# Modules over Boolean Like Semiring of Fractions

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# Modules over Boolean Like Semiring of Fractions



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## ADDIS ABABA UNIVERSITY

## DEPARTMENT OF MATHEMATICS

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# Abstract

The concept of Boolean like rings is originally due to A.L.Foster, in 1946. Later, in 1982, V. Swaminathan has extensively studied the geometry of Boolean like rings. Recently in 2011, Venkateswarlu et al introduced the notion of Boolean like semirings by generalizing the concept of Boolean like rings of Foster. K.Venkateswarlu, B.V.N. Murthy, and Y. Yitayew have also made an extensive study of Boolean like semirings.

This work is a continued study of the theory of Boolean like semirings by introducing and investigating the notions; Boolean like semirings of fractions and Modules over Boolean like semiring of fractions. A technique of constructing fractions of Boolean like semirings is introduced and the fractions of Boolean like semirings obtained are precisely the Boolean like rings of A. L. Foster. In addition, various characterizations of different classes of ideals (namely, prime, 2-potent prime, weakly prime, primary, weakly primary, almost primary, semi prime and 2-absorbing) in Boolean like semiring of fractions are considered in the sense of extended and contracted ideals in $S^{-1}R$ and in R. In this case, it has been proved that every ideal of $S^{-1}R$ is an extended ideal but every ideal of $R$ is not in general contracted. Thus, certain conditions that amount an ideal of $R$ to be contracted are identified. A correspondence theorem between certain classes of ideals of R disjoint from a multiplicative sub set S of R and ideals of $S^{-1}R$ is stated and proved.

On the other hand, the notion of Modules over Boolean like semirings is introduced and studied. It is noted that unlike the theory of Modules in rings, left and right modules structurally found to behave differently in the sense of getting similar results in both classes. It is shown that right modules are zero symmetric where as left modules need not be. In line to this, it is stated and proved that every module over a Boolean like semiring is a disjoint union of mutually isomorphic zero symmetric modules. Further, generalizing the results obtained for ideals of R ( in this dissertation as well as in the works of other authors), certain characterizations of prime and generalized prime sub modules are studied. Finally, a method of constructing fractions of modules over Boolean like semirings is introduced and shown that $S^{-1}M$ is a Boolean like semiring module over $S^{-1}R$.

# List of Publications

- ‡ **Ketsela Hailu**, Berhanu Bekele, Zelalem Teshome, K. Venkateswarlu; Boolean like semi ring of fractions;*International Journal of Mathematical Archive*, 3(4), 2012, 1554 - 1560.
- ‡ **Ketsela Hailu**, Berhanu Bekele, K. Venkateswarlu; Certain Special classes of Ideals in Boolean like semi ring of Fractions; *International Journal of Mathematical Archives*, 3(8), 2012, 3178-3183.
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Ketsela Hailu Demissie
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November 12, 2015.

# Contents

# Chapter 1

# Introduction

The notion of Boolean rings has been generalized by different authors in many ways. For instance, P-rings of N.H.McCoy, and D.Montgomery [32], $P_k$ - rings introduced by McCoy [31] and by A.L. Foster [13], Associate rings by I. Sussman [41], $P_1$ and $P_2$ rings introduced by N.V Subrahmanyam [40] , Pre P-rings of Abian and McWorter [1], quasi Boolean rings and generalized quasi Boolean rings by J.Luh, J.Wang and L.Chung [28], Periodic and quasi periodic rings studied by H.E.Bell [7] and Boolean like rings of A.L. Foster [12] are some of the ring theoretic generalizations of Boolean rings.

Among these , Boolean like rings of A.L. Foster preserve many of the formal properties of Boolean rings. While many of the ring theoretic generalizations of Boolean rings are semi simple ( with out non zer nilpotent element), a Boolean like ring of A.L. Foster is not semi simple. Foster has also given a method of construction of Boolean like rings by abstract synthesis of Boolean rings and zero rings (a ring of characteristic two in which $ab = 0$ for all elements a

and b of the ring) using prime ideals of the Boolean ring.
In 1982, V. Swaminathan [42] further studied on the structure of Boolean like rings and established new results . He came up with a general method of construction of Boolean like rings which improved the construction of Boolean like rings by abstract synthesis due to Foster.
In the sense of axiomatic relations, we observe the following while comparing the classes of Boolean rings with Boolean like rings;

1. We know that the idempotent property of every element in a Boolean ring R leads to the identity $a+a=0$ for every element a of R. But , the converse of this implication does not hold true in general.
2. Also, in a Boolean ring $ab(1-a)(1-b)=0$ for all elements a and b of R. In fact this identity is also a consequence of the idempotent property, but still the converse may not hold.
3. It has been noted in [12] that the axioms $ab(1-a)(1-b)=0$ and $a+a=0$ are independent. Moreover, the combination of the two does not guarantee idempotency.
4. Every Boolean ring is commutative but a commutative ring need not be Boolean .
5. Foster's ring axiomatic definition of Boolean like rings generalizes Boolean rings by replacing the strong condition of idempotency by the weaker axioms $ab(1-a)(1-b)=0$ and $a+a=0$ in a commutative ring with unity.

So, the question of weakening some of the axioms of a Boolean like ring of Foster [12] to obtain a more general and stronger system

that might preserve many of the results in a Boolean like ring still sounds quite reasonable.
In 2011 , K. Venkateswarlu et al introduced the notion of Boolean like semirings [BLSR] in [44] by generalizing the notion of Boolean like rings of A.L. Foster [12] . The way Boolean like semirings are obtained answers the question we posed earlier. Boolean like semirings have been introduced from Boolean like rings by a method analogous to the method of generalizing Boolean rings to Boolean like rings, in [44]. The following observations may give a better insight;

1. A Boolean like ring is a commutative ring with unity whereas a near ring $(R, +, \cdot)$ is a system in which $(R, +)$ is only a group satisfying one side distributive property and $(R, \cdot)$ is a semi group. It is very easy to see how far general a near ring from a commutative ring with unity is !

2. In a Boolean like ring, $ab(1 - a)(1 - b) = 0$ and $a + a = 0$ hold true independently for all elements a and b. Further, the axioms $ab(1-a)(1-b) = 0$ and $ab = ab(a+b+ab)$are equivalent. How ever, the later need not imply the former in a near ring.

3. In a near ring, the axioms $ab = ab(a + b + ab)$ and $a + a = 0$ are independent.

4. Thus, Venkateswarlu et al in [44] introduced the notion of Boolean like semirings as a generalization of Boolean like rings only by making use of the independent axioms $ab = ab(a + b + ab)$ and $a + a = 0$ on a left near ring.In fact, Boolean like semirings are also special classes of left near rings.

Further, Venkateswarlu and B.V.N Murthy have extensively studied on Boolean like semirings in [8] , [9], [45], [46] and in [47].

This dessertation is devoted to a continued study on Boolean like semirings of Venkateswarlu et al in [8] , [9] [44], [45], [46] and in [47]. The study focusses on two main problems. The first problem concentrates on a method of obtaining a quotient structure (fractions ) of Boolean like semirings and study some of it's properties. As it is well known, localization has many significances. It serves as a unifying idea in commutative ring theory, it preserves exactness. By this, localization plays an important role in homological algebra which in turn is key to modern development. Localization also preserves the Noetherian property in commutative rings. It is the generalization of the process in the construction of new rings (ring of fractions or rings of quotients). The technique is also applied in proving some important results about unique factorizations.
Keeping this in view, it is meaningful and interesting to move towards a study on localizing structures of generalized systems such as Boolean like semirings of [44] and our work charters the way. In this part of the dessertation, we further investigate on the ideal properties of the quotient Boolean like semiring. In particular, we study the contraction and extension of ideals in $R$ and in $S^{-1}R$. The results obtained in this line affirm that the quotient structure of R preserves many of the generalized prime ideal structures such as weakly prime, primary, almost primary, semi prime , 2-potent and 2-absorbing.
The second problem concentrates on the study of the theory of mod-

ules over Boolean like semirings and their quotients. The concept of modules over near rings has been studied by many authors in different ways such as radical of near ring modules by James C. Beidlman in [20] and near rings and near modules by D. Scott in [37] . But their study was on left near modules for right near rings or right near modules for left near rings. Later in 1988, Gary Ross introduced the notion of left near modules for left near rings in [16] where he introduced a new line of research which diversified the theory of near ring modules.
Imitating the line of thought of Gary Ross, we introduce the notion of Boolean like semiring modules (left Boolean like semiring module over left Boolean like semirings ) in addition to Beidlman's and Scott's approach.
In chapter 2, we present the basic background materials for ready references in the forth coming chapters.

Chapter 3 investigates a method of constructing Boolean like semiring of fractions. While moving from R to $S^{-1}R$, the situation has been found natural as analogously expected as in the theory of rings and their fractions. That is, the Boolean like semiring of fractions, $S^{-1}R$, obtained is precisely the Boolean like ring of A.L. Foster in [12]. In fact, the condition of weak commutativity we assumed is a necessary condition to obtain the result. To substantiate this, we have provided with an example of a Boolean like semiring R which is not weak commutative and it's quotient $S^{-1}R$ is not a Boolean like ring of A.L. Foster.

Chapter 4 is devoted for the study of ideals in Boolean like semiring of fractions. Analogous to the theory of rings, the notions of extended and contracted ideals are discussed here and a number of similar results have been established. Further in this sense, certain characterizations of different classes of prime ideals are studied. In addition we have proved a correspondance theorem between the prime ideals of R disjoint fom S and the ideals of $S^{-1}R$.

In chapter 5, we introduce the notion of modules over Boolean like semirings and provide various basic properties. In addition, we have identified various structural differences between left and right modules over Boolean like semirings which showed the theory of modules over near rings goes in a more diversified way as compared to that of modules over rings or semirings. For instance, every right module is zero symetric. Whereas , the left modules are not zero symmetric in general. On the other hand, every left module is a disjoint union of mutually isomorphic zero symmetric left modules. These structural differences are clearly noted in this part of the study along with illustrative examples.

In chapter 6, a study on certain classes of generalized prime sub modules such as prime, weakly prime, primary, semi prime and 2 - absorbing. This part of our study generalizes the theory of prime ideals in Boolean like semiring R to the theory of modules over R. Further we have characterized the relation between these different classes of sub modules structures. Moreover the notion of quotient of sub modules is introduced and studied.

The last chapter is devoted for fractions of modules over Boolean like semirings. In this, a method of constructing modules of fractions is introduced. Further, certain sub module structures of the module of fractions are studied.

Finally, we point out that in the process of producing this thesis, the following papers were written: [22], [23] and [24] and they form a basis of the chapters 3,4 and 5.

# Chapter 2

# Background Materials

In this chapter we collect some background materials on basics of rings and modules, Boolean like rings , Boolean like semirings and near rings that are cornerstones for our work in the preceding chapters.

## 2.1 Rings and Modules

### 2.1.1 Definitions and properties

**Definition 2.1.1.** [43] *A non empty set with two binary operations,*$(R, +, \cdot)$, *is called a ring if it satisfies the following conditions:*

1. $(R, +)$ *is an Abelian group;*
2. $(R, \cdot)$ *is a semigroup;*
3. $a(b + c) = ab + ac$, *for all elements* $a, b, c$ *of R;*
4. $(a + b)c = ac + bc$ , *for all elements* $a, b, c$ *of R;*

**Definition 2.1.2.** [43] *A ring R is called commutative if* $ab = ba$ *for all elements a and b of R.*

**Definition 2.1.3.** [43] *A Boolean ring is a ring in which every element is idempotent.*

**Remark 2.1.1.** *It is clear that every Boolean ring is commutative with characteristic 2.*

**Definition 2.1.4.** [43] *Let R be a ring. A (left) R modue is an additive Abelian group A together with a function* $f : R \times A \longrightarrow A$*, mapping* $(r, a) \rightarrow ra$ *such that for all* $r, s \in R$ *and* $a, b \in A$*:*

1. $r(a + b) = ra + rb$;
2. $(r + s)a = ra + sa$
3. $(rs)a = r(sa)$
4. *In addition, A is called a unitary R-module if R is unitary and* $1_R a = a, \forall a \in A$.

### 2.1.2 Rings and Modules of fractions

The well- known construction of the field of fractions of a commutative integral domain was generalized to arbitrary commutative rings. While extending this construction to non-commutative rings, one finds that this cant be possible always. But, one can give necessary and sufficient conditions for the existence of a ring of fractions. Such a condition was found by Ore in [35] for the case of a skew-field of fractions of a domain. The existence of a total ring of fractions of an arbitrary ring was considered by K. Asano in [5]. General rings of fractions were studied by Elizarov in [11] , and a systematic theory of rings and modules of fractions was developed by P.Gabriel in [14] in connection with his theory of general rings of quotients.

### Ring of fractions

Let R be a commutative ring and let S be a subset of R which is closed under multiplication, with 1 in S. The concept of constructing ring of fractions $S^{-1}R$ by a generalization of the well-known construction of a field of fractions for an integral domain is as follows. Consider all pairs (a, s) with a in R, s in S. Put $(a, s) \sim (b, t)$ if there exists u in S such that $u(ta - sb) = 0$.
It is a routine matter to verify that $\sim$ is an equivalence relation. Let $\frac{a}{s}$ denote the equivalence class containing (a, s). It is then possible to define

$$\frac{a}{s} + \frac{b}{t} = \frac{ta + sb}{st}, \frac{a}{s}\frac{b}{t} = \frac{ab}{st}$$

and this makes $S^{-1}R = \{\frac{a}{s}/a \in R, s \in S\}$ a commutative ring. We have the following results;

**Definition 2.1.5.** [26] *A non empty subset of a ring R with 1 is said to be multiplicative,if* $1 \in S$ *and* $a, b \in S$ *implies* $ab \in S$.

**Example 2.1.1.**

1. The set of all elements of a ring R that are not divisors of zero (whenever exists) form a multiplicative set S ,
2. The set of all non zero elements of an integral domain form a multiplicative set S,
3. The set of all units of a ring R form a multiplicative set S,
4. If P is a prime ideal of a ring R , then the set $S = R \setminus P$ is a multiplicative set

**Theorem 2.1.1.** [6] Let R be a commutative ring and S be a multiplicatively closed subset of R. Define a relation $\sim$ on $R \times S$ by $(r_1, s_1) \sim (r_2, s_2)$ if and only if there exists an element s in S such that $ss_2r_1 = ss_1r_2$ . Then $\sim$ is an equivalence relation.

**Theorem 2.1.2.** [6] *Let R be a commutative ring and S be a multiplicatively closed subset of R. Define the binary operations*

$$+ : \frac{a}{s} + \frac{b}{t} = \frac{ta + sb}{st}, \ \cdot : \frac{a}{s}.\frac{b}{t} = \frac{ab}{st}$$

*Then the ring of fractions, $(S^{-1}R, +, .)$ is a commutative ring with unity.*

**Theorem 2.1.3.** Let R be a commutative ring, S be a multiplicatively closed subset in R and I be an ideal of R.Then the set $S^{-1}I = \{\frac{a}{s}/a \in I, s \in S\}$, called the **extension of I** , denoted by $\mathbf{I^e}$, is an ideal of $S^{-1}R$.

**Theorem 2.1.4.** [6] *Let I be an ideal of a commutative ring R and S be a multiplicative subset of R. Then,*

1. $I^{ec} = \cup(I : s) = \{r \in R/sr \in I$ *for some* $s \in S\}$.
2. $I^e = S^{-1}R$ *if and only if* $I \cap S \neq \varnothing$

**Theorem 2.1.5.** [6] *Let R be a commutative ring and S be a multiplicatively closed subset of R. Then every ideal of $S^{-1}R$ is an extended ideal (i.e. $I^{ce} = I$)*

**Theorem 2.1.6.** [6] *Let S be a multiplicative set and R be a commutative ring. Let I and J be ideals of R then,*

1. $S^{-1}(I + J) = S^{-1}I + S^{-1}J$,
2. $S^{-1}(IJ) = (S^{-1}I)(S^{-1}J)$
3. $S^{-1}(I \cap J) = (S^{-1}I) \cap (S^{-1}J)$
4. $S^{-1}r(I) = r(S^{-1}I)$ , *where $r(I)$ is the radical of I.*

**Corollary 2.1.1.** [6] *If N is the nil radical of a commutative ring R then $S^{-1}N$ is the nil radical of $S^{-1}R$.*

**Theorem 2.1.7.** [6] *Let P be a prime ideal of a commutative ring*

*R and S be a multiplicative subsetof R such that $P \cap S = \varnothing$. Then $P^{ec} = P$*

**Theorem 2.1.8.** [6] *Let J be a prime ideal of $S^{-1}R$ then*

$$J^c = \{x \in R | f(x) \in J\}$$

*is a prime ideal of R.*

**Theorem 2.1.9.** [6] *Let S be a multiplicative subset of a ring R. The mapping $P \longrightarrow P^e$ is a one to one correspondence from the set of all prime ideals of R disjoint from S to the set of all prime ideals of $S^{-1}R$*

## Modules of fractions

In this we collect certain results on modules of fractions over a commutative ring of fractions.

**Theorem 2.1.10.** [6] *Let S be a multiplicative set in a ring R and let M be a left module over R. Define the relation $\sim$ on $S \times M$ by ,*

$$(s_1, m_1) \sim (s_2, m_2)$$

*if and only if there exists s in S such that $ss_2m_1 = ss_1m_2$. Then $\sim$ is an equivalence relation.*

**Theorem 2.1.11.** [6] *Let S be a multiplicative set in a ring R and let M be an R-module. Define the binary operations '+'and '$\cdot$' as;*

$$+ : S^{-1}M \times S^{-1}M \longrightarrow S^{-1}M \ \text{by}$$

$$\frac{m_1}{s_1} + \frac{m_2}{s_2} = \frac{s_2m_1 + s_1m_2}{s_1s_2}$$

*and,*

$$\cdot : S^{-1}R \times S^{-1}M \longrightarrow S^{-1}M \ \text{by}$$

$$\frac{r}{s}\frac{m}{t} = \frac{rm}{st}$$

*Then* $S^{-1}M$ *is an* $S^{-1}R$ *module.*

## 2.2 Near rings and Near modules

We have the following results on near rings and near modules,

### 2.2.1 Near Rings - Definition and Properties

**Definition 2.2.1.** [*15*] *A set R together with the two binary operations of addition and multiplication (written as (R,+,.)) is called a near-ring if the following conditions are satisfied;*

1. $(R, +)$ *is a group,*
2. $(R, .)$ *is a semi group*
3. *At least one of the following two distributive conditions hold;*
   (a) $a(b + c) = ab + ac$
   (b) $(a + b)c = ac + bc$ $\forall a, b, c \in R$.

**Remark 2.2.1.**

1. If 3(a) holds, then R is called a left near-ring and if 3(b) holds,then R is called a right near-ring.
2. R is called a near-ring with identity if R has multiplicative identity.
3. If R is a right near-ring, then it is always true that $0r = 0, \forall r \in R$
   If it is also true that $r0 = 0$ $\forall r \in R$, then R is referred to as a

zero symmetric right near-ring.

4. If $(R, +)$ is an Abelian group, then R is called an Abelian near-ring.

**Definition 2.2.2.** [*15*] *Let N be a (right) near ring.*

*1.* $N_0 = \{n \in N \mid n0 = 0\}$ *is called the zero symmetric part of N.*

*2.* $N_c = \{n \in N \mid nn^{'} = n, \forall n^{'} \in N\}$ *is called the constant part of N.*

**Definition 2.2.3.** [*15*]

1. A near ring N is called zero symmetric if $N = N_0$.

2. A near ring N is called constant if $N = N_c$ , otherwise it is called a non constant near ring

3. An element of a right near ring N is called distributive if for all $n, n^{'} \in N, d(n + n^{'}) = dn + dn^{'}$

4. Let $N_d = \{d \in N \mid$ d is distributive $\}$,

   (a) If $N = N_d$ , N is said to be distributive.

   (b) A near ring N with the property that $N_d$ generates ( N, +) is called a distributively generated near ring .

### 2.2.2 Near Modules

Just like the way a near ring is generalized from a ring, the notion of a near module (also called an N - group where N is near ring) is also a generalization of a module over a ring. More specifically, a near module is defined on an arbitrary group (which need not be Abelian) and required to satisfy atleast one of the distributive properties.

**Definition 2.2.4.** [*37*] *Let N be a right near ring . A left R modue is an additive group M together with a function* $f : R \times M \longrightarrow M$*, mapping* $(r, a) \rightarrow ra$ *such that for all* $r, s \in R$ *and for all* $a \in M$*:*

1. $(r + s)a = ra + sa$
2. $(rs)a = r(sa)$
3. *In addition, A is called unitary R-module if R is unitary and* $1_R a = a, \forall a \in A$.

**Remark 2.2.2.** *A right module over a right near ring can also be analogously defined.*

**Definition 2.2.5.** [*37*] *Let M be a left module over a right near ring N. A sub group B of M is called a sub module if* $NB \subseteq B$.

## 2.3 Boolean like semirings

A Boolean like semiring is a special class of near ring. It is also a generalization of Boolean like rings of A.L.Foster [12]. As clearly indicated in Venkateswarlu et al in [44], the structure of a Boolean like semiring is independent of rings, Boolean semirings of Subrahmanyam or semiring structures. We recall the following from [12] and [44]

### 2.3.1 Definitions and Basic properties

**Definition 2.3.1.** [*12*] A commutative ring with unity is called a Boolean like ring if it is of characteristic two and satisfying the condition $xy(1-x)(1-y) = 0$ for all elements x and y of the ring.

**Remark 2.3.1.**

1. In a Boolean ring R, $a(a-1) = 0$ implies $a+a = 0$ for all elements of R. But these two are independent in Boolean like rings.
2. In a Boolean ring, $a(1-a) = 0$ implies $ab(1-a)(1-b) = 0$, but the converse is not true in general.
3. In a Boolean like ring, the conditions $ab(1-a)(1-b) = 0$ and $ab = ab(a+b+ab)$, are equivalent.

**Theorem 2.3.1.** [*12*] *If $\eta_1$ and $\eta_2$ are two nilpotent elements of a Boolean like ring R, then $\eta_1\eta_2 = 0$.*

**Definition 2.3.2.** [*44*] A non empty set R together with two binary operations $+$ and $\cdot$ satisfying the following conditions is called a Boolean like semiring;

1. *$(R,+)$ is an Abelian group;*
2. *$(R,.)$ is a semi group;*
3. *$a(b+c) = ab+ac, \forall a,b,c \in R$;*
4. *$a+a = 0, \forall a \in R$;*
5. *$ab = ab(a+b+ab), \forall a,b \in R$.*

**Remark 2.3.2.**

1. *A system $(R,+,\cdot)$ satisfying conditions 1, 2 and 3 is called a left near ring.*
2. *Conditions 4 and 5 are independent as clearly shown in the following examples.*

**Example 2.3.1.** *Let K = {0,1,2,3} where $+$ and $\cdot$ are defined as* *in the following tables;*

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

| · | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 0 | 2 | 0 | 2 |
| 3 | 0 | 1 | 2 | 3 |

*Then,* $(K, +, \cdot)$ *is a left near ring in which* $ab = ab(a + b + ab)$ *but for every a in K,* $a + a = 0$ *is not true.*

**Example 2.3.2.** *Let* $R = \{0, a, b, c\}$ *with the two binary operations defined on R as follows;*

| + | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | a | b | c |
| a | a | 0 | c | b |
| b | b | c | 0 | a |
| c | c | b | a | 0 |

| . | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | b | c |
| b | 0 | a | b | c |
| c | 0 | a | b | c |

*In this example,* $a + a = 0$ *but the identity* $ab = ab(a + b + ab)$ *is not true for every element of R. For instance,* $ac = c \neq a = ac(a + c + ac)$.

**Lemma 2.3.1.** [44] *Let R be a Boolean like semiring. Then for every a in R:*

1. $a0 = 0$
2. $a^4 = a^2$ *, which is called weak idempotent law,*
3. $a^{2n} = a^2$*, for any positive integer n.*
4. $a^n = a$ *or* $a^2$ *or* $a^3 \quad \forall a \in R$.

**Definition 2.3.3.** [44] *A Boolean like semiring R is called weak commutative if* $abc = acb$ *for all elements a , b and c of R.*

**Lemma 2.3.2.** [44] *If R is a weak commutative Boolean like semiring, then for any a,b and c in R;*

1. $0a = 0$,
2. $(ab)^n = a^n b^n$ ;
3. $(a + b)a = (a + b)ab$ *if* $a^2 = 0$;

The following are examples of Boolean like semirings,

**Example 2.3.3.** *Let* $R = \{0, a, b, c\}$ *with '+' and '·' defined by the following tables.*

| + | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | a | b | c |
| a | a | 0 | c | b |
| b | b | c | 0 | a |
| c | c | b | a | 0 |

| . | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | 0 | a |
| b | 0 | 0 | 0 | 0 |
| c | 0 | c | 0 | c |

**Example 2.3.4.** *Let* $R = \{0, a, b, c, d, e, f, 1\}$. *Define '+' and ' · ' on R by the following tables.*

| + | 0 | a | b | c | d | e | f | 1 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | a | b | c | d | e | f | 1 |
| a | a | 0 | c | b | e | d | 1 | f |
| b | b | c | 0 | a | f | 1 | d | e |
| c | c | b | a | 0 | 1 | f | e | d |
| d | d | e | f | 1 | 0 | a | b | c |
| e | e | d | 1 | f | a | 0 | c | b |
| f | f | 1 | d | c | b | c | 0 | a |
| 1 | 1 | f | e | d | c | b | a | 0 |

| · | 0 | a | b | c | d | e | f | 1 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | 0 | 0 | 0 | a | a | a | a |
| b | 0 | 0 | b | b | 0 | 0 | b | b |
| c | 0 | 0 | b | b | a | a | c | c |
| d | 0 | 0 | 0 | 0 | d | d | d | d |
| e | 0 | a | 0 | a | d | e | d | e |
| f | 0 | 0 | b | b | d | d | f | f |
| 1 | 0 | a | b | c | d | e | f | 1 |

### 2.3.2 Ideals in Boolean like semirings

**Definition 2.3.4.** [44] *A subset $I$ of a Boolean like semiring $R$ is said to be an ideal if;*

1. $(I, +)$ *is a subgroup of (R,+);*
2. $RI \subseteq I$;
3. $(r + a)s + rs \in I \ \forall r, s \in R, a \in I$.

**Theorem 2.3.2.** [44] *The set , $N(R)$, of all nilpotent elements in a weak commutative Boolean like semiring form an ideal.*

**Theorem 2.3.3.** [44] *If I is an ideal of a weak commutative Boolean like semiring R, then the radical of I,*

$$r(I) = \{x \in R | x^n \in I \text{ for some positive integer } n\}$$

*is an ideal of R.*

**Definition 2.3.5.** [44] *A non empty subset S of a Boolean like semiring R is called a sub Boolean like semiring if S itself is Boolean like semiring.*

**Remark 2.3.3.** [44] *In a Boolean like semiring R, every ideal is a Sub Boolean like semiring but not conversely.*

**Theorem 2.3.4.** [44] *If R is a Boolean like semiring and I is an ideal of R then the set*

$$(R : I) = \{x \in R | rx \in I \ \forall r \in R\}$$

*is a left ideal of R and $I \subseteq (R : I)$ . Further $(R : I)$ is an ideal of R if R is weak commutative*

**Theorem 2.3.5.** [44] *If I and J are ideals of a Boolean like semiring R then $I + J$ is an ideal of R.*

**Theorem 2.3.6.** [44] *If I and J are ideals of a Boolean like semiring R then $I \cap J$ is an ideal of R.*

**Theorem 2.3.7.** [44] *If I and J are ideals of Boolean like semiring R then,*

1. $I \subseteq r(I)$,
2. $r(I \cap J) = r(I) \cap r(J)$,
3. $I \subseteq J \Rightarrow r(I) \subseteq r(J)$,
4. $r(r(I)) = r(I)$,
5. $r(I + J) = r(r(I) + r(J))$,
6. $r(IJ) = r(I \cap J)$.
7. *If R is a weak commutative Boolean like semiring with right unity 1 then $r(I) = R$ if and only if $I = R$*

**Definition 2.3.6.** [44] *If R and S are Boolean like semirings ,then a mapping $f : R \longrightarrow S$ is said to be homomorphism of R into S if*

1. $f(a + b) = f(a) + f(b)$ .
2. $f(ab) = f(a)f(b)\ \forall a, b \in R$

**Theorem 2.3.8.** [44] *Let R and S be Boolean like semi rings . Let I be an ideal of S and let $f : R \longrightarrow S$ be a homomorphism , then $f^{-1}(I)$ is an ideal of R, called the contraction of I and is denoted by $I^c$.*

**Theorem 2.3.9.** [44] *Let I and J be two ideals of a Boolean like semiring R then,*

1. $(I^c + J^c) \subseteq (I + J)^c$,
2. $(I)^c = (I^c \cap J^c)$,
3. $I^c J^c \subseteq (IJ)^c$,

4. $(I : J)^c \subseteq (I^c : J^c)$,

5. $[r(I)]^c = r(I^c)$.

**Remark 2.3.4.** *Every homomorphic image of a Boolean like semiring is Boolean like semiring.*

**Theorem 2.3.10.** [44] *If I is an ideal of a Boolean like semiring R then the quotient,*

$$R/I = \{x + I/x \in R\}$$

*is also Boolean like semiring.*

**Theorem 2.3.11.** [44] *If $f : R \longrightarrow S$ is a homomorphism of Boolean like semirings then the kernel of f*

$$Kerf = \{x \in R | f(x) = 0\}$$

*is an ideal of R.*

**Definition 2.3.7.** [44] *Let R be a weak commutative Boolean like semiring . Let I and J be ideals of R. Then their ideal quotient denoted by $(I : J)$, is is defined by*

$$(I : J) = \{x \in R/Jx \subseteq I\}$$

.

**Theorem 2.3.12.** [44] *If R is a weak commutative Boolean like semiring and I, J and K are the ideals of R then the following hold*

1. $(I : J) = \{xR/Jx \in I\}$ *is an ideal of R ,*

2. $I \subseteq (I : J)$,

3. $((I : J) : K) = (I : JK)$,

4. $(\cap I_i : J) = \cap_i (I_i : J)$.

**Theorem 2.3.13.** [44] *Let R be a weak commutative Boolean like* *semiring and I be an ideal of R. Then the set*

$$AnnI = \{a \in R | sa = 0 \ \forall s \in I\}$$

*called the annihilator of I, is an ideal of R.*

**Definition 2.3.8.** [*48*] *A proper ideal P of a Boolean like semi ring R is called prime if* $xy \in P$ *implies either* $x \in P$ *or* $y \in P$.

# Chapter 3

# Boolean Like Semiring of Fractions

In this chapter we investigate the theory of Boolean like semirings by introducing the notion of quotients of Boolean like semiring. We have proved that the fractions of Boolean like semirings are precisely the Boolean like rings of A.L. Foster [12] .

## 3.1 Construction of Boolean like semiring of fractions

**Definition 3.1.1.** *A non empty subset $S$ of a Boolean like semiring $R$ is called multiplicatively closed whenever $a,b \in S$ implies $ab \in S$ $\forall a, b, \in R$.*

We establish the following results which will be used in the sequel.

**Theorem 3.1.1.** *Let $R$ be a weak commutative Boolean like semiring.Then for all $a$ and $b$ in $R$,*

$$ab^2 + a^2b = ab + (ab)^2.$$

*Proof.*

$$\begin{aligned} ab^2 + a^2b &= abb + aab \\ &= abb + aba \quad \text{by weak commutativity of R,} \\ &= ab(b + a) \quad \text{since R is left distributive,} \\ &= ab(b + a + ab + ab) \qquad \text{since } Cha(R) = 2 \\ &= ab(a + b + ab) + (ab)^2 \quad \text{ since R is a BLSR.} \\ &= ab + (ab)^2 \end{aligned}$$

□

The following Theorem is a generalization of Theorem 2.3.1 .

**Theorem 3.1.2.** *Let R be a weak commutative Boolean like semiring.Then for all r, a and b in R,*

$$r(a + a^2)(b + b^2) = 0.$$

*Proof.* In [8] and [44] , it has been noted that for any element 'a' of R, $(a + a^2)$ is nilpotent. Now,

$$\begin{aligned} r(a + a^2)(b + b^2) &= r(a + a^2)b + r(a + a^2)b^2 \\ &= rb(a + a^2) + rb^2(a + a^2) \\ &= rba + rba^2 + rb^2a + rb^2a^2) \\ &= (rba + rb^2a^2) + (rba^2 + rb^2a) \\ &= r[ba + (ba)^2] + r[ba^2 + b^2a] \\ &= r[(ba + (ba)^2) + (ba^2 + b^2a)] \\ &= r0 \;\; \text{by Theorem 3.1.1 and Char(R) = 2.} \\ &= 0 \end{aligned}$$

□

In his doctoral dissertation in [8] , B.V.N. Murthy indicated that the zero symmetric property of Boolean like semirings can be obtained only under the assumption of weak commutativity. But, in the following Theorem we prove the property holds true in every Boolean like semiring.

**Theorem 3.1.3.** *Every Boolean like semiring is zero symmetric.*

*Proof.* Let $r \in R$. Clearly $r0 = 0$. To show $0r = 0$,

$$\begin{aligned} 0r &= (r0)r \qquad \text{since } r0 = 0. \\ &= (r0)(r + 0 + r0) \\ &= r0 \qquad \text{since R is a BLSR} \end{aligned}$$

□

**Theorem 3.1.4.** *Let R be a weak commutative Boolean like semiring and S be a multiplicatively closed sub set of R. Define a relation $\sim$ on $R \times S$ by $(r_1, s_1) \sim (r_2, s_2)$ if and only if there exists an element $s \in S$ such that $s(s_1r_2 + s_2r_1) = 0$. Then $\sim$ is an equivalence relation.*

*Proof.* Let $(r, s) \in R \times S$. Then for any $t \in S$,
$t(sr + sr) = ts(r + r) = ts(0) = 0$. Hence $\sim$ is reflexive.
To prove $\sim$ is symmetric, let $(r_1, s_1) \sim (r_2, s_2)$, then there exists t in S such that $t(s_2r_1+s_1r_2) = 0$. Hence $t(s_1r_2+s_2r_1) = 0$. Consequently we have $(r_2, s_2) \sim (r_1, s_1)$ . Which proves $\sim$ is symmetric.
Finally, let $(r_1, s_1) \sim (r_2, s_2)$ and $(r_2, s_2) \sim (r_3, s_3)$ for $r_1, r_2, r_3 \in R$ and $s_1, s_2, s_3 \in S$. Then, $(r_1, s_1) \sim (r_2, s_2)$ then there exists $t_1 \in S$ such that

$$t_1[s_2r_1 + s_1r_2] = 0 \tag{3.1.1}$$

and $(r_2, s_2) \sim (r_3, s_3)$ then there exists $t_2 \in S$ such that

$$t_2[s_3r_2 + s_2r_3] = 0 \tag{3.1.2}$$

Now from 3.1.1 and 3.1.2, we have
$t_1[s_2r_1 + s_1r_2] = 0 = t_2[s_3r_2 + s_2r_3]$
$\Rightarrow t_1[s_2r_1 + s_1r_2]s_3 = 0 = t_2[s_3r_2 + s_2r_3]s_1$ by Theorem 3.1.3
$\Rightarrow t_1s_3[s_2r_1 + s_1r_2] = 0 = t_2s_1[s_3r_2 + s_2r_3]$ by WC of R ,
$\Rightarrow t_2t_1s_3[s_2r_1 + s_1r_2] = 0 = t_2s_1[s_3r_2 + s_2r_3]t_1$,
applying Theorem 3.1.3 on $t_2$ and $t_1$ ,
$\Rightarrow t_2t_1s_3[s_2r_1 + s_1r_2] = 0 = t_2t_1s_1[s_3r_2 + s_2r_3]$ , by weak commutativity of R,
$\Rightarrow t_2t_1s_3[s_2r_1 + s_1r_2] + t_2t_1s_1[s_3r_2 + s_2r_3] = 0$
$\Rightarrow (t_2t_1s_3)(s_2r_1) + (t_2t_1s_3)(s_1r_2) + (t_2t_1s_1)(s_3r_2) + (t_2t_1s_1)(s_2r_3) = 0$
$\Rightarrow [(t_2t_1s_3)(s_2r_1)+(t_2t_1s_1)(s_2r_3)]+[(t_2t_1s_3)(s_1r_2)+(t_2t_1s_1)(s_3r_2)] = 0$
$\Rightarrow [(t_2t_1s_3s_2)r_1 + (t_2t_1s_1s_2)r_3] + [(t_2t_1s_3s_1)r_2 + (t_2t_1s_1)(s_3r_2)] = 0$
$\Rightarrow [(t_2t_1s_2s_3)r_1 + (t_2t_1s_2s_1)r_3] + [(t_2t_1s_1s_3)r_2 + (t_2t_1s_1)(s_3r_2)] = 0$
$\Rightarrow [(t_2t_1s_2)(s_3r_1)+(t_2t_1s_2)(s_1r_3)]+[(t_2t_1s_1)(s_3r_2)+(t_2t_1s_1)(s_3r_2)] = 0$
$\Rightarrow (t_2t_1s_2)[s_3r_1 + s_1r_3] + (t_2t_1s_1)[s_3r_2 + s_3r_2] = 0$ by repeatedly applying weak commutativity and semigroup property of R ,
$\Rightarrow (t_2t_1s_2)[s_3r_1 + s_1r_3] + (t_2t_1s_1)(0) = 0$ since charactersitc of R is 2 ,
$\Rightarrow (t_2t_1s_2)[s_3r_1 + s_1r_3] = 0$
Thus there exists $t = t_2t_1s_2$ so that $(t)[s_3r_1 + s_1r_3] = 0$. Therefore, $\sim$ is transitive and consequently it is an equivalence relation on $R \times S$. □

**Notation:** From Theorem 3.1.4, we denote the equivalence class containing $(r, s)$ in $R \times S$ by $\frac{r}{s}$ and the set of all equivalence classes of $R \times S$ by $S^{-1}R$.

The following are basic properties which will be repeatedly used in the sequel.

**Lemma 3.1.1.** Let R be a weak commutative Boolean like semiring and S be a multiplicatively closed subset of R. Then, for all $r, r_1, r_2$ *in R and* $s_1, s_2, s, s^{'}, t$ *in S;*

*a) If* $0 \notin S$ *and S has no zero divisors, then* $(r_1, s_1) \sim (r_2, s_2)$ *if and only if* $s_2 r_1 = s_1 r_2$,

*b)* $\frac{r}{s} = \frac{tr}{ts} = \frac{tr}{st} = \frac{rt}{ts} = \frac{rt}{st}$,

*c)* $\frac{rs}{s} = \frac{rs^{'}}{s^{'}}$,

*d)* $\frac{s}{s} = \frac{s^{'}}{s^{'}}$,

*e)* $\frac{r_1 r_2}{s} = \frac{r_2 r_1}{s}$,

*f)* $\frac{r}{s_1 s_2} = \frac{r}{s_2 s_1}$,

*g)* $\frac{r}{s} = \frac{0}{s}$ if and only if there exists t in S such that $tr = 0$.

*Proof.*

a) Let $(r_1, s_1) \sim (r_2, s_2)$ then,there exists t in S such that $t(s_2 r_1 + s_1 r_2) = 0$ which implies $s_2 r_1 + s_1 r_2 = 0$ since 0 is not in S and S has no zero divisor. Consequently, $s_2 r_1 = s_1 r_2$ since R is of characteristic 2.

b) Let $q \in S$. Then, $q(tsr + str) = q(ts)r + q(str) = q(st)r + q(str)$ ( since R is weak commutative) $= q(str + str) = q0 = 0$ so that $\frac{r}{s} = \frac{tr}{ts}$. And the remaining equalities of fractions can be shown by simillar techniques.

c) Let $t \in S$. Then, $t(s^{'}rs) = t(s^{'}sr) = t(sr)s^{'} = ts(rs^{'})$. Hence we have $\frac{rs}{s} = \frac{rs^{'}}{s^{'}}$.

d) Let $t \in S$. Then, $t(s^{'}s) = t(ss^{'})$, since R is weak commutative $\Rightarrow t(s^{'}s) + t(ss^{'}) = 0 \Rightarrow (s,s) \sim (s^{'},s^{'}) \Rightarrow \frac{s}{s} = \frac{s^{'}}{s^{'}}$.

e) Let $t \in S$. Then, $ts(r_1r_2) = ts(r_2r_1)$, hence the result.

f) Let $t \in S$., *then by weak commutativity of R it follows that* $t(s_2s_1r) = t(s_ns_2r)$ so that the result follows.

g) Suppose $\frac{r}{s} = \frac{0}{s}$. Then by Definition, there exists q in S such that $q(sr + s0) = 0$. Equivalently, $(qs)r = 0$. Thus choose $t = qs$ so that $tr = 0$. Conversely, let $tr = 0$ for r in R and t in S. Then, $\frac{r}{s} = \frac{tr}{ts}$ by part b). So, $\frac{r}{s} = \frac{0}{ts} = \frac{t0}{ts} = \frac{0}{s}$.

□

**Theorem 3.1.5.** Let S be a multiplicatively closed sub set of a weak commutative Boolean like semiring R. For $r_1, r_2, \in R$ and $s_1, s_2 \in S$, Define the operations + and $\cdot$ *on* $S^{-1}R$ as follows:

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{s_2r_1 + s_1r_2}{s_1s_2}$$

*and*

$$\frac{r_1}{s_1}\frac{r_2}{s_2} = \frac{r_1r_2}{s_1s_2}.$$

Then $(S^{-1}R, +, \cdot)$ is a Boolean like semiring.

*Proof.* First we prove + and $\cdot$ are well defined. For $r_i \in R$ and $s_i \in S$ where $i = 1, 2, 3, 4$, let $\frac{r_1}{s_1} = \frac{r_3}{s_3}$ and $\frac{r_2}{s_2} = \frac{r_4}{s_4}$. Then to prove + is well defined, we show $\frac{s_2r_1+s_1r_2}{s_1s_2} = \frac{s_4r_3+s_3r_4}{s_3s_4}$. We claim that for some t in S,

$$t[(s_3s_4)(s_2r_1 + s_1r_2) + (s_1s_2)(s_4r_3 + s_3r_4)] = 0.$$

But by Definition of our assumptions, $\frac{r_1}{s_1} = \frac{r_3}{s_3}$ implies there exists $t_1 \in S$ such that $t_1(s_3r_1 + s_1r_3) = 0$ and $\frac{r_2}{s_2} = \frac{r_4}{s_4}$ implies there exists

$t_2 \in S$ such that $t_2(s_2r_4 + s_4r_2) = 0$

Now consider, $(t_2t_1)[(s_3s_4)(s_2r_1 + s_1r_2) + (s_1s_2)(s_4r_3 + s_3r_4)]$

$= (t_2t_1)(s_3s_4)(s_2r_1)+(t_2t_1)(s_3s_4)(s_1r_2)+(t_2t_1)(s_1s_2)(s_4r_3)+(t_2t_1)(s_1s_2)(s_3r_4)$

$= (t_2t_1)(s_3s_4s_2)r_1+(t_2t_1)(s_3s_4s_1)r_2+(t_2t_1)(s_1s_2s_4)r_3+(t_2t_1)(s_1s_2s_3)r_4$

$= (t_2t_1)(s_4s_2s_3)r_1+(t_2t_1)(s_3s_1s_4)r_2+(t_2t_1)(s_2s_4s_1)r_3+(t_2t_1)(s_1s_3s_2)r_4$

$= (t_2t_1)(s_4s_2)(s_3r_1)+(t_2t_1)(s_3s_1)(s_4r_2)+(t_2t_1)(s_2s_4)(s_1r_3)+(t_2t_1)(s_1s_3)(s_2r_4)$

$= (t_2t_1)(s_2s_4)(s_3r_1)+(t_2t_1)(s_1s_3)(s_4r_2)+(t_2t_1)(s_2s_4)(s_1r_3)+(t_2t_1)(s_1s_3)(s_2r_4)$

$= (t_2t_1)(s_2s_4)(s_3r_1 + s_1r_3) + (t_2t_1)(s_1s_3)(s_4r_2 + s_2r_4)$

$= t_2(s_2s_4)[t_1(s_3r_1 + s_1r_3)] + [t_2(s_4r_2 + s_2r_4]t_1(s_1s_3)$

$= t_2(s_2s_4)[0] + [0]t_1(s_1s_3)$

$= 0$

Now, choose $t = t_2t_1$. Clearly $t \in S$. Hence the claim.

To prove $\cdot$ is well defined, let $\frac{r_1}{s_1} = \frac{r_3}{s_3}$ which implies there exits $t_1 \in S$ such that $t_1(s_3r_1 + s_1r_3) = 0$ and let $\frac{r_2}{s_2} = \frac{r_4}{s_4}$ which implies there exists $t_2 \in S$ such that $t_2(s_2r_4 + s_4r_2) = 0$

We claim $s[(s_3s_4)(r_1r_2) + (s_1s_2)(r_3r_4)] = 0$ for some $s \in S$

$\Rightarrow t_1(s_3r_1 + s_1r_3) = 0 = t_2(s_2r_4 + s_4r_2)$

$\Rightarrow t_1(s_3r_1 + s_1r_3)(s_4r_2) = 0 = t_2(s_2r_4 + s_4r_2)(s_1r_3)$

$\Rightarrow t_1(s_3r_1 + s_1r_3)(s_4r_2)t_2 = 0 = t_1t_2(s_2r_4 + s_4r_2)(s_1r_3)$

$\Rightarrow t_1t_2(s_3r_1 + s_1r_3)(s_4r_2) = 0 = t_1t_2(s_2r_4 + s_4r_2)(s_1r_3)$

$\Rightarrow t_1t_2(s_4r_2)(s_3r_1 + s_1r_3) = 0 = t_1t_2(s_1r_3)(s_2r_4 + s_4r_2)$

$\Rightarrow t_1t_2(s_4r_2)(s_3r_1 + s_1r_3) + t_1t_2(s_1r_3)(s_2r_4 + s_4r_2) = 0$

$$\Rightarrow t_1t_2(s_4r_2)(s_3r_1)+t_1t_2(s_4r_2)(s_1r_3) + t_1t_2(s_1r_3)(s_2r_4) + t_1t_2(s_1r_3)(s_4r_2) = 0$$

$$\Rightarrow t_1t_2(s_4r_2s_3)r_1 + t_1t_2(s_4r_2)(s_1r_3) + t_1t_2(s_1r_3s_2)r_4 + t_1t_2(s_1r_3)(s_4r_2) = 0$$

$$\Rightarrow t_1t_2(s_4s_3r_2)r_1 + t_1t_2(s_4r_2)(s_1r_3) + t_1t_2(s_1s_2r_3)r_4 + t_1t_2(s_1r_3)(s_4r_2) = 0$$

$$\Rightarrow t_1t_2(s_4s_3)(r_2r_1) + t_1t_2(s_4r_2)(s_1r_3) + t_1t_2(s_1s_2)(r_3r_4) + t_1t_2(s_4r_2)(s_1r_3) = 0$$

$$\Rightarrow t_1t_2(s_4s_3)(r_1r_2) + t_1t_2(s_4r_2)(s_1r_3) + t_1t_2(s_1s_2)(r_3r_4) + t_1t_2(s_4r_2)(s_1r_3) = 0$$

$$\Rightarrow [t_1t_2(s_4s_3)(r_1r_2) + t_1t_2(s_1s_2)(r_3r_4] + [t_1t_2(s_4r_2)(s_1r_3) + +t_1t_2(s_4r_2)(s_1r_3)] = 0$$
$$\Rightarrow [t_1t_2(s_4s_3)(r_1r_2) + t_1t_2(s_1s_2)(r_3r_4)] + [0] = 0$$
$$\Rightarrow t_1t_2(s_4s_3)(r_1r_2) + t_1t_2(s_1s_2)(r_3r_4) = 0$$
$$\Rightarrow t_1t_2[(s_4s_3)(r_1r_2) + (s_1s_2)(r_3r_4)] = 0$$

Thus , choose $s = t_1t_2$. Hence both $+$ and $\cdot$ are well defined. Before proceeding to the next, we observe the following :

**Remark 3.1.1.** *For* $\frac{r_1}{s}, \frac{r_2}{s} \in S^{-1}R$,

$$\frac{r_1}{s} + \frac{r_2}{s} = \frac{sr_1 + sr_2}{ss} = \frac{s(r_1 + r_2)}{ss} = \frac{r_1 + r_2}{s} \quad \textit{(by Lemma 3.1.1)}$$

We show $S^{-1}R$ is an Abelian group,
Let $\frac{r_1}{s_1}, \frac{r_2}{s_2}, \frac{r_3}{s_3} \in S^{-1}R$,. Then;
$\frac{r_1}{s_1} + [\frac{r_2}{s_2} + \frac{r_3}{s_3}] = \frac{r_1}{s_1} + [\frac{s_3r_2+s_2r_3}{s_2s_3}]$

$$
\begin{aligned}
&= \frac{(s_2s_3)r_1 + s_1(s_3r_2 + s_2r_3)}{s_1(s_2s_3)} \\
&= \frac{s_2(s_3r_1) + s_1(s_3r_2) + s_1(s_2r_3)}{s_1(s_2s_3)} \\
&= \frac{s_2(s_3r_1)}{s_1(s_2s_3)} + \frac{s_1(s_3r_2)}{s_1(s_2s_3)} + \frac{s_1(s_2r_3)}{s_1(s_2s_3)} \\
&= \frac{s_2r_1}{s_1s_2} + \frac{s_1r_2}{s_1s_2} + \frac{r_3}{s_3} \text{ (by Lemma 3.1.1)} \\
&= [\frac{s_2r_1 + s_1r_2}{s_1s_2}] + \frac{r_3}{s_3} \\
&= \frac{r_1}{s_1} + [\frac{r_2}{s_2} + \frac{r_3}{s_3}] \\
&= [\frac{r_1}{s_1} + \frac{r_2}{s_2}] + \frac{r_3}{s_3}
\end{aligned}
$$

which implies + is associative. And for any $\frac{r}{s} \in S^{-1}R$, $\frac{r}{s} + \frac{0}{s} = \frac{r}{s} = \frac{0}{s} + \frac{r}{s}$. Hence $\frac{0}{s}$ is an identity element for $S^{-1}R$ with respect to +. Moreover since R is of characteristic 2 and $\frac{r}{s} + \frac{r}{s} = \frac{r+r}{s} = \frac{0}{s}$, every element in $S^{-1}R$ is its own inverse. And ,

$$
\begin{aligned}
\frac{r}{s} + \frac{q}{t} &= \frac{tr + sq}{st} \\
&= \frac{sq + tr}{st} = \frac{s'(sq + tr)}{s'(st)} \text{ , } (s' \in S) \\
&= \frac{s'(sq + tr)}{s'(ts)} = \frac{q}{t} + \frac{r}{s}.
\end{aligned}
$$

Hence we obtain $S^{-1}R$ is an Abelian group.

To show $S^{-1}R$ is a semi group, Let $\frac{r_1}{s_1}, \frac{r_2}{s_2}, \frac{r_3}{s_3} \in S^{-1}R$, then,

$$
\begin{aligned}
\frac{r_1}{s_1}[\frac{r_2}{s_2}\frac{r_3}{s_3}] &= \frac{r_1}{s_1}[\frac{r_2r_3}{s_2s_3}] \\
&= \frac{r_1(r_2r_3)}{s_1(s_2s_3)}
\end{aligned}
$$

$$= \frac{(r_1r_2)r_3}{(s_1s_2)s_3}$$
$$= [\frac{r_1r_2}{s_1s_2}]\frac{r_3}{s_3}$$
$$= [\frac{r_1}{s_1}\frac{r_2}{s_2}]\frac{r_3}{s_3}.$$

Hence, $(S^{-1}R, \cdot)$ is a semi group. Further,

$$\frac{r_1}{s_1}[\frac{r_2}{s_2} + \frac{r_3}{s_3}] = \frac{r_1}{s_1}[\frac{s_3r_2 + r_1(s_2r_3}{s_2s_3}]$$
$$= [\frac{r_1(s_3r_2 + s_2r_3)}{s_1(s_2s_3)}]$$
$$= \frac{r_1(s_3r_2) + r_1(s_2r_3)}{s_1(s_2s_3)}$$
$$= \frac{r_1(s_3r_2)}{s_1(s_2s_3)} + \frac{r_1(s_2r_3)}{s_1(s_2s_3)}$$
$$= \frac{r_1r_2}{s_1s_2} + \frac{r_1r_3}{s_1s_3}$$

Also, for $\frac{r}{s} \in S^{-1}R, \frac{r}{s} + \frac{r}{s} = \frac{r+r}{s} = \frac{0}{s}$. Consequently $S^{-1}R$ is of characteristic 2.

Finally, if $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in S^{-1}R$, , we claim
$[\frac{r_1}{s_1}\frac{r_2}{s_2}] = [\frac{r_1}{s_1}\frac{r_2}{s_2}][\frac{r_1}{s_1} + \frac{r_2}{s_2} + \frac{r_1}{s_1}\frac{r_2}{s_2}] = [\frac{r_1}{s_1}\frac{r_2}{s_2}][\frac{s_2r_1+s_1r_2+r_1r_2}{s_1s_2}]$.

Now let $t \in S$. Then consider;

$t[(s_1s_2)(r_1r_2)(s_2r_1 + s_1r_2 + r_1r_2) + (s_1s_2)(s_1s_2)(r_1r_2)]$

$= t[(s_1s_2)(r_1r_2)(s_2r_1) + (s_1s_2)(r_1r_2)(s_1r_2) +$

$(s_1s_2)(r_1r_2)(r_1r_2) + (s_1s_2)(s_1s_2)(r_1r_2)]$

$= t[(s_1s_2)(r_1r_2)(r_1s_2) + (s_1s_2)(r_1r_2)(s_1r_2) +$

$(s_1s_2)(r_1r_2)(r_1r_2) + (s_1s_2)(r_1r_2)(s_1s_2)]$

$= t(s_1s_2)(r_1r_2)[(r_1s_2 + s_1r_2 + r_1r_2) + (s_1s_2)]$

$= t(s_1s_2)(r_1r_2)[(r_1(s_2 + r_2) + s_1(r_2 + s_2))]$

$= t(s_1s_2)(r_1r_2)[(r_1(s_2 + r_2) + t(s_1s_2)(r_1r_2)s_1(r_2 + s_2))]$

$= t(s_1s_2)(r_1r_2)[(s_2 + r_2)r_1 + t(s_1s_2)(r_1r_2)(r_2 + s_2)s_1]$

$= t(s_1s_2)(r_1r_2)(s_2 + r_2)(r_1 + s_1)$
$= t(s_1s_2r_1)r_2(s_2 + r_2)(r_1 + s_1)$
$= t(s_1r_1s_2)r_2(s_2 + r_2)(r_1 + s_1)$
$= t(s_1r_1)(s_2r_2)(s_2 + r_2)(r_1 + s_1)$
$= t(s_1r_1)(s_1 + r_1)(s_2r_2)(s_2 + r_2)$
$= t[(s_1r_1)s_1 + (s_1r_1)r_1(s_2r_2)s_2 + (s_2r_2)r_2]$
$= t[(s_1^2r_1 + s_1r_1^2)(s_2^2r_2 + s_2r_2^2)]$ by weak commutativity of R,
$= t[s_1r_1 + (s_1r_1)^2][s_2r_2 + (s_2r_2)^2]$ (by Theorem 3.1.1 )
$= 0$ (by Theorem 3.1.2)
Thus we have $\frac{r_1}{s_1}\frac{r_2}{s_2} = \frac{r_1}{s_1}\frac{r_2}{s_2}[\frac{r_1}{s_1} + \frac{r_2}{s_2} + \frac{r_1}{s_1}\frac{r_2}{s_2}]$ □

**Theorem 3.1.6.** $(S^{-1}R, +, \cdot)$ is a Boolean like ring .

*Proof.* Let $\frac{r_1}{s_1}, \frac{r_2}{s_2}, \frac{r}{s} \in S^{-1}R, s \in S$,then

$$\begin{aligned}\frac{r_1}{s_1}\cdot\frac{r_2}{s_2} &= \frac{r_1r_2}{s_1s_2}\\ &= \frac{sr_1r_2}{ss_1s_2}\\ &= \frac{sr_2r_1}{ss_2s_1}\\ &= \frac{r_2r_1}{s_2s_1}\\ &= \frac{r_2}{s_2}\cdot\frac{r_1}{s_1}\end{aligned}$$

This proves $\cdot$ is commutative over $S^{-1}R$. And for any $t \in S$, $\frac{r}{s} = \frac{rt}{st} = \frac{r}{s}\cdot\frac{t}{t} = \frac{t}{t}\cdot\frac{r}{s}$. As a result $\frac{t}{t}, t \in S$ is the unity of $S^{-1}R$. Moreover, right distributive property follows from the left distributive property as well as from the commutative property of '$\cdot$' over $S^{-1}R$.

Finally, for $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in S^{-1}R$,

$$\begin{aligned}
[\frac{r_1}{s_1} \cdot \frac{r_2}{s_2}][\frac{s_1}{s_1} + \frac{r_1}{s_1}][\frac{s_2}{s_2} + \frac{r_2}{s_2}] &= [\frac{r_1}{s_1} \cdot \frac{r_2}{s_2}][(\frac{s_1 + r_1}{s_1})(\frac{s_2 + r_2}{s_2})] \\
&= [\frac{r_1 r_2}{s_1 s_2}][(\frac{s_1 + r_1}{s_1})(\frac{s_2 + r_2}{s_2})] \\
&= [\frac{r_1 r_2}{s_1 s_2}][\frac{(s_1 + r_1)(s_2 + r_2)}{s_1 s_2}] \\
&= \frac{(r_1 r_2)(s_1 + r_1)(s_2 + r_2)}{(s_1 s_2)(s_1 s_2)} \\
&= \frac{s(s_1 s_2)(r_1 r_2)(s_1 + r_1)(s_2 + r_2)}{s(s_1 s_2)(s_1 s_2)(s_1 s_2)} \\
&= \frac{s(s_1 s_2 r_1) r_2 (s_1 + r_1)(s_2 + r_2)}{s(s_1 s_2)(s_1 s_2)(s_1 s_2)} \\
&= \frac{s(s_1 r_1)(s_2 r_2)(s_1 + r_1)(s_2 + r_2)}{s(s_1 s_2)(s_1 s_2)(s_1 s_2)} \\
&= \frac{s(s_1 r_1)(s_1 + r_1)(s_2 r_2)(s_2 + r_2)}{s(s_1 s_2)(s_1 s_2)(s_1 s_2)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{s[(s_1 r_1)s_1 + (s_1 r_1)r_1][(s_2 r_2)s_2 + (s_2 r_2)r_2]}{s(s_1 s_2)(s_1 s_2)(s_1 s_2)} \\
&= \frac{s[(s_1^2 r_1) + s_1 r_1^2][s_2^2 r_2 + s_2 r_2^2]}{s(s_1 s_2)(s_1 s_2)(s_1 s_2)} \\
&= \frac{s[(s_1 r_1) + (s_1 r_1)^2][s_2 r_2 + (s_2 r_2)^2]}{s(s_1 s_2)(s_1 s_2)(s_1 s_2)} \text{ (by Theorem 3.1.1 )} \\
&= \frac{0}{s(s_1 s_2)(s_1 s_2)(s_1 s_2)} \quad \text{(by Theorem 3.1.2)} \\
&= \frac{0}{s_3}, \quad \text{where } s_3 = s(s_1 s_2)(s_1 s_2)(s_1 s_2) \in S.
\end{aligned}$$

Therefore we have that $(\mathrm{S}^{-1}R, +, \cdot)$ is a Boolean like ring. □

## 3.2 Examples

In here , we furnish different Examples that can explain the situation more briefly.

**Example 3.2.1.** *Let* $R = \{0, a, b, c\}$ *with '+' and '·' defined by the following tables.*

| + | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | a | b | c |
| a | a | 0 | c | b |
| b | b | c | 0 | a |
| c | c | b | a | 0 |

| . | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | 0 | a |
| b | 0 | 0 | 0 | 0 |
| c | 0 | c | 0 | c |

Clearly $(R, +, .)$ is a weak commutative Boolean like semiring and if $S = \{a, c\}$ is a multiplicative sub set of R, $S^{-1}R = \{\frac{0}{a} = \frac{0}{c} = \frac{b}{a} = \frac{b}{c}, \frac{a}{a} = \frac{c}{c} = \frac{c}{a} = \frac{a}{c}\}$ is a 2 element Boolean ring which is a special class of a Boolean like ring.

**Example 3.2.2.** *Let* $R = \{0, a, b, c\}$ *with '+' and '·' defined by the following tables.*

| + | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | a | b | c |
| a | a | 0 | c | b |
| b | b | c | 0 | a |
| c | c | b | a | 0 |

| . | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | 0 | a | a |
| b | 0 | 0 | b | b |
| c | 0 | a | b | c |

In this, $(R, +, \cdot)$ is a Boolean like semiring which is not weak commutative. If $S = \{c\}$ , we get $S^{-1}R = \{0 = \frac{0}{c}, \frac{a}{c}, \frac{b}{c}, \frac{c}{c} = 1\}$. In this

Example, even if we define a multiplication operation as in Theorem 3.1.5, we observe that $S^{-1}R$ is not a Boolean like ring of Foster [12] since it is not commutative. For instance,

$$\frac{a}{c} = \frac{a}{c}.\frac{b}{c} \neq \frac{b}{c}.\frac{a}{c} = \frac{0}{c}$$

.

**Example 3.2.3.** *Let $R = \{0, a, b, c, d, e, f, 1\}$. Define '+' and ' $\cdot$ ' on R by the following tables.*

| + | 0 | a | b | c | d | e | f | 1 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | a | b | c | d | e | f | 1 |
| a | a | 0 | c | b | e | d | 1 | f |
| b | b | c | 0 | a | f | 1 | d | e |
| c | c | b | a | 0 | 1 | f | e | d |
| d | d | e | f | 1 | 0 | a | b | c |
| e | e | d | 1 | f | a | 0 | c | b |
| f | f | 1 | d | c | b | c | 0 | a |
| 1 | 1 | f | e | d | c | b | a | 0 |

| $\cdot$ | 0 | a | b | c | d | e | f | 1 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | 0 | 0 | 0 | a | a | a | a |
| b | 0 | 0 | b | b | 0 | 0 | b | b |
| c | 0 | 0 | b | b | a | a | c | c |
| d | 0 | 0 | 0 | 0 | d | d | d | d |
| e | 0 | a | 0 | a | d | e | d | e |
| f | 0 | 0 | b | b | d | d | f | f |
| 1 | 0 | a | b | c | d | e | f | 1 |

In this Example, $(R, +, \cdot)$ is a Boolean like semiring which is not weak commutative since $0 = e(da) \neq e(ad) = a$. Moreover , if $S = \{e\}$ is a multiplicative sub set of R, $S^{-1}R$ will be the four element set $\{\frac{0}{e} = \frac{b}{e}, \frac{e}{e} = \frac{1}{e}, \frac{a}{e} = \frac{c}{e}, \frac{d}{e} = \frac{f}{e}.\}$
But , if we define the multiplication operation on $S^{-1}R$ in the natural (classical) way as in Theorem 3.1.5, we observe that multiplication will not be well defined. For Example, observe that $\frac{d}{e} = \frac{f}{e}$ and $\frac{a}{e} = \frac{c}{e}$ whereas $\frac{a}{e}.\frac{d}{e} = \frac{ad}{e} = \frac{a}{e} \neq \frac{b}{e} = \frac{fc}{e} = \frac{f}{e}.\frac{c}{e}$.

**Remark 3.2.1.**

a. Weak commutativity is a necessary condition to construct Boolean like semiring of fractions.

b. In this method of constructing $S^{-1}R$, so far , it is an open question whether one can get an appropriate Definition for multiplication in the absence of weak commutativity in order that $S^{-1}R$ would be a Boolean like ring or even a Boolean like semiring.

c. The commutativity of multiplication we obtained in Theorem 3.1.6 mainly emanates from the weak commutativity of R.

# Chapter 4

# Ideals in Boolean like semiring of fractions

In this chapter, we extensively study various characterizations of different classes of ideals in Boolean like semiring of fractions. Namely, we study the notions of extended and contracted ideals of R and $S^{-1}R$. Further, we study on different structures of prime ideals and establish conditions in which $I^{ec} = I$.

## 4.1 Extended and Contracted ideals

**Theorem 4.1.1.** *Let $R$ be a weak commutative Boolean like semiring and let $S$ be a multiplicatively closed subset in $R$. Define a map $f : R \longrightarrow S^{-1}R$ by $f(r) = \frac{rs}{s}, r \in R, s \in S$. Then;*

1. *$f$ is a homomorphism,*
2. *If $0 \notin S$ and $S$ contains no zero divisors, then $f$ is a monomorphism.*

*Proof.* $f$ is a well defined map since for any $t, s_1, s_2 \in S$, and $p, q \in$

$R$, with $p = q$ implies $p + q = 0$,

$$\Rightarrow t(s_2s_1(p+q)) = 0,$$
$$\Rightarrow ts_2s_1p = ts_2s_1q,$$
$$\Rightarrow ts_2ps_1 = ts_1qs_2,$$
$$\Rightarrow \frac{ps_1}{s_1} = \frac{qs_2}{s_2},$$
$$\Rightarrow f(p) = f(q).$$

Also,

1. $f(r+q) = \frac{(r+q)s}{s} = \frac{s(r+q)}{s} = \frac{sr}{s} + \frac{sq}{s} = \frac{rs}{s} + \frac{qs}{s} = f(r) + f(q)$. And $f(pq) = \frac{(pq)s}{s} = \frac{(pq)s^2}{s^2} = \frac{ps}{s}.\frac{qs}{s} = f(p).f(q)$.
2. Let $x, y \in R$ such that $f(x) = f(y) \Rightarrow \frac{xs}{s} = \frac{ys}{s} \Rightarrow$ there exists $t \in S$ such that $t(sxs + sys) = 0 \Rightarrow ts^2(x+y) = 0 \Rightarrow x + y = 0$, since S has no zero divisors and $ts^2 \in S$. $\Rightarrow x = y$.

□

**Theorem 4.1.2.** *Let $R$ be a weak commutative Boolean like semiring , $S$ be a multiplicatively closed subset of $R$ and $I$ be an ideal of $R$. Then, the set $S^{-1}I = \{\frac{a}{s} | a \in I, s \in S\}$ is an ideal of $S^{-1}R$.*

*Proof.* Clearly, $\frac{0}{s} \in S^{-1}I$ and hence $S^{-1}I$ is non empty. Let $t_1, t_2 \in S^{-1}I$. Which implies we can write ;
$t_1 = \frac{x}{p}, t_2 = \frac{y}{q}$; for $x, y \in I$, and $p, q \in S$.
$\Rightarrow t_1 + t_2 = \frac{x}{p} + \frac{y}{q} = \frac{qx+py}{pq} \in S^{-1}I \quad since \quad qx + py \in I$
$\Rightarrow t_1 + t_2 \in S^{-1}I$. Hence, $S^{-1}I$ is a sub group of $S^{-1}R$.
And, let $\lambda \in S^{-1}R$ and $\beta \in S^{-1}I$ so that $\lambda = \frac{r}{s}$, and $\beta = \frac{x}{q}$ for some $r \in R, x \in I, s, q \in S$, which implies $\lambda\beta = \frac{r}{s}\frac{x}{q} = \frac{rx}{sq} \in S^{-1}I$ since $rx \in I$ and $sq \in S$. As a result $RI \subseteq I$.
Further, let $\lambda_1, \lambda_2 \in S^{-1}R, \beta \in S^{-1}I$, then

$\lambda_1 = \frac{r_1}{s_1}, \lambda_2 = \frac{r_2}{s_2}, \beta = \frac{x}{q}$, for some $r_1, r_2 \in R, s_1, s_2, q \in S, x \in I$. Consequently,

$$\begin{aligned}
(\lambda_1 + \beta)\lambda_2 + \lambda_1\lambda_2 &= (\frac{r_1}{s_1} + \frac{x}{q})\frac{r_2}{s_2} + \frac{r_1}{s_1}\frac{r_2}{s_2} \\
&= \frac{(qr_1 + s_1x)}{s_1q}\frac{r_2}{s_2} + \frac{r_1r_2}{s_1s_2} \\
&= \frac{(qr_1 + s_1x)r_2}{s_1qs_2} + \frac{r_1r_2}{s_1s_2} \\
&= \frac{s_1s_2[(qr_1 + s_1x)r_2] + [s_1qs_2(r_1r_2)]}{(s_1qs_2)s_1s_2} \\
&= \frac{(s_1s_2)[(qr_1 + s_1x)r_2] + (s_1s_2)(qr_1)r_2}{(s_1s_2)qs_1s_2} \\
&= \frac{(s_1s_2)[(qr_1 + s_1x)r_2 + (qr_1)r_2]}{(s_1s_2)(qs_1s_2)} \\
&= \frac{(qr_1 + s_1x)r_2 + (qr_1)r_2}{(qs_1s_2)} \\
&\in S^{-1}I, \\
&\text{since } (qr_1 + s_1x)r_2 + (qr_1)r_2 \in I, (qs_1s_2) \in S.
\end{aligned}$$

As a result, $S^{-1}I$ is an ideal of $S^{-1}R$. □

**Definition 4.1.1.** *If $I$ is an ideal of a Boolean like semiring $R$ and $f : R \longrightarrow S^{-1}R$ is a homomorphism, the ideal $f(I) = S^{-1}I$, denoted by $I^e$, is called the extension of $I$.*

In view of Theorem 4.1.2, we have the following results;

**Lemma 4.1.1.** *Let $I$ be an ideal of $R$ and $S$ be a multiplicatively closed subset of $R$. Then, $\frac{r}{s} \in S^{-1}I$ if and only if there exists $t \in I$ such that $tr \in I$, for all $r \in R, s \in S$.*

*Proof.* Let $\frac{r}{s} \in S^{-1}I \Rightarrow \frac{r}{s} = \frac{a}{q}, a \in I, q \in S \Rightarrow$ there exists $m \in S$ such that $m[qr + sa = 0] \Rightarrow m(qr) = m(sa) = (ms)a \in I$ , thus choose $t = mq \in S$. Conversely, if $tr \in I$ for some $t \in S, r \in R, \frac{r}{s} = \frac{tr}{ts} \in S^{-1}I$ follows. □

**Theorem 4.1.3.** *Let S be a multiplicative subset of a weak commutative Boolean like semiring R and I and J be ideals of R then,*

1. $I \subseteq J \Rightarrow S^{-1}I \subseteq S^{-1}J$
2. $S^{-1}(I+J) = S^{-1}I + S^{-1}J$
3. $S^{-1}(IJ) = (S^{-1}I)(S^{-1}J)$
4. $S^{-1}(I \cap J) = (S^{-1}I) \cap (S^{-1}J)$
5. $S^{-1}r(I) = r(S^{-1}I)$

*Proof.* 1. Let $\frac{x}{s} \in S^{-1}I$. Then there exists t in S such that $tx \in I \subseteq J$.
$\Rightarrow tx \in J$
$\Rightarrow \frac{tx}{ts} \in S^{-1}J$
$\Rightarrow \frac{x}{s} \in S^{-1}J$
$\Rightarrow S^{-1}I \subseteq S^{-1}J.$

2. Since $I \subseteq I + J$ and $J \subseteq I + J$, and $I + J$ is an ideal, we have $S^{-1}I \subseteq S^{-1}(I+J)$ and $S^{-1}J \subseteq S^{-1}(I+J)$. Thus we have $S^{-1}I + S^{-1}J \subseteq S^{-1}(I+J)$.

To prove the other way containment,

$$\text{let } \frac{a}{s} \in S^{-1}(I+J) \Rightarrow ta \in I+J \text{ for some t in S .}$$
$$\Rightarrow ta = a_1 + a_2 \text{ where } a_1 \in I \text{ and } a_2 \in J$$
$$\Rightarrow \frac{a}{s} = \frac{ta}{ts} = \frac{a_1+a_2}{ts} = \frac{a_1}{ts} + \frac{a_2}{ts}$$
$$\in S^{-1}I + S^{-1}J$$

3. Let $m \in S^{-1}(IJ)$ . Then,

$$\begin{aligned} m &= \frac{x}{s} \text{ for some } x \in IJ \\ &\Rightarrow x = \sum a_i b_i \text{ for some } a_i \in I \text{ and } b_i \in J. \\ &\Rightarrow \frac{a_i}{s} \in S^{-1}I \text{ and } \frac{b_i}{s} \in S^{-1}J \\ &\Rightarrow m = \frac{x}{s} = \frac{\sum a_i b_i}{s} = \sum \frac{a_i b_i}{s} \\ &= \sum \frac{a_i b_i s}{s^2} = \sum (\frac{a_i}{s}).(\frac{b_i}{s}s) \in (S^{-1}I)(S^{-1}J) \end{aligned}$$

Hence $S^{-1}(IJ) \subseteq (S^{-1}I)(S^{-1}J)$.
Now, let $y \in (S^{-1}I)(S^{-1}J)$. Then
$y = \sum \frac{a_i}{s_i}\frac{b_i}{n_i}$ for some $\frac{a_i}{s_i} \in S^{-1}I$ and $\frac{b_i}{n_i} \in S^{-1}J$

$$\begin{aligned} &\Rightarrow y = \sum \frac{a_i b_i}{q_i} \text{ where } q_i = s_i n_i \\ &\Rightarrow y = \frac{\sum a_i b_i}{q} \text{ where } q = \prod q_i \\ &\Rightarrow y \in S^{-1}(IJ) \end{aligned}$$

4. Clearly $S^{-1}(I \cap J) \subseteq (S^{-1}I) \cap (S^{-1}J)$. To prove the other way containment,

$$\begin{aligned} \text{Let } \frac{x}{s} \in (S^{-1}I) \cap (S^{-1}J) &\Rightarrow \frac{x}{s} \in (S^{-1}I) \text{ and } \frac{x}{s} \in (S^{-1}J) \\ &\Rightarrow \exists t_1, t_2 \in S \ni t_1 x \in I, t_2 x \in J \\ &\Rightarrow (t_1 x)t_2 \in I, t_1(t_2 x) \in J \\ &\Rightarrow (t_1 t_2)x \in I, (t_1 t_2)x \in J \\ &\Rightarrow (t_1 t_2)x \in I \cap J \\ &\Rightarrow \frac{x}{s} \in S^{-1}(I \cap J) \end{aligned}$$

5. Let $\frac{x}{s} \in S^{-1}r(I) \Rightarrow tx \in r(I), t \in S. \Rightarrow (tx)^n \in I$ for some natural

number n. $\Rightarrow t^n x^n \in I \Rightarrow \frac{x^n}{s^n} = \frac{t^n x^n}{t^n s^n} \in S^{-1}I \Rightarrow [\frac{x}{s}]^n \in S^{-1}I \Rightarrow \frac{x}{s} \in r(S^{-1}I)$

The other way, let $\frac{x}{s} \in r(S^{-1}I) \Rightarrow [\frac{x}{s}]^k \in S^{-1}I$ for some natural number k $\Rightarrow \frac{x^k}{s^k} \in S^{-1}I \Rightarrow mx^k \in I$ for some m in S. $m^k x^k \in I$ since I is ideal. $\Rightarrow (mx)^k \in I \Rightarrow mx \in r(I) \Rightarrow \frac{mx}{ms} \in S^{-1}r(I) \Rightarrow \frac{x}{s} \in S^{-1}r(I)$

□

**Lemma 4.1.2.** *Let I be an ideal of a weak commutative Boolean like semiring R and S be a multiplicative subset of R. Then, $I^e = S^{-1}R$ if and only if $I \cap S \neq \emptyset$.*

*Proof.* Suppose $I^e = S^{-1}R \Rightarrow \frac{s}{s} \in I^e$ for $s \in S \Rightarrow ts \in I$ for some t in S $\Rightarrow I \cap S \neq \varnothing$ since $ts \in S$ . Conversely, if I$\cap S \neq \varnothing$, *let* s $\in I \cap S$ $\Rightarrow \frac{s}{s} \in I^e$and *let* $\frac{r}{s} \in S^{-1}R \Rightarrow \frac{r}{s} = \frac{r}{s}\frac{s}{s} \in I^e$ *since* I$^e$ is an ideal. □

**Theorem 4.1.4.** *Let J be an ideal of $S^{-1}R$ and $f : R \longrightarrow S^{-1}R$ be a homomorphism which maps $r \rightarrow \frac{rs}{s}$. Then, the set $f^{-1}(J) = \{x \in R | f(x) \in J\}$ is an ideal of R.*

*Proof.* Clearly, $0 \in f^{-1}J$ and hence $f^{-1}(J)$ is non empty. Let $x, y \in f^{-1}(J) \Rightarrow f(x + y) = f(x) + f(y) \in J$ since J is an ideal. Consequently, $x + y \in f^{-1}(J)$ and thus $f^{-1}J$ is a sub group of $R$. Next,let $r \in R, x \in f^{-1}J \Rightarrow f(rx) = f(r)f(x) \in J$ since $f(x) \in J$ and J is an ideal $\Rightarrow rx \in f^{-1}(J)$. Finally, let $x \in f^{-1}(J), r, s \in R \Rightarrow f[(r+x)s+rs] = f((r+x)s+f(rs) = (f(r)+f(x))f(s)+f(r)f(s) \in J$ since J is an ideal and $f(x) \in J$. Hence, $f^{-1}(J)$ is an ideal of R. □

**Definition 4.1.2.** *The ideal $f^{-1}(J)$ is called the contraction of J to*

*$R$ and is denoted by $J^c$.*

We recall the following definition from Y. Yitayew in [48].

**Definition 4.1.3.** *A proper ideal $P$ of a Boolean like semiring $R$ is called prime if $x \in P$ or $y \in P$ whenever $xy \in P$ for $x, y \in R$.*

**Theorem 4.1.5.** *Let $S$ be a multiplicatively closed subset of a weak commutative Boolean like semiring $R$. If $P$ is a prime ideal of $R$ such that $P \cap S = \emptyset$, then $S^{-1}P$ is a prime ideal of $S^{-1}R$.*

*Proof.* Clearly $S^{-1}P \neq S^{-1}R$ and $S^{-1}P$ is an ideal of $S^{-1}R$. Let $t_1, t_2 \in S^{-1}R$ such that $t_1 t_2 \in S^{-1}P$, then $t_1 = \frac{r_1}{s_1}, t_2 = \frac{r_2}{s_2}$, for some $r_1, r_2 \in R, s_1, s_2 \in S$.

$$\begin{aligned}
\frac{r_1}{s_1}\frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2} \in S^{-1}P \Rightarrow \frac{r_1 r_2}{s_1 s_2} &= \frac{a}{s} \text{ for some } \ a \in P, s \in S. \\
&\Rightarrow q[s(r_1 r_2) + (s_1 s_2)a] = 0 \\
&\Rightarrow q[s(r_1 r_2)] = q[(s_1 s_2)a] \in P \\
&\Rightarrow (qs)(r_1 r_2) \in P \\
&\Rightarrow qs \in P \text{ or } r_1 r_2 \in P \\
&\Rightarrow r_1 r_2 \in P \text{ since } qs \in S \text{ and } \ P \cap S = \emptyset \\
&\Rightarrow r_1 \in P \text{ or } \ r_2 \in P. \\
&\Rightarrow \frac{r_1}{s_1} \in S^{-1}P \text{ or } \frac{r_2}{s_2} \in S^{-1}P.
\end{aligned}$$

Thus, $S^{-1}P$ is a prime ideal of $S^{-1}R$. □

**Theorem 4.1.6.** *Let $R$ be a weak commutative Boolean like semiring . If $J$ is a prime ideal of $S^{-1}R$, then $J^c$ is a prime ideal of $R$ with $J^c \cap S = \emptyset$, where $S$ is a multiplicatively closed subset of $R$.*

*Proof.* By Theorem 4.1.4, it follows that $J^c$ is an ideal. Next, we claim that $J^c$ is a proper ideal. For if $J^c$ is not proper, then

$$J^c = R \Rightarrow f(x) \in J, \forall x \in R.$$

Now, s $\in S \Rightarrow f(s) \in J \Rightarrow \frac{ss}{s} = \frac{s^2}{s} \in J \Rightarrow \frac{s^2}{s} = \frac{s^2}{s}\frac{s}{s^2} = \frac{s}{s} \in J$ which contradicts to the hypothesis that $J$ is a proper ideal of $S^{-1}R$. Now , suppose $x, y \in R$ such that

$$\begin{aligned} xy \in J^c &\Rightarrow f(xy) \in J \\ &\Rightarrow f(x)f(y) \in J \\ &\Rightarrow f(x) \in J \text{ or } f(y) \in J, \text{ since J is a prime ideal} \\ &\Rightarrow x \in J^c \text{ or } y \in J^c, \text{ hence } J^c \text{ is a prime ideal} \end{aligned}$$

Finally, for if $J^c \cap S \neq \emptyset$, let t $\in$J$^c \cap S \Rightarrow t \in J^c$ and $t \in S \Rightarrow f(t) \in J$ and $t \in S \Rightarrow \frac{ts}{s} \in J$ for some $s \in S$ and $ts \in S \Rightarrow \frac{s}{s} = \frac{ts}{s}\frac{s}{ts} \in J$, which contradicts the hypothesis that P is prime and hence proper. □

## 4.2 Certain Generalized Prime Ideals

### 4.2.1 Primary and almost primary ideals in Boolean like semiring of fractions

We start with the following definitions;

**Definition 4.2.1.** *A proper ideal P of a Boolean like semiring R is called primary if* $x \in P$ *or* $y^2 \in P$*, whenever* $xy \in P$ *for* $x, y \in R$.

**Definition 4.2.2.** *A proper ideal P of a Boolean like semiring R is called almost primary if* $x \in P$ *or* $y^2 \in P$*, whenever* $xy \in P - P^2$ *for* $x, y \in R$.

**Remark 4.2.1.**

1. A primary ideal is almost primary, but not conversely.
2. In a Boolean like semiring R , for every element a , $a^n = a$ or $a^n = a^2$ or $a^n = a^3$ for any integer $n \geq 4$. Thus, it is appropriate to define primary and almost primary ideals in in Boolean like semirings as in the above way than the ordinary definition of primary or almost primary ideals of rings.

**Theorem 4.2.1.** *Let P be a primary ideal of a Boolean like semiring R and S be a multiplicative subset of R such that $P \cap S = \emptyset$. Then, $S^{-1}P$ is a primary ideal of $S^{-1}R$.*

*Proof.* It is clear that $S^{-1}P$ is a proper ideal of S$^{-1}R$.
$Now,$ let $\frac{r}{s}, \frac{p}{q} \in S^{-1}R$ such that $\frac{r}{s}\frac{p}{q} \in S^{-1}P$.
If $\frac{r}{s} \in S^{-1}P$, then we are through. Otherwise, by Lemma 4.1.1 , $tr \notin P, \forall t \in S$. But,

$$\frac{rp}{sq} = \frac{r}{s}\frac{p}{q} \in S^{-1}P \Rightarrow m(rp) \in P, \text{ for some m in S,}$$
$$\Rightarrow (mr)p \in P$$
$$\Rightarrow p^2 \in P \text{ since } mr \notin P, \text{ and P is primary}$$
$$\Rightarrow \frac{p^2}{q^2} \in S^{-1}P$$
$$\Rightarrow (\frac{p}{q})^2 \in S^{-1}P.$$

Consequently, $S^{-1}P$ is a primary ideal of $S^{-1}R$. □

**Theorem 4.2.2.** *If J is a primary ideal of $S^{-1}R$, then $J^c$ is a primary ideal of R with $J^c \cap S = \emptyset$.*

*Proof.* Let $x, y \in R$ such that xy $\in J^c.Then, Moreover; for if$ J$^c \cap S \neq \emptyset$, let t $\in$J$^c \cap S$

$$\Rightarrow t \in J^c \text{ and } t \in S,$$
$$\Rightarrow f(t) \in J \quad \text{and } t \in S,$$
$$\Rightarrow \frac{ts}{s} \in J \quad \text{for some} \quad s \in S \quad \text{and } ts \in S$$
$$\Rightarrow \frac{s}{s} = \frac{ts}{s}\frac{s}{ts} \in J,$$

which contradicts the hypothesis that P is prime and hence proper

□

**Theorem 4.2.3.** *If Q is a primary ideal of a weak commutative Boolean like semiring R such that Q is disjoint from a multiplicative subset S of R, then* $S^{-1}r(Q)$ *is a prime ideal of* $S^{-1}R$

*Proof.* First we claim $r(Q)$ is prime. Let $xy \in r(Q)$ so that $(xy)^n \in Q$ for some $n \in \mathbb{N}$. Since R is a Boolean like semiring, $n \in \{1, 2, 3\}$.

If $n = 1$, then $xy \in Q \Rightarrow x \in Q$ or $y^2 \in Q$ since Q is primary so that $x \in r(Q)$ or $y \in r(Q)$.

If $n = 2$, then $(xy)^2 \in Q \Rightarrow x^2y^2 \in Q \Rightarrow x^2 \in Q$ or $y^2 = (y^2)^2 \in Q$ since Q is primary so that $x \in r(Q)$ or $y \in r(Q)$ .

If $n = 3$, then $(xy)^3 \in Q \Rightarrow x^3y^3 \in Q \Rightarrow x^3 \in Q$ or $y^2 = (y^3)^2 \in Q$ since Q is primary so that $x \in r(Q)$ or $y \in r(Q)$ .

Hence the radical of a primary ideal is prime.

Now, $S^{-1}r(Q) = r(S^{-1}Q$ by part 5 of Theorem 4.1.3. So, by Theorem 4.2.1 , we have $S^{-1}Q$ is primary.

So, that $S^{-1}r(Q)$ is prime by the first part of this proof. □

**Theorem 4.2.4.** *Let P be an almost primary ideal of a Boolean like semiring R and S be a multiplicative subset of R such that* $P \cap S = \emptyset$.

*Then , $S^{-1}P$ is an almost primary ideal of $S^{-1}R$.*

*Proof.* Let $\frac{r}{s}, \frac{q}{t} \in S^{-1}R$ such that $\frac{r}{s}\frac{q}{t} \in S^{-1}P - [S^{-1}P]^2$. If $\frac{r}{s} \in S^{-1}P$, then we are through. Otherwise,$mr \notin P, \forall m \in S$. But,

$$\begin{aligned}
\frac{r}{s}\frac{q}{t} \in S^{-1}P - [S^{-1}P]^2 \Rightarrow \frac{rq}{st} &\in S^{-1}P - [S^{-1}P]^2 \\
&\Rightarrow \lambda(rq) \in P, \quad \text{for some } \lambda \in S \quad \text{and} \\
&\lambda(rq) \notin P^2 \text{ by Lemma 4.1.1;} \\
&\Rightarrow \lambda(rq) \in P - P^2 \\
&\Rightarrow (\lambda r)q \in P \Rightarrow q^2 \in P \quad \text{since } \lambda r \notin P, \\
&\Rightarrow \frac{q^2}{t^2} \in S^{-1}P, \\
&\Rightarrow [\frac{q}{t}]^2 \in S^{-1}P,
\end{aligned}$$

Thus, $S^{-1}P$ is almost primary. □

**Theorem 4.2.5.** *Let $J$ be an almost primary ideal of $S^{-1}R$ such that $(J^2)^c \subseteq (J^c)^2$, then $J^c$ is an almost primary ideal of R. Moreover $J^c \cap S = \emptyset$.*

*Proof.* In [8], it is shown that $(J^c)^2 \subseteq (J^2)^c$. Thus, combining with the hypothesis we have $(J^c)^2 = (J^2)^c$. Moreover, $J^c$ is an ideal of R. Now, let $x, y \in R$ such that $xy \in J^c - (J^c)^2$. Which implies $xy \in J^c$ and $xy \notin (J^c)^2 = (J^2)^c$ so that $f(xy) = f(x)f(y) \in J$ and $f(x)f(y) \notin J^2$. Consequently we have $f(x) \in J$ or $(f(y))^2 = f(y^2) \in J$ since J is almost primary ideal of R. Which implies $x \in J^c$ or $y^2 \in J^c$. Hence $J^c$ is an almost primary ideal of R.
The second part of the Theorem follows in the same way to Theorem 4.2.2. □

### 4.2.2 Weakly Prime and Weakly primary Ideals in Boolean like semiring of fractions

**Definition 4.2.3.** *A proper ideal $I$ of a Boolean like semiring $R$ is called*

1. *Weakly prime if $0 \neq xy \in I$ implies $x \in I$ or $y \in I$ for $x, y \in R$.*
2. *Weakly primary if $0 \neq xy \in I$ implies $x \in I$ or $y^2 \in I$ for $x, y \in R$.*

**Remark 4.2.2.** *clearly , prime $\Rightarrow$ weakly prime $\Rightarrow$ weakly primary. But the converse need not be true in general.*

**Theorem 4.2.6.** *Let $P$ be a weakly prime ideal of a Boolean like semiring $R$ disjoint from a multiplicative subset $S$ of $R$. Then $S^{-1}P$ is a weakly prime ideal of $S^{-1}R$.*

*Proof.* Clearly $S^{-1}P$ is a proper ideal of $S^{-1}R$. Let P be a weakly prime ideal of R such that $0 \neq \frac{r_1 r_2}{s_1 s_2} \in P^e$. So, $s[r_1 r_2] \in P$ for some $s \in S$ and $t(r_1 r_2) \neq 0$ $\forall t \in S$. Which implies, $0 \neq [sr_1]r_2 \in P$ . Hence $sr_1 \in P$ or $r_2 \in P$. If $sr_1 \in P$ then $\frac{r_1}{s} = \frac{sr_1}{ss} \in S^{-1}P$ and if $r_2 \in P$, then $\frac{r_2}{s_2} \in S^{-1}P$. In any case, the result holds. □

**Theorem 4.2.7.** *Let $S$ be a multiplicative subset of $R$ such that $S$ has no zero divisor. If $J$ is a a weakly prime ideal of $S^{-1}R$, then $J^c$ is a weakly prime ideal of $R$. Moreover $J^c \cap S = \emptyset$.*

*Proof.* Let $x, y \in R$ such that $0 \neq xy \in J^c$. Then, since S has no zero divisor, $t(xy) \neq 0$ for all t in S.Now,

$$\begin{aligned} 0 \neq xy \in J^c &\Rightarrow f(xy) \in J \\ &\Rightarrow 0 \neq f(x)f(y) \in J \end{aligned}$$

since f is a monomorphism by Theorem 4.1.1 $\Rightarrow f(x) \in J$ or $f(y) \in J$ since J is weakly prime ideal,

$\Rightarrow x \in J^c$ or $y \in J^c$, hence the result.

Further, for if $J^c \cap S \neq \emptyset$, let $t \in J^c \cap S \Rightarrow f(t) \in J$ and $t \in S \Rightarrow \frac{t^2}{t} \in J \Rightarrow \frac{t}{t} = \frac{t^2}{t}\frac{t}{t^2} \in J$ which contradicts to the hypothesis that J is weakly prime and hence proper.Consequently $J^c \cap S = \emptyset$ □

**Theorem 4.2.8.** *Let P be a weakly primary ideal of a Boolean like semiring R disjoint from a multiplicative subset S of R. Then $S^{-1}P$ is a weakly primary ideal of $S^{-1}R$.*

*Proof.* Clearly $S^{-1}P$ is a proper ideal of $S^{-1}R$. Let P be a weakly primary ideal of R such that $0 \neq \frac{r_1 r_2}{s_1 s_2} \in S^{-1}P$. So, $s[r_1 r_2] \in P$ for some $s \in S$ and $t(r_1 r_2) \neq 0 \ \forall t \in S$ . Which implies, $0 \neq [sr_1]r_2 \in P$ . Hence $sr_1 \in P$ or $r_2^2 \in P$. If $sr_1 \in P$ then $\frac{r_1}{s} = \frac{sr_1}{ss} \in S^{-1}P$ and if $(r_2)^2 \in P$, then $[\frac{r_2}{s_2}]^2 = \frac{r_2^2}{s_2^2} \in S^{-1}P$. In any case, the result holds. □

**Theorem 4.2.9.** *Let S be a multiplicative subset of R such that S has no zero divisor.If J is a weakly primary ideal of $S^{-1}R$, then $J^c$ is a weakly primary ideal of R. Moreover $J^c \cap S = \emptyset$.*

*Proof.* Let $x, y \in R$ such that $0 \neq xy \in J^c$. Then, since S has no zero divisor, $t(xy) \neq 0$ for all t in S.Now,

$$0 \neq xy \in J^c \Rightarrow f(xy) \in J$$

$$\Rightarrow 0 \neq f(x)f(y) \in J$$

since f is a monomorphism by Theorem 4.1.1

$$\Rightarrow f(x) \in J \text{ or } [f(y)]^2 = f(y)f(y) = f(y^2) \in J$$

since J is weakly primary ideal,

$$\Rightarrow x \in J^c \text{ or } y^2 \in J^c, \text{ hence the result.}$$

Further, for if $J^c \cap S \neq \emptyset$, let $t \in J^c \cap S \Rightarrow f(t) \in J$ and $t \in S \Rightarrow \frac{t^2}{t} \in J \Rightarrow \frac{t}{t} = \frac{t^2}{t}\frac{t}{t^2} \in J$ which contradicts to the hypothesis that J is weakly primary and hence proper.Consequently $J^c \cap S = \emptyset$ □

### 4.2.3 Semiprime and 2-absorbing Ideals of Boolean like semiring of fractions

**Definition 4.2.4.** *A proper ideal I of a Boolean like semiring is called*

1. *semiprime if $x \in I$ whenever $x^2 \in I \forall x \in R$.*
2. *quasi prime if $x \in I$ whenever $x^3 \in I \forall x \in R$.*

**Lemma 4.2.1.** *In Boolean like semiring, an ideal is semiprime if and only if it is quasi prime.*

*Proof.* Let P be a semiprime ideal of a Boolean like semiring R such that $x^3 \in P$ for some x in R. Then $x^2 = (x^3)(x^3) = x^6 = x^2x^4 = x^2x^2 = x^2 \in P \Rightarrow$x $\in P$ (since P is semiprime) Hence P is quasi prime. The other way, let P be a quasi prime ideal such that $x^2 \in P$. Thus, $x^3 = x.x^2 \in P$ (since P is an ideal) Hence $x \in P$ (since P is quasi prime). Therefore, P is semiprime. □

**Theorem 4.2.10.** *If I is a semiprime ideal of a weak commutative Boolean like semiring R disjoint from a multiplicative subset S of R, then $I^e$ is a semiprime ideal of $S^{-1}R$*

*Proof.* Let I be a semiprime ideal and $\frac{r}{s} \in S^{-1}R$ such that $\frac{r}{s} \notin I^e$. Then $tr \notin I \ \forall t \in S$ so that $(tr)^2 = t^2r^2 \notin I$ since I is semiprime. Thus, $\frac{t^2r^2}{t^2s^2} \notin I^e$. Therefore, $\frac{r^2}{s^2} = [\frac{r}{s}]^2 \notin I^e$. □

**Theorem 4.2.11.** *If J is a semiprime ideal of $S^{-1}R$, then $J^c$ is a semiprime ideal of R. Moreover, $J^c \bigcap S = \emptyset$.*

*Proof.* Let x be in R such that $x^2 \in J^c$ . Thus, $f(x^2) \in J$ which implies $f(x)f(x) \in J$. So, $[f(x)]^2 \in J$ . Hence, $f(x) \in J$ since J is semiprime Thus, $x \in J^c$

Further, for if $J^c \cap S \neq \emptyset$, let $t \in J^c \cap S \Rightarrow f(t) \in J$ and $t \in S \Rightarrow \frac{t^2}{t} \in J \Rightarrow \frac{t}{t} = \frac{t^2}{t}\frac{t}{t^2} \in J$ which contradicts to the hypothesis that J is semiprime and hence proper.Consequently $J^c \cap S = \emptyset$ □

**Definition 4.2.5.** *A proper ideal I of R is called 2-absorbing if $ab \in I$ or $bc \in I$ or $ac \in I$ for all a,b and c in R, whenever $abc \in I$.*

**Theorem 4.2.12.** *Let P be a 2-absorbing ideal of R and S be a multiplicatively closed subset of R such that $P \bigcap S = \emptyset$. Then $S^{-1}P$ is a 2-absorbing ideal of $S^{-1}R$.*

*Proof.* Let $\frac{r_1}{s_1}, \frac{r_2}{s_2}, \frac{r_3}{s_3} \in S^{-1}R$ such that $\frac{r_1}{s_1}\frac{r_2}{s_2}\frac{r_3}{s_3} \in S^{-1}P$,

$$\Rightarrow \frac{r_1r_2r_3}{s_1s_2s_3} \in S^{-1}P$$

$$\Rightarrow t(r_1r_2r_3) \in P \text{ for some t in S}$$

$$\Rightarrow (tr_1)r_2r_3 \in P \text{ for some t in S}$$

$$\Rightarrow t(r_1)r_2 \in P, \text{ or } r_2r_3 \in P \text{ or } tr_1r_3 \in P.$$

$$\Rightarrow \frac{r_1r_2}{s_1s_2} \in S^{-1}P \text{ or } \frac{r_2r_3}{s_2s_3} \in S^{-1}P \text{ or } \frac{r_1r_3}{s_1s_3} \in S^{-1}P.$$

Hence in any case the result holds. □

**Theorem 4.2.13.** *If J is a 2-absorbing ideal of $S^{-1}R$, then $J^c$ is a 2-absorbing ideal of R. Moreover, $J^c \bigcap S = \emptyset$.*

*Proof.* Let a,b and c be elements of R such that $abc \in J^c$ and J be a 2-absorbing ideal of $S^{-1}R$.

$\Rightarrow f(abc) = f(a)f(b)f(c) \in J,$

$\Rightarrow f(ab) = f(a)f(b) \in J$ or $f(ac) = f(a)f(c) \in J,$ or $f(bc) = f(b)f(c) \in J.$

$\Rightarrow ab \in J^c$ or $ac \in J^c$ or $bc \in J^c$,

so that $J^c$ is a 2-absorbing ideal of R.
Further, for if $J^c \cap S \neq \emptyset$, let $t \in J^c \cap S \Rightarrow f(t) \in J$ and $t \in S \Rightarrow \frac{t^2}{t} \in J \Rightarrow \frac{t}{t} = \frac{t^2}{t}\frac{t}{t^2} \in J$ which contradicts to the hypothesis that J is 2-absorbing and hence proper.Consequently $J^c \cap S = \emptyset$ □

## 4.3 Structure of Ideals in R and in $S^{-1}R$.

**Theorem 4.3.1.** *Let S be a multiplicative subset of a weak commutative Boolean like semiring. Then, every ideal of $S^{-1}R$ is an extended ideal.*

*Proof.* Let $J$ be an ideal of $S^{-1}R$ and let $J^c = I$. Then we claim $J = I^e$. That is $J = (J^c)^e = J^{ce}$.
Let $y \in I^e$. Then there exists $x \in I = J^c$ such that $f(x) = y$. Which implies, $y = f(x) \in J$. Consequently we have $I^e \subseteq J$.
To prove the other way containment, let $\frac{r}{s} \in J$. Then consider $f(r) = \frac{rs}{s} = \frac{rs^2}{s^2} = \frac{r}{s}\frac{s^2}{s} \in J$ since J is an ideal. Thus we have $f(r) \in J$. Which means $r \in J^c = I$. So that $\frac{r}{s} \in I^e$. Therefore, $J \subseteq I^e$.
So we get $I^e = (J^c)^e = J$ and hence every ideal of $S^{-1}R$ is an extended ideal. □

Now we ask if every ideal of R is a contracted ideal of $S^{-1}R$. But, in the following we observe that this is not always possible and the next two Theorems are devoted to show this.

**Theorem 4.3.2.** *Let I be an ideal of a weak commutative Boolean like semiring R and S be a multiplicative subset of R. Then the set* $\bigcup_{s\in S}(I : s) = \{r \in R | sr \in I, s \in S\}$ *is an ideal of R.*

*Proof.* Clearly $0 \in \cup(I : s)$ and hence $\cup(I : s) \neq \emptyset$. Let $x, y \in \cup(I : s)$ then there exists $t_1, t_2 \in S$ such that $t_1x \in I, t_2y \in I \Rightarrow t_1t_2(x + y) = t_1t_2x + t_1t_2y \in I \Rightarrow x + y \in \cup(I : s)$ (since each summand belongs to I and I is an ideal). Moreover, $r \in R, x \in \cup(I : s) \Rightarrow s(rx) = s(xr) = (sx)r \in I$ for some s corresponding to x. $\Rightarrow rx \in \cup(I : s)$. Finally, if $x \in \cup(I : s), p, q \in R$ with $sx \in I$, $s[(p+x)q+pq)] = (sp+sx)q+s(pq) = (sp+sx)q+(sp)q \in I$, since I is an ideal of R. As a result $\cup(I : s)$ is an ideal of R. □

**Theorem 4.3.3.** *Let I be an ideal of a weak commutative Boolean like semiring R and s be an element of a multiplicative subset S of R. Then,* $I^{ec} = \bigcup_{s\in S}(I : s) = \{r \in R | sr \in I \textit{ for some } s \in S\}$.

*Proof.* Let $r \in I^{ec}$. Then $f(r) = \frac{rs}{s} \in I^e$ for some $s \in S$. Then, $t(rs) \in I$ for some $t \in S$. Hence $(ts)r \in I$ since R is weak commutative. So, $r \in \cup(I : s)$ since $ts \in S$. To prove the other way, let $r \in \cup(I : s)$ so that $sr \in I$ for some s. And $f(r) = \frac{sr}{s} \in I^e$ as a result, $r \in I^{ec}$. Consequently, we have $I^{ec} = \cup_{s\in S}(I : s)$.

**Remark 4.3.1.** In Theorem 4.3.3 we proved that $I^{ec} = \cup_{s\in S}(I : s)$, and clearly $I \subseteq \cup_{s\in S}(I : s)$. Further, we observe that the other way containment need not hold in general. For instance, refering Example 3.2.2, if we take the zero ideal I, then $\cup_{s\in S}(I : s) = \{0, a\}$

which affirms equality need not hold.

So, with this motivation, it will be quite interesting to ask in what conditions would equality of the containment be established? In the next section, we will answer this question by characterizing the ideal I of R.

## 4.4 Conditions for which $\mathbf{I}^{ec} = I$

**Definition 4.4.1.** *A proper ideal I of a Boolean like semiring R is called 2-potent prime if* $x \in I$ *or* $y \in I$ *whenever* $xy \in I^2$ $\forall x, y \in R$.

We have the following results;

**Theorem 4.4.1.** *Let P be a 2-potent prime ideal of a weak commutative Boolean like semiring R and S be a multiplicative subset of R such that* $P \cap S = \varnothing$. *Then* $P^{ec} = P$.

*Proof.* Clearly $P \subseteq P^{ec}$. Let $y \in P^{ec}$. So, $ty \in P$ for some $t \in S$. Hence, $[ty]^2 \in P^2$ which implies $t^2y^2 \in P^2$ so that $t^2 \in P$ or $y^2 \in P$ (since P is 2-potent prime) as a result, $y^2 \in P$ (since $t^2$ is not in P .Hence, $(y^2)^2 = y^2 \in P^2$ .Therefore, $y \in P$. □

**Remark 4.4.1.** Observe that Theorem 4.4.1 may not hold true for a 2-potent prime ideal of a ring. The property of weak idempotency, which is a consequence of the axiom $ab = ab(a+b+ab)$ of a Boolean like semiring, breaks the tie in this proof.

**Corollary 4.4.1.** $P^{ec} = P$ *for every prime ideal P of R disjoint from a multiplicative subset S.*

*Proof.* Follows from Theorem 4.4.1 since every prime ideal is 2-potent prime. □

**Theorem 4.4.2.** *Let P be a weakly primary ideal of a Boolean like semiring R such that $P \cap S = \varnothing$. If P is semiprime and S has no zero divisors, then $P^{ec} = P$.*

*Proof.* Clearly, $P \subseteq P^{ec}$ and let $y \in P^{ec}$. Then there exists some t in S such that $ty \in P$. If $0 \neq ty$, then $y^2 \in P$ since $t \notin P$ and $y \in P$ since P is semiprime. If $0 = ty$, then $0 = y \in P$ since S has no zero divisor. In any case $P^{ec} = P$. □

**Corollary 4.4.2.** *Let P be a weakly prime ideal of a Boolean like semiring R such that $P \cap S = \varnothing$. If S has no zero divisors, then $P^{ec} = P$.*

*Proof.* The result follows from Theorem 4.4.2 since every weakly prime ideal is weakly primary. □

**Theorem 4.4.3.** *Let P be a primary ideal of a Boolean like semiring R such that $P \cap S = \varnothing$. Then $P^{ec} = P$ if P is semiprime.*

*Proof.* Clearly $P \subseteq P^{ec}$. Let $y \in P^{ec}$ .Which implies, $sy \in P$ for some $s \in S$ .Hence, $s \in P$ or $y^2 \in P$ (since I is primary ). So, $y^2 \in P$, since s is not in P. Therefore, $y \in P$ (since P is semiprime). □

**Lemma 4.4.1.** *If Q is a primary ideal of a Boolean like semiring R disjoint from S, then $[r(Q)]^{ec} = r(Q)$*

*Proof.* Since the radical of a primary ideal is prime and every prime ideal is 2-potent prime, the result follows by Theorem 4.4.1. □

**Theorem 4.4.4.** *[**Correspondence Theorem**]* Let S be a multiplicatively closed subset of a weak commutative Boolean like semiring R. Then,

1. There is a one - to - one correspondence between the 2-potent prime ideals of $S^{-1}R$ and the 2-potent prime ideals of R disjoint from S.
2. There is a one - to - one correspondence between the prime ideals of $S^{-1}R$ and the prime ideals of R disjoint from S.
3. If S has no zero divisor, there is a one - to - one correspondence between the weakly primary ideals of $S^{-1}R$ and the weakly primary ideals of R disjoint from S.
4. If S has no zero divisor, there is a one - to - one correspondence between the weakly prime ideals of $S^{-1}R$ and the weakly prime ideals of R disjoint from S.
5. There is a one - to - one correspondence between the primary ideals of $S^{-1}R$ and the primary ideals of R that are semiprime and disjoint from S.

*Proof.* If we define a mapping from the set of 2-potent ,prime , weakly prime , weakly primary or primary ideals of R disjoint from S to the set of respectively 2-potent , prime, weakly prime , weakly primary or primary ideals of $S^{-1}R$ , such a map will be onto by Theorem 4.3.3. Moreover, the one to oneness of the map for each of the statments in order from 1 - 5 follows respectively from Theorem 4.4.1, Corrolary 4.4.1, Theorem 4.4.2, Corrolary 4.4.2 and from Theorem 4.4.3. Consequently the result is true. □

# Chapter 5

# Boolean like semiring Modules

The theory of modules over near rings has been studied by many authors in many ways. Their study was left neaR-modules for right near rings or right neaR-modules for left near rings. But in [16], Gary Ross has introduced the notion of left neaR-modules for left near rings in which he introduced a new line of research that diversified the theory of near ring modules. Imitating the line of thought of Gary Ross, we introduce the notion of left and right modules over Boolean like semirings.

In modules over rings, there is a duality between right and left modules in the sense that one can obtain a similar result in both classes of a module. But, in this study we observe that such a duality can not be obtained in general between the left and right modules. Rather, we observe certain structural differences between left and right modules over Boolean like semirings. Thus, this chapter is devoted to point out some of these structural differences by studying various properties of a left and right modules that can seldom be dually obtained in the other class of module.

Throughout this chapter, where no confusion can arise , we refer by a left (right) R-module we meant to say a left(right) Boolean like semiring module.

## 5.1 Structure of left and right modules

**Definition 5.1.1.** Let R be a Boolean like semiring and (M,+) be an Abelian group. Then M is called ;

1. A left R-module if, for all $m_1, m_2 \in M$ and $r, s \in R$ ,there exists a map $\cdot R \times M \longrightarrow M$ such that;
   (a) $r(m_1 + m_2) = rm_1 + rm_2$, and
   (b) $(rs)m = r(sm)$.
2. A right R-module if, for all $m \in M$ and $r, s \in R$ , there exists a map $\cdot : M \times R \longrightarrow M$ such that,
   (a) $m(r + s) = mr + ms$, and
   (b) $m(rs) = (mr)s$.
3. A subgroup N of a left (right) R-module M is called a submodule of M if $RM \subseteq N(MR \subseteq N)$.
4. A subgroup N of a right R-module M is called an R-ideal of M if $(m + n)r - mr \in N, \forall m \in M, n \in N, r \in R$.
5. A left R-module M is called zero symmetric if $r0_M = 0_M = 0_R m$.

### 5.1.1 Basic properties

We begin with the following,

**Lemma 5.1.1.** Let M be a right R-module. Then ,

1. $m0_R = 0_M, \forall m \in M$;
2. $0_M r = 0_M, \forall r \in R$;
3. Every R-ideal of M is a submodule of M.

*Proof.* 1. $m0_R = m(0_R + 0_R) = m0_R + m0_R \Rightarrow m0_R = 0_M$

2. $0_M r = (m0_R)r = m(0_R r) = m0_R = 0_M$
3. Let N be an R-ideal of M. Then, $\forall n \in N, m \in M, r \in R, (m + n)r - mr \in N$. In particular for $m = 0$, $(m+n)r - mr = nr \in N, \forall n \in N, r \in R$ . Consequently, $NR \subseteq N$. Hence N is a submodule of M.

□

**Remark 5.1.1.**

*1.* The first two properties of Lemma 5.1.1 imitate the zero symmetric nature of Boolean like semirings, and hence we say that a right R-module is zero symmetric. But, in general, the class of left R-modules is not zero symmetric.

*2.* The notion of an R - ideal coincides with a submodule in the case of left R-modules. However, they do not coincide in general on left near ring modules mainly because the modules need not be Abelian groups.

**Definition 5.1.2.** *An R-module M is called of characteristic 2 if* $m + m = 0 \ \forall m \in M$.

**Lemma 5.1.2.** *For a right R-module M and* $m \in M$*, the set* $mR = \{mr/r \in R\} = N$ *is a submodule of M with characteristic 2.*

*Proof.* Clearly N is non empty subgroup of M and $(mr)R \subseteq mR = N$. And for $n = mr \in N$ for some r in R, we have $n+n = mr+mr = m(r+r) = m0 = 0$. Hence N is a submodule of characteristic 2. □

*To substantiate the given definitions and the indicated properties of left and right R-modules, we provide the following Examples;*

**Example 5.1.1.** *Every Boolean like semiring is a module over itself.*

**Example 5.1.2.** *Let $R = \{0, a, b, c\}$ and define '+' and '.' by the following first two tables,and take $M = \{0, 1, 2, 3\}$ and define '+' on M to be addition modulo four and a map $\mu : R \times M \longrightarrow M$ by the last table;*

| + | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | a | 0 | c | b |
| b | b | c | 0 | a |
| c | c | b | a | 0 |

| . | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | 0 | a | a |
| b | 0 | 0 | b | b |
| c | 0 | a | b | c |

| μ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| a | 0 | 1 | 2 | 3 |
| b | 0 | 1 | 2 | 3 |
| c | 0 | 1 | 2 | 3 |

*Thus we obtain a left module M over the Boolean like semiring R and we have the following observations;*

1. For every non zero m in M, $0m \neq 0$. Thus, Example 5.1.2 illustrates that all left R-modules need not be zero symmetric.
2. Observe that characteristic of M is not equal to 2 since $1 + 1 = 3 + 3 = 2 \neq 0$ which indicates all R-modules need not be of characterstic 2.
3. In Lemma 5.1.2 above we have shown that in any right R-module, we can always get a structure of submodule with char-

acterstic 2. Where as in this Example of a left module, observe that the set $N = \{Rm, m \in M\}$ is not even a subgroup because if $m = 3, Rm = \{3\}$ but $3 + 3 = 2 \notin R3$. Further, characterstic of $Rm$ is not equal to 2 since $0 \neq a3 + a3 = 2 = a1 + a1$. As a result, in the class of left modules it is not always possible to get a submodule structure of the type $N = Rm$; which in turn evidences the absence of duality of properties in left and right module structures over R.

4. We also note that the structure of R- modules is distinct from the structure of semi modules (modules over semirings). For instance, in this Example , one of the properties of a left semi module $(a+b)m = am+bm$ is not true in this case and therefore not all R-modules are semi modules. Moreover a semi module need not be an Abelian group and hence not an R-module in general.

**Example 5.1.3.** *Let $R = \{0, a, b, c\}$ and define '+' and '.' as in Example 5.1.2 , and define '+' on M to be addition modulo four and a map $\mu : R \times M \longrightarrow M$ by the following table;*

| $\mu$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | 0 | 0 | 0 |
| b | 0 | 2 | 2 | 0 |
| c | 0 | 2 | 2 | 0 |

*In this Example, we see that M is a zero symmetric left R-module.*

**Example 5.1.4.** *Let $R = \{0, a, b, c\}$ be the Boolean like semiring* *defined in the Example 5.1.2. Let* $M = \{0, 1, 2, 3\}$ *in which* $'+'$ *and* $\mu^{'} : M \times R \longrightarrow M$ *are defined in the following tables as;*

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 0 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 0 |

| $\mu^{'}$ | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 2 | 2 |
| 2 | 0 | 0 | 2 | 2 |
| 3 | 0 | 0 | 3 | 3 |

Clearly M is a zero symmetric right R-module.

In Remark 5.1.1 we have noted that left R-modules are not in general zero symmetric. This indicates that unlike to the theory of modules over rings, left and right modules over Boolean like semirings lack duality . With this motivation, we have further investigated if there are sub structures of left R-modules that possess the duality of right R-modules in the sense of zero symmetry and obtained an affirmative result. The following subsection is devoted to this investigation.

### 5.1.2 Compatible sets in left Boolean like semiring modules

**Definition 5.1.3.** *A left (right) Boolean like semiring module M is said to be of :*

1. *Type I if and only if* $0_R M \neq 0_M (0_M R \neq 0_M)$ ,
2. *Type II if and only if* $0_R M = 0_M (0_M R = 0_M)$.

**Remark 5.1.2.**

1. By Lemma 5.1.1, every right module over a Boolean like semiring is of type II,
2. Example 5.1.2 substantiates there are are left Boolean like semiring modules that are not of type II.

**Definition 5.1.4.** *A subset N of a left R module M is called compatible set if* $0n_1 = 0n_2$ , $\forall n_1, n_2 \in S$.

**Lemma 5.1.3.** *Let M be a left Boolean like semiring module. Define a relation* $\sim$ *on M by* $m_1 \sim m_2$ *if and only if* $0m_1 = 0m_2$. *Then* $\sim$ *is an equivalence relation.*

*Proof.* The proof is straightforward. □

Now consider an equivalence class N of M determined by $\sim$. Then, there exists $n \in M$ such that $0n_i = n$ $,\forall n_i \in N$.

**Remark 5.1.3.**

1. $n \in N$ *since* $0n = 0(0n_i) = (00)n_i = 0n_i = n$,
2. *For any* $r \in R$, $rn = r(0n_i) = (r0)n_i = 0n_i = n$,
3. For any $r \in R, a \in N, 0(ra) = (0r)a = 0a = n$ which implies $ra \in N$. Consequently $RN \subseteq N$.
   But, N need not be closed with '+'. That is,
   $0(n + n) = 0n + 0n = 0n$ need not hold in general for any left R-module..
   So, we cant find a subgroup structure with the induced operation '+' and hence we look for an appropriate operation to be defined on N so that we can get a subgroup structure of R.

*Now, define an operation $*$ on $N$ by $a * b = a + b - n$. Then,*

$$\begin{aligned} 0(a+b-n) &= 0a + 0b - 0n \\ &= n + n - n \\ &= n \end{aligned}$$

*Which implies that $a + b - n \in N$ and clearly $*$ is well defined.*

**Theorem 5.1.1.** *The structure $(N, *)$ is an Abelian group.*

*Proof.* Since $M \neq \emptyset, N$ contains at least a single element and hence $N \neq \emptyset$. Also from part 3 of Remark 5.1.3, we have that $(N, *)$ is a subgroup. Next,

1. Let $a, b, c \in N$ Then,

$$\begin{aligned} (a * b) * c &= (a + b - n) * c \\ &= (a + b - n) + c - n \\ &= a + (b + c - n) - n \\ &= a + (b * c) - n \\ &= a * (b * c) \end{aligned}$$

2. Let $a \in N$ be any element. Then, $a * n = a + n - n = a = n + a - n = n * a$. Hence n is the zero of N.

3. For every $a \in N$ there exists $(n + n - a) \in N$ such that ,

$$\begin{aligned} a * (n + n - a) &= a + (n + n - a) - n \\ &= n = (n + n - a) + a - n \\ &= (n + n - a) * a. \end{aligned}$$

   Thus $(n + n - a)$ is the inverse of a in N.

4. For any a and b in N, $a * b = (a + b - n) = (b + a - n) = b * a$

Thus $(N, *)$ is an Abelian group. □

**Theorem 5.1.2.** *N is a type II left R-module.*

*Proof.* Let $\mu : R \times N \longrightarrow N$ with $(r, a) \mapsto ra$ be the induced map. Then, for r in R and a in N, $0(ra) = (0r)a = 0a = n$ implies $ra \in N$. Moreover, if $r, s \in R$ and $a, b \in N$, then,

$$\begin{aligned} r(a * b) &= r(a + b - n) \\ &= ra + rb - rn \\ &= ra + rb - n \\ &= ra * rb \end{aligned}$$

and (rs)a = r(sa) follows from (3) of Remark 5.1.3 . Hence (N,*) is a module over R.
Further, since $0a = n$, ( which is the zero of N) for all a in N, $0N = n$ and hence N is of type II. □

**Theorem 5.1.3.** *The equivalence classes of M determined by $\sim$ are mutually isomorphic.*

*Proof.* Let N and P be two classes of M determined by $\sim$ with corresponding zero elements n and p respectively.
Define a map $T : N \longrightarrow P$ by $T(a) = a - n + p$. Since

$$\begin{aligned} 0(a - n + p) &= 0a - 0n + 0p \\ &= n - n + p \\ &= p \end{aligned}$$

whch implies $a - n + p \in P$.

Moreover, if $a_1 = a_2$, $T(a_1) = T(a_2)$ follows from the definition of T. Further,

1. Let $a, b \in N$. Then,

$$\begin{aligned} T(a * b) &= T(a + b - n) \\ &= (a + b - n) - n + p \\ &= (a - n + p) + b - n \\ &= (a - n + p) + (b - n + p) - p \\ &= T(a) + T(b) - p \\ &= T(a) * T(b) \end{aligned}$$

   And if $r \in R$ and $a \in N$, then

$$\begin{aligned} T(ra) &= ra - n + p \\ &= ra - rn + rp \quad \text{(by (2) of Remark 5.1.3 )} \\ &= r(a - n + p) \\ &= rT(a). \end{aligned}$$

   Hence T is a module homomorphism.

2. To prove T is one to one, let $T(a) = T(b) \Rightarrow a - n + p = b - n + p \Rightarrow a = b$.

3. Let $y \in P$ be any element. Then there exists $y - p + n \in N$ such that $T(y - p + n) = (y - p + n) - n + p = y$.

As a result, T is onto and hence an isomorphism.
Therefore, any two classes of M determined by $\sim$ are mutually isomorphic.

□

**Theorem 5.1.4.** *Every left module over a Boolean like semiring is a disjoint union of mutually isomorphic R-modules of type II.*

*Proof.* Follows from Lemma 5.1.2 and Theorems 5.1.1, 5.1.2, 5.1.3. □

**Corollary 5.1.1.** *Every R-module is a disjoint union of mutually isomorphic R-modules of type II.*

*Proof.* If M is a left R-module of type I, then the result follows by Theorem 5.1.4. If M is a right R-module, the result follows by Remark 5.1.2. □

## 5.2 Congruence relations and Annihilators

### 5.2.1 Congruence relation

*Let $\sim$ be a relation on the left R-module M with respect to the subgroup N of M. Define $\sim$ on M by $m_1 \sim m_2$ if and only if $m_1 - m_2 \in N$. Then, $\sim$ is an equivalence relation.*

**Definition 5.2.1.** *Let M be a left R-module. A congruence relation $\theta$ on M is an equivalence relation on M satisfying; $(m, m^{'}) \in \theta \Rightarrow (m + n, m^{'} + n) \in \theta$ and $(rm, rm^{'}) \in \theta \forall m, m^{'}, n \in M$ and $\forall r \in R$*

*The following theormem guarantees the existance of a submodule structure in an R-module M.*

**Theorem 5.2.1.** *Let $\theta$ be an equivalence relation $\sim$ on a unitary R-module M and $\Theta = \{(m, m^{'}) \in M \times M$ such that $(m, m^{'}) \in \theta\}$. If $\theta$ is a congruence relation on M, then $\Theta$ is an R submodule of $M \times M$*

*and the maps; $p_1$ and $p_2 : \Theta \longrightarrow M$ defined by $p_1(m, m^{'}) = m$ and $p_2(m, m^{'}) = m^{'}$ are homomorphisms of R submodules.*

*Proof.* Clearly $M \times M$ is a module over R. Since $\theta$ is an equivalence relation, $\Theta$ contains the pair (0,0) and hence it is non empty. And let $x = (m_1, m_2)$ and $y = (m_3, m_4)$ be elements of $\Theta$.
Then by the congruence relation, $(m_1 + m_3, m_2 + m_3) \in \Theta$ and $(m_2 + m_3, m_2 + m_4) \in \Theta \Rightarrow (m_1 + m_3, m_2 + m_4) = x + y \in \Theta$ since $\Theta$ is a congruence relation. Hence $\Theta$ is a subgroup.
Next, let $r \in R, x = (m_1, m_2) \in \Theta$. Since $\Theta$ is a congruence relation, $(rm_1, rm_2) \in \Theta \Rightarrow rx \in \Theta$ so that $\Theta$ is an R submodule of $M \times M$.
Clearly, $p_1, p_2$ are well defined maps. To show $p_1, p_2$ are homomorphisms; let $x = (m_1, m_2), y = (m_3, m_4) \in \Theta$. Then,

$$\begin{aligned} p_1(x + y) &= p_1((m_1 + m_3, m_2 + m_4)) \\ &= m_1 + m_3 \\ &= p_1((m_1, m_2)) + p_1((m_3, m_4)). \end{aligned}$$

And for $r \in R, x = (m_1, m_2) \in \Theta$, we have $p_1(rx) = p_1((rm_1, rm_2)) = rm_1 = rp_1((m_1, m_2)) = rp_1(x)$. Hence $p_1$ is a module homomorphism. Similarly , $p_2$ is also a module homomorphism. □

Thus we observe that there is a one to one correspondence between a congruence relation on a set M due to a subgroup N and the partition $\wp$ of M. In this direction, we have the following result.

**Theorem 5.2.2.** *Let M be an R-module. Let $\sim$ be an equivalence relation on M and $M/\sim$ be the associated partition on M. Then,*

1. *If $\sim$ is a congruence relation on M then $M/\sim$ is an R-module by* $\overline{m} + \overline{m'} = \overline{m + m'}$ *and* $r\overline{m} = \overline{rm}$

2. *If $M/\sim$ is an R-module, and the canonical map, $f : M \longrightarrow M/\sim$ with $m \mapsto \overline{m}$ is a homomorphism of R-modules, then $\sim$ is a congruence relation.*

*Proof.* To prove the the Theorem, we first show that + and . are well defined. Let $\overline{m_1} = \overline{m_1'}$ and $\overline{m_2} = \overline{m_2'}$ , then

$$\begin{aligned} \overline{m_1} + \overline{m_2} &= \overline{m_1 + m_2} \\ &= \overline{m_1' + m_2} \qquad \text{by the congruence relation} \\ &= \overline{m_1' + m_2'} \\ &= \overline{m_1'} + \overline{m_2'}. \end{aligned}$$

And, let $\overline{m_1} = \overline{m_1'}$ and $r_1 = r_2$. Then,

$$\begin{aligned} r_1\overline{m_1} &= \overline{r_1 m_1} \\ &= \overline{r_1 m_1'} \qquad \text{by the congruence relation} \\ &= \overline{r_2 m_1'} \\ &= r_2\overline{m_1'}. \end{aligned}$$

Thus both operations are well defined.
Clearly, $M/\sim$ is an Abelian group. Now,

$$\begin{aligned} r(\overline{m_1} + \overline{m_2}) &= r\overline{m_1 + m_2} \\ &= \overline{rm_1 + rm_2} \\ &= \overline{rm_1} + \overline{rm_2} \\ &= r\overline{m_1} + r\overline{m_2} \end{aligned}$$

and $(rs)\overline{m} = \overline{(rs)m} = \overline{r(sm)} = (r\overline{sm}) = r[s\overline{m}]$ . Hence $M/\sim$ is an R-module.

To prove the second part, let $m_1 \sim m_2, m \in M$, so that $\overline{m_1} = \overline{m_2}$ and which implies $f(m_1) = f(m_2)$. Then,

$$\begin{aligned}\overline{m_1 + m} &= f(m_1 + m)\\ &= f(m_1) + f(m)\\ &= f(m_2) + f(m)\\ &= f(m_2 + m)\\ &= \overline{m_2 + m}\end{aligned}$$

and;
$\overline{rm_1} = f(rm_1) = rf(m_1) = r\overline{m_1} = r\overline{m_2} = \overline{rm_2}$ □

### 5.2.2 Annihilators

In this section we provide certain results that follow from annihilating sets of an R-module M. We start with the following.

**Definition 5.2.2.** *Let M a right R-module . The annihilating set of M , denoted by A(M), is defined as;* $A(M) = \{r \in R | mr = 0 \ \forall m \in M, \}$

*Similarly, the annihilating set of an element m of an R module M is defined as,* $A(m) = \{r \in R | mr = 0, m \in M\}$

**Remark 5.2.1.** Note that $A(0) = R$, which is not the case in general for a right neaR-module. This result is true for a Boolean like semiring module because of the zero symmetric property of Boolean like semirings which in turn emanates from the axiom $ab = ab(a + b + ab)$ for all a and b in R.

*We have the following results,*

**Lemma 5.2.1.** *Let M be a right R-module . Then the set A(M) is a right ideal of R.*

*Proof.* Clearly A(M) contains 0 and hence it is non empty. Let $m \in M$ and $x, y \in A(M)$ hence $mx + my = 0$. Since M is an R-module we have $m(x + y) = 0$ which implies $x + y \in A(M)$. Hence A(M)is a subgroup of R. Next , let $m \in M, x \in A(M)$ and $r, s \in R$, then

$$\begin{aligned} m[(r + x)s + rs] &= m(r + x)s + m(rs) \\ &= (mr + mx)s + m(rs) \\ &= m(rs) + m(rs) \\ &= m(rs + rs) \\ &= m0 = 0 \\ &\Rightarrow (r + x)s + rs \in A(M). \end{aligned}$$

Thus $A(M)$ is a right ideal of R.

**Remark 5.2.2.**

1. If a Boolean like semiring R satisfies the condition $abc = bac \, \forall a, b, c \in R$ which is called the left weak commutative property, then $A(M)$ becomes a left ideal and hence an ideal of R.

   *Proof.* Let $x \in A(M)$ and $r \in R$. Then

   $$\begin{aligned} m(rx) &= m[rx(r + x + rx] \\ &= m(rxr) + m(rxx) + m(rxrx) \\ &= m(xrr) + m(xrx) + m(xrrx) \\ &= (mx)rr + (mx)rx + (mx)rrx \\ &= 0rr + 0rx + 0rrx = 0 \text{ hence } rx \in A(M). \end{aligned}$$

□

2. A right R-module M is called unitary if R is unitary with the right unity element 1 and $m1 = m, \forall m \in M$. If M is a unitary R-module and R is a weak commutative ,then A(M) is also a left ideal and hence an ideal of R.

   *Proof.* Let $x \in A(M)$ and $r \in R$, then $m(rx) = (m1)rx = m(1rx) = m(1xr) = (m1)xr = m(xr) = (mx)r = 0r = 0$. Hence $rx \in A(M)$ □

3. If M is a left R-module, $Ann(M)$ may be empty. Also, let alone to be an ideal, it may not have a subgroup structure.

**Example 5.2.1.** *In Example 5.1.2 above,* $A(M) = \emptyset$

# Chapter 6

# Certain classes of generalized prime submodules

Many authors such as [2, 3, 29, 38] have generalized the concepts of prime, primary, weakly prime, and 2-absorbing generalized prime ideals of commutative rings (also some authors for arbitrary rings) to prime, primary, weakly prime and 2-absorbing modules and submodules over commutative rings or rings. Similar generalizations have also been given in [17, 21, 36] for semi modules of semirings. But, in this line very few has been done on prime and primary modules by S. Juglal and N. J. Groenewald in [39] on modules over near rings. So, in this chapter, we will provide our findings on certain generalized prime sub modules over Boolean like semirings in line to the authors such as [36, 39].

## 6.1 Prime and Weakly Prime submodules

We start with identifying certain ideal structures due to submodules.

**Definition 6.1.1.** *Let $M$ be a right $R$ module and $N$ be a submodule* *of M. Then the quotient of M to N, denoted by* $(M : N)$*, is defined by the set* $(N : M) = \{r \in R | Mr \subseteq N\}$.

**Lemma 6.1.1.** *If N is a submodule of a right R - module M, then* $(N : M)$ *is a right subgroup of R.*

*Proof.* Clearly $0 \in (N : M)$ and hence $(N : M) \neq \emptyset$. Let $r, s \in (N : M)$

$$\Rightarrow mr, ms \in N, \forall m \in M.$$
$$\Rightarrow m(r + s) = mr + ms \in N$$
$$\Rightarrow r + s \in (N : M).$$

And let $r \in R, x \in (N : M)$

$$\Rightarrow Mx \subseteq N$$
$$\Rightarrow Mxr \subseteq Nr \subseteq N$$
$$\Rightarrow Mxr \subseteq N$$
$$\Rightarrow xr \in (N : M)$$

□

**Lemma 6.1.2.** *If N is a submodule of a right R - module M, then* $(N : M)$ *is a left ideal of R.*

*Proof.* Clearly $(N : M)$ is a subgroup of R. Let $x \in (N : M)$ and $r \in R$. Now, $Mr \subseteq M \Rightarrow Mrx \subseteq Mx \subseteq N \Rightarrow rx \in (N : M) \Rightarrow R(N : M) \subseteq (N : M)$ □

**Lemma 6.1.3.** *If N is an R-ideal of a right R module M, then* $(N : M)$ *is an ideal of R.*

*Proof.* Here we prove only the third axiom of an ideal of R. Let $x \in$(N:M). Then, for $r, s \in R, m \in M, m[(r+x)s+rs] = (mr+mx)s+m(rs) = (mr+mx)s+(mr)s \in N$ since N is an R-ideal of M. Which implies $(r+x)s+rs \in (N:M)$ □

**Definition 6.1.2.** *A proper submodule N of M is called prime if for every $r \in R$ and $m \in M$ such that $mr \in N$ implies either $Mr \subseteq N$ (equivalently $r \in (N:M)$) or $m \in N$ and M itself is called prime if 0 is a prime submodule of M.*

**Remark 6.1.1.** *If M is a prime R module, then every submodule of M will be prime.*

**Example 6.1.1.** *A prime ideal of a Boolean like semiring R is prime submodule of R.*

**Example 6.1.2.** *Let $R = \{0, a, b, c\}$ and define '+' and '.' by the following tables,*

| + | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | a | 0 | c | b |
| b | b | c | 0 | a |
| c | c | b | a | 0 |

| . | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | 0 | a | a |
| b | 0 | 0 | b | b |
| c | 0 | a | b | c |

*Clearly $(R, +, .)$ is a Boolean like semiring. Take M = {0,1,2,3} and define '+' on M to be addition modulo four and a map $\mu : M \times R \longrightarrow M$ by the following table;*

| $\mu$ | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 2 | 2 |
| 2 | 0 | 0 | 2 | 2 |
| 3 | 0 | 0 | 0 | 0 |

*In this example, the submodule N = {0,2} of M is a prime submodule.*

**Example 6.1.3.** *Let* $R = \{0, a, b, c\}$ *be the Boolean like semiring defined in example 6.1.2. Let* $M = \{0, 1, 2, 3\}$ *be a module over R in which '+' and* $\mu^{'}$ *are defined as in the following tables ;*

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 0 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 0 |

| $\mu^{'}$ | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 2 | 2 |
| 2 | 0 | 0 | 2 | 2 |
| 3 | 0 | 0 | 3 | 3 |

*The submodule* $N^{'} = \{0, 2\}$ *is not prime submodule since,* $1b = 2 \in N^{'}$ *but neither* $1 \in N^{'}$ *nor* $Mb = \{0, 2, 3\}$ *is contained in* $N^{'}$*.*

**Theorem 6.1.1.** *If N is a prime R - ideal of an R - module M, the set* $(N : M)$ *is also a prime ideal of R.*

*Proof.* Clearly , $(N : M)$ is an ideal of R. Let $x, y \in R$ such that $xy \in (N : M) \Rightarrow m(xy) \in N, \forall m \in M. \Rightarrow (mx)y \in N \Rightarrow mx \in N$ or $y \in (N : M)$.(since N is a prime submodule of M). If $y \in (N : M)$, then we are done. Otherwise, $mx \in N \Rightarrow Mx \subseteq N \Rightarrow x \in (N : M)$. □

**Remark 6.1.2.** *The converse of this Theorem is not true in general.*

**Example 6.1.4.** *Let $R$ and $M$ be as in example 6.1.3 above. Then clearly $M \times M$ is a module over $R$ by component wise addition and multiplication. And if we consider $N = \{(0,0),(2,2)\}$ which is a submodule of $M \times M$, then $(N : M \times M) = \{0, a\}$ which is a Prime ideal of R. Where as, N is not a prime submodule of $M \times M$ since $(2,1)b = (2,2) \in N$ but neither $(2,1) \in N$ nor $b \in (N : M \times M)$*

**Lemma 6.1.4.** *If $N$ is a prime R-ideal of an $R$ - module R, then $KJ \subseteq N \Rightarrow K \subseteq N$ or $J \subseteq (N : M)$ for any submodule $K$ of $M$ and any ideal $J$ of R.*

*Proof.* Let $k \in K, j \in J \Rightarrow kj \in KJ \subseteq N \Rightarrow kj \in N \Rightarrow k \in N$ or $j \in (N : M) \Rightarrow K \subseteq N$ or $J \subseteq (N : M)$ □

**Definition 6.1.3.** *A module M over a Boolean like semiring R is called multiplication module if for every submodule N of M, there exists an ideal $I$ of $R$ such that $N = MI$*

**Example 6.1.5.** *Referring back to examples 6.1.2 and 6.1.3, the module defined in example 6.1.2 is multiplication whereas the module in example 6.1.3 is not multiplication since we cant find an ideal for the submodule $N = \{0, 3\}$ such that $N = MI$.*

**Remark 6.1.3.** *If M is multiplication module such that $N = MI$ for an $R$ -ideal N of M, then $N = MI \subseteq M(N : I) \subseteq N \Rightarrow N = M(N : M)$*

**Theorem 6.1.2.** *Let $M$ be a multiplication module and $N$ be an R-ideal of M. Then N is prime if and only if $(N : M)$ is prime.*

*Proof.* The forward statement holds by Theorem 6.1.1. To prove conversely, let K be any submodule of M and J be any ideal of R such that $KJ \subseteq N$. Since M is multiplication, there exists an ideal I of R such that $K = MI$. Which implies, $M(IJ) = KJ \subseteq N$ so that $IJ \subseteq (N : M)$. So, $I \subseteq (N : M)$ or $J \subseteq (N : M)$. If $J \subseteq (N : M)$ then we are through. Otherwise, $I \subseteq (N : M) \Rightarrow MI \subseteq M(N : M) = N$ □

**Remark 6.1.4.** *If N is a prime R - ideal M, and R is weak commutative, then $\sqrt{(N : M)}$ is a prime ideal of R.*

**Lemma 6.1.5.** *If M is a prime R- module, for any $m \in M$, $(0 : m) = (0 : M)$*

*Proof.* Since M is a prime module, 0 is a prime submodule of M. That is, $mx = 0 \Rightarrow m = 0$ or $x \in (0 : M)$.
Clearly, $(0 : M) \subseteq (0 : m)$. Let $y \in (0 : m) \Rightarrow my = 0 \Rightarrow m = 0$ or $y \in (0 : M)$ In any case, $(0 : m) \subseteq (0 : M)$ □

**Theorem 6.1.3.** *N is a prime submodule of an R module M if and only if $N \times M$ is a prime R submodule of $M \times M$.*

*Proof.* ($\Longrightarrow$) Let $m_1, m_2 \in M, r \in R$ such that $(m_1, m_2)r \in N \times M$

$$\begin{aligned}
&\Rightarrow (m_1 r, m_2 r) \in N \times M \\
&\Rightarrow m_1 r \in N \\
&\Rightarrow m_1 \in N \text{ or } r \in (N : M) \\
&\Rightarrow m_1 \in N \text{ or } Mr \subseteq N \\
&\Rightarrow m_1 \in N \text{ or } Mr \times Mr \subseteq N \times N \subseteq N \times M \\
&\Rightarrow (M \times M)r \subseteq N \times M \\
&\Rightarrow r \in (N \times M : M \times M)
\end{aligned}$$

($\Longleftarrow$) Let $m \in M, r \in R$ such that $mr \in N$

$$\Rightarrow (mr, m^{'}r) \in N \times M \text{ for any } m^{'} \in M$$
$$\Rightarrow (m, m^{'})r \in N \times M$$
$$\Rightarrow (m, m^{'}) \in N \times M \text{ or } r \in (N \times M : M \times M)$$
$$\Rightarrow m \in N \text{ or } rM \subseteq N$$
$$\Rightarrow m \in N \text{ or } r \in (N : M)$$

□

**Remark 6.1.5.** *The following is an easy consequence of Theorem 6.1.3 for an R module M and a prime submodule N hence given with out proof;*

**Corollary 6.1.1.** *$M \times N$ is a prime $R$ submodule of $M \times M$.*

**Theorem 6.1.4.** *Let $N_1$ and $N_2$ be submodules of an $R$ module $M$ . Then, $N_1$ and $N_2$ are prime submodules of $M$ whenever $N_1 \times N_2$ is prime submodule of $M \times M$.*

*Proof.* 1. Let $m_1, m_2 \in M; r, s \in R$ such that $m_1 r \in N_1, m_2 s \in N_2$. Then, $(m_1 r, n_2 r) = (m_1, n_2)r \in N_1 \times N_2$ for any $n_2 \in N_2$. So, by primeness of $N_1 \times N_2$, we get $(m_1, n_2) \in N_1 \times N_2$ or $r \in (N_1 \times N_2 : M \times M)$. Consequently , $m_1 \in N_1$ or $r \in (N_1 : M)$ which implies $N_1$ is prime. Similarly, we get $N_2$ is prime.

**Corollary 6.1.2.** *$N$ is a prime submodule of $M$ if and only if $N \times N$ is a prime submodule of $M \times M$.*

*Proof.* ($\Longrightarrow$) Suppose N is prime and let $(m_1, m_2)r \in N \times N$, for $m_1, m_2 \in M$ and $r \in R$. Then we have $m_1 r \in N$ and $m_2 r \in N$ so that $m_1 \in N$ or $r \in (N : M)$ and $m_2 \in N$ or

$r \in (N : M)$. If $m_1, m_2 \in N$ or $r \in (N : M)$ we are through. Also in any of the remaining cases at least $r \in (N : M)$ and hence $r \in (N \times N : M \times M)$ as required. Hence $N \times N$ is prime. The converse follows from Theorem 6.1.4. □

**Definition 6.1.4.** *A proper submodule $N$ of an $R$ - module $M$ is called weakly prime if for $m \in M$ and $r \in R$, $0 \neq mr \in N$ implies $m \in N$ or $r \in (N : M)$*

**Example 6.1.6.** *The zero submodule of any module over a Boolean like semiring is weakly prime.*

**Remark 6.1.6.** *Every prime submodule is weakly prime.*

**Definition 6.1.5.** *An element $m$ of an $R$ module $M$ is called torsion free if $ann(m) = \{0\}$ and $M$ is called torsion free if every element of $M$ is tosion free.*

**Theorem 6.1.5.** *If $M$ is a torsion free R-module, a submodule $N$ is weakly prime if and only if $N$ is prime.*

*Proof.* ($\Longrightarrow$) Suppose N is a weakly prime submodule of M such that $mx \in N, m \in M, x \in R$. If $mx \neq 0$ then we are done. Otherwise, $mx = 0$ implies $x = 0$ follows by the hypothesis that M is torsion free. Which implies $x \in (N : M)$, hence the result.

($\Longleftarrow$) The converse is always true since every prime submodule is weakly prime. □

## 6.2 Semiprime and Primary Submodules

**Definition 6.2.1.** *A submodule $N$ of an $R$ module $M$ is called semiprime if $mr \in N$ whenever $mr^2 \in N$ for $m \in M, r \in R$*

**Remark 6.2.1.** *Every prime submodule is semiprime but not conversly.*

**Example 6.2.1.** *In example 6.1.3,* $N' = \{0, 2\}$ *is semiprime but not prime.*

The following are easy consequences of Definition 6.2.1.

**Theorem 6.2.1.** *If* $N$ *is a semiprime* $R$*-ideal of* $M$*. Then* $(N : M)$ *is a semiprime ideal of* $R$*.*

**Remark 6.2.2.** *The converse of Theorem 6.2.1 is not true in general. The following example illustrates this.*

**Example 6.2.2.** *Consider* $R = \{0, a, b, c\}$ *in which addition and multiplication are defined as; Let* $R = \{0, a, b, c\}$ *and define '+' and '.' by the following tables,*

| + | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | a | 0 | c | b |
| b | b | c | 0 | a |
| c | c | b | a | 0 |

| . | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | 0 | 0 | 0 |
| b | 0 | 0 | b | b |
| c | 0 | 0 | b | b |

*Clearly* $(R, +, \cdot)$ *is a Boolean like semiring. And take* $M$ *as a module over itself. Consider the ideal* $I = \{0\}$ *which is not semiprime since* $a^2 = 0 \in I$ *but* $a \notin I$*. But,* $(I : R) = (0 : R) = \{0, a\}$ *is a semiprime ideal of* $R$*.*

**Lemma 6.2.1.** *If* $N$ *is a semiprime* $R$*-ideal of* $M$ *and* $R$ *is weak commutative, then* $\sqrt{(N : M)}$ *is a semiprime ideal of* $R$*.*

**Theorem 6.2.2.** $N_1$ *and* $N_2$ *are semiprime submodules of* $M$ *if and only if* $N_1 \times N_2$ *is semiprime submodule of* $M \times M$*.*

*Proof.* $(\Longrightarrow)$ Let $(m_1, m_2) \in M \times M$ and $r \in R$ such that

$(m_1, m_2)r^2 \in N_1 \times N_2$

$\Rightarrow (m_1 r^2, m_2 r^2) \in N_1 \times N_2$

$\Rightarrow m_1 r^2 \in N_1, m_2 r^2 \in N_2$

$\Rightarrow m_1 r \in N_1, m_2 r \in N_2$ since $N_1$ and $N_2$ are semiprime,

$\Rightarrow (m_1 r, m_2 r) \in N_1 \times N_2$

$\Rightarrow (m_1, m_2)r \in N_1 \times N_2$. Hence $N_1 \times N_2$ is semiprime.

($\Longleftarrow$) Let $m_1, m_2 \in M$ and $r_1, r_2 \in R$ such that $m_1 r_1^2 \in N_1$, and $m_2 r_2^2 \in N_2$

$\Rightarrow (m_1 r_1^2, n r_1^2) \in N_1 \times N_2$ for some $n \in N_2$

$\Rightarrow (m_1, n) r_1^2 \in N_1 \times N_2$

$\Rightarrow (m_1, n) r_1 \in N_1 \times N_2$ since $N_1 \times N_2$ is semiprime,

$\Rightarrow (m_1 r_1, n r_1) \in N_1 \times N_2$

$\Rightarrow m_1 r \in N_1$. Hence $N_1$ is semiprime. Similarly, $N_2$ is semiprime.

□

**Definition 6.2.2.** *A proper submodule N of an R - module M is called primary if $m \in N$ or $x \in \sqrt{(N:M)}$ whenever $mx \in N$ for any $m \in M, x \in R$*

**Theorem 6.2.3.** *If N is a primary R-ideal of M, then (N:M) is a primary ideal of R.*

*Proof.* Let $xy \in (N:M) \Rightarrow m(xy) \in N \Rightarrow (mx)y \in N \Rightarrow mx \in N$ or $y \in \sqrt{(N:M)} \Rightarrow y \in (N:M)$ or $x \in (N:M)$.In any case , we are through. □

**Remark 6.2.3.** *(a) Every prime submodule is primary but not conversely. There are examples of primary ideals that are* *not prime and they can be considered as submodules of R as a module over itself.*

(b) *The converse of Theorem 6.2.3 is not true in general.*

**Example 6.2.3.** *In example 6.1.4, $(N : M)$ is primary but $N$ is not primary since $(2,1)b \in N$ but neither $(2,1) \in N$ nor $b \in (N : M \times M.) = \{0, a\}$.*

**Lemma 6.2.2.** *$N$ is primary if and only if $r^2 \in (N : M)$ whenever $mr \in N$ and $m \in M \setminus N$.*

*Proof.* ($\Longrightarrow$) Let $mr \in N \Rightarrow r \in \sqrt{(N : M)} \Rightarrow r^n \in (N : M), n \in \{1, 2, 3\}$ since every element r of R has such a form. $\Rightarrow r \in (N : M)$ or $r^2 \in (N : M)$ or $r^3 \in (N : M)$. In any case the result holds and the converse statement holds trivially. □

From example 6.1.4 above, we observe that N is not semiprime even if $(N : M)$ is semiprime, ideal N of M. Thus we give the following characterizations of N .

**Theorem 6.2.4.** *Let N be an R-ideal of an R module M. If $(N : M)$ is semiprime ideal of R, then N is prime if and only if N is primary.*

*Proof.* The forward proof is trivial. To prove the converse, let $mr \in N$ for some $m \in M, r \in R$. If $m \in N$, we are done. Otherwise, $r^2 \in (N : M) \Rightarrow r \in (N : M)$ (since $(N : M)$ is semiprime). Hence the result. □

**Corollary 6.2.1.** *If N is a semiprime submodule, then N is prime if and only if N is primary.*

**Theorem 6.2.5.** *If N is primary R-ideal then, N is prime if* *and only if $(N : M)$ is semiprime.*

*Proof.* The forward implication always holds true. To prove the converse, suppose $mr \in N$, which implies $m \in N$ or $r^2 \in (N : M)$ so that $m \in N$ or $r \in (N : M)$ since $(N : M)$ is semiprime. Hence N is prime. □

The following are consequences of Theorem 6.2.5.

**Corollary 6.2.2.** *(a) If N is primary R-ideal then, N is prime if and only if $(N : M)$ is prime.*

*(b) If N is primary R-ideal then, N is prime if and only if N is semiprime.*

## 6.3 2-absorbing submodules

**Definition 6.3.1.** *A proper submodule N of M is called 2-absorbing if for $r, s \in R, m \in M$ such that $m(rs) \in N$ implies $rs \in (N : M)$ or $mr \in N$ or $ms \in N$.*

**Remark 6.3.1.** *Every prime submodule is 2-absorbing but not conversely For instance;*

**Example 6.3.1.** *Take R and M as in Example 6.1.3 and consider the submodule $N = \{0, 2\}$ which is a 2- absorbing submodule but not prime.*

**Theorem 6.3.1.** *If N is a 2-absorbing R-ideal of M, then $(N : M)$ is also a 2 - absorbing ideal of R.*

*Proof.* Let $x, y, z \in R$ such that $xyz \in (N : M)$. Then for $m \in M, m(xyz) in N \Rightarrow (mx)yz \in N \Rightarrow yz \in (N : M)$ or $m(xy) \in N$. or $m(xz) \in N$ (since N is 2-absorbing.) If $yz \in (N :$

$M$) then we are done. Otherwise, $m(xy) \in N$ or $m(xz) \in N$. $\Rightarrow$xy $\in (N : M)$ or $xz \in (N : M)$. □

**Corollary 6.3.1.** *If N is a 2-absorbing R-ideal of a weak commutative R - module M, then $\sqrt{(N : M)}$ is also a 2-absorbing ideal of an R .*

*Proof.* Let $xyz \in \sqrt{(N : M)} \Rightarrow (xyz)^n \in (N : M)$ for some $n \in \{1, 2, 3\}$, since every element a in a Boolean like semiring is in the form of $a$ or $a^2$ or $a^3$.
If $n = 1$, the result follows since $(N : M)$ is 2-absorbing. Let $(xyz)^2 \in (N : M) \Rightarrow x^2y^2z^2 \in (N : M)$
$\Rightarrow x^2y^2 \in (N : M)$ or $y^2z^2 \in (N : M)$ or $x^2z^2 \in (N : M)$,
$\Rightarrow (xy)^2 \in (N : M)$ or $(xz)^2 \in (N : M)$ or $(yz)^2 \in (N : M)$
$\Rightarrow xy \in \sqrt{(N : M)}$ or $xz \in \sqrt{(N : M)}$ or $yz \in \sqrt{(N : M)}$
For n=3, $(xyz)^3 \in (N : M) \Rightarrow (xyz)^2 = (xyz)^4 = (xyz)^3(xyz) \in (N : M)$ since $(N : M)$ is an ideal of R. Hence the result follows by the case $n = 2$. □

**Theorem 6.3.2.** *Let $N_1, N_2$ be submodules of an R module M . If $N_1, N_2$ are prime submodules then, $N_1 \times N_2$ will be 2-absorbing.*

*Proof.* Let $(m_1, m_2)rs \in N_1 \times N_2$ where $m_1, m_2 \in M; r, s \in R$. Then, $(m_1rs, m_2rs) \in N_1 \times N_2$ so that $m_1rs \in N_1$ and $m_2rs \in N_2$. Which implies, $m_1 \in N_1$ or $rs \in (N_1 : M)$ and $m_2 \in N_2$ or $rs \in (N_2 : M)$ (by primness of $N_1$ and $N_2$) Thus, we obtain the following cases ;

(a) $m_1 \in N_1$ and $m_2 \in N_2$; or

(b) $m_1 \in N_1$ and $rs \in (N_2 : M)$; or

(c) $rs \in (N_1 : M)$ and $m_2 \in M_2$; or

(d) $rs \in (N_1 : M)$ and $rs \in (N_2 : M)$. For the first and last cases, the result follows trivially. From the second case it follows that, $m_1 \in N_1$ and $r \in (N_2 : M)$ or $s \in (N_2 : M)$ since $(N_2 : M)$ is a prime ideal. Hence we have $m_1 r \in N_1$ and $m_2 r \in N_2$ or $m_1 s \in N_1$ and $m_2 s \in N_2$. Thus $(m_1, m_2)r \in N_1 \times N_2$ or $(m_1, m_2)s \in N_1 \times N_2$. Therefore $N_1 \times N_2$ is 2-absorbing submodule of $M \times M$. And similarly, the result also holds for the third case.

□

# Chapter 7

# Modules of Fractions

In this chapter we introduce a method of constructing modules of fractions for a left Boolean like semiring module M and study structures of submodules of the fractions. Even though we imitate the method followed to construct modules of fractions over commutative rings, due to the lack of commutative property as well as right distributive property in Boolean like semirings, much of the results demanded a different technique and additional identities.

On the other hand, we have observed that a similar method can not lead to get right modules of fractions. This is mainly due to the lack of right distributive property of R. We begin with the following,

## 7.1 Construction of Modules of Fractions

**Theorem 7.1.1.** *Let S be a multiplicative subset of a weak commutative Boolean like semiring R. Define a relation* $\sim$ *on* $S \times M$ *by;* $(s_1, m_1) \sim (s_2, m_2)$ *if and only if* $t[s_2m_1] = t[s_1m_2]$ *for some* *t in S. Then $\sim$ is an equivalence relation.*

*Proof.* Clearly $\sim$ is reflexive and symmetric. To prove $\sim$ is transitive, let $(s_1, m_1) \sim (s_2, m_2)$ and $(s_2, m_2) \sim (s_3, m_3)$. Which implies $t_1[s_2m_1] = t_1[s_1m_2]$ and, $t_2[s_3m_2] = t_2[s_2m_3]$ $for some$ $t_1, t_2 \in S$. We claim there exists some t in S such that $t[s_3m_1] = t[s_1m_3]$
If we choose $t = t_1^2t_2s_2 \in S$ , then,

$$\begin{aligned}
t(s_3m_1) &= (t_1^2t_2s_2)(s_3m_1) \\
&= (t_1t_1t_2)(s_2s_3m_1), \text{by associativity of R;} \\
&= t_1(t_2t_1)(s_3s_2m_1) = (t_1t_2s_3)(t_1s_2m_1), \text{by WC of R;} \\
&= (t_1t_2s_3)(t_1s_1m_2), \text{by hypothesis;} \\
&= t_1(t_2s_3)(t_1s_1m_2) \\
&= (t_1t_1s_1)(t_2s_3m_2) \\
&= (t_1^2s_1)(t_2s_2m_3), \text{by hypothesis;} \\
&= (t_1^2t_2s_2)(s_1m_3) \\
&= t(s_1m_3)
\end{aligned}$$

□

We denote the equivalence class containing (s,m) by $\frac{m}{s}$ and the set of all equivalence classes by $S^{-1}M$. The following are easy consequences of the above theorem and we have stated all without proof.

**Lemma 7.1.1.** *Let M be a left Boolean like semiring module,then*

*(a)* $\frac{m}{s} = \frac{s'm}{s's}$ *for all* $m \in M, s', s \in S$

*(b)* $\frac{m}{s} = \frac{s'm}{ss'}$ $\forall m \in M, s', s \in S$

(c) $\frac{sm}{s} = \frac{s^{'}m}{s^{'}}$ $\forall m \in M, s^{'}, s \in S$

(d) $\frac{(s_1 s_2)m}{s} = \frac{(s_2 s_1)m}{s}$ $\forall m \in M, s_1, s_2 \in S$

(e) $\frac{m}{s_1 s_2} = \frac{m}{s_2 s_1}$ $\forall m \in M, s_1, s_2 \in S$

(f) $\frac{0}{s} = \frac{0}{s^{'}}$ $\forall s^{'}, s \in S$

(g) $\frac{m}{s} = \frac{0}{s^{'}}$ *if and only if* $tm = 0$ *for some* $t$ *in* $S$.

(h) *For all* $r \in R$ *and* $m \in M$, $r(-m) = r(0 - m) = r0 - rm = -rm$

**Theorem 7.1.2.** *Let $S$ be a multiplicative set in a weak commutative Boolean like semiring $R$ and $M$ be a left $R$-module . Define the operations '+': $S^{-1}M \times S^{-1}M \to S^{-1}M$ and '.' : $S^{-1}R \times S^{-1}M \to S^{-1}M$ by;* $\frac{m_1}{s_1} + \frac{m_2}{s_2} = \frac{s_2 m_1 + s_1 m_2}{s_1 s_2}$; $\frac{r}{s} . \frac{m}{t} = \frac{rm}{st}$ *Then $S^{-1}M$ is an $S^{-1}R$ module .*

*Proof.* we prove in two steps as follows;

(a) First we prove '+' is well defined. Let $\frac{m_1}{s_1} = \frac{m_1^{'}}{s_1^{'}}$ and $\frac{m_2}{s_2} = \frac{m_2^{'}}{s_2^{'}}$ Which implies $t_1(s_1^{'} m_1) = t_1(s_1 m_1^{'})$ and $t_2(s_2^{'} m_2) = t_2(s_2 m_2^{'})$ for some $t_1, t_2 \in S$. We claim $\frac{s_2 m_1 + s_1 m_2}{s_1 s_2} = \frac{s_2^{'} m_1^{'} + s_1^{'} m_2^{'}}{s_1^{'} s_2^{'}}$ .
Now choose $t = t_1^2 t_2 \in S$ then,

$$\begin{aligned}
& t(s_1^{'} s_2^{'})[s_2 m_1 + s_1 m_2] \\
& = (t_1^2 t_2)(s_1^{'} s_2^{'})(s_2 m_1) + (t_1^2 t_2)(s_1^{'} s_2^{'})(s_1 m_2) \\
& = (t_1 t_2)(t_1 s_1^{'})(s_2^{'} s_2) m_1 + t_1^2 (t_2 s_1^{'})(s_2^{'} s_1) m_2 \\
& = t_1(t_2 s_2^{'} s_2) t_1(s_1^{'} m_1) + t_1^2 (s_1^{'} s_1) t_2(s_2^{'} m_2) \\
& = (t_1^2 t_2)(s_1 s_2)(s_2^{'} m_1^{'}) + (t_1^2 t_2)(s_1 s_2)(s_1^{'} m_2^{'}) \\
& = (t_1^2 t_2)(s_1 s_2)[(s_2^{'} m_1^{'}) + (s_1^{'} m_2^{'}) \\
& = t(s_1 s_2)[s_2^{'} m_1^{'} + s_1^{'} m_2^{'}].
\end{aligned}$$

With the given definition of '+', we have

**Remark 7.1.1.** $\frac{m_1}{s} + \frac{m_2}{s} = \frac{m_1+m_2}{s}$

To prove '+' is associative, let $\frac{m_1}{s_1}, \frac{m_2}{s_2}, \frac{m_3}{s_3} \in S^{-1}M$, then

$$\begin{aligned}
\frac{m_1}{s_1} + [\frac{m_2}{s_2} + \frac{m_3}{s_3}] &= \frac{m_1}{s_1} + [\frac{s_3m_2 + s_2m_3}{s_2s_3}] \\
&= \frac{s_2s_3m_1 + s_1s_3m_2 + s_1s_2m_3}{s_1s_2s_3} \\
&= \frac{s_2s_3m_1 + s_1s_3m_2}{s_1s_2s_3} + \frac{s_1s_2m_3}{s_1s_2s_3} \\
&= \frac{s_2m_1 + s_1m_2}{s_1s_2} + \frac{m_3}{s_3} \\
&= [\frac{m_1}{s_1} + \frac{m_2}{s_2}] + \frac{m_3}{s_3}
\end{aligned}$$

Moreover, $\frac{0}{s}, s \in S^{-1}M$ is the identity element , and for every element $\frac{m}{s} \in S^{-1}M$, there is a corresponding element $\frac{-m}{s} \in S^{-1}M$, such that $\frac{-m}{s} + \frac{m}{s} = \frac{0}{s}$. Hence every element is invertible. Further,

$$\begin{aligned}
\frac{m_1}{s_1} + \frac{m_2}{s_2} &= \frac{s_2m_1 + s_1m_2}{s_1s_2} \\
&= \frac{s_1m_2 + s_2m_1}{s_1s_2} \\
&= \frac{s_1m_2}{s_1s_2} + \frac{s_2m_1}{s_1s_2} \\
&= \frac{m_2}{s_2} + \frac{m_1}{s_1}
\end{aligned}$$

Hence, $(S^{-1}M, +)$ is an Abelian group.

(b) Next we prove $S^{-1}M$, is an $S^{-1}R$ module. To show '.' is well defined, Let $\frac{r_1}{s_1} = \frac{r_2}{s_2}$ and $\frac{m}{s} = \frac{m^{'}}{s^{'}}$ for some $r_1, r_2 \in R$ , $m, m^{'} \in M$ and $s, s_1, s_2, s^{'} \in S$. As a result, there exist elements $t_1, t_2 \in S$ such that $t_1(s_2r_1) = t_1(s_1r_2)$ and $t_2(s^{'}m) = t_2(sm^{'})$.

Now choose $t = t_1^2 t_2 \in S$ then,

$$\begin{aligned} t[(s_2 s^{'})r_1 m] &= (t_1^2 t_2)(s_2 s^{'})(r_1 m) \\ &= t_1(t_2 s^{'})[t_1(s_2 r_1)]m \\ &= t_1(t_2 s^{'})[t_1(s_1 r_2)]m \\ &= t_1[t_1(s_1 r_2)]t_2(s^{'})m \\ &= t_1[t_1(s_1 r_2)]t_2(sm^{'}) \\ &= t_1^2 t_2(s_1 r_2)(sm^{'}) \\ &= (t_1^2 t_2)(s_1 s)(r_2 m^{'}) \\ &= t[(s_1 s)(r_2 m^{'})]. \end{aligned}$$

Hence '.' is well defined. Next;

i. let $\frac{r}{s} \in S^{-1}R$ and $\frac{m_1}{s_1}, \frac{m_2}{s_2} \in S^{-1}M$, then

$$\begin{aligned} \frac{r}{s}[\frac{m_1}{s_1} + \frac{m_2}{s_2}] &= \frac{r}{s}[\frac{s_2 m_1 + s_1 m_2}{s_1 s_2}] \\ &= \frac{r[s_2 m_1 + s_1 m_2]}{s s_1 s_2} \\ &= \frac{r s_2 m_1}{s s_1 s_2} + \frac{r s_1 m_2}{s s_1 s_2} \\ &= \frac{r m_1}{s s_1} + \frac{r m_2}{s s_2} \\ &= \frac{r}{s}\frac{m_1}{s_1} + \frac{r}{s}\frac{m_2}{s_2} \end{aligned}$$

ii. For any $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in S^{-1}R$ and $\frac{m}{s} \in S^{-1}M$, $[\frac{r_1}{s_1}\frac{r_2}{s_2}]\frac{m}{s} = \frac{r_1}{s_1}[\frac{r_2}{s_2}\frac{m}{s}]$.
Thus $S^{-1}M$ is a module over the Boolean like semiring $S^{-1}R$.

Consider the following example.

**Example 7.1.1.** *Let $R = \{0, a, b, c\}$ with '+' and '$\cdot$' defined by the following tables.*

| + | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | a | b | c |
| a | a | 0 | c | b |
| b | b | c | 0 | a |
| c | c | b | a | 0 |

| . | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | 0 | a |
| b | 0 | 0 | 0 | 0 |
| c | 0 | c | 0 | c |

Clearly $(R, +, \cdot)$ is a weak commutative Boolean like semiring. And let $M = \{0, 1, 2, 3\}$ in which '+' is defined by addition modulo four and '$\mu'$ : $R \times M \longrightarrow M$ is defined by the following table,

| $\mu$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| a | 0 | 1 | 2 | 3 |
| b | 0 | 1 | 2 | 3 |
| c | 0 | 1 | 2 | 3 |

Then M is a left R modue. If we take $S = \{c\}$ then, $S^{-1}M = \{\frac{0}{c}, \frac{1}{c}, \frac{2}{c}, \frac{3}{c}\}$.

**Remark 7.1.2.**

(a) Even though $S^{-1}R$ is a Boolean like ring, $S^{-1}M$ is only a Boolean like semiring module over $S^{-1}R$. For instance observe in example 7.1.1 that,

$$[\frac{a}{c} + \frac{b}{c}]\frac{1}{c} = \frac{1}{c}$$

whereas,

$$\frac{a}{c}\frac{1}{c} + \frac{b}{c}\frac{1}{c} = \frac{2}{c},$$

and the two classes are distinct, which shows that left distributive property still fails to hold.

(b) $S^{-1}M$ is a unitary R-module.

(c) As compared to the theory of modules over rings, we observe the following differences,

i. If M is a unitary module over a ring R of characterstic 2, then $Ch(M) = 2$. But in this case as shown in example 7.1.1, even if $S^{-1}M$ is a unitary module over a Boolean like semiring , it is not of characterstic 2. For example,

$$\frac{3}{c} + \frac{3}{c} = \frac{3+3}{c} = \frac{2}{c} \neq \frac{0}{c}$$

.

ii. In modules over a ring, the right and left modules of fractions exist together. But in our case , even if it happens to know the right and left modules of fractions exist , both may not exist together under the same circumstances.

**Notation:** We use the following notations: $S^{-1}M := M_s, S^{-1}R := R_s$ and $\frac{r}{s} := r_s, \frac{m}{s} := m_s$

## 7.2 Structure of submodules in $S^{-1}M$

The following Lemma is an easy consequence of definitions and hence stated with out proof.

**Lemma 7.2.1.** *Let N be a submodule of a Boolean like semiring module M. Then,* *(a)* $N_s$ *is a submodule of* $M_s$*,*

*(b)* $m_s \in N_s$ *if and only if* $tm \in N$ *for some* $t \in S$*,*

*(c) If* $N \cap S \neq \emptyset$ *then,* $N_s = M_s$*.*

**Lemma 7.2.2.** *Let* $M$ *and* $N$ *be submodules of an R-module , then*

*(a)* $M \cap N$ *is a submodule and* $(M \cap N)_s = M_s \cap N_s$

*(b)* $(A(M))_s = A(M_s)$*.*

*Proof.* (a) Clearly $M \cap N$ is a submodule of R. Now, Let $u_s \in (M \cap N)_s \Leftrightarrow tu \in M \cap N$, for some t in S. $\Leftrightarrow tu \in M, tu \in N \Leftrightarrow u_s \in M_s, u_s \in N_s \Leftrightarrow u_s \in (N_s \cap M_s)$

(b) $(\subseteq)$ Let $r_s \in (A(M))_s$

$$\begin{aligned}
&\Rightarrow (tr)m = 0 \forall m \in M \\
&\Rightarrow t(rm) = 0 \\
&\Rightarrow (rm)_s = 0_s \\
&\Rightarrow r_s m_s = 0_s \\
&\Rightarrow r_s \in A(M_s)
\end{aligned}$$

$(\supseteq)$ Let $r_s \in A(M_s)$

$$\begin{aligned}
&\Rightarrow r_s m_s = 0_s \\
&\Rightarrow (rm)_s = 0_s \\
&\Rightarrow t(rm) = 0 \quad \text{for some } t \in S. \\
&\Rightarrow (tr)m = 0
\end{aligned}$$

$$\Rightarrow tr \in A(M)$$
$$\Rightarrow (tr)_{ts} \in (A(M))_s$$
$$\Rightarrow r_s \in (A(M))_s$$

□

**Lemma 7.2.3.** *Let M be a left R-module of type II and N be a submodule of M. Then,* $(N : M)_s = (N_s : M_s)$.

*Proof.* Suppose

$$\begin{aligned} r_s \in (N : M)_s &\Leftrightarrow tr \in (N : M) \\ &\Leftrightarrow (tr)M \subseteq N \\ &\Leftrightarrow (tr)_{ts}M_s \subseteq N_s \\ &\Leftrightarrow r_s \in (N_s : M_s) \end{aligned}$$

□

**Theorem 7.2.1.** *Let M be an R-module and N be a submodule of M.Then,*

*(a) If N is a prime such that* $N \cap S = \emptyset$ *then so is* $N_s$ .

*(b) If N is primary such that* $N \cap S = \emptyset$ *then so is* $N_s$.

*Proof.* (a) Suppose for any m in M and r in R, $r_s m_s \in N_s \Longrightarrow t(rm) \in N$ for some $t \in S$. $\Longrightarrow (tr)m \in N \Longrightarrow (tr)M \subseteq N$ or $m \in N \Longrightarrow r_s M_s \subseteq N_s$ or $m_s \in N_s$.

(b) Suppose N is a primary submodule of M and $r_s m_s \in N_s$ for any $r_s \in R_s$ and $m_s \in M_s$. $\Rightarrow t(rm) \in N$ for some t in S. $\Rightarrow (tr)m \in N \Rightarrow (tr)^2 M \subseteq N$ or $m \in N$. If $(tr)^2 M \subseteq N$, then $t^2 r^2 M \subseteq N$ follows from weak commutativity of R.

$\Rightarrow (t^2r^2)_{t^2s}M_s \subseteq N_s \Rightarrow r_s^2 M_s \subseteq N_s$ and if $m \in N$, then, $m_s \in N_s$. □

**Theorem 7.2.2.** *If N is a 2- absorbing submodule of a module M such that $S \cap N = \emptyset$, then $N_s$ is also a 2-absorbing submodule of $M_s$.*

*Proof.* Let $m_s \in M_s, r_{s_1}, q_{s_2} \in R_s$ such that $(rqm)_{ss_1s_2} \in N_s$. $\Rightarrow \exists t \in S \ni (rqt)m \in N \Rightarrow rm \in N$ or $(qt)m \in N$ or $r(qt) \in (N : M)$ (Since N is 2 - absorbing.)$\Rightarrow (rm)_{ss_1} \in N_s$ or $(qtm)_{ss_2t} = (qm)_{ss_2} \in N_s$ or $r(qt)_{s_1(s_2t)} = (rq)_{s_1s_2} \in (N_s : M_s)$. In any case, the result holds.

# Future Plan

## Some emanating Questions

We pose the following questions that can arise from the works in this dissertation;

**[Question 1]** Imitating Swaminathans method of constructing Boolean Like rings by abstract synthesis as in [42] from product of a Boolean Ring and a unitary module over the Boolean ring, is it possible to abstractly synthesize Boolean like semirings of Venkateswarlu et al [44], that is not a Boolean like ring, from product of a Boolean like ring and a unitary module over it?
(**OR**): In any way possible, can we abstractly synthesize a Boolean like semiring?

**[Question 2]** Is it possible to imitate Ore's method of construction of ring of fractions as in [35] inorder to get Boolean like semiring of fractions? And if possible, how would the result we obtained in chapter 3 be compared to the new one?

**[Question 3]** If $S_1$ and $S_2$ are distinct multiplicative sets of a weak commutative Boolean like semiring, is $S_1^{-1}R$ isomorphic to $S_2^{-1}R$

?

[**Question 4**] Can we further characterize different generalized classes of prime sub module and settle converse statments of certain theorems in chapter 6 or produce counter examples to show the the converses fail to hold.

[**Question 5**] Is it possible to construct right modules of fractions ?

[**Question 6**] Can we generalize the notion of extended and contracted ideals to the class of sub modules ?

[**Question 7**] Can we generalize the correspondence theorem in theorem 4.4.4 to different classes of prime sub modules?

[**Question 8**] Which of the existing theories in rings and modules can be extended to the theory of Boolean like semirings and modules over Boolean like semirings?

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# Annex - A : First Paper

BOOLEAN LIKE SEMI RING

OF

FRACTIONS

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# BOOLEAN LIKE SEMI RING OF FRACTIONS

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*ABSTRACT*

*In this paper we construct the fractions of a Boolean like semi ring and establish that Boolean like semi ring of fraction is Boolean like ring of Foster [1].*

***Key words**: Boolean like semi ring, Boolean like ring, ring of fractions.*

***Mathematics Subject Classification:** 16Y30, 16 Y 60.*

## INTRODUCTION

The concept of Boolean like rings is originally due to Foster A.L. [1]. Later Swarminathan [6, 7] has extensively studied the geometry of Boolean like rings. Recently Venkateswarlu et al [8] introduced the notion of Boolean like semi rings by generalizing the concept of Boolean like rings of Foster. Venkateswarlu and Murthy [8, 9,10 & 11] have made an extensive study of Boolean like semi rings to name some:ideals, Prime ideals, Maximal ideals, nil radical and Jacobson radical of Boolean like semi rings. Further they have generalized most of the concepts of commutative theory of rings to the class of Boolean like semi rings. In fact it is observed in [8 ] that Boolean like semi rings are special classes of left near rings. In this paper we further investigate the theory of Boolean like semi rings by introducing the notion of fractions of Boolean like semi rings and prove that fractions of Boolean like semi rings are precisely the Boolean like rings of Foster [1]. This paper is divided into two sections. The first section is devoted to collect certain definition and results concerning Boolean like semi rings from [8]. In section 2 we introduce the notion of fractions of Boolean like rings and prove that every Boolean like semi ring of fractions is a Boolean like ring of Foster (see Theorem 2.9)

## 1. PRELIMINARIES

Here, we recall certain definitions and results on Boolean like semi rings from [8].

**Definition 1.1:** A non empty set R together with two binary operations + and ·satisfying the following conditions is called Boolean like semi ring;

1. (R, +) is an abelian group;
2. (R, · ) is a semi group;
3. a · (b + c ) = a.b + a.c;
4. a + a = 0 for all a in R;
5. ab(a + b + ab ) = ab for all a, b, c ∈ R.

**Lemma 1.2:** Let R be a Boolean like semi ring. Then a.0 = 0 for all a in R.

**Definition 1.3:** A Boolean like semi ring R is said to be weak commutative if abc = acb for all a, b and c ∈ R.

**Lemma 1.4:** Let R be weak commutative Boolean like semi ring. Then 0. a = 0 for all a ∈ R.

**Lemma 1.5:** Let R be weak commutative Boolean like semi ring and let m and n be integers. Then, (i) $a^m a^n = a^{m+n}$ (ii)$(a^m)^n = a^{mn}$ (iii) $(ab)^n = a^n b^n$

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## 2. CONSTRUCTION OF BOOLEAN LIKE SEMI RING OF FRACTIONS

Here we introduce the construction of $S^{-1}$R from a Boolean like semi ring R.

**Definition2.1:** A non empty subset S of R is called multiplicatively closed whenever a, b ∈S implies ab ∈S

Now we prove the following two lemmas which we use in the sequel.

**Lemma 2.2:** Let R be a weak commutative Boolean like semi ring. Then $ba^2 + b^2a = ba + (ba)^2$ ∀a, b ∈ R.

**Proof:** $ba^2 + b^2a$ = baa + bba = baa + bab (by 3 of def 1.1)

= ba( a + b) (by 3 of def 1.1)

= ba (b + a) (by 1 of def 1.1)

= ba( b + a + <u>ba + ba</u> ) (by 4 of def 1.1)

= bab + baa + baba + baba (by 3 of def 1.1)

= ba (b + a + ba ) + $(ba)^2$ (by 1of def 1.1 and lemma 1.5)

= ba + $(ba)^2$

**Lemma 2.3:** Let R be a weak commutative Boolean like semi ring. Then

$$c\,(a + a^2)(b + b^2) = 0\ \forall a, b, c \in R.$$

**Proof:** Consider c $(a+a^2)(b + b^2)$ = c $(a+a^2)b$ + c $(a+a^2)\, b^2$ (by 3 of def 1.1)

= cb $(a+a^2)$ + c $b^2$ $(a+a^2)$ (by def 1.3)

= c [b $(a+a^2)$ + $b^2$ $(a+a^2)$ ] (by 3of def 1.1)

= c[ba + $ba^2$ + $b^2a$ + $b^2a^2$) ] (by 3 of def 1.1)

= c[ (ba + $b^2a^2$ ) + ($ba^2 + b^2a$) ]

= c[ (ba + $(ba)^2$ ) + ($ba^2 + b^2a$) ] (by lemma 1.5)

= c[ ba + $(ba)^2$ + (ba + $(ba)^2$) ] (by lemma 2.2)

= c0 (by 4 of def 1.1)

= 0 (by lemma1.2)

**Remark 2.4:** In [8], it is observed that a + $a^2$ is a nilpotent element in a Boolean like semi ring R.

**Theorem 2.5:** Let R be a weak commutative Boolean like semi ring and S be a multiplicatively closed subset of R. Define a relation ∼ on R x S by $(r_1, s_1) \sim (r_2, s_2)$ if and only if ∃ s∈ S such that $s(s_1 r_2 + s_2 r_1) = 0$ . Then ∼ is an equivalence relation.∼

**Proof:** Let (r, s) ∈ R x S. Then for any t in S, from 4 of definition 1.1 and lemma 1.2 we have that t (sr + sr ) = t0 = 0.

Hence (r, s) ∼ (r, s).Thus ∼ is reflexive.

Now suppose $(r_1, s_1) \sim (r_2, s_2)$. Then $t\,(s_1r_2 + s_2r_1) = 0$ for some t in S.

$\Rightarrow$ $t\,(s_2r_1 + s_1r_2) = 0$

$\Rightarrow$ $(r_2, s_2) \sim (r_1, s_1)$. Thus ∼ is symmetric.

 

Finally let $(r_1, s_1)$,$(r_2, s_2)$ and $(r_3, s_3) \in R \times S$ such that $(r_1, s_1) \sim (r_2, s_2)$ and $(r_2, s_2) \sim (r_3, s_3)$. Then $t[s_1r_2+s_2r_1] = 0 = t'[s_2r_3+s_3r_2]$ for some t and $t'$ in S

$\Rightarrow t[s_1r_2+s_2r_1]s_3 = 0 = t'[s_2r_3+s_3r_2]s_1$ (by lemma 1.4)

$\Rightarrow ts_3[s_1r_2+s_2r_1] = 0 = t's_1[s_2r_3 + s_3r_2]$ (by def 1.3)

$\Rightarrow t'ts_3[s_1r_2+s_2r_1] = 0 = t's_1[s_2r_3 + s_3r_2]t$ (by def 2.3 & lemma 1.4)

$\Rightarrow t't\,s_3(s_1r_2)+ t't\,s_3(s_2r_1) + t'ts_1(s_2r_3) + t'ts_1(s_3r_2) = 0$

$\Rightarrow [t'ts_1(s_3r_2)+ t't\,s_1(s_3r_2)] + [t'ts_2(s_1r_3) + t'ts_2(s_3r_1)] = 0$

$\Rightarrow t'ts_2(s_3r_1) + t'ts_2(s_1r_3) = 0$

$\Rightarrow t'ts_2[s_3r_1 +s_1r_3] = 0$ (by def 1.3)

$\Rightarrow t'ts_2[s_1r_3+s_3r_1] = 0$ (by 1 of def 1.1 where $tt's_2$ is in S.)

Hence we have some $t'ts_2 \in S$ such that $(r_1, s_1) \sim (r_3, s_3)$. Thus $\sim$ is an equivalence relation.

**Remark 2.6:** From the preceding theorem we denote the equivalence class containing (r, s) in R x S by $\frac{r}{s}$ and the set of all equivalence classes by $S^{-1}$R.

**Lemma 2.7:** Let R be a weak commutative Boolean like semi ring and S be a multiplicatively closed subset of R. Then

(i) If $0 \notin S$ and R has no zero divisors, then $(r_1, s_1) \sim (r_2, s_2)$ if and only if $s_1r_2 = s_2r_1$

(ii) $\frac{r}{s} = \frac{rt}{st} = \frac{tr}{st} = \frac{tr}{ts}$ for all r in R and for all s, t in S

(iii) $\frac{rs}{s} = \frac{rs'}{s'}$ for all r in R and for all s, s′ in S.

(iv) $\frac{s}{s} = \frac{s'}{s'}$ for all s, $s'$ in S.

(v) If $0 \in S$, then $S^{-1}$R contains exactly one element.

**Proof:** Routine.

**Theorem 2.8:** Let S be a multiplicatively closed subset in a weak commutative Boolean like semi ring R. Define binary operations + and · on $S^{-1}$R as follows:

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{s_2r_1+ s_1r_2}{s_1s_2} \quad \text{and} \quad \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1r_2}{s_1s_2}$$

Then ($S^{-1}$R, +, · ) is a Boolean like semi ring.

**Proof:** To prove + and · are well defined let $\frac{r_1}{s_1} = \frac{r_1'}{s_1'}$ , $\frac{r_2}{s_2} = \frac{r_2'}{s_2'}$. This implies $t[s_1'r_1+s_1r_1'] = 0 = t'[s_2'r_2+s_2r_2']$ for some t, $t'$, in S. [*]

First we prove that $\frac{s_2r_1+ s_1r_2}{s_1s_2} = \frac{s_2'r_1'+ s_1'r_2'}{s_1's_2'}$ [i.e. $\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1'}{s_1'} + \frac{r_2'}{s_2'}$ ]

Consider $t't\,[s_1's_2'(s_2r_1 + s_1r_2) + s_1s_2(s_2'r_1' + s_1'r_2')]$

$= t't(s_1's_2')(s_2r_1) + t't(s_1's_2')(s_1r_2) + t't(s_1s_2)(s_2'r_1') + t't(s_1s_2)(s_1'r_2')$ (by 3 of def 1.1)

$= t't(s_2's_2)(s_1'r_1) + t't(s_1's_1)(s_2'r_2) + t't(s_2's_2)(s_1r_1') + t't(s_1's_1)(s_2r_2')$ (by def 1.3)

$= (t't)\,(s_2's_2)[(s_1'r_1) +(s_1r_1')] + (t't)\,(s_1's_1)[(s_2'r_2) + (s_2r_2')]$

$= t'(s_2's_2)t[(s_1'r_1) +(s_1r_1')] + t'[(s_2'r_2) + (s_2r_2')]\,t(s_1's_1)$ (by def 1.3)

$= t'(s_2's_2)(0) + (0)\, t\, (s_1's_1)$ (by [*] above)

$= 0 + 0$ (by lemmas 1.2 & 1.4)

$= 0$

Hence + is well defined. Next we prove, $s[(s_1's_2')r_1r_2 + s_1s_2\,(r_1'r_2')] = 0$ for some s in S.

Now, $\frac{r_1}{s_1} = \frac{r_1'}{s_1'}$ , $\frac{r_2}{s_2} = \frac{r_2'}{s_2'}$ implies,

$t[s_1'r_1 + s_1r_1'] = 0 = t'[s_2'r_2 + s_2 r'_2]$ for some t and $t'$ in S.

$\Rightarrow t[s_1'r_1 + s_1r_1']\, s_2'r_2 = 0 = t'[s_2'r_2 + s_2r_2']s_1r_1'$ (by lemma 1.4)

$\Rightarrow tt'[s_1'r_1 + s_1r_1']\, s_2'r_2 = 0 = tt'[s_2'r_2 + s_2r_2']s_1r_1'$ (by lemma 1.2 & 1.4)

$\Rightarrow tt'\,(s_2'r_2)\,[s_1'r_1 + s_1r_1'] + tt'(s_1r_1')\,[s_2'r_2 + s_2r_2'] = 0$ (by def 1.3)

$\Rightarrow tt'\,(s_2'r_2)\,[s_1'r_1] + tt'\,(s_2'r_2)\,[s_1r_1'] + tt'(s_1r_1')\,[s_2'r_2] + tt'(s_1r_1')\,[s_2r_2'] = 0$ (by 3 of def 1.1)

$\Rightarrow tt'(s_1's_2')\,[r_1\, r_2] + tt'[s_1r_1']\,(s_2'r_2) + tt'(s_1r_1')[s_2'r_2] + tt'[s_1s_2](r_1'r_2') = 0$ (by def 1.3)

$\Rightarrow tt'(s_1's_2')\,[r_1\, r_2] + tt'[s_1s_2](r_1'r_2') = 0$ (by 4 of def 1.1)

$\Rightarrow tt'\,[(s_1's_2')\,[r_1\, r_2] + [s_1s_2](r_1'r_2')\,] = 0$

Thus choose s = $tt'$ which is also in S. Hence multiplication is also well defined. We observe the following trivial fact, before proceeding to the next.

Before proceeding to the next,

$\frac{r_1}{s} + \frac{r_2}{s} = \frac{r_1+r_2}{s}$ for any $r_1$, $r_2$ in R and s in S (from lemma 2.7 and theorem 2.8)

**Now we claim that ( $S^{-1}$R , + ) is an abelian group.**

**A.** + is associative. Let $\frac{r_1}{s_1}, \frac{r_2}{s_2}, \frac{r_3}{s_3} \in s^{-1}$R. Then,

$\frac{r_1}{s_1} + [\frac{r_2}{s_2} + \frac{r_3}{s_3}] = \frac{r_1}{s_1} + [\frac{s_3r_2 + s_2r_3}{s_2s_3}] = \frac{(s_2s_3)r_1 + s_1(s_3r_2 + s_2r_3)}{s_1(s_2s_3)}$

$= \frac{(s_2r_1)s_3 + (s_1r_2)s_3 + (s_1s_2)r_3}{(s_1s_2)s_3} = \frac{(s_2r_1)s_3}{(s_1s_2)s_3} + \frac{(s_1r_2)s_3}{(s_1s_2)s_3} + \frac{(s_1s_2)r_3}{(s_1s_2)s_3}$ (by lemma 2.7)

$= \frac{(s_2r_1 + s_1r_2)s_3}{(s_1s_2)s_3} + \frac{(s_1s_2)r_3}{(s_1s_2)s_3} = \frac{s_2r_1 + s_1r_2}{s_1s_2} + \frac{r_3}{s_3}$ (by lemma 2.7)

$= [\frac{r_1}{s_1} + \frac{r_2}{s_2}] + \frac{r_3}{s_3}$. Hence associative.

**B.** Existence of additive identity (zero element)

For any $\frac{r}{s}$ in $s^{-1}$R, $\frac{r}{s} + \frac{0}{s} = \frac{r+0}{s}$ (by remark 3.4)

$= \frac{r}{s} = \frac{0+r}{s} = \frac{0}{s} + \frac{r}{s}$ .

Hence $\frac{0}{s}$ is the additive identity for any s in S.

 

**C.** + is commutative.

Let $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in S^{-1}R$, then $\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{s_2 r_1 + s_1 r_2}{s_1 s_2} = \frac{s_1 r_2 + s_2 r_1}{s_1 s_2}$ (by 1of def 1.1)

$= \frac{s(s_1 r_2 + s_2 r_1)}{s(s_1 s_2)}$, (by lemma 2.7 for any s in S)

$= \frac{s(s_1 r_2 + s_2 r_1)}{s(s_2 s_1)} = \frac{s_1 r_2 + s_2 r_1}{s_2 s_1}$ (by def 1.3)

$= \frac{r_2}{s_2} + \frac{r_1}{s_1}$

**D.** Existence of additive inverse:

Let $\frac{r}{s}$ be in $S^{-1}R$. Then $\frac{r}{s} + \frac{r}{s} = \frac{r+r}{s} = \frac{0}{s}$. (by #)

$= \frac{s(s_1 r_2 + s_2 r_1)}{s(s_2 s_1)} = \frac{s_1 r_2 + s_2 r_1}{s_2 s_1}$ (by def 1.3)

$= \frac{r_2}{s_2} + \frac{r_1}{s_1}$

Hence ($S^{-1}$R, +) is an abelian group.

**E. ($S^{-1}$R, · ) is a semi group**.

Let $\frac{r_1}{s_1}, \frac{r_2}{s_2}, \frac{r_3}{s_3} \in S^{-1}R$, then $\frac{r_1}{s_1} \cdot [\frac{r_2}{s_2} \cdot \frac{r_3}{s_3}] = \frac{r_1}{s_1} \cdot [\frac{r_2 r_3}{s_2 s_3}] = \frac{r_1(r_2 r_3)}{s_1(s_2 s_3)} = \frac{(r_1 r_2) r_3}{(s_1 s_2) s_3}$ (by 2 of def 1.1)

$= [\frac{r_1 r_2}{s_1 s_2}] \cdot \frac{r_3}{s_3} = [\frac{r_1}{s_1} \cdot \frac{r_2}{s_2}] \cdot \frac{r_3}{s_3}$

**F. (.) is distributive over +**

Let $\frac{r_1}{s_1}, \frac{r_2}{s_2}, \frac{r_3}{s_3} \in S^{-1}R$, then

$\frac{r_1}{s_1} \cdot [\frac{r_2}{s_2} + \frac{r_3}{s_3}] = \frac{r_1}{s_1} \cdot [\frac{s_3 r_2 + s_2 r_3}{s_2 s_3}] = \frac{r_1(s_3 r_2 + s_2 r_3)}{s_1(s_2 s_3)} = \frac{r_1(s_3 r_2) + r_1(s_2 r_3)}{s_1(s_2 s_3)} = \frac{r_1(r_2 s_3) + r_1(r_3 s_2)}{(s_1 s_2) s_3}$

$= \frac{(r_1 r_2) s_3 + (r_1 r_3) s_2}{s_1(s_2 s_3)}$ (by def 1.3)

$= \frac{(r_1 r_2) s_3}{(s_1 s_2) s_3} + \frac{(r_1 r_3) s_2}{(s_1 s_3) s_2} = \frac{(r_1 r_2)}{(s_1 s_2)} + \frac{(r_1 r_3)}{(s_1 s_3)}$

$= [\frac{r_1}{s_1} \cdot \frac{r_2}{s_2}] + [\frac{r_1}{s_1} \cdot \frac{r_3}{s_3}]$

**G. Characteristic of $S^{-1}$R is 2.**

Let $\frac{r}{s}$ be in $S^{-1}R$, then $\frac{r}{s} + \frac{r}{s} = \frac{r+r}{s} = \frac{0}{s}$

**H.** Let $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in S^{-1}R$. Then $\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} [\frac{r_1}{s_1} + \frac{r_2}{s_2} + \frac{r_1}{s_1} \cdot \frac{r_2}{s_2}]$

Claim: $\frac{r_1 r_2}{s_1 s_2} = \frac{r_1 r_2}{s_1 s_2} [\frac{s_2 r_1 + s_1 r_2 + r_1 r_2}{s_1 s_2}]$

Let t∈S be any element. Then consider;

$t[(s_1 s_2)(r_1 r_2)(s_2 r_1 + s_1 r_2 + r_1 r_2) + (s_1 s_2)(s_1 s_2)(r_1 r_2)]$

$= t[(s_1 s_2)(r_1 r_2)(s_2 r_1 + s_1 r_2 + r_1 r_2) + (s_1 s_2))(r_1 r_2)(s_1 s_2)]$ (by def 1.3)

$= t(s_1 s_2)(r_1 r_2)[(s_2 r_1 + s_1 r_2 + r_1 r_2) + (s_1 s_2)]$

 

$= t(s_1s_2)(r_1r_2)[r_1s_2 + s_1r_2 + r_1r_2 + s_1s_2]$ (by def 1.3)

$= t(s_1s_2)(r_1r_2)[r_1(s_2 + r_2) + s_1(r_2 + s_2)]$ (by 3 of def 1.1)

$= t(s_1s_2)(r_1r_2)(r_1 + s_1)(r_2 + s_2)$

$= t(s_1r_1)(s_2r_2)(s_1 + r_1)(s_2 + r_2)$ (by def 1.3)

$= t(s_1r_1)(s_1 + r_1)(s_2r_2)(s_2 + r_2)$ (by def 1.3)

$= t(s_1r_1 + (s_1r_1)^2)(s_2r_2 + (s_2r_2)^2)$

$= 0$ (by lemma 2.2).

Hence we have that $\frac{r_1r_2}{s_1s_2} = \frac{r_1r_2}{s_1s_2}[\frac{s_2r_1+s_1r_2+r_1r_2}{s_1s_2}]$ which proves 5 of def1.1

Thus $(S^{-1}R, +, \cdot)$ is a Boolean like semi ring.

Now we recall the following definition of A. L. Foster [1] regarding Boolean like ring

**Definition [1]:** A Boolean like ring R is a commutative ring with unity and is of characterstic 2 in which ab(1+a)(1+b) = 0 for all a, b in R

**Theorem 2.9: $(S^{-1}R, +, .)$ is a Boolean like ring.**

**Proof:** Let $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in S^{-1}R$. Then $\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1r_2}{s_1s_2} = \frac{s(r_1r_2)}{s(s_1s_2)}$ (by lemma 2.7)

$= \frac{s(r_2r_1)}{s(s_2s_1)} = \frac{(r_2r_1)}{(s_2s_1)}$ (by lemma 2.7)

$= \frac{r_2r_1}{s_2s_1} = \frac{r_2}{s_2} \cdot \frac{r_1}{s_1}$

Thus $\cdot$ is commutative.

Hence left distributive property also holds from commutative property of $\cdot$ and F of theorem 2.8

**i.** Multiplicative identity,

Let $\frac{r}{s}$ be any element. Then,

$\frac{r}{s} \cdot \frac{s}{s} = \frac{rs}{ss} = \frac{r}{s}$ (by lemma 2.7)

$= \frac{sr}{ss} = \frac{s}{s} \cdot \frac{r}{s}$ (by lemma 2.7). Hence $\frac{s}{s}$ is the multiplicative identity.

**ii.** Let $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in S^{-1}R$. Then,

$\frac{r_1}{s_1} \cdot \frac{r_2}{s_2}[\frac{s_1}{s_1} + \frac{r_1}{s_1}][\frac{s_2}{s_2} + \frac{r_2}{s_2}] = \frac{(r_1r_2)(s_1+r_1)(s_2+r_2)}{(s_1s_2)(s_1s_2)}$ (by lemma 2.7)

$= \frac{s(s_1s_2)(r_1r_2)(s_1+r_1)(s_2+r_2)}{s(s_1s_2)(s_1s_2)(s_1s_2)}$ (by lemma 2.7)

$= \frac{s(s_1r_1)(s_1+r_1)(s_2r_2)(s_2+r_2)}{s(s_1s_2)(s_1s_2)(s_1s_2)}$

$= \frac{0}{s'}$ where $s' = s(s_1s_2)(s_1s_2)(s_1s_2)$ (by lemma 2.3)

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# Annex - B : Second Paper

Certain Special classes

of

Ideals in Boolean like Semi ring of Fractions

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# Certain Special classes of ideals in Boolean like Semi ring of Fractions

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***ABSTRACT***

*In this paper we introduce the notions of Extended, Primary and Almost Primary ideals of Boolean like semi ring of fractions. Further we obtain certain properties regarding extended ideals in this class. Also we prove that the contraction of an almost primary (and hence primary and prime) ideal is also almost primary (primary and prime respectively) in a Boolean like semi ring (a special class of near ring).*

***Key words:*** *Boolean like semi ring of fractions, Extended ideal, Contracted ideal, Almost Primary ideal.*

***Mathematics Subject Classification:*** *16Y30, 16 Y 60.*

## INTRODUCTION

Foster A.L. [2] introduced the notion of Boolean like ring R as a commutative ring with unity and is of characteristic 2 in which ab (1+ a) (1+b) = 0 for all a, b ∈ R. Later in [4] Venkateswarlu et al generalized the notion of Boolean like ring by introducing the concept of Boolean like semi ring. Boolean like semi ring is a special class of near ring. Venkateswarlu and Murthy [1, 5, 7] made an extensive study of the class of Boolean like semi rings and also have studied certain properties of contraction of ideals in [8].

Recently the present authors have introduced the notion Boolean like semi ring of fractions in [4] and it was proved that every Boolean like semi ring of fraction is a Boolean like ring. However there are many interesting facts about the class of Boolean like semi ring of fractions which do not subsume the properties of Boolean like rings.

This paper is divided into 3 sections of which the first sections is devoted for recollecting certain definitions and results concerning Boolean like semi rings and as well as Boolean like semi ring of fractions. In section 2, we introduce the concept of extended ideal in a Boolean like semi ring of fractions and prove that $S^{-1}I$ is an ideal of $S^{-1}R$ .Further, pre image of any ideal of $S^{-1}R$ is also an ideal in the Boolean like semi ring R (see definition 2.3). Also we observe in theorem 2.4 that $S^{-1}P$ is a prime ideal of $S^{-1}R$ whenever P is a prime ideal in R which is disjoint from S. In theorem 2.8 we prove $f^{-1}(J)$ is a prime ideal of R and $f^{-1}(J) \cap S = \emptyset$ provided J is a prime ideal in $S^{-1}R$ with right unity.

In section 3, we introduce the notions of primary and almost primary ideal in Boolean like semi ring of fractions and further we prove that $S^{-1}P$ is a almost primary (hence primary) ideal of $S^{-1}R$ whenever P is almost primary (respectively primary) in R (3.5 & 3.7). Finally we establish in theorems 3.6 & 3.8 that the contraction of an ideal J is almost primary (hence primary) in a Boolean like semi ring R whenever J is almost primary (primary) ideal of $S^{-1}R$.

## 1. PRELIMINARIES.

In this, we recall certain definitions and results on Boolean like semi rings and Boolean like semi ring of fractions from [2], [4] and [6]

**Definition 1.1** A non empty set R together with two binary operations + and · satisfying the following conditions is called Boolean like semi ring;

1. (R, +) is an abelian group.
2. (R, ·) is a semi group.
3. a · (b + c ) = a.b + a.c

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4. a + a = 0 for all a ∈ R.
5. ab (a + b + ab) = ab for all a, b, c ∈ R.

**Lemma 1.2** Let R be a Boolean like semi ring. Then a.0 = 0 for all a in R.

**Definition 1.3** A Boolean like semi ring R is said to be weak commutative if abc = acb for all a, b and c ∈ R.

**Lemma 1.4** Let R be weak commutative Boolean like semi ring and let m and n be integers. Then
(i) $a^m a^n = a^{m+n}$
(ii) $(a^m)^n = a^{mn}$
(iii) $(ab)^n = a^n b^n$ for all a, b ∈ R

**Definition 1.5**. A non empty subset I of R is said to be an ideal if

1 (I, +) is a sub group of (R, +), i.e, for a, b ∈ R ⇒ a + b ∈ R
2. ra ∈ R for all a ∈ I , r ∈ R ,i.e RI ⊆I
3. (r+a)s + rs ∈ I for all r, s ∈ R , a ∈ I

**Definition 1.6** A non empty subset S of a Boolean like semi ring R is called multiplicatively closed whenever a, b ∈ S implies ab ∈S.

**Lemma 1.7** Let R be a weak commutative Boolean like semi ring and S be a multiplicatively closed subset of R. Then

1. $\frac{r}{s} = \frac{rt}{st} = \frac{tr}{st} = \frac{tr}{ts}$ for all r in R and for all s, t in S
2. $\frac{rs}{s} = \frac{rs'}{s'}$ for all r in R and for all s, s′ in S.
3. $\frac{s}{s} = \frac{s'}{s'}$ for all s,$s'$ in S.

**Theorem 1.8.** Let S be a multiplicatively closed sub set in a weak commutative Boolean like semi ring R. Define binary operations + and · on $S^{-1}$R as follows:

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{s_2 r_1 + s_1 r_2}{s_1 s_2} \quad \text{and} \quad \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}$$ for $s_1, s_2$ ∈ S and $r_1, r_2$ ∈ R

Then ($S^{-1}$R, +, · ) is a Boolean like semi ring.

**Definition 1.9.** If R and R′ are Boolean like semi rings, a mapping f: R→R′ is said to be a Boolean like semi ring homomorphism (or simply homomorphism) of R into R ′ if

f(a + b) = f (a) + f (b) and f (ab) = f(a)f(b) for all a, b ∈ R.

**Definition 1.10.** Let R and S be two Boolean like semi rings. If I is an ideal of S and f : R→S is a homomorphism then $f^{-1}$( I ) is an ideal of R, called the contraction of I and is denoted by $I^c$

## 2. EXTENDED IDEALS IN BOOLEAN LIKE SEMI RING OF FRACTIONS

We begin with the following

**Theorem 2.1** Let R be a weak commutative Boolean like semi ring and let S be a multiplicatively closed subset in R. Define f: R →$S^{-1}$R by f(r) = $\frac{rs}{s}$ where s∈S. Then

1. f is a homomorphism.
2. If 0∉S and S contain no divisors of zero then f is a monomorphism.

**Proof.** To prove 1, Let r, t ∈ R. By repeated application of 1 of lemma 1.7, we have

f (r + t) = $\frac{(r+t)s}{s} = \frac{s(r+t)}{s} = \frac{rs}{s} + \frac{ts}{s}$ = f(r) f(t) and f (rt) = $\frac{(rt)s}{s} = \frac{s(rt)s}{ss} = \frac{rs}{s} . \frac{ts}{s}$ = f(r) f(t).

Hence f is a homomorphism.

Now let r, t ∈R su1.1ch that f(r) = f(t). Then $\frac{rs}{s} = \frac{ts}{s}$

⇒ there exists m ∈ S such that m[s(rs) + s(ts)] = 0 and hence $ms^2[r + t] = 0$

⇒ r + t = 0 since $ms^2 \in S$ and S doesn't contain (non zero) zero divisors. Thus r = t.

Hence f is a monomorphism.

**Theorem 2.2**. Let R be a weak commutative Boolean like semi ring, S be a multiplicatively closed subset in R and I be an ideal of R. Let $S^{-1}I = \{\frac{a}{s} / a \in I, s \in S\}$. Then $S^{-1}I$ is an ideal of $S^{-1}R$.

**Proof.** Since 0 ∈ I, it is clear that $\frac{0}{s} \in S^{-1}I$ and hence $S^{-1}I$ is non empty.

Let $t_1, t_2 \in S^{-1}I$. Then $t_1 = \frac{a_1}{s_1}$, $t_2 = \frac{a_2}{s_2}$ for some $a_1, a_2 \in I$ and $s_1, s_2 \in S$.

⇒ $t_1 + t_2 = \frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{s_2 a_1 + s_1 a_2}{s_1 s_2} \in S^{-1}I$ since $s_2 a_1 + s_1 a_2 \in I$.

Further let $\lambda \in S^{-1}R$ and $\beta \in S^{-1}I$. Then $\lambda = \frac{r}{s}$ and $\beta = \frac{a}{s'}$ for some r ∈ R and a ∈ I and s, $s' \in S$

Thus $\lambda\beta = \frac{r}{s}\frac{a}{s'} = \frac{ra}{ss'} \in S^{-1}I$.

Finally, Let $\lambda_1, \lambda_2 \in S^{-1}R$ and $\beta \in S^{-1}I$, then $\lambda_1 = \frac{r_1}{s_1}$, $\lambda_2 = \frac{r_2}{s_2}$ and $\beta = \frac{a}{s}$ for some $r_1, r_2 \in R$, a ∈ I and $s, s_1, s_2 \in S$. By using the definitions of 1.1, 1.5 and lemma 1.7, we have

$$[\lambda_1 + \beta]\lambda_2 + \lambda_1\lambda_2 = [\frac{r_1}{s_1} + \frac{a}{s}]\frac{r_2}{s_2} + \frac{r_1}{s_1}.\frac{r_2}{s_2} = [\frac{sr_1 + s_1 a}{s_1 s}]\frac{r_2}{s_2} + \frac{r_1 r_2}{s_1 s_2}$$
$$= \frac{s_1 s_2[sr_1 + s_1 a]r_2 + (s_1 s)s_2(r_1 r_2)}{(s_1 s)s_2(s_1 s_2)}$$
$$= \frac{[s_1 s_2][sr_1 + s_1 a]r_2 + (sr_1)r_2)]}{[s_1 s_2]s(s_1 s_2)}$$
$$= \frac{[sr_1 + s_1 a]r_2 + (sr_1)r_2]}{s(s_1 s_2)} \in S^{-1}I$$

Since by the definition 1.5, we have that $[sr_1 + s_1 a]r_2 + (sr_1)r_2] \in I$

Hence $S^{-1}I$ is an ideal of $S^{-1}R$.

In view of the above theorem , we have the following

**Definition 2.3**. If I is an ideal of R and f: R →$S^{-1}R$ is a homomorphism then f (I) = $S^{-1}I$ is an ideal of $S^{-1}R$, called the Extension of I and is denoted by $I^e$.

**Theorem 2.4.** Let P be a prime ideal of a weak commutative Boolean like semi ring R such that P∩S = ∅ then $S^{-1}P$ is a prime ideal of $S^{-1}R$.

**Proof.** First we show $S^{-1}P \neq S^{-1}R$. Suppose $S^{-1}P = S^{-1}R$

Then $\frac{s}{s} \in S^{-1}P$ since $\frac{s}{s} \in S^{-1}R$.

⇒ $\frac{s}{s} = \frac{a}{s'}$ for some a ∈ P and $s' \in S$.

⇒ $m[sa + s's] = 0$ for some m ∈ S.

⇒ $ms's$ = msa and msa ∈ P since a∈P.

⇒ $ms's \in P \cap S$

$\Rightarrow$ P$\cap$S $\neq \emptyset$ contradiction to our supposition. Hence $S^{-1}P \neq S^{-1}R$.

It is clear from the above theorem 2.2, that $S^{-1}P$ is an ideal of $S^{-1}R$.

Next, suppose $t_1, t_2 \in S^{-1}R$ such that $t_1t_2 \in$P. Then $t_1 = \frac{r_1}{s_1}$ and $t_2 = \frac{r_2}{s_2}$.

$\Rightarrow \frac{r_1 r_2}{s_1 s_2} = \frac{a}{s}$ for some a in P and s in S.

$\Rightarrow s'(s_1s_2a + sr_1r_2) = 0$ for some s′ in S.

$\Rightarrow (s's)(r_1r_2) = (s's_1s_2)a \in P$

$\Rightarrow s's \in P$ or $r_1r_2 \in P$ . But $s's \notin P$ for $P \cap S = \emptyset$ and $s's \in S$. Hence $r_1r_2 \in P$.

$\Rightarrow r_1 \in P$ or $r_2 \in P$.

$\Rightarrow \frac{r_1}{s_1} \in S^{-1}P$ or $\frac{r_2}{s_2} \in S^{-1}P$.

$\Rightarrow t_1 \in S^{-1}P$ or $t_2 \in S^{-1}P$.

Hence $S^{-1}P$ is a prime ideal of $S^{-1}R$.

**Theorem 2.5** Let J be an ideal of $S^{-1}R$. Let f: R $\rightarrow S^{-1}R$ be the homomorphism given by f(r) = $\frac{rs}{s}$.

Then $f^{-1}(J) = \{x \in R \,/\, f(x) \in J\}$ is an ideal of R.

**Proof.** Since f (0) = 0 $\in$ J, we have $0 \in f^{-1}(J)$. Hence $f^{-1}(J)$ is non empty.

Let x, y $\in f^{-1}(J)$, then f(x+ y) = f(x) + f(y) $\in$ J. Hence x + y $\in f^{-1}(J)$. Also let r $\in$ R and x $\in f^{-1}$ (J).

Then f(rx) = f(r) f(x) $\in$ J since f(x) $\in$ J. Thus rx $\in f^{-1}$ (J) .Finally let r, s $\in$ R and x $\in^{-1}$(J). Then

f [( r + x)s + rs] = f[(r+x)s] + f(rs) = [f(r) +f(x)]f(s) + f(r)f(s) lies in J since J is an ideal of $S^{-1}R$.

Hence $f^{-1}J$ is an ideal of R.

**Definition 2.6.** The ideal $f^{-1}J$ is called the contraction of J to R and denoted by $J^c$.

**Definition 2.7.** Let R be a Boolean like semi ring. The element 1 in R is called right unity if a1 = a for all a $\in$R.

**Theorem 2.8** Let R be a weak commutative Boolean like semi ring with right unity. Let J be a prime ideal of $S^{-1}R$ . Then $f^{-1}(J)$ is a prime ideal of R and $f^{-1}(J) \cap S = \emptyset$

**Proof.** By theorem 2.5, $f^{-1}(J)$ is an ideal of R. Now claim $f^{-1}(J)$ is proper in R.

Suppose $f^{-1}(J)$ is not proper in R. Then $f^{-1}(J)$ = R. Hence f(x) $\in$ J for all x $\in$ R.

$\Rightarrow f(1) = \frac{1s}{s} \in J \Rightarrow \frac{s1}{s} \in J$ ( by lemma 1.7 )

$\Rightarrow \frac{s}{s} \in J$ which contradicts that J is prime.

Now let $t_1, t_2 \in R \ni t_1t_2 \in f^{-1}(J)$

$\Rightarrow f(t_1t_2) \in J \Rightarrow f(t_1) f(t_2) \in J$

$\Rightarrow f(t_1) \in J$ or $f(t_2) \in J$ since J is a prime ideal of $S^{-1}R$

$\Rightarrow t_1 \in f^{-1}(J)$ or $t_2 \in f^{-1}(J)$. Hence $f^{-1}(J)$ is a prime ideal of R.

 

Finally, for if $f^{-1}(J) \cap S \neq \emptyset$ let $s \in f^{-1}(J) \cap S$

$\Rightarrow s \in f^{-1}(J)$ and $s \in S$ and hence $f(s) \in J$ and $s \in S$

$\Rightarrow \frac{ss'}{s'} = \frac{s''}{s'} \in J$ and $s \in S$ for some $s'$ in S and $s'' = ss'$.

$\Rightarrow \frac{s}{s} = (\frac{s''}{s'}) \frac{s'}{s''} \in J$, for all $s \in S$ and since J is an ideal. But this contradicts the hypothesis that J is prime. Hence the theorem.

## 3. PRIMARY AND ALMOST PRIMARY IDEALS IN BOOLEAN LIKE SEMI RING OF FRACTIONS

We recall from [ 5 ] with the following

**Definition 3.1[5].** A proper ideal p of a Boolean like semi ring R is called primary if $x \in P$ or $y^2 \in P$ whenever $xy \in P$ for every $x, y \in R$.

**Definition 3.2** A proper ideal p of a Boolean like semi ring R is called almost primary if $x \in P$ or $y^2 \in P$ whenever $xy \in P - P^2$ for every $x, y \in R$.

**Remark 3.3.**

**(a).** It is clear that every primary ideal is almost primary but not conversely.

**(b)**. In a Boolean like semi ring R, $a^n = a$ or $a^2$ or $a^3$ for any $n \geq 1$.Hence in definition 3.1 and 3.2 , it is appropriate to define primary and almost primary in the above fashion instead of defining the usual way as in the case of rings.

**Theorem 3.4**. Let S be a multiplicative set and I be an ideal in a Boolean like semi ring R. Then for all $r \in R, s \in S$ $\frac{r}{s} \in S^{-1}I \Leftrightarrow mr \in I$ for some $m \in S$.

**Proof.** Let $\frac{r}{s} \in S^{-1}I$. Then $\frac{r}{s} = \frac{a}{t}$ for some $a \in I$ and $t \in S$.Then $n[tr + sa] = 0$ for some n in S. Hence $(nt)r = (ns)a \in I$. Hence choose $m = nt$ in S.

Conversely, if mr is in I for some m in S, we have $\frac{r}{s} = \frac{mr}{ms} \in S^{-1}I$.

**Theorem 3.5** Let P be a primary ideal of a Boolean like semi ring R and S be a multiplicative subset of R such that $P \cap S = \emptyset$. Then $S^{-1}P$ is a primary ideal of $S^{-1}R$.

**Proof.** By theorem 2.2 we have that $S^{-1}P$ is an ideal of $S^{-1}R$. And it is routine to verify that $S^{-1}P$ is a proper ideal.

Now let $\frac{r_1}{s_1}, \frac{r_2}{s_2}$ be in $S^{-1}R$ such that $\frac{r_1 r_2}{s_1 s_2} \in S^{-1}P$. Then if $\frac{r_1}{s_1} \in S^{-1}P$ we are done. Otherwise, by theorem 3.4, $mr_1 \notin P$ for all m in S. But $\frac{r_1 r_2}{s_1 s_2} \in S^{-1}P$ implies $\frac{r_1 r_2}{s_1 s_2} = \frac{a}{s}$ for some a in P and s in S.

$\Rightarrow t[sr_1r_2] = t[s_1s_2a] = (ts_1s_2)a \in P$ since P is an ideal and $a \in P$

$\Rightarrow (tsr_1)r_2 \in P$

$\Rightarrow r_2^2 \in P$ since $ts_1 \in S$, $mr_1 \notin P$ for all $m \in S$ , hence $tsr_1 \notin P$ and P is primary.

$\Rightarrow \frac{r_2^2}{s_2^2} \in S^{-1}P$. Hence $S^{-1}P$ is a primary ideal in $S^{-1}R$.

**Theorem 3.6** Let J be a primary ideal of $S^{-1}R$. Then $J^c = \{r \in R / f(r) \in J\}$ is also a primary ideal of R. Moreover $J^c \cap S = \emptyset$.

**Proof.** Clearly $J^c$ is a proper ideal of R. Now let $r_1, r_2 \in R$ such that $r_1 r_2 \in J^c$.

 

$\Rightarrow f(r_1r_2) \in J$

$\Rightarrow f(r_1)f(r_2) \in J. \Rightarrow f(r_1) \in J$ or $f(r_2^2) = [f(r_2)]^2 \in J$ since f is homomorphism and J is primary.

$\Rightarrow r_1 \in J^c$ or $r_2^2 \in J^c$. Hence $J^c$ is primary. Now suppose that $J^c \cap S \neq \emptyset$. Then there exists $s \in J^c \cap S$. Letting $s'' = ss'$ then $\frac{s''}{s'} = \frac{ss'}{s'} = f(s) \in J$. Thus $\frac{s'}{s'} = (\frac{s''}{s'})(\frac{s'}{s''}) \in J$, which is a contradiction to the fact that J is primary and hence proper.

**Theorem 3.7.** Let P be an almost primary ideal of a Boolean like semi ring R and S be a multiplicative subset of R such that $P \cap S = \emptyset$. Then $S^{-1}P$ is an almost primary ideal of $S^{-1}R$.

**Proof.** Clearly $S^{-1}P$ is a proper ideal of $S^{-1}R$.

Let $t_1, t_2 \in S^{-1}R$ such that $t_1t_2 \in S^{-1}P - (S^{-1}P)^2$ . Then $t_1 = \frac{r_1}{s_1}$, $t_2 = \frac{r_2}{s_2}$.

Then $t_1t_2 = \frac{r_1}{s_1}\frac{r_2}{s_2} \in S^{-1}P - (S^{-1}P)^2$. Now, if $t_1 = \frac{r_1}{s_1}$ lies in $S^{-1}P$ we are done.

Otherwise let $\frac{r_1}{s_1} \notin S^{-1}P$. Hence by theorem 3.4 we have $mr_1 \notin P$ for every m in S.

Finally we end this by the following

**Theorem 3.8** Let J be an almost primary ideal of $S^{-1}R$. Then $J^c = \{r \in R / f(r) \in J\}$ is also an almost primary ideal of R. Moreover $J^c \cap S = \emptyset$.

**Proof.** in the similar lines of the above theorem and by the definition of almost primary ideal.

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# Annex - C : Third Paper

ON CERTAIN GENERALIZED PRIME IDEALS
IN BOOLEAN LIKE SEMIRING OF
FRACTIONS

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# ON CERTAIN GENERALIZED PRIME IDEALS IN BOOLEAN LIKE SEMIRING OF FRACTIONS

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Abstract. In this paper, we introduce the notions of semiprime ideals, 2-potent prime ideals, weakly prime ideals and weakly primary ideals in a Boolean like semiring of fractions. Further, we obtain various results concerning the notions.

**Key Words:** Prime, primary, weakly primary, ideals, Boolean like semirings.
**2010 Mathematics Subject Classification:** Primary: 16Y30; 16Y60.

## 1. Introduction

The concept of Boolean like ring $R$ was introduced by Foster in [3]. It is defined as a commutative ring with unity and of characteristic 2 in which $ab(1+a)(1+b) = 0$ for every $a, b \in R$. Later in [7] Venkateswarlu et. al. introduced the concept of Boolean like semirings by generalizing the concept of Boolean like ring of Foster. In fact, Boolean like semirings are special classes of near rings [4]. Venkateswarlu et al have made an extensive study of the class of Boolean like semirings in [8 ,9].
Recently in [5], Ketsela et al have introduced the notion of Boolean like semiring of fractions and proved that every Boolean like semiring of fraction is a Boolean like ring of Foster. However there are many interesting facts about the class of ideals in Boolean like semiring of fractions which do not subsume the properties of ideals in commutative rings. Some of the facts have been established in [6]. Now, in this paper we study the

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properties of special classes of ideals in Boolean like semiring of fractions.

This paper is divided into two sections of which the first section is devoted for collecting certain definitions and results concerning Boolean like semirings and as well as Boolean like semiring of fractions. In section 2, we introduce the notions of 2-potent prime, weakly prime , weakly primary, quasi primary and almost primary ideals of a Boolean like semiring of fractions. Also, we prove that $S^{-1}I$ is semiprime (resp., 2-potent prime, weakly prime, weakly primary, quasi prime) if $I$ is semiprime (resp., 2-potent prime, weakly prime, weakly primary, quasi prime).

## 2. Preliminaries

In this section, we we recall certain definitions and results concerning Boolean like semirings from [5], [6], [7] and [8].

**Definition 2.1.** A non empty set $R$ together with two binary operations $+$ and $\cdot$ satisfying the following conditions is called a Boolean like semiring;

(1) $(R, +)$ is an abelian group;
(2) $(R, \cdot)$ is a semi group;
(3) $a(b + c) = a.b + a.c$;
(4) $a + a = 0$ ;
(5) $ab(a + b + ab) = ab$ for all $a, b, c \in R$.

**Theorem 2.2.** *Let $R$ be a Boolean like semiring. Then $a.0 = 0$ for every $a \in R$.*

**Definition 2.3.** A Boolean like semiring $R$ is said to be weak commutative if $abc = acb$ for every $a, b, c \in R$.

**Lemma 2.4.** *Let $R$ be weak commutative Boolean like semiring and let $m$ and $n$ be two integers. Then,*

(1) $a^m a^n = a^{m+n}$ *;*
(2) $(a^m)^n = a^{mn}$*;*
(3) $(ab)^n = a^n b^n$ *for every* $a, b \in R$.

**Definition 2.5.** Let $R$ be a weak commutative Boolean like semiring. A nonempty subset $S$ of $R$ is called multiplicatively closed whenever $a, b \in S$ implies $ab \in S$.

**Theorem 2.6.** *Let $R$ be a weak commutative Boolean like semiring and $S$ a multiplicatively closed subset of $R$. Define a relation $\sim$ on*

*$R \times S$ by $(r_1, s_1) \sim (r_2, s_2)$ if and only if there exists $s \in S$ such that $s(s_1r_2 + s_2r_1) = 0$. Then $\sim$ is an equivalence relation.*

**Lemma 2.7.** *Let $R$ be a weak commutative Boolean like semiring and $S$ a multiplicatively closed subset of $R$. Then, for every $r \in R$ and $s, s', t \in S$ we have;*

(1) $\frac{r}{s} = \frac{rt}{st} = \frac{tr}{st} = \frac{tr}{st}$;
(2) $\frac{rs}{s} = \frac{rs^{'}}{s^{'}}$ ;
(3) $\frac{s}{s} = \frac{s^{'}}{s^{'}}$.

**Theorem 2.8.** *Let $S$ be a multiplicatively closed subset in a weak commutative Boolean like semiring $R$. Define binary operations '+' and '.' on $S^{-1}R$ as follows:*

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{s_2r_1+s_1r_2}{s_1s_2} \quad \text{and} \quad \frac{r_1}{s_1}\frac{r_2}{s_2} = \frac{r_1r_2}{s_1s_2}.$$

*Then $(S^{-1}R, +, .)$ is a Boolean like ring.*

**Definition 2.9.** A Boolean like ring $R$ is a commutative ring with unity and is of characteristic 2 in which $ab(1+a)(1+b) = 0$ for every $a, b \in R$.

**Theorem 2.10.** $(S^{-1}R, +, .)$ *in Theorem 3 is a Boolean like ring.*

**Definition 2.11.** A subset $I$ of a Boolean like semiring $R$ is said to be an ideal of $R$ if;

(1) $(I, +)$ is a subgroup of $(R, +)$;
(2) $ra \in R$ for every $a \in I, r \in R$;
(3) $(r + a)s + rs \in I$ for every $r, s \in R, a \in I$.

**Definition 2.12.** If $R$ and $R^{'}$ are Boolean like semirings, a mapping $f : R \longrightarrow R^{'}$ is said to be a homomorphism of $R$ into $R^{'}$ if $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all $a$ and $b$ in $R$.

**Definition 2.13.** Let R and $R^{'}$ be two Boolean like semirings. If $I$ is an ideal of $R^{'}$ and $f : R \to R^{'}$ is a homomorphism, then $f^{-1}(I)$ is an ideal of $R$, called the contraction of $I$ and is denoted by $I^c$

**Theorem 2.14.** *Let $R$ be a weak commutative Boolean like semiring, $S$ a multiplicatively closed subset of $R$ and $I$ an ideal of $R$. Let $S^{-1}I = \{\frac{a}{s} | a \in I, s \in S\}$. Then $S^{-1}I$, called the extension of $I$ and denoted by $I^e$, is an ideal of $S^{-1}R$.*

**Theorem 2.15.** *Let $P$ be a prime ideal of a weak commutative Boolean like semiring $R$ such that $P \cap S = \varnothing$. Then $S^{-1}P$ is a prime ideal of $S^{-1}R$.*

**Theorem 2.16.** *Let $R$ be a weak commutative Boolean like semiring with right unity and let $J$ be a prime ideal of $S^{-1}R$. Then $J^c$ is a prime ideal of $R$ and $J^c \cap S = \varnothing$*

**Definition 2.17.** A proper ideal $P$ of a Boolean like semiring $R$ is called primary if $x \in P$ or $y^2 \in P$ whenever $xy \in P$ for every $x, y \in R$.

**Definition 2.18.** A proper ideal $P$ of a Boolean like semiring $R$ is called almost primary if $x \in P$ or $y^n \in P$ whenever $xy \in P \setminus P^2$ for every $x, y \in R$.

*Remark* 2.19. In a Boolean like semiring $R$, $a^n = a$ or $a^2$ or $a^3$ for evert $a \in R$.

**Theorem 2.20.** *Let $S$ be a multiplicatively closed subset and $I$ an ideal in a Boolean like semiring $R$. Then $\frac{r}{s}, r \in R, s \in S$; is in $S^{-1}I$ if and only if $mr \in I$ for some $m \in S$*

**Theorem 2.21.** *Let $P$ be a primary ideal of a Boolean like semiring $R$ and $S$ be a multiplicatively closed subset of $R$ such that $P \cap S = \emptyset$. Then $S^{-1}P$ is a primary ideal of $S^{-1}R$.*

**Theorem 2.22.** *Let $J$ be a primary ideal of $S^{-1}R$. Then $J^c$ is also a primary ideal of $R$. Moreover $J^c \cap S = \emptyset$.*

**Theorem 2.23.** *Let $P$ be an almost primary ideal of a Boolean like semiring $R$ and $S$ a multiplicatively closed subset of $R$ such that $P \cap S = \emptyset$. Then $S^{-1}P$ is an almost primary ideal of $S^{-1}R$*

**Theorem 2.24.** *Let $J$ be an almost primary ideal of $S^{-1}R$. Then $J^c$ is also an almost primary ideal of $R$. Moreover $J^c \cap S = \varnothing$ .*

**Definition 2.25.** An ideal $I$ of a Boolean like semiring $R$ is called semiprime if $x \in I$ whenever $x^2 \in I$ for every $x \in R$.

**Definition 2.26.** A proper ideal $P$ of a Boolean like semiring $R$ is called weakly prime if $0 \neq xy \in P$ implies $x \in P$ or $y \in P$ for every $x, y \in R$.

**Definition 2.27.** A proper ideal $P$ of a Boolean like semiring $R$ is called weakly primary if $0 \neq xy \in P$ implies $x \in P$ or $y^2 \in P$ for every $x, y \in R$.

## 3. Special Classes of Ideals

In this section, we study certain generalizations of prime ideals namely primary, almost primary, weakly prime, weakly primary ,semiprime, quasi prime and 2-potent ideals of Boolean like semiring of fractions. We begin with the following result.

**Theorem 3.1.** *Let $I$ be an ideal of a weak commutative Boolean like semiring $R$ and $S$ be a multiplicatively closed subset of $R$. Then,*

(1) $I^{ec} = \cup(I : s) = \{r \in R | sr \in I \text{ for some } s \in S\}$.
(2) $I^e = S^{-1}R$ *if and only if* $I \cap S \neq \varnothing$.

*Proof.*
1. Let $r \in I^{ec}$. Then $f(r) = \frac{rs}{s} \in I^e$ for some $s \in S$. Then, $t(rs) \in I$ for some $t \in S$. Hence $(ts)r \in I$ since R is weak commutative. So, $r \in \cup(I : s)$ since $ts \in S$. To prove the other way, let $r \in \cup(I : s)$ so that $sr \in I$ for some s. And $f(r) = \frac{sr}{s}$ $\in I^e$ as a result, $r \in I^{ec}$
2. Suppose $I^e = S^{-1}R$ which implies $\frac{s}{s} \in I^e$ for $s \in S$. Hence $ts \in I$ for some t in S so that $I \cap S \neq \varnothing$ since $ts \in S$ Conversely, if $I \cap S \neq \varnothing$ , let $s \in I \cap S$ . Thus, $\frac{s}{s} \in I^e$ and let $\frac{r}{s} \in S^{-1}R$ so that $\frac{r}{s} = \frac{r}{s}\frac{s}{s} \in I^e$ since $I^e$ is an ideal.

□

Now we study certain characterizations in which $I^{ec} = I$.

**Definition 3.2.** A proper ideal $I$ of a Boolean like semiring is 2-potent prime if $x \in I$ or $y \in I$ whenever $xy \in I^2$ for every $x, y \in R$.

**Theorem 3.3.** *Let $P$ be a 2-potent prime ideal of a weak commutative Boolean like semiring $R$ and $S$ a multiplicatively closed subset of $R$ such that $P \cap S = \varnothing$. Then $P^{ec} = P$.*

*Proof.* Clearly $P \subseteq P^{ec}$. Let $y \in P^{ec}$. So, $ty \in P$ for some $t \in S$. Hence, $[ty]^2 \in P^2$ which implies $t^2y^2 \in P^2$; so that $t^2 \in P$ or $y^2 \in P$ (since P is 2-potent prime). As a result, $y^2 \in P$ (since $t^2$ is not in $P$). Hence, $(y^2)^2 = y^2 \in P^2$. Therefore, $y \in P$. □

*Remark* 3.4. $P^{ec} = P$ for every prime ideal $P$ of $R$ disjoint from $S$ since every prime ideal is 2-potent prime.

Now we introduce the notion of quasi prime ideals in Boolean like semirings. We begin with the following.

**Definition 3.5.** An ideal $I$ of a Boolean like semiring $R$ is called quasi prime if $x \in I$ whenever $x^3 \in I$ for every $x \in R$.

**Lemma 3.6.** *In Boolean like semiring, an ideal is semiprime if and only if it is quasi prime.*

*Proof.* Let $P$ be a semiprime ideal of a Boolean like semiring $R$ such that $x^3 \in P$ for some $x$ in $R$. Then $x^2 = (x^3)(x^3) = x^6 = x^2x^4 = x^2x^2 = x^2 \in P$ implies $x \in P$ (since $P$ is semiprime). Hence $P$ is quasi prime. Conversely, let $P$ be a quasi prime ideal such that $x^2 \in P$. Thus, $x^3 = x.x^2 \in P$. Hence $x \in P$ (since $P$ is quasi prime). Therefore, $P$ is semiprime. □

**Corollary 3.7.** *Let $P$ be a primary ideal of a Boolean like semiring $R$ such that $P \cap S = \varnothing$. Then $P^{ec} = P$ if $P$ is semiprime or quasi prime.*

*Proof.* Clearly $P \subseteq P^{ec}$. Let $y \in P^{ec}$. Then, $sy \in P$ for some $s \in S$. Hence, $s \in P$ or $y^2 \in P$ (since $I$ is primary). So, $y^2 \in P$ since $s$ is not in $P$. Thus, $y \in P$. □

**Theorem 3.8.** *If $I$ is a semiprime ideal of a weak commutative Boolean like semiring $R$ and $S$ is a multiplicatively closed subset of $R$, then $I^e$ is a semiprime ideal of $S^{-1}R$.*

*Proof.* Let $I$ be a semiprime ideal and $\frac{r}{s} \in S^{-1}R$ such that $\frac{r}{s} \notin I^e$. Then $tr \notin I \ \forall t \in S$ so that $(tr)^2 = t^2r^2 \notin I$ since I is semiprime. Thus, $\frac{t^2r^2}{t^2s^2} \notin I^e$. Therefore, $\frac{r^2}{s^2} \notin I^e$. □

**Theorem 3.9.** *If $J$ is a semiprime ideal of $S^{-1}R$, then $J^c$ is a semiprime ideal of $R$.*

*Proof.* Let x be in R such that $x^2 \in J^c$ . Thus, $f(x^2) \in J$ which implies $f(x)f(x) \in J$. So, $[f(x)]^2 \in J$ . Hence, $f(x) \in J$ since J is semiprime Thus, $x \in J^c$. □

*Remark* 3.10.
a) Every 2-potent prime ideal is semiprime.
b) $I^e$ is a 2-potent prime ideal of $S^{-1}R$ if I is a 2-potent prime ideal of R.
c) $J^c$ is a 2-potent prime ideal of R if J is a 2-potent prime ideal of $S^{-1}R$.
d) $I^{ec}$ is a semiprime ideal of R if I is a semiprime.

**Theorem 3.11.** *Let $P$ be a weakly primary ideal of a Boolean like semiring $R$ and $S$ be a multiplicative subset of $R$. Then $P^e$ is a weakly primary ideal of $S^{-1}R$.*

*Proof.* Let P be a weakly primary ideal of $R$ such that $0 \neq \frac{r_1r_2}{s_1s_2} \in P^e$. So, $s[r_1r_2] \in P$ and $s(r_1r_2) \neq 0 \ \forall s \in S$ . Which implies, $[sr_1]r_2 \in P$

. Hence $sr_1 \in P$ or $r_2^2 \in P$. If $sr_1 \in P$ then $\frac{r_1}{s} = \frac{sr_1}{ss} \in P^e$ and if $(r_2)^2 \in P$, then $[\frac{r_2}{s_2}]^2 = \frac{r_2^2}{s_2^2} \in P^e$. In any case, the result holds. □

*Remark* 3.12.
i) $P^e$ is a weakly prime ideal of $S^{-1}R$ whenever P is weakly prime.
ii) The contraction of a weakly primary (and hence weakly prime) ideal of a Boolean like semiring $R$ is weakly primary (weakly prime) if $R$ is a domain.

**Theorem 3.13.** *If $Q$ is a primary ideal of a Boolean like semiring $R$ such that $P = r(Q)$ and $Q$ is disjoint from a multiplicative subset $S$ of $R$, then $S^{-1}P$ is a prime ideal of $S^{-1}R$*

*Proof.* Since the radical of a primary ideal is prime, $P$ is prime. Thus the extension $S^{-1}P$ of a prime ideal is also prime in $S^{-1}R$. □

**Lemma 3.14.** *If $Q$ is a primary ideal of a Boolean like semiring $R$ disjoint from $S$, then $[r(Q)]^{ec} = r(Q)$.*

*Proof.* The proof is straightforward. □

**Lemma 3.15.** *Let $I$ and $J$ be ideals of a Boolean like semiring $R$. Then $S^{-1}I \subseteq S^{-1}J$ if $I \subseteq J$ and the converse will be true if $J$ is a 2-potent prime ideal of $R$ disjoint from $S$.*

**Theorem 3.16.** *Let $S$ be a multiplicatively closed subset of a Boolean like semiring $R$ and $I$ and $J$ be ideals of $R$. Then,*

a) $S^{-1}(I + J) = S^{-1}I + S^{-1}J$
b) $S^{-1}(IJ) = (S^{-1}I)(S^{-1}J)$
c) $S^{-1}(I \cap J) = (S^{-1}I) \cap (S^{-1}J)$
d) $S^{-1}r(I) = r(S^{-1}I)$

*Proof.* a) Since $I \subseteq I + J$ and $J \subseteq I + J$, and $I + J$ is an ideal, we have $S^{-1}I \subseteq S^{-1}(I + J)$ and $S^{-1}J \subseteq S^{-1}(I + J)$. Thus we have $S^{-1}I + S^{-1}J \subseteq S^{-1}(I + J)$. Conversely, let $\frac{a}{s} \in S^{-1}(I + J)$. Then we have, $ta \in I + J$ for some $t$ in $S$. So, $ta = a_1 + a_2$ where $a_1 \in I$ and $a_2 \in J$. And hence, $\frac{a}{s} = \frac{ta}{ts} = \frac{a_1+a_2}{ts} = \frac{a_1}{ts} + \frac{a_2}{ts} \in S^{-1}I + S^{-1}J$.

b) Let $m \in S^{-1}(IJ)$. So that $m = \frac{x}{s}$ for some $x \in IJ$. Hence, $x = \sum a_i b_i$ for some $a_i$ in $I$ and $b_i$ in $J$. So, $\frac{a_i}{s} \in S^{-1}I$ and $\frac{b_i}{s} \in S^{-1}J$. Thus, $m = \frac{x}{s} = \frac{\sum a_i b_i}{s} = \sum \frac{a_i b_i}{s} = \sum \frac{a_i b_i s}{s^2} = \sum (\frac{a_i}{s}).(\frac{b_i}{s} s) \in (S^{-1}I)(S^{-1}J)$ Hence $S^{-1}(IJ) \subseteq (S^{-1}I)(S^{-1}J)$.

To show that $(S^{-1}I)(S^{-1}J) \subseteq S^{-1}(IJ)$, let $y \in (S^{-1}I)(S^{-1}J)$. Then $y = \sum \frac{a_i}{s_i}\frac{b_i}{n_i}$ for some $\frac{a_i}{s_i} \in S^{-1}I$ and $\frac{b_i}{n_i} \in S^{-1}J$. So $y = \sum \frac{a_i b_i}{q_i}$ where $q_i = s_i n_i$. Thus, $y = \frac{\sum a_i b_i}{q}$ where $q = \prod q_i$. As a result, $y \in S^{-1}(IJ)$.

c) Clearly $S^{-1}(I \cap J) \subseteq (S^{-1}I) \cap (S^{-1}J)$. Let $\frac{x}{s} \in (S^{-1}I) \cap (S^{-1}J)$. Then, $\frac{x}{s} \in (S^{-1}I)$ and $\frac{x}{s} \in (S^{-1}J)$. This implies, $\exists t_1, t_2 \in S \ni t_1 x \in I, t_2 x \in J$. Hence, $(t_1 x)t_2 \in I, t_1(t_2 x) \in J$ since $I$ and $J$ are ideals. So, $(t_1 t_2)x \in I, (t_1 t_2)x \in J$ since $R$ is weak commutative. Thus,$(t_1 t_2)x \in I \cap J$ . As a result, $\frac{x}{s} \in S^{-1}(I \cap J)$.

d) Let $\frac{x}{s} \in S^{-1}r(I)$. Then, $tx \in r(I), t \in S$. So that, $(tx)^n \in I$ for some natural number $n$. Hence, $t^n x^n \in I$ so that $\frac{x^n}{s^n} = \frac{t^n x^n}{t^n s^n} \in S^{-1}I$. Thus, $[\frac{x}{s}]^n \in S^{-1}I$. Consequently, $\frac{x}{s} \in r(S^{-1}I)$. Conversely, let $\frac{x}{s} \in r(S^{-1}I)$. So that $[\frac{x}{s}]^k \in S^{-1}I$ for some natural number $k$. Hence, $\frac{x^k}{s^k} \in S^{-1}I \Rightarrow mx^k \in I$ for some $m$ in $S$. $m^k x^k \in I$ since $I$ is ideal. As a result, $(mx)^k \in I$. So, $mx \in r(I)$. Thus, $\frac{mx}{ms} \in S^{-1}r(I)$. Consequently, $\frac{x}{s} \in S^{-1}r(I)$.

□

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