Thermal diffusivity is a characteristic of a material that measures how quickly heat is conducted through the material .

***Formula**:*
$$\alpha = \frac{k}{\rho c},$$
where:

- k represents the material's thermal conductivity.
- ρ is the density of the material.
- c denotes the specific heat capacity.

[32]

12. Initial Conditions

The state of the system at the initial time ($t = 0$). Example:

$$u(x, 0) = g(x),$$

where $g(x)$ represents the initial condition [24].

13. Exponential Order

A function $f(t)$ is said to be of exponential order α if there exist positive constants M and α such that

$$|f(t)| \leq Me^{\alpha t} \quad \text{for all } t \geq 0.$$

This property ensures that the function does not grow faster than an exponential function with rate α [15].

14. Laplace Transform

Definition 6. *The **Laplace transform** is an integral transform used to convert a function of time, typically denoted as $f(t)$, into a function of a complex variable s, denoted as $F(s)$. It is particularly useful in solving differential equations by transforming them into algebraic equations.*

The Laplace transform of a function $f(t)$, where $t \geq 0$, is defined as:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\, dt,$$

where s is a complex variable.

In this definition:

- t is the time-domain variable.
- s is the complex frequency-domain variable.
- $\mathcal{L}\{f(t)\}$ denotes the Laplace transform of the function $f(t)$.

Inverse Laplace Transform

The inverse Laplace transform is a technique used to revert a function from its frequency-domain representation back to its original time-domain form. It is denoted as:

$$u(t) = \mathcal{L}^{-1}\{U(s)\},$$

where $U(s)$ is the Laplace-transformed function, and $u(t)$ is the corresponding time-domain function [20].

Existence Theorem for the Laplace Transform

Theorem: *If $f(t)$ is a function that is piecewise continuous on the interval $[0, \infty)$ and grows no faster than an exponential function of order α, then its Laplace transform, $\mathcal{L}\{f(t)\}$, exists for values of s where the real part $\Re(s)$ is greater than α [30].*

Proof. *Let's demonstrate that the existence of Laplace transforms under these conditions.*

The Laplace transform is defined by the integral:

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt,$$

which converges when the real part of s, denoted as $\Re(s)$, exceeds α.

Because $f(t)$ is piecewise continuous, the integral can be divided into smaller intervals where $f(t)$ is continuous, while separately accounting for any jumps or discontinuities.

$$\int_0^{\infty} e^{-st} f(t)\, dt = \sum_{i=0}^{n} \int_{a_i}^{b_i} e^{-st} f(t)\, dt + \int_{b_n}^{\infty} e^{-st} f(t)\, dt,$$

where $0 = a_0 < b_0 = a_1 < b_1 = \cdots < a_n < b_n < \infty$.

On each interval $[a_i, b_i]$, where $f(t)$ is continuous, the integral

$$\int_{a_i}^{b_i} e^{-st} f(t)\, dt$$

is well-defined and converges for every s.

For sufficiently large values of t, particularly when $t \geq b_n$, the exponential growth condition $|f(t)| \leq Me^{\alpha t}$ applies. This results in the following estimation:

$$\left| \int_{b_n}^{\infty} e^{-st} f(t)\, dt \right| \leq M \int_{b_n}^{\infty} e^{-(\Re(s)-\alpha)t}\, dt.$$

This last integral converges when $\Re(s) > \alpha$. To see why, consider:

$$\int_{b_n}^{\infty} e^{-(\Re(s)-\alpha)t}\, dt = \left[\frac{e^{-(\Re(s)-\alpha)t}}{-(\Re(s)-\alpha)} \right]_{b_n}^{\infty} = \frac{e^{-(\Re(s)-\alpha)b_n}}{\Re(s)-\alpha},$$

which is a finite value as long as $\Re(s) > \alpha$.

Because each integral over the finite intervals converges, and the integral from b_n to infinity also converges when $\Re(s) > \alpha$, the entire integral converges as well.

Therefore, if $f(t)$ is piecewise continuous and of exponential order α, the Laplace transform $\mathcal{L}\{f(t)\}$ exists for $\Re(s) > \alpha$. This concludes the proof of the existence theorem for the Laplace transform.

Uniqueness Theorem for the Laplace Transform

Theorem 1. *Suppose two functions $f(t)$ and $g(t)$ have the same Laplace transform $F(s)$ for sufficiently large values of $\Re(s)$. Then $f(t)$ and $g(t)$ are equal almost everywhere, except possibly on a null set.[30].*

Proof. *Suppose $f(t)$ and $g(t)$ are two functions such that their Laplace transforms satisfy:*

$$F(s) = \mathcal{L}\{f(t)\}(s) = \mathcal{L}\{g(t)\}(s)$$

for sufficiently large values of $\Re(s)$. We need to show that $f(t) = g(t)$ almost everywhere on $[0, \infty)$.

The Laplace transform of a function $h(t)$ is defined as:

$$\mathcal{L}\{h(t)\}(s) = \int_0^\infty e^{-st} h(t)\, dt$$

for $s > \sigma$, where σ is a real number such that the integral converges.

Given that $F(s) = G(s)$ for sufficiently large $\Re(s)$, we can express this as:

$$\int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-st} g(t)\, dt$$

for those values of s.

We can rearrange this to obtain:

$$\int_0^\infty e^{-st} (f(t) - g(t))\, dt = 0$$

According to the properties of the Laplace transform, if the Laplace transform of a function is zero for all s in a half-plane, then that function must be zero almost everywhere in the corresponding time domain. Thus, we have:

$$\mathcal{L}\{f(t) - g(t)\}(s) = 0$$

for sufficiently large $\Re(s)$.

Therefore, by the uniqueness of the Laplace transform, we conclude that:

$$f(t) - g(t) = 0 \quad \text{almost everywhere on } [0, \infty).$$

This implies that $f(t) = g(t)$ almost everywhere except possibly on a null set.

Thus, we have proven that if two functions have the same Laplace transform, they must be equal almost everywhere.

□

Properties of Laplace Transform

The Laplace transform is a significant mathematical tool with numerous essential properties. It is extensively used in engineering, physics, and applied mathematics to analyze the linearity of time-invariant systems. The following are some fundamental properties of the Laplace transform:

1. ***Linearity**: The transform of Laplace preserves linearity. For any functions $g(t)$ and $f(t)$, and constants a and b, the linearity property can be expressed as:*

$$\mathcal{L}\{ag(t) + bf(t)\} = a\mathcal{L}\{g(t)\} + b\mathcal{L}\{f(t)\},$$

   where $\mathcal{L}$ denotes the Laplace transform .

2. ***Frequency Shift**: If a function $f(t)$ has a Laplace transform $F(s)$, then multiplying $f(t)$ by an exponential factor e^{at} results in a shifted Laplace transform:*

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a),$$

   indicating that the transform of the shifted function $e^{at}f(t)$ is $F(s-a)$.

3. ***Differentiation in the Time Domain**: When the Laplace transform is applied to the derivative of a function $f(t)$, if $f(t)$ has a Laplace transform $F(s)$, then the transform of the derivative $f'(t)$ is:*

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

   For higher-order derivatives, the relationship generalizes to:

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0),$$

   where $f^{(n)}(t)$ represents the n-th derivative of $f(t)$.

4. ***Integration in the Time Domain:** If a function $f(t)$ has a Laplace transform $F(s)$, then the Laplace transform of the integral of $f(t)$ from 0 to t is given by:*

$$\mathcal{L}\left\{\int_0^t f(\nu)d\nu\right\} = \frac{F(s)}{s}\}.$$

5. ***Standard form of Laplace Transform***

Function	*Laplace Transform*
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
e^{at}	$\frac{1}{s-a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$

These are the standard Laplace transforms for the given functions:

- *1 transforms to* $\frac{1}{s}$.
- t *transforms to* $\frac{1}{s^2}$.
- e^{at} *transforms to* $\frac{1}{s-a}$.
- $\sin(at)$ *transforms to* $\frac{a}{s^2+a^2}$.
- $\cos(at)$ *transforms to* $\frac{s}{s^2+a^2}$.

[16]

6. ***Standard form of Inverse Laplace Transform***

Laplace Transform	*Inverse Laplace Transform*
$\frac{1}{s}$	1
$\frac{1}{s^2}$	t
$\frac{1}{s-a}$	e^{at}
$\frac{a}{s^2+a^2}$	$\sin(at)$
$\frac{s}{s^2+a^2}$	$\cos(at)$

This table lists the inverse Laplace transforms corresponding to the Laplace transforms provided

- $\frac{1}{s}$ *inversely transforms to 1.*
- $\frac{1}{s^2}$ *inversely transforms to* t.
- $\frac{1}{s-a}$ *inversely transforms to* e^{at}.
- $\frac{a}{s^2+a^2}$ *inversely transforms to* $\sin(at)$.
- $\frac{s}{s^2+a^2}$ *inversely transforms to* $\cos(at)$.

[16]

15. Series Solution

Definition 7. *A solution of a differential equation expressed as an infinite sum of functions.*

Example:

$$u(t) = \sum_{n=0}^{\infty} u_n(t),$$

where each $u_n(t)$ *is determined iteratively [17].*

16. Iterative Solution

Definition 8. *A method of solving equations by repeated approximation, progressively getting closer to the exact solution .*

17. Adomian Decomposition Method

Definition 9. *A method to solve differential equations by decomposing the solution into a series of functions.*

$$u(t) = \sum_{n=0}^{\infty} u_n(t),$$

where each $u_n(t)$ represents a component of the series.

Adomian Polynomials: *These polynomials are employed to represent nonlinear terms within the decomposition process. They are defined by:*

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} N \left(\sum_{k=0}^{\infty} \lambda^k u_k \right) \Bigg|_{\lambda=0}.$$

[7]

18. Laplace Adomian Decomposition Method (LADM)

Definition 10. *Combines the Laplace transform with ADM to solve differential equations, converting the problem into the Laplace domain for simplification and then using ADM to solve for the solution iteratively.*

Steps*:*

1. ***Apply Laplace Transform*** *to the PDE.*
2. ***Incorporate Initial Conditions****.*
3. ***Decompose Solution*** *using ADM.*
4. ***Solve Iteratively*** *in the Laplace domain.*
5. ***Apply the inverse Laplace transform*** *to retrieve the solution in the time domain.*

19. Laplacian Operator (∇^2)

Definition 11. *A differential operator that represents the sum of all unmixed second partial derivatives.*

Formula*:*

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Use*: The diffusion term is described by this expression in the heat equation.[31].*

Chapter 3

Laplace Adomian Decomposition Method

3.1 Adomian Decomposition Method

This section starts with a complete explanation of Adomian's method and its properties for convergence. Following this, the section focuses on two special applications of the technique: nonlinear and differential equations. The first application is for nonlinear equations, where an effective iterative procedure for such equations will be developed. A second application uses Adomian's method, which is applied numerically for differential equations [1, 2].

3.1.1 Method Description

The Adomian Decomposition Method (ADM) is a powerful technique widely used for solving various types of differential equations, including linear and nonlinear ordinary differential equations (ODEs), partial differential equations (PDEs), algebraic equations, integral equations, and integro-differential equations [3]. The method begins with an equation of the form:

$$Lu + Nu + Ru = g,$$

where L denotes a linear operator, N represents a nonlinear operator, R is the residual term, and g is the forcing function. Assuming the linear operator L has an inverse L^{-1}, the solution u can be expressed as an infinite series:

$$u = \sum_{n=0}^{\infty} u_n,$$

or equivalently,

$$u = u_0 + u_1 + u_2 + \cdots ,$$

where each term u_n is determined recursively. The nonlinear term $N(u)$ is expanded using Adomian polynomials, which are expressed as an infinite series:

$$N(u) = \sum_{n=0}^{\infty} A_n,$$

with the Adomian polynomials A_n defined recursively by:

$$A_n = u_0 + u_1 + \cdots + u_n.$$

Substituting these polynomials into the original equation and treating R as a linear operator leads to a formal recursive algorithm:

$$\begin{aligned} u_0 &= L^{-1}g, \\ u_1 &= -L^{-1}R(u_0) - L^{-1}A_0, \\ u_2 &= -L^{-1}R(u_1) - L^{-1}A_1, \\ &\vdots \end{aligned}$$

Example of Adomian Polynomials

The Adomian polynomials, denoted as A_n, are constructed to represent the increasing complexities of terms in the expansion of $N(u)$. For instance:

$$\begin{aligned} A_0 &= N(u_0), \\ A_1 &= f'(u_0)u_1, \\ A_2 &= f'(u_0)u_2 + \frac{1}{2!}f''(u_0)u_1^2, \\ A_3 &= f'(u_0)u_3 + \frac{2}{2!}f''(u_0)u_1u_2 + \frac{1}{3!}f'''(u_0)u_1^3, \\ &\vdots \end{aligned}$$

These polynomials provide a systematic technique for solving linear and nonlinear equations, which is essential for the iterative process of the Adomian Decomposition Method [4].

3.1.2 Convergence of the Adomian Decomposition Method

The convergence of the Adomian Decomposition Method (ADM) can be established by utilizing the Banach Fixed-Point Theorem, which is also known as the Contraction Mapping Theorem. This fundamental theorem provides the essential conditions for the convergence of iterative methods, which are crucial for solving functional equations and form the foundation of ADM [12, 33]. ADM has been effectively applied to a variety of problems, ranging from differential equations to integral equations, demonstrating its robustness and versatility [5].

Theorem 2. *Consider the nonlinear differential equation*

$$u - N(u) = f,$$

where $N(u)$ is a nonlinear operator and f is a given function. The solution u is expressed as a series:

$$u = \sum_{n=0}^{\infty} u_n,$$

where u_n are the terms generated by the Adomian Decomposition Method. The nonlinear operator $N(u)$ is expanded as:

$$N(u) = \sum_{n=0}^{\infty} A_n,$$

where A_n are the Adomian polynomials. The components u_n are determined by the following recursive scheme:

$$u_0 = f,$$

$$u_{n+1} = A_n(u_0, u_1, \ldots, u_n), \quad n \geq 0.$$

If there exists a constant $\alpha < 1$ such that for all $n \geq 0$,

$$\|s_{n+1} - s_n\| \leq \alpha \|s_n - s_{n-1}\|,$$

where $s_n = \sum_{k=0}^{n} u_k$, then the series $\sum_{n=0}^{\infty} u_n$ converges to the exact solution u of the equation $u - N(u) = f$.

Proof. *In the Adomian Decomposition Method (ADM), the solution u is expressed as a series, represented by the following form:*

$$u = \sum_{n=0}^{\infty} u_n.$$

The nonlinear term $N(u)$ is represented as an infinite series:

$$N(u) = \sum_{n=0}^{\infty} A_n,$$

where A_n denotes the Adomian polynomials [7]. By substituting this series into the functional equation, we obtain the following recursive scheme:

$$u_0 = f,$$

$$u_{n+1} = A_n(u_0, u_1, \ldots, u_n), \quad n \geq 0.$$

To apply the Banach Fixed-Point Theorem, it is necessary to show that the operator T, defined by ADM, is a contraction. Define the iterative sequence $s_n = u_0 + u_1 + \ldots + u_n$. The ADM iteration can then be written as:

$$s_{n+1} = N(u_0 + s_n).$$

To ensure that T is a contraction, there must exist a constant $\alpha < 1$ such that:

$$\|s_{n+1} - s_n\| = \|u_{n+1}\| \leq \alpha \|u_n\| \leq \alpha^2 \|u_{n-1}\| \leq \ldots \leq \alpha^{n+1} \|u_0\| \quad .$$

We now show that the sequence $\{s_n\}$ is Cauchy:

$$\|s_m - s_n\| = \|(s_m - s_{m-1}) + (s_{m-1} - s_{m-2}) + \cdots + (s_{n+1} - s_n)\| .$$

$$\leq \|s_m - s_{m-1}\| + \|s_{m-1} - s_{m-2}\| + \cdots + \|s_{n+1} - s_n\|.$$

$$\leq \alpha^m \|u_0\| + \alpha^{m-1} \|u_0\| + \cdots + \alpha^{n+1} \|u_0\|.$$

$$\leq (\alpha^{n+1} + \alpha^{n+2} + \cdots) \|u_0\| = \frac{\alpha^{n+1}}{1-\alpha} \|u_0\|.$$

Since $\alpha < 1$, the sequence $\{s_n\}$ approaches a limit s within H. Consequently, solving the equation $u - N(u) = f$ is equivalent to finding s such that $s = N(u_0 + s)$. Given that N is continuous, we can then deduce

$$N(u_0 + s) = N\left(\lim_{n\to\infty} (u_0 + s_n)\right) = \lim_{n\to\infty} N(u_0 + s_n) = \lim_{n\to\infty} s_{n+1} = s.$$

Thus, the solution of the equation is $s = \sum_{n=0}^{\infty} u_n$, and the Banach Fixed-Point Theorem guarantees the convergence of the sequence. [33].

3.1.3 Applying the Adomian Decomposition Method to solve PDE

The Adomian Decomposition Method (ADM) is a widely recognized technique for addressing partial differential equations . In this section, we will solve a specific PDE using ADM, demonstrating its effectiveness.

Consider the following PDE:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

with the initial condition given by:

$$u(x, 0) = f(x)$$

Solution. *Assume that the solution $u(x, t)$ can be represented as an infinite series:*

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t)$$

Substituting this series into the PDE, we obtain:

$$\frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty} u_n(x, t)\right) + \frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} u_n(x, t)\right) = 0$$

Simplifying the series yields:

$$\sum_{n=0}^{\infty} \left(\frac{\partial u_n}{\partial t} + \frac{\partial u_n}{\partial x} \right) = 0$$

Next, we derive recursive relations for each term in the series:

$$\frac{\partial u_{n+1}}{\partial t} + \frac{\partial u_n}{\partial x} = 0$$

For $n = 0$*:*

$$\frac{\partial u_0}{\partial t} + \frac{\partial u_0}{\partial x} = 0$$

Given the initial condition $u_0(x, 0) = f(x)$*, the solution is* $u_0(x, t) = f(x-t)$*.*
For $n = 1$*:*

$$\frac{\partial u_1}{\partial t} + \frac{\partial u_0}{\partial x} = 0$$

$$\frac{\partial u_1}{\partial t} - f'(x - t) = 0$$

Solving this, we obtain $u_1(x, t) = tf'(x - t)$*.*
For $n = 2$*:*

$$\frac{\partial u_2}{\partial t} + \frac{\partial u_1}{\partial x} = 0$$

$$\frac{\partial u_2}{\partial t} + tf''(x - t) = 0$$

Solving this, we obtain $u_2(x, t) = \frac{t^2}{2!} f''(x - t)$*.*

In the end, summing the series terms yields an approximate solution: Finally, summing the series terms provides an approximate solution:

$$u(x, t) \approx u_0(x, t) + u_1(x, t) + u_2(x, t) + \cdots$$

$$u(x, t) \approx f(x - t) + tf'(x - t) + \frac{t^2}{2!} f''(x - t) + \cdots$$

By following these steps, ADM generates a series solution for the PDE. While this example is simple, the method is particularly powerful for more complex, nonlinear PDEs [21].

3.2 Laplace Adomian Decomposition Method

Laplace Transform

The Laplace transform is a highly effective integral transform used across various fields of mathematics and engineering. It transforms differential equations into algebraic equations in the complex frequency domain, simplifying the process of solving linear differential equations with specified initial conditions. [19].

Adomian Decomposition Method (ADM)

The Adomian Decomposition Method (ADM) is an approach that simplifies nonlinear problems by decomposing them into a sequence of linear sub-problems. It represents the solution as an infinite series, with each term calculated recursively using Adomian polynomials. This method is especially advantageous because it addresses the nonlinearity of the equations directly, without relying on approximations [22].

3.2.1 Method Description

Consider a general nonlinear (Partial Differential equation (PDE)) of the form:

$$L[u] + R[u] + N[u] = H(x, y),$$

subject to the initial conditions:

$$u(x, 0) = f(x), \quad u_y(x, 0) = g(x),$$

where:

- ***L*** *represents a linear operator,*
- ***R[u]*** *includes additional linear terms,*
- ***N[u]*** *is a nonlinear operator, and*
- ***H(x, y)*** *is the source term.*

Solution. *First, apply the Laplace transform with respect to y:*

$$\mathcal{L}[L[u]] + \mathcal{L}[R[u]] + \mathcal{L}[N[u]] = \mathcal{L}[H(x, y)].$$

By using the properties of the Laplace transform, this equation can be rewritten as:

$$s^2 u(x, s) - su(x, 0) - u_y(x, 0) + \mathcal{L}[R[u]] + \mathcal{L}[N[u]] = \mathcal{L}[H(x, y)].$$

Next, we rearrange the equation to solve for $u(x, s)$:

$$s^2 u(x, s) = sf(x) + g(x) + \mathcal{L}[H(x, y) - N[u] - R[u]].$$

Thus,

$$u(x,s) = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2}\mathcal{L}[H(x,y) - N[u] - R[u]].$$

To find $u(x,y)$, we apply the inverse Laplace transform:

$$u(x,y) = f(x) + yg(x) + \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[H(x,y) - N[u] - R[u]]\right].$$

Assuming the solution can be expressed as the series:

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y),$$

The nonlinear term $N[u]$ is expanded using Adomian polynomials A_n as follows:

$$N[u] = \sum_{n=0}^{\infty} A_n,$$

where the Adomian polynomials A_n are given by:

$$A_n = \frac{1}{n!}\,\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\Bigg|_{\lambda=0}.$$

By substituting the series into the transformed equation and comparing terms, we derive the following recursive relation:

$$u_0(x,y) = f(x) + yg(x),$$

$$u_{n+1}(x,y) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[H(x,y) - A_n - R[u_n]]\right].$$

This recursive relation allows us to iteratively compute the terms $u_n(x,y)$, thereby constructing the series solution for $u(x,y)$.

3.2.2 Application to Partial Differential Equations

Here, we apply the method described above to solve specific partial differential equations (PDEs). By following the recursive scheme, we can approximate solutions to nonlinear PDEs efficiently.

This refinement is based on the methods and approaches detailed in standard references on partial differential equations and their solutions [23].

Example 2. *Solve the nonlinear partial differential equation:*

$$\frac{\partial u(x,t)}{\partial t} + u\frac{\partial u(x,t)}{\partial x} = x + xt^2$$

with the initial condition:

$$u(x,0) = 0.$$

Solution. *First, we take the **Laplace transform** of the PDE with respect to t, assuming $u(x,t)$ has a Laplace transform $U(x,s)$. The Laplace transform of the time derivative is:*

$$\mathcal{L}\left\{\frac{\partial u(x,t)}{\partial t}\right\} = sU(x,s) - u(x,0).$$

Given that $u(x,0) = 0$, this simplifies to:

$$\mathcal{L}\left\{\frac{\partial u(x,t)}{\partial t}\right\} = sU(x,s).$$

Now take the Laplace transform of both sides of the PDE:

$$sU(x,s) + \mathcal{L}\left\{u\frac{\partial u(x,t)}{\partial x}\right\} = \mathcal{L}\{x + xt^2\}.$$

The Laplace transform of the right-hand side is:

$$\mathcal{L}\{x + xt^2\} = \frac{x}{s} + \frac{2x}{s^3}.$$

Thus, the transformed PDE becomes:

$$sU(x,s) + \mathcal{L}\left\{u\frac{\partial u(x,t)}{\partial x}\right\} = \frac{x}{s} + \frac{2x}{s^3}.$$

*Now, we apply the **Adomian Decomposition Method (ADM)** to manage the nonlinear term. We assume the solution $u(x,t)$ can be expressed as a series:*

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t),$$

where $u_n(x,t)$ are the components of the solution that need to be determined.

*Similarly, the nonlinear term $u\frac{\partial u(x,t)}{\partial x}$ is decomposed into **Adomian polynomials** $A_n(x,t)$, where:*

$$u\frac{\partial u(x,t)}{\partial x} = \sum_{n=0}^{\infty} A_n(x,t).$$

The zeroth-order equation is obtained by substituting the series solution into the transformed PDE and collecting the $n = 0$ terms:

$$sU_0(x,s) = \frac{x}{s} + \frac{2x}{s^3}.$$

Solving for $U_0(x,s)$:

$$U_0(x,s) = \frac{x}{s^2} + \frac{2x}{s^4}.$$

Now, take the inverse Laplace transform of $U_0(x,s)$ to find $u_0(x,t)$:

$$u_0(x,t) = \mathcal{L}^{-1}\left\{\frac{x}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{2x}{s^4}\right\}.$$

Using standard Laplace transform tables:

$$u_0(x,t) = xt + \frac{xt^3}{3}.$$

The first-order equation involves the first Adomian polynomial A_0, which is:

$$A_0 = u_0\frac{\partial u_0}{\partial x}.$$

We already have $u_0(x,t) = xt + \frac{xt^3}{3}$, so compute A_0:

$$A_0 = \left(xt + \frac{xt^3}{3}\right)\cdot\left(t + \frac{t^3}{3}\right) = xt^2 + \frac{xt^4}{3} + \frac{xt^6}{9}.$$

Taking the Laplace transform of A_0:

$$\mathcal{L}[A_0] = x\left(\frac{2!}{s^3} + \frac{4\cdot 4!}{3\cdot s^5} + \frac{6!}{9\cdot s^7}\right).$$

The first-order equation is then:

$$sU_1(x,s) = -\frac{x}{s}\mathcal{L}[A_0].$$

Solving for $U_1(x,s)$:

$$U_1(x,s) = -\frac{2x}{s^4}.$$

Taking the inverse Laplace transform:

$$u_1(x,t) = -\frac{xt^5}{15}.$$

Next, we compute the second Adomian polynomial A_1 using $u_0(x,t)$ and $u_1(x,t)$:

$$A_1 = u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x}.$$

Since $u_0(x,t) = xt + \frac{xt^3}{3}$ and $u_1(x,t) = -\frac{xt^5}{15}$, compute A_1:

$$A_1 = (xt + \frac{xt^3}{3})\cdot\left(-\frac{t^5}{15}\right) + \left(-\frac{xt^5}{15}\right)\cdot(t) = -\frac{xt^6}{15} + \textit{higher-order terms.}$$

Taking the Laplace transform of A_1:

$$\mathcal{L}\{A_1\} = \mathcal{L}\left\{-\frac{xt^6}{15}\right\} = -\frac{6x}{15s^7}.$$

The second-order equation is:

$$sU_2(x,s) = -\mathcal{L}\{A_1\} = -\frac{6x}{15s^7}.$$

Solving for $U_2(x,s)$:

$$U_2(x,s) = -\frac{6x}{15s^8}.$$

Taking the inverse Laplace transform:

$$u_2(x,t) = -\frac{xt^8}{8!}.$$

Summing up the components $u_0(x,t)$, $u_1(x,t)$, and $u_2(x,t)$:

$$u(x,t) \approx u_0(x,t) + u_1(x,t) + u_2(x,t) = xt + \frac{xt^3}{3} - \frac{xt^5}{15} - \frac{xt^8}{8!}.$$

The initial condition is $u(x,0) = 0$. Substituting $t = 0$ into the approximate solution:

$$u(x,0) = x(0) + \frac{x(0)^3}{3} - \frac{x(0)^5}{15} - \frac{x(0)^8}{8!} = 0.$$

Thus, the initial condition is satisfied.

The approximate solution to the PDE using the ***Laplace-Adomian Decomposition Method*** *is:*

$$u(x,t) \approx xt + \frac{xt^3}{3} - \frac{xt^5}{15} - \frac{xt^8}{8!}.$$

Chapter 4

Laplace-Adomian Decomposition Method to Solve Nonlinear Heat Equation

Solving the Nonlinear Heat Equation Using the Laplace-Adomian Decomposition Method (LADM)

The Laplace-Adomian Decomposition Method (LADM) is an effective approach for tackling nonlinear differential equations. This section provides a detailed, step-by-step guide on applying LADM to solve a nonlinear heat equation. The nonlinear heat equation is given by:

$$u_t = \alpha u_{xx} + f(x)$$

Subject to the initial condition:

$$u(x, 0) = g(x)$$

Solution. *Applying the Laplace transform to both sides of the equation:*

$$\mathcal{L}\{u_t\} = \mathcal{L}\{\alpha u_{xx} + f(x)\}$$

This becomes:

$$\mathcal{L}\{u_t\} = \alpha\mathcal{L}\{u_{xx}\} + \mathcal{L}\{f(x)\}$$

Using the properties of the Laplace transform, we get:

$$sU(x, s) - u(x, 0) = \alpha\mathcal{L}\{u_{xx}\} + \mathcal{L}\{f(x)\}$$

Now, substituting the initial condition (4.2) *into* (4.3), *we obtain:*

$$sU(x,s) - g(x) = \alpha\mathcal{L}\{u_{xx}\} + \mathcal{L}\{f(x)\}$$

Using Adomian polynomials to handle the nonlinear term $f(u)$:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

Then,

$$f(u) = f\left(\sum_{n=0}^{\infty} u_n\right) = \sum_{n=0}^{\infty} A_n$$

Next, we construct the solution iteratively.
For $n = 0$:

$$sU_0(x,s) - g(x) = \alpha\mathcal{L}\{u_{0xx}(x,s)\}$$

Solving this, we get:

$$U_0(x,s) = \frac{g(x)}{s}$$

For $n \geq 1$:

$$sU_n(x,s) = \alpha\mathcal{L}\{u_{nxx}(x,s)\} + \mathcal{L}\{A_{n-1}\}$$

Now, we apply the inverse Laplace transform.
For $n = 0$:

$$\mathcal{L}^{-1}\{U_0(x,s)\} = \mathcal{L}^{-1}\left\{\frac{g(x)}{s}\right\}$$

$$U_0(x,t) = g(x)$$

For $n \geq 1$:

$$\mathcal{L}^{-1}\{sU_n(x,s)\} = \mathcal{L}^{-1}\{\alpha\mathcal{L}\{u_{nxx}(x,s)\}\} + \mathcal{L}^{-1}\{A_{n-1}\}$$

Summing the terms, we get the final solution.

Example 3. *Find the solution to the nonlinear heat equation:*

$$u_t = u_{xx} + u^2$$

with the initial condition $u(x,0) = e^{-x^2}$.

Solution. *We start by applying the Laplace transform w.r.t time t. Let $U(x,s)$ represent the Laplace transform of $u(x,t)$. This transformation transforms the partial differential equation into an algebraic equation in terms of $U(x,s)$:*

$$sU(x,s) - e^{-x^2} = U_{xx}(x,s) + \mathcal{L}\{u^2\},$$

where $\mathcal{L}\{u^2\}$ represents the Laplace transform of u^2.

To handle the nonlinear term u^2, we decompose $u(x,t)$ into a series using Adomian polynomials:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

This decomposition allows us to express u^2 *as a series of Adomian polynomials* A_n*:*

$$u^2 = \sum_{n=0}^{\infty} A_n,$$

where each A_n *is defined by the Adomian decomposition method.*

We also assume that $U(x,s)$ *can be expressed as a series:*

$$U(x,s) = \sum_{n=0}^{\infty} U_n(x,s).$$

We solve for each $U_n(x,s)$ *iteratively:*

For $n = 0$

$$sU_0(x,s) - e^{-x^2} = U_{0,xx}(x,s).$$

Solving this, we obtain:

$$U_0(x,s) = \frac{e^{-x^2}}{s}.$$

For $n \geq 1$

$$sU_n(x,s) = U_{n,xx}(x,s) + \mathcal{L}\{A_{n-1}\}.$$

Using the known $U_{n-1}(x,s)$*, we can find* $U_n(x,s)$*.*

After finding $U_n(x,s)$*, we apply the inverse Laplace transform to retrieve* $u_n(x,t)$*:*

For $n = 0$

$$\mathcal{L}^{-1}\{U_0(x,s)\} = \mathcal{L}^{-1}\left\{\frac{e^{-x^2}}{s}\right\},$$

which gives:

$$u_0(x,t) = e^{-x^2}.$$

For $n = 1$

$$sU_1(x,s) = U_{1,xx}(x,s) + \mathcal{L}\{A_0\}.$$

Since $A_0 = u_0^2 = (e^{-x^2})^2 = e^{-2x^2}$*, we have:*

$$sU_1(x,s) = U_{1,xx}(x,s) + \mathcal{L}\{e^{-2x^2}\}.$$

Solving this, we find:

$$U_1(x,s) = \frac{\mathcal{L}\{e^{-2x^2}\}}{s}.$$

Applying the inverse Laplace transform:

$$u_1(x,t) = \mathcal{L}^{-1}\left\{\frac{\mathcal{L}\{e^{-2x^2}\}}{s}\right\}.$$

For $n = 2$

$$sU_2(x,s) = U_{2,xx}(x,s) + \mathcal{L}\{A_1\}.$$

A_1 *involves terms from* u_0 *and* u_1*, requiring computation based on the previous terms.*

Finally, summing the series terms, the solution $u(x,t)$ *can be expressed as:*

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

Example 4. *To solve the nonlinear heat equation:*

$$u_t = u_{xx} + u^3$$

with the modified initial condition:

$$u(x,0) = \cos(ax),$$

Solution. *We begin by applying the Laplace transform with respect to time* t*. Let* $U(x,s)$ *denote the Laplace transform of* $u(x,t)$*. This process converts the partial differential equation into an algebraic equation involving* $U(x,s)$*:*

$$sU(x,s) - u(x,0) = U_{xx}(x,s) + \mathcal{L}\{u^3\},$$

Substituting the initial condition $u(x,0) = \cos(ax)$*:*

$$sU(x,s) - \cos(ax) = U_{xx}(x,s) + \mathcal{L}\{u^3\}.$$

Assume the solution $u(x,t)$ *can be expressed as an infinite series:*

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

Then, the nonlinear term u^3 can be decomposed using Adomian polynomials:

$$u^3 = \left(\sum_{n=0}^{\infty} u_n\right)^3 = \sum_{n=0}^{\infty} A_n,$$

where the first few Adomian polynomials are:

- $A_0 = u_0^3$
- $A_1 = 3u_0^2 u_1$
- $A_2 = 3u_0^2 u_2 + 3u_0 u_1^2$
- $A_3 = 3u_0^2 u_3 + 6u_0 u_1 u_2 + u_1^3$

[6]

Solve for Each $U_n(x,s)$ Iteratively
Expand $U(x,s)$ as:

$$U(x,s) = \sum_{n=0}^{\infty} U_n(x,s).$$

Each $U_n(x,s)$ can be found iteratively.
For** $n = 0$**:
The equation for $U_0(x,s)$ is:

$$sU_0(x,s) - \cos(ax) = U_{0,xx}(x,s).$$

Solving this gives:

$$U_0(x,s) = \frac{\cos(ax)}{s}.$$

For $n \geq 1$, the equation is:

$$sU_n(x,s) = U_{n,xx}(x,s) + \mathcal{L}\{A_{n-1}\}.$$

For example, using $A_0 = u_0^3 = \left(\frac{\cos(ax)}{s}\right)^3$, we have:

$$\mathcal{L}\{A_0\} = \frac{\cos^3(ax)}{s^3}.$$

Thus, for $U_1(x,s)$:

$$sU_1(x,s) = U_{1,xx}(x,s) + \frac{\cos^3(ax)}{s^3}.$$

Once each $U_n(x,s)$ is determined, apply the inverse Laplace transform to find $u_n(x,t)$.
Finally, sum the series:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

The first few terms are:

$$u(x,t) = \cos(ax) + \frac{t^3}{6}\cos^3(ax) + \frac{t^5}{120}\cos^5(ax) + \cdots.$$

This series represents the solution to the nonlinear heat equation with the initial condition $u(x,0) = \cos(ax)$.

Solving the Porous Medium Equation (PME) Using the Laplace-Adomian Decomposition Method (LADM)

The Porous Medium Equation (PME) is given by:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2(u^2(x,t))}{\partial x^2}.$$

For $m = 2$, the PME simplifies to:

$$\frac{\partial u(x,t)}{\partial t} = 2u(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} + 2\left(\frac{\partial u(x,t)}{\partial x}\right)^2.$$

With Initial Condition:

$$u(x,0) = 1 + x^2.$$

Solution. *We apply the Laplace transform concerning t:*

$$\mathcal{L}\left\{\frac{\partial u(x,t)}{\partial t}\right\} = \mathcal{L}\left\{2u(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} + 2\left(\frac{\partial u(x,t)}{\partial x}\right)^2\right\}.$$

This simplifies to:

$$s\tilde{u}(x,s) - u(x,0) = 2\frac{\partial^2 \tilde{u}(x,s)}{\partial x^2} + 2\mathcal{L}\left\{\left(\frac{\partial u(x,t)}{\partial x}\right)^2\right\}.$$

Substituting $u(x,0) = 1 + x^2$:

$$s\tilde{u}(x,s) - (1 + x^2) = 2\frac{\partial^2 \tilde{u}(x,s)}{\partial x^2} + 2\mathcal{L}\left\{\left(\frac{\partial u(x,t)}{\partial x}\right)^2\right\}.$$

Decomposition Using ADM:
Express $u(x,t)$ as:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

Thus:

$$\tilde{u}(x,s) = \sum_{n=0}^{\infty} \tilde{u}_n(x,s).$$

$u^2(x,t)$*Expand as:*

$$u^2(x,t) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n),$$

where A_n are Adomian polynomials.
Iteration for** $n = 0$**:

$$s\tilde{u}_0(x,s) - u_0(x) = 0,$$

yielding:

$$\tilde{u}_0(x,s) = \frac{u_0(x)}{s}.$$

The inverse Laplace transform gives:

$$u_0(x,t) = 1 + x^2.$$

Iteration for** $n = 1$**:
For the first iteration, compute:

$$s\tilde{u}_1(x,s) = 2\mathcal{L}\left\{u_0(x)\frac{\partial^2 u_0(x)}{\partial x^2}\right\} + 2\mathcal{L}\left\{\left(\frac{\partial u_0(x)}{\partial x}\right)^2\right\}.$$

Given $u_0(x) = 1 + x^2$:

$$\frac{\partial u_0(x)}{\partial x} = 2x \quad \textit{and} \quad \frac{\partial^2 u_0(x)}{\partial x^2} = 2.$$

Thus:

$$\mathcal{L}\left\{u_0(x)\frac{\partial^2 u_0(x)}{\partial x^2}\right\} = \mathcal{L}\left\{(1+x^2)\cdot 2\right\} = \frac{2(1+x^2)}{s}.$$

$$\mathcal{L}\left\{\left(\frac{\partial u_0(x)}{\partial x}\right)^2\right\} = \mathcal{L}\left\{(2x)^2\right\} = \frac{4x^2}{s}.$$

Therefore:

$$s\tilde{u}_1(x,s) = \frac{2(1+x^2)}{s} + \frac{4x^2}{s}.$$

Simplifying:

$$s\tilde{u}_1(x,s) = \frac{2+6x^2}{s}.$$

Thus:

$$\tilde{u}_1(x,s) = \frac{2+6x^2}{s^2}.$$

Applying the inverse Laplace transform yields:

$$u_1(x,t) = 2t + 6tx^2.$$

Combining the initial condition and First Iterations:

$$u(x,t) = u_0(x) + u_1(x,t).$$

Therefore:

$$u(x,t) \approx (1+x^2) + (2t+6tx^2).$$

This simplifies to:

$$u(x,t) \approx 1 + x^2 + 2t + 6tx^2.$$

The Laplace-Adomian Decomposition Method (LADM) provides the following approximation for the Porous Medium Equation (PME) solution:

$$u(x,t) \approx 1 + x^2 + 2t + 6tx^2.$$

Solving the Fast Diffusion Equation Using the Laplace-Adomian Decomposition Method (LADM)

This section addresses the solution of the Fast Diffusion Equation (FDE) using the Laplace-Adomian Decomposition Method (LADM).

Example 5. *Consider the Fast Diffusion Equation (FDE):*

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 \left(u^{1/2}(x,t)\right)}{\partial x^2},$$

where $u(x,t)$ represents the density of the medium, x is the spatial coordinate, t is time, and the exponent $m = \frac{1}{2}$. The initial condition is:

$$u(x,0) = \sin(x).$$

Solution. *Applying the Laplace transform with respect to t on both sides of the FDE, we obtain:*

$$\mathcal{L}\left\{\frac{\partial u(x,t)}{\partial t}\right\} = \mathcal{L}\left\{\frac{\partial^2\left(u^{1/2}(x,t)\right)}{\partial x^2}\right\}.$$

This simplifies to:

$$s\tilde{u}(x,s) - u(x,0) = \frac{\partial^2\tilde{U}(x,s)}{\partial x^2},$$

where $\tilde{u}(x,s)$ is the Laplace transform of $u(x,t)$, and $\tilde{U}(x,s)$ is the Laplace transform of $u^{1/2}(x,t)$.

Substituting the initial condition $u(x,0) = \sin(x)$, we obtain:

$$s\tilde{u}(x,s) - \sin(x) = \frac{\partial^2\tilde{U}(x,s)}{\partial x^2}.$$

Next, decompose $u(x,t)$ using the Adomian Decomposition Method:

Express $u(x,t)$ as an infinite series:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t),$$

and its Laplace transform:

$$\tilde{u}(x,s) = \sum_{n=0}^{\infty} \tilde{u}_n(x,s).$$

Similarly, expand the nonlinear term $u^{1/2}(x,t)$ as:

$$u^{1/2}(x,t) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n),$$

where A_n are Adomian polynomials representing the nonlinear terms.

For the zeroth iteration, solve:

$$s\tilde{u}_0(x,s) - \sin(x) = 0,$$

which yields:

$$\tilde{u}_0(x,s) = \frac{\sin(x)}{s}.$$

By applying the inverse Laplace transform, we obtain:

$$u_0(x,t) = \sin(x).$$

First Iteration:

$$s\tilde{u}_1(x,s) = \frac{\partial^2 \tilde{U}_0(x,s)}{\partial x^2},$$

where:

$$\tilde{U}_0(x,s) = \frac{\sin(x)}{s}.$$

Thus:

$$\frac{\partial^2 \tilde{U}_0(x,s)}{\partial x^2} = -\frac{\sin(x)}{s}.$$

Therefore:

$$s\tilde{u}_1(x,s) = -\frac{\sin(x)}{s},$$

which simplifies to:

$$\tilde{u}_1(x,s) = -\frac{\sin(x)}{s^2}.$$

The inverse Laplace transform yields:

$$u_1(x,t) = -t\sin(x).$$

Second Iteration:
Compute:

$$s\tilde{u}_2(x,s) = \frac{\partial^2 \tilde{U}_1(x,s)}{\partial x^2},$$

where:

$$\tilde{U}_1(x,s) = -\frac{\sin(x)}{s^2}.$$

Thus:

$$\frac{\partial^2 \tilde{U}_1(x,s)}{\partial x^2} = -\frac{\sin(x)}{s^2}.$$

Therefore:

$$s\tilde{u}_2(x,s) = \frac{\sin(x)}{s^2},$$

which simplifies to:

$$\tilde{u}_2(x,s) = \frac{\sin(x)}{s^3}.$$

The inverse Laplace transform gives:

$$u_2(x,t) = \frac{t^2 \sin(x)}{2}.$$

Combining these results:

$$u(x,t) \approx \sin(x) - t\sin(x) + \frac{t^2 \sin(x)}{2} - \frac{t^3 \sin(x)}{6} + \frac{t^4 \sin(x)}{24}.$$

In general, the n-th term in the series is:

$$u_n(x,t) = \frac{t^n \sin(x)}{n!}.$$

Thus, the full solution is:

$$u(x,t) = \sin(x)\left(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} + \cdots\right),$$

where each term $u_n(x,t)$ is calculated iteratively, resulting in the solution to the Fast Diffusion Equation with the specified initial condition.

Conclusion

The Laplace Adomian Decomposition Method (LADM) marks a significant advancement in computational mathematics, particularly for addressing nonlinear heat equations. By integrating the Laplace transform with the Adomian decomposition technique, LADM provides a systematic and efficient strategy for approximating solutions to nonlinear heat transfer problems.

Effective Nonlinear Term Handling: *A major advantage of LADM lies in its ability to decompose the nonlinear terms of heat equations using Adomian polynomials. This approach allows for the breakdown of complex nonlinearities into more manageable components, which facilitates the iterative construction of the solution in series form.*

Versatility and Applicability: *LADM is versatile and can be applied to a broad range of nonlinear heat equations, accommodating various boundary and initial conditions common in heat transfer problems. Whether dealing with nonlinearities due to material properties, radiation, or convective heat transfer, LADM provides a structured framework for effectively addressing these challenges.*

Accuracy and Efficiency: *The method is notable for delivering accurate solutions while maintaining computational efficiency. By converting the problem into the Laplace domain, LADM simplifies the differential equations into algebraic forms, thereby streamlining the solution process and reducing computational effort compared to purely numerical methods.*

Challenges and Considerations: *Despite its many benefits, LADM requires careful attention to convergence issues, particularly when handling highly nonlinear terms or boundary conditions that may impact series convergence. Robust numerical techniques and computational tools are essential for validating and optimizing the solutions derived using LADM.*

Future Directions

The ongoing development of LADM continues to expand its applicability and improve its performance in solving nonlinear heat equations. Future research may focus on refining convergence criteria, extending the method to more complex boundary conditions, and exploring applications in multidimensional heat transfer problems.

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