BESSEL FUNCITON AND THE MODIFIED BESSEL FUNCTION

A Seminar Report

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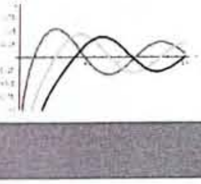

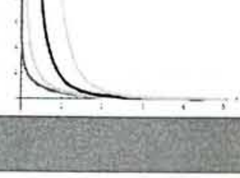

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Tesfaye Teferi

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Introduction

Bessel function is defied for a first time by the mathematician *Daniel Bernoulli* and generalized by *Friedrich Bessel*. A differential equation of the form

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\alpha^2}{x^2}\right)y = 0$$

Where α is arbitrary real or complex number is called a *Bessel equation* and it's solution is known as *Bessel function*. Bessel functions are also called cylinder function or cylindrical harmonic function because they are found in the solution to Laplace's equation in cylindrical coordinated.

Bessel equation arises in problems involving vibrations, or heat conduction in regions possessing circular symmetry; therefore Bessel function have many application in physics and engineering in connection with the propagation of waves, elasticity, fluid motion and especially in many problem of potential theory and diffusion involving cylindrical symmetry.

This seminar report consists three chapters. The first chapter remained about the power series, second order linear differential equation, singularity point, Sturm-Liouville problem and then gamma function which help to express factorial.

In the second chapter i will discuss about the Bessel equation and its solution which is Bessel functions. I will also discuss about properties of Bessel function and some recurrence formula. There is also some plot of Bessel function. The third chapter discuss about the modified Bessel equation and it's solution, which is the special case of the Bessel equation.

Chapter 1
Definitions and Some Preliminary Concepts

1.1 Power Series

An infinite series of the form

$$\sum_{n=0}^{\infty} c_n(x-x_0)^n = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \cdots \qquad (1)$$

where c's are independent of x is called a *power series* in x about the point x_0 .A common special case is the series about $x_0 = 0$

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots \ . \qquad (2)$$

Let's apply the ratio tests to determine the convergence of the series in equation (1)

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-x_0)^{n+1}}{c_n(x-x_0)^n}\right| = |x-x_0|\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = l|x-x_0|$$

Where $l = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$. Therefore, we see that the series converges absolutely for $|x-x_0| < \frac{1}{l} = R$ and diverges (in the absolute sense) for $|x-x_0| > \frac{1}{l} = R$. The range of x for which the series converges, $x_0 - R < x < x_0 + R$ is called the *interval of convergence* of the series and R is called its *radius of convergences*.

There are three possible situations for a power series $\sum_{n=0}^{\infty} c_n(x-x_0)^n$, namely

i. It converges for all values of x; or
ii. It converges for values of x in an open interval $(x_0 - R, x_0 + R)$, but not outside the closed interval $[x_0 - R, x_0 + R]$; or
iii. It converges only for $= x_0$.

In case 1, the interval of convergence is $(-\infty, +\infty)$; in case 2, it is $(x_0 - R, x_0 + R)$ and possibly one or both of its endpoints; and in case 3, it is only the point $x = x_0$. The radius or convergence in each case is ∞, R, and 0 respectively.

Let

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

Since each of the terms $u_n(x) = c_n x^n$ is continuous function of x and $f(x)$ converges uniformly for $-s \le x \le s$, where $0 < s < R$, $f(x)$ must be a continuous function in the interval of uniform convergence, with $u_n(x)$ continuous and $f(x)$ uniformly convergent, we find that the differentiated series is a power series with continuous functions and the same radius of convergence as the original series. Therefore our power series may be differentiated or integrated as often as desired within the interval of uniform convergence.

We now establish that the power series representation is unique. If

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \ , -R_c < x < R_c$$
$$= \sum_{n=0}^{\infty} d_n x^n \ , -R_d < x < R_d \qquad (3)$$

with overlapping intervals of convergence, including the origin, then

$$c_n = d_n$$

For all n; that is, we assume two(different) power series representations and then proceed to show that the two are actually identical.

From equation (3)

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} d_n x^n \ , -R < x < R \qquad (4)$$

where R is the smaller of R_c, R_d. By setting x=0 to eliminate all but the constant terms, we obtain $c_0 = d_0$ now, exploiting the differentiability of our power series, we differentiate equation (4), getting

$$\sum_{n=0}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} n d_n x^{n-1}$$

We again set x=0 to isolate the new constant terms and find $c_1 = d_1$. By repeating this process n times, we get $c_n = d_n$. Which shows that the two series coincide. Therefore our power series representation is unique.

1.2 Second Order Linear Equation

The general second order linear differential equation is

$$A(x)\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

Or, more simply

$$A(x)y'' + P(x)y' + Q(x)y = R(x) \qquad (5)$$

where $A(x)y''$, $P(x), Q(x), and\ R(x)$ are functions of x alone (or perhaps constants).

If $(x) = 0$, then equation (5) reduces to

$$A(x)y'' + P(x)y' + Q(x)y = 0 \qquad (6)$$

and it is called *homogeneous linear equation*. If $R(x)$ is not identically zero, then equation (5) is said to be *non-homogeneous*.

Definition 1.1: The n-function $f_1, f_2, f_3, \cdots, f_n$ are called *linearly dependent* on $a \leq x \leq b$ if and only if there exist constants $c_1, c_2, c_3, \cdots, c_n$ not all zero such that $c_1 f_1 + c_2 f_2 + c_3 f_3 + \cdots + c_n f_n = 0$ for all $x \in [a, b]$; and they are *linearly independent* on $a \leq x \leq b$ if and only if $c_1 f_1 + c_2 f_2 + c_3 f_3 + \cdots + c_n f_n = 0$ implies that $c_1 = c_2 = c_3 = \cdots = c_n = 0$.

Definition 1.2: let $f_1, f_2, f_3, \cdots, f_n$ be a real functions in which each has $(n-1)$ derivatives on $[a, b]$. The determinant

$$\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the *Wronskian* of the n-functions $f_1, f_2, f_3, \cdots, f_n$ and is denoted by $W(f_1, f_2, f_3, \cdots, f_n)(x)$.

Now, let consider the homogeneous linear equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots a_{n-1}(x)y' + a_n(x)y = 0 \tag{7}$$

where $a_0, a_1, \cdots a_{n-1}$, and a_n are continuous on $[a, b]$.

Theorem 1.1: A necessary and sufficient condition that n-solutions $f_1, f_2, f_3, \cdots, f_n$ of the nth order homogeneous linear equation (5) be linearly independent on $[a, b]$ is that the value of the Wronskian is non-zero for all $x \in [a, b]$.

Theorem 1.2: If $y_1(x)$ and $y_2(x)$ are linearly independent solution of the homogeneous equation (6), then $c_1 y_1(x) + c_2 y_2(x)$ is the general solution of equation (6).

1.3 Sturm-Liouville Problems

Consider the boundary value problem which consists of

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]y = 0 \tag{8}$$

with supplementary equations

$$\begin{cases} A_1 y(a) + A_2 y'(a) = 0 \\ B_1 y(b) + B_2 y'(b) = 0 \end{cases} \tag{9}$$

where p, q, and r are real functions such that p has a continuous derivative, q and r are continuous and $p(x) > 0$ and $r(x) > 0$for all $x \in [a, b]$; and λ is a parameter independent of x. A_1, A_2, B_1, and B_2 are real constant.

The supplementary equation (9) is also called *separated boundary conditions*. This type of boundary problem is called *Sturm-Liouville problem* (Regular Sturm-Liouville problem). The value of the parameter in equation (8) for which there exist non-trivial solutions of the problem is called *Eigenvalues* of the problem. The corresponding non-trivial solutions are called *Eigenfunction* of the problem.

Example 1.1: Find the eigenvalues and eigenfunction of the Sturm-Liouville boundary value problem given by

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right]+\left[\frac{\lambda}{x}\right]y=0$$

with $y'(1)=0\,, y'(e^{2\pi})=0$; where we assume the parameter $\lambda \geq 0$.

Solutions: - We consider separately the cases $\lambda=0$ and $\lambda>0$.

Case I: If $\lambda=0$ we have

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right]=0$$

Then the solution becomes $y=c_1 ln|x|+c_2$ and $y'=\frac{c_1}{x}$

Now from the given condition

$$y'(1)=\frac{c_1}{1}=0 \Rightarrow c_1=0$$
$$y'(e^{2\pi})=\frac{c_1}{e^{2\pi}}=0 \Rightarrow c_1=0$$

i.e. $c_1=0$ and nothing about c_2.

Then we have $y=c_2 \neq 0$.

Case II: If $\lambda>0$

By Euler method we get the solution

$$y=c_1 cos\left(\sqrt{\lambda}\, ln|x|\right)+c_2 \sin\left(\sqrt{\lambda}\, ln|x|\right)$$

Then using the boundary conditions we get $c_2=0$ and $c_1 \sin\left(2\pi\sqrt{\lambda}\right)=0$. Choose $c_1 \neq 0$, then $2\pi\sqrt{\lambda}=n\pi \;\; (n=1,2,3,\cdots)$

$$\Rightarrow \lambda=\frac{n^2}{4} \quad (n=1,2,3,\cdots)$$

Therefore, the eigenvalues is given by $\lambda_n=\frac{n^2}{4} \;\; (n=1,2,3,\cdots)$ and the corresponding Eigenfunction is given by $y_n=c_n cos\left(\frac{n}{2}\, ln|x|\right)$, $(n=1,2,3,\cdots)$ and $c_n \neq 0$.

1.3.1 Orthogonality of Eigenfunction

Definition 1.3: Two functions f and g are called *orthogonal* with respect to the weight function r which is piecewise continuous and positive on the interval $a \leq x \leq b$ if and only if

$$\int_a^b f(x)g(x)r(x)dx = 0 .$$

Example 1.2: The functions $\sin x$ and $\sin 2x$ are orthogonal with respect to the weight function having the constant value 1 on the interval $0 \leq x \leq \pi$, for

$$\int_0^{\pi}(\sin x)(\sin 2x)(1)dx = \left.\frac{2\sin^3 x}{3}\right|_0^{\pi} = 0$$

Definition 1.4: Let $\{\phi_n\}$,n=1, 2, 3,... be an infinite set of functions defined on the interval $a \leq x \leq b$. The set $\{\phi_n\}$ is called an *orthogonal system* with respect to the weight function r on $a \leq x \leq b$ if every two distinct functions of the set are orthogonal with respect to r on $a \leq x \leq b$. That is, the set $\{\phi_n\}$ is orthogonal with respect to r on $a \leq x \leq b$,

$$\int_a^b \phi_n(x)\phi_m(x)r(x)dx = 0 \text{ for } m \neq n.$$

Note: We consider in our discussion of Orthogonality the weight function r to have a constant value 1 on the interval of the problem. But it is not concluded that $r(x) = 1$ for all x.

1.4 Ordinary Points and Singular Points of Differential Equations

Consider the linear differential equation

$$A(x)y''(x) + P(x)y'(x) + Q(x)y(x) = 0 \tag{10}$$

where $A(x), P(x)$, and $Q(x)$ are polynomials containing no common factors. Suppose we want to solve (10) in some interval containing the point x_0. If $(x_0) \neq 0$, then the point x_0 is called an *ordinary point*.

In this case, we can divide equation (8) by $A(x)$ to obtain

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \tag{11}$$

where $p(x) = P(x)/A(x)$ and $q(x) = Q(x)/A(x)$ are continuous in the neighborhood of x_0. The functions $p(x)$and $q(x)$ in equation (11) will usually be ratios of polynomials, if $A(x), P(x)$, and $Q(x)$ are polynomials, in which case they will have series expansions about the point x_0 . x_0 is an ordinary point of equation (11) if $p(x)$and $q(x)$are analytic at x_0, then we can actually relax the condition that $A(x), P(x)$, and $Q(x)$ are in equation (10) be polynomials to the condition that $p(x)$and $q(x)$are analytic at x_0: that is, they have convergent series expansion at x_0.

A point that is not an ordinary point of a differential equation is called a *singular point*. For the case in which $A(x), P(x),$ and $Q(x)$ in equation (10) are polynomials with no common factors, a singular point is a point at which $A(x) = 0$. The functions $p(x) = P(x)/A(x)$ and $q(x) = Q(x)/A(x)$ diverges at a singular point $x = x_0$ because $A(x) = 0$[provided that $A(x), P(x),$ and $Q(x)$ have no common factors involving x_0]. The nature of the solution in the neighborhood of a singular point depends critically upon how strongly $p(x)$and $q(x)$ diverges there.

In particular, if $p(x)$and $q(x)$ diverges at $x = x_0$, but

$$\lim_{x\to x_0}(x-x_0)p(x) = \text{finite and } \lim_{x\to x_0}(x-x_0)^2 q(x) = \text{finite} \tag{12}$$

Then we are able to find solutions. Equation (12) mean that $p(x)$and $q(x)$ do not diverge more strongly then $1/x - x_0$ and $1/(x-x_0)^2$, respectively. The point $x = x_0$ in this case is called a regular singular point. If the singular point is not a *regular singular point*, it is called an *irregular singular point.*

Note: 1. The functions $P(x),$ and $Q(x)$ in equation (10) do not necessarily have to be polynomials. For example consider the equation

$$x^2y'' + (\sin x)\ y' + 2(\cos x\,)\,y = 0$$

In this case, the point $x = 0$ is a regular singular point because

$$\lim_{x\to 0} xp(x) = \lim_{x\to 0}\frac{\sin x}{x} = 1 \text{ and also } \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} 2\cos x = 2$$

2. We can obtain a series solution about a regular singular point x_0 by assuming a solution of the form

$$y(x) = (x-x_0)^r \textstyle\sum_{n=0}^{\infty} c_n\,(x-x_0)^n$$

where r may or may not be a positive integer.

1.5 Solution near Regular Singular Points

A power series method may not work for solution about regular singular point. In this case we shall assume an expansion of the form

$$y(x) = \textstyle\sum_{n=0}^{\infty} c_n\, x^{n+r} \tag{13}$$

If r happens to be an integer, the equation (13) is just a power series, but often r will not be an integer. The general series given by equation (13) is called a *Frobenius series*, and the use of such a series to find solutions about regular singular points is called the *method of Frobenius.*

To determine the coefficients in the Frobenius series, we substitute equation (13) into the differential equation and equate the coefficients of the various power of x to zero.

1.6 Gama Functions

The expression $n!$ equal to $1.2.3.\dots.n$, occurs when we enumerate permutations and combinations of things, such as the number of ways that N molecules can be distributed over n molecular quantum states. In the 1700s, Euler introduced a function that yields $n!$ when n is a positive integer, the function is called the *gamma function* and is defined by the integral expression,

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \quad \text{when } n \text{ is positive.}$$

1.6.1 Property of Gamma Functions

$$\Gamma(n+1) = n\Gamma(n)$$

In order to prove this relation, let us consider the integral

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx$$

Integrating it by parts, we get

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx$$

$$= [e^{-x}x^n]_0^\infty - \int_0^\infty e^{-x} x^{n-1} dx$$

$$= n\int_0^\infty e^{-x} x^{n-1} dx = n\Gamma(n)$$

Hence $\Gamma(n+1) = n\Gamma(n)$ (14)

From (14) it is evident that if the value of $\Gamma(n)$ is known for n between two successive positive integers, the value $\Gamma(n)$ for any positive value of n can be determined by the successive application of (14).

Now replacing n by $n-1$ in (12) we get

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

Similarly $\Gamma(n-1) = (n-2)\Gamma(n-2)$ e.t.c. hence (10) yields

$$\Gamma(n+1) = n(n-1)(n-2)\cdots 3.2.1\,\Gamma(1)$$

But by definition $\Gamma(1) = \int_0^\infty e^{-x}\, dx = 1$

Therefore, $\Gamma(n+1) = n(n-1)(n-2)\cdots 3.2.1 = n!$ provided that n is a positive integer.

Note: $\Gamma(-n) = \infty$ when $n = 0$ or a positive integer.

Chapter 2

Bessel's Equation and Bessel Functions

2.1 Introduction

Definition 2.1: A differential equation of the form

$$y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right)y = 0$$

or $$x^2y'' + xy' + (x^2 - n^2)y = 0 \tag{1}$$

is called *Bessel's equation* of order n, where n is non-negative constant. The solution of this equation is called *Bessel function.*

Let us show how Bessel's differential equation is obtained from Laplace's equation $\nabla^2 u = 0$ expressed in cylindrical coordinates (ρ, ϕ, z).

Laplace's equation in cylindrical coordinates is given by

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{2}$$

If we assume a solution of the form $u = \mathrm{P}\Phi Z$, where P is a function of ρ, Φ is a function of ϕ and Z is a function of z, then (2) becomes

$$\mathrm{P}''\Phi Z + \frac{1}{\rho}\mathrm{P}'\Phi Z + \frac{1}{\rho^2}\mathrm{P}\Phi'' Z + \mathrm{P}\Phi Z'' = 0 \tag{3}$$

where the prime denotes derivatives with respect to the particular independent variable involved. Dividing equation (3) by $\mathrm{P}\Phi Z$ yields

$$\frac{\mathrm{P}''}{\mathrm{P}} + \frac{1}{\rho}\frac{P'}{\mathrm{P}} + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} + \frac{Z''}{Z} = 0 \tag{4}$$

Equation (4) can be written as

$$\frac{\mathrm{P}''}{\mathrm{P}} + \frac{1}{\rho}\frac{P'}{\mathrm{P}} + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} = -\frac{Z''}{Z}$$

Since the right side depends on Z while the left side depends on ρ and ϕ, it follows that each side must be a constant, say $-\lambda^2$. Thus we have

$$\frac{\mathrm{P}''}{\mathrm{P}} + \frac{1}{\rho}\frac{P'}{\mathrm{P}} + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} = -\lambda^2 \tag{5}$$

and $$Z'' - \lambda^2 Z = 0$$

If we now multiply both sides of equation (5) by ρ^2 it becomes

$$\rho^2 \frac{P''}{P} + \rho \frac{P'}{P} + \frac{\Phi''}{\Phi} = -\lambda^2 \rho^2$$

Which can be written as

$$\rho^2 \frac{P''}{P} + \rho \frac{P'}{P} + \lambda^2 \rho^2 = -\frac{\Phi''}{\Phi}$$

Since the write side depends only on Φ , while the left side depends only on P , it follows that each side must be a constant, say μ^2 . Thus we have

$$\rho^2 \frac{P''}{P} + \rho \frac{P'}{P} + \lambda^2 \rho^2 = \mu^2 \tag{6}$$

and $$\Phi'' + \mu^2 \Phi = 0$$

The equation (6) can be written as

$$\rho^2 P'' + \rho P' + (\lambda^2 \rho^2 - \mu^2) P = 0 \tag{7}$$

which is Bessel's differential equation with P instead of y, ρ instead of x and μ instead of n.

If we let $\lambda\rho = x$ in equation (7), we get

$$x^2 y'' + xy' + (x^2 - \mu^2) y = 0$$

We have

$$\frac{dP}{d\rho} = \frac{dP}{dx}\frac{dx}{d\rho} = \frac{dP}{dx}\lambda = \lambda \frac{dy}{dx}$$

Where $y(x)$, or briefly y, represents that function of x which P(ρ) becomes when $\rho = x/\lambda$.

Similarly

$$\frac{d^2P}{d\rho^2} = \frac{d}{d\rho}\left(\frac{dP}{d\rho}\right) = \frac{d}{dx}\left(\lambda \frac{dy}{dx}\right)\frac{dx}{d\rho} = \frac{d}{dx}\left(\lambda \frac{dy}{dx}\right)\lambda = \lambda^2 \frac{d^2y}{dx^2}$$

Then equation (7) becomes

$$\left(\frac{x}{\lambda}\right)^2 \lambda^2 \frac{d^2y}{dx^2} + \left(\frac{x}{\lambda}\right)\lambda \frac{dy}{dx} + (x^2 - \mu^2) y = 0$$

or $$x^2 y'' + xy' + (x^2 - \mu^2) y = 0$$

hence the equation (1) in the definition of Bessel equation is obtained, for example, from Laplace's equation $\nabla^2 u = 0$ expressed in cylindrical coordinates (ρ, ϕ, z).

Note: Bessel's equation depends only on n^2 and not on n alone, thus if $J_n(x)$ is solution then $J_{-n}(x)$ is also a solution. Thus we assume a non-negative constant n i.e. $n \geq 0$.

2.2 Solution of Bessel's Equation

Equation (1) is an ordinary differential equation of the second order. The coefficients $P(x) = \frac{1}{x}$ and $Q(x) = \left(1 - \frac{n^2}{x^2}\right)$ are continuous except at $x = 0$; and also $xP(x) = 1$ and $x^2Q(x) = (x^2 - n^2)$ are both analytic at $x = 0$ and so $x = 0$ is a regular singular point of the differential equation (1); and all other values of x are ordinary points; so we can use Frobenius method

Let the series solution of (1) be of the form

$$y = \sum_{m=0}^{\infty} c_m \, x^{k+m} \, , c_0 \neq 0$$

where the first term is nonzero and k is some arbitrary constant. Then differentiating it we get

$$y' = \sum_{m=0}^{\infty} c_m \, (k+m) x^{k+m-1}$$

$$y'' = \sum_{m=0}^{\infty} c_m \, (k+m)(k+m-1) x^{k+m-2}$$

Then substituting in (1), we get

$$x^2 \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) \, x^{k+m-2} + x \sum_{m=0}^{\infty} c_m (k+m) \, x^{k+m-1} + \left(x^2 - n^2\right) \sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

$$\sum_{m=0}^{\infty} c_m (k+m)(k+m-1) \, x^{k+m} + \sum_{m=0}^{\infty} c_m (k+m) \, x^{k+m} + \sum_{m=0}^{\infty} c_m x^{k+m+2} - n^2 \sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

$$\sum_{m=0}^{\infty} c_m \left[(k+m)(k+m-1) + (k+m) - n^2\right] x^{k+m} + \sum_{m=0}^{\infty} c_m x^{k+m+2} = 0$$

$$\sum_{m=0}^{\infty} c_m \left[(k+m)^2 - n^2\right] x^{k+m} + \sum_{m=0}^{\infty} c_m x^{k+m+2} = 0 \qquad (8)$$

By uniqueness of power series the coefficients of each power of x on the left hand side of equation(8) must vanish individually, equating to zero the coefficients of the smallest power of x, namely x^k of (8) gives the indicial equation

$$c_0(k^2 - n^2) = 0$$

Since $c_0 \neq 0$, $k = n$ and $k = -n$ which is the required *indicial roots.*

Let as consider two cases

Case I: $k = n, n \geq 0$

Then from (8) which is equal with

$$\sum_{m=0}^{\infty}\{c_m[(k+m)^2 - n^2] + c_{m-2}\}\, x^{k+m} = 0$$

Thus we have

$$c_m[(k+m)^2 - n^2] + c_{m-2} = 0 \tag{9}$$

Now since $k = n$

$$c_m[(n+m)^2 - n^2] + c_{m-2} = 0$$

$$\text{i.e. } c_m[m(2n+m)] + c_{m-2} = 0$$

Hence $c_m = \frac{-c_{m-2}}{m(2n+m)}$ (10)

This is a two-term recurrence relation. Putting $m = 1$, (4) satisfies that $c_1 = 0$ (since $c_{-1} = 0$). Further (10) also shows that $c_3 = c_5 = \cdots = 0$

i.e. all $c's$ with odd subscripts are zero.

To obtain the remaining coefficients, let put $m = 2, 4, 6, \cdots$ in (4), then

$$c_2 = \frac{-c_0}{2(2n+2)}$$

$$c_4 = \frac{-c_2}{4(2n+4)} = c_2 = \frac{c_0}{2.4(2n+2)(2n+4)}$$

$$c_6 = \frac{-c_4}{6(2n+6)} = \frac{-c_0}{2.4.6(2n+2)(2n+4)(2n+6)}$$

$$\vdots$$

Thus putting this value in $y = \sum_{m=0}^{\infty} c_m\, x^{k+m}$ we gets

$$y = c_0 x^n\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \frac{x^6}{2.4.6(2n+2)(2n+4)(2n+6)} + \cdots\right]$$

which we can also write as

$$y = c_0 x^n\left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^4.1.2(n+1)(n+2)} - \frac{x^6}{2^6.1.2.3.(n+1)(n+2)(n+3)} + \cdots\right]$$

$$y = c_0 x^n\left[1 - \frac{(x/2)^2}{1!(n+1)} + \frac{(x/2)^4}{2!(n+1)(n+2)} - \frac{(x/2)^6}{3!(n+1)(n+2)(n+3)} + \cdots\right] \tag{11}$$

On looking at the terms in the last series we note that the denominators contain factorial, i.e. $1!, 2!, \cdots$;also present in these denominator are $(n+1), (n+1)(n+2), \cdots$ which would becomes factorial, i.e. $(n+1)!, (n+2)!, \cdots$ if we multiply each of them by $n!$. A particular solution supplying these factorials and introducing $\frac{x}{2}$ instead of x in the factor is obtained by choosing

$$c_0 = \frac{1}{2^n n!}$$

In which case (5) becomes

$$y = \left(\frac{x}{2}\right)^n \left[\frac{1}{n!} - \frac{(x/2)^2}{1!(n+1)!} + \frac{(x/2)^4}{2!(n+2)!} - \frac{(x/2)^6}{3!(n+3)!} + \cdots\right]$$

Now using gamma function to generalize factorials we have

$$y = \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \frac{(x/2)^2}{1!\,\Gamma(n+2)} + \frac{(x/2)^4}{2!\,\Gamma(n+3)} - \frac{(x/2)^6}{3!\,\Gamma(n+4)} + \cdots\right] \tag{12}$$

which is solution of Bessel equation for all $n \geq 0$, and it is called the *Bessel function of the first kind of order* n and let denote it by $J_n(x)$. Thus

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} \tag{13}$$

Case II: If $k = -n, n > 0$

It is not necessary to repeat all the above steps. Let replace n by $-n$ in (6), then we have

$$J_{-n}(x) = \left(\frac{x}{2}\right)^{-n} \left[\frac{1}{\Gamma(-n+1)} - \frac{(x/2)^2}{1!\,\Gamma(-n+2)} + \frac{(x/2)^4}{2!\,\Gamma(-n+3)} - \frac{(x/2)^6}{3!\,\Gamma(-n+4)} + \cdots\right] \tag{14}$$

or

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{-n+2r} \tag{15}$$

Now if n is positive but is not an integer, the solution (15) is not bounded (i.e. $J_{-n}(x)$ will be infinite), while $J_n(x)$ is finite, this is because of $J_n(x)$ is contains a positive power of x only, on the other hand $J_{-n}(x)$ contains a negative power of x. This implies that the two solutions are linearly independent.

Therefore $J_n(x)$ and $J_{-n}(x)$ are two independent solution of (1) when n is not an integer.

Note: 1.The general solution of Bessel equation when n is not integer is given by $y = c_1 J_n(x) + c_2 J_{-n}(x)$ where c_1 and c_2 are arb itrary constant.

2.The radius of convergence of the series $\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma\,(\pm n+r+1)}\left(\frac{x}{2}\right)^{\pm n+2r}$ is easily found by ration test $\frac{1}{R}=\lim_{r\to\infty}\frac{r!\,\Gamma\,(\pm n+r+1)}{(r+1)!\,\Gamma\,(\pm n+r+2)}=\lim_{r\to\infty}\frac{1}{(r+1)!(\pm n+r+2)}=0$

Hence $R=\infty$, and hence equation (13) and (15) converge for all x.

2.3 Relation Between $J_n(x)$ and $J_{-n}(x)$

Theorem 1: If n is a non-negative integer $J_{-n}(x)=(-1)^n J_n(x)$

Proof: By definition we have

$$J_{-n}(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma\,(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

Since $n>0$, $\Gamma\,(-n+r+1)$is infinite for $r=0,1,2,\cdots,n-1$. Thus r must be taken form n to infinite. i.e.

$$J_{-n}(x)=\sum_{r=n}^{\infty}\frac{(-1)^r}{r!\,\Gamma\,(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^{m+n}}{(m+n)!\,\Gamma\,(m+1)}\left(\frac{x}{2}\right)^{-n+2(m+n)}$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m(-1)^n}{m!\,\Gamma\,(m+n+1)}\left(\frac{x}{2}\right)^{2m+n}$$

$$=(-1)^n\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma\,(m+n+1)}\left(\frac{x}{2}\right)^{2m+n}$$

$$=(-1)^n J_n(x)$$

Note: 1. The second solution simply reproduced the first, we have failed to construct a second independent solution for Bessel's equation by this series technique when n is an integer

2. $J_n(-x)=(-1)^n J_n(x)$ for $n=0,1,2,\cdots$

Proof:- $J_n(-x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma\,(n+r+1)}\left(\frac{-x}{2}\right)^{n+2r}$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r(-1)^{n+2r}}{r!\,\Gamma\,(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$=(-1)^n\sum_{r=0}^{\infty}\frac{(-1)^{3r}}{r!\,\Gamma\,(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}=(-1)^n J_n(x)$$

3. J_n is even if n is even and odd if n is odd.

2.4 Properties of Bessel Function

2.4.1 Graph and Properties of Bessel Function $J_0(x)$ and $J_1(x)$

Let us now attempt to gain some familiarity with the Bessel function. We can start with the simplest case when n=0. The Bessel function of the first kind in the case is given by

$$J_0(x) = 1 - \frac{x^2}{2} + \frac{x^4}{2^2 4^2} - \cdots$$

After laborious calculation we can tabulated the results:

$J_0(0) = 1,\ J_0(1) = 0.77,\ J_0(2) = -0.22,\ J_0(3) = -0.26, \cdots$

We may graph $J_0(x)$ for $x \geq 0$ and obtain that shown in Fig 1. The graph for $x < 0$ is easily obtained since it is symmetrical to the y-axis. It will be seen that the graph is oscillatory in character. The graph also reveals that there are roots of the equation
$J_0(x) = 0$, also called *zeros* of $J_0(x)$, obtained as points of intersection of the graph with the x-axis. Investigation shows that there are infinitely many roots which are all real and positive.

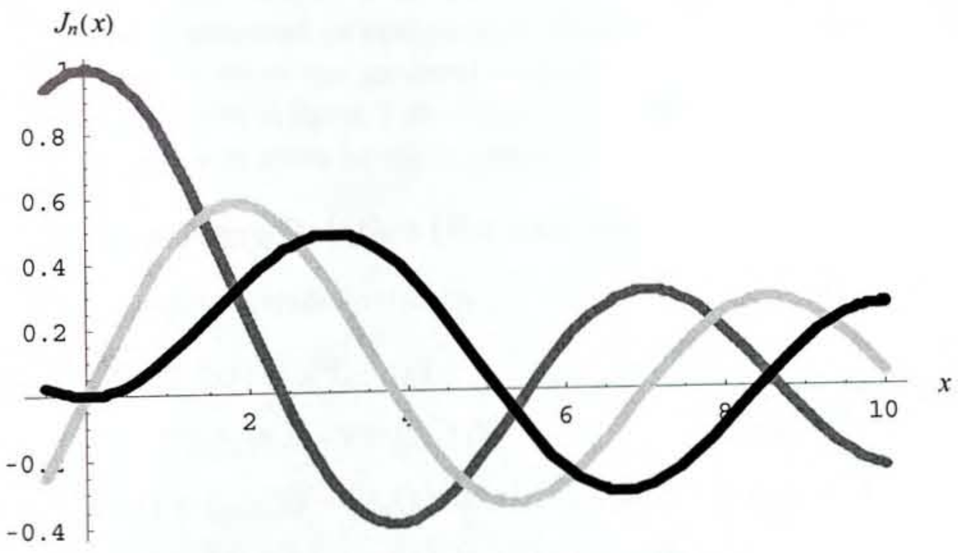

Figure 1:- Plots of $J_0(x)$, $J_1(x)$ and $J_2(x)$

Matimatica Program for the plot of Bessel function $J_0(x)$, $J_1(x)$ and $J_2(x)$

```
Plot[Evaluate[Table[BesselJ[n,x],{n,0,2}]],{x,-2,15},PlotRange->All,PlotStyle->{{Thickness[0.009],RGBColor[0,0.75,0.25]},{Thickness[0.009],RGBColor[1,0.3,0.2]},{Thickness[0.009],RGBColor[1,0.75,0]}}];
```

Note: The roots of $J_n(x) = 0$ are all real, and that between any two successive positive roots of $J_n(x) = 0$ there is precisely one root of $J_{n+1}(x) = 0$. Let us see by table some first few positive zeros of $J_0(x)$ and $J_1(x)$.

Zeros of	1	2	3	4	5
$J_0(x)$	2.4048	5.5201	8.6537	11.7915	14.9309
$J_1(x)$	3.8317	7.0156	10.1735	13.3237	16.4706
$J_2(x)$	5.1356	8.4172	11.6198	14.7690	17.9598

Table 1 Positive zeros of some Bessel function

It is of interest that if we take the difference between successive zeros of $J_0(x)$ we obtain $3.1153, 3.1336, 3.1378, \cdots$, suggesting the conjecture that these difference approach $\pi = 3.14\cdots$. A similar observation on the difference of successive zeros of $J_1(x)$, $J_2(x), \cdots$also leads to such a conjecture. In general the difference of successive zeros of $J_n(x)$ approaches π. Since the successive zeros of $\sin x$ or $\cos x$ differ by π the approximate description of Bessel function $J_n(x)$ as a “damped sine wave” is further warranted.

The wave characteristics of Bessel function seem very much like the shapes of water waves which could be generated for example by dropping a stone into the middle of a large puddle of water. The water waves thus generated would resemble the surface of revolution generated by revolving the curves of figure 1 about the y-axis. Indeed it does turn out that in the theory of hydrodynamics such waves having the shapes of Bessel function do arise.

2.4.2 Recurrence Relation (Formula) for J_n

Let us see some recurrence formula for J_n

1. $\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$
2. $\frac{d}{dx}\{x^{-n} J_n(x)\} = -x^{-n} J_{n+1}(x)$
3. $J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$ or $xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)$
4. $J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$ or $xJ_n'(x) = -xJ_{n+1}(x) + nJ_n(x)$
5. $J_n'(x) = \frac{1}{2}\{J_{n-1}(x) - J_{n+1}(x)\}$ or $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$
6. $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$ or $2nJ_n(x) = x\{J_{n-1}(x) + J_{n+1}(x)\}$

Proof: 1. By definition of $J_n(x)$ we have

$$x^n J_n(x) = x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+n+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+n+1)} \left(\frac{x}{2}\right)^{2n+2r}$$

Then $\frac{d}{dx}\{x^n J_n(x)\} = \frac{d}{dx} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+n+1)} \left(\frac{x}{2}\right)^{2n+2r}$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+n+1)} \frac{d}{dx} \left(\frac{x}{2}\right)^{2n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{1}{2}\right)^{2n+2r}}{r!\,\Gamma(r+n+1)} (2n+2r) x^{2n+2r-1}$$

$$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n-1+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= x^n J_{n-1}(x)$$

2. $\frac{d}{dx}\{x^{-n} J_n(x)\} = \frac{d}{dx}\left\{x^{-n} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+n+1)} \left(\frac{x}{2}\right)^{n+2r}\right\}$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{1}{2}\right)^{n+2r}}{r!\,\Gamma(r+n+1)} \frac{d}{dx}\left(x^{2r}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{1}{2}\right)^{n+2r} 2r x^{2r-1}}{r(r-1)!\,\Gamma(r+n+1)}$$

$$= -x^{-n} \sum_{r=0}^{\infty} \frac{(-1)^r}{(r-1)!\,\Gamma(r+n+1)} \left(\frac{x}{2}\right)^{n+1+2r}$$

$$= -x^{-n} J_{n+1}(x)$$

3. From recurrence formula (1), i.e. $\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$

$$n x^{n-1} J_n(x) + x^n J_n'(x) = x^n J_{n-1}(x)$$
$$J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$$

4. From recurrence formula (2), i.e. $\frac{d}{dx}\{x^{-n} J_n(x)\} = -x^{-n} J_{n+1}(x)$

$$-n x^{-n-1} J_n(x) + x^{-n} J_n'(x) = -x^{-n} J_{n+1}(x)$$
$$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

5. From recurrence formula (3) and (4) i.e. $\begin{cases} J_n'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x) \\ J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x) \end{cases}$

$$2J_n'^{(x)} = J_{n-1}(x) - J_{n+1}(x)$$

$$J_n'(x) = \frac{1}{2}\{J_{n-1}(x) - J_{n+1}(x)\}$$

6. From recurrence formula (3) and (4) i.e. $\begin{cases} J_n'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x) \\ J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x) \end{cases}$

$$0 = \frac{2n}{x}J_n(x) - J_{n-1}(x) - J_{n+1}(x)$$

$$\text{Hence } J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n\ (x)$$

Based on this recursion formula let do some example:

Example 2.1: Prove that

i. $\frac{1}{2}xJ_n(x) = (n+1)J_{n+1}(x) - (n+3)J_{n+3}(x) + (n+5)J_{n+5}(x) + \cdots$

ii. $J_n'(x) = \frac{2}{x}\left[\frac{n}{2}J_n(x) - (n+2)J_{n+2}(x) + (n+4)J_{n+4}(x) - \cdots\right]$

Solution: i. Recurrence relation (6) is $2nJ_n(x) = x\{J_{n-1}(x) + J_{n+1}(x)\}$

Replacing n by $n+1$ in this relation we have

$$2(n+1)\ J_{n+1}(x) = x\{J_{n+1\,-1}(x) + J_{n+1\,+1}(x)\}$$

$$\text{i.e. } \frac{1}{2}xJ_n(x) = (n+1)J_{n+1}(x) - \frac{1}{2}xJ_{n+2}(x) \tag{16}$$

again replacing n by $n+2$ in (16) we have

$$\frac{1}{2}xJ_{n+2}(x) = (n+3)J_{n+3}(x) - \frac{1}{2}xJ_{n+4}(x) \tag{17}$$

Putting the value of $\frac{1}{2}xJ_{n+2}(x)$ from (17) in (16) we get

$$\frac{1}{2}xJ_n(x) = (n+1)J_{n+1}(x) - (n+3)J_{n+3}(x) + \frac{1}{2}xJ_{n+4}(x) \tag{18}$$

Replacing n by $n+4$ in (17) we have

$$\frac{1}{2}xJ_{n+4}(x) = (n+5)J_{n+5}(x) - \frac{1}{2}xJ_{n+6}(x)$$

Thus (18) becomes,

$$\frac{1}{2}xJ_n(x) = (n+1)J_{n+1}(x) - (n+3)J_{n+3}(x) + (n+5)J_{n+5}(x) - \cdots$$

ii. From recurrence formula (3), i.e. $xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)$

$$J_n'(x) = \frac{1}{x}\{-nJ_n(x) + xJ_{n-1}(x)\} = J_{n-1}(x) - \frac{n}{x}J_n(x)$$

But from result (i) for $n-1$ we have

$$J_{n-1}(x) = \frac{2}{x}\{nJ_n(x) - (n+2)J_{n+2}(x) + (n+4)J_{n+4}(x) - \cdots\}$$

Then we get

$$J_n'(x) = -\frac{n}{x}J_n(x) + \frac{2}{x}[nJ_n(x) - (n+2)J_{n+2}(x) + (n+4)J_{n+4}(x) - \cdots]$$

Example 2.2: Prove that $\int_0^x t\{J_n(t)\}^2\,dt = \frac{1}{2}x^2\{J_n^2(x) - J_{n-1}(x)J_{n+1}(x)\}$

Solution: $\frac{d}{dt}\left[\frac{t^2}{2}\{J_n^2(t) - J_{n-1}(t)J_{n+1}(t)\}\right] = t\{J_n^2(t) - J_{n-1}(t)J_{n+1}(t)\} + \frac{t^2}{2}\{2J_n(t)J_n'(t) - J_{n+1}'(t)J_{n+1}(t) - J_{n-1}(t)J_{n+1}'(t)\}$

$$= t\{J_n^2(t) - J_{n-1}(t)J_{n+1}(t)\} + \frac{t^2}{2}\left\{2J_n(t)\frac{1}{2}J_{n-1}(t) - J_{n+1}(t)\right\}$$

$$-J_{n+1}(t)\left\{\frac{n-1}{t}J_{n-1}(t) - J_{n+1}(t)\right\} - J_{n-1}(t)\left\{J_n(t)\frac{n-1}{t}J_{n+1}(t)\right\}$$

$= t\,J_n^2(t)$ Using recurrence formula (3), (4) and (5)

i.e. $t\,J_n^2(t) = \frac{d}{dt}\left[\frac{t^2}{2}\{J_n^2(t) - J_{n-1}(t)J_{n+1}(t)\}\right]$

Integrating both side w.r.t. 'x' from 0 to x we get

$$\int_0^x tJ_n^2(t)\,dt = \left[\frac{t^2}{2}\{J_n^2(t) - J_{n-1}(t)J_{n+1}(t)\}\right]_{t=0}^{x}$$

$$= \frac{1}{2}x^2\{J_n^2(x) - J_{n-1}(x)J_{n+1}(x)\}$$

2.4.3 Integration and Recurrence Relation

Problem 2.1: If $n > -1$, show that $\int_0^x t^n\,J_n(t)dt = x^{n+1}\,J_{n+1}(x)$

Solution: From recurrence formula $\frac{d}{dt}[t^n\,J_n(t)] = t^n\,J_{n-1}(t)$ by substituting n by $n+1$, we have $\frac{d}{dt}[t^{n+1}\,J_{n+1}(t)] = t^{n+1}\,J_n(t)$ then integrating w.r.t. 'x' from 0 to x we get

$$\int_0^x t^{n+1}\,J_n(t)\,dt = [t^{n+1}\,J_{n+1}(t)]_{t=0}^{x} = x^{n+1}\,J_{n+1}(x)$$

Problem 2.2: Show that

i. $\int_0^x t^{-n} J_{n+1}(t)dt = \frac{1}{2^n \Gamma(n+1)} - x^{-n} J_n(x)$ for n>1

ii. $\int_0^\infty t^{-n} J_{n+1}(t)dt = \frac{1}{2^n \Gamma(n+1)}$ for $n > -1/2$

Solution: i. From recurrence formula

$$\frac{d}{dt}\{t^{-n} J_n(t)\} = -t^{-n} J_{n+1}(t) \tag{19}$$

Integrating (19) w.r.t. 'x' from 0 to x, we have

$$[t^{-n} J_n(t)]_{t=0}^{x} = -\int_0^x t^{-n} J_{n+1}(t)\, dt$$

$$x^{-n} J_n(x) - \lim_{t\to 0} \{t^{-n} J_n(t)\} = -\int_0^x t^{-n} J_{n+1}(t)\, dt \tag{20}$$

But $$\lim_{t\to 0} \{t^{-n} J_n(t)\} = \lim_{t\to 0} \frac{1}{t^n} \frac{t^n}{\Gamma(n+1)}\left[1 - \frac{t^2}{2.2.(n+1)} + \cdots\right] = \frac{1}{2^n \Gamma(n+1)}$$

Thus (20) becomes

$$x^{-n} J_n(x) - \frac{1}{2^n \Gamma(n+1)} = -\int_0^x t^{-n} J_{n+1}(t)\, dt$$

i.e. $$\int_0^x t^{-n} J_{n+1}(t)dt = \frac{1}{2^n \Gamma(n+1)} - x^{-n} J_n(x)$$

ii. Integrating (19) w.r.t. 'x' from 0 to ∞, we have

$$[t^{-n} J_n(t)]_{t=0}^{\infty} = -\int_0^\infty t^{-n} J_{n+1}(t)\, dt$$

$$\lim_{t\to 0} \{t^{-n} J_n(t)\} - \lim_{t\to 0} \{t^{-n} J_n(t)\} = -\int_0^\infty t^{-n} J_{n+1}(t)\, dt \tag{21}$$

But we know that for larger interval of x the approximate value of $J_n(x)$ is

$$J_n(x) \sim \left(\frac{2}{\pi x}\right)^{1/2} \cos\left\{x - (n + 1/2)\frac{\pi}{2}\right\}, n > 1/2$$

Hence

$$\lim_{t\to\infty} \{t^{-n} J_n(t)\} = 0$$

Thus (21) becomes

$$\int_0^\infty t^{-n} J_{n+1}(t)dt = \frac{1}{2^n \Gamma(n+1)} \text{ for } n > -1/2$$

Remark: An integration of the form $\int x^m J_n(x)dx$ for $m + n \geq 0$ can be completely integrated if $m + n$ is odd integer, while if $m + n$ is even, then the integral can be put in the terms of $\int J_0(x)dx$.

2.5 Bessel Function of Non-integer Order

In our development of Bessel function of integer order, we had in (13) $a_0 = \frac{1}{2^n n!}$ which we can write as $a_0 = \frac{1}{2^n \Gamma(n+1)}$. This suggest that for non-integer α, we choose $a_0 = \frac{1}{2^\alpha \Gamma(\alpha+1)}$;
Following exactly the same procedure as for the integer order, we find the non-integer order Bessel function is given by

$$J_\alpha(x) = \sum_{m0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{\alpha+2m}$$

In fact this formula can be used for both integer and non-integer α.

2.5.1 Bessel Equation of Half-integer Order

For $n = 1/2$, equation (1) yields

$$x^2 y'' + xy' + (x^2 - {}^1/_4)y = 0 \tag{22}$$

Substituting in $y = \sum_{m0}^{\infty} x^{k+m}$, we obtain

$$\sum_{m0}^{\infty} c_m[(k+m)(k+m-1) + (k+m) - {}^1/_4]x^{k+m} + \sum_{m=0}^{\infty} c_m x^{k+m+2} = 0$$

$$\left(k^2 - \frac{1}{4}\right) c_0 x^k + \left((k+1)^2 - \frac{1}{4}\right) c_1 x^{k+1} + \sum_{m=2}^{\infty} \left\{\left[(k+m)^2 - \frac{1}{4}\right] c_m + c_{m-2}\right\} + x^{k+m} = 0 \tag{23}$$

The roots of the indicial equation are $k = {}^1/_2$and $k = -{}^1/_2$; hence the roots differ by an integer. The recurrence relation is

$$[(k+m)^2 - {}^1/_4]c_m = -c_{m-2} \; for \; m \geq 2 \tag{24}$$

Now for $k = {}^1/_2$, we find the coefficient of x^{k+1} in (23) that $c_1 = 0$. Hence from (24) we obtain that $c_3 = c_5 = \cdots = c_{2m+1} \cdots = 0$.

Further

$$c_m = \frac{-c_{m-2}}{m(m+1)} \text{ for } m = 2,4,6,\cdots$$

Or letting $m = 2r$

$$c_{2r} = \frac{-c_{2r-2}}{2r(2r+1)} \text{ for } r = 1,2,3,\cdots$$

Thus

$$c_2 = \frac{-c_0}{3.2} = -\frac{c_0}{3!}$$

$$c_4 = \frac{-c_2}{5.4} = \frac{c_0}{5!}$$

$$\vdots$$

Hence, taking $c_0 = 1$, we obtain

$$y_1(x) = x^{1/2}\left[1 + \sum_{r=1}^{\infty} \frac{(-1)^r x^{2r}}{(2r+1)!}\right]$$

$$= x^{-1/2}\left[1 + \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r+1}}{(2r+1)!}\right] \quad ,x > 0 \tag{25}$$

We recognize the sum in equation (25) as Taylor series expansion of $\sin x$. The Bessel function of the first kind of order one-half, denoted by $J_{1/2}$ is defined as $\left(\frac{2}{\pi}\right)^{1/2} y_1$. Thus

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x \text{ for } x > 0.$$

Or in short

$$J_{1/2}(x) = \sum_{r=1}^{\infty} \frac{(-1)^r}{r!\,\Gamma\left(r+\frac{3}{2}\right)} \left(\frac{x}{2}\right)^{2r+\frac{1}{2}}$$

$$= \frac{1}{\Gamma\left(\frac{3}{2}\right)}\left(\frac{x}{2}\right)^{\frac{1}{2}} - \frac{1}{\Gamma\left(\frac{5}{2}\right)}\left(\frac{x}{2}\right)^{\frac{5}{2}} + \frac{1}{\Gamma\left(\frac{7}{2}\right)}\left(\frac{x}{2}\right)^{\frac{9}{2}} + \cdots$$

$$= \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)\sqrt{\pi}}\left[1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \cdots\right]$$

$$= \left(\frac{2}{\pi x}\right)^{1/2}\left[x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right]$$

$$= \left(\frac{2}{\pi x}\right)^{1/2} \sin x$$

Similarly for $k = -1/2$, we obtain

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x \text{ for } x > 0.$$

Note: 1. The general solution of equation (22) is $y = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$ where c_1 and c_2 are arbitrary constants.

2. Using the recurrence relations,

$$J_{n+1} = \frac{n}{x} J_n - J_n' \quad and \; J_{n-1} = \frac{n}{x} J_n + J_n'$$

We can find $J_{n+1/2}$ for any integer n.

Example 2.3: Express $J_{3/2}(x)$ in terms of $\sin x$ and $\cos x$.

Solution: From the above recursion formula for $n = 1/2$, we have

$$J_{3/2}(x) = \frac{1/2}{x} J_{1/2}(x) - J_{1/2}{}'(x)$$

$$= \frac{1/2}{x}\left(\frac{2}{\pi x}\right)^{1/2} \sin x - \left(-1/2\right)\left(\frac{2}{\pi}\right)^{1/2} x^{-3/2} \sin x + \left(\frac{2}{\pi x}\right)^{1/2} \cos x$$

$$= 2^{-1/2}\left[\pi^{-1/2}x^{-3/2} \sin x + \pi^{-1/2}x^{-3/2} \sin x - \pi^{-1/2} \cos x\right]$$

$$= \left(\frac{2}{\pi}\right)^{1/2} x^{-3/2} \sin x - \left(\frac{2}{\pi}\right)^{1/2} x^{-1/2} \cos x$$

$$= \left(\frac{2}{\pi}\right)^{1/2}\left(x^{-3/2} \sin x - x^{-1/2} \cos x\right)$$

$$= \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

2.6 Bessel Functions of the First Kind of Order n

Definition 2.2: Bessel function of the first kind of order n is a function defined by

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n}$$

and denoted by $J_n(x)$ where n is any non-negative constant.

Remark: When there is no confusion regarding the variable, we shall write J_n for $J_n(x)$ and J_n' for $\frac{d}{dx}(J_n(x))$.

2.6.1 Generating Function for $J_n(x)$

Now consider the function $exp\left\{\frac{1}{2}x\left(t - \frac{1}{t}\right)\right\}$ or $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$, we can expand this function in a Laurent series in power of t,

$$exp\left\{\frac{1}{2}x\left(t - \frac{1}{t}\right)\right\} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

Let show this result:

$$exp\left\{\frac{1}{2}x\left(t-\frac{1}{t}\right)\right\} = e^{xt/2 - x/2t} = e^{xt/2} * e^{-x/2t}$$

$$= \left\{\left[1+\left(\frac{xt}{2}\right)+\frac{1}{2!}\left(\frac{xt}{2}\right)^2+\frac{1}{3!}\left(\frac{xt}{2}\right)^3+\cdots\frac{1}{n!}\left(\frac{xt}{2}\right)^n+\cdots\right]\right.$$

$$\left.\left[1+\frac{1}{2!}\left(-\frac{x}{2t}\right)^2+\frac{1}{3!}\left(-\frac{x}{2t}\right)^3+\cdots+\frac{1}{n!}\left(-\frac{x}{2t}\right)^n+\cdots\right]\right\}$$

$$= \left\{\left[1+\left(\frac{x}{2}\right)t+\frac{1}{2!}\left(\frac{x}{2}\right)^2 t^2+\cdots\frac{1}{n!}\left(\frac{x}{2}\right)^n t^n+\cdots\right]\right.$$

$$\left.\left[1-\left(\frac{x}{2}\right)t^{-1}+\frac{1}{2!}\left(\frac{x}{2}\right)^2 t^{-2}+\cdots(-1)^n\frac{1}{n!}\left(\frac{x}{2}\right)^n t^{-n}+\cdots\right]\right\} \quad (26)$$

The coefficients of t^n in product (26) is obtained by multiplying the coefficients of $t^n, t^{n+1}, \cdots$in the 1st bracket with the coefficient of $t^0, t^{-1}, t^{-2}\cdots$ in the 2nd bracket respectively and is thus

$$= \frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+2)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{(n+4)!}\left(\frac{x}{2}\right)^{n+4} - \cdots$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!(n+r)!}\left(\frac{x}{2}\right)^{2r+n}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n} = J_n(x)$$

Again the coefficient of t^{-n} in the product (26) is obtained by multiplying the coefficient of $t^{-n}, t^{-(n+1)}, \cdots$ of the second bracket with the coefficient of $t^0, t^1, t^2\cdots$in the first bracket respectively and is thus

$$= (-1)^n\frac{1}{n!}\left(\frac{x}{2}\right)^n + (-1)^{n+1}\frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+1}\frac{x}{2} + (-1)^{n+3}\frac{1}{(n+3)!}\left(\frac{x}{2}\right)^{n+3}\left(\frac{x}{2}\right)^2 + \cdots$$

$$= (-1)^n\left[\left(\frac{x}{2}\right)^n\frac{1}{n!} - \left(\frac{x}{2}\right)^{n+2}\frac{1}{(n+1)!} + \left(\frac{x}{2}\right)^{n+4}\frac{1}{(n+2)!2!} - \cdots\right]$$

$$= (-1)^n J_n(x)$$

Thus the coefficients of $t^{-n} = (-1)^n J_n(x)$

Therefore $J_n(x)$=$(-1)^n \times$ the coefficients of t^{-n}.

Finally, in the product (26) the coefficient of t^0 is obtained by multiplying the coefficients of $t^0, t^1, t^2, \cdots$ in the first bracket with the coefficient of $t^0, t^{-1}, t^{-2}, \cdots$in the second bracket and is thus

$$= 1 - (x/2)^2 + (x/2)^4 \left(\frac{1}{2!}\right)^2 - (x/2)^6 \left(\frac{1}{3!}\right)^2 + \cdots$$

$$= 1 - x^2/2 + x^4/2^2 4^2 - \cdots = J_0(x)$$

We observe that the coefficients of $t^0, (t - t^{-1}), (t^2 - t^{-2}), \cdots, (t^n + (-1)^n t^{-n})$ are $J_0(x), J_1(x), \cdots, J_n(x), \cdots$ respectively. Then (26) gives

$$exp\left\{\frac{1}{2}x\left(t - \frac{1}{t}\right)\right\} = J_0(x) + [t - t^{-1}]J_1(x) + \cdots + [(t^n + (-1)^n t^{-n}]J_n(x) + \cdots$$

$$= \sum_{n=-\infty}^{\infty} t^n J_n(x) \text{ as } J_{-n}(x) = (-1)J_n(x)^n$$

Or we can show as

$$exp\left\{\frac{1}{2}x\left(t - \frac{1}{t}\right)\right\} = e^{xt/2} e^{-x/2t}$$

$$= \sum_{r=0}^{\infty} \frac{(1/2 xt)^r}{r!} \sum_{k=0}^{\infty} \frac{\left(-\frac{1/2 x}{t}\right)^k}{k!}$$

$$= \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (1/2 x)^{k+r} t^{r-k}}{r!k!}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{(n+k)!k!} t^n$$

$$= \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

Note: The function $exp\left\{\frac{1}{2}x\left(t - \frac{1}{t}\right)\right\}$ is the *generating function* for Bessel function of the first kind.

2.6.2 The Bessel Function Satisfies Bessel's Equation

Problem 2.3 Use the generating function $exp\left\{\frac{1}{2}x\left(t - \frac{1}{t}\right)\right\} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$ to show that $J_n$satisfies Bessel's equation

$$x^2 y'' + xy' + (x^2 - n^2)y = 0$$

Solution: We must show that

$$\sum_{n=-\infty}^{\infty} [x^2 J_n''(x) + xJ_n'(x) + (x^2 - n^2)t^n J_n(x)]t^n = 0$$

To get the appropriate terms in the sum we will differentiate the generating function with respect to x and t. Now differentiating w.r.t x

$$\frac{1}{2}\left(t - \frac{1}{t}\right) exp\left\{\frac{1}{2}x\left(t - \frac{1}{t}\right)\right\} = \sum_{n=-\infty}^{\infty} t^n J_n'(x)$$

$$\frac{1}{4}\left(t - \frac{1}{t}\right)^2 exp\left\{\frac{1}{2}x\left(t - \frac{1}{t}\right)\right\} = \sum_{n=-\infty}^{\infty} t^n J_n''(x)$$

Also differentiating w.r.t t

$$\frac{1}{2}x\left(1+{}^{1}/_{t^2}\right)exp\left\{\frac{1}{2}x\left(t-\frac{1}{t}\right)\right\}=\sum_{n=-\infty}^{\infty}nt^{n-1}J_n(x)$$

$$\frac{1}{2}x\left(t+\frac{1}{t}\right)exp\left\{\frac{1}{2}x\left(t-\frac{1}{t}\right)\right\}=\sum_{n=-\infty}^{\infty}nt^{n}J_n(x)$$

Then,

$$\frac{1}{2}x\left(1-\frac{1}{t^2}\right)exp\left\{\frac{1}{2}x\left(t-\frac{1}{t}\right)\right\}+\frac{1}{4}x^2\left(t+\frac{1}{t}\right)^2 exp\left\{\frac{1}{2}x\left(t-\frac{1}{t}\right)\right\}=\sum_{n=-\infty}^{\infty}=n^2t^{n-1}J_n(x)$$

$$\frac{1}{2}x\left(t-\frac{1}{t}\right)exp\left\{\frac{1}{2}x\left(t-\frac{1}{t}\right)\right\}+\frac{1}{4}x^2\left(t+\frac{1}{t}\right)^2 exp\left\{\frac{1}{2}x\left(t-\frac{1}{t}\right)\right\}=\sum_{n=-\infty}^{\infty}n^2t^{n}J_n(x)$$

Then,

$$\sum_{n=-\infty}^{\infty}[x^2J_n''(x)+xJ_n'(x)+(x^2-n^2)t^nJ_n(x)]t^n$$
$$=\left[\frac{1}{4}x^2\left(t-\frac{1}{t}\right)^2+\frac{1}{2}x\left(t-\frac{1}{t}\right)+x^2-\frac{1}{2}x\left(t-\frac{1}{t}\right)-\frac{1}{4}x^2\left(t+\frac{1}{t}\right)^2\right]exp\left\{\frac{1}{2}x\left(t-\frac{1}{t}\right)\right\}=0$$

Thus, $x^2J_n''(x)+xJ_n'(x)+(x^2-n^2)t^nJ_n(x)=0$

Hence J_n satisfies Bessel equation.

2.6.3 Equations Reducible to Bessel's equation

In many problems, we come across differential equation, which by means or a suitable substitution can be reduced to Bessel's equation. Consider the differential equation

$$x^2y''+1-2\alpha xy'+\xi\beta^2\gamma^2x^{2\gamma}+(\alpha^2-n^2\gamma^2)y=0 \tag{27}$$

where α,β,γ and n are real constants.

On changing the dependent variable y and the independent variable x by means of the substitution

$y=ux^{\alpha}$ and $t=t^{\gamma}$, we get

$$t^2\frac{d^2u}{dt^2}+t\frac{du}{dt}+(\beta^2t^2-n^2)u=0 \tag{28}$$

Now substituting $z=\beta t$ in (28), it reduces to

$$z^2\frac{d^2u}{dz^2}+z\frac{du}{dz}+(z^2-n^2)u=0 \tag{29}$$

Which is Bessel's equation, and has for its solution

$u=c_1J_n(z)+c_2J_{-n}(z)$ when n is non-integer.

Replacing z by βt in the above solution, we get the solution of (28) as

$$u=c_1J_n(\beta t)+c_2J_{-n}(\beta t) \tag{30}$$

Finally, replacing u by $yx^{-\alpha}$ and t by x^{γ} in (30), we get

$$y = x^{\alpha}[c_1 J_n(\beta x^{\gamma}) + c_2 Y_n(\beta x^{\gamma})]$$

Which is the a solution of (27)

Example 2.4: Solve the equation

$$x^2 y'' + xy' + \frac{1}{4}(x - n^2)y = 0 \text{ in terms of Bessel function.}$$

Solution: Let use the substitution $x = t^2$, then comparing with the equation (27), we get

$$\alpha = 0, \beta = 1, \gamma = \frac{1}{2}$$

Hence, the solution become

$$y = c_1 J_n(\sqrt{x}) + c_2 J_{-n}(\sqrt{x}) \text{ or } y = c_1 J_n(\sqrt{x}) + c_2 Y_n(\sqrt{x})$$

depending on n being a non-negative or an integer.

2.7 Bessel's Function of the Second Kind of Order n

When n is an integer, J_n and J_{-n} are not linearly independent. To determine the second linearly independent solution of the Bessel equation, let define

$$Y_n(x) = \begin{cases} \dfrac{J_n(x)\cos(n\pi) - J_{-n}(x)}{\sin(n\pi)} & \text{when } n \text{ is not an integer} \\ \lim\limits_{\mu \to n} \dfrac{J_\mu(x)\cos(\mu\pi) - J_{-\mu}(x)}{\sin(\mu\pi)} & \text{when } n \text{ is an integer} \end{cases} \tag{31}$$

The function Y_n is called the *Bessel function of the second kind of order* n. It is also known as the *Neumann function*.

Now, when n is a non-integer Y_n is clearly a solution of the Bessel equation, since it is a linearly combination of two linearly independent solutions $J_n(x)$ and $J_n(x)$.

For integer , $\mu = n$ and n=0,1,2,…., equation (25) becomes

$$Y_n(x) = \lim_{\mu \to n} \frac{J_\mu(x)\cos(\mu\pi) - J_{-\mu}(x)}{\sin(\mu\pi)}$$

The limit is indeterminate form of 0/0, since $\cos(n\pi) = (-1)^n$, $\sin(n\pi) = 0$ and $J_n(x) = (-1)^n J_n(x)$.

Then by L' Hospital rule

$$Y_n(x) = \lim_{\mu\to n} \frac{\frac{\partial}{\partial\mu}[J_n(x)\cos(\mu\pi) - J_{-\mu}(x)]}{\frac{\partial}{\partial\mu}[\sin(\mu\pi)]}$$

$$= \left[\frac{-\pi\sin(\mu\pi)\, J_\mu(x) + \cos(\mu\pi)\frac{\partial}{\partial\mu}J_\mu(x) - \frac{\partial}{\partial\mu}J_{-\mu}(x)}{\pi\cos(\mu\pi)}\right]_{\mu=n}$$

$$= \frac{1}{\pi}\left[\frac{\partial}{\partial\mu}J_\mu(x) - (-1)^n\frac{\partial}{\partial\mu}J_{-\mu}(x)\right]_{\mu=n}$$

Let now show that $Y_n(x)$ so defined is a solution of the Bessel's equation. By definition, J_μ and $J_{-\mu}$ respectively, satisfies the following differential equation:

$$x^2 J_\mu''(x) + xJ_\mu'(x) + (x^2-\mu^2)J_\mu(x) = 0$$

$$x^2 J_{-\mu}''(x) + xJ_{-\mu}'(x) + (x^2-\mu^2)J_{-\mu}(x) = 0$$

Differentiating w.r.t. μ we have

$$x^2\frac{d^2}{dx^2}\left(\frac{\partial J_\mu}{\partial\mu}\right) + x\frac{d}{dx}\left(\frac{\partial J_\mu}{\partial\mu}\right) + (x^2-\mu^2)\frac{\partial J_\mu}{\partial\mu} - 2\mu J_\mu = 0$$

$$x^2\frac{d^2}{dx^2}\left(\frac{\partial J_{-\mu}}{\partial\mu}\right) + x\frac{d}{dx}\left(\frac{\partial J_{-\mu}}{\partial\mu}\right) + (x^2-\mu^2)\frac{\partial J_{-\mu}}{\partial\mu} - 2\mu J_{-\mu} = 0$$

Multiplying the second equation by $(-1)^n$ and substituting it from the first equation, we have

$$x^2\frac{d^2}{dx^2}\left(\frac{\partial J_\mu}{\partial\mu} - (-1)^n\frac{\partial J_{-\mu}}{\partial\mu}\right) + x\left(\frac{\partial J_\mu}{\partial\mu} - (-1)^n\frac{\partial J_{-\mu}}{\partial\mu}\right) + (x^2-\mu^2)\left[\frac{\partial J_\mu}{\partial\mu} - (-1)^n\frac{\partial J_{-\mu}}{\partial\mu}\right] - 2\mu\left(J_\mu - (-1)^n J_{-\mu}\right) = 0$$

Taking the limit $\to n$, the last term drops out because $J_n - (-1)^n J_{-n} = 0$.

Clearly the Neumann function expressed in (28) satisfies the Bessel's equation.

Neumann function has a logarithm term, since

$$\frac{\partial J_\mu}{\partial\mu} = \frac{\partial}{\partial\mu}\left[x^\mu\sum_{r=0}^{\infty}\frac{(-1)^n}{r!\Gamma(r+\mu+1)}\left(\frac{x}{2}\right)^{2r}\right]$$

$$= \frac{\partial x^\mu}{\partial\mu}\sum_{r=0}^{\infty}\frac{(-1)^n}{r!\Gamma(r+\mu+1)}\left(\frac{x}{2}\right)^{2r} + x^\mu\frac{\partial}{\partial\mu}\sum_{r=0}^{\infty}\frac{(-1)^n}{r!\Gamma(r+\mu+1)}\left(\frac{x}{2}\right)^{2r}$$

and

$$\frac{\partial x^\mu}{\partial\mu} = \frac{\partial}{\partial\mu}\left[e^{\mu\ln x}\right] = e^{\mu\ln x}\ln x = x^\mu\ln x$$

thus $Y_n(x)$ contains a term $J_n(x)(\ln x)$. The first three order of Neumann function are shown in figure 1

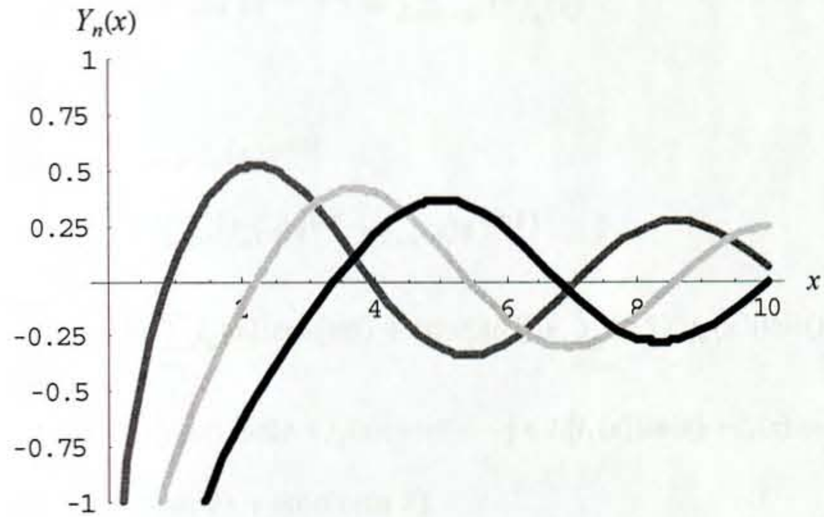

Figure 1 Neumann function $Y_0(x), Y_1(x), Y_2(x)$.

Mathematical program for the plot of Bessel function of the second kind

```
Plot[Evaluate[Table[BesselY[n,x],{n,0,2}]],{x,0,10},PlotRange->{-1,.55}, PlotStyle->
{{Thickness[0.016],RGBColor[0,0.75,0.25]},{Thickness[0.016],RGBColor[1,0.75,0]},
{Thickness[0.016],RGBColor[1,0.1,0.1]}}];
```

Problem 4: Show that $J_n(x)$ and $Y_n(x)$ are linearly independent for all n.

Proof: let see the Wronskian

$$W[J_n, Y_n] = \begin{vmatrix} J_n & J_n(x)\cos(n\pi) - J_{-n}(x)\csc(n\pi) \\ J'_n & J_n'(x)\cot(n\pi) - J_n'(x)\csc(n\pi) \end{vmatrix}$$

$$= \cot(n\pi)\begin{vmatrix} J_n & J_n \\ J'_n & J'_n \end{vmatrix} - \csc(n\pi)\begin{vmatrix} J_n & J_{-n} \\ J'_n & J'_{-n} \end{vmatrix}$$

$$= -\csc(n\pi) - \frac{2}{\pi x}\sin(n\pi)$$

$$= \frac{-2}{\pi x} \neq 0$$

Hence, the function $J_n(x)$ and $Y_n(x)$ are linearly independent.

Note: The general solution of the Bessel Equation can be written as

$$y_n(x) = c_1 J_n(x) + c_2 Y_n(x)$$

2.8 Integral Representation of Bessel Function

A particular useful and powerful way of treating Bessel function employs integral representations. We now that $e^{\frac{1}{2}x(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$

If we substitute $t = e^{i\theta}$

We get $e^{ixsin\,\theta} = \sum_{n=-\infty}^{\infty} J_n(x)\, e^{in\theta}$

$$= J_0(x) + \sum_{n=1}^{\infty}\left[J_n(x)e^{in\theta} + J_{-n}(x)e^{-in\theta}\right]$$

$$= J_0(x) + \sum_{n=1}^{\infty} J_n(x)[cos(n\theta) + isin(n\theta)] + \sum_{n=1}^{\infty} (-1)^n J_n(x)[cos(n\theta) + isin(n\theta)]$$

$$= J_0(x) + 2\left[J_2(x)\, cos2\theta + J_4(x)cos4\theta + \cdots\right] + 2i\left[J_1(x)\sin(\theta) + J_3(x)\sin(3\theta) + \cdots\right]$$

Since $e^{ixsin\,\theta} = cos(xsin\,\theta) + isin(xsin\,\theta)$

$$cos(xsin\,\theta) = J_0(x) + 2[J_2(x)\, cos2\theta + J_4(x)cos4\theta + \cdots] \tag{32}$$

and

$$sin(xsin\,\theta) = 2[J_1(x)\sin(\theta) + J_3(x)\sin(3\theta) + \cdots] \tag{33}$$

The coefficients $J_n(x)$ can be readily obtained as follow:

We shall use

$$\int_0^{\pi} \cos m\theta \cos n\theta\, d\theta = \int_0^{\pi} \sin m\theta \sin n\theta\, d\theta = \begin{cases} \pi/2 & if\ m = n \\ 0 & if\ m \neq n \end{cases} \tag{34}$$

Multiplying both sides of (32) by $cos\,(n\theta)$ and integrating w.r.t. θ from 0 to π and using (34), we obtain

$$\int_0^{\pi} \cos(xsin\,\theta)\cos n\theta\, d\theta = \begin{cases} \pi J_n(x) & if\ n\ is\ even \\ 0 & if\ n\ is\ odd \end{cases}$$

Similarly, multiplying both sides (32) by $sin\,(n\theta)$ and integrating w.r.t. θ from 0 to π and using (34), we obtain

$$\int_0^{\pi} \sin(xsin\,\theta)\sin n\theta\, d\theta = \begin{cases} 0 & if\ n\ is\ even \\ \pi J_n(x) & if\ n\ is\ odd \end{cases}$$

Adding these equations, we obtain

$$\pi J_n(x) = \int_0^\pi [\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta]\, d\theta$$

$$= \int_0^\pi \cos(x\sin\theta - n\theta)\, d\theta$$

or $J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta - n\theta)\, d\theta$

Let now see the case when = 0 ;

Problem 2.5: Show that $J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\, d\theta = \frac{1}{\pi}\int_0^\pi \cos(x\cos\theta)\, d\theta$

Solution: Integrating (33) w.r.t. 'θ' between the limits 0 to π and using the result $\int_0^\pi \cos\, n\theta\, d\theta = 0$, if n is even integer, we have that

$$\int_0^\pi \cos(x\sin\theta)\, d\theta = J_0(x)\int_0^\pi d\theta + 0 + 0 + \cdots = J_0(x)\pi$$

i.e. $$J_0(x)\pi = \int_0^\pi \cos(x\sin\theta)\, d\theta$$

also replacing θ by $\left(\theta - \frac{\pi}{2}\right)$ in (33) and integrating w.r.t. 'θ' between the limits 0 to π we get

$$\int_0^\pi \cos(x\cos\theta)\, d\theta = J_0(x)\pi - 0 - 0 - \cdots$$

or $J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\cos\theta)\, d\theta$

2.9 Bessel Function as Eigenfunction of Sturm-Liouville Problems

2.9.1 Boundary Conditions of Bessel's Equation

By itself Bessel's equation is not a Sturm-Liouville equation. There is no way for it to satisfy any given boundary condition. However, the closely related equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2x^2 - n^2)y = 0 \tag{35}$$

is a Sturm- Liouville equation. It can be shown that

$$y(x) = J_n(\lambda x)$$

is a solution of this equation. Let z = λx, then

$$\frac{dy}{dx} = \frac{dJ_n(z)}{dz}\frac{dz}{dx} = \lambda\frac{dJ_n(z)}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\lambda\frac{dJ_n(z)}{dz}\right] = \lambda^2\frac{d^2J_n(z)}{dz^2}$$

Then equation (35) becomes

$$x^2\lambda^2\frac{d^2J_n(z)}{dz^2}+x\lambda\frac{dJ_n(z)}{dz}+(\lambda^2x^2-n^2)J_n(z)=z^2\frac{d^2J_n(z)}{dz^2}+z\frac{dJ_n(z)}{dz}+(z^2-n^2)J_n(z)=0$$

Hence $z^2\frac{d^2J_n(z)}{dz^2}+z\frac{dJ_n(z)}{dz}+(z^2-n^2)J_n(z)=0$

which is the regular Bessel's equation. Thus $J_n(z)$ is a solution of this equation implies that $J_n(\lambda x)$ is a solution of equation (35).

We can rewrite equation (35) as

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right]+\left(\lambda^2x-\frac{n^2}{x}\right)y=0 \qquad (36)$$

Equation (36) together with a boundary condition at $x=c$ constitute a Sturm-Liouville problem in the interval of $0\le x\le c$. The general boundary condition is of the form

$$Ay(c)+By'(c)=0$$

where A and B are two constants. If $B=0$, it is known as the *Dirichlet condition.* If $A=0$, it is known as the *Neumann condition.*

This means that only those values of λ that satisfy the equation

$$AJ_n(\lambda c)-B\frac{dJ_n(\lambda x)}{dx}\bigg|_{x=c}=0$$

are acceptable. Since Bessel functions have oscillatory character, there are infinite numbers of λ that satisfy this equation. These values of λ are the Eigenvalues of the problem. For example, if $B=0, n=0, c=2$, then

$$J_0(2\lambda)=0$$

The jth root of this equation, labeled λ_{0j} , can be founded from the table of zeros of $J_n(x)$ as

$$\lambda_{01}=\frac{2.4048}{2}=1.2924\,,\lambda_{02}=\frac{5.5201}{2}=2.7601,\ \text{e.t.c.}$$

2.9.2 Orthogonality of Bessel Functions

Corresponding to the set of Eigenvalue $\{\lambda_{nj}\}$, the Eigenfunction are $\{J_n(\lambda_{nj}x)\}$. These Eigenfunction form a complete set and they are orthogonal to each other with respect to the weight function s, that is

$$\int_0^c J_n(\lambda_{ni}s)J_n(\lambda_{nk}s)s\,ds=0\ \text{ if } \lambda_{ni}\neq\lambda_{nk}$$

Let proof this result;

Since $J_n(x)$ is a solution of the Bessel's equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0 \tag{37}$$

therefore putting $x=\lambda_{ni}s$ and calling $y=u$ we get

$$\frac{1}{{\lambda_{ni}}^2}\frac{d^2u}{ds^2}+\frac{1}{\lambda_{ni}s}\frac{1}{\lambda_{ni}}\frac{du}{dx}+\left(1-\frac{n^2}{{\lambda_{ni}}^2s^2}\right)u=0$$

Multiplying throughout by ${\lambda_{ni}}^2s^2$

$$s^2\frac{d^2u}{ds^2}+s\frac{du}{dx}+\left({\lambda_{ni}}^2s^2-n^2\right)u=0 \tag{38}$$

Similarly putting $x=\lambda_{nk}s$ and calling $y=v$ in equation (38), we have

$$s^2\frac{d^2v}{ds^2}+s\frac{dv}{dx}+\left({\lambda_{nk}}^2s^2-n^2\right)v=0 \tag{39}$$

If we multiply (38) by $\frac{v}{s}$, and (39), by $\frac{u}{s}$ and subtract we get

$$s\left(v\frac{d^2u}{ds^2}-su\frac{d^2v}{ds^2}\right)+\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)+\left({\lambda_{ni}}^2-{\lambda_{nk}}^2\right)suv=0$$

or

$$\frac{d}{ds}\left\{s\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)\right\}+\left({\lambda_{ni}}^2-{\lambda_{nk}}^2\right)suv=0 \tag{40}$$

Now, $u=J_n(\lambda_{ni}s)$ and $v=J_n(\lambda_{nk}s)$ and then integrating (40) between the limits 0 to c, we get

$$s\{(J_n(\lambda_{nk}s)J_n'(\lambda_{ni}s)\lambda_{ni}-J_n(\lambda_{ni}s)J_n'(\lambda_{nk}s)\lambda_{nk})\}|_0^c-\int_0^c({\lambda_{ni}}^2-{\lambda_{nk}}^2)J_n(\lambda_{ni}s)J_n(\lambda_{nk}s)s=0$$

since $J_n(\lambda_{nk}c)=0$, $J_n(\lambda_{ni}c)=0$, we have that

$$\left({\lambda_{ni}}^2-{\lambda_{nk}}^2\right)\int_0^c J_n(\lambda_{ni}s)J_n(\lambda_{nk}s)s=0$$

since $\lambda_{ni}\neq\lambda_{nk}$, we have that

$$\int_0^c J_n(\lambda_{ni}s)J_n(\lambda_{nk}s)sds=0$$

Chapter 3

The Modified Bessel Function

3.1 The Modified Bessel's Equation

Bessel's differential equation of n^{th} order is given by

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0 \tag{1}$$

Now, putting $x = it$ or $t = -ix$, we have

$$\frac{dx}{dt}=i\,, i.e. \frac{dt}{dx}=\frac{1}{i}$$

Thus $\frac{dy}{dx}=\frac{1}{i}\frac{dy}{dt}$

$$\frac{d^2y}{dx^2}=-\frac{d^2y}{dt^2}$$

Thus (1) becomes

$$-\frac{d^2y}{dt^2}-\frac{1}{t}\frac{dy}{dt}+\left(1+\frac{n^2}{t^2}\right)y=0 \text{ or } \frac{d^2y}{dt^2}+\frac{1}{t}\frac{dy}{dt}-\left(1+\frac{n^2}{t^2}\right)y=0$$

which we call it the *modified Bessel's equation.*

Definition 3.1: A differential equation of the form

$$\frac{d^2y}{dt^2}+{}^{1}/_{t}\frac{dy}{dt}-\left(1+{}^{n^2}/_{t^2}\right)y=0$$

is called the *modified Bessel equation*.

Since $y = c_1J_n(x) + c_2Y_n(x)$ (where c_1 and c_2 are arbitrary constant) is the general solution of the Bessel equation, therefore the solution of the modified Bessel equation is obtained by putting $x = it$, i.e,

$$y = c_1J_n(it) + c_2Y_n(it)$$

We know that

$$J_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma\,(n+r+1)}\left(\frac{x}{2}\right)^{2r+n}$$

Now putting $x = it$

$$J_n(it) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma\,(n+r+1)} \left(\frac{it}{2}\right)^{2r+n}$$

$$= (i)^n \sum_{r=0}^{\infty} \frac{1}{r!\,\Gamma\,(n+r+1)} \left(\frac{t}{2}\right)^{2r+n}$$

or $$(i)^{-n} J_n(it) = \sum_{r=0}^{\infty} \frac{1}{r!\,\Gamma\,(n+r+1)} \left(\frac{t}{2}\right)^{2r+n}$$

Let denote $I_n(t) = (i)^{-n} J_n(it)$ which is known as the modified function of the first kind of order n. thus replacing t by x, we get

$$I_n(x) = (i)^{-n} J_n(ix) = \sum_{r=0}^{\infty} \frac{1}{r!\,\Gamma\,(n+r+1)} \left(\frac{x}{2}\right)^{2r+n}$$

Note: Since J_n and $J_{-n}$are linearly independent solution when n is not an integer, I_n and $I_{-n}$are linearly independent solution to the modified Bessel equation when n is not an integer, this is because

$$W[I_n, I_{-n}] = \begin{vmatrix} I_n & I_{-n} \\ I'_n & I'_{-n} \end{vmatrix}$$

$$= \begin{vmatrix} (i)^{-n} J_n(ix) & (i)^n J_{-n}(ix) \\ i(i)^{-n} J_n(ix) & i(i)^n J_{-n}(ix) \end{vmatrix}$$

$$= i \begin{vmatrix} J_n(ix) & J_{-n}(ix) \\ J_n(ix) & J_{-n}(ix) \end{vmatrix}$$

$$= i\left(\frac{-2}{i\pi x}\right) \sin(n\pi) = \frac{-2}{\pi x} \sin(n\pi)$$

Thus $I_n$and $I_{-n}$are linearly independent.

In order to have a second linearly independent solution when n is an integer, we define the modified Bessel function of the second kind as

$$K_n(x) = \begin{cases} \frac{\pi}{2} \frac{[I_{-n}(x) - I_n(x)]}{\sin(n\pi)} & \textit{when n is not an integer} \\ \lim\limits_{\mu \to n} \frac{\pi}{2} \frac{[I_{-\mu}(x) - I_\mu(x)]}{\sin(\mu\pi)} & \textit{when n is an integer} \end{cases}$$

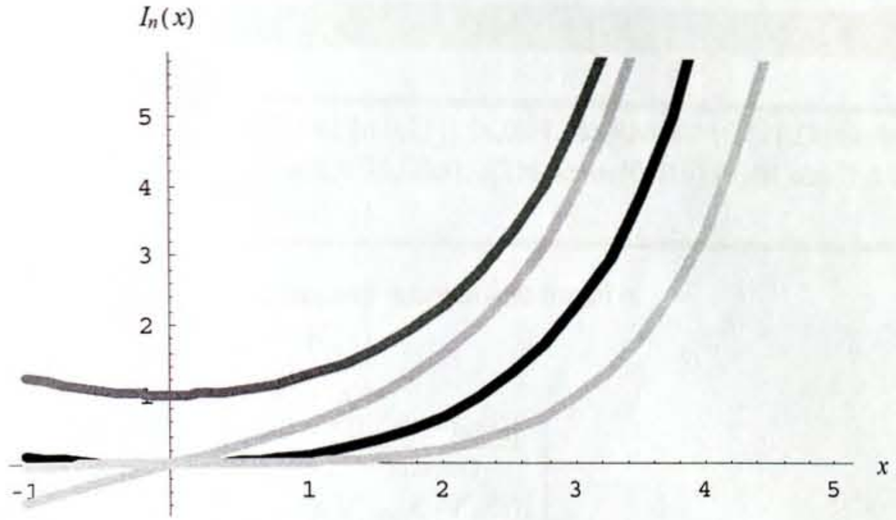

Figure 2. Neumann function $I_0(x), I_1(x), I_3(x)$, and $I_4(x)$.

Mathematica program for the plot of Neumann function

```
Plot[Evaluate[Table[BesselI[n,x],{n,0,3}]],{x,-1,5},AxesLabel→TraditionalForm/@{x,BesselI[n,x]},PlotStyle→{{Thickness[0.016],RGBColor[0,0.75,0.25]},{Thickness[0.016],RGBColor[1,0.75,0]},{Thickness[0.016],RGBColor[0.3,0.1,0.1]},{Thickness[0.016],RGBColor[1,0.71,0.5]}}]
```

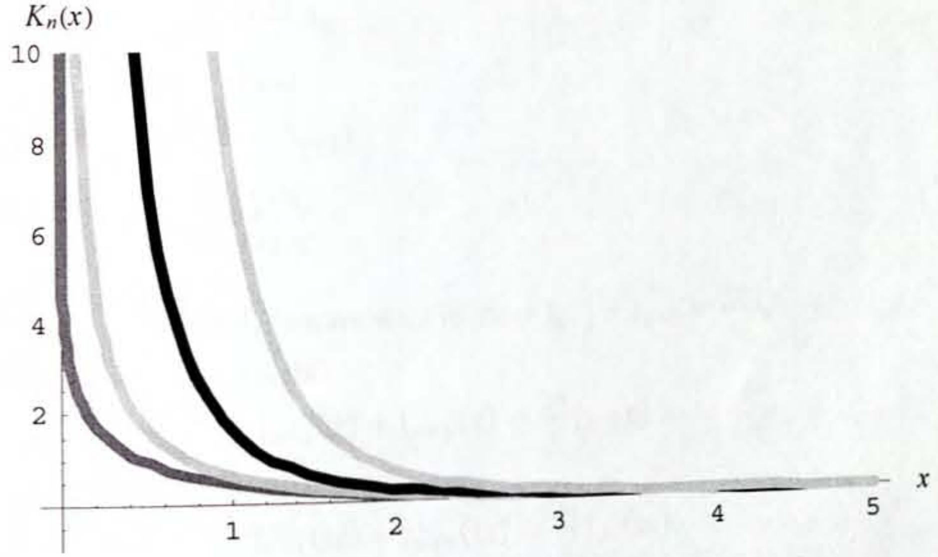

Figure 3. Modified Bessel equation of the second kind $K_0(x), K_1(x), K_3(x)$, and $K_4(x)$.

Mathematica program for the plot of Modified Bessel equation of the second kind

```
Plot[Evaluate[Table[BesselK[n,x],{n,0,2}]],{x,0,5},PlotRange->{0,10},PlotStyle->
{{Thickness[0.016],RGBColor[0,0.75,0.25]},{Thickness[0.016],RGBColor[1,0.75,0]},{Thick
ness[0.016],RGBColor[1,0.1,0.1]}}];
```

Problem 3.1: I_n *and* K_n are linearly independent for all n.

Proof: $W[I_n, K_n] = \begin{vmatrix} I_n & K_n \\ I'_n & K'_n \end{vmatrix}$

$$= \begin{vmatrix} I_n & \frac{\pi}{2}\frac{[I_{-n}(x)-I_n(x)]}{\sin(n\pi)} \\ I'_n & \frac{\pi}{2}\frac{[I'_{-n}(x)-I'_n(x)]}{\sin(n\pi)} \end{vmatrix}$$

$$= \frac{\pi}{2}\csc(n\pi)\left[\begin{vmatrix} I_n & I_{-n} \\ I'_n & I'_{-n} \end{vmatrix} - \begin{vmatrix} I_n & I_n \\ I'_n & I'_n \end{vmatrix}\right]$$

$$= \frac{\pi}{2}\csc(n\pi)\left(\frac{-2}{\pi}\right)\sin(n\pi) = \frac{-1}{x} \neq 0$$

Hence the general solution of the modified Bessel equation is $y = c_1 I_n(x) + c_2 K_n(x)$, where c_1 and c_2 are arbitrary constant.

3.2 Recursion Formula for the Modified Bessel Equation

1. $A_{n-1} - A_{n+1} = \frac{2n}{x} A_n$
2. $A'_n = \frac{n}{x} A_n + A_{n+1}$
3. $A'_n = \frac{1}{2}(A_{n-1} + A_{n+1})$
4. $A'_n = A_{n-1} - \frac{n}{x} A_n$

Where A stands for either I or K.

Proof: 1. Let A stands for I, then we want to show $I_{n-1} - I_{n+1} = \frac{2n}{x} I_n$

Consider the recurrence formula

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n\ (x)$$

Replacing x by ix we obtain

$$J_{n-1}(ix) + J_{n+1}(ix) = \frac{2n}{x} J_n\ (ix)$$

By definition $I_n(x) = (i^{-n})J_n(ix)$ or $J_n(ix) = (i^n)I_n(x)$, thus

$$(i^{n+1})I_{n+1}(x) = -\frac{2ni}{x} i^n I_n(x) - i^{n-1} I_n(x)$$

Dividing by i^{n+1}then we get

$$I_{n-1}(x) - I_{n+1}(x) = \frac{2n}{x} I_n(x)$$

Problem 3.2: Show that

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sinh x$$

$$I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cosh x$$

Solution: Since we know that $I_n(x) = (i)^{-n}J_n(ix)$

Thus $I_{1/2}(x) = i^{-1/2}J_{1/2}(ix) = i^{-1/2}\sqrt{\frac{2}{i\pi x}}\sinh(ix) = \sqrt{\frac{2}{\pi x}}\sinh x$

From the recursion formula (4) we have that $I'_n = I_{n-1} - \frac{n}{x}I_n$

thus $I_{-1/2}(x) = I_{1/2}'^{(x)} - \frac{1}{2x}I_{1/2}(x) = \frac{-1}{2}\sqrt{\frac{2}{\pi}}x^{-3/2}\sinh x + \sqrt{\frac{2}{\pi x}}\cosh x + \frac{1}{2x}\sqrt{\frac{2}{\pi x}}\sinh x$

$$= \sqrt{\frac{2}{\pi x}}\cosh x$$

Problem 3.3: Find $K_{1/2}$ and $K_{-1/2}$.

Solution: Since $K_{1/2}(x) = \frac{\pi}{2}\left(I_{-1/2}(x) - I_{1/2}(x)\right)$

$$= \frac{\pi}{2}\left[\sqrt{\frac{2}{\pi x}}\cosh x - \sqrt{\frac{2}{\pi x}}\sinh x\right] = \sqrt{\frac{\pi}{2x}}e^{-x}$$

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