QUANTUM ANALYSIS OF COHERENTLY DRIVEN THREE-LEVEL LASERS

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ABSTRACT

We present a detailed derivation of the equations of evolution for the reduced density operators (or the master equations) of coherently driven degenerate and nondegenerate three-level lasers. Employing these equations we investigate the squeezing and statistical properties of the light generated by both types of lasers. In both cases, we find that the linear gain coefficient and the strength of the driving radiation highly affects the squeezing and the statistical properties of the light.

CONTENTS

1. INTRODUCTION

In classical electrodynamics, the electric field vector describing a light wave may be decomposed into two components with time dependence cosωt and sinωt. The fluctuations in either of these components can be reduced to arbitrary order independent of the fluctuations in the other component. Likewise in quantum optics, the electric field operator describing a single-mode radiation can be decomposed into two component operators ($\hat{a}_+$ and $\hat{a}_-$), referred to as quadrature operators. Unlike in classical electrodynamics, in quantum optics the fluctuations in the quadrature operators can not be reduced independently to arbitrary order. For a single-mode radiation in any state, the product of the fluctuations in the two quadratures satisfy the Heisenberg uncertainty relation $\Delta a_+ \Delta a_- \geq 1$ [1].

In particular, for some class of light states called minimum uncertainty states, the uncertainty product is equal to the vacuum level, which represent the standard quantum limit to reduction of noise in a signal. One such a minimum uncertainty state is a coherent state, the closest quantum counter-part to a classical radiation [2]. In this state, the product of the fluctuations is minimum and randomly distributed between the two quadratures.

Other minimum uncertainty states are also possible, with less fluctuations in one quadrature than the coherent state at the expense of increased fluctuations in the other quadrature. Such a state is said to be a squeezed state and a light mode in this state is called a squeezed light. A squeezed light has some potential applications [1, 3, 4, 5]. A coherent beam of laser light is used in present optical communication system, in which the ultimate limit to noise reduction is given by the vacuum level. However, if a beam of squeezed light is used instead to transmit information, the quantum noise level could be reduced below the vacuum level in one of the quadratures. A squeezed light can also be used in the detection of weak signal (such as gravitational wave).

Quantum optical systems called two-photon devices, in which two photons are generated or destroyed at a time, are in principle the best possible sources of squeezed light [6]. This is mainly becuase of the presence of nonlinear terms in the creation and annihilation operators in the Hamiltonian describing such a system. One such two-photon device is a parametric oscillator, which is an important source of squeezed light [7-9]. It has been also shown that a three-level laser under certain conditions generates squeezed light [10, 11].

As described in reference [10], a three-level laser, which is the focus of this thesis, is a system consisting of three-level atoms in a cascade configuration, where the crucial role is played by atomic coherence. In such a system the atomic coherence is introduced either by initially preparing the atoms in coherent superposition of the upper and the lower levels or by coupling these two levels by a strong coherent radiation [11-13].

More precisely, in a coherently prepared or a coherently driven three-level laser, three-level atoms are injected at a constant rate into a cavity coupled to a vacuum reservoir via a single port mirror. When an atom decays from the upper level to the lower level via the intermediate level, two photons are generated and two photons will be destroyed in the reverse process. This makes the three-level laser a two-photon device, which is in principle the best source of squeezed light. If the two photons are identical the system is referred to as a degenerate three-level laser, otherwise it is a nondegenerate three-level laser.

It is also found that the degree of squeezing depends on the atomic coherence, the cavity damping constant, and the linear gain coefficient (a quantity describing the amplification of the cavity radiation by the injected atoms) [10,11].

In this thesis we study a three-level laser in which the atomic coherence is introduced by a coherent driving radiation. Using the method used in reference [10], we analyze the squeezing and statistical properties of the light generated by coherently driven degenerate and nondegenerate three-level lasers.

2. COHERENTLY DRIVEN DEGENERATE THREE-LEVEL LASER

A coherentely driven three-level laser may be defined as a quantum optical system in which three-level atoms in a cascade configuration, with the top and bottom levels coupled by a strong radiation, are injected at a certain rate into a cavity coupled to a vacuum reservoir via a single port mirror.

We denote the upper, intermediate, and bottom levels by $|a\rangle$, $|b\rangle$, and $|c\rangle$, respectively as indicated in figure [2.1]. We assume that the atoms can make a transition from $|a\rangle$ to $|b\rangle$ or from $|b\rangle$ to $|c\rangle$ with direct transition from $|a\rangle$ to $|c\rangle$ to be dipole forbidden. If the $|a\rangle \to |b\rangle$ and the $|b\rangle \to |c\rangle$ transitions are at resonance, then we have a degenerate three-level laser. Under certain conditions such a system generates squeezed light [10, 11].

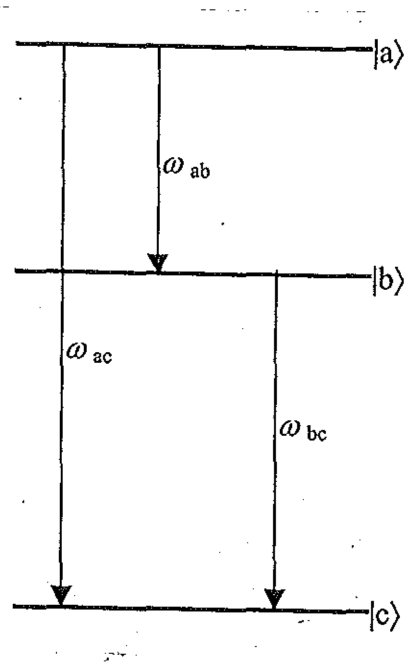


Figure [2.1]. Three level atom in a cascade configuration.

In this chapter we seek to derive the equation of evolution of the density operator for the cavity mode, in the linear and adiabatic approximations. Applying this equation we obtain the equations of evolution of the first and the second order moments for the cavity mode operators. The steady state solutions of the resulting equation are then used to calculate the quadrature variance, the squeezing spectrum, and the variance of the photon number. Moreover, applying the same solutions, we also determine the antinormally-ordered charcterstic function with the aid of which the Q function is obtained. Finally, the Q function is used to calculate the photon number distribution.

2.1. The master equation

Assume the atoms are injected into a single mode cavity at a certain constant rate r_a and removed after some time τ (see Fig. [2.2]). The transitions from $|a\rangle$ to $|b\rangle$ and from $|b\rangle$ to $|c\rangle$ produce a pair of photons with the same frequency.

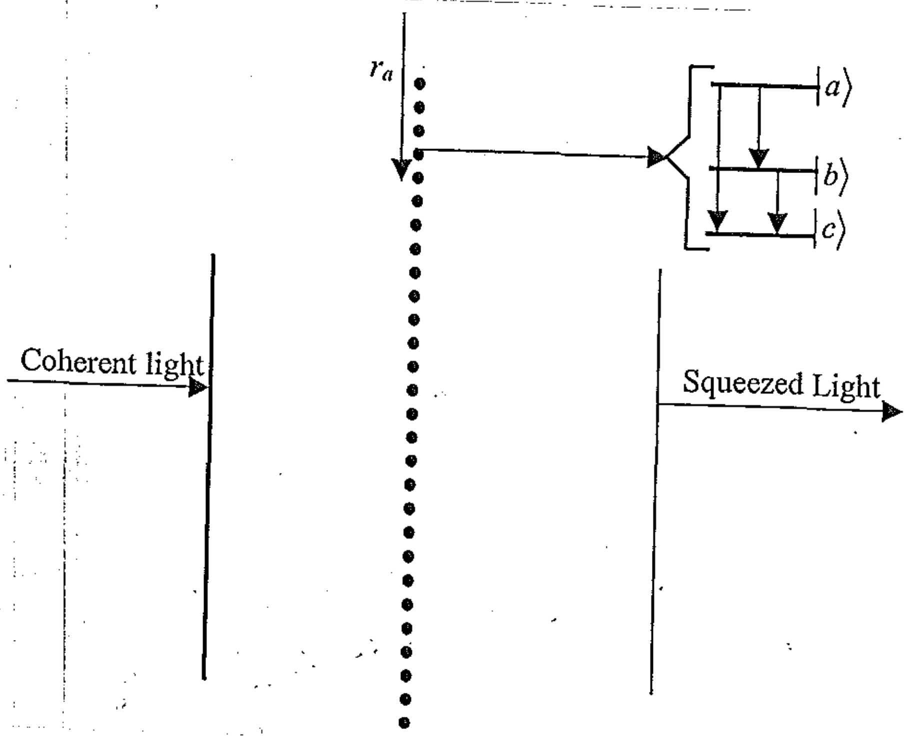


Figure [2.2] A three-level laser with external driving radiation.

The Hamiltonian describing the coupling between $|a\rangle$ and $|c\rangle$ by the driving radiation can be expressed as [2],

$$H' = ig' \left(\hat{c}^\dagger |c\rangle\langle a| e^{i(\omega' - \omega_{ac})t} - \hat{c}|a\rangle\langle c| e^{-i(\omega' - \omega_{ac})t} \right), \tag{2.1}$$

where $\hat{c}$ is the annihilation operator for the driving radiation, ω' is its frequency and $g' = \left(\frac{\omega'}{2\hbar\epsilon_0 V}\right)^{1/2} \mathbf{d}_{ac}.\hat{e}$ is the coupling constant between the atom and radiation. At resonance ($\omega' = \omega_{ac}$) the Hamiltonian reduces to

$$H' = ig' \left(\hat{c}^\dagger |c\rangle\langle a| - \hat{c}|a\rangle\langle c| \right) \tag{2.2}$$

For a strong driving radiation the number of photons is very large that we can treat the radiation classically, i.e, $\hat{c}$ can be replaced by ε, assumed to be complex and constant. Expressing ε as

$$\varepsilon = |\varepsilon| e^{-i\phi}, \tag{2.3}$$

the Hamiltonian can be put in the form

$$H' = -i\frac{\Omega}{2} \left(|a\rangle\langle c| e^{-i\phi} - |c\rangle\langle a| e^{i\phi} \right), \tag{2.4a}$$

where

$$\Omega = 2g'|\varepsilon| \tag{2.4b}$$

is a quantity proportional to the amplitude of the driving radiation.

On the other hand, the Hamiltonian describing the interaction between a three-level atom and the cavity mode is

$$H'' = ig \left(\hat{a}^\dagger |b\rangle\langle a| e^{i(\omega - \omega_{ab})t} - \hat{a}|a\rangle\langle b| e^{-i(\omega - \omega_{ab})t} \right)$$

$$+ig \left(\hat{a}^\dagger |c\rangle\langle b| e^{i(\omega - \omega_{bc})t} - \hat{a}|b\rangle\langle c| e^{-i(\omega - \omega_{bc})t} \right), \tag{2.5}$$

where $\hat{a}$ is the annihilation operator for the cavity mode and ω is its frequency. At resonance this Hamiltonian reduces to

$$H'' = ig\left(\hat{a}^{\dagger}|b\rangle\langle a| - \hat{a}|a\rangle\langle b| + \hat{a}^{\dagger}|c\rangle\langle b| - \hat{a}|b\rangle\langle c|\right). \tag{2.6}$$

Therefore, the total interaction Hamiltonian of the system is

$$H = ig\left(\hat{a}^{\dagger}|b\rangle\langle a| - \hat{a}|a\rangle\langle b| + \hat{a}^{\dagger}|c\rangle\langle b| - \hat{a}|b\rangle\langle c|\right) - i\frac{\Omega}{2}\left(|a\rangle\langle c|e^{-i\phi} - |c\rangle\langle a|e^{i\phi}\right). \tag{2.7}$$

We take the intial state of a single atom to be

$$|\psi_A(0)\rangle = |a\rangle. \tag{2.8}$$

Hence the density operator for a single atom before it is injected into the cavity has the form

$$\hat{\rho}_A(0) = |a\rangle\langle a|. \tag{2.9}$$

Let $\hat{\rho}_{AR}(t, t_j)$ be the density operator for the radiation plus a single atom at time t, with the atom injected into the cavity at an earlier time t_j. As it is metioned at the begining of this section, the atoms are injected into the cavity at a constant rate r_a and removed from the cavity after a certain time τ. Therefore, it is easy to see that

$$t - \tau \leq t_j \leq t. \tag{2.10}$$

The density operator for all the atoms in the cavity plus the radiation at time t can then be expressed as

$$\hat{\rho}_{AR}(t) = r_a \sum_j \hat{\rho}_{AR}(t, t_j)\Delta t_j, \tag{2.11}$$

where $r_a \Delta t_j$ represents the number of atoms injected into the cavity at time t_j. Converting the summation to integration, we have

$$\hat{\rho}_{AR}(t) = r_a \int_{t-\tau}^{t} \hat{\rho}_{AR}(t, t')dt'. \tag{2.12}$$

Differentiating both sides with respect to t and employing the identity

$$\frac{d}{dx}\int_a^x f(x,y)dy = f(x,x) - f(x,a) + \int_a^x \frac{\partial}{\partial x} f(x,y)dy, \tag{2.13}$$

we can rewrite (2.12) as

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a[\hat{\rho}_{AR}(t,t) - \hat{\rho}_{AR}(t,t-\tau)] + r_a \int_{t-\tau}^{t} \frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t')dt'. \tag{2.14}$$

We note that $\hat{\rho}_{AR}(t,t)$ is the density operator at time t for the radiation plus an atom injected at the same time t, whereas $\hat{\rho}_{AR}(t,t-\tau)$ reperesents the density operator for an atom plus the radiation at time t, with the atom injected into the cavity at $(t-\tau)$ and being removed from the cavity at time t.

Since the atomic and the radiation variables are uncorrelated at the instant the atom is injected into or removed from the cavity, we can write

$$\hat{\rho}_{AR}(t,t) = \hat{\rho}_A(t,t)\hat{\rho}(t) \tag{2.15a}$$

and

$$\hat{\rho}_{AR}(t,t-\tau) = \hat{\rho}_A(t,t-\tau)\hat{\rho}(t), \tag{2.15b}$$

where $\hat{\rho}(t)$ is the density operator for the radiation at time t and $\hat{\rho}_A(t,t')_{(t'=t \text{ or } t-\tau)}$ is the density operator at time t for an atom injected at t'. Moreover, it is not difficult to see that

$$\hat{\rho}_A(t,t) = \hat{\rho}_A(0). \tag{2.16}$$

Substituting Eqs. (2.15) and (2.16) into Eq. (2.14), we obtain

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a[\hat{\rho}_A(0) - \hat{\rho}_A(t,t-\tau)]\hat{\rho}(t) + r_a \int_{t-\tau}^{t} \frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t')dt' \tag{2.17}$$

When the interaction of the cavity mode with a vacuum reservoir is not taken into account, the density operator $\hat{\rho}_{AR}(t,t')$ evolves in time according to

$$\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t') = -i[\hat{H}, \hat{\rho}_{AR}(t,t')], \tag{2.18}$$

so that employing (2.12), we get

$$r_a \int_{t-\tau}^{t} \frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t')dt' = -i[\hat{H}, \hat{\rho}_{AR}(t)]. \tag{2.19}$$

In view of this result Eq. (2.17) takes the form

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a[\hat{\rho}_A(0) - \hat{\rho}_A(t, t-\tau)]\hat{\rho}(t) - i[\hat{H}, \hat{\rho}_{AR}(t)]. \tag{2.20}$$

Now taking the trace over the atomic variables and noting that

$$Tr_A\hat{\rho}_A(0) = Tr_A\hat{\rho}_A(t, t-\tau) = 1, \tag{2.21}$$

we find

$$\frac{d}{dt}\hat{\rho}(t) = -iTr_A[\hat{H}, \hat{\rho}_{AR}(t)]. \tag{2.22}$$

Furthermore, on taking into account the interaction of the cavity mode with a vacuum reservoir via a single port-mirror, the equation of evolution of the reduced density operator can be written as [10]

$$\frac{d}{dt}\hat{\rho}(t) = -iTr_A[\hat{H}, \hat{\rho}_{AR}(t)] + \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}), \tag{2.23}$$

where κ is the cavity damping constant. On account of expression (2.7), we note that

$$\begin{aligned} Tr_A[\hat{H}, \hat{\rho}_{AR}(t)] = {} & ig\left[\hat{a}^\dagger Tr_A(|b\rangle\langle a|\hat{\rho}_{AR} + |c\rangle\langle b|\hat{\rho}_{AR}) - \hat{a}Tr_A(|a\rangle\langle b|\hat{\rho}_{AR} + |b\rangle\langle c|\hat{\rho}_{AR})\right] \\ & + ig\left[-Tr_A(\hat{\rho}_{AR}|b\rangle\langle a| + \hat{\rho}_{AR}|c\rangle\langle b|)\hat{a}^\dagger + Tr_A(\hat{\rho}_{AR}|a\rangle\langle b| + \hat{\rho}_{AR}|b\rangle\langle c|)\hat{a}\right] \\ & - i\frac{\Omega}{2}\left[Tr_A(e^{-i\phi}|a\rangle\langle c|\hat{\rho}_{AR} - e^{i\phi}|c\rangle\langle a|\hat{\rho}_{AR}) + Tr_A(e^{-i\phi}\hat{\rho}_{AR}|a\rangle\langle c| - e^{i\phi}\hat{\rho}_{AR}|c\rangle\langle a|)\right], \end{aligned} \tag{2.24}$$

so applying the cyclic property of the trace operation, we gets

$$Tr_A[\hat{H}, \hat{\rho}_{AR}(t)] = ig\left[\hat{a}^\dagger\hat{\rho}_{ab} - \hat{\rho}_{ab}\hat{a}^\dagger + \hat{a}^\dagger\hat{\rho}_{bc} - \hat{\rho}_{bc}\hat{a}^\dagger - \hat{a}\hat{\rho}_{ba} + \hat{\rho}_{ba}\hat{a} - \hat{a}\hat{\rho}_{cb} - \hat{\rho}_{cb}\hat{a}\right], \tag{2.25a}$$

where

$$\hat{\rho}_{\alpha\beta} = \langle\alpha|\hat{\rho}_{AR}(t)|\beta\rangle, \tag{2.25b}$$

with $\alpha, \beta = a, b, c$.

In view of Eqs. (2.23) and (2.25a), the master equation for the cavity mode can be put in the form

$$\frac{d}{dt}\hat{\rho}(t) = g\left[\hat{a}^{\dagger}\hat{\rho}_{ab} - \hat{\rho}_{ab}\hat{a}^{\dagger} + \hat{a}^{\dagger}\hat{\rho}_{bc} - \hat{\rho}_{bc}\hat{a}^{\dagger} - \hat{a}\hat{\rho}_{ba} + \hat{\rho}_{ba}\hat{a} - \hat{a}\hat{\rho}_{cb} + \hat{\rho}_{cb}\hat{a}\right]$$
$$+\frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{a}). \tag{2.26}$$

In addition, multiplying (2.20) on the left by $\langle\alpha|$ and on the right by $|\beta\rangle$, we see that

$$\frac{d}{dt}\hat{\rho}_{\alpha\beta}(t) = r_a[\langle\alpha|\hat{\rho}_A(0)|\beta\rangle - \langle\alpha|\hat{\rho}_A(t, t-\tau)|\beta\rangle]\hat{\rho}(t) - i\langle\alpha|[\hat{H}, \hat{\rho}_{AR}(t)]|\beta\rangle - \gamma\hat{\rho}_{\alpha\beta}, \tag{2.27}$$

where the last term is added to account for the damping of the atoms due to spontaneous emission, with the atomic decay rate γ considered to be the same for all the three levels.

Assuming the atoms are removed from the cavity after they have decayed to a level other than the middle or the lower level, it can be easily verified that

$$\langle\alpha|\hat{\rho}_A(t, t-\tau)|\beta\rangle = 0. \tag{2.28}$$

On account of this, Eq. (2.27) reduces to

$$\frac{d}{dt}\hat{\rho}_{\alpha\beta}(t) = r_a\langle\alpha|\hat{\rho}_A(0)|\beta\rangle\hat{\rho}(t) - i\langle\alpha|[\hat{H}, \hat{\rho}_{AR}(t)]|\beta\rangle - \gamma\hat{\rho}_{\alpha\beta}. \tag{2.29}$$

Moreover, with the aid of (2.7) and (2.9), Eq. (2.29) can be expressed as

$$\frac{d}{dt}\hat{\rho}_{\alpha\beta}(t) = r_a\delta_{\alpha a}\delta_{a\beta}\rho(t) + g\left[\hat{a}^{\dagger}(\delta_{\alpha b}\hat{\rho}_{a\beta} + \delta_{\alpha c}\hat{\rho}_{b\beta}) - \hat{a}(\delta_{\alpha a}\hat{\rho}_{b\beta} + \delta_{\alpha b}\hat{\rho}_{c\beta})\right]$$
$$+g\left[(\hat{\rho}_{\alpha a}\delta_{b\beta} + \hat{\rho}_{\alpha b}\delta_{c\beta})\hat{a} - (\hat{\rho}_{\alpha b}\delta_{a\beta} + \hat{\rho}_{\alpha c}\delta_{b\beta})\hat{a}^{\dagger}\right] - \gamma\hat{\rho}_{\alpha\beta}$$

$$-\frac{\Omega}{2}\left[e^{-i\phi}\delta_{\alpha a}\hat{\rho}_{c\beta} - e^{i\phi}\delta_{\alpha c}\hat{\rho}_{a\beta} - e^{-i\phi}\hat{\rho}_{\alpha a}\delta_{c\beta} + e^{i\phi}\hat{\rho}_{\alpha c}\delta_{a\beta}\right]. \tag{2.30}$$

It then follows that

$$\frac{d}{dt}\hat{\rho}_{aa}(t) = r_a\hat{\rho} - g(\hat{a}\hat{\rho}_{ba} + \hat{\rho}_{ab}\hat{a}^\dagger) - \frac{\Omega}{2}(e^{-i\phi}\hat{\rho}_{ca} + e^{i\phi}\hat{\rho}_{ac}) - \gamma\hat{\rho}_{aa}, \tag{2.31}$$

$$\frac{d}{dt}\hat{\rho}_{bb}(t) = g(\hat{a}^\dagger\hat{\rho}_{ab} - \hat{a}\hat{\rho}_{cb} - \hat{\rho}_{bc}\hat{a}^\dagger + \hat{\rho}_{ba}\hat{a}) - \gamma\hat{\rho}_{bb}, \tag{2.32}$$

$$\frac{d}{dt}\hat{\rho}_{cc}(t) = g(\hat{a}^\dagger\hat{\rho}_{bc} + \hat{\rho}_{cb}\hat{a}) + \frac{\Omega}{2}(e^{i\phi}\hat{\rho}_{ac} + e^{-i\phi}\hat{\rho}_{ca}) - \gamma\hat{\rho}_{cc}, \tag{2.33}$$

$$\frac{d}{dt}\hat{\rho}_{ab}(t) = g(-\hat{a}\hat{\rho}_{bb} + \hat{\rho}_{aa}\hat{a} - \hat{\rho}_{ac}\hat{a}^\dagger) - \frac{\Omega}{2}e^{-i\phi}\hat{\rho}_{cb} - \gamma\hat{\rho}_{ab}, \tag{2.34}$$

$$\frac{d}{dt}\hat{\rho}_{ac}(t) = g(-\hat{a}\hat{\rho}_{bc} + \hat{\rho}_{ab}\hat{a}) - \frac{\Omega}{2}(e^{-i\phi}\hat{\rho}_{cc} - e^{-i\phi}\hat{\rho}_{aa}) - \gamma\hat{\rho}_{ac}, \tag{2.35}$$

$$\frac{d}{dt}\hat{\rho}_{bc}(t) = g(\hat{a}^\dagger\hat{\rho}_{ac} - \hat{a}\hat{\rho}_{cc} + \hat{\rho}_{bb}\hat{a}) + \frac{\Omega}{2}e^{-i\phi}\hat{\rho}_{ba} - \gamma\hat{\rho}_{bc}. \tag{2.36}$$

We seek to carry out our analysis in the linear approximation and in the good-cavity limit $\kappa << g, \gamma$. The linear approximation is achieved by dropping the g terms in Eqs. (2.31), (2.32), (2.33) and (2.35). In the good cavity limit the cavity mode variables changes slowely compared with the atomic variables. As a result of this, the atomic variables will reach steady state in a relatively short period, so that the time derivatives of such variables can be set to zero. This procedure may be refered to as the adiabatic approximation scheme [10]. Thus upon dropping the g terms and applying the adiabatic approximation scheme, Eqs. (2.31), (2.32), (2.33), and (2.35) reduce to

$$r_a\hat{\rho} - \frac{\Omega}{2}(e^{i\phi}\hat{\rho}_{ac} + e^{-i\phi}\hat{\rho}_{ca}) - \gamma\hat{\rho}_{aa} = 0, \tag{2.37}$$

$$\hat{\rho}_{bb} = 0, \tag{2.38}$$

$$\frac{\Omega}{2}(e^{i\phi}\hat{\rho}_{ac} + e^{-i\phi}\hat{\rho}_{ca}) - \gamma\hat{\rho}_{cc} = 0, \tag{2.39}$$

$$-\frac{\Omega}{2}(\hat{\rho}_{cc} - \hat{\rho}_{aa})e^{-i\phi} - \gamma\hat{\rho}_{ac} = 0. \tag{2.40}$$

After simultaneously solving Eqs. (2.37), (2.39) and (2.40), we obtain

$$\hat{\rho}_{aa} = \frac{r_a\hat{\rho}}{\gamma\left(1+\frac{\Omega^2}{\gamma^2}\right)}\left(1+\frac{\Omega^2}{2\gamma^2}\right), \tag{2.41}$$

$$\hat{\rho}_{cc} = \frac{r_a\hat{\rho}}{\gamma\left(1+\frac{\Omega^2}{\gamma^2}\right)}\left(\frac{\Omega^2}{2\gamma^2}\right), \tag{2.42}$$

$$\hat{\rho}_{ac} = \frac{r_a\hat{\rho}}{\gamma\left(1+\frac{\Omega^2}{\gamma^2}\right)}\left(\frac{\Omega}{2\gamma}\right)e^{-i\phi}, \tag{2.43}$$

$$\hat{\rho}_{ca} = \frac{r_a\hat{\rho}}{\gamma\left(1+\frac{\Omega^2}{\gamma^2}\right)}\left(\frac{\Omega}{2\gamma}\right)e^{i\phi}. \tag{2.44}$$

Taking into account these results and applying the adiabatic approximation once more, expressions (2.34) and the complex conjugate of (2.36) are found to be

$$\frac{gr_a\hat{\rho}}{\gamma\left(1+\frac{\Omega^2}{\gamma^2}\right)}\left[\left(1+\frac{\Omega^2}{2\gamma^2}\right)\hat{\rho}\hat{a} - \frac{\Omega}{2\gamma}e^{-i\phi}\hat{\rho}\hat{a}^\dagger\right] - \frac{\Omega}{2}e^{-i\phi}\hat{\rho}_{cb} - \gamma\hat{\rho}_{ab} = 0, \tag{2.45}$$

$$\frac{gr_a\hat{\rho}}{\gamma\left(1+\frac{\Omega^2}{\gamma^2}\right)}\left[\frac{\Omega}{2\gamma}e^{i\phi}\hat{\rho}\hat{a} - \frac{\Omega^2}{2\gamma^2}\hat{\rho}\hat{a}^\dagger\right] + \frac{\Omega}{2}e^{i\phi}\hat{\rho}_{ab} - \gamma\hat{\rho}_{cb} = 0. \tag{2.46}$$

From these two equations, we get

$$\hat{\rho}_{ab} = \frac{gr_a\hat{\rho}}{\gamma^2\beta}\left[\left(1+\frac{\Omega^2}{4\gamma^2}\right)\hat{\rho}\hat{a} - \frac{\Omega}{2\gamma}\left(1-\frac{\Omega^2}{2\gamma^2}\right)e^{-i\phi}\hat{\rho}\hat{a}^\dagger\right], \tag{2.47}$$

$$\hat{\rho}_{cb} = \frac{gr_a\hat{\rho}}{\gamma^2\beta}\left[\frac{\Omega}{\gamma}\left(1+\frac{\Omega^2}{4\gamma^2}\right)e^{i\phi}\hat{\rho}\hat{a} - \frac{3\Omega^2}{4\gamma^2}\hat{\rho}\hat{a}^\dagger\right], \tag{2.48}$$

where

$$\beta = \left(1+\frac{\Omega^2}{\gamma^2}\right)\left(1+\frac{\Omega^2}{4\gamma^2}\right). \tag{2.49}$$

The expressions for $\hat{\rho}_{ba}$ and $\hat{\rho}_{bc}$ can be obtained by taking the complex conjugate of the expressions for $\hat{\rho}_{ab}$ and $\hat{\rho}_{cb}$, respectively.

Finally, on substituting Eqs. (2.47), (2.48) and their complex conjugates into Eq. (2.26), the equation of evolution of the density operator for the cavity mode takes the form

$$\frac{d}{dt}\hat{\rho}(t) = \frac{A}{2\beta}\left[p(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}) + q(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a})\right]$$

$$+\frac{A}{2\beta}\left[\eta(\hat{a}^{\dagger}\hat{\rho}\hat{a}^{\dagger}-\hat{\rho}\hat{a}^{\dagger 2})+z(\hat{a}^{\dagger}\hat{\rho}\hat{a}^{\dagger}-\hat{a}^{\dagger 2}\hat{\rho})+\eta^{*}(\hat{a}\hat{\rho}\hat{a}-\hat{a}^{2}\hat{\rho})+z^{*}(\hat{a}\hat{\rho}\hat{a}-\hat{\rho}\hat{a}^{2})\right], \tag{2.50a}$$

where

$$A=\frac{2g^{2}r_{a}}{\gamma^{2}}, \tag{2.50b}$$

is the linear gain coeffecient,

$$p=1+\frac{\Omega^{2}}{4\gamma^{2}}, \tag{2.50c}$$

$$q=\frac{\beta\kappa}{A}+\frac{3\Omega^{2}}{4\gamma^{2}}, \tag{2.50d}$$

$$\eta=-\frac{\Omega}{2\gamma}\left(1-\frac{\Omega^{2}}{2\gamma^{2}}\right)e^{-i\phi}, \tag{2.50e}$$

$$z=-\frac{\Omega}{\gamma}\left(1+\frac{\Omega^{2}}{4\gamma^{2}}\right)e^{-i\phi}. \tag{2.50f}$$

When there is no driving radiation ($\Omega=0$) Eq. (2.50a) reduces to the master equation of a two-level laser operating below thereshold. It is also important to note that the presence of the quadratic terms $\hat{a}^{\dagger 2}$ and $\hat{a}^{2}$ in the master equation is a signature that the system under consideration can generate squeezed light.

2.2. Quadrature Variances

The expectation value of any operator $\hat{A}$ in the Schrodinger picture evolves in time according to

$$\frac{d}{dt}\langle\hat{A}\rangle=Tr\left(\frac{d\hat{\rho}}{dt}\hat{A}\right). \tag{2.51}$$

Therefore, employing Eq. (2.50a), the cyclic property of the trace operation, and the identities

$$[\hat{a},f(\hat{a},\hat{a}^{\dagger}]=\frac{\partial f}{\partial\hat{a}^{\dagger}}, \tag{2.52a}$$

$$[\hat{a}^{\dagger},f(\hat{a},\hat{a}^{\dagger}]=-\frac{\partial f}{\partial\hat{a}}, \tag{2.52b}$$

it can be easily verified that

$$\frac{d}{dt}\langle\hat{a}\rangle=\frac{A}{2\beta}[(p-q)\langle\hat{a}\rangle+(\eta-z)\langle\hat{a}^{\dagger}\rangle], \tag{2.53}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle = \frac{A}{2\beta}[2(p-q)\langle\hat{a}^\dagger\hat{a}\rangle + (\eta - z)\langle\hat{a}^{\dagger 2}\rangle + (\eta^* - z^*)\langle\hat{a}^2\rangle + 2p], \tag{2.54}$$

$$\frac{d}{dt}\langle\hat{a}^2\rangle = \frac{A}{\beta}[(p-q)\langle\hat{a}^2\rangle + (\eta - z)\langle\hat{a}^\dagger\hat{a}\rangle - z]. \tag{2.55}$$

The equations of evolution for $\langle\hat{a}^\dagger\rangle$ and $\langle\hat{a}^{\dagger 2}\rangle$ can be obtained by taking the complex conjugates of Eqs. (2.53) and (2.55).

At steady state the time derivateves in equations (2.53), (2.54) and (2.55) vanish and we are left with

$$(p-q)\langle\hat{a}\rangle_{ss} + (\eta - z)\langle\hat{a}^\dagger\rangle_{ss} = 0, \tag{2.56}$$

$$2(p-q)\langle\hat{a}^\dagger\hat{a}\rangle_{ss} + (\eta - z)\langle\hat{a}^{\dagger 2}\rangle_{ss} + (\eta^* - z^*)\langle\hat{a}^2\rangle_{ss} + 2p = 0, \tag{2.57}$$

$$(p-q)\langle\hat{a}^2\rangle_{ss} + (\eta - z)\langle\hat{a}^\dagger\hat{a}\rangle_{ss} - z = 0, \tag{2.58}$$

where the subscript ss stands for steady state. Taking the complex conjugates of Eqs. (2.56) and (2.58) we get

$$(p-q)\langle\hat{a}^\dagger\rangle_{ss} + (\eta^* - z^*)\langle\hat{a}\rangle_{ss} = 0, \tag{2.59}$$

$$(p-q)\langle\hat{a}^{\dagger 2}\rangle_{ss} + (\eta^* - z^*)\langle\hat{a}^\dagger\hat{a}\rangle_{ss} - z^* = 0. \tag{2.60}$$

Upon simultaneously solving Eqs. (2.56) to (2.60), we readily obtain

$$\langle\hat{a}\rangle_{ss} = 0, \tag{2.61}$$

$$\langle\hat{a}^\dagger\rangle_{ss} = 0, \tag{2.62}$$

$$\langle\hat{a}^\dagger\hat{a}\rangle_{ss} = \frac{p(q-p) + z^*z - z^*\eta}{(p-q)^2 - |\eta - z|^2}, \tag{2.63}$$

$$\langle\hat{a}^2\rangle_{ss} = \frac{p\eta - qz}{(p-q)^2 - |\eta - z|^2}, \tag{2.64}$$

and

$$\langle\hat{a}^{\dagger 2}\rangle_{ss} = \frac{p\eta^* - qz^*}{(p-q)^2 - |\eta - z|^2}. \tag{2.65}$$

For $\Omega = 0$, it is not difficult to see that $p = 1$, $q = \frac{\kappa}{A}$ and $\eta = z = 0$, so that

$$\langle \hat{a}^\dagger \hat{a} \rangle_{ss} = \frac{A}{\kappa - A}, \tag{2.66}$$

which is the steady state mean phopton number of a two-level laser operating below thereshold.

The quadrature operators for a single-mode radiation, are defined as

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a} \tag{2.67a}$$

and

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}) \tag{2.67b}$$

Based on these definitions the variance of the quadrature operators can be expressed as

$$\Delta \hat{a}_\pm^2 = 1 + 2(\langle \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle) \pm \left(\langle \hat{a}^2 \rangle + \langle \hat{a}^{\dagger 2} \rangle - \langle \hat{a} \rangle^2 - \langle \hat{a}^\dagger \rangle^2 \right). \tag{2.68a}$$

For the system under consideration $\langle \hat{a} \rangle$ and $\langle \hat{a}^\dagger \rangle$ vanish at steady state, so that the quadrature variance at steady state reduce to

$$\Delta \hat{a}_\pm^2 = 1 + 2\langle \hat{a}^\dagger \hat{a} \rangle_{ss} \pm [\langle \hat{a}^2 \rangle_{ss} + \langle \hat{a}^{\dagger 2} \rangle_{ss}]. \tag{2.68b}$$

Moreover, on account of Eqs. (2.63), (2.64) and (2.65), it can be easily verified that

$$\Delta \hat{a}_\pm^2 = \frac{q^2 - p^2 + z^* z - \eta^* \eta \pm [p(\eta + \eta^*) - q(z + z^*)]}{(p - q)^2 - |\eta - z|^2}. \tag{2.69}$$

Again for $\Omega = 0$ this will reduce to the steady state quadrature variaces of a two-level laser operating below thereshold.

Fig. [2.3] shows that the cavity mode is in squeezed state for a wide range of (Ω/γ) and a relatively better squeezing is observed for $10 \leq (\Omega/\gamma) \leq 20$. It is also important to note that the degree of squeezing increases with the linear gain coefficient. To achieve nearly perfect squeezing one should consider sufficiently large values of the linear gain coefficient.

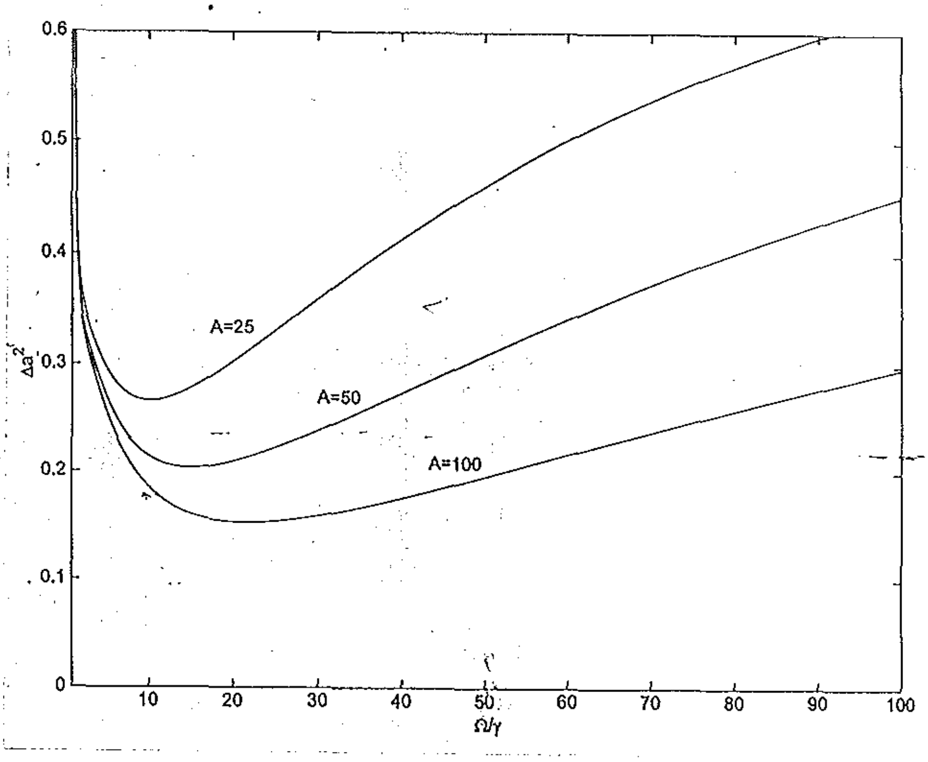


Fig. [2.3] Plots of the quadrature variance $\Delta\hat{a}_-^2$ versus Ω/γ for $\kappa = 0.8$, $\phi = 0$ and for different values of the linear gain coefficient.

2.3. Squeezing Spectrum

We next seek to determine the squeezing spectrum of the cavity mode. To this end, the equation of evolution of the expectation value of $\hat{a}$, which is obtained when Eq. (2.53) and its complex conjugate are decoupled, is given by

$$\frac{d^2}{dt^2}\langle\hat{a}(t)\rangle - \frac{A}{\beta}(p-q)\frac{d}{dt}\langle\hat{a}(t)\rangle + \left(\frac{A}{2\beta}\right)^2\left[(p-q)^2 - |\eta - z|^2\right]\langle\hat{a}(t)\rangle = 0. \qquad (2.70a)$$

The solution of this equation can be expressed as

$$\langle \hat{a}(t)\rangle = C_1 e^{\mu_+ t} + C_2 e^{\mu_- t}. \tag{2.70b}$$

The expression of $\langle \hat{a}^\dagger(t)\rangle$, which can be obtained by substituting Eq. (2.70b) and its time derivative into Eq. (2.53), is found to be

$$\langle \hat{a}^\dagger(t)\rangle = \frac{|\eta - z|}{\eta - z}\left[C_1 e^{\mu_+ t} - C_2 e^{\mu_- t}\right], \tag{2.70c}$$

where

$$\mu_\pm = \frac{A(p-q)}{2\beta} \pm \frac{A|\eta - z|}{2\beta}. \tag{2.70d}$$

After determining the constants C_1 and C_2 from the initial conditions, Eqs. (2.70b) and (2.70c) will take the form

$$\langle \hat{a}(t)\rangle = \frac{1}{2}\langle \hat{a}(t_0)\rangle \left[e^{\mu_+(t-t_0)} + e^{\mu_-(t-t_0)}\right]$$
$$+\frac{1}{2}\langle \hat{a}^\dagger(t_0)\rangle \frac{|\eta - z|}{\eta^* - z^*}\left[e^{\mu_+(t-t_0)} - e^{\mu_-(t-t_0)}\right] \tag{2.70e}$$

and

$$\langle \hat{a}^\dagger(t)\rangle = \frac{1}{2}\langle \hat{a}^\dagger(t_0)\rangle \left[e^{\mu_+(t-t_0)} + e^{\mu_-(t-t_0)}\right]$$
$$+\frac{1}{2}\langle \hat{a}(t_0)\rangle \frac{|\eta - z|}{\eta - z}\left[e^{\mu_+(t-t_0)} - e^{\mu_-(t-t_0)}\right]. \tag{2.70f}$$

Replacing t by $t+\tau$ and t_0 by t we get

$$\langle \hat{a}(t+\tau)\rangle = \frac{1}{2}\langle \hat{a}(t)\rangle \left[e^{\mu_+\tau} + e^{\mu_-\tau}\right] + \frac{1}{2}\langle \hat{a}^\dagger(t)\rangle \frac{|\eta - z|}{\eta^* - z^*}\left[e^{\mu_+\tau} - e^{\mu_-\tau}\right] \tag{2.70g}$$

and

$$\langle \hat{a}^\dagger(t+\tau)\rangle = \frac{1}{2}\langle \hat{a}^\dagger(t)\rangle \left[e^{\mu_+\tau} + e^{\mu_-\tau}\right] + \frac{1}{2}\langle \hat{a}(t)\rangle \frac{|\eta - z|}{\eta - z}\left[e^{\mu_+\tau} - e^{\mu_-\tau}\right]. \tag{2.70h}$$

In view of these results, the time dependent expectation values of the quadrature operators can be expressed as

$$\langle \hat{a}_+(t+\tau)\rangle = \frac{1}{2}[\langle \hat{a}(t)\rangle + \langle \hat{a}^\dagger(t)\rangle]\left(e^{\mu_+\tau} + e^{\mu_-\tau}\right)$$

$$+\frac{1}{2}|\eta - z|\left[\frac{\langle \hat{a}(t)\rangle}{\eta - z} + \frac{\langle \hat{a}^\dagger(t)\rangle}{\eta^* - z^*}\right]\left(e^{\mu_+\tau} - e^{\mu_-\tau}\right), \tag{2.71a}$$

$$\langle \hat{a}_-(t+\tau)\rangle = \frac{i}{2}[\langle \hat{a}^\dagger(t)\rangle - \langle \hat{a}(t)\rangle]\left(e^{\mu_+\tau} + e^{\mu_-\tau}\right)$$

$$+\frac{i}{2}|\eta - z|\left[\frac{\langle \hat{a}(t)\rangle}{\eta - z} - \frac{\langle \hat{a}^\dagger(t)\rangle}{\eta^* - z^*}\right]\left(e^{\mu_+\tau} - e^{\mu_-\tau}\right). \tag{2.71b}$$

The sqeezing spectrum of the output radiation is expressible as

$$S_\pm^{out}(\omega) = 1 + 2\mathrm{Re}\int_0^\infty \langle :\hat{a}_\pm^{out}(t), \hat{a}_\pm^{out}(t+\tau):\rangle_{ss} e^{i\omega\tau} d\tau, \tag{2.72a}$$

where

$$\langle :\hat{A}, \hat{B}:\rangle = \langle :\hat{A}\hat{B}:\rangle - \langle :\hat{A}:\rangle\langle :\hat{B}:\rangle, \tag{2.72b}$$

and :: stands for normal ordering. In Eq. (2.72a) one is added to account for the normal ordering. For a cavity mode coupled to ordinary vacuum reservoir the output variables are related to the intra cavity variables by [10]

$$\hat{a}_\pm^{out}(t) = \sqrt{\kappa}\hat{a}_\pm(t). \tag{2.73}$$

With this the two-time correlation function in Eq. (2.72a) takes the form

$$\langle :\hat{a}_\pm^{out}(t), \hat{a}_\pm^{out}(t+\tau):\rangle_{ss} = \kappa\langle :\hat{a}_\pm(t), \hat{a}_\pm(t+\tau):\rangle_{ss}, \tag{2.74}$$

Moreover, recalling that for the system under consideration the expectation values of both $\hat{a}_+$ and $\hat{a}_-$ vanish at steady state, we can rewrite Eq. (2.74) in the form

$$\langle :\hat{a}_\pm^{out}(t), \hat{a}_\pm^{out}(t+\tau):\rangle_{ss} = \kappa\langle :\hat{a}_\pm(t)\hat{a}_\pm(t+\tau):\rangle_{ss}. \tag{2.75}$$

Applying the quantum regression theorem and employing Eqs. (2.71a) and (2.71b), Eq. (2.75) can be written as

$$\langle : \hat{a}_{\pm}^{out}(t), \hat{a}_{\pm}^{out}(t+\tau) : \rangle_{ss} = \frac{\kappa}{2}[2\langle \hat{a}^{\dagger}\hat{a}\rangle_{ss} \pm (\langle \hat{a}^{\dagger 2}\rangle_{ss} + \langle \hat{a}^{2}\rangle_{ss})]\left(e^{\mu_{+}\tau} + e^{\mu_{+}\tau}\right)$$

$$+\frac{\kappa}{2}[\pm 2\langle \hat{a}^{\dagger}\hat{a}\rangle_{ss}\cos\phi + \langle \hat{a}^{\dagger 2}\rangle_{ss}e^{-i\phi} + \langle \hat{a}^{2}\rangle_{ss}e^{i\phi}]\left(e^{\mu_{+}\tau} - e^{\mu_{+}\tau}\right). \quad (2.76)$$

or

$$\langle : \hat{a}_{\pm}^{out}(t), \hat{a}_{\pm}^{out}(t+\tau) : \rangle_{ss} = \kappa[B_{\pm}e^{\mu_{+}\tau} + D_{\pm}e^{\mu_{-}\tau}], \quad (2.77a)$$

where

$$B_{\pm} = \langle \hat{a}^{\dagger}\hat{a}\rangle_{ss}(1 \pm \cos\phi) + \frac{\langle \hat{a}^{\dagger 2}\rangle_{ss}}{2}\left(e^{-i\phi} \pm 1\right) + \frac{\langle \hat{a}^{2}\rangle_{ss}}{2}\left(e^{i\phi} \pm 1\right) \quad (2.77b)$$

and

$$D_{\pm} = \langle \hat{a}^{\dagger}\hat{a}\rangle_{ss}(1 \mp \cos\phi) - \frac{\langle \hat{a}^{\dagger 2}\rangle_{ss}}{2}\left(e^{-i\phi} \mp 1\right) - \frac{\langle \hat{a}^{2}\rangle_{ss}}{2}\left(e^{i\phi} \mp 1\right), \quad (2.77c)$$

are real parameters introduced for convenience. With the aid of Eqs. (2.63), (2.64), and (2.65), the expressions for these parameters can be written in a more simplyfied form.

$$B_{\pm} = \frac{p(q-p) + z^{*}z - z^{*}\eta + p|\eta| - q|z|}{(p-q)^{2} - |\eta - z|^{2}}(1 \pm \cos\phi), \quad (2.78a)$$

$$D_{\pm} = \frac{p(q-p) + z^{*}z - z^{*}\eta - p|\eta| + q|z|}{(p-q)^{2} - |\eta - z|^{2}}(1 \mp \cos\phi). \quad (2.78b)$$

On substituting Eq. (2.77a) into Eq. (2.72a), we get

$$S_{\pm}^{out}(\omega) = 1 + 2\mathrm{Re}\int_{0}^{\infty} \kappa B_{\pm}e^{(\mu_{+}+i\omega)\tau}d\tau + 2\mathrm{Re}\int_{0}^{\infty} \kappa D_{\pm}e^{(\mu_{-}+i\omega)\tau}d\tau. \quad (2.79b)$$

Restricted to the values of (Ω/γ) for which both μ_{+} and μ_{-} are negative, the integrals can be easily evaluated to give

$$S_{\pm}^{out}(\omega) = 1 + 2\kappa B_{\pm}\mathrm{Re}\left(\frac{1}{-\mu_{+} - i\omega}\right) + 2\kappa D_{\pm}\mathrm{Re}\left(\frac{1}{-\mu_{-} - i\omega}\right) \quad (2.80a)$$

or

$$S_{\pm}^{out}(\omega) = 1 - 2\kappa\left[\frac{\mu_+ B_\pm}{\mu_+^2 + \omega^2} + \frac{\mu_- D_\pm}{\mu_-^2 + \omega^2}\right]. \tag{2.80b}$$

We recall that η and z are complex numbers of the same phase angle ϕ, so that $|\eta - z| = |\eta| - |z|$. Hence the expressions of the real parameters in Eqs. (2.78a) and (2.78b) reduce to

$$B_\pm = -\frac{p - |z|}{p - q + |\eta - z|}(1 \pm \cos\phi) \tag{2.81}$$

and

$$D_\pm = -\frac{p + |z|}{p - q - |\eta - z|}(1 \pm \cos\phi). \tag{2.82}$$

On account of Eqs. (2.81) and (2.82), Eq. (2.80b) takes the form

$$S_{\pm}^{out}(\omega) = 1 + 2\kappa\left[\frac{\mu_+(p - |z|)(1 \pm \cos\phi)}{(\mu_+^2 + \omega^2)[p - q + |\eta - z|]} + \frac{\mu_-(p + |z|)(1 \mp \cos\phi)}{(\mu_-^2 + \omega^2)[p - q - |\eta - z|]}\right]. \tag{2.83}$$

Substituting the expressions for μ_+, μ_- and choosing $\omega = 0$, we readily obtain

$$S_{\pm}^{out}(0) = 1 + \frac{4\beta\kappa}{A}\left[\frac{(p - |z|)(1 \pm \cos\phi)}{[p - q + |\eta - z|]^2} + \frac{(p + |z|)(1 \mp \cos\phi)}{[p - q - |\eta - z|]^2}\right]. \tag{2.84a}$$

Moreover, for $\phi = 0$, this reduces to

$$S_{+}^{out}(0) = 1 + \frac{8\beta\kappa}{A}\frac{p - |z|}{[p - q + |\eta - z|]^2} \tag{2.84b}$$

and

$$S_{-}^{out}(0) = 1 + \frac{8\beta\kappa}{A}\frac{p + |z|}{[p - q - |\eta - z|]^2}. \tag{2.84c}$$

As it be can seen from Fig. [2.4], a relatively better squeezing is observed for values of (Ω/γ) near zero and small values of the linear gain coefficient. On the other hand, almost perfect squeezing can be obtained for large values of the linear gain coefficient and (Ω/γ).

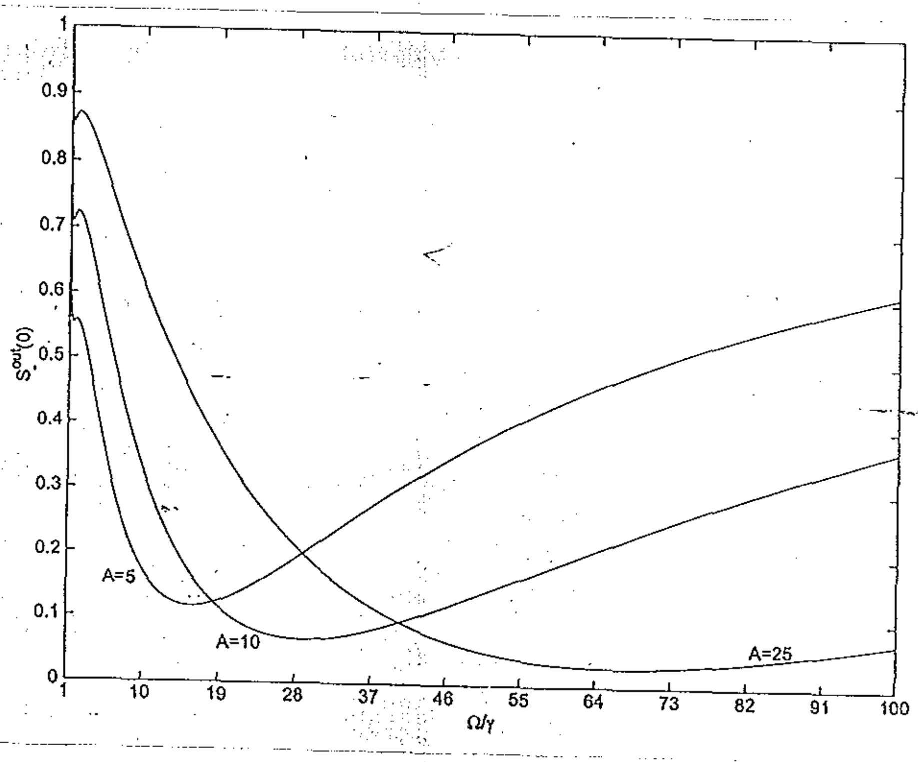


Fig. [2.4] plots of the squeezing spectrum $S_{-}^{out}(0)$ versus Ω/γ for $\kappa = 0.8$, $\phi = 0$ and for different values of the linear gain coefficient.

2.4. Photon Statistics

A. *The variance of the photon number*

In this section we will calculate the variance of the photon number at steady state for the cavity mode under consideration. To this end, we note that for a single-mode radiation the variance of the photon number can be expressed as

$$\Delta n^2 = \bar{n} + \langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle_{ss} - \bar{n}^2, \tag{2.85a}$$

where according to Eq. (2.63) the mean photon number is given by

$$\bar{n} = \frac{p(q-p) + z^*z - z^*\eta}{(p-q)^2 - |\eta - z|^2}. \tag{2.85b}$$

In order to use the known results of Eqs. (2.64) and (2.65), it is necessary to express $\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle_{ss}$ in terms of the steady state expectation values of $\hat{a}^{\dagger 2}$ and $\hat{a}^2$. To get the desired expression, we note that for $\hat{a}_1$,$\hat{a}_2$, $\hat{a}_3$, and $\hat{a}_4$ are all Gaussian random variables, one can write

$$\langle \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4\rangle = \langle \hat{a}_1\hat{a}_2\rangle\langle \hat{a}_3\hat{a}_4\rangle + \langle \hat{a}_1\hat{a}_3\rangle\langle \hat{a}_2\hat{a}_4\rangle + \langle \hat{a}_1\hat{a}_4\rangle\langle \hat{a}_2\hat{a}_3\rangle. \tag{2.86}$$

Since the cavity mode operator $\hat{a}$ is a Gaussian rondom variable,

$$\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle_{ss} = \langle \hat{a}^{\dagger 2}\rangle_{ss}\langle \hat{a}^2\rangle_{ss} + 2\langle \hat{a}^\dagger\hat{a}\rangle_{ss}^2, \tag{2.87}$$

so that the variance of the photon number takes the form

$$\Delta n^2 = \bar{n} + \bar{n}^2 + \langle \hat{a}^{\dagger 2}\rangle_{ss}\langle \hat{a}^2\rangle_{ss}. \tag{2.88}$$

Now in view of Eqs. (2.64), (2.65), and (2.85b), it can be easily verified that

$$\Delta n^2 = \frac{[(p-q)^2 - |\eta - z|^2]\,(pq - z^*\eta) + 2(p^2\eta^*\eta + q^2z^*z)}{[(p-q)^2 - |\eta - z|^2]^2}. \tag{2.89}$$

This reduces to the variance of the photon number for a two-level laser operating below thereshold if we set Ω to zero.

From Fig. [2.5](a) we see that the variance of the photon number is greater than the mean photon number. This confirms that the photon statistics is super-Poissonian. In addition, in Fig. [2.5](b) we have plotted the uncertainty in the photon number Δn versus Ω/γ.

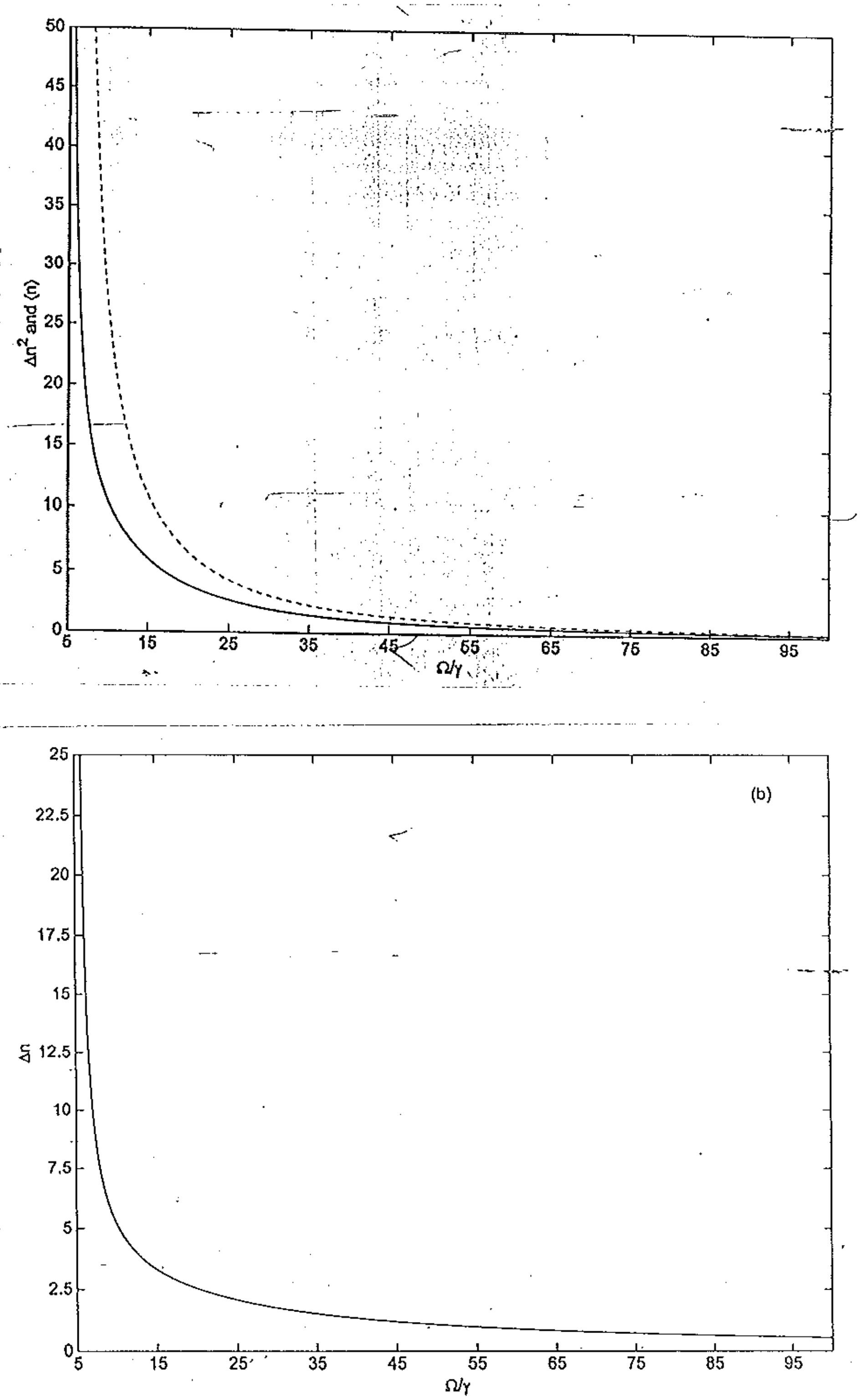


Fig. [2.5] Plots of (a) $\bar{n}$ (solid) and Δn^2 (dot), (b) the uncertainty in the photon number Δn versus Ω/γ, for $A = 25$ and $\kappa = 0.8$.

B. *The photon number distribution*

Next we seek to determine the photon number distribution of the cavity mode. To this end, we note that the Q function for a single cavity mode can be expressed as

$$Q(\alpha^*, \alpha) = \frac{1}{\pi^2} \int d^2z \phi(z^*, z) \exp[\alpha z^* - \alpha^* z], \tag{2.90a}$$

where $\phi(z^*, z)$ is the antinormally ordered characteristic function defined by

$$\phi(z^*, z) = \langle e^{-z^* \hat{a}} e^{z \hat{a}^\dagger} \rangle. \tag{2.90b}$$

Applying the Baker-Hausdroff identity, this can be expressed in the form

$$\phi(z^*, z) = e^{-z^* z/2} \langle \exp(z\hat{a}^\dagger - z^* \hat{a}) \rangle. \tag{2.91}$$

At steady state $\hat{a}$ is a Gaussian random variable with zero mean, so that in view of the derivation given in appendix C of reference [10], expression (2.91) can be written as

$$\phi(z^*, z) = e^{-z^* z/2} \exp\left[\frac{1}{2} \langle (z\hat{a}^\dagger - z^* \hat{a})^2 \rangle_{ss}\right]. \tag{2.92}$$

It then follows that

$$\phi(z^*, z) = \exp\left[-(\langle \hat{a}^\dagger \hat{a} \rangle_{ss} + 1) z^* z + (\langle \hat{a}^{\dagger 2} \rangle_{ss} z^2 + \langle \hat{a}^2 \rangle_{ss} z^{*2})/2\right]. \tag{2.93}$$

Introducing a real parameter a and a complex parameter b given by

$$a = (\langle \hat{a}^\dagger \hat{a} \rangle_{ss} + 1) \tag{2.94a}$$

and

$$b = \langle \hat{a}^{\dagger 2} \rangle_{ss}, \tag{2.94b}$$

we can rewrite Eq. (2.93) in the form

$$\phi(z^*, z) = \exp\left[-a z^* z + (b z^2 + b^* z^{*2})/2\right]. \tag{2.95}$$

With the aid of this result expression (2.90a) takes the form

$$Q(\alpha^*, \alpha) = \frac{1}{\pi^2} \int d^2z \exp\left[-az^*z - \alpha^* z + \alpha z^* + (bz^2 + b^* z^{*2})/2\right]. \tag{2.96}$$

Employing the relation

$$\int \frac{d^2z}{\pi} \exp\left[-az^*z + bz + cz^* + Az^2 + Bz^{*2}\right] = \frac{1}{\sqrt{a^2 - 4AB}} \exp\left[\frac{abc + Ac^2 + Bb^2}{a^2 - 4AB}\right], \tag{2.97a}$$

the integral can be easily evaluated to give

$$Q(\alpha^*, \alpha) = \frac{(u^2 - v^*v)^{1/2}}{\pi} \exp\left[-u\alpha^*\alpha + (v\alpha^2 + v^*\alpha^{*2})/2\right], \tag{2.97b}$$

where

$$u = \frac{a}{a^2 - b^*b} \tag{2.97c}$$

and

$$v = \frac{b}{a^2 - b^*b}. \tag{2.97d}$$

The photon number distribution for a single-mode radiation is expressible as[10]

$$P(n) = \frac{\pi}{n!} \frac{\partial^{2n}}{\partial \alpha^n \partial \alpha^{*n}} \left[Q(\alpha^*, \alpha) e^{\alpha^* \alpha}\right]_{\alpha^* = \alpha = 0}. \tag{2.98}$$

With the aid of this expression and Eq. (2.97a), the steady state photon number distribution for the system under consideration can be expressed as

$$P(n) = \frac{\sqrt{u^2 - v^*v}}{n!} \frac{\partial^{2n}}{\partial \alpha^n \partial \alpha^{*n}} \exp\left[(1-u)\alpha^*\alpha + (v\alpha^2 + v^*\alpha^{*2})/2\right]_{\alpha^* = \alpha = 0}. \tag{2.99}$$

Expanding the exponetial function in power series, we have

$$P(n) = \frac{\sqrt{u^2 - v^*v}}{n!} \frac{\partial^{2n}}{\partial \alpha^n \partial \alpha^{*n}} \sum_{ijk} \frac{(1-u)^i v^{*j} v^k}{2^{j+k} i! j! k!} (\alpha^*)^{i+2j} (\alpha)^{i+2k} |_{\alpha^* = \alpha = 0}. \tag{2.100}$$

Carrying out the differentiation and applying the condition $\alpha^* = \alpha = 0$, we get

$$P(n) = \frac{\sqrt{u^2 - v^*v}}{n!} \sum_{ijk} \frac{(1-u)^i v^{*j} v^k (i+2j)!(i+2k)!}{2^{j+k} i! j! k! (i+2j-n)!(i+2k-n)!} \delta_{i+2j,n} \delta_{i+2k,n}. \tag{2.101}$$

Now in view of the property of the Kronecker delta symbole, we see that

$$i = n - 2k \tag{2.102a}$$

and

$$j = k, \tag{2.102b}$$

so that the photon number distribution reduces to

$$P(n) = n!\sqrt{u^2 - v^*v} \sum_{k=0}^{[n]} \frac{(1-u)^{n-2k}(v^*v)^k}{2^{2k}k!^2(n-2k)!}, \tag{2.103}$$

where $[n] = n/2$ for n is even and $[n] = (n-1)/2$ for n is odd.

This distribution function has the same form as the probability of observing n siginal photons in a degenerate parametric oscillator. This shows that under specific conditions, a coherently driven degenerate three-level laser acts as a parametric oscillator.

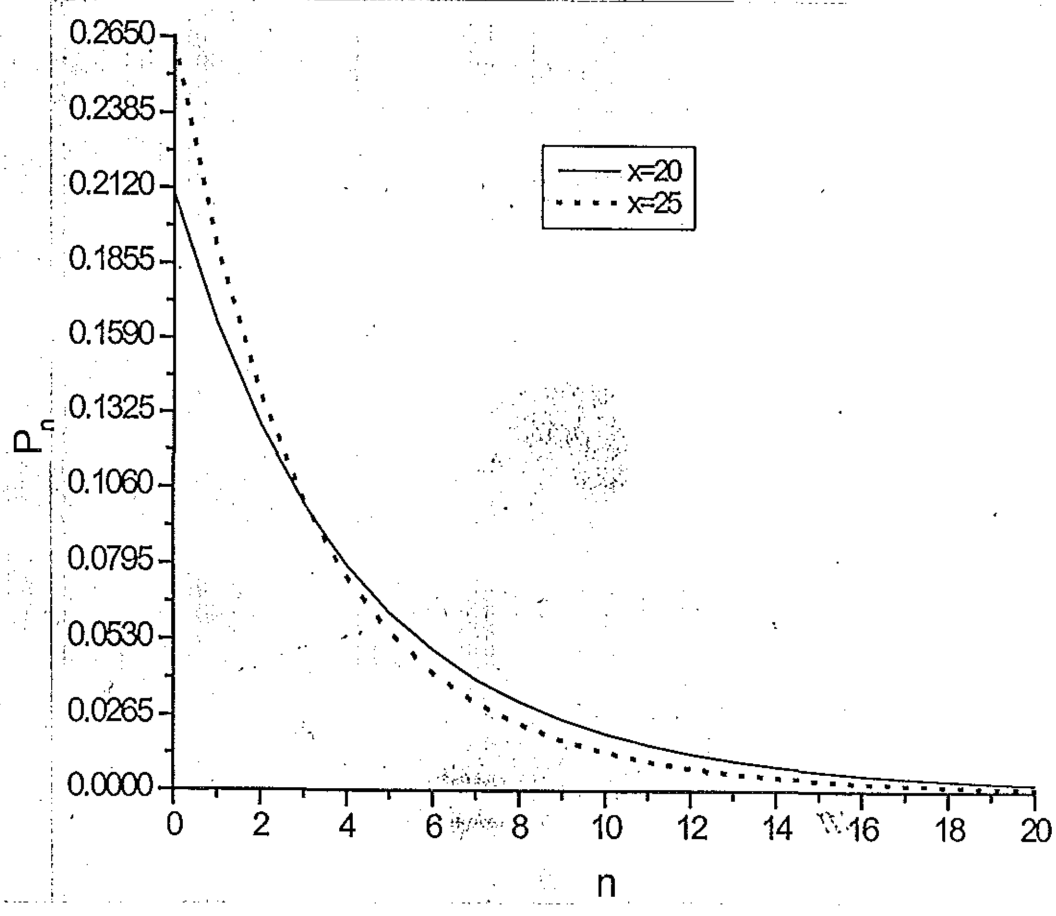


Fig. [2.6] Plots of $P(n)$ versus n, for $A = 25$, $\kappa = 0.8$ $\Omega/\gamma = 20$ and $\Omega/\gamma = 25$.

In Fig. [2.6] we have ploted the photon number distribution versus the photon number for two values of Ω/γ. Both graphs show that the distribution function decreases with increasing photon number.

3. COHERENTLY DRIVEN NONDEGENERATE THREE-LEVEL LASER

In the previous chapter we have considered a three-level laser in which the transition frequencies ω_{ab} and ω_{bc} are the same. In the case the transition from the top to the bottom level via the intermediate level leads to the generation of two identical photons. In this chapter we consider the case in which these transition frequencies are not equal, so that two different photons will be generated as the atom decays from the top to the bottom level via the intermediate level. Here we consider the case in which the top and the bottom levels are coupled by a strong external radiation, which was also the case in the previous chapter. We call such quantum optical system coherently driven nondegenerate three-level laser.

Following a similar procedure as in the degenerate case, we will determine the quadrature variance, the squeezing spectrum, the variance of the photon number, and the photon number distribution for the two cavity modes.

3.1. The master Equation

We now consider a three-level atom in a cascade configuration (see Fig. [2.1]) with $\omega_{ab} \neq \omega_{bc}$. Again treating the driving radiation classically, the interaction Hamiltonian between a single atom and the radiation is given by Eq. (2.4a). The interaction Hamiltonian of the atom and the two cavity modes, is expressed as

$$\hat{H}'' = ig(\hat{a}^{\dagger}|b\rangle\langle a| - \hat{a}|a\rangle\langle b| + \hat{b}^{\dagger}|c\rangle\langle b| - \hat{b}|b\rangle\langle c|), \tag{3.1}$$

where $\hat{a}$ is the annihilation operator for the cavity mode with frequency ω_{ab} and $\hat{b}$ is the annihilation operators for the cavity mode with frequency ω_{bc}, g is the coupling constant between the atom and cavity modes, assumed to be the same for the two modes.

The total interaction Hamiltonian is then

$$\hat{H} = ig(\hat{a}^\dagger|b\rangle\langle a| - \hat{a}|a\rangle\langle b| + \hat{b}^\dagger|c\rangle\langle b| - \hat{b}|b\rangle\langle c|) - i\frac{\Omega}{2}(e^{-i\phi}|a\rangle\langle c| - e^{i\phi}|c\rangle\langle a|). \tag{3.2}$$

Following a similar procedure as in the degenerate case, the equation of evolution for the reduced density operator is found to be

$$\frac{d\hat{\rho}}{dt} = -iTr_A[\hat{H}, \hat{\rho}_{AR}(t)] + \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a} + 2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}), \tag{3.3}$$

where κ is the cavity damping constant, assumed to be the same for the two modes.

In view of Eq. (3.2), the commutator $[\hat{H}, \hat{\rho}_{AR}(t)]$ can be expressed as

$$\begin{aligned}[\hat{H}, \hat{\rho}_{AR}] = ig[&\hat{a}^\dagger|b\rangle\langle a|\hat{\rho}_{AR} - \hat{a}|a\rangle\langle b|\hat{\rho}_{AR} + \hat{b}^\dagger|c\rangle\langle b|\hat{\rho}_{AR} - \hat{b}|b\rangle\langle c|\hat{\rho}_{AR} \\ &-\hat{\rho}_{AR}(t)|b\rangle\langle a|\hat{a}^\dagger + \hat{\rho}_{AR}|a\rangle\langle b|\hat{a} - \hat{\rho}_{AR}|c\rangle\langle b|\hat{b}^\dagger + \hat{\rho}_{AR}|b\rangle\langle c|\hat{b}] \\ &-i\frac{\Omega}{2}[e^{-i\phi}|a\rangle\langle c|\hat{\rho}_{AR} - e^{i\phi}|c\rangle\langle a|\hat{\rho}_{AR} - e^{-i\phi}\hat{\rho}_{AR}|a\rangle\langle c| + e^{i\phi}\hat{\rho}_{AR}|c\rangle\langle a|],\end{aligned} \tag{3.4}$$

so that

$$Tr_A[\hat{H}, \hat{\rho}_{AR}(t)] = ig[\hat{a}^\dagger\hat{\rho}_{ab} - \hat{\rho}_{ab}\hat{a}^\dagger - \hat{a}\hat{\rho}_{ba} + \hat{\rho}_{ba}\hat{a} + \hat{b}^\dagger\hat{\rho}_{bc} - \hat{\rho}_{bc}\hat{b}^\dagger - \hat{b}\hat{\rho}_{cb} + \hat{\rho}_{cb}\hat{b}], \tag{3.5a}$$

where

$$\hat{\rho}_{\alpha\beta} = \langle\alpha|\hat{\rho}_{AR}|\beta\rangle, \tag{3.5b}$$

with α, β = a,b,c. On substuting Eq. (3.5a) into Eq. (3.3), we readily obtain

$$\begin{aligned}\frac{d\hat{\rho}}{dt} = ig[&\hat{a}^\dagger\hat{\rho}_{ab} - \hat{\rho}_{ab}\hat{a}^\dagger - \hat{a}\hat{\rho}_{ba} + \hat{\rho}_{ba}\hat{a} + \hat{b}^\dagger\hat{\rho}_{bc} - \hat{\rho}_{bc}\hat{b}^\dagger - \hat{b}\hat{\rho}_{cb} + \hat{\rho}_{cb}\hat{b}] \\ &+\frac{\kappa}{2}[2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a} + 2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}].\end{aligned} \tag{3.6}$$

On the other hand, employing Eq. (2.29), with $\hat{H}$ given by Eq. (3.2), it can be easily verified that

$$\frac{d\hat{\rho}_{\alpha\beta}}{dt} = r_a\hat{\rho}\delta_{\alpha a}\delta_{a\beta} + g[\hat{a}^\dagger\hat{\rho}_{a\beta}\delta_{\alpha b} - \hat{a}\hat{\rho}_{b\beta}\delta_{\alpha a} - \hat{\rho}_{\alpha b}\hat{a}^\dagger\delta_{a\beta} + \hat{\rho}_{\alpha a}\hat{a}\delta_{b\beta} + \hat{b}^\dagger\hat{\rho}_{b\beta}\delta_{\alpha c} - \hat{b}\hat{\rho}_{c\beta}\delta_{\alpha b}$$

$$-\hat{\rho}_{\alpha c}\hat{b}^\dagger\delta_{b\beta} + \hat{\rho}_{\alpha b}\hat{b}\delta_{c\beta}] - \frac{\Omega}{2}[e^{-i\phi}\hat{\rho}_{c\beta}\delta_{\alpha a} - e^{i\phi}\hat{\rho}_{a\beta}\delta_{\alpha c} - e^{-i\phi}\hat{\rho}_{\alpha a}\delta_{c\beta} + e^{i\phi}\hat{\rho}_{\alpha c}\delta_{a\beta}] - \gamma\hat{\rho}_{\alpha\beta}. \quad (3.7)$$

With the the aid of this we get

$$\frac{d\hat{\rho}_{aa}}{dt} = r_a\hat{\rho} - g[\hat{a}\hat{\rho}_{ba} + \hat{\rho}_{ab}\hat{a}^\dagger] - \frac{\Omega}{2}[e^{i\phi}\hat{\rho}_{ac} + e^{-i\phi}\hat{\rho}_{ca}] - \gamma\hat{\rho}_{aa}, \quad (3.8)$$

$$\frac{d\hat{\rho}_{bb}}{dt} = g[\hat{a}^\dagger\hat{\rho}_{ab} + \hat{\rho}_{ba}\hat{a} - \hat{b}\hat{\rho}_{cb} - \hat{\rho}_{bc}\hat{b}^\dagger] - \gamma\hat{\rho}_{bb}, \quad (3.9)$$

$$\frac{d\hat{\rho}_{cc}}{dt} = g[\hat{a}^\dagger\hat{\rho}_{bc} + \hat{\rho}_{cb}\hat{b}] + \frac{\Omega}{2}[e^{i\phi}\hat{\rho}_{ac} + e^{-i\phi}\hat{\rho}_{ca}] - \gamma\hat{\rho}_{cc}, \quad (3.10)$$

$$\frac{d\hat{\rho}_{ab}}{dt} = g[-\hat{a}\hat{\rho}_{bb} + \hat{\rho}_{aa}\hat{a} - \hat{\rho}_{ac}\hat{b}^\dagger] - \frac{\Omega}{2}e^{-i\phi}\hat{\rho}_{cb} - \gamma\hat{\rho}_{ab}, \quad (3.11)$$

$$\frac{d\hat{\rho}_{ac}}{dt} = g[-\hat{a}\hat{\rho}_{bc} + \hat{\rho}_{ab}\hat{b}] - \frac{\Omega}{2}[\hat{\rho}_{cc} - \hat{\rho}_{aa}]e^{-i\phi} - \gamma\hat{\rho}_{ac}, \quad (3.12)$$

$$\frac{d\hat{\rho}_{bc}}{dt} = g[\hat{a}^\dagger\hat{\rho}_{ac} - \hat{b}\hat{\rho}_{cc} - \hat{\rho}_{bb}\hat{b}] + \frac{\Omega}{2}e^{-i\phi}\hat{\rho}_{ba} - \gamma\hat{\rho}_{bc}. \quad (3.13)$$

In the linear and adiabatic approximations these matrix elements of the density operator take the form

$$\hat{\rho}_{aa} = \frac{r_a\hat{\rho}}{\gamma\left(1 + \frac{\Omega^2}{\gamma^2}\right)}\left(1 + \frac{\Omega^2}{2\gamma^2}\right), \quad (3.14)$$

$$\hat{\rho}_{bb} = 0, \quad (3.15)$$

$$\hat{\rho}_{cc} = \frac{r_a\hat{\rho}}{\gamma\left(1 + \frac{\Omega^2}{\gamma^2}\right)}\left(\frac{\Omega^2}{2\gamma^2}\right), \quad (3.16)$$

$$\hat{\rho}_{ac} = \frac{r_a\hat{\rho}}{\gamma\left(1 + \frac{\Omega^2}{\gamma^2}\right)}\left(\frac{\Omega}{2\gamma}\right)e^{-i\phi}, \quad (3.17)$$

$$\hat{\rho}_{ab} = \frac{r_a\hat{\rho}}{\gamma^2\beta}\left[\left(1 + \frac{\Omega^2}{4\gamma^2}\right)\hat{\rho}\hat{a} - \frac{\Omega}{2\gamma}\left(1 - \frac{\Omega^2}{2\gamma^2}\right)e^{-i\phi}\hat{\rho}\hat{b}^\dagger\right], \quad (3.18)$$

and

$$\hat{\rho}_{bc} = \frac{r_a\hat{\rho}}{\gamma^2\beta}\left[\frac{\Omega}{\gamma}\left(1 + \frac{\Omega^2}{4\gamma^2}\right)e^{-i\phi}\hat{a}^\dagger\hat{\rho} - \frac{3\Omega^2}{4\gamma^2}\hat{b}\hat{\rho}\right], \quad (3.19)$$

where β is still given by Eq. (2.49).

On substituting Eqs. (3.18), (3.19), and their copmlex conjugates into Eq. (3.6), we get

$$\frac{d}{dt}\hat{\rho}(t) = \frac{A}{2\beta}\left[p(2\hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^{\dagger} - \hat{a}\hat{a}^{\dagger}\hat{\rho}) + \frac{\beta\kappa}{A}(2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{a}) + q(2\hat{b}\hat{\rho}\hat{b}^{\dagger} - \hat{b}^{\dagger}\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^{\dagger}\hat{b})\right]$$

$$+\frac{A}{2\beta}\left[\eta(\hat{a}^{\dagger}\hat{\rho}\hat{b}^{\dagger} - \hat{\rho}\hat{a}^{\dagger}\hat{b}^{\dagger}) + z(\hat{a}^{\dagger}\hat{\rho}\hat{b}^{\dagger} - \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{\rho}) + \eta^{*}(\hat{b}\hat{\rho}\hat{a} - \hat{a}\hat{b}\hat{\rho}) + z^{*}(\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b})\right], \quad (3.20)$$

where A, p, q, η and z are defined by Eqs. (2.50b)-(2.50f)

As the quadratic terms $\hat{a}^{\dagger 2}$ and $\hat{a}^2$ in Eq. (2.50a) show the possiblity of the generating squeezed light by a coherently driven degenerate three-level laser, the mixed terms $\hat{a}\hat{b}$ and $\hat{a}^{\dagger}\hat{b}^{\dagger}$ in Eq. (3.20), indicate the possiblity of the generating squeezed light by a coherently driven nondegenerate three-level laser.

3.2. Quadrature Variances

With the aid of Eq. (3.20) and the relation $\frac{d}{dt}\langle\hat{A}\rangle = Tr(\frac{d\rho}{dt}\hat{A})$, it can be verified that

$$\frac{d}{dt}\langle\hat{a}\rangle = \frac{A}{2\beta}\left[(p - \frac{\beta\kappa}{A})\langle\hat{a}\rangle + \eta\langle\hat{b}^{\dagger}\rangle\right], \quad (3.21)$$

$$\frac{d}{dt}\langle\hat{b}\rangle = -\frac{A}{2\beta}\left[q\langle\hat{b}\rangle + z\langle\hat{a}^{\dagger}\rangle\right], \quad (3.22)$$

$$\frac{d}{dt}\langle\hat{a}^2\rangle = \frac{A}{\beta}\left[(p - \frac{\beta\kappa}{A})\langle\hat{a}^2\rangle + \eta\langle\hat{a}\hat{b}^{\dagger}\rangle\right], \quad (3.23)$$

$$\frac{d}{dt}\langle\hat{b}^2\rangle = -\frac{A}{\beta}\left[q\langle\hat{b}^2\rangle + \eta\langle\hat{a}^{\dagger}\hat{b}\rangle\right], \quad (3.24)$$

$$\frac{d}{dt}\langle\hat{a}^{\dagger}\hat{a}\rangle = \frac{A}{2\beta}\left[2(p - \frac{\beta\kappa}{A})\langle\hat{a}^{\dagger}\hat{a}\rangle + \eta\langle\hat{a}^{\dagger}\hat{b}^{\dagger}\rangle + \eta^{*}\langle\hat{a}\hat{b}\rangle + 2p\right], \quad (2.25)$$

$$\frac{d}{dt}\langle\hat{b}^{\dagger}\hat{b}\rangle = -\frac{A}{2\beta}\left[2q\langle\hat{b}^{\dagger}\hat{b}\rangle + z\langle\hat{a}^{\dagger}\hat{b}^{\dagger}\rangle + z^{*}\langle\hat{a}\hat{b}\rangle\right], \quad (2.26)$$

$$\frac{d}{dt}\langle\hat{a}\hat{b}\rangle = \frac{A}{2\beta}\left[(p - q - \frac{\beta\kappa}{A})\langle\hat{a}\hat{b}\rangle + \eta\langle\hat{b}^{\dagger}\hat{b}\rangle - z\langle\hat{a}^{\dagger}\hat{a}\rangle - z\right], \quad (3.27)$$

$$\frac{d}{dt}\langle\hat{a}^{\dagger}\hat{b}\rangle = \frac{A}{2\beta}\left[(p - q - \frac{\beta\kappa}{A})\langle\hat{a}^{\dagger}\hat{b}\rangle - z\langle\hat{a}^{\dagger 2}\rangle + \eta^{*}\langle\hat{b}^2\rangle\right]. \quad (2.28)$$

The equations of evolution for $\langle\hat{a}^\dagger\rangle$, $\langle\hat{b}^\dagger\rangle$, $\langle\hat{a}^{\dagger 2}\rangle$, $\langle\hat{b}^{\dagger 2}\rangle$, $\langle\hat{a}^\dagger\hat{b}^\dagger\rangle$, and $\langle\hat{a}\hat{b}^\dagger\rangle$ can be obtained by taking the complex conjugates of Eqs. (3.21), (3.22), (3.23), (3.24), (3.27), and (3.28).

At steady state the time derivatives of all the above operators vanish and the steady state expectation values are found to be

$$\langle\hat{a}\rangle_{ss} = 0, \tag{3.29}$$

$$\langle\hat{b}\rangle_{ss} = 0, \tag{3.30}$$

$$\langle\hat{a}^2\rangle_{ss} = 0, \tag{3.31}$$

$$\langle\hat{b}^2\rangle_{ss} = 0, \tag{3.32}$$

$$\langle\hat{a}^\dagger\hat{b}\rangle_{ss} = 0, \tag{3.33}$$

$$\langle\hat{a}\hat{b}\rangle_{ss} = \frac{-qz\frac{\beta\kappa}{A}}{(p-q-\frac{\beta\kappa}{A})[q(p-\frac{\beta\kappa}{A})-\eta z^*]}, \tag{3.34}$$

$$\langle\hat{a}^\dagger\hat{a}\rangle_{ss} = \frac{\eta z^*(p-q)-pq(p-q-\frac{\beta\kappa}{A})}{(p-q-\frac{\beta\kappa}{A})[q(p-\frac{\beta\kappa}{A})-\eta z^*]}, \tag{3.35}$$

$$\langle\hat{b}^\dagger\hat{b}\rangle_{ss} = \frac{z^*z\frac{\beta\kappa}{A}}{(p-q-\frac{\beta\kappa}{A})[q(p-\frac{\beta\kappa}{A})-\eta z^*]}. \tag{3.36}$$

At this point it is important to note that for $\Omega = 0$, $\langle\hat{b}^\dagger\hat{b}\rangle_{ss} = 0$. This is an expected result since there is equal probability for the atoms to make a transition from $|b\rangle$ to $|c\rangle$ and from $|c\rangle$ to $|b\rangle$. However, if we consider the $|a\rangle$ to $|b\rangle$ and $|b\rangle$ to $|a\rangle$ transitions, we note that the atoms are initially prepared to be in the upper level, so that we expect more transitions from $|a\rangle$ to $|b\rangle$ than from $|b\rangle$ to $|a\rangle$. Therefore, there is a net gain in the number of photons of mode $\hat{a}$. It is also important to note that $\langle\hat{a}^\dagger\hat{a}\rangle_{ss}$ reduces to the steady state photon number for a two-level laser operating below thereshold, for $\Omega = 0$.

Once we have obtained the steady state solutions, we now proceed to calculate the quadrature variances of the cavity modes under consideration. Considering a two-mode radiation, the quadrature operators are defined as

$$\hat{c}_+ = \frac{1}{\sqrt{2}}[\hat{a}^\dagger + \hat{a} + \hat{b}^\dagger + \hat{b}], \tag{3.37a}$$

$$\hat{c}_- = \frac{i}{\sqrt{2}}[\hat{a}^\dagger - \hat{a} + \hat{b}^\dagger - \hat{b}]. \tag{3.37b}$$

With the aid of these definitions and recalling that for the system under consideration $\langle\hat{a}\rangle_{ss}$, $\langle\hat{b}\rangle_{ss}$, $\langle\hat{a}^2\rangle_{ss}$, $\langle\hat{b}^2\rangle_{ss}$ and $\langle\hat{a}^\dagger\hat{b}\rangle_{ss}$ vanish, the quadrature variances are found to be

$$\Delta c_\pm^2 = 1 + \langle\hat{a}^\dagger\hat{a}\rangle_{ss} + \langle\hat{b}^\dagger\hat{b}\rangle_{ss} \pm [\langle\hat{a}\hat{b}\rangle_{ss} + \langle\hat{a}^\dagger\hat{b}^\dagger\rangle_{ss}]. \tag{3.38}$$

In view of Eqs. (3.34) to (3.36) and the complex conjugate of Eq. (3.34), expression (3.38) takes the form

$$\Delta c_\pm^2 = \frac{\frac{\beta\kappa}{A}[z^*(\eta + z) - q(p - q - \frac{\beta\kappa}{A}) \mp q(z^* + z)]}{(p - q - \frac{\beta\kappa}{A})[q(p - \frac{\beta\kappa}{A}) - \eta z^*]}. \tag{3.39}$$

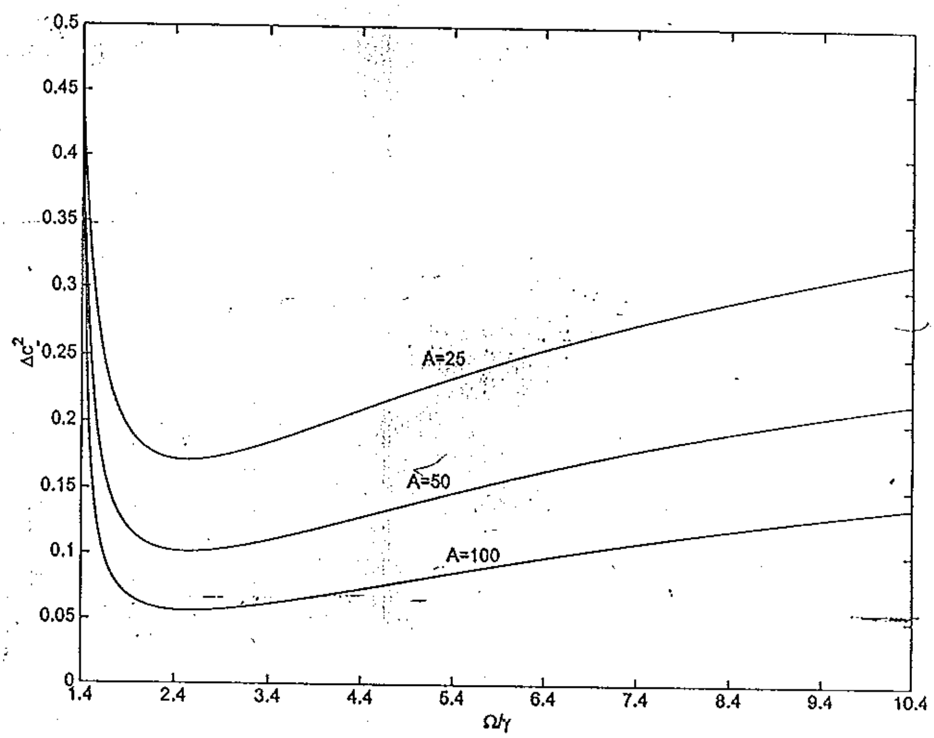


Fig. [3.1] Plot of the quadrature variance Δc_-^2 versus Ω/γ for $\kappa = 0.8$, $\phi = 0$ and for different values of the linear gain coefficient.

As it can be seen from Fig. [3.1], the two-mode light generated by a coherently driven nondegenerate three-level laser is in squeezed state for wide range of Ω/γ, with a relatively better squeezing is observed in the vicinity of $\Omega/\gamma = 2$. It is also important to note that the degree of squeezing increases with increasing linear gain coefficient and perfect squeezing can be achieved if sufficiently large values of the linear gain coefficient are used.

3.3. Squeezing Spectrum

Combining Eq. (3.21) and the complex conjugate of Eq. (3.22), the equation of evolution for $\langle\hat{a}\rangle$ is found to be

$$\frac{d^2}{dt^2}\langle\hat{a}\rangle = \frac{A}{2\beta}\left[(p-q-\frac{\beta\kappa}{A})\frac{d}{dt}\langle\hat{a}\rangle + \left(q(p-\frac{\beta\kappa}{A}) - \eta z^*\right)\langle\hat{a}\rangle\right]. \tag{3.40}$$

The solution of this equation can be expressed as

$$\langle\hat{a}(t)\rangle = C_1 e^{\mu_+ t} + C_2 e^{\mu_- t}, \tag{3.41a}$$

where

$$\mu_\pm = \frac{A}{4\beta}\left[p-q-\frac{\beta\kappa}{A} \pm \sqrt{(p+q-\frac{\beta\kappa}{A})^2 - \eta z^*}\right] \tag{3.41b}$$

and C_1, C_2 are constants to be determined from initial conditions.

Differentiating Eq. (3.41a) with respect to time, we get

$$\frac{d}{dt}\langle\hat{a}\rangle = \mu_+ C_1 e^{\mu_+ t} + \mu_- C_2 e^{\mu_- t}. \tag{3.42}$$

On substuting Eqs. (3.41a) and (3.42) into Eq. (3.21), we readily obtain

$$\langle\hat{b}^\dagger(t)\rangle = mC_1 e^{\mu_+ t} + nC_2 e^{\mu_- t}, \tag{3.43a}$$

with

$$m = \frac{1}{2\eta}\left[-(p+q-\frac{\beta\kappa}{A}) + \sqrt{(p+q-\frac{\beta\kappa}{A})^2 - \eta z^*}\right], \tag{3.43b}$$

$$n = \frac{1}{2\eta}\left[-(p+q-\frac{\beta\kappa}{A}) - \sqrt{(p+q-\frac{\beta\kappa}{A})^2 - \eta z^*}\right], \tag{3.43c}$$

are complex constants introduced for convenience.

Interms of the values of $\langle\hat{a}\rangle$ and $\langle\hat{b}^\dagger\rangle$ at the initial time, the constants C_1 and C_2 are found to be

$$C_1 = \frac{1}{m-n}(-n\langle\hat{a}(t_0)\rangle + \langle\hat{b}^\dagger(t_0)\rangle)e^{-\mu_+ t_0} \tag{3.44a}$$

and

$$C_2 = \frac{1}{m-n}(m\langle\hat{a}(t_0)\rangle - \langle\hat{b}^\dagger(t_0)\rangle)e^{-\mu_- t_0}. \tag{3.44b}$$

With the aid of these we can rewrite Eqs. (3.41a) and (3.43a) in the form

$$\langle\hat{a}(t)\rangle = \frac{1}{m-n}(-n\langle\hat{a}(t_0)\rangle + \langle\hat{b}^\dagger(t_0)\rangle)e^{\mu_+(t-t_0)}$$
$$+\frac{1}{m-n}(m\langle\hat{a}(t_0)\rangle - \langle\hat{b}^\dagger(t_0)\rangle)e^{-\mu_-(t-t_0)} \tag{3.45}$$

and

$$\langle\hat{b}^\dagger(t)\rangle = \frac{m}{m-n}(-n\langle\hat{a}(t_0)\rangle + \langle\hat{b}^\dagger(t_0)\rangle)e^{\mu_+(t-t_0)}$$
$$+\frac{n}{m-n}(m\langle\hat{a}(t_0)\rangle - \langle\hat{b}^\dagger(t_0)\rangle)e^{-\mu_-(t-t_0)}. \tag{3.46}$$

The expressions for $\langle\hat{a}^\dagger(t)\rangle$ and $\langle\hat{b}(t)\rangle$ can be determined by taking the complex conjugates of the expressions for $\langle\hat{a}(t)\rangle$ and $\langle\hat{b}^\dagger(t)\rangle$, respectively, i.e,

$$\langle\hat{a}^\dagger(t)\rangle = \frac{1}{m^*-n^*}(-n^*\langle\hat{a}^\dagger(t_0)\rangle + \langle\hat{b}(t_0)\rangle)e^{\mu_+(t-t_0)}$$
$$+\frac{1}{m^*-n^*}(m^*\langle\hat{a}^\dagger(t_0)\rangle - \langle\hat{b}(t_0)\rangle)e^{-\mu_-(t-t_0)} \tag{3.47}$$

and

$$\langle\hat{b}(t)\rangle = \frac{m^*}{m^*-n^*}(-n^*\langle\hat{a}^\dagger(t_0)\rangle + \langle\hat{b}(t_0)\rangle)e^{\mu_+(t-t_0)}$$
$$+\frac{n^*}{m^*-n^*}(m^*\langle\hat{a}^\dagger(t_0)\rangle - \langle\hat{b}(t_0)\rangle)e^{-\mu_-(t-t_0)}. \tag{3.48}$$

In view of Eq. (3.37) and Eqs. (3.45)-(3.48), the expectation values of the quadrature operators for the two cavity modes are found to be

$$\langle\hat{c}_+(t)\rangle = \frac{1}{\sqrt{2}}e^{\mu_+(t-t_0)}\left[\frac{1+m}{m-n}(\langle\hat{b}^\dagger(t_0)\rangle - n\langle\hat{a}(t_0)\rangle) + \frac{1+m^*}{m^*-n^*}(\langle\hat{b}(t_0)\rangle - n^*\langle\hat{a}^\dagger(t_0)\rangle)\right]$$
$$+\frac{1}{\sqrt{2}}e^{\mu_-(t-t_0)}\left[\frac{1+n}{m-n}(m\langle\hat{a}(t_0)\rangle - \langle\hat{b}^\dagger(t_0)\rangle) + \frac{1+n^*}{m^*-n^*}(m^*\langle\hat{a}^\dagger(t_0)\rangle - \langle\hat{b}(t_0)\rangle)\right] \quad (3.49a)$$

and

$$\langle\hat{c}_-(t)\rangle = \frac{i}{\sqrt{2}}e^{\mu_+(t-t_0)}\left[\frac{1-m}{m-n}(n\langle\hat{a}(t_0)\rangle - \langle\hat{b}^\dagger(t_0)\rangle) - \frac{1-m^*}{m^*-n^*}(n^*\langle\hat{a}^\dagger(t_0)\rangle - \langle\hat{b}(t_0)\rangle)\right]$$
$$+\frac{i}{\sqrt{2}}e^{\mu_-(t-t_0)}\left[\frac{1-n}{m-n}(\langle\hat{b}^\dagger(t_0)\rangle - m\langle\hat{a}(t_0)\rangle) - \frac{1-n^*}{m^*-n^*}(\langle\hat{b}(t_0)\rangle - m^*\langle\hat{a}^\dagger(t_0)\rangle)\right]. \quad (3.49b)$$

Replacing t_0 by t and t by $(t+\tau)$, we can rewrite Eqs. (3.49a) and (3.49b) in the form

$$\langle\hat{c}_+(t+\tau)\rangle = \frac{1}{\sqrt{2}}e^{\mu_+\tau}\left[\frac{1+m}{m-n}(\langle\hat{b}^\dagger(t)\rangle - n\langle\hat{a}(t)\rangle) + \frac{1+m^*}{m^*-n^*}(\langle\hat{b}(t)\rangle - n^*\langle\hat{a}^\dagger(t)\rangle)\right]$$
$$+\frac{1}{\sqrt{2}}e^{\mu_-\tau}\left[\frac{1+n}{m-n}(m\langle\hat{a}(t)\rangle - \langle\hat{b}^\dagger(t)\rangle) + \frac{1+n^*}{m^*-n^*}(m^*\langle\hat{a}^\dagger(t)\rangle - \langle\hat{b}(t)\rangle)\right] \quad (3.50a)$$

and

$$\langle\hat{c}_-(t+\tau)\rangle = \frac{i}{\sqrt{2}}e^{\mu_+\tau}\left[\frac{1-m}{m-n}(n\langle\hat{a}(t)\rangle - \langle\hat{b}^\dagger(t)\rangle) - \frac{1-m^*}{m^*-n^*}(n^*\langle\hat{a}^\dagger(t)\rangle - \langle\hat{b}(t)\rangle)\right]$$
$$+\frac{i}{\sqrt{2}}e^{\mu_-\tau}\left[\frac{1-n}{m-n}(\langle\hat{b}^\dagger(t)\rangle - m\langle\hat{a}(t)\rangle) - \frac{1-n^*}{m^*-n^*}(\langle\hat{b}(t)\rangle - m^*\langle\hat{a}^\dagger(t)\rangle)\right]. \quad (3.50b)$$

The sqeezing spectrum of the two cavity modes is expressible as

$$S_\pm^{out}(\omega) = 2\mathrm{Re}\int_0^\infty \langle c_\pm^{out}(t), c_\pm^{out}(t+\tau)\rangle_{ss} e^{i\omega\tau} d\tau. \quad (3.51)$$

On account of Eqs. (2.72b), (2.73), (3.29), (3.30), and applying the fundamental commutation relation $[\hat{a}(t), \hat{a}^\dagger(t')] = [\hat{b}(t), \hat{b}^\dagger(t')] = \delta(t-t')$, the two-time correlation function in Eq. (3.51), takes the form

$$\langle c_\pm^{out}(t), c_\pm^{out}(t+\tau)\rangle_{ss} = \delta(\tau) + \kappa\langle: \hat{c}_\pm(t)\hat{c}_\pm(t+\tau) :\rangle_{ss}. \quad (3.52)$$

In view of Eq. (3.52), the squeezing spectrum can be put in the form

$$S_{\pm}^{out}(\omega) = 1 + 2\kappa \mathrm{Re} \int_0^{\infty} \langle : c_{\pm}(t), c_{\pm}(t+\tau) : \rangle_{ss} e^{i\omega\tau} d\tau. \tag{3.53}$$

Now applying the quantum regression theorem to Eq. (3.50) and taking into consideration Eqs. (3.31), (3.32), (3.33), we get

$$\begin{aligned}
\langle : \hat{c}_{\pm}(t)\hat{c}_{\pm}(t+\tau) : \rangle_{ss} &= \frac{1}{2} e^{\mu_{+}\tau} \left(\frac{1 \pm m}{m-n}\right) \left[-n\langle \hat{a}^{\dagger}\hat{a}\rangle_{ss} + \langle \hat{a}^{\dagger}\hat{b}^{\dagger}\rangle_{ss} \pm (\langle \hat{b}^{\dagger}\hat{b}\rangle_{ss} - n\langle \hat{a}\hat{b}\rangle_{ss})\right] \\
&+ \frac{1}{2} e^{\mu_{+}\tau} \left(\frac{1 \pm m^{*}}{m^{*}-n^{*}}\right) \left[-n^{*}\langle \hat{a}^{\dagger}\hat{a}\rangle_{ss} + \langle \hat{a}\hat{b}\rangle_{ss} \pm (\langle \hat{b}^{\dagger}\hat{b}\rangle_{ss} - n^{*}\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\rangle_{ss})\right] \\
&+ \frac{1}{2} e^{\mu_{-}\tau} \left(\frac{1 \pm n}{m-n}\right) \left[m\langle \hat{a}^{\dagger}\hat{a}\rangle_{ss} - \langle \hat{a}^{\dagger}\hat{b}^{\dagger}\rangle_{ss} \pm (m\langle \hat{a}\hat{b}\rangle_{ss} - \langle \hat{b}^{\dagger}\hat{b}\rangle_{ss})\right] \\
&+ \frac{1}{2} e^{\mu_{-}\tau} \left(\frac{1 \pm n^{*}}{m^{*}-n^{*}}\right) \left[m^{*}\langle \hat{a}^{\dagger}\hat{a}\rangle_{ss} - \langle \hat{a}\hat{b}\rangle_{ss} \pm (m^{*}\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\rangle_{ss} - \langle \hat{b}^{\dagger}\hat{b}\rangle_{ss})\right].
\end{aligned} \tag{3.54}$$

With this result substituted into Eq. (3.53), we get

$$S_{\pm}^{out}(\omega) = 1 + 2\kappa Re \int_0^{\infty} B_{\pm} e^{(\mu_{+}+i\omega)\tau} d\tau + 2\kappa Re \int_0^{\infty} D_{\pm} e^{(\mu_{-}+i\omega)\tau} d\tau, \tag{3.55}$$

where

$$\begin{aligned}
B_{\pm} &= \frac{1 \pm m}{2(m-n)} \left[-n\langle \hat{a}^{\dagger}\hat{a}\rangle_{ss} + \langle \hat{a}^{\dagger}\hat{b}^{\dagger}\rangle_{ss} \pm (\langle \hat{b}^{\dagger}\hat{b}\rangle_{ss} - n\langle \hat{a}\hat{b}\rangle_{ss})\right] \\
&+ \frac{1 \pm m^{*}}{2(m^{*}-n^{*})} \left[-n^{*}\langle \hat{a}^{\dagger}\hat{a}\rangle_{ss} + \langle \hat{a}\hat{b}\rangle_{ss} \pm (\langle \hat{b}^{\dagger}\hat{b}\rangle_{ss} - n^{*}\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\rangle_{ss})\right]
\end{aligned} \tag{3.56a}$$

and

$$\begin{aligned}
D_{\pm} &= \frac{1 \pm n}{2(m-n)} \left[m\langle \hat{a}^{\dagger}\hat{a}\rangle_{ss} - \langle \hat{a}^{\dagger}\hat{b}^{\dagger}\rangle_{ss} \pm (m\langle \hat{a}\hat{b}\rangle_{ss} - \langle \hat{b}^{\dagger}\hat{b}\rangle_{ss})\right] \\
&+ \frac{1 \pm n^{*}}{2(m^{*}-n^{*})} \left[m^{*}\langle \hat{a}^{\dagger}\hat{a}\rangle_{ss} - \langle \hat{a}\hat{b}\rangle_{ss} \pm (m^{*}\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\rangle_{ss} - \langle \hat{b}^{\dagger}\hat{b}\rangle_{ss})\right].
\end{aligned} \tag{3.56b}$$

For those values of Ω/γ for which both μ_{+} and μ_{-} are negative, the integrals in Eq. (3.55) can be easily evaluated to give

$$S_{\pm}^{out}(\omega) = 1 + 2\kappa Re \left[\frac{B_{\pm}}{-\mu_{+} - i\omega} + \frac{D_{\pm}}{-\mu_{-} - i\omega}\right] \tag{3.57a}$$

or

$$S_{\pm}^{out}(\omega) = 1 - 2\kappa \left[\frac{B_{\pm}\mu_{+}}{\mu_{+}^{2} + \omega^{2}} + \frac{D_{\pm}\mu_{-}}{\mu_{-}^{2} + \omega^{2}} \right]. \tag{3.57b}$$

We note that for $\omega = 0$, expression (3.57b) reduces to

$$S_{\pm}^{out}(\omega) = 1 - \frac{2\kappa}{\mu_{+}\mu_{-}} \left[B_{\pm}\mu_{-} + D_{\pm}\mu_{+} \right]. \tag{3.57c}$$

With the aid of Eq. (3.56), we can write

$$\begin{aligned}
2\left[B_{\pm}\mu_{-} + D_{\pm}\mu_{+}\right] &= \frac{1}{m-n}\left[\langle\hat{a}^{\dagger}\hat{a}\rangle_{ss}\left[\mu_{+}m - \mu_{-}n \pm mn(\mu_{+} - \mu_{-})\right]\right] \\
&+ \frac{1}{m-n}\left[\langle\hat{b}^{\dagger}\hat{b}\rangle_{ss}\left[\mu_{-}m - \mu_{+}n \pm (\mu_{-} - \mu_{+})\right] + \langle\hat{a}\hat{b}\rangle_{ss}\left[mn(\mu_{+} - \mu_{-}) \pm (\mu_{+}m - \mu_{-}n)\right]\right] \\
&+ \frac{1}{m-n}\left[\langle\hat{a}^{\dagger}\hat{b}^{\dagger}\rangle_{ss}\left[\mu_{-} - \mu_{+} \pm (\mu_{-}m - \mu_{+}n)\right]\right] + \frac{1}{m^{*}-n^{*}}\left[\langle\hat{a}^{\dagger}\hat{a}\rangle_{ss}\left[\mu_{+}m^{*} - \mu_{-}n^{*} \pm m^{*}n^{*}(\mu_{+} - \mu_{-})\right]\right] \\
&+ \frac{1}{m^{*}-n^{*}}\left[\langle\hat{b}^{\dagger}\hat{b}\rangle_{ss}\left[\mu_{-}m^{*} - \mu_{+}n^{*} \pm (\mu_{-} - \mu_{+})\right] + \langle\hat{a}\hat{b}\rangle_{ss}\left[\mu_{-} - \mu_{+} \pm (\mu_{-}m^{*} - \mu_{+}n^{*})\right]\right] \\
&+ \frac{1}{m^{*}-n^{*}}\left[\langle\hat{a}^{\dagger}\hat{b}^{\dagger}\rangle_{ss}\left[m^{*}n^{*}(\mu_{+} - \mu_{-}) \pm (\mu_{+}m^{*} - \mu_{-}n^{*})\right]\right].
\end{aligned} \tag{3.58}$$

Applying Eq. (3.41b), (3.43b), and (3.43c), one can easily verify that

$$m - n = \frac{1}{\eta}\sqrt{(p + q - \frac{\beta\kappa}{A})^{2} - 4z^{*}\eta}, \tag{3.59}$$

$$\mu_{+} - \mu_{-} = \frac{A}{2\beta}\sqrt{(p + q - \frac{\beta\kappa}{A})^{2} - 4z^{*}\eta}, \tag{3.60}$$

$$\mu_{+}\mu_{-} = -\frac{A^{2}}{4\beta^{2}}\left[q(p - \frac{\beta\kappa}{A}) - 4z^{*}\eta\right], \tag{3.61}$$

$$m\mu_{+} - n\mu_{-} = -\frac{Aq}{2\beta\eta}\sqrt{(p + q - \frac{\beta\kappa}{A})^{2} - 4z^{*}\eta}, \tag{3.62a}$$

$$n\mu_{+} - m\mu_{-} = -\frac{A(\frac{\beta\kappa}{A})}{2\beta\eta}\sqrt{(p + q - \frac{\beta\kappa}{A})^{2} - 4z^{*}\eta}, \tag{3.62b}$$

$$mn(\mu_{+} - \mu_{-}) = \frac{Az^{*}}{2\beta\eta}\sqrt{(p + q - \frac{\beta\kappa}{A})^{2} - 4z^{*}\eta}. \tag{3.63}$$

On substituting Eqs. (3.59), (3.60), (3.62), (3.63) and their complex conjugates into Eq. (3.58), we obtain

$$2\left[B_{\pm}\mu_{-} + D_{\pm}\mu_{+}\right] = \frac{A}{2\beta}\left[\langle\hat{a}^{\dagger}\hat{a}\rangle_{ss}[-2q \pm (z^{*} + z)] + \langle\hat{b}^{\dagger}\hat{b}\rangle_{ss}[2(p - \frac{\beta\kappa}{A}) \mp (\eta + \eta^{*})]\right]$$

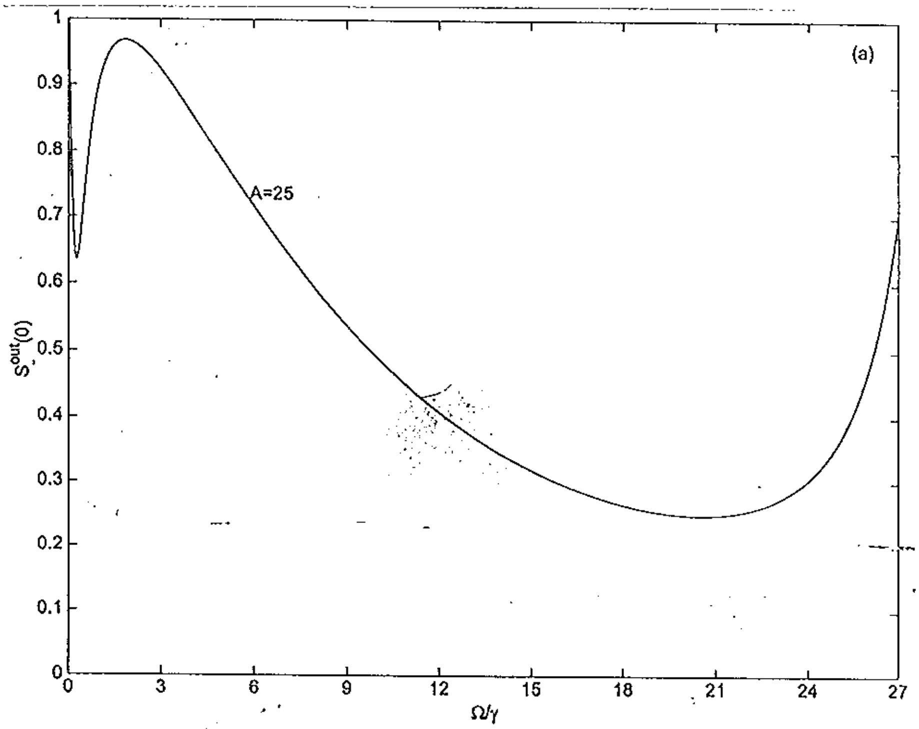
$$+\frac{A}{2\beta}\left[\langle\hat{a}\hat{b}\rangle_{ss}(z^*-\eta^*)+\langle\hat{a}^\dagger\hat{b}^\dagger\rangle_{ss}(z-\eta)\pm(\langle\hat{a}\hat{b}\rangle_{ss}+\langle\hat{a}^\dagger\hat{b}^\dagger\rangle_{ss})(p-\frac{\beta\kappa}{A}-q)\right]. \qquad (3.64a)$$

Moreover, on account of Eqs. (3.34), (3.35), (3.36) and the complex conjugate of Eq. (3.34), this reduces to

$$2\left[B_\pm\mu_- + D_\pm\mu_+\right] = \frac{A}{2\beta}\frac{\frac{2\beta\kappa}{A}z^*z + 2q(pq - z^*\eta) \pm (z+z^*)\left[z^*\eta - q(p+\frac{\beta\kappa}{A})\right]}{\left[q(p-\frac{\beta\kappa}{A}) - z^*\eta\right]}. \qquad (3.64b)$$

Finally, combination of Eqs. (3.61), (3.64b), and Eq. (3.57c) gives

$$S_\pm^{out}(0) = 1 + \frac{4\beta\kappa}{A}\frac{\frac{\beta\kappa}{A}z^*z + q(pq - z^*\eta) \pm \frac{1}{2}(z+z^*)\left[z^*\eta - q(p+\frac{\beta\kappa}{A})\right]}{\left[q(p-\frac{\beta\kappa}{A}) - z^*\eta\right]^2}. \qquad (3.65)$$



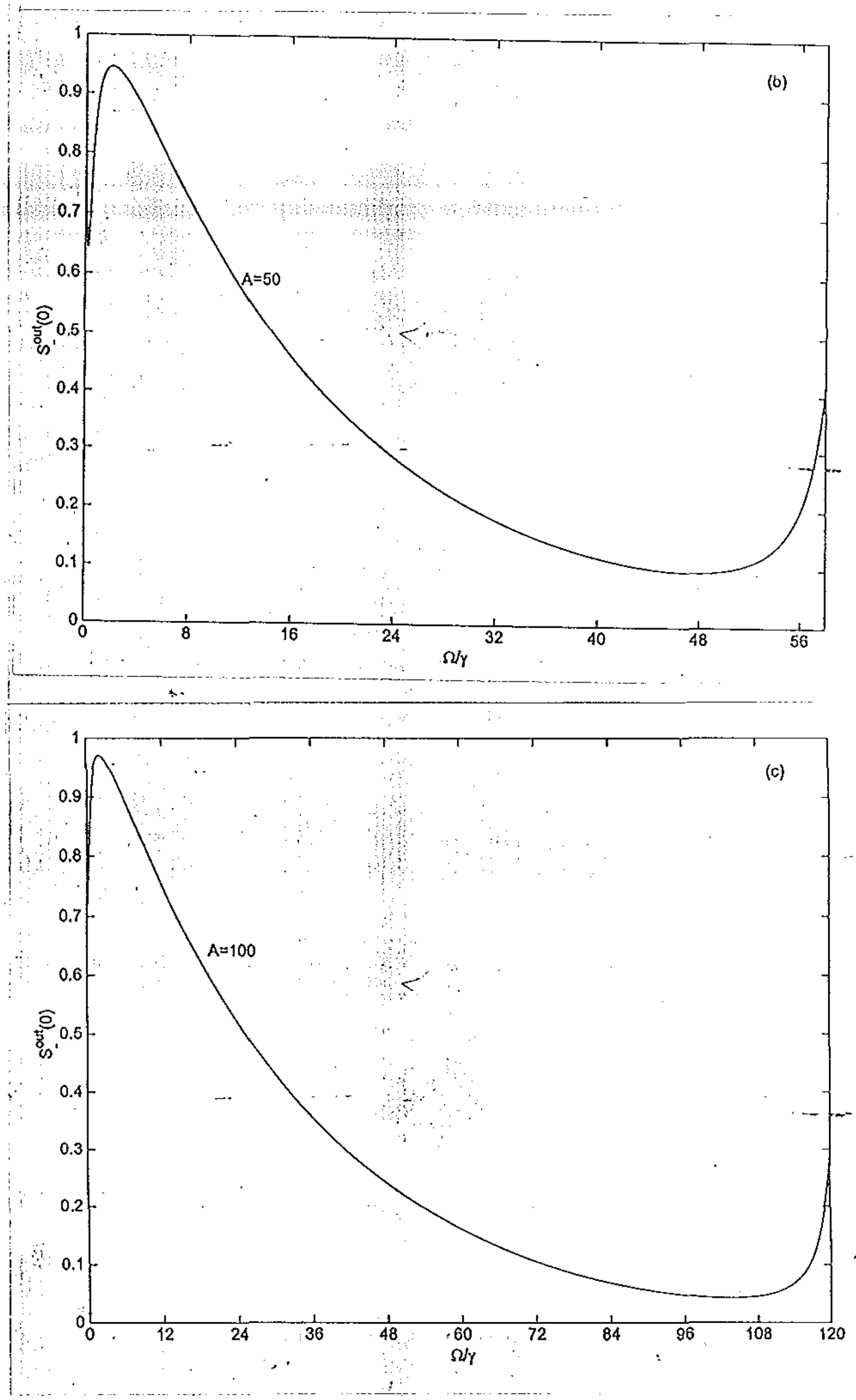


Fig. [3.2] Plots of the squeezing spectrum $S_{-}^{out}(0)$ versus Ω/γ for $\kappa = 0.8$, $\phi = 0$, (a) $A = 25$, (b) $A = 50$ and (c) $A = 100$.

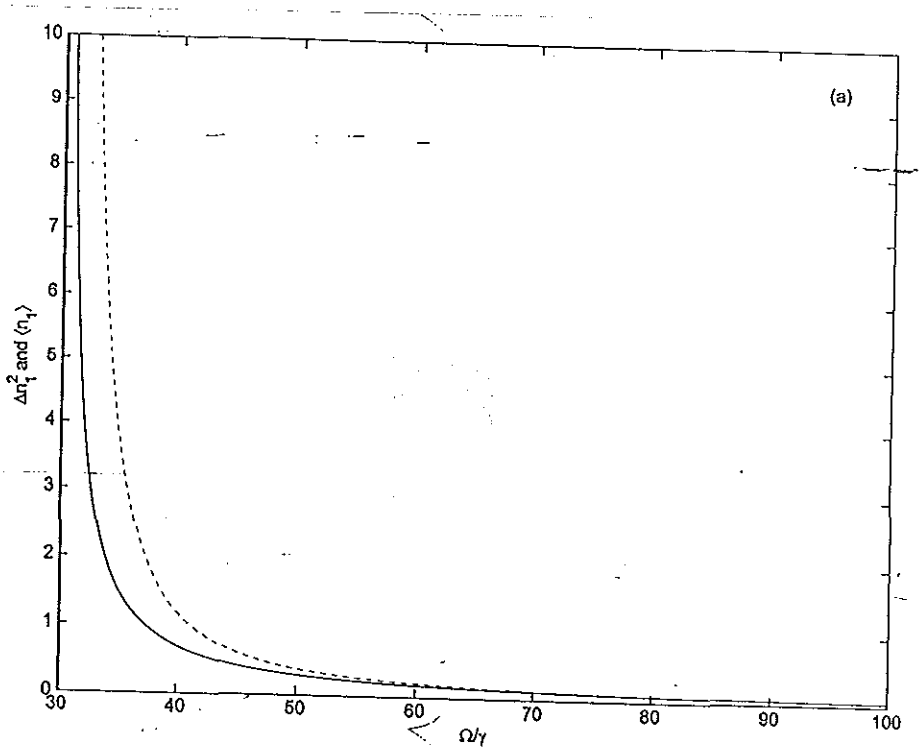
On account of Eqs. (3.31), (3.32), (3.67a), and (3.67b) it can be easily verified that

$$\Delta n_1^2 = \frac{\beta\kappa}{A}\frac{\left[(q-p)(pq-\eta z^*)+\frac{\beta\kappa}{A}pq\right]\left[z^*\eta - q(p-\frac{\beta\kappa}{A}-q)\right]}{\left(p-q-\frac{\beta\kappa}{A}\right)^2\left[q(p-\frac{\beta\kappa}{A})-\eta z^*\right]^2} \tag{3.69a}$$

and

$$\Delta n_2^2 = \frac{\beta\kappa}{A}z^*z\frac{\left[(p-q-\frac{\beta\kappa}{A})\left(q(p-\frac{\beta\kappa}{A})-\eta z^*\right)+\frac{\beta\kappa}{A}z^*z\right]}{\left(p-q-\frac{\beta\kappa}{A}\right)^2\left[q(p-\frac{\beta\kappa}{A})-\eta z^*\right]^2}. \tag{3.69b}$$

From figure [3.3] we observe that for the indicated range of Ω/γ the variance of the photon number is greater than the mean photon number for both cavity modes. This clearly shows that the photon statistics is super-Poissonian.



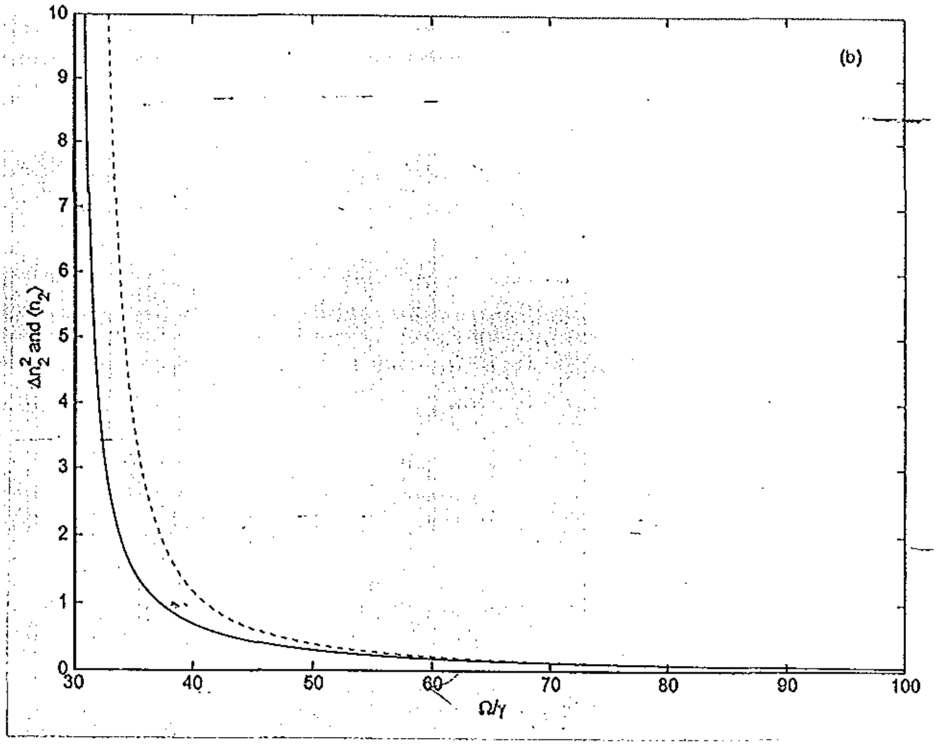


Fig. [3.3] plots of (a) the mean photon number $\bar{n}_1$ (solid) and Δn_1^2 (dot), (b) the mean photon number $\bar{n}_2$ (solid) and Δn_2^2 (dot), versus Ω/γ for $A = 25$ and $\kappa = 0.8$.

B. *Photon number distribution*

Now we proceed to calculate the steady state photon number distribution of the two cavity modes. To this end, we note that for a two-mode radiation, the antinormally ordered characteristic function is defined as

$$\phi(z, \eta) = \langle e^{-z^*\hat{a}} e^{z\hat{a}^\dagger} e^{-\eta^*\hat{b}} e^{\eta\hat{b}^\dagger} \rangle. \tag{3.70}$$

Applying the Baker-Hausdroff identity, this can be written in the form

$$\phi(z, \eta) = \exp\left[-\frac{1}{2}(z^*z + \eta^*\eta)\right] \langle \exp(z\hat{a}^\dagger - z^*\hat{a} + \eta\hat{b}^\dagger - \eta^*\hat{b}) \rangle. \tag{3.71}$$

As we have mentioned in section **A**, both $\hat{a}$ and $\hat{b}$ are at steady state Gaussian random variables. Therefore, the characteristic function can be written in the more appealing form as [10]

$$\phi(z,\eta) = \exp\left[-\frac{1}{2}(z^*z+\eta^*\eta)\right]\exp\left[\frac{1}{2}\langle(z\hat{a}^\dagger - z^*\hat{a} + \eta\hat{b}^\dagger - \eta^*\hat{b})^2\rangle_{ss}\right] \qquad (3.72a)$$

or

$$\phi(z,\eta) = \exp\left[-\frac{1}{2}(z^*z+\eta^*\eta)\right]\exp\left[\frac{1}{2}\langle z^2\hat{a}^{\dagger 2} - z^{*2}\hat{a}^2 + \eta^2\hat{b}^{\dagger 2} - \eta^{*2}\hat{b}^2 - 2z^*z\hat{a}^\dagger\hat{a}\rangle_{ss}\right]$$

$$\times\exp\left[\frac{1}{2}\langle -2\eta^*\eta\hat{b}^\dagger\hat{b} - z^*z - \eta^*\eta + 2z^*\eta^*\hat{a}\hat{b} + 2z\eta\hat{a}^\dagger\hat{b}^\dagger - 2z\eta^*\hat{a}^\dagger\hat{b} - 2z^*\eta\hat{a}\hat{b}^\dagger\rangle_{ss}\right]. \qquad (3.72b)$$

Taking into account Eqs. (3.31), (3.33), and (3.34), expression (3.72b) can be expressed as

$$\phi(z,\eta) = \exp\left[-(\langle\hat{a}^\dagger\hat{a}\rangle_{ss}+1)z^*z - (\langle\hat{b}^\dagger\hat{b}\rangle_{ss}+1)\eta^*\eta + \langle\hat{a}^\dagger\hat{b}^\dagger\rangle_{ss}z\eta + \langle\hat{a}\hat{b}\rangle_{ss}z^*\eta^*\right] \qquad (3.73)$$

or

$$\phi(z,\eta) = \exp\left[-az^*z - b\eta^*\eta + cz\eta + c^*z^*\eta^*\right], \qquad (3.74)$$

where

$$a = \langle\hat{a}^\dagger\hat{a}\rangle_{ss} + 1, \qquad (3.75a)$$

$$b = \langle\hat{b}^\dagger\hat{b}\rangle_{ss} + 1, \qquad (3.75b)$$

and

$$c = \langle\hat{a}^\dagger\hat{b}^\dagger\rangle_{ss}. \qquad (3.75c)$$

With the aid of Eq. (3.74), the steady state Q function for the two-mode radiation can be written as

$$Q(\alpha,\beta) = \frac{1}{\pi^4}\int d^2z d^2\eta \exp\left[-az^*z - b\eta^*\eta + cz\eta + c^*z^*\eta^*\right]$$

$$\times \exp\left[z^*\alpha - z\alpha^* + \eta^*\beta - \eta\beta^*\right]. \tag{3.76}$$

Inorder to carry out the integrations, we may put this in the form

$$Q(\alpha,\beta) = \frac{1}{\pi^2}\int \frac{d^2\eta}{\pi}\exp(-b\eta^*\eta + \eta^*\beta - \eta\beta^*)$$
$$\times \int \frac{d^2z}{\pi}\exp\left[-az^*z + (c\eta - \alpha^*)z + (\alpha + c^*\eta^*)z^*\right]. \tag{3.77}$$

Employing Eq. (2.97a) with $A = B = 0$, the integration over z can be evaluated to give

$$Q(\alpha,\beta) = \frac{1}{a\pi^2}\exp\left(\frac{-\alpha^*\alpha}{a}\right)\int \frac{d^2\eta}{\pi}\exp\left[-(b - \frac{c^*c}{a})\eta^*\eta + (\frac{c\alpha}{a} - \beta^*)\eta + (\beta - \frac{c^*\alpha^*}{a})\eta^*\right]. \tag{3.78}$$

Similarly, carrying out the integration over η we get

$$Q(\alpha,\beta) = \frac{1}{\pi^2(ab - c^*c)}\exp\left[-\frac{b}{ab - c^*c}\alpha^*\alpha - \frac{a}{ab - c^*c}\beta^*\beta + \frac{c}{ab - c^*c}\alpha\beta + \frac{c^*}{ab - c^*c}\alpha^*\beta^*\right]. \tag{3.79}$$

This can be rewritten as

$$Q(\alpha,\beta) = \frac{ur - v^*v}{\pi^2}\exp\left[-u\alpha^*\alpha - r\beta^*\beta + v\alpha\beta + v^*\alpha^*\beta^*\right], \tag{3.80}$$

in which

$$u = \frac{b}{ab - c^*c}, \tag{3.81a}$$

$$r = \frac{a}{ab - c^*c}, \tag{3.81b}$$

and

$$v = \frac{c}{ab - c^*c}. \tag{3.81c}$$

For a two-mode radiation the photon number distribution $P(n,m)$, which is the joint probability to observe n photons of one type and m photons of the other type, can be expressed in terms of the Q function as [10]

$$P(n,m) = \frac{\pi^2}{n!m!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}\frac{\partial^{2m}}{\partial\beta^{*m}\partial\beta^m}\left[Q(\alpha,\beta)e^{\alpha^*\alpha+\beta^*\beta}\right]_{\alpha=\alpha^*=\beta=\beta^*=0}. \tag{3.82}$$

Hence on account of Eq. (3.80), the steady state photon number distribution for coherently driven nondegenerate three-level laser is given by

$$P(n,m) = \frac{ur - v^*v}{n!m!} \frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^{n}} \frac{\partial^{2m}}{\partial\beta^{*m}\partial\beta^{m}} \exp\left[(1-u)\alpha^*\alpha + (1-r)\beta^*\beta\right]$$

$$\times \exp\left[v\alpha\beta + v^*\alpha^*\beta^*\right]|_{\alpha=\alpha^*=\beta=\beta^*=0}. \qquad (3.83)$$

Expanding the exponential functions in power series, this can be written as

$$P(n,m) = \frac{ur - v^*v}{n!m!} \sum_{ijkl} \frac{(1-u)^i(1-r)^j v^k v^{*l}}{i!j!k!l!} \frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^{n}} \left[\alpha^{(i+k)}\alpha^{*(i+l)}\right]$$

$$\times \frac{\partial^{2m}}{\partial\beta^{*m}\partial\beta^{m}} \left[\beta^{(j+k)}\beta^{*(j+l)}\right]|_{\alpha=\alpha^*=\beta=\beta^*=0}. \qquad (3.84)$$

On carrying out the defferentiation and applying the conditions $\alpha = \alpha^* = \beta = \beta^* = 0$, we get

$$P(n,m) = \frac{ur - v^*v}{n!m!} \sum_{ijkl} \frac{(1-u)^i(1-r)^j v^k v^{*l}(i+k)!(i+l)!(j+k)!(j+l)!}{i!j!k!l!(i+k-n)!(i+l-n)!(j+k-m)!(j+l-m)!}$$

$$\times \delta_{i+k,n}\delta_{i+l,n}\delta_{j+k,m}\delta_{j+l,m}. \qquad (3.85)$$

From the property of the Kronecker delta symbole, we have

$$i = n - k, \qquad (3.86a)$$

$$j = m - k, \qquad (3.86b)$$

and

$$l = k, \qquad (3.86c)$$

so that the photon number distribution takes the form

$$P(n,m) = n!m!(ur - v^*v) \sum_{k=0}^{\mathrm{Min}(m,n)} \frac{(1-u)^{n-k}(1-r)^{m-k}(v^*v)^k}{k!^2(n-k)!(m-k)!}. \qquad (3.87)$$

Introducing the notation

$$p = \mathrm{Min}(m, n), \tag{3.88}$$

Eq. (3.87) can be rewritten as

$$P(n, m) = n!m!(ur - v^*v) \sum_{k=0}^{p} \frac{(1-u)^{n-k}(1-r)^{m-k}(v^*v)^k}{k!^2(n-k)!(m-k)!}. \tag{3.89}$$

This distribution function has the same form as the distribution function for siginal-idler modes produced by a nondegenerate parametric oscillator.

Fig. [3.4] is a plot of $P(n, n)$ versus n for two different values of Ω/γ. As it can be clearly seen from this graph the distribution function decreases with increasing photon number.

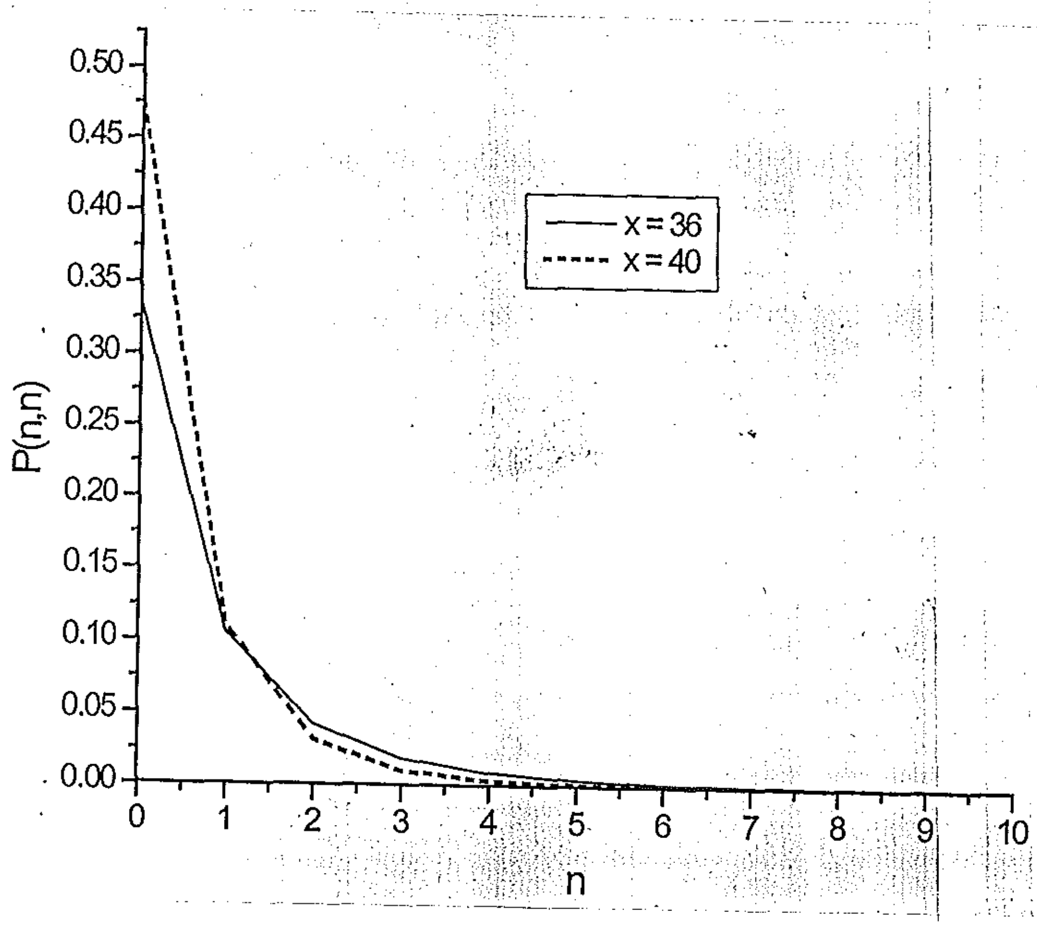


Fig. [3.4] Plots of $P(n, n)$ versus n for $A = 25$, $\kappa = 0.8$, $\Omega/\gamma = 36$ and $\Omega/\gamma = 40$

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