Mass spectroscopy of ground and excited states of $c\bar{c}$ mesons in the frame work of Bethe Salpeter equation

A Dissertation Submitted to

the Department of Physics

Addis Ababa University

In Partial Fulfilment of

the Requirement of the Degree of

Doctor of Philosophy in Physics

(**High energy physics**)

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Addis Ababa, Ethiopia

January 2018

DECLARATION

I hereby declare that this PhD dissertation is my original work and has not been presented for a degree in any other university, and that all sources of material used for the dissertation have been duly acknowledged.

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Place and date of submission:

Addis Ababa University

Department of Physics

January 2018

Addis Ababa University
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by

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Acknowledgement

I am deeply indebted to my supervisor Dr. Shashank Bhatnagar for his guidance, encouragement, invaluable comments, patience and friendly approach during this work.

I am greatly acknowledge Samara University and Addis Ababa University for their financial support during my study.I also wishe to thank Chandigarh University for the hospitality during my visit during August-October 2017.

There are no words to describe the appreciation and gratitude I feel for my parents and families most importantly of my mother and my wife. Their wonderful example of dedication and commitment has provided me with a lifetime of inspiration. And my mom, special thanks are for you. My mother is next to God for me, and whatever I am today is because of the moral values and commitments taught by her.

Abstract

An important role in applications of quantum chromodynamics to hadronic physics is played by charmonium and bottomomium which are built up of a heavy quark and heavy antiquark. In this work, we calculate the mass spectrum of charmonium for 1P, ..., 3P states of 0^{++} , 1^{++} and 1^{+-}, as well as for 1S, ..., 4S states of 0^{-+}, and 1S, ..., 5D states of 1^{--} along with the two-photon decay widths of ground and first excited state of 0^{++} quarkonia for the process, $0^{++} \rightarrow \gamma\gamma$ in the framework of a QCD motivated Bethe-Salpeter Equation. In this 4×4 BSE framework, the coupled Salpeter equations are first shown to decouple for the confining part of interaction, under heavy-quark approximation, and analyically solved, and later the one-gluon-exchange interaction is perturbatively incorporated leading to mass spectral equations for various quarkonia. The analytic forms of wave functions obtained are used for calculation of two-photon decay widths of χ_{c0}. Our results are in reasonable agreement with data (where ever available) and other models. We derive the mass spectrum of the meson anallytically by decoupling the coupled integral equation showing that the incorporation of coulomb potential purterbatively brings our numerical value much more closer to the experimental value and lift the degeneracy of the spectrum.

To the memory of my late son,

Imran Lmenew

June 6, 2005 - April 26, 2009 E.C

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1

Introduction

The study of elementary particles and their interaction has brought about a better understanding of our universe from sub-nuclear scales to cosmological scales. There are four fundamental interactions observed in nature: strong, electromagnetic, weak, and gravitational in order of decreasing strength respectively. To each of these forces, there belongs a physical theory.

Quantum Chromodynamics (QCD) is the correct gauge theory for strong interaction dynamics. The physical degrees of freedom are quarks and gluons. Gluons are the quanta of gauge fields which are in some ways analogous to photons in QED. The crucial difference is that gluons couple directly to each other, where as photons do not. The QCD effective coupling constant $\alpha_s(Q^2)$ decreases as the momentum transfer scale Q increases (asymptotic freedom). This allows to make perturbative calculations in α_s at high energies. At low energies, it develops an intrinsic scale, usually referred as $\mathit{\Lambda}_{QCD}$ that is $\Lambda \sim \Lambda_{QCD}$ is the characteristic scale of QCD, which provides the main contribution to the masses of most hadrons. At scales Q$\sim \Lambda_{QCD}$, $\alpha_s(Q) \sim 1$ and perturbation theory cannot be used. Investigations must be carried out using nonperturbative techniques.The QCD coupling constant $\alpha_s(Q^2)$ is small at large Q^2. A consequence of the self-coupling of gluons is that the force between quarks does not vanish at large distances but becomes a constant

$F_{QCD} \approx 800 MeV fm^{-}1 \approx 15,000N$. This is in contrast to the coulomb force between two charges which goes as $\sim \frac{1}{r^2}$ and hence vanishes at large distances. It would, thus, require a vast amount of energy to separate quarks out to a macroscopic distance. Indeed, long before the separation becomes large enough to measure, the energy stored in the gluon field ”sparks” into quark-antiquark pairs. This phenomenon of QCD explains why freely propagating quarks are not observed in nature, and it is called confinement. An important role in applications of Quantum Chromodynamics (QCD) to hadronic physics is played by charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$), which are built up of a heavy quark and heavy anti-quark. By definition, heavy quark has a mass m, which is large in comparison to the typical hadronic scale, Λ_{QCD}. Quarkonia are characterized by at least three widely separated scales [1]: the hard scale (the mass m of heavy quarks), the soft scale (relative momentum q$\sim$ mv), and the ultra soft scale (typical kinetic energy, $E \sim mv^2$ of heavy quark and anti-quark). The appearance of all these scales in the dynamics of heavy quarkonium makes its quantitative study extremely difficult. Quarkonium systems are crucially important to improve our understanding of QCD. They probe all the energy regions of QCD from hard region, where perturbative QCD dominates, to the low energy region, where non-perturbative effects dominate. Quarkonium states are thus a unique laboratory where our understanding of non-perturbative QCD may be tested.

There has been a renewed interest in recent years in spectroscopy of these heavy hadrons in charm and beauty sectors, which was primarily due to experimental facilities the world over such as BABAR, Belle, CLEO, DELPHI, BES etc.[2-6], which have been providing accurate data on $c\bar{c}$, and $b\bar{b}$ hadrons with respect to their masses and decays. In the process many new states have been discovered such as $\chi_{b0}(3P)$, $\chi_{c0}(2P)$, $\chi(3915)$, $\chi(4260)$, $\chi(4360)$, $\chi(4430)$, $\chi(4660)$ [7]. The data strongly suggests that among new resonances may be exotic four-quark states, or hybrid states with gluonic degrees of freedom in addition to $c\bar{c}$ pair, or loosely bound states of heavy hadrons, i.e. charmonium molecules. Further, there are also open questions about the quantum number assignments of some of these states such as X(3915) (as to whether it is $\chi_{c0}(2P)$ or $\chi_{c2}(2P)$[7,8]. Thus charmonium offers us intriguing puzzles. However, since the mass spectrum and the decays of all these bound states of heavy quarks can be tested experimentally, theoretical studies on them may throw valuable insight about the heavy quark dynamics and lead to a deeper understanding of QCD further. Studies on mass spectrum of these hadrons is particularly important, since it throws light on the $Q\bar{Q}$ potential, since the long range confinement potential has not been derived from QCD sofar. Further, though these states appear to be simple, however, their production mechanism is still not properly understood. These mesons are involved in a number of reactions which are of great importance for study of Cabibbo-Kobayashi-Maskawa (CKM) matrix and CP violation. In this thesis we also study the two-photon decays of scalar quarkonia, χ_{c0}. These decays are sensitive probes of quarkonium wave functions.

Many non-perturbative approaches, such as Effective field theory[9], Lattice QCD[10-12], Chiral perturbation theory[13], QCD sum rules[14, 15], N.R.QCD[16], dynamical-equation based approaches like Schwinger-Dyson equation and Bethe-Salpeter equation (BSE)[17-26], and potential models[27-32] have been proposed to deal with the long distance property of QCD. We wish to mention that amongst all these approaches, effective field theories provide new tools and definite predictions concerning heavy quarkonium and decays. Recent progress in understanding of non-relativistic field theories make it possible to go beyond phenomenological models, and for the first time face the possibility of providing unified description of all aspects of heavy quarkonium physics. This allows us to use quarkonium as a bench mark for understanding of QCD, and for precise determination of Standard Model parameters (e.g. heavy quark masses, QCD coupling constant, α_s), and for new physics searches.

All this opens up new challenges in theoretical understanding of heavy hadrons and also provide an important tool for exploring the structure of these simplest bound states in QCD and for studying the non-perturbative (long distance) behavior of strong interactions. In this work, we now do the full mass spectral problem of charmonium for 1P, ..., 4P states of 0^{++} , 1^{++}, 1^{+-}, as well as for 1S, ..., 4S states of 0^{-+}, and 1^{--} along with the two-photon decay widths of ground and first excited state of scalar quarkonia, χ_{c0} for the process, $0^{++} \rightarrow \gamma\gamma$ in the framework of a QCD motivated Bethe-Salpeter Equation under Covariant Instantaneous Ansatz (CIA), employing the full BS kernel comprising both, the long-range confinement, and the short range one-gluon-exchange (coulomb) interactions. We do understand that in $Q\bar{Q}$ quarkonia, the constituents are close enough to each other to warrant a more accurate treatment of the coulomb term. Though for $b\bar{b}$ systems, the coulomb term will be extremely dominant in comparison to confining term, and should not be treated perturbatively. However, seeing our mass spectral results for $c\bar{c}$ systems, it may not be so unreasonable to treat the coulomb term perturbatively for $c\bar{c}$ systems. This is specially so for orbital excitations of these states, where the centrifugal effects[33] ensure that the $c-\bar{c}$ separation is large enough to feel the effect of confining term more strongly than the coulomb term. We further wish to state that some of earlier works[33-35] have treated the OGE(coulomb) term perturbatively for charmed mesons and baryons, while some works[36, 37] did not take into account the importance of coulomb term for heavy quarkonium systems. Thus, in the present work, the coupled Salpeter equations for scalar (0^{++}), pseudovector(1^{+-}) and axial vector (1^{++}) quarkonia are first shown to decouple for the confining part of interaction, under heavy-quark approximation, and the analytic forms of mass spectral equations are worked out, which are then solved in approximate harmonic oscillator basis to obtain the unperturbed wave functions for various states of these quarkonia. We then incorporate the one-gluon-exchange perturbatively into the unperturbed spectral equation, and obtain the full spectrum. The wave functions of scalar (0^{++}) quarkonia, are then used to calculate their two-photon decay widths. We further extend the mass spectral calculations of pseudoscalar (0^{-+}), and vector (1^{--}) quarkonia with the perturbative inclusion of one-gluon-exchange effects. The approximations used in this analytic treatment of the confining interaction are shown to be fully under control. In this thesis, we calculate the mass spectrum, weak decay constants and two photon decay widths of ground, radially and orbitally excited states of 0^{++}, 1^{++}, 1^{+-}, 0^{-+} and 1^{--} charmonium and bottomonium, along with the spectra and transition amplitudes of heavy equal mass scalar and axial vector mesons using the formulation of Bethe-Salpeter equation under covariant Instantaneous Ansatz (CIA).

This thesis is organized as follows: Chapter 2 deals with the derivation of Bethe Salpeter equation. It also deals with the derivation of Salpeter equations under CIA. Chapter 3 deals with the spectroscopy of ground and excited states of 0^{++}(scalar), 1^{++}(axial vector) and 1^{+-} $c\bar{c}$ quarkonia. Chapter 4 deals with the spectroscopy of 0^{-+} (pseudoscalar) and 1^{--} (vector) $c\bar{c}$ quarkonia. Chapter 5 deals with two photon decays of 0^{++} $c\bar{c}$ quarkonia. Chapter 6 deals with Summary and Conclusions.

2

Bethe Salpeter Equation

The Bethe-Salpeter equation describes the bound states of a two-body (two particles) quantum mechanical system in a relativistically covariant formalism. Examples of two- particle systems described by the Bethe-Salpeter equation are positronium which is the bound state of an electron-positron pair, the exciton which is the bound state of an electron-hole pair, meson which is the quark-anti quark pair in bound state etc. Since the particle pair described by the Bethe-Salpeter equation is in a bound state, the particles can interact infinitely often. So that this situation cannot be described by the summation of a few Feynman diagrams. The Bethe-Salpeter formalism is generally accepted to represent the appropriate framework for the description of bound states within relativistic quantum field theory. Within this formalism, a bound state is described by its Bethe-Salpeter amplitude. In principle, this bound state amplitude should be found as a solution of the homogeneous Bethe-Salpeter equation. However, apart from a very few special cases such as the famous Wick-Cutkosky model which describes the interaction of two scalar particles by exchange of a massless scalar particle,the Bethe-Salpeter equation turns out to be practically not tractable. One of the main reasons for this fact is the appearance of time- like variables in the equation of motion. Consequently people usually consider some 3-dimensional reduction of the Bethe-Salpeter equation. The most popular among these 3-dimensional reductions is based on the assumption that the interaction between the bound-state constituents is instantaneous in the center-of momentum frame of the bound state. The result of this is called the ”Salpeter equation” or ”instantaneous Bethe-Salpeter equation”. This equation may be formulated as eigenvalue problem for the mass M of the bound state. The Salpeter amplitude is obtained from the Bethe-Salpeter amplitude by equating the time variables of the involved bound-state constituents. To derive the Bethe-Salpeter equation first it is necessary to start from the one particle Dirac propagator, then generalize to the two particle propagator.

2.1 Single particle Dirac propagator In External Field

To explain the mathematical concept, we start with Schrodinger equation. We recall that the Schrodinger quation of a particle($\hbar = 1$) is given by

$$[\frac{i\partial}{\partial t} - H]\psi(x_2) = 0, \tag{2.1}$$

where the total Hamiltonian of the particle is given by

$$H = \frac{-\nabla^2}{2m} + V. \tag{2.2}$$

This equation is valid for the non relativistic two body problem, where m is the reduced mass

$$m = \frac{m_1 m_2}{m_1 + m_2}. \tag{2.3}$$

The solution of the above equation for the particle propagating from space time point x_2 to space time point x_1 takes the form

$$\psi(x_1) = \int d^4x K(x_1, x_2)\psi(x_2), t_1 > t_2, \tag{2.4}$$

where the propagator(greens function) satisfies

$$[\frac{i\partial}{\partial t} - H]K(x_1, x_2) = i\delta^{(4)}(x_1 - x_2), \tag{2.5}$$

where the propagator $K(x_1, x_2)$ is interpreted as the probabilty amplitude for a particle wave originating at the space time point x_2 to propagate to the space time point x_1. This is propagator equation for Schrodinger particles which moves only forward in time. If we replace the Schrodinger Hamiltonian by the Dirac Hamiltonain

$$H = \vec{\alpha}.\vec{p} + \beta m, \tag{2.6}$$

one can obtain the free Dirac particle equation in the absence of electromagnetic field $A_\mu(x)$ for the state function $\psi(x_2)$ in the form

$$[i\frac{\partial}{\partial t} - \vec{\alpha}.\vec{p} - \beta m]\psi(x_2) = 0. \tag{2.7}$$

This is free Dirac equation in the abscence of electromagnetic field $A_\mu(x)$. $\vec{\alpha}$ and β are 4×4 Dirac matrices and Eq. (2.5) becomes Dirac equation for the propagator. In the presence of electromagnetic field $A_\mu(x)$, Eq. (2.7) takes different form. We note that the interaction Hamiltonian between the atomic electrons and the radiation field $A_\mu(x)$ is assumed to be obtained from the minimal coupling prescription as

$$\begin{aligned} \vec{p} &\longrightarrow \vec{p} - e\vec{A} \\ \frac{i\partial}{\partial t} &\longrightarrow \frac{\partial}{\partial t} - e\phi. \end{aligned} \tag{2.8}$$

Upon substitution of Eq.(2.8) into free Dirac Particle equation of Eq. (2.7), one can obtain the Dirc equation in the presence of electromagnetic field $A_\mu(x)$ as

$$[i\frac{\partial}{\partial t} - e\phi - \vec{\alpha}.\vec{p} + e\vec{\alpha}.\vec{A} - \beta m]\psi(x_2) = 0. \tag{2.9}$$

To make this equation covariant, lets multiply from the left on both sides by $(-\beta)$. It then follows that

$$[-i\beta\frac{\partial}{\partial t} + e\beta\phi + \beta\vec{\alpha}.\vec{p} - e\beta\vec{\alpha}.\vec{A}) + m]\psi(x_2) = 0. \tag{2.10}$$

Using the fact that

$$\beta = \gamma_4,$$
$$\gamma_k = -i\beta\alpha_k, \tag{2.11}$$

where $k = 1, 2, 3$, we obtain

$$[\gamma_\mu \partial_\mu - ie\gamma_\mu A_\mu + m]\psi(x_2) = 0. \tag{2.12}$$

This is the covariant form of Dirac equation in the presence of electromagnetic field $A_\mu(x)$. To solve this equation , we consider the unit source problem and the full Dirac propagator satisfies the equation

$$[\gamma_\mu \partial_\mu - ie\gamma_\mu A_\mu + m]K(x_1, x_2) = -i\delta^{(4)}(x_1 - x_2), \tag{2.13}$$

where $K(x_1, x_2)$ is the full Dirac propagator. For a free wave equation($e = 0$), this equation reduces to

$$[\gamma_\mu \partial_\mu + m]K_0(x_1, x_2) = -i\delta^{(4)}(x_1 - x_2), \tag{2.14}$$

where $K_0(x_1, x_2)$ is the free Dirac particle propagator.

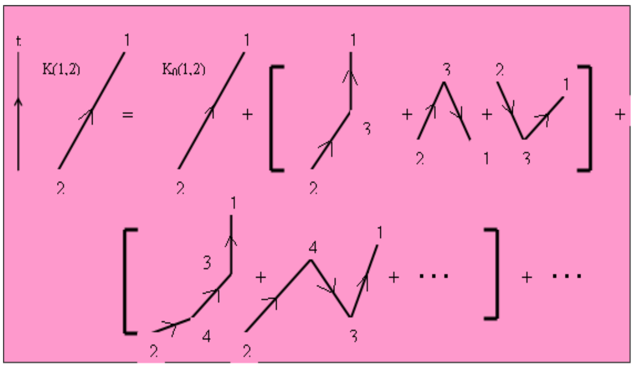

Figure 2.1: Diagrammatical interpretation of one particle propagator

On the other hand, Eq.(2.13)can be rewritten in the form

$$[\gamma_\mu \partial_\mu + m]K(x_1, x_2) = -i\delta^{(4)}(x_1 - x_2) + ie\gamma_\mu A_\mu K(x_1, x_2), \tag{2.15}$$

where the propagation is from x_2 to x_1. On account of the relation

$$\int_{-\infty}^{\infty} f(x)\delta(x-a) = f(a) \tag{2.16}$$

and

$$\int \delta(a-x)\delta(x-b)dx = \delta(a-b) \tag{2.17}$$

and introducing space time points x_3, where $t_2 < t_3 < t_1$, Eq. (2.15) becomes

$$[\gamma_\mu\partial_\mu + m]\kappa(x_1,x_2) = -i\int d^4x_3\delta^{(4)}(x_1-x_3)\delta^{(4)}(x_3-x_2) + ie\int d^4x_3\delta^4(x_3-x_1)\gamma_\mu A_\mu(x_3)\kappa(x_3,x_2). \tag{2.18}$$

From Eq.(2.14), we can deduce the equation satisfied by a free propagator $K_0(x_1, x_3)$ as

$$[\gamma_\mu\partial_\mu + m]K_0(x_1,x_3) = -i\delta^{(4)}(x_1-x_3). \tag{2.19}$$

Then, Eq. (2.18) takes the form

$$\begin{aligned}[\gamma_\mu\partial_\mu + m]K(x_1,x_2) = & \int d^4x_3[\gamma_\mu\partial_\mu + m]K_0(x_1,x_3)\delta^4(x_3-x_2) \\ & -e\int d^4x_3[\gamma_\mu\partial_\mu + m]K_0(x_1,x_3)\gamma_\mu A_\mu(x_3)K(x_3,x_2)\end{aligned} \tag{2.20}$$

Dividing this equation by $\gamma_\mu\partial_\mu + m$ on both sides and carrying out the integration for the first term on the right hand side using the identity given by Eq. (2.16) , we get

$$K(x_1,x_2) = K_0(x_1,x_2) - e\int d^4x_3K_0(x_1,x_3)\gamma_\mu A_\mu(x_3)K(x_3,x_2). \tag{2.21}$$

Introducing an intermediate space time point x_4 where $t_2 < t_4 < t_3 < t_1$, one can write $K(x_3, x_2)$ in the same mannor to be

$$K(x_3,x_2) = K_0(x_3,x_2) - e\int d^4x_4K_0(x_3,x_4)\gamma_\mu A_\mu(x_4)K(x_4,x_2). \tag{2.22}$$

plugging Eq. (2.22) into Eq. (2.21), the full propagator takes the form

$$\begin{aligned}K(x_1,x_2) = & K_0(x_1,x_2) - e\int d^4x_3K_0(x_1,x_3)\gamma_\mu A_\mu(x_3)K_0(x_3,x_2) \\ & +(-e)^2\int d^4x_3\int d^4x_4K_0(x_1,x_3)\gamma_\mu A_\mu(x_3)K_0(x_3,x_4) \\ & \times\gamma_\mu A_\mu(x_4)K(x_4,x_2) + ...\end{aligned} \tag{2.23}$$

Since the propagating particle is Dirac particle, both direction of times ($\uparrow e^-$ and $\downarrow e^+$) are meaningful. In the coming subsection we generalize one particle propagator to two particle propagator where the particles are mutually interacting.

2.2 Two Particle Bound State Propagator With Particles In Mutual Interaction

Now we consider the simultaneous propagation of two free Dirac particles. Particle one propagates from space time point x_3 to space time point x_1 through path a and particle two propagates from space time point point x_4 to space time point x_2 through path b. The free propagator for simultaneous propagation of two free

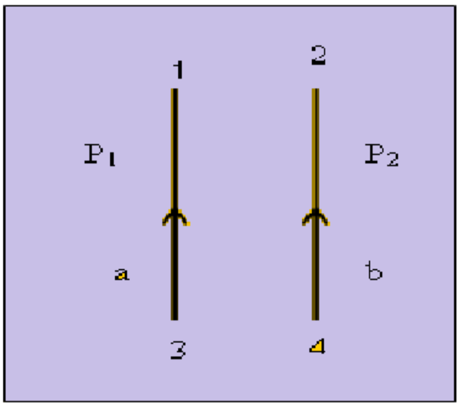

Figure 2.2: Simultaneous propagation of two free particles.

particles is defined by

$$K_{0ab}(x_1, x_2; x_3, x_4) = K_{0a}(x_1, x_3)K_{0b}(x_2, x_4). \tag{2.24}$$

where $K_{0a}(x_1, x_3)$ and $K_{0b}(x_2, x_4)$ are the one particle free Dirac propagator shown in the figer 2.2 below.

Let us consider the two particles interacting through a single photon exchange.To find the form of the propagator for these two interacting particles, lets consider the simplest case (Moller interaction $e^- + e^- \longrightarrow e^- + e^-$). Now lets assume the interaction between a and b be coulomb interaction which is given by Coulomb potential

$$V(\vec{r}) = \frac{e^2}{4\pi\vec{r}_{ab}}, \tag{2.25}$$

where $\vec{r}_{ab} = \vec{r}_a - \vec{r}_b$ is the separation between the two particles. But the Coulomb potential in Eq.(2.25) is non-covariant. To write it in a covariant form, we first write $\frac{e^2}{\vec{r}_{ab}}$ in momentum space by the Fourier transformation. The three dimensional Fourier transform of the Coulomb potential is given by

$$V(\vec{q}) = \frac{e^2}{4\pi}\int d^3\vec{r}\frac{e^{-i\vec{q}.\vec{r}}}{\mid \vec{r}_{ab} \mid}, \tag{2.26}$$

where $\vec{q} = \vec{p}_a - \vec{p}_b$ is the momentum transfer between the two electrons. To carry out this integration, let us consider spherical polar coordinates in which the vector $\vec{q}$ of the Fourier transform lies along the polar axis $(\theta = 0)$ So that, we write

$$V(\vec{q}) = \frac{e^2}{4\pi}\int e^{-i\vec{q}.\vec{r}} r dr \sin\theta d\theta d\phi. \tag{2.27}$$

Direct integration over the spherical coordinate ϕ gives 2π. We then have

$$V(\vec{q}) = \frac{e^2}{2}\int_0^\infty r dr \int_0^\pi e^{-iqr\cos\theta}\sin\theta d\theta. \tag{2.28}$$

In view of the fact that

$$\sin\theta e^{-iqr\cos\theta} = \frac{1}{iqr}\frac{d}{d\theta}[e^{-iqr\cos\theta}], \tag{2.29}$$

one can obtained

$$V(\vec{q}) = \frac{e^2}{2}\int_0^\infty dr[\frac{e^{iqr} - e^{-iqr}}{iq}]. \tag{2.30}$$

This integral has a problem at large distances i.e it does not converge. We repair this infrared divergence by the insertion of a convergence factor in the original integral that is we replace $\frac{1}{r} \rightarrow \frac{e^{-\lambda r}}{r}$, where λ is a small positive number that will be taken to

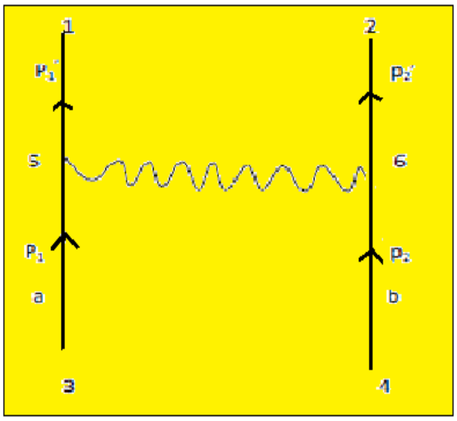

Figure 2.3: Simultaneous propagation with mutual interaction through a photon.

zero at the end of the calculation. With this alteration, the Fourier transformation given by Eq. (2.27) takes the form

$$V(\vec{q}) = \frac{e^2}{4\pi} \int d^3\vec{r} e^{-\lambda r} \frac{e^{-i\vec{q}.\vec{r}}}{\mid \vec{r}_{ab} \mid}. \tag{2.31}$$

Following the above steps, one can obtain

$$V(\vec{q}) = \frac{e^2}{2iq} \int_0^\infty dr[e^{-(\lambda-iq)r} - e^{-(\lambda+iq)r}]. \tag{2.32}$$

Carrying out the integration and taking the limt of λ to be zero, we obtain

$$V(\vec{q}) = \frac{e^2}{q^2}. \tag{2.33}$$

Still this equation is not in its covariant form. Finally by using the property of the Dirac gamma matrices, it is possible to write this potential invariantly as

$$V(\vec{q}) = \frac{e^2\gamma_4^a\gamma_4^b}{\mid \vec{q} \mid^2}. \tag{2.34}$$

Now we can write the covariant generalization of Eq. (2.34) as

$$V(\vec{q}) = \frac{e^2\gamma_\mu^a\gamma_\nu^b}{(\vec{p}_a - \vec{p}_b)^2}. \tag{2.35}$$

This can be put in the form

$$V(\vec{q}) = ie_a\gamma_\mu^a \left[\frac{-i}{q^2}\right] e_b\gamma_\nu^b. \tag{2.36}$$

The right hand side of Eq. (2.36) is Moller interaction in momentum space in the form

$$V(\vec{q}) = ie_a\gamma^a_\mu D^{\mu\nu}_F(x_5, x_6)e_b\gamma^b_\nu. \tag{2.37}$$

In two particle case we further generalized in the form

$$V(\vec{q}) = I(x_5, x_6), \tag{2.38}$$

where

$$I(x_5, x_6) = ie_a\gamma^a_\mu[D^{\mu\nu}_F(x_5, x_6)]e_b\gamma^b_\nu, \tag{2.39}$$

is the interaction potential and $D^{\mu\nu}_F(x_5, x_6)$ is the photon propagator joining space time points x_5 and x_6 which is given by

$$D^{\mu\nu}_F(x_5, x_6) = -i\delta_{\mu\nu}\int \frac{d^4q}{(2\pi)^4}(\frac{1}{q^2})e^{iq(x_5-x_6)}. \tag{2.40}$$

Here x_5 and x_6 are intermediate space time points at which a photon connects to two fermion lines a and b such that $t_3<t_5<t_1$ and $t_4<t_6<t_2$. To generalize a single particle propagator to two particle propagator, we replace

$$e\gamma_\mu A_\mu \rightarrow ie_a\gamma^a_\mu D^{\mu\nu}_F(x_5, x_6)e_b\gamma^b_\nu. \tag{2.41}$$

For general case of one quantum exchange between two interacting particles in mutual interaction, $D^{\mu\nu}_F$ would be the corresponging gauge field propagator. That means D_F denotes the photon propagator in QED and the gluon propagator in QCD. Then the first order correction to two particle propagator due to simutaneous interaction of both Dirac particle through exchange of a quantum is given by

$$K_{1ab}(x_1, x_2; x_3, x_4) = \int d^4x_5d^4x_6K_{0ab}(x_1, x_2; x_5, x_6)[ie_a\gamma^a_\mu D^{\mu\nu}_F(x_5, x_6)e_b\gamma^b_\nu]K_{0ab}(x_5, x_6; x_3, x_4). \tag{2.42}$$

This can be put in the form

$$K_{1ab}(x_1, x_2; x_3, x_4) = \int d^4x_5 d^4x_6 K_{0ab}(x_1, x_2; x_5, x_6) I(x_5, x_6) K_{0ab}(x_5, x_6; x_3, x_4), \quad (2.43)$$

where $I(x_5, x_6)$ plays the role of $\gamma_\mu A_\mu$ for two particle propagator.

Now let us consider the interaction of two particles through the exchange of two photons as shown in the figure (2.4) below. Introducing another intermediate space time points x_7 and x_8 between points x_5, x_3 and x_6, x_4 for the interaction of the two particles through exchange of two photons, the second order correction for the simultaneous propagation from x_3 to x_5 and x_4 to x_6 of the two-particle propagator becomes

$$\begin{aligned} K_{2ab}(x_1, x_2; x_3, x_4) = \quad & \int d^4x_5 \int d^4x_6 \int d^4x_7 \int d^4x_8 K_{0ab}(x_1, x_2; x_5, x_6) I(x_5, x_6) \\ & \times K_{0ab}(x_5, x_6; x_7, x_8) I(x_7, x_8) K_{0ab}(x_7, x_8; x_3, x_4), \quad (2.44) \end{aligned}$$

where $t_3 < t_7 < t_5 < t_1$ and $t_4 < t_8 < t_6 < t_2$. If we iterate upto n photon exchange, the

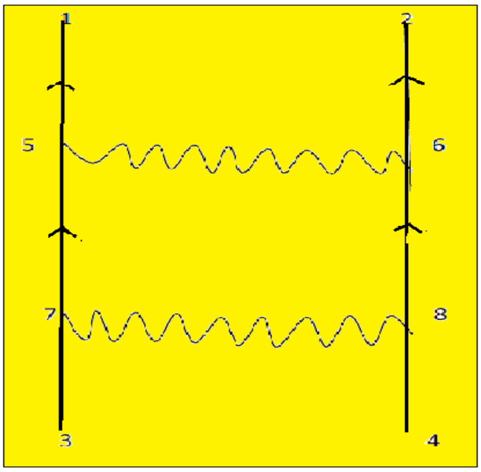

Figure 2.4: simultaneous propagation with mutual interaction through two photons.

full propagator for the two particles in a sequential action of Moller interaction in

Ladder approximation is given by

$$
\begin{aligned}
K_{ab}(x_1,x_2;x_3,x_4) = \quad & K_{0ab}(x_1,x_2;x_3,x_4) \\
& + \int d^4x_5 \int d^4x_6 K_{0ab}(x_1,x_2;x_5,x_6) I(x_5,x_6) K_{oab}(x_5,x_6;x_3,x_4) \\
& + \int d^4x_5 \int d^4x_6 \int d^4x_7 \int d^4x_8 K_{oab}(x_1,x_2;x_5,x_6) I(x_5,x_6) \\
& \times K_{0ab}(x_5,x_6;x_7,x_8) I(x_7,x_8) K_{0ab}(x_7,x_8;x_3,x_4) + ... \qquad (2.45)
\end{aligned}
$$

This can be put in the form

$$
K_{ab} = K_{0ab} + K_{1ab} + K_{2ab} + ... \qquad (2.46)
$$

One can see that the solution contains an infinite series of interaction. For many practical puposes, one restricts oneself to the lowest order of the interaction kernel that is one photon exchange. This prescription is called ladder approximation in which only one photon is exchanged at a given time, but this process can be repeated an arbitrary number of times. Now, Eq. (2.45) further can be put exactly in the form

$$
\begin{aligned}
K_{ab}(x_1,x_2;x_3,x_4) = \quad & K_{0ab}(x_1,x_2;x_3,x_4) \\
& + \int d^4x_5 \int d^4x_6 K_{0ab}(x_1,x_2;x_5,x_6) \\
& \times I(x_5,x_6) K_{0ab}(x_5,x_6;x_3,x_4). \qquad (2.47)
\end{aligned}
$$

For simplicity, we assume that the two particles 1 and 2 are distinguishable, since otherwise exchange graphs would appear.

For particle antiparticle system, in the expansion in terms of Feynman graphs, the direction of the arrow of one fermion line is inverted in Figure 2.5 below. [1]

[1]For simplicity, we assume the two particles are distinguishable otherwise exchange graphs would appear

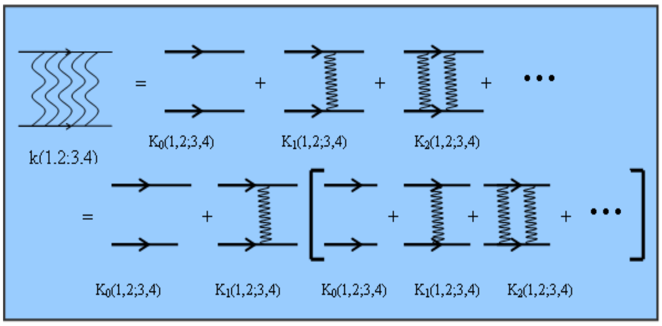

Figure 2.5: Ladder approximation graphs for two interacting particles propagation.

2.3 Bethe Salpeter two particle Wave Function

We note that in non relativistic quantum mechanics of a sigle particle propagating from space time point x_2 to space time point x_1, if the wave function at x_2 is known, then the wave function at x_1 is given by

$$\psi(x_1) = \int d^3x_2 K(x_1, x_2)\psi(x_2). \tag{2.48}$$

The a bove condtion can be applied for a system of particles propagating simultaneously from space time point x_3, x_4 to space time point x_1, x_2, that is if the wave function of the bound state at x_3, x_4 is known, then the wave function at x_1, x_2 is given by

$$\psi(x_1, x_2) = \int d^3x_3 \int d^3x_4 K(x_1, x_2; x_3, x_4)\psi(x_3, x_4). \tag{2.49}$$

Plugging Eq. (2.47) into Eq. (2.49), one can obtan

$$\psi(x_1, x_2) = \psi_0(x_1, x_2) + \int d^4x_5 \int d^4x_6 K_0(x_1, x_2; x_5, x_6) I(x_5, x_6)\psi(x_5, x_6), \tag{2.50}$$

which is a complicated inhomogeneous integral equation. $\psi_0(x_1, x_2)$ is a free two particle wave function. For two particles forming a bound state after interaction, then $\psi_0(x_1, x_2)$ will vanish and the integral equation becomes homogeneous which is expressible in the form

$$\psi(x_1,x_2) = \int d^4x_5 \int d^4x_6 K_0(x_1,x_2;x_5,x_6)I(x_5,x_6)\psi(x_5,x_6). \tag{2.51}$$

This is the wave function for two bound particles which is also called Bethe Salpeter equation.

2.4 Bethe Salpeter Equation For $q\bar{q}$ bound state

The Bethe-Salpeter equation given in Eq. (2.51) can also be writen in another form. To derive Bethe Salpeter equation for fermionic quarks, lets operate this equation from the left hand side by the free Dirac operators $(\gamma_\mu\partial^a_\mu + m_a)(\gamma_\mu\partial^b_\mu + m_b)$. It then follows that

$$\begin{aligned}(\gamma_\mu\partial^a_\mu + m_a)(\gamma_\mu\partial^b_\mu + m_b)\psi(x_1,x_2) = \quad & \int d^4x_5 \int d^4x_6 (\gamma_\mu\partial^a_\mu + m_a)K_o(x_1,x_5) \\ & (\gamma_\mu\partial^b_\mu + m_b)K_0(x_2,x_6)I(x_5,x_6)\psi(x_5,x_6),\end{aligned} \tag{2.52}$$

where we use Eq. (2.24) for the free Propagator.

Since the one particle propagators obey the relation

$$\begin{aligned}(\gamma_\mu\partial^a_\mu + m_a)K_0(x_1,x_5) &= -i\delta^4(x_1-x_5), \\ (\gamma_\mu\partial^b_\mu + m_b)K_0(x_2,x_6) &= -i\delta^4(x_2-x_6),\end{aligned} \tag{2.53}$$

Eq. (2.52), takes the form

$$(\gamma_\mu\partial_{1\mu}+m_1)(\gamma_\mu\partial_{2\mu}+m_2)\psi(x_1,x_2) = -\int d^4x_5 \int d^4x_6 \delta^4(x_1-x_5)\delta^4(x_2-x_6)I(x_5,x_6)\psi(x_5,x_6), \tag{2.54}$$

where we replace $a \to 1$ and $b \to 2$ for particle one and particle two respectively.

In this form the Bethe Salpeter equation is an integro-differential equation. For practical purposes there is another useful form of this equation, obtained by transforming it into momentum space. Integrating over the integration variable x_5 followed by the integration variable x_6 using the identity given by Eq. (2.16), we get

$$(\gamma_\mu \partial_{1\mu} + m_1)(\gamma_\mu \partial_{2\mu} + m_2)\psi(x_1, x_2) = -I(x_1, x_2)\psi(x_1, x_2), \tag{2.55}$$

where $I(x_1, x_2)$ is an interaction potential given in Eq. (2.39). However, for particle antiparticle (quark antiquark) interaction, the interaction potential becomes negative in Eq. (2.37) due to opposite charges of the interacting particles antiparticle. Consequently, the right hand side of Eq. (2.55) becomes positive. Lets define $I(x_1, x_2) = -iK(x_1, x_2)$, where $K(x_1, x_2)$ is an interaction kernel and is defined by

$$K(x_1, x_2) = e\gamma_\mu^1 D_F^{\mu\nu}(x_1, x_2) e\gamma_\mu^2. \tag{2.56}$$

In view of this equation, Eq.(2.55) for quark antiquark can be put in the form

$$(\gamma_\mu^1 \partial_{1\mu} + m_1)(\gamma_\mu^2 \partial_{2\mu} + m_2)\psi(x_1, x_2) = iK(x_1, x_2)\psi(x_1, x_2). \tag{2.57}$$

This equation is called Bethe Salpeter equation in coordinate representation. Here $K(x_1, x_2)$ is not the same as the propagator. Rather it is the interaction kernel given by Eq. (2.56). For practical purpose it is easier to work with Bethe Salpeter equation (BSE) in momentum space. To do so, let us consider the hadron moves with four momentum P_μ. In center of mass frame, the center of mass coordinate is given by

$$\bar{X} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}. \tag{2.58}$$

For equal mass system $m_1 = m_2 = m$. Then the center of mass coordinate becomes

$$\bar{X} = \frac{x_1 + x_2}{2}. \tag{2.59}$$

On the other hand the relative separation of the two particle is defined by

$$x = x_1 - x_2. \tag{2.60}$$

From the above equation, we write

$$\partial_{1\mu} = \frac{\partial}{\partial x_{1\mu}} = \frac{\partial \bar{X}}{\partial x_{\mu 1}}\frac{\partial}{\partial \bar{X}} + \frac{\partial x}{\partial x_{\mu 1}}\frac{\partial}{\partial x} = \frac{1}{2}\partial_{\bar{X}\mu} + \partial_{x\mu}$$
$$\partial_{2\mu} = \frac{\partial}{\partial x_{2\mu}} = \frac{\partial \bar{X}}{\partial x_{\mu 2}}\frac{\partial}{\partial \bar{X}} + \frac{\partial x}{\partial x_{\mu 2}}\frac{\partial}{\partial x} = \frac{1}{2}\partial_{\bar{X}\mu} - \partial_{x\mu} \tag{2.61}$$

substituting this equation into Eq. (2.57), we get

$$(\frac{1}{2}\gamma_\mu\partial_{\bar{x}\mu} + \gamma_\mu\partial_{x\mu} + m)(\frac{1}{2}\gamma_\mu\partial_{\bar{x}\mu} - \gamma_\mu\partial_{x\mu} + m)\psi(\bar{X}, x) = iK(\bar{X}, x)\psi(\bar{X}, x). \tag{2.62}$$

For the case of no interaction, one writes the two particle wave function in a variable separable manner and the interaction kernel as

$$\psi(\bar{X}, x) = \psi(\bar{X})\psi(x)$$
$$K(\bar{X}, x) = K(x). \tag{2.63}$$

One can put the above equation in the form

$$(\frac{1}{2}\gamma_\mu\partial_{\bar{x}\mu} + \gamma_\mu\partial_{x\mu} + m)(\frac{1}{2}\gamma_\mu\partial_{\bar{x}\mu} - \gamma_\mu\partial_{x\mu} + m)e^{iP\bar{X}}\psi(x) = iK(x)e^{iPX}\psi(x), \tag{2.64}$$

where without interaction the explicit form of the center of mass spinor is plane wave function given by

$$\psi(\bar{X}) = \frac{1}{\sqrt{V}}e^{iP.\bar{X}}. \tag{2.65}$$

Here we take $K(x)$ as a function of x only. This is because we assume that the interaction between the particles depends on the relative separation between them . Differentiating with respect to $\bar{X}_\mu$, we obtain

$$(\frac{i}{2}\gamma_\mu P_\mu + \gamma_\mu\partial_{x\mu} + m)(\frac{i}{2}\gamma_\mu P_\mu - \gamma_\mu\partial_{x\mu} + m)\psi(P)\psi(x) = iK(x)\psi(P)\psi(x). \tag{2.66}$$

Lets take the inverse Fourier transform of $\psi(x)$ and $K(x)$ to express them in momentum space as,

$$\psi(x) = \frac{1}{(2\pi)^4}\int d^4q e^{iq.x}\psi(q),$$
$$K(x) = \frac{1}{(2\pi)^4}\int d^4k e^{ik.x}K(k), \tag{2.67}$$

where q is the relative momentum of the two particles.

Plugging this equation into the above and operating on the wave function, we write

$$\int d^4q[i(\frac{1}{2}\not{P}+\not{q})+m][i(\frac{1}{2}\not{P}-\not{q})+m]e^{iq.x}\psi(P)\psi(q) = \frac{i}{(2\pi)^4}\int d^4q'\int d^4k'e^{i(q'+k').x}\psi(P)\psi(q')K(k'). \tag{2.68}$$

To simplify this equation, lets define

$$q = k' + q', \tag{2.69}$$

where q' is a constant or a given value. We then see that

$$d^4q = d^4k'. \tag{2.70}$$

In view of these two equations, Eq. (2.68) takes the form

$$[i(\frac{1}{2}\not{P}+\not{q})+m][i(\frac{1}{2}\not{P}-\not{q})+m]\psi(P,q) = \frac{i}{(2\pi)^4}\int d^4q'\psi(P,q')K(q,q'). \tag{2.71}$$

But even if one restricts oneself to the simplest case of the ladder approximation Eq. (2.47), the structure of Eq. (2.71) is still so complicated in finding exact solutions. This can be rewritten as

$$(i\not{p}_1+m)(i\not{p}_2+m)\psi(P,q) = \frac{i}{(2\pi)^4}\int d^4q'K(q,q')\psi(P,q'), \tag{2.72}$$

where the above equation with the labeling 1 and 2 depicting the two particles is the 16×1 representation of Bethe-Salpeter equation. Here, $K(q,q')$ is the interaction kernel between the quark and anti-quark, and $p_{1,2}$ are the momenta of the quark and anti-quark, which are related to the internal 4-momentum q and total momentum P of hadron of mass M as $p_{1,2\mu} = \hat{m}_{1,2}P_\mu \pm q_\mu$, where $\hat{m}_{1,2} = \frac{1}{2}\big[1 \pm \frac{(m_1^2-m_2^2)}{M^2}\big]$ are the Wightman-Garding (WG) definitions of masses of individual quarks which ensure that $P.q = 0$ on the mass shells of either quarks,even when $m_1 = m_2$. However for equal mass mesons, ($m_1 = m_2 = m$), we have $\hat{m}_1 = \hat{m}_2 = 1$. Then $p_{1,2}$ becomes,

$$\begin{aligned} p_1 &= \frac{1}{2}P + q, \\ p_2 &= \frac{1}{2}P - q. \end{aligned} \tag{2.73}$$

The total momentum of the hadron and the relative momentum of the quarks is given by

$$\begin{aligned} P &= p_1 + p_2 \\ 2q &= p_1 - p_2. \end{aligned} \tag{2.74}$$

The Bethe-Salpeter equation given in Eq. (2.72) can be put in 4× 4 representation of $\psi(P,q)$in the form,

$$S_F^{-1}(p_1)\psi(P,q)S_F^{-1}(p_2) = \frac{i}{(2\pi)^4}\int d^4q' K(q,q')\psi(P,q'), \tag{2.75}$$

where the two quark propagators sandwich the 4D BS wave function $\psi(P,q)$, and the two inverse quark propagators are

$$S_F^{-1}(p_{1,2}) = i(i\not{p}_{1,2} + m_{1,2}). \tag{2.76}$$

This is Bethe Salpeter equation in momentum representation for $q\bar{q}$ system. Furthermore the numerical solution of the Bethe-Salpeter equation is an important tool for calculating bound states in the realm of elementary particle physics (quark-antiquark systems) called mesons. However, to solve this equation, we employ the covariant Instantaneous Ansatz shown in the next section.

2.5 Bethe Salpeter Equation Under Covariant Instantaneous Ansaz for Quarks

Bethe−Salpeter (BS) equation is a very good tool to treat various relativistic bound state systems. For a quark antiquark system of masses m_1 and m_2, respectivrly, the four dimensional (4D) Bethe Salpeter equation for fermionic quark antiquark system written in 4×4 representation of 4D Bethe Salpeter wave function $\Psi(P, q)$ is given by Eq. (2.75), where $K(q, q')$ is the interaction kernel between the quark and antiquark, and p_1 and p_2 are their respective momenta. The constituents interact to produce a hadron of mass M and four momentum P_μ. The individual momenta p_1 and p_2 of constituents are related to the internal four momentum(relative four momentum) q and the total momentum P of the hadron(of mass M) by Eq. (2.73) In momentum space, the kernel $K(q, q')$ depends on the relative momentum q and q' of the initial and final states of the bound state (hadron). It is convenient to express the internal momentum of the hadron q_μ as the sum of two parts

$$q_\mu = \hat{q}_\mu + \sigma P_\mu, \tag{2.77}$$

where $\hat{q}_\mu$ is the transverse component and is defined by

$$\hat{q}_\mu = q_\mu - \frac{(q.P)P_\mu}{P^2} \tag{2.78}$$

which is orthogonal to the total hadron momentum P_μ $(\hat{q}.P = 0)$. σP_μ is the longitudinal component of 4-momentum q_μ given by

$$\sigma P_\mu = \frac{q.P}{P_{\ \mu}} P^2 \tag{2.79}$$

which is parallel to P_μ. Hence one can decompose the relative momentum as

$$q_\mu = (\hat{q}, iM\sigma), \tag{2.80}$$

where the transverse component $\hat{q}$ is an effective three dimensional(3D) vector while the longitudinal component $M\sigma$ plays the role of the fourth component and is like the time component. The 4D volume element of the relative momenta q' can be written in an invariant form as

$$d^4q' = d^3\hat{q}' M d\sigma, \tag{2.81}$$

Under covariant instantaneous Ansatz(CIA), the Bethe−Salpeter interaction kernel $K(q, q')$ depends only on the spatial components, of the relative momenta q and q' involved, that is, it takes the form

$$K(q, q') = K(\hat{q}, \hat{q}'). \tag{2.82}$$

With the ansatz Eq. (2.82), the longitudinal component $M\sigma$ of q_μ does not appear in the form $K(\hat{q}, \hat{q}')$ of the kenel. In the instantaneous approximation the Bethe Salpeter equation may be reduced to an equation of motion for a Salpeter wave function (amplitude) $\psi(\hat{q})$. In momentum space, the salpeter amplitude $\psi(\hat{q})$ is obtained by integrating the 4-D Bethe Salpeter amplitude $\psi(P, q)$ over the time component $M\sigma$ of the relative momentum . To do that, lets plug Eq. (2.81) and (2.82) into Eq. (2.75) and obtain

$$S_F^{-1}(p_1)\Psi(P, q)S_F^{-1}(-p_2) = \int \frac{d^3\hat{q}'}{(2\pi)^3} K(\hat{q}, \hat{q}')\Psi(\hat{q}'), \tag{2.83}$$

where we define a three dimensional wave function as

$$\psi(\hat{q}) = \frac{i}{2\pi}\int_{-\infty}^{\infty} M d\sigma \psi(P, q'). \tag{2.84}$$

Note that P plays only the role of a parameter. Eq. (2.83) is called covariant form of Salpeter equation and can also be put in the form

$$S_F^{-1}(p_1)\Psi(P, q)S_F^{-1}(-p_2) = \Gamma(\hat{q}), \tag{2.85}$$

where

$$\Gamma(\hat{q}) = \int \frac{d^3\hat{q}'}{(2\pi)^3} K(\hat{q}, \hat{q}')\Psi(\hat{q}') \tag{2.86}$$

plays the role of hadron quark vertex function, that describe mesons as bound quark antiquark pairs. For the form of a kernel $K(\hat{q}, \hat{q}') = \frac{1}{4}(\vec{\lambda}_1.\vec{\lambda}_2\gamma_\mu \otimes \gamma_\mu)V(\hat{q}, \hat{q}')$, one can express $\Gamma(\hat{q})$ as

$$\Gamma(\hat{q}) = \frac{1}{4}(\vec{\lambda}_1.\vec{\lambda}_2)\frac{1}{(2\pi)^3}\int d^3\hat{q}'\gamma_\mu\Psi(\hat{q}')\gamma_\mu V(\hat{q}, \hat{q}'), \tag{2.87}$$

Following[27], we can to a good approximation express $\gamma_\mu\psi(\hat{q}')\gamma_\mu \approx \Theta\psi(\hat{q}')$ where Θ involves spin-spin interactions above, that factor out of the RHS of hadron quark vertex on taking the dominant Dirac structure in $\psi(\hat{q}')$ in calculation of Θ. We can rewrite the hadron vertex function $\Gamma(\hat{q})$ as

$$\Gamma(\hat{q}) = \frac{1}{4}(\vec{\lambda}_1.\vec{\lambda}_2)\frac{\Theta}{(2\pi)^3}\int d^3\hat{q}'\Psi(\hat{q}')V(\hat{q},\hat{q}'). \tag{2.88}$$

Then the 4D Bethe Salpeter wave function can be obtained by multiplying $S_F(p_1)$ from the left and $S_F(-p_2)$ from the right hand side of Eq. (2.85) and becomes

$$\Psi(P,q) = S_F(p_1)\Gamma(\hat{q})S_F(-p_2). \tag{2.89}$$

Here $S_F(p)$ is the usual fermion propagator of the quarks and is given by

$$S_F(\pm p_{1,2}) = -i\frac{\mp i\not{p}_{1,2} + m_{1,2}}{\Delta_{1,2}}, \tag{2.90}$$

where

$$\Delta_{1,2} = m_{1,2}^2 + p_{1,2}^2 \tag{2.91}$$

is the inverse propagator. The actual evaluation of the integral over the time component of the relative momentum q of the tensor product of the Fermion propagator in the integrated Bethe Salpeter equation (2.89) requires the knowledge of the dependency of Fermion propagator $S_F(p)$ on $M\sigma$. We now decompose the propagators of the two constituents. In the center of mass system, we can express

$$P = (0, iE) = (0, iM), \tag{2.92}$$

and using the orthogonality relation $P.\hat{q} = 0$, the denominator of the propagator for a Fermion constituent quark for unequal mass quark can be put in the form

$$p_1^2 + m_1^2 = -[(\hat{m}_1 M + M\sigma - \omega_1)(\hat{m}_1 M + M\sigma + \omega_1)]. \tag{2.93}$$

In a similar manner, the numerator can be put in the form

$$-i(-i\not{p}_1 + m_1) = -[(\hat{m}_1 M + M\sigma + \omega_1)\Lambda_1^+(\hat{q}) + (\hat{m}_1 M + M\sigma - \omega_1)\Lambda_1^-(\hat{q})], \quad (2.94)$$

where we define

$$\Lambda_1^{\pm}(\hat{q}) = \frac{1}{2\omega_1}[\frac{\not{P}\omega_1}{M} \pm (im_1 + \not{\hat{q}})] \quad (2.95)$$

as the energy projection operators for the positive and negative energies of particle one. On account of Eq. (2.93) and Eq. (2.94), the propagator for the quark constituent takes the form

$$S_F(p_1) = \frac{\Lambda_1^+(\hat{q})}{\hat{m}_1 M + M\sigma - \omega_1} + \frac{\Lambda_1^-(\hat{q})}{\hat{m}_1 M + M\sigma + \omega_1}. \quad (2.96)$$

Following a similar procedure, the denominator for the propagator of the antiquark Fermion can be put in the form

$$p_2^2 + m_2^2 = -[(\hat{m}_2 M - M\sigma - \omega_2)(\hat{m}_2 M - M\sigma + \omega_2)]. \quad (2.97)$$

On the other hand, the numerator can be put in the form

$$-i(i\not{p}_2 + m_2) = (\hat{m}_2 M - M\sigma + \omega_2)\Lambda_2^+(\hat{q}) + (\hat{m}_2 M - M\sigma - \omega_2)\Lambda_2^-(\hat{q}), \quad (2.98)$$

where we define

$$\Lambda_2^{\pm}(\hat{q}) = \frac{1}{2\omega_2}[\frac{\not{P}\omega_2}{M} \mp (im_2 + \not{\hat{q}})] \quad (2.99)$$

as the positive and negative energy projection operators for the antiquark (particle 2) respectively. In view of Eq. (2.97) and Eq. (2.96), the propagator for the antiquark can be decomposed as

$$S_F(-p_2) = \frac{-\Lambda_2^+(\hat{q})}{(\hat{m}_2 M - M\sigma - \omega_2)} + \frac{-\Lambda_2^-(\hat{q})}{(\hat{m}_2 M - M\sigma + \omega_2)}. \quad (2.100)$$

In general the propagators of constituent quarks can be decomposed as

$$S_F(\pm p_j) = \frac{I(j)\Lambda_j^+(\hat{q})}{I(j)M\sigma + \hat{m}_j M - \omega_j} + \frac{I(j)\Lambda_j^+(\hat{q})}{I(j)M\sigma + \hat{m}_j M + \omega_j}, \quad (2.101)$$

with

$$\omega_j = \sqrt{m_j^2 + \hat{q}^2} \tag{2.102}$$

and

$$\Lambda_j^{\pm}(\hat{q}) = \frac{1}{2\omega_j}\left[\frac{\not{P}\omega_j}{M} \pm I(j)(im_j + \not{\hat{q}})\right], \tag{2.103}$$

where $j = 1, 2$ for quarks and antiquarks respectively and $I(j) = (-1)^{j+1}$ and $\Lambda_j^{\pm}(\hat{q})$ satisfy the relation

$$\begin{aligned}
&\Lambda_j^{\pm}(\hat{q}) + \Lambda_j^{\mp}(\hat{q}) = \frac{\not{P}}{M}, \\
&\Lambda_j^{\pm}(\hat{q})\frac{\not{P}}{M}\Lambda_j^{\pm}(\hat{q}) = -\Lambda_j^{\pm}(\hat{q}), \\
&\Lambda_j^{\pm}(\hat{q})\frac{\not{P}}{M}\Lambda_j^{\mp}(\hat{q}) = 0.
\end{aligned} \tag{2.104}$$

Due to these equations, $\Lambda_j^{\pm}$ may be considered as projection operators, since in the rest frame $\vec{P} = 0$ they turn to the energy projection operator and they form a complete set. Now we are in a position to derive Salpeter equations. To do so, we plug Eq .(2.101) into Eq.(2.89) and obtain

$$\begin{aligned}
\psi(P,q) = \quad & \frac{-\Lambda_1^+(\hat{q})\Gamma(\hat{q})\Lambda_2^+(\hat{q})}{(\hat{m}_1 M - \omega_1 + M\sigma)(\hat{m}_2 M - \omega_2 - M\sigma)} + \frac{-\Lambda_1^+(\hat{q})\Gamma(\hat{q})\Lambda_2^-(\hat{q})}{(\hat{m}_1 M - \omega_1 + M\sigma)(\hat{m}_2 M + \omega_2 - M\sigma)} \\
& + \frac{-\Lambda_1^-(\hat{q})\Gamma(\hat{q})\Lambda_2^+(\hat{q})}{(\hat{m}_1 M + \omega_1 + M\sigma)(\hat{m}_2 M - \omega_2 - M\sigma)} + \frac{-\Lambda_1^-(\hat{q})\Gamma(\hat{q})\Lambda_2^-(\hat{q})}{(\hat{m}_1 M + \omega_1 + M\sigma)(\hat{m}_2 M + \omega_2 - M\sigma)}
\end{aligned} \tag{2.105}$$

Multiplying Eq. (2.105) from the left by $\Lambda_1^{\pm}(\hat{q})\frac{\not{P}}{M}$ and from the right hand side by $\frac{\not{P}}{M}\Lambda_2^{\pm}(\hat{q})$ and applying the relation given by Eq. (2.106) to the right hand side of this equation, we get

$$(\hat{m}_1 M \mp \omega_1 + M\sigma)(\hat{m}_2 M \mp \omega_2 - M\sigma)\psi^{\pm\pm}(P,q) = -\Lambda_1^{\pm}(\hat{q})\Gamma(\hat{q})\Lambda_2^{\pm}(\hat{q}), \tag{2.106}$$

where all of the four combinations of signs are admitted. The Bethe-Salpeter equation in Eq. (2.89) then becomes a system of four projected equations where we define the 4D projected wave functions as

$$\psi^{\pm\pm}(P,q) = \Lambda_1^{\pm}(\frac{\not{P}}{M})\psi(P,q)(\frac{\not{P}}{M})\Lambda_2^{\pm}(\hat{q}). \tag{2.107}$$

Integrating on both sides of Eq. (2.106) over $Md\sigma$, one can, in a complete analogous way as Eq. (2.84) obtain

$$\psi^{\pm\pm}(\hat{q}) = \frac{-i}{2\pi}\int_{-\infty}^{\infty} Md\sigma \frac{\Lambda_1^{\pm}(\hat{q})\Gamma(\hat{q})\Lambda_2^{\pm}(\hat{q})}{(\hat{m_1}M \mp \omega_1 + M\sigma)(\hat{m_2}M \mp \omega_2 - M\sigma)}, \tag{2.108}$$

where we define the 3D projected wave function by

$$\psi^{\pm\pm}(\hat{q}) = \frac{i}{2\pi}\int_{-\infty}^{\infty} Md\sigma\psi^{\pm\pm}(P,q). \tag{2.109}$$

To make equation (2.108) unique, we have to determine how to treat the poles when integrating over $d\sigma$. We know that, the rules that we have already used frequently, which states that the condition of causality can be full filled using $i\epsilon$ prescription due to which we can write

$$\psi^{\pm\pm}(\hat{q}) = \frac{-i\Gamma^{\pm\pm}(\hat{q})}{2\pi}\int_{-\infty}^{\infty}\frac{Md\sigma}{(\frac{1}{2}M \mp \omega_1 + M\sigma \pm i\epsilon)(\frac{1}{2}M \mp \omega_2 - M\sigma \pm i\epsilon)}, \tag{2.110}$$

where we define

$$\Gamma^{\pm\pm}(\hat{q}) = \Lambda_1^{\pm}(\hat{q})\Gamma(\hat{q})\Lambda_2^{\pm}(\hat{q}). \tag{2.111}$$

The σ integration can be performed using the theorem of residue, where the integration path is closed by a half circle in the upper or lower half plane as shown in figure below. The integrand has two poles at

$$\sigma_1^{\pm} = -\frac{1}{2} \pm \frac{\omega_1}{M} \mp i\epsilon, \sigma_2^{\pm} = \frac{1}{2} \mp \frac{\omega_2}{M} \pm i\epsilon. \tag{2.112}$$

As we have already emphasized, both signs in Eq. (2.110) can be chosen independently. If both signs of ϵ in Eq. (2.112) is equal, then both poles are in the same half plane. Closing the contour either above or below real axis leads to

$$\psi^{+-} = \psi^{-+} = 0. \tag{2.113}$$

On the opposit case, to carry out the remaing integral for complex variable σ, we use the fact that

$$\int_{-\infty}^{\infty} d\sigma \frac{1}{\sigma - a \mp i\epsilon}\frac{1}{\sigma + b \pm i\epsilon} = \pm\frac{2\pi i}{a+b}. \tag{2.114}$$

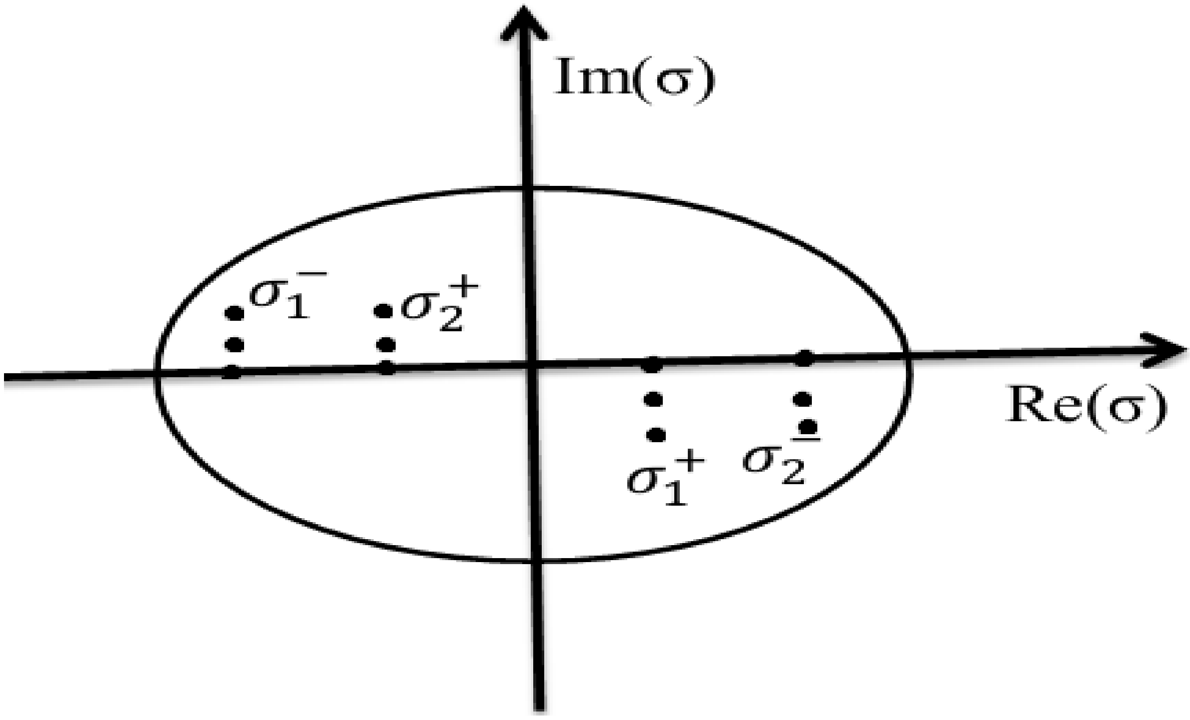

Figure 2.6: Contour integration in σ plane.

So that, in view of this equation, we obtain

$$\psi^{++}(\hat{q}) = \frac{-\Gamma^{++}(\hat{q})}{M - \omega_1 - \omega_2}, \tag{2.115}$$

and

$$\psi^{--}(\hat{q}) = \frac{\Gamma^{--}(\hat{q})}{M + \omega_1 + \omega_2}. \tag{2.116}$$

On account of Eq. (2.111), this can be put in the form

$$\psi^{++}(\hat{q}) = \frac{-\Lambda_1^+(\hat{q})\Gamma(\hat{q})\Lambda_2^+(\hat{q})}{M - \omega_1 - \omega_2} \tag{2.117}$$

and

$$\psi^{--}(\hat{q}) = \frac{\Lambda_1^-(\hat{q})\Gamma(\hat{q})\Lambda_2^-(\hat{q})}{M + \omega_1 + \omega_2}. \tag{2.118}$$

In view of Eq. (2.112), (2.116) and (2.18), the complete wave function can be put in the form

$$\psi(\hat{q}) = \psi^{++} + \psi^{+-} + \psi^{-+} + \psi^{--}. \tag{2.119}$$

This complete wave function then obeys the equation

$$\psi(\hat{q}) = \frac{-\Lambda_1^+(\hat{q})\Gamma(\hat{q})\Lambda_2^+(\hat{q})}{M - \omega_1 - \omega_2} + \frac{\Lambda_1^-(\hat{q})\Gamma(\hat{q})\Lambda_2^-(\hat{q})}{M + \omega_1 + \omega_2}. \tag{2.120}$$

Here, M is the eigenvalue. Applying the complete set of the projection operators $\Lambda_j^{\pm}(\hat{q})\frac{\not{P}}{M}$ from the left and $\frac{\not{P}}{M}\Lambda_j^{\pm}(\hat{q})$ from the right hand side, we obtain obtain

$$
\begin{aligned}
\Lambda_1^{+}(\hat{q})\frac{\not{P}}{M}\psi(\hat{q})\frac{\not{P}}{M}\Lambda_2^{+}(\hat{q}) &= -\frac{\Lambda_1^{+}(\hat{q})\frac{\not{P}}{M}\Lambda_1^{+}(\hat{q})\Gamma(\hat{q})\Lambda_2^{+}(\hat{q})\frac{\not{P}}{M}\Lambda_2^{+}(\hat{q})}{M-\omega_1-\omega_2} + \frac{\Lambda_1^{+}(\hat{q})\frac{\not{P}}{M}\Lambda_1^{-}(\hat{q})\Gamma(\hat{q})\Lambda_2^{-}(\hat{q})\frac{\not{P}}{M})\Lambda_2^{+}(\hat{q})}{M+\omega_1+\omega_2}\\
\Lambda_1^{+}(\hat{q})\frac{\not{P}}{M}\psi(\hat{q})\frac{\not{P}}{M}\Lambda_2^{-}(\hat{q}) &= -\frac{\Lambda_1^{+}(\hat{q})\frac{\not{P}}{M}\Lambda_1^{+}(\hat{q})\Gamma(\hat{q})\Lambda_2^{+}(\hat{q})\frac{\not{P}}{M}\Lambda_2^{-}(\hat{q})}{M-\omega_1-\omega_2} + \frac{\Lambda_1^{+}(\hat{q})\frac{\not{P}}{M}\Lambda_1^{-}(\hat{q})\Gamma(\hat{q})\Lambda_2^{-}(\hat{q})\frac{\not{P}}{M}\Lambda_2^{-}(\hat{q})}{M+\omega_1+\omega_2}\\
\Lambda_1^{-}(\hat{q})\frac{\not{P}}{M}\psi(\hat{q})\frac{\not{P}}{M}\Lambda_2^{+}(\hat{q}) &= -\frac{\Lambda_1^{-}(\hat{q})\frac{\not{P}}{M}\Lambda_1^{+}(\hat{q})\Gamma(\hat{q})\Lambda_2^{+}(\hat{q})\frac{\not{P}}{M}\Lambda_2^{+}(\hat{q})}{M-\omega_1-\omega_2} + \frac{\Lambda_1^{-}(\hat{q})\frac{\not{P}}{M}\Lambda_1^{-}(\hat{q})\Gamma(\hat{q})\Lambda_2^{-}(\hat{q})\frac{\not{P}}{M}\Lambda_2^{+}(\hat{q})}{M+\omega_1+\omega_2}\\
\Lambda_1^{-}(\hat{q})\frac{\not{P}}{M}\psi(\hat{q})\frac{\not{P}}{M}\Lambda_2^{-}(\hat{q}) &= -\frac{\Lambda_1^{-}(\hat{q})\frac{\not{P}}{M}\Lambda_1^{+}(\hat{q})\Gamma(\hat{q})\Lambda_2^{+}(\hat{q})\frac{\not{P}}{M}\Lambda_2^{-}(\hat{q})}{M-\omega_1-\omega_2} + \frac{\Lambda_1^{-}(\hat{q})\frac{\not{P}}{M}\Lambda_1^{-}(\hat{q})\Gamma(\hat{q})\Lambda_2^{-}(\hat{q})\frac{\not{P}}{M}\Lambda_2^{-}(\hat{q})}{M+\omega_1+\omega_2}
\end{aligned}
\tag{2.121}
$$

In view of Eq. (2.103),and Eq. (2.106), we get the full Salpeter equations

$$
\begin{aligned}
&[M-\omega_1-\omega_2]\psi^{++}(\hat{q}) = -\Lambda_1^{+}(\hat{q})\Gamma(\hat{q})\Lambda_2^{+}(\hat{q}),\\
&[M+\omega_1+\omega_2]\psi^{--}(\hat{q}) = \Lambda_1^{-}(\hat{q})\Gamma(\hat{q})\Lambda_2^{-}(\hat{q}),\\
&\psi^{+-}(\hat{q}) = \psi^{-+}(\hat{q}) = 0.
\end{aligned}
\tag{2.122}
$$

For a meson composed of equal mass quark-antiquark $(m_1 = m_2 = m)$ we have $\omega_1 = \omega_2 = \omega$, the Salpeter equation takes the form

$$
\begin{aligned}
&[M-2\omega]\psi^{++}(\hat{q}) = -\Lambda_1^{+}(\hat{q})\Gamma(\hat{q})\Lambda_2^{+}(\hat{q}),\\
&[M+2\omega]\psi^{--}(\hat{q}) = \Lambda_1^{-}(\hat{q})\Gamma(\hat{q})\Lambda_2^{-}(\hat{q}),\\
&\psi^{+-}(\hat{q}) = \psi^{-+}(\hat{q}) = 0.
\end{aligned}
\tag{2.123}
$$

From these three equation, we see that the last equation do not contain eigenvalue M, so essentially they act as equation of constraints on the Bethe Salpeter wave function $\psi(\hat{q})$. Thus to obtain the mass spectral equation, we have to start with four equations to solve the instantaneous Bethe Salpeter equation. Here $\psi^{++}(\hat{q})$ and $\psi^{--}(\hat{q})$ are the positive and negative energy components of the wave function respectively. In the next chapter, we are going to deal with mass spectrum of different mesons with use of Salpeter equations.

3

Spectroscopy of ground,and excited states of 0^{++}, 1^{++} and 1^{+-} $c\bar{c}$ quarkonia

3.1 Derivation of Bethe Salpeter wave function and the mass spectral equation of scalar meson

In the constituent quark model, we treat a meson as a bound quark-antiquark pair, $q\bar{q}$. In this picture the q and the $\bar{q}$ both have spin half (spin-$\frac{1}{2}$) . These can combine to give either total spin $\vec{S}$ = 0, or total spin $\vec{S}$ = 1. In addition to the total spin, we can have orbital angular momentum L between the $q\bar{q}$ pair. Then, the $\vec{L}$ and $\vec{S}$ can combine to give total angular momentum $\vec{J}$ = $\vec{L} \oplus \vec{S}$, where $\vec{J}$ =$|\vec{J} - \vec{S}|$, $|\vec{L} - \vec{S} + 1|$, . . . , $|\vec{L} + \vec{S}|$. The states can be written in spectroscopic notation as $^{2S+1}L_J$, where the numbers $l = 0, 1, 2, 3, 4, 5, ...$ are designated by S, P, D, F, G, H, . . ., respectively. Using the quarks quantum number, we are then able to use L, S and J to construct the J^{PC} quantum numbers of the mesons. Let us start with parity, P. Mathematically, parity is the reflection operator. Fermions and antifermions have intrinsic opposite parity. This leads to the parity of a meson being: $P = (-1)^{l+1}$. This parity is conserved in a reaction. We consider the decay $A \rightarrow B + C$, where there is orbital angular momentum L between B and C. Parity conservation says that $P(A) = P(B).P(C).(-1)^L$.

The next quantum number that we are going to see is charge conjugation, C, which represents a transformation of the particle into its antiparticle. This reverses several properties of the particle such as charge. Now this Charge conjugation is given by $C = (-1)^{l+S}$. For the two ground states $(L = 0)$ with $J^{PC} = 0^{-+}$ and $J^{PC} = 1^{--}$, they are expressed in the form ${}^1S_0$ and ${}^3S_1$, respectively; and for the orbitally excited states $(L = 1)$ with $J^{PC} = 0^{++}, 1^{++}, 2^{++}$ and 1^{+-}, they are symbolized as ${}^3P_0, {}^3P_1, {}^3P_2$ and ${}^1P_1$, respectively. The lowest states are given in the table 3.1 below for quarkonia

states(${}^{2S+1}L_J$)	S	L	J	P	C	J^{PC}	Mesons	Name
${}^1S_0$	0	0	0	$-$	$+$	0^{-+}	η_c, η_b	Pseudoscalar
${}^3S_1$	1	0	1	$-$	$-$	1^{--}	J/ψ, Υ	vector
${}^1P_1$	0	1	1	$+$	$-$	1^{+-}	h_c, h_b	Axial vector(C=+)
${}^3P_0$	1	1	0	$+$	$+$	0^{++}	χ_{co}, χ_{b0}	Scalar
${}^3P_1$	1	1	1	$+$	$+$	1^{++}	χ_{c1}, χ_{b1}	Axial vector(C=-)
${}^3P_2$	1	1	2	$+$	$+$	2^{++}	χ_{c2}, χ_{b2}	Tensor
${}^1D_2$	0	2	2	$-$	$+$	2^{-+}		
${}^3D_1$	1	2	1	$-$	$-$	1^{--}	ψ, Υ	vector
${}^3D_2$	1	2	2	$-$	$-$	2^{--}		
${}^3D_3$	1	2	3	$-$	$-$	3^{--}		

Table 3.1: The Quarkonium states as a function of L, S and J for heavy mesons. These then correspond to the named mesons of the specified J^{PC}.

The general decomposition of the instantaneous 4D Bethe Salpeter (BS) wave function in terms of various Dirac structures and scalar functions $f_i(\hat{q})$ multiplying them for the state 0^{++} is given by[39]

$$\psi^s(P,q) = f_1(P,q) - i\not{P}f_2(P,q) - i\not{q}f_3(P,q) - [\not{P},\not{q}]f_4(P,q), \tag{3.1}$$

where, $f_i(q^2, q.P, P^2)$ are the Lorentz scalar amplitudes multiplying the various Dirac structures in the BS wave function $\psi(P, q)$. By using the naive power counting rule, it can be shown that the Dirac structure associated with amplitudes f_1 and f_2 are leading, while the structures associated with f_3 and f_4 are subleading and would contribute lesser to meson observable calculations in comparison to leading Dirac structures associated to f_1 and f_2. Among the two leading Dirac structures associated with amplitudes f_1 and f_2, the structure associated with f_1 is dominant. In the center of mass frame, where $q_\mu = (\hat{q}, 0)$, we can then write the general decomposition of the instantaneous BS wave function for scalar meson $(J^{pc} = 0^{++})$, of dimensionality M in the center of mass frame for equal mass system is taken as[39]

$$\psi(\hat{q}) = Mf_1(\hat{q}) - i\not{P}f_2(\hat{q}) - i\not{\hat{q}}f_3(\hat{q}) - \frac{2\not{P}\not{\hat{q}}}{M}f_4(\hat{q}), \tag{3.2}$$

where we use the fact that $\not{\hat{q}}\not{P} = -\not{P}\not{\hat{q}}$. M is the mass of the scalar meson. We now use the constraint equations

$$\psi^{+-}(\hat{q}) = 0 \tag{3.3}$$

to reduce the four independent scalar function to two independent scalar wave functions. On account of Eq. (2.109), one can write it as

$$\Lambda_1^+(\hat{q})(\frac{\not{P}}{M})\psi(\hat{q})(\frac{\not{P}}{M})\Lambda_2^-(\hat{q}) = 0. \tag{3.4}$$

Substituting from Eq. (2.104) and (3.2) into Eq. (3.4) and taking the trace on both sides and evaluating the trace using trace theorems and by taking the sum of the four trace values, one can obtain

$$\begin{aligned} &M(\omega_1\omega_2 + m_1m_2 - \hat{q}^2)f_1(\hat{q}) + M[m_1\omega_2 + m_2\omega_1]f_2(\hat{q}) \\ &+\hat{q}^2[m_1 + m_2]f_3(\hat{q}) + 2\hat{q}^2[\omega_1 - \omega_2]f_4(\hat{q}) = 0. \end{aligned} \tag{3.5}$$

On the other hand, we use the last constraint equation

$$\psi^{-+}(\hat{q}) = 0, \tag{3.6}$$

to reduce the four independent scalar function to two independent scalar wave functions. On account of Eq. (2.109), one can write it as

$$\Lambda_1^-(\hat{q})(\frac{\not{P}}{M})\psi(\hat{q})(\frac{\not{P}}{M})\Lambda_2^+(\hat{q}) = 0. \tag{3.7}$$

Substituting from (2.104) and Eq. (3.2) into Eq. (3.7) and taking the trace once again on both sides and evaluating the trace using trace theorems and by taking the sum of the four trace values, one can obtain

$$\begin{aligned} &M(\omega_1\omega_2 + m_1 m_2 - \hat{q}^2)f_1(\hat{q}) - M[m_1\omega_2 + m_2\omega_1]f_2(\hat{q}) \\ &+\hat{q}^2[m_1 + m_2]f_3(\hat{q}) - 2\hat{q}^2[\omega_1 - \omega_2]f_4(\hat{q}) = 0. \end{aligned} \tag{3.8}$$

We now add Eq. (3.5) and Eq. (3.8) and arrive at

$$f_1(\hat{q}) = \frac{-\hat{q}^2(m_1 + m_2)f_3(\hat{q})}{M(\omega_1\omega_2 + m_1 m_2 - \hat{q}^2)}. \tag{3.9}$$

Similarly subtracting Eq. (3.8) from Eq. (3.5), we get

$$f_2(\hat{q}) = \frac{-2\hat{q}^2(\omega_1 - \omega_2)f_4(\hat{q})}{M(m_1\omega_2 + m_2\omega_1)}. \tag{3.10}$$

For equal mass system, these two equations are reduced to

$$\begin{aligned} f_1(\hat{q}) &= \frac{-\hat{q}^2 f_3(\hat{q})}{Mm}, \\ f_2(\hat{q}) &= 0. \end{aligned} \tag{3.11}$$

So, we can apply the obtained constraints Eq. (3.11) to Eq. (3.2) and rewrite the relativistic wave function of state (0^{++}) in the form

$$\psi(\hat{q}) = \left[\frac{-\hat{q}^2}{m} - i\not{\hat{q}}\right]f_3(\hat{q}) - \frac{2\not{P}\not{\hat{q}}}{M}f_4(\hat{q}). \tag{3.12}$$

One can see that the Instantaneous BS wave function of scalar (0^{++}) state is determined by only two independent functions f_3 and f_4. For a kernel with spin dependence of the form, $\gamma_\mu \otimes \gamma_\mu$, we can write

$$\Gamma(\hat{q}) = \int \frac{d^3\hat{q}'}{(2\pi)^3} V(\hat{q},\hat{q}')\gamma_\mu\psi(\hat{q})\gamma_\mu, \tag{3.13}$$

where V is spatial part of the kernel. We can then a good approximation express $\gamma_\mu \psi(\hat{q})\gamma_\mu \approx \Theta\psi(\hat{q})$, where Θ involves the spin-spin interactions alone, that factor out of the right hand side of the hadron quark vertex on taking the dominant Dirac structures in $\psi(\hat{q})$ in the calculation of Θ, and we can write $\Gamma(\hat{q}) = \Theta \int \frac{d^3\hat{q}'}{(2\pi)^3} V(\hat{q},\hat{q}')\psi(\hat{q})$

Putting Eq. (2.104), Eq. (2.89), Eq. (2.108) and the wave function in Eq. (3.12) into the first two Salpeter equations in Eq. (2.123) and by taking the trace on both sides for equal quark antiquark, one can obtain

$$
\begin{aligned}
(M-2\omega)\left[f_3(\hat{q})+\frac{2mf_4(\hat{q})}{\omega}\right] &= \frac{1}{\omega^2\hat{q}^2}\int\frac{d^3\hat{q}}{(2\pi)^3}K(\hat{q},\hat{q}')\Big[\hat{q}^2\hat{q}'^2 f_3(\hat{q}') - m^2\hat{q}.\hat{q}' f_3(\hat{q}') - 2m\omega\hat{q}\hat{q}' f_4(\hat{q}')\\
(M+2\omega)\left[f_3(\hat{q})-\frac{2mf_4(\hat{q})}{\omega}\right] &= \frac{1}{\omega^2\hat{q}^2}\int\frac{d^3\hat{q}}{(2\pi)^3}K(\hat{q},\hat{q}')\Big[-\hat{q}^2\hat{q}'^2 f_3(\hat{q}') + m^2\hat{q}.\hat{q}' f_3(\hat{q}') - 2m\omega\hat{q}\hat{q}' f_4
\end{aligned}
$$

Solution of this equations need information about the Bethe Salpeter kernel, $K(\hat{q},\hat{q}')$ which is taken to be one gluon exchange like as regards the color $\frac{1}{4}\vec{\lambda}_1.\vec{\lambda}_2$ and spin $(\gamma_\mu \otimes \gamma_\mu)$ dependence, and has a scalar part $V(\hat{q},\hat{q}')$. The kernel can be written as

$$
\begin{aligned}
K(\hat{q},\hat{q}') &= \frac{1}{4}\vec{\lambda}_1.\vec{\lambda}_2(\gamma_\mu\otimes\gamma_\mu)V(\hat{q},\hat{q}'),\\
V(\hat{q},\hat{q}') &= \frac{4\pi\alpha_s}{(\hat{q}-\hat{q}')^2} + \frac{3}{4}\omega_{q\bar{q}}^2\int d^3\vec{r}\left[r^2(1+4\hat{m}_1\hat{m}_2A_0M_>^2r^2)^{-\frac{1}{2}} - \frac{C_o}{\omega_o^2}\right]e^{i(\hat{q}-\hat{q}').\vec{r}},\\
M_> &= M_{max}(M, m_1+m_2),
\end{aligned}
\tag{3.15}
$$

where $\omega_{q\bar{q}}^2 = 4M_>\hat{m}_1\hat{m}_2\omega_0^2\alpha_s(M_>^2)$ is the flavor dependent spring constant, and $\alpha_s(M_>^2) = \frac{12\pi}{(33-2N_f)\log(\frac{M_>^2}{\Lambda^2})}$ is the QCD coupling constant. here the color factor is given by

$$
\frac{1}{2}\vec{\lambda}_1.\frac{1}{2}\vec{\lambda}_2 = \frac{-4}{3} \tag{3.16}
$$

and N_f is the number of quark flavors with mass less than $\sqrt{Q^2}$.Here $\omega^2_{q\bar{q}}$ is the flavour dependent spring constant. The proportionality of $\omega^2_{q\bar{q}}$ on $\alpha_s(Q^2)$ is need to provide a more direct QCD motivation to confinement. This assumption further facilitates a flavor variation in $\omega^2_{q\bar{q}}$.

On the other hand, ω_0^2 is postulated as a universal spring constant which is common to all flavors. Here in the expression for $V(\hat{q},\hat{q}')$, the constant term $\frac{C_0}{\omega_0^2}$ is designed to take account of the correct zero point energy while A_0 term $(A_0 \ll 1)$ simulates an effect of an almost linear confinement for heavy quark sectors (large m_1, m_2) while retaining the harmonic form for light quark sectors (small m_1, m_2) as is believed to be true for QCD. Hence the term $r^2(1+4A_0\hat{m}_1\hat{m}_2M^2_>r^2)^{-1/2}$ in the above expression is responsible for effecting a smooth transition from harmonic $(q\bar{q})$ to linear $(Q\bar{Q})$ confinement. The smallness of $A_0(A_0 = 0.0283)$ ensures that for light flavors, its structure is dominated by the harmonic form which amounts to setting $A_0 = 0$. On the other hand ω_0 may be regarded as a reduced spring constants of the confining interaction for light quarks for which the constant A_0 is not important. It controls the confinement scale for an integrated view of the different flavor sectors of hadron spectra. Thus the scalar part $V(\hat{q},\hat{q}')$ involves both the OGE term V_{OGE} arising from the one gluon exchange, as well as the confining term V_c. We first ignore the V_{OGE} term and work only with the confining part, V_c. In view of Eq. (3.15) and (3.16), Eq. (3.14) can be put in the form

$$(M-2\omega)\left[f_3(\hat{q})+\frac{2mf_4(\hat{q})}{\omega}\right] = \frac{-4\theta_s}{3\omega^2\hat{q}^2}\int\frac{d^3\hat{q}'}{(2\pi)^3}V(\hat{q},\hat{q}')\Big[\hat{q}^2\hat{q}'^2f_3(\hat{q}')-m^2\hat{q}.\hat{q}'f_3(\hat{q}')-2m\omega\hat{q}\hat{q}'f_4(\hat{q}'$$

$$(M+2\omega)\left[f_3(\hat{q})-\frac{2mf_4(\hat{q})}{\omega}\right] = \frac{-4\theta_s}{3\omega^2\hat{q}^2}\int\frac{d^3\hat{q}'}{(2\pi)^3}V(\hat{q},\hat{q}')\Big[-\hat{q}^2\hat{q}'^2f_3(\hat{q}')+m^2\hat{q}.\hat{q}'f_3(\hat{q}')-2m\omega\hat{q}\hat{q}'f$$

where the spin dependence of the interaction is contained in the factor, $\Theta_s = \gamma_\mu\psi(\hat{q})\gamma_\mu$. If we take the parameter A_0 which correspond to case of light meson$(q\bar{q})$, since $A_0 \ll 1$, the square root factor in the denominators $k = (1 + A_0 M^2 r^2)^{-\frac{1}{2}} = 1$. we take the scalar part of the confining potential $V_c(\hat{q}, \hat{q}')$. Therefore, the scalar potential takes the form

$$V_c(\hat{q}, \hat{q}') = \frac{3}{4}\omega_{q\bar{q}}^2 \int d^3\vec{r}(r^2 - \frac{C_0}{\omega_0^2})e^{i(\hat{q}-\hat{q}').\vec{r}}. \tag{3.18}$$

The second term in the right hand side is nothing but the Fourier integral representation of the delta function. We then put the above equation in the form

$$V_c(\hat{q}, \hat{q}') = -\frac{3}{4}\omega_{q\bar{q}}^2 \frac{C_0}{\omega_0^2}(2\pi)^3\delta^3(\hat{q} - \hat{q}') + \frac{3}{4}\omega_{q\bar{q}}^2 \int d^3\vec{r} r^2 e^{i(\hat{q}-\hat{q}').\vec{r}}. \tag{3.19}$$

On the other hand, the second term in the right hand side can be put in the form

$$r^2 = \lim_{a\to 0}[-\frac{d^3 f(r)}{da^3}], \tag{3.20}$$

where

$$f(r) = \frac{e^{-ar}}{r}. \tag{3.21}$$

In three dimensional space a function $f(r)$ possesses spherical symmetry, so that $f(\vec{r}) = f(r)$. Lets find the Fourier transform of $f(r)$. To do so lets choose spherical polar coordinates in which the vector $\hat{q} - \hat{q}'$ of the Fourier transform lies along the polar axis $(\theta = 0)$. We then write

$$d^3\vec{r} = r^2 \sin\theta dr d\theta d\phi,$$

$$(\hat{q} - \hat{q}').\vec{r} = (\hat{q} - \hat{q}')r\cos\theta. \tag{3.22}$$

The Fourier transform of the function is then given by

$$f(\hat{q} - \hat{q}') = \int f(r)e^{i(\hat{q}-\hat{q}').\vec{r}}d^3\vec{r}. \tag{3.23}$$

In view of Eq. (3.22), this equation can be put in the form

$$f(\hat{q} - \hat{q}') = 2\pi \int_0^\infty r^2 f(r)dr \int_0^\pi d\theta \sin\theta e^{i(\hat{q}-\hat{q}')r\cos\theta}. \tag{3.24}$$

Now one can use the fact that

$$\sin\theta e^{i(\hat{q}-\hat{q'})r\cos\theta} = \frac{-1}{i(\hat{q}-\hat{q'})r}\frac{d}{d\theta}\left(e^{i(\hat{q}-\hat{q'})r\cos\theta}\right) \tag{3.25}$$

to carry through the angular integration over θ and reduce the multiple integral to a one dimensional integral over the radial coordinate. It then follows that

$$f(\hat{q}-\hat{q'}) = -2\pi\int_0^\infty r^2 f(r)dr\int_0^\pi \frac{1}{i(\hat{q}-\hat{q'})r}d(e^{i(\hat{q}-\hat{q'})r\cos\theta}). \tag{3.26}$$

Now carrying out the integration over the angular part, one can write

$$f(\hat{q}-\hat{q'}) = 2\pi\int_0^\infty r^2 f(r)dr[\frac{e^{i(\hat{q}-\hat{q'})r} - e^{-i(\hat{q}-\hat{q'})r}}{i(\hat{q}-\hat{q'})r}]. \tag{3.27}$$

Now we can apply the exponential definition of the sine function

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{3.28}$$

to obtain the equation

$$f(\hat{q}-\hat{q'}) = 4\pi\int_0^\infty dr f(r) r^2 \frac{\sin\left[(\hat{q}-\hat{q'})r\right]}{(\hat{q}-\hat{q'})r}. \tag{3.29}$$

plugging the expression for $f(r)$, one can get

$$f(\hat{q}-\hat{q'}) = 4\pi\int_0^\infty dr e^{-ar}\sin(\hat{q}-\hat{q'})r. \tag{3.30}$$

using the fact that

$$\sin(\hat{q}-\hat{q'})r = Im e^{i(\hat{q}-\hat{q'})r}, \tag{3.31}$$

one can write

$$f(\hat{q}-\hat{q'}) = \frac{4\pi}{(\hat{q}-\hat{q'})}Im\int_0^\infty e^{-(a-i(\hat{q}-\hat{q}'))r}dr. \tag{3.32}$$

carrying out the integration, we get

$$f(\hat{q}-\hat{q'}) = \frac{4\pi}{(\hat{q}-\hat{q'}}Im\left[\frac{a+i(\hat{q}-\hat{q}')}{a^2+(\hat{q}-\hat{q}')^2}\right]. \tag{3.33}$$

Now we take the imaginary part and is found to be

$$f(\hat{q}) = \frac{4\pi}{a^2 + (\hat{q} - \hat{q}')^2}. \tag{3.34}$$

In view of this, one can write

$$r^2 = \lim_{a\to 0} \frac{d^3}{da^3}[\frac{-4\pi}{a^2 + (\hat{q} - \hat{q}')^2}]. \tag{3.35}$$

It then follows that, the scalar potential takes the form

$$V_c(\hat{q}, \hat{q}') = \frac{3}{4}\omega_{q\bar{q}}^2\big[\lim_{a\to 0} \frac{d^3}{da^3}[\frac{-4\pi}{a^2 + (\hat{q} - \hat{q}'^2)}] - \frac{C_0}{\omega_0^2}(2\pi)^3\delta^3(\hat{q} - \hat{q}')\big]. \tag{3.36}$$

We remember the fact that

$$(2\pi)^3\vec{\nabla}_{\hat{q}}^2\delta^3(\hat{q} - \hat{q}') = \lim_{a\to 0} \frac{d^3}{da^3}[\frac{4\pi}{a^2 + (\hat{q} - \hat{q}')^2}], \tag{3.37}$$

where the differentiation here is with respect to $\hat{q}$, while $\hat{q}'$ keept constant. Now, one can put the scalar potential in the form

$$V_c(\hat{q}, \hat{q}') = \frac{-3}{4}\omega_{q\bar{q}}^2(2\pi)^3\big[\vec{\nabla}_{\hat{q}}^2 + \frac{C_0}{\omega_0^2}\big]\delta^3(\hat{q} - \hat{q}'). \tag{3.38}$$

This can be rewritten as

$$\begin{aligned} V_c(\hat{q}, \hat{q}') &= \frac{-3}{4}(2\pi)^3\overline{V}_c\delta^3(\hat{q} - \hat{q}'), \\ \overline{V}_c &= \omega_{q\bar{q}}^2\big[\vec{\nabla}_{\hat{q}}^2 + \frac{C_0}{\omega_0^2}\Big]. \end{aligned} \tag{3.39}$$

On account of this, the integral equation given by Eq. (3.17)takes the form

$$(M - 2\omega)\Big[f_3(\hat{q}) + \frac{2mf_4(\hat{q})}{\omega}\Big] = \frac{\theta_s\overline{V}_c}{\omega^2\hat{q}^2}\int d^3\hat{q}'\Big[-\hat{q}^2\hat{q}'^2 f_3(\hat{q}') + m^2\hat{q}.\hat{q}' f_3(\hat{q}') + 2m\omega\hat{q}\hat{q}' f_4(\hat{q}')\Big]\delta^3(\hat{q} -$$

$$(M + 2\omega)\Big[f_3(\hat{q}) - \frac{2mf_4(\hat{q})}{\omega}\Big] = \frac{\theta_s\overline{V}_c}{\omega^2\hat{q}^2}\int d^3\hat{q}'\Big[\hat{q}^2\hat{q}'^2 f_3(\hat{q}') - m^2\hat{q}.\hat{q}' f_3(\hat{q}') + 2m\omega\hat{q}\hat{q}' f_4(\hat{q}')\Big]\delta^3(\hat{q} - \hat{q}'$$

Carrying out the integration applying the identity given by Eq. (2.16), we get

$$\begin{aligned} (M - 2\omega)\Big[f_3(\hat{q}) + \frac{2mf_4(\hat{q})}{\omega}\Big] &= \frac{\theta_s\overline{V}_c}{\omega^2}\Big[\hat{q}^2 f_3(\hat{q}) - m^2 f_3(\hat{q}) - 2m\omega f_4(\hat{q})\Big] \\ (M + 2\omega)\Big[f_3(\hat{q}) - \frac{2mf_4(\hat{q})}{\omega}\Big] &= \frac{\theta_s\overline{V}_c}{\omega^2}\Big[-\hat{q}^2 f_3(\hat{q}) + m^2 f_3(\hat{q}) - 2m\omega f_4(\hat{q})\Big] \end{aligned} \tag{3.40}$$

Taking the sum of this two equation, we get

$$Mf_3(\hat{q}) - 4mf_4(\hat{q}) = \frac{-2m\theta_s\overline{V}_c f_4(\hat{q})}{\omega}. \tag{3.41}$$

Similarly, subtracting the second equation from the first, we get

$$\frac{2mMf_4(\hat{q})}{\omega} - 2\omega f_3(\hat{q}) = \frac{-\theta_s\overline{V}_c(m^2 - \hat{q}^2)f_3(\hat{q})}{\omega^2}. \tag{3.42}$$

Now solving for $f_3(\hat{q})$ from Eq. (3.41), we get

$$f_3(\hat{q}) = \frac{1}{M}\left[4mf_4(\hat{q}) - \frac{2m\theta_s\overline{V}_c f_4(\hat{q})}{\omega}\right]. \tag{3.43}$$

On the other hand, solving for $f_4(\hat{q})$ from Eq. (3.42), we get

$$f_4(\hat{q}) = \frac{\omega}{2mM}\left[2\omega f_3(\hat{q}) - \frac{\theta_s\overline{V}_c(m^2 - \hat{q}^2)f_3(\hat{q})}{\omega^2}\right]. \tag{3.44}$$

Plugging Eq. (3.43) and Eq. (3.44) into Eq. (3.40)and if we employ the heavy mass approximation, $\omega \approx m$ on the right hand side of the two algebraic equations which is valid since in the confinement region, the relative momentum between heavy quarks in the bound state can be considered small,and hence the heavy-quark is supposed to be moving with non-relativistic speeds and making $\hat{q}^2 - m^2 = -m^2$, these decoupled equations can be expressed as

$$\begin{aligned}
\left[\frac{M^2}{4} - \omega^2\right]f_3(\hat{q}) &= -m\theta_s\overline{V}_c f_3(\hat{q}) + \frac{\theta_s^2\overline{V}_c^2 f_3}{4} \\
\left[\frac{M^2}{4} - \omega^2\right]f_4(\hat{q}) &= -m\theta_s\overline{V}_c f_4(\hat{q}) + \frac{\theta_s^2\overline{V}_c^2 f_4}{4}
\end{aligned} \tag{3.45}$$

Using the input parameters, C_0 = 0.21, ω_0 = 0.15 GeV.,Λ = 0.200 GeV., and A_0 = 0.01, along with the input quark masses m_c = 1.490 GeV. and m_b = 5.070 GeV., we try to determine the numerical values of $\Omega_s = m\theta_s\omega_{q\bar{q}}^2$ and $\Omega_s' = \frac{\theta_s^2\omega_{q\bar{q}}^4}{4}$ for scalar mesons χ_{c0} and χ_{b0} in the right hand side of Eq. (3.45), and their percentage ratio in Table 3.2 below. Now one can see that $\omega_{q\bar{q}}^4 \ll \omega_{q\bar{q}}^2$ and hence the second term is negligble. In

	Ω_s	Ω'_s	$\frac{\Omega'_s}{\Omega_s}\%$
χ_{co}	0.1213	0.0016	1.315

Table 3.2: Numerical values of coefficients

comparison to the first term on the right hand side of Eq. (3.45), second term can be droped. To derive the mass spectrum, we put the spatial part, $\overline{V}_c$ in Eq. (3.45) and obtain

$$\begin{aligned}\left[\frac{M^2}{4}-m^2-\hat{q}^2\right]f_3(\hat{q}) &= -m\theta_s\omega_{q\bar{q}}^2[\vec{\nabla}_{\hat{q}}^2+\frac{C_0}{\omega_0^2}]f_3(\hat{q})\\ \left[\frac{M^2}{4}-m^2-\hat{q}^2\right]f_4(\hat{q}) &= -m\theta_s\omega_{q\bar{q}}^2[\vec{\nabla}_{\hat{q}}^2+\frac{C_0}{\omega_0^2}]f_4(\hat{q}).\end{aligned} \tag{3.46}$$

From these two equations we see that, $f_3(\hat{q})$ and $f_4(\hat{q})$ satisfy the same equation . Thus, we take $f_3(\hat{q}) \approx f_4(\hat{q}) \approx \phi(\hat{q})$. With the use of the above equality of amplitudes, we re-express f_3 in terms of f_1 and the complete wave function $\psi_s(\hat{q})$ can be expressed as

$$\psi_s(\hat{q}) = \left[M + i\frac{mM\not{\hat{q}}}{\hat{q}^2} - \frac{2\not{P}\not{\hat{q}}}{M}\right]\phi_s(\hat{q}). \tag{3.47}$$

We can reduce this equation into the equation of a simple quantum mechanical 3D$-$harmonic oscillator with coefficients depending on the hadron mass M and total quantum number N . The wave function satisfies the 3D BSE:

$$\left[\frac{M^2}{4}-m^2-\hat{q}^2\right]\phi(\hat{q}) = -m\theta_s\omega_{q\bar{q}}^2[\vec{\nabla}_{\hat{q}}^2+\frac{C_0}{\omega_0^2}]\phi(\hat{q}). \tag{3.48}$$

This can be put in the form

$$(\frac{M^2}{4}-m^2-\hat{q}^2)\phi(\hat{q}) = -\beta_s^4[\vec{\nabla}_{\hat{q}}^2+\frac{C_0}{\omega_0^2}]\phi(\hat{q}), \tag{3.49}$$

where

$$\beta^4 = 4m\omega_{q\bar{q}}^2, \tag{3.50}$$

and the spin-spin interaction term is taken to be $\Theta_s = \gamma_\mu \psi(\hat{q})\gamma_\mu$. Therefore, Eq. (3.49) takes the form

$$\vec{\nabla}^2_{\hat{q}}\phi(\hat{q}) + [\frac{E}{\beta^4} - \frac{\hat{q}^2}{\beta^4}]\phi(\hat{q}) = 0, \tag{3.51}$$

where we define the total energy to be

$$E = \frac{M^2}{4} - m^2 + \frac{\beta^4 C_0}{\omega_0^2}. \tag{3.52}$$

Putting the expression for the Laplacian operator in spherical coordinate, we get

$$\frac{d^2\phi(\hat{q})}{d\hat{q}^2} + \frac{2}{\hat{q}}\frac{d\phi(\hat{q})}{d\hat{q}} - \frac{l(l+1)\phi(\hat{q})}{\hat{q}^2} + (\frac{E}{\beta^4} - \frac{\hat{q}^2}{\beta^4})\phi(\hat{q}) = 0, \tag{3.53}$$

where l is the orbital quantum number with the values $l = 0, 1, 2, 3, ...$

This is nothing but second order linear homogeneous differential equation whose solution can be found by using the power series method. The wave function satisfies the 3D Bethe Salpeter equation for equal mass scalar heavy meson. This is a 3D harmonic oscillator equation where $l = 0, 1, 2, 3, ...$ corresponds to S, P, D wave states respectively. Let the solution of this equation be

$$\phi(\hat{q}) = h(\hat{q})e^{-\frac{\hat{q}^2}{2\beta^2}}, \tag{3.54}$$

where $h(\hat{q})$ is function which must not affect the asymptotic behavior of ϕ. In the confinement region, $\hat{q} \to 0$ and the quarks bound together. Now by taking the first and second derivative, we get

$$\begin{aligned} \frac{d\phi}{d\hat{q}} &= \left[h'(\hat{q}) - \frac{\hat{q}h(\hat{q})}{\beta^2}\right]e^{\frac{-\hat{q}^2}{2\beta^2}}, \\ \frac{d^2\phi}{d\hat{q}^2} &= \left[h''(\hat{q}) - \frac{2\hat{q}h'(\hat{q})}{\beta^2} + (\frac{\hat{q}^2}{\beta^4} - \frac{1}{\beta^2})h(\hat{q})\right]e^{\frac{-\hat{q}^2}{2\beta^2}}. \end{aligned} \tag{3.55}$$

Plugging these two equation into the above homogeneous differential equation, one can obtain

$$h''(\hat{q}) + (\frac{2}{\hat{q}} - \frac{2\hat{q}}{\beta^2})h'(\hat{q}) + \left(\frac{E}{\beta^4} - \frac{3}{\beta^2} - \frac{L(L+1)}{\hat{q}^2}\right)h(\hat{q}) = 0. \tag{3.56}$$

Here the primed notation indicates the first and second derivative with respect to $\hat{q}$. This is nothing but second order linear homogeneous differential equation whose solution can be found by using a generalized power series (Frobenius) method. Let the solution be

$$h(\hat{q}) = \hat{q}^k \sum_{n=0}^{\infty} a_n \hat{q}^n = \sum_{n=0}^{\infty} a_n \hat{q}^{k+n}, \tag{3.57}$$

where k is a number that may be real or complex and $a_0 \neq 0$. Now taking the derivative of the series with respect to $\hat{q}$, we get

$$\begin{aligned} \frac{dh(\hat{q})}{d\hat{q}} &= \sum_{n=0}^{\infty} a_n (k+n) \hat{q}^{k+n-1} \\ \frac{d^2 h(\hat{q})}{d\hat{q}^2} &= \sum_{n=0}^{\infty} a_n (k+n)(k+n-1) \hat{q}^{k+n-2}. \end{aligned} \tag{3.58}$$

Putting these two equation into the above differential equation, we get

$$\sum_{n=0}^{\infty} \left\{ \Big[(k+n)(k+n-1) + 2(k+n) - l(l+1) \Big] a_n \hat{q}^{k+n-2} + \left[\frac{E}{\beta^4} - \frac{3}{\beta^2} - \frac{2(k+n)}{\beta^2} \right] a_n \right\} \hat{q}^{k+n} = 0. \tag{3.59}$$

By setting $n = 0$, we get the coefficient of $\hat{q}^{k-2}$, the lowest power of $\hat{q}$ appearing on the left hand side. Since each coefficient has to vanish for this equation to hold true, we write

$$\big[k(k+1) - l(l+1) \big] a_0 = 0. \tag{3.60}$$

Since $a_0 \neq 0$, then for this equation holds true, we should have

$$k^2 + k - l(l+1) = 0. \tag{3.61}$$

This is quadratic equation for the index k and has two roots which are found to be

$$k = l, k = -(l+1). \tag{3.62}$$

The second lowest power appearing in above equation is $\hat{q}^{k-1}$ for $n = 1$ in the first summation. Then the requirement that the coefficient of $\hat{q}^{k-1}$ vanish yields

$$[k(k+1) + 2(k+1) - l(l+1)] a_1 = 0. \tag{3.63}$$

Since the coefficient $(k+1)(k+1)-l(l+1)$ will not vanish, then for this equation to satisfy, we require $a_1 = 0$. In order to write all these series in terms of the coefficients of $\hat{q}^{k+n}$ lets change or set(shift) $n = n+2$ in the first summation and keep $n = n$ in the second summation (since both summation are independent). We then write

$$\sum_{n=0}^{\infty}\left\{a_{n+2}\left[(k+n+2)(k+n+1)+2(k+n+2)-l(l+1)\right]+a_n\left[\frac{E}{\beta^4}-\frac{3}{\beta^2}-\frac{2(k+n)}{\beta^2}\right]\right\}\hat{q}^{k+n}=0. \tag{3.64}$$

This can be rewritten in the form

$$\sum_{n=0}^{\infty}\left\{a_{n+2}\left[(k+n+2)(k+n+3)-l(l+1)\right]+a_n\left[\frac{E}{\beta^4}-\frac{\left[2(k+n)+3\right]}{\beta^2}\right]\right\}\hat{q}^{k+n}=0. \tag{3.65}$$

Now equating the general coefficient of $\hat{q}^{k+n}$ in this equation, we get

$$a_{n+2}\left[(k+n+2)(k+n+3)-l(l+1)\right]+a_n\left[\frac{E}{\beta^4}-\frac{\left[2(k+n)+3\right]}{\beta^2}\right]=0. \tag{3.66}$$

From this equation one can obtain the recursion relation between the coefficients a_{n+2} and a_n as

$$a_{n+2}=\frac{\left[\frac{2(k+n)+3}{\beta^2}-\frac{E}{\beta^4}\right]a_n}{(k+n+2)(k+n+3)-l(l+1)}. \tag{3.67}$$

This is a two term recurrence relation. It is obvious that if we start with a_o, the recursion relation leads to the even coefficients $a_2, a_4, ...etc$ and ignores $a_1, a_2, ...$ etc. Since a_1 is arbitrary, one can set equal to zero and from the recursion relation, we get $a_1 = a_3 = a_5 = ... = 0$. That is all the odd power coefficients vanish. Now if we take the difference of the two indicial roots ,we get

$$\left|k_1 - k_2\right| = 2l+1. \tag{3.68}$$

Since $l \geq 0$, the two indicial roots differ by an integer or unity. So that the recurrence relation corresponding to the larger of the two indicial roots always leads to a solution of the ordinary differential equation. Therefore, taking the larger root of the indicial equation, one can write the recurrence relation in the form

$$a_{n+2} = \frac{\left[\frac{2(n+l)+3}{\beta^2} - \frac{E}{\beta^4}\right] a_n}{(n+l+2)(n+l+3) - l(l+1)}. \tag{3.69}$$

We remember that for scalar meson, $l = 1$, then the recurrence relation becomes

$$a_{n+2} = \frac{\left[\frac{2(n+1)+3}{\beta^2} - \frac{E}{\beta^4}\right] a_n}{(n+3)(n+4) - 2}. \tag{3.70}$$

This recurrence relation leads to

$$\begin{aligned} a_2 &= \frac{[\frac{5}{\beta^2} - \frac{E}{\beta^4}] a_0}{(3)(4) - 2}, \\ a_4 &= \frac{[\frac{9}{\beta^2} - \frac{E}{\beta^4}] a_2}{(5)(6) - 2}, \\ a_6 &= \frac{[\frac{13}{\beta^2} - \frac{E}{\beta^4}] a_4}{(7)(8) - 2}, \\ & ... \\ & .... \end{aligned} \tag{3.71}$$

This requires that all coefficients a_{2n+1} have to vanish. In general the recurrence relation can be put in the form

$$a_{2n+2} = \frac{\left[\frac{2(2n+l)+3}{\beta^2} - \frac{E}{\beta^4}\right] a_{2n}}{(2n+l+2)(2n+l+3) - l(l+1)}, \tag{3.72}$$

where the principal quantum number n is given by $n = 0, 1, 2, 3, 4, ...$ Now the power series solution must terminate. For this to happen there must occur some "highest" n (call it n'), such that the recursion formula spits out $a_{2n'+2} = 0$. This will truncate the series $h(\hat{q})$ for physicaly acceptable solutions, then the recursion relation requires that

$$\left[\frac{2(2n'+l)+3}{\beta^2} - \frac{E}{\beta^4}\right] = 0. \tag{3.73}$$

From this one can obtained the energy eigenvalue equation

$$E_N = 2\beta^2 \left[N + \frac{3}{2}\right], \tag{3.74}$$

where $N = 2n' + l$ and N is known as the principal quantum number and n as the radial quantum number. $n = 0, 1, 2, 3....$ The allowed values of N are integers, $N = l, l + 2, l + 4...$. The orbital quantum number l can have values $l = 0, 1, 2, ..., .$ To each value of $n = 0, 1, 2, 3, ...$ of N will then correspond to a polynomial of order $2n + l$ in $\hat{q}$ and an odd physically acceptable wave function $\phi(\hat{q})$ given by the above equation for the first orbitally excited states. For the allowed values of E, the recursion formula leads to

$$a_{2n+2} = \frac{\left[-\frac{4}{\beta^2}(n' - n)\right]a_{2n}}{(2n + l + 2)(2n + l + 3) - l(l + 1)}. \tag{3.75}$$

Now the power series solution can be written in the form

$$h_N(\hat{q}) = \sum_{n=0}^{n'} a_{2n}\hat{q}^{2n+l}. \tag{3.76}$$

In view of this the un normalized wave function(eigenfunction) can be put in the form

$$\phi_N(\hat{q}) = h_N(\hat{q})e^{-\frac{\hat{q}^2}{2\beta^2}}. \tag{3.77}$$

Now from the eigenvalue equation one can derive the mass spectral equation. We then have

$$\frac{1}{2\beta^2}[\frac{M^2}{4} - m^2 + \frac{\beta^4 c_o}{\omega_0^2}] = N + \frac{3}{2}. \tag{3.78}$$

From the mass spectral equation one can see that, the mass spectral equation depends on not only by the principal quantum number n but also the orbital quantum number l. Since we are dealing with the first orbitally excited state with quantum number $J^{pc} = 0^{++}$(p-wave) for $l = 1$ then we use the corresponding recursion relation for scalar meson.

If $n' = 0$ there is one term in the power series

$$h_1(\hat{q}) = a_0\hat{q}. \tag{3.79}$$

For $n'=0, n=0$ yields from recursion relation

$$a_2 = 0. \tag{3.80}$$

If $n'=1$, there are two terms in the series

$$h_3(\hat{q}) = a_0\hat{q} + a_2\hat{q}^3. \tag{3.81}$$

If $n'=1, n=0$ yields from the recursion relation

$$a_2 = \frac{-2a_0}{5\beta^2}. \tag{3.82}$$

If $n'=1$, $n=1$ yields

$$a_4 = 0. \tag{3.83}$$

It then follows that

$$h_3(\hat{q}) = \frac{a_0}{5\beta_s^2}(5\beta_s^2\hat{q} - 2\hat{q}^3). \tag{3.84}$$

If $n'=2$, there are three terms in the series

$$h_5(\hat{q}) = a_0\hat{q} + a_2\hat{q}^3 + a_4\hat{q}^5. \tag{3.85}$$

If $n'=2, n=0$ yields from the recursion relation

$$a_2 = \frac{[\frac{-4}{\beta^2}(2-0)]a_0}{(3)(4)-2} = \frac{-4a_0}{5\beta^2}. \tag{3.86}$$

If $n'=2, n=1$ yields from the recursion relation

$$a_4 = \frac{[\frac{-4}{\beta^2}(2-1)]a_2}{(5)(6)-2} = \frac{4a_0}{35\beta_s^4}. \tag{3.87}$$

If $n'=2, n=2$ yields

$$a_6 = \frac{[-\frac{4}{\beta_s^2}(2-2)]a_4}{(7)(8)-2} = 0. \tag{3.88}$$

Then the power series takes the form

$$h_5(\hat{q}) = a_0\left[\hat{q} - \frac{4\hat{q}^3}{5\beta_s^2} + \frac{4\hat{q}^5}{35\beta_s^4}\right]. \tag{3.89}$$

If $n' = 3$, there are four terms in the series

$$h_7(\hat{q}) = a_0\hat{q} + a_2\hat{q}^3 + a_4\hat{q}^5 + a_6\hat{q}^7. \tag{3.90}$$

For $n' = 3, n = 0$ yields from the recursion relation

$$a_2 = \frac{[-\frac{4}{\beta_s^2}(3-0)]a_0}{(3)(4)-2} = \frac{-6a_0}{5\beta_s^2}. \tag{3.91}$$

For $n' = 3, n = 1$ yields

$$a_4 = \frac{[-\frac{4}{\beta_s^2}(3-1)]a_2}{(5)(6)-2} = \frac{12a_0}{35\beta_s^4}. \tag{3.92}$$

For $n' = 3, n = 2$ yields

$$a_6 = \frac{[-\frac{4}{\beta_s^2}(3-2)]a_4}{(7)(8)-2} = \frac{-24a_0}{945\beta_s^6}. \tag{3.93}$$

For $n' = 3, n = 3$ yields

$$a_8 = \frac{[-\frac{4}{\beta_s^2}(3-3)]a_6}{(9)(10)-2} = 0. \tag{3.94}$$

Then the polynomial takes the form

$$h_7(\hat{q}) = a_0\hat{q}\big[1 - \frac{6\hat{q}^2}{5\beta_s^2} + \frac{12\hat{q}^4}{35\beta_s^4} - \frac{24\hat{q}^6}{945\beta_s^6}\big]. \tag{3.95}$$

From the energy eigen value equation, the first orbitally excited state energy eigenvalue for $n' = 0, 1, 2, 3$ respectively can be found

$$\begin{aligned} E_1 &= 5\beta^2, \\ E_3 &= 9\beta^2, \\ E_5 &= 13\beta^2, \\ E_7 &= 17\beta^2. \end{aligned} \tag{3.96}$$

We see that to each of the discrete value E_N of the energy, given by the energy eigen value equation, there corresponds one and only one physically acceptable eigen function. Therefore, the energy eigenfunction for radially excited state can be put in the form

$$\phi_1(1p,\hat{q}) = N_{1s}\hat{q}e^{-\frac{\hat{q}^2}{2\beta_s^2}},$$
$$\phi_3(2p,\hat{q}) = N_{3s}\hat{q}\left[1-\frac{2\hat{q}^2}{5\beta_s^2}\right]e^{-\frac{\hat{q}^2}{2\beta_s^2}},$$
$$\phi_5(3p,\hat{q}) = N_{5s}\hat{q}\left[1-\frac{4\hat{q}^2}{5\beta_s^2}+\frac{4\hat{q}^4}{35\beta_s^4}\right]e^{-\frac{\hat{q}^2}{2\beta_s^2}},$$
$$\phi_7(4p,\hat{q}) = N_{7s}\hat{q}\left[1-\frac{6\hat{q}^2}{5\beta_s^2}+\frac{12\hat{q}^4}{35\beta_s^4}-\frac{24\hat{q}^6}{945\beta_s^6}\right]e^{-\frac{\hat{q}^2}{2\beta_s^2}}. \quad (3.97)$$

where N_s is the 3D scalar normalizer. But for parameter $A_0 \neq 0$ i.e if we reintroduce the factor k in the denominator, we can write the complete potential $V(\hat{q},\hat{q'})$ as

$$V(\hat{q},\hat{q'}) = \frac{-3}{4}\omega_{q\bar{q}}^2(2\pi)^3\left[\frac{\vec{\nabla}_{\hat{q}}^2}{\sqrt{1-A_0M^2\vec{\nabla}_{\hat{q}}^2}}+\frac{C_0}{\omega_0^2}\right]\delta^3(\hat{q}-\hat{q'}), \quad (3.98)$$

where we have used $(-\vec{\nabla}_{\hat{q}}^2)$ as the representation of r^2 in momentum space and $\hat{m}_1 = \hat{m}_2 = \frac{1}{2}$ for equal mass quarks. To derive the mass spectrum, we put Eq. (3.98) into Eq. (3.40) for scalar meson. However, to solve the equation with $A_0 \neq 0$, we follow an approximate analytical procedure where, we treat k as a correction factor due to small value of parameter, $A_0(A_0 \ll 1)$ while we work in approximate harmonic oscillator basis due to its transparency in bringing out the dependence of mass spectral equations on the total quantum number N, with little compromise on numerical accuracy. In this connection, we wish to mention that recently, harmonic oscillator basis has also been widely employed to study heavy quarkonia using a light front quark model. The latter is achieved through the effective replacement $k = (1-A_0M^2\vec{\nabla}_{\hat{q}}^2)^{\frac{-1}{2}} = [1+2A_0(2n+l+\frac{3}{2})]^{\frac{-1}{2}}$ which is quite valid for $c\bar{c}$ and $b\bar{b}$ systems. With this, we can reduce the above integral equation to equation of a simple quantum mechanical 3D harmonic oscillator with coefficients depending on the hadron mass M and total quantum number N and orbital quantum number

L. In view of this Eq. (3.40) can be put in the form

$$(M-2\omega)\Big[f_3(\hat{q})+\frac{2mf_4(\hat{q})}{\omega}\Big] = \frac{\theta_s\overline{V}_c}{\omega^2\hat{q}^2}\int d^3\hat{q}\Big[-\hat{q}^2\hat{q}'^2f_3+m^2\hat{q}.\hat{q}'f_3+2m\omega\hat{q}\hat{q}'f_4\Big]\delta^3(\hat{q}-\hat{q}')$$
$$(M+2\omega)\Big[f_3(\hat{q})-\frac{2mf_4(\hat{q})}{\omega}\Big] = \frac{\theta_s\overline{V}_c}{\omega^2\hat{q}^2}\int d^3\hat{q}\Big[\hat{q}^2\hat{q}'^2f_3-m^2\hat{q}.\hat{q}'f_3+2m\omega\hat{q}\hat{q}'f_4\Big]\delta^3(\hat{q}-\hat{q}') \quad (3.99)$$

where we define

$$\overline{V}_c=\omega_{q\bar{q}}^2\big[k\vec{\nabla}_{\hat{q}}^2+\frac{C_0}{\omega_0^2}\big]. \quad (3.100)$$

Carrying out the integration and following similar procedure as we did a bove for light scalr meson, we obtain

$$\Big[\frac{M^2}{4}-m^2-\hat{q}^2\Big]\phi(\hat{q})=\frac{-4m\omega_{q\bar{q}}^2}{\sqrt{1+2A_0(N+\frac{3}{2})}}\Big[\vec{\nabla}_{\hat{q}}^2+\frac{C_o}{\omega_0^2}\sqrt{1+2A_0(2n+l+\frac{3}{2})}\Big]\phi(\hat{q}) \quad (3.101)$$

This can be rewritten in the form

$$\Big[\frac{M^2}{4}-m^2-\hat{q}^2\Big]\phi(\hat{q})=-\beta_s^4\Big[\vec{\nabla}_{\hat{q}}^2+\frac{C_0}{\omega_0^2}\sqrt{1+2A_0(N+\frac{3}{2})}\Big]\phi(\hat{q}), \quad (3.102)$$

where we define the gaussian parameter for a meson composed of heavy quark antiquark to be

$$\beta_s^4=\frac{4m\omega_{q\bar{q}}^2}{\sqrt{1+2A_0(N+\frac{3}{2})}}. \quad (3.103)$$

Now Eq. (3.102) can be put in the form

$$\vec{\nabla}_{\hat{q}}^2\phi(\hat{q})+(\frac{E}{\beta_s^4}-\frac{\hat{q}^2}{\beta_s^4})\phi(\hat{q})=0, \quad (3.104)$$

where the total energy eigenvalue of the scalar heavy meson is defined by

$$E=\frac{M^2}{4}-m^2+\frac{\beta^4C_0}{\omega_0^2}\sqrt{1+2A_0(N+\frac{3}{2})}. \quad (3.105)$$

If we look at Eq. (3.104 and Eq. (3.51)), one can see that the two equations are equal and they are second order differential equation so that they have the same solution with different gaussian parameter and different eigenvalue. Therefore,the energy eigenfunction for the first orbitally excited state can be put in the form

$$\begin{aligned}
\phi(1p,\hat{q}) &= N_s\hat{q}e^{-\frac{\hat{q}^2}{2\beta_s^2}},\\
\phi(2p,\hat{q}) &= N_s\frac{\hat{q}}{5\beta_s^2}\left[5\beta_s^2-2\hat{q}^2\right]e^{-\frac{\hat{q}^2}{2\beta_s^2}},\\
\phi(3p,\hat{q}) &= N_s\frac{\hat{q}}{35\beta_s^4}\left[35\beta_s^4-28\beta_s^2\hat{q}^2+4\hat{q}^4\right]e^{-\frac{\hat{q}^2}{2\beta_s^2}},\\
\phi(4p,\hat{q}) &= N_s\hat{q}\left[1-\frac{6\hat{q}^2}{5\beta_s^2}+\frac{12\hat{q}^4}{35\beta_s^4}-\frac{24\hat{q}^6}{945\beta_s^6}\right]e^{-\frac{\hat{q}^2}{2\beta_s^2}}.
\end{aligned} \tag{3.106}$$

3.2 3D Normalizer

We require that the eigenfunctions in Eq. (3.106) be normalized to unity so that

$$\int |\phi(\hat{q})|^2 d^3\hat{q} = 1. \tag{3.107}$$

Now the 3D volume element can be put in the form $d^3\hat{q} = |\hat{q}|^2 d|\hat{q}| \sin\theta d\theta d\phi$. In view of this one can write

$$\int_0^\infty |\phi|^2\hat{q}^2 d\hat{q}\int_0^\pi sin\theta d\theta\int_0^{2\pi} d\phi = 1. \tag{3.108}$$

Carrying out the integration for the angular part and inview of Eq. (3.106), the ground state of the scalar meson normalizer can be calculated as

$$4\pi N_{1s}^2\int_0^\infty \hat{q}^4 e^{-\frac{\hat{q}^2}{\beta_S^2}} d\hat{q} = 1. \tag{3.109}$$

Carrying out the integration using the relation

$$\begin{aligned}
\int_0^\infty x^{2n}e^{-ax^2}dx &= \frac{\Gamma(n+\frac{1}{2})}{2a^{n+\frac{1}{2}}},\\
\int_0^\infty x^{2n+1}e^{-ax^2}dx &= \frac{n!}{2a^{n+1}},
\end{aligned} \tag{3.110}$$

where the gamma function is defined by

$$\Gamma(n+\frac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}. \tag{3.111}$$

Here n is an integer (0, 1, 2, 3, ...) and a > 0. In view of this, one can obtain

$$N_{1s} = (\frac{2}{3})^{\frac{1}{2}} \frac{1}{(\beta_s^{\frac{5}{2}})(\pi^{\frac{3}{4}})}, \tag{3.112}$$

where β_s is given by Eq. (3.103).

Following similar procedure, one can obtain the 3D normilizer for the first and second radially excited state $\phi(2p, \hat{q})$ to be

$$\begin{aligned}
N_{2s} &= (\frac{5}{3})^{\frac{1}{2}} \frac{1}{(\beta_s^{\frac{5}{2}})(\pi^{\frac{3}{4}})}, \\
N_{3s} &= (\frac{35}{12})^{\frac{1}{2}} \frac{1}{(\beta_s^{\frac{5}{2}})(\pi^{\frac{3}{4}})}, \\
N_{4s} &= (\frac{1890}{46359})^{\frac{1}{2}} \frac{1}{(\beta_s^{\frac{5}{2}})(\pi^{\frac{3}{4}})}.
\end{aligned} \tag{3.113}$$

In view of this the normalized eigenfunction is found to be

$$\begin{aligned}
\phi(1p, \hat{q}) &= (\frac{2}{3})^{\frac{1}{2}} \frac{1}{(\beta_s^{\frac{5}{2}})(\pi^{\frac{3}{4}})} \hat{q} e^{-\frac{\hat{q}^2}{2\beta_s^2}}, \\
\phi(2p, \hat{q}) &= (\frac{5}{3})^{\frac{1}{2}} \frac{1}{(\beta_s^{\frac{5}{2}})(\pi^{\frac{3}{4}})} \hat{q} \left[1 - \frac{2\hat{q}^2}{5\beta_s^2}\right] e^{-\frac{\hat{q}^2}{2\beta_s^2}}, \\
\phi(3p, \hat{q}) &= (\frac{35}{12})^{\frac{1}{2}} \frac{1}{(\beta_s^{\frac{5}{2}})(\pi^{\frac{3}{4}})} \hat{q} \left[1 - \frac{4\hat{q}^2}{5\beta_s^2} + \frac{4\hat{q}^4}{35\beta_s^4}\right] e^{-\frac{\hat{q}^2}{2\beta_s^2}}, \\
\phi(4p, \hat{q}) &= (\frac{1890}{46359})^{\frac{1}{2}} \frac{1}{(\beta_s^{\frac{5}{2}})(\pi^{\frac{3}{4}})} \hat{q} \left[1 - \frac{6\hat{q}^2}{5\beta_s^2} + \frac{12\hat{q}^4}{35\beta_s^4} - \frac{24\hat{q}^6}{945\beta_s^6}\right] e^{-\frac{\hat{q}^2}{2\beta_s^2}}.
\end{aligned} \tag{3.114}$$

Several interesting features emerge from the examination of the eigenfunctions $\phi_N(np, \hat{q})$ i.e, the bound state eigen function is zero for only p state $(l = 1)$ and different from zero for s state $l = 0$ at $\hat{q} = 0$ and the wave functions are forced to remain large over distances (interaction momenta).

Now, treating the mass spectral equation, Eq.(3.104) as the unperturbed equation, with the unperturbed wave functions, $\phi(nP, \hat{q})$ for scalar mesons in Eq.(3.114), we now incorporate the coulomb term in this equation. Then the above mass spectral equation can be written as:

$$E_s \phi_s(\hat{q}) = [-\beta_s^4 \vec{\nabla}_{\hat{q}}^2 + \hat{q}^2 + V_{coul}^s]\phi(\hat{q}), \tag{3.115}$$

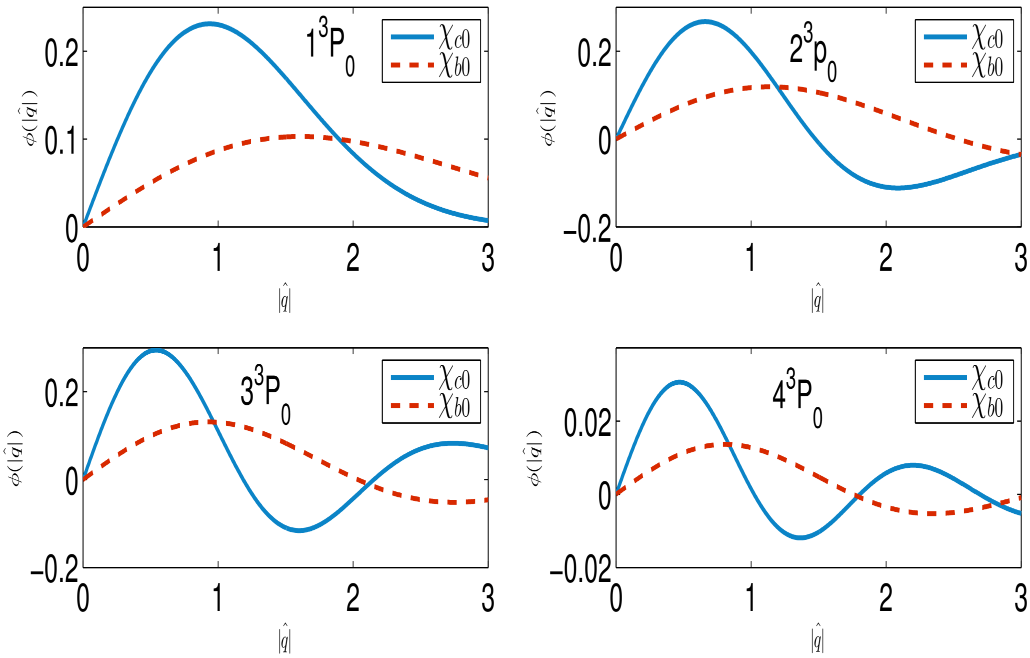

Figure 3.1: The graphs the unperturbed wave functions of the ground, first, second and the third excited states with quantum number $J^{pc} = 0^{++}$ for $c\bar{c}$ and $b\bar{b}$ states of scalar meson .

Now, treating the coulomb term as a perturbation to the unperturbed mass spectral equation, we can write the complete mass spectra for the ground and excited states for equal mass heavy scalar (0^{++}) mesons using first order perturbation theory as:

$$\frac{1}{2\beta_s^2}\left[\frac{M^2}{4} - m^2 + \frac{\beta_s^4 C_0}{\omega_0^2}\sqrt{1 + 2A_0(N + \frac{3}{2})}\right] + \gamma < V^s_{coul} >= N + \frac{3}{2}, \qquad (3.116)$$

where $N = 2n + l, n = 0, 1, 2, 3, ...$ and $< V^s_{coul} >$ is the expectation value of V^s_{coul} between the unperturbed states of given quantum numbers n (with $l = 1$) for scalar mesons, and has been weighted by a factor of $\gamma_s = (C_0^2)(\beta^2)/3$ to have the coulomb term dimensionally consistent with the harmonic term and its expectation values for the 1P, 2P, 3P and 4P states are

$$\begin{aligned}
&< 1P \mid V^s_{coul} \mid 1P >= -\frac{128\pi\alpha_s}{9\beta_s^2},\\
&< 2P \mid V^s_{coul} \mid 2P >= -\frac{64\pi\alpha_s}{18\beta_s^2},\\
&< 3P \mid V^s_{coul} \mid 3P >= -\frac{3712\pi\alpha_s}{213\beta_s^2},\\
&< 4P \mid V^s_{coul} \mid 4P >= -\frac{1152\pi\alpha_s}{25\beta_s^2}.
\end{aligned} \tag{3.117}$$

From the mass spectral equation, one can see that, the mass spectra depends not only on the principal quantum number n, but also the orbital quantum number l.

3.3 Numerical calculation of mass spectral for heavy scalar meson

We are now in aposition to calculate the mass spectrum using the mass spectral equation given by Eq. (3.116) for $c\bar{c}$.

We now give the mass spectrum in BSE-CIA of ground and excited states of χ_{c0} with quantum numbers $J^{PC} = 0^{++}$ in GeV. It is seen that the set of input parameters is not unique. It is observed that the mass spectra of mesons of various J^{PC} is practically insensitive to a range of variations of parameter $\omega_0 \in [0.120 - 0.180]$ GeV. as long as $\frac{C_0}{\omega_0}$is a constant, and reasonably good fits are obtained for the parameter ratio,$\frac{C_0}{\omega_0}$= 9.3333GeV^2 , with other parameters Λ = 0.250GeV.,A_0 = 0.01, and m_c = 1.490GeV. Thus for ω_0= 0.120GeV, we have, C_0 = 0.1344, while for ω_0=0.180GeV., we have C_0 = 0.3024. However for comparison with experiment, we take the values ω_0=0.145GeV., and C_0 = 0.18688, along with Λ=0.250 GeV., A_0 = 0.01, and m_c=1.490 GeV.

From the mass spectral equation, one can see that, the mass spectra depends not only on the principal quantum number N, but also the orbital quantum number l. We are now in a position to calculate the numerical values for mass spectral of heavy equal mass scalar meson with the input parameters of our model.

	M($\omega_0 = 0.130$)	M($\omega_0 = 0.1450$)	M($\omega_0 = 0.160$)	Exp[7]
$M_{\chi_{c0}}(1P_0)$	3.3967	3.4474	3.4995	3.4140±0.0003
$M_{\chi_{c0}}(2P_0)$	4.0967	4.1552	4.2039	
$M_{\chi_{c0}}(3P_0)$	4.7204	4.8463	4.9687	
$M_{\chi_{c0}}(4P_0)$	5.2961	5.5186	5.7308	

Table 3.3: Mass spectrum in BSE-CIA of ground and excited states of χ_{c0} with quantum numbers $J^{PC} = 0^{++}$ in GeV. units with a range of variations of parameter $\omega_0 \in [0.120 - 0.180]$ GeV. for the parameter ratio, $\frac{C_0}{\omega_0}$= 9.3333GeV^2 , and other parameters Λ = 0.250GeV.,A_0 = 0.01, and m_c = 1.490GeV. However for comparison with experiment, we take the value ω_0=0.145GeV. and C_0 = 0.186889

	BSE-CIA	Expt[7]	Pot.Model[28]	BSE[55]	RQM[29]
$M_{\chi_{c0}}(1^3p_0)$	3.43577	3.4140±0.00031	3.440	3.414	3.413
$M_{\chi c0}(2^3p_0)$	4.14366		3.920	3.8368	3.870
$M_{\chi c0}(3^3p_0)$	4.86479			4.1401	4.3010
$M_{\chi c0}(4^3p_0)$	5.565				

Table 3.4: Mass spectrum of the ground and excited states of $_c0$ in the BSE-CIA with quantum numbers $J^{PC} = 0^{++}$in units of GeV with the above set of parameters(Λ = 0.250GeV.,A_0 = 0.01, m_c = 1.490GeV., ω_0=0.145GeV., and C_0 = 0.186889) along with data and the results of other models.

3.4 Spectroscopy of Axial vector meson(1^{++})

The general form for the relativistic salpeter wave function of $^3P_1$ state which $J^{pc} = 1^{++}$ for equal mass system of dimensionality M is given by

$$\begin{aligned}\psi(\hat{q}) = \quad & \gamma_5[\gamma_\mu + \frac{P_\mu \not{P}}{M^2}][iMf_1(\hat{q}) + \not{P}f_2(\hat{q}) - \not{\hat{q}}f_3(\hat{q}) + 2i\frac{\not{P}\not{\hat{q}}}{M}f_4(\hat{q})] \\ & +\gamma_5[\hat{q_\mu}f_3(\hat{q}) + \frac{2i\hat{q_\mu}\not{P}}{M}f_4(\hat{q})]. \end{aligned} \tag{3.118}$$

From this one can see that the Dirac covariants M associated with the amplitude f_1 and f_2 are the most dominant one in comparison to all the other Dirac covariants. In addition to this $\frac{P_\mu \not{P}}{M^2}$ is very small and contributes less for meson observable so that one can neglect it. Then the Salpeter wave function takes the form[39]

$$\psi(\hat{q}) = \gamma_5[iM\gamma_\mu f_1(\hat{q}) + \gamma_\mu \not{P} f_2(\hat{q})]. \tag{3.119}$$

Plugging this wave function into the first two equations of Eq. (2.123) as the scalar meson and taking trace on both sides as well as following the steps as for the scalar meson, we get

$$[\frac{M^2}{4} - m^2 - \hat{q}^2]f_1(\hat{q}) = -\omega\theta_A\omega_{q\bar{q}}^2[\vec{\nabla}_{\hat{q}}^2 + \frac{c_0}{\omega_0^2}]f_1(\hat{q}) + \frac{\theta_A^2\omega_{q\bar{q}}^4}{4}[\vec{\nabla}_{\hat{q}}^2 + \frac{c_0}{\omega_0^2}]^2 f_1(\hat{q})$$
$$[\frac{M^2}{4} - m^2 - \hat{q}^2]f_2(\hat{q}) = -\omega\theta_A\omega_{q\bar{q}}^2[\vec{\nabla}_{\hat{q}}^2 + \frac{c_0}{\omega_0^2}]f_2(\hat{q}) + \frac{\theta_A^2\omega_{q\bar{q}}^4}{4}[\vec{\nabla}_{\hat{q}}^2 + \frac{c_0}{\omega_0^2}]^2 f_2(\hat{q}) \tag{3.120}$$
$$\tag{3.121}$$

From these two equation one can see that both f_1 and f_2 satisfy the same equation and hence we can approximately write $f_1 \sim f_2 \sim \phi$. On the other hand $\omega_{q\bar{q}}^4 \ll \omega_{q\bar{q}}^2$ and hence we can forget the second term on the right hand side. Moreover, in the low energy limit, we can use $\omega \approx m$ which is true for heavy quarkonium. It then follows that

$$[\frac{M^2}{4} - m^2 - \hat{q}^2]\phi(\hat{q}) = -2m\omega_{q\bar{q}}^2[\vec{\nabla}_{\hat{q}}^2 + \frac{c_0}{\omega_0^2}]\phi(\hat{q}). \tag{3.122}$$

This can be put in the form

$$[\frac{M^2}{4} - m^2 - \hat{q}^2]\phi(\hat{q}) = -\beta_A^4[\kappa\vec{\nabla}_{\hat{q}}^2 + \frac{c_0}{\omega_0^2}]\phi(\hat{q}), \tag{3.123}$$

which can in turn be expressed in an identical form as scalar meson as

$$E\phi_A(\hat{q}) = [-\beta_A^4\vec{\nabla}_{\hat{q}}^2 + \hat{q}^2]\phi_A(\hat{q}), \tag{3.124}$$

where we define the gaussian parameter for a meson composed of heavy quark antiquark to be

$$\beta_A^4 = \frac{2m\omega_{q\bar{q}}^2}{\sqrt{1 + 2A_0(N + \frac{3}{2})}}. \tag{3.125}$$

Putting the expression for the Laplacian operator in spherical coordinate, we get

$$\frac{d^2\phi(\hat{q})}{d\hat{q}^2} + \frac{2}{\hat{q}}\frac{d\phi(\hat{q})}{d\hat{q}} - \frac{l(l+1)\phi(\hat{q})}{\hat{q}^2} + (\frac{E}{\beta_A^4} - \frac{\hat{q}^2}{\beta_A^4})\phi(\hat{q}) = 0. \tag{3.126}$$

This is the same equation as the scalar meson with β_s^4 replaced by β_A^4. with the total energy eigenvalue given by

$$E = \frac{M^2}{4} - m^2 + \frac{\beta_A^4 C_0}{\omega_0^2}\sqrt{1 + 2A_0(N + \frac{3}{2})}. \tag{3.127}$$

Following exactly the same steps as in the scalar meson case, we can express the energy eigen values of the system as:

$$E_N = 2\beta_A^2(N + \frac{3}{2}), \tag{3.128}$$

where $N = 2n' + l$, $n = 0, 1, 2, ...$ and $l = 0, 1, 2, 3, ...$

If we look at Eq. (3.126), one can see that this equation is second order differential equation so that its solution is the same with that of the scalar meson case but with different gaussian parameter and different eigenvalue. Therefore,the energy eigenfunction for the first orbitally excited state of the axial vector meson can be put in the form

$$\begin{aligned}
\phi(1p,\hat{q}) &= N_A\hat{q}e^{-\frac{\hat{q}^2}{2\beta_h^2}},\\
\phi(2p,\hat{q}) &= N_A\frac{\hat{q}}{5\beta_h^2}\left[5\beta_h^2-2\hat{q}^2\right]e^{-\frac{\hat{q}^2}{2\beta_h^2}},\\
\phi(3p,\hat{q}) &= N_A\frac{\hat{q}}{35\beta_h^4}\left[35\beta_h^4-28\beta_h^2\hat{q}^2+4\hat{q}^4\right]e^{-\frac{\hat{q}^2}{2\beta_h^2}},\\
\phi(4p,\hat{q}) &= N_A\frac{\hat{q}}{945\beta^6}\left[945\beta_h^6-1134\beta_h^4\hat{q}^2+324\beta_h^2\hat{q}^4-24\hat{q}^6\right]e^{-\frac{\hat{q}^2}{2\beta_h^2}}, \quad (3.129)
\end{aligned}$$

where N_A is the scalar 3D normalizer for axial vector meson. Solving for the 3D normalizer the eigen function can be put in the form

$$\begin{aligned}
\phi(1p,\hat{q}) &= (\frac{2}{3})^{\frac{1}{2}}\frac{1}{(\beta_A^{\frac{5}{2}})(\pi^{\frac{3}{4}})}\hat{q}e^{-\frac{\hat{q}^2}{2\beta_A^2}},\\
\phi(2p,\hat{q}) &= (\frac{5}{3})^{\frac{1}{2}}\frac{1}{(\beta_A^{\frac{5}{2}})(\pi^{\frac{3}{4}})}\hat{q}\left[1-\frac{2\hat{q}^2}{5\beta_A^2}\right]e^{-\frac{\hat{q}^2}{2\beta_A^2}},\\
\phi(3p,\hat{q}) &= (\frac{35}{12})^{\frac{1}{2}}\frac{1}{(\beta_A^{\frac{5}{2}})(\pi^{\frac{3}{4}})}\hat{q}\left[1-\frac{4\hat{q}^2}{5\beta_A^2}+\frac{4\hat{q}^4}{35\beta_A^4}\right]e^{-\frac{\hat{q}^2}{2\beta_A^2}},\\
\phi(4p,\hat{q}) &= (\frac{1890}{46359})^{\frac{1}{2}}\frac{1}{(\beta_A^{\frac{5}{2}})(\pi^{\frac{3}{4}})}\hat{q}\left[1-\frac{6\hat{q}^2}{5\beta_A^2}+\frac{12\hat{q}^4}{35\beta_A^4}-\frac{24\hat{q}^6}{945\beta_A^6}\right]e^{-\frac{\hat{q}^2}{2\beta_A^2}}. \quad (3.130)
\end{aligned}$$

We have put the plot of wave function for $b\bar{b}$ mesons for acadamic interest.

Now, the perturbative inclusion of the coulomb term will be the same for axial vector meson in the same form as for the scalar case, except that the inverse range parameter β_s has got to be replaced by β_A. The plots of wave functions for axial vector quarkonia are thus similar to the corresponding plots of scalar quarkonia. Now, treating the above mass spectral equation, Eq. (3.127) as the unperturbed equation, with the unperturbed wave functions, $\phi_A(nP,\hat{q})$ for axial vector mesons in Eq. (3.130), we now incorporate the coulomb term in this equation. Then the above mass spectral equation can be written as:

$$E_A\phi_A(\hat{q}) = [-\beta_A^4\vec{\nabla}_{\hat{q}}^2+\hat{q}^2+V_{coul}^A]\phi_A(\hat{q}). \quad (3.131)$$

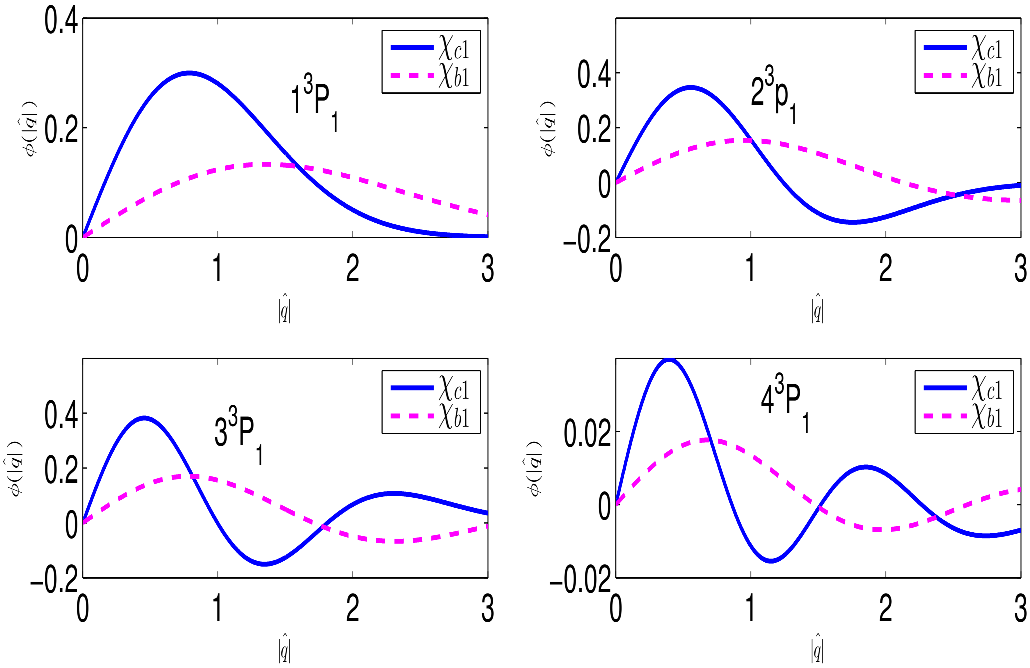

Figure 3.2: The graphs of the unperturbed wave functions of the ground, first, second and the third excited states with quantum number $J^{pc} = 1^{++}$ for $c\bar{c}$ (solid line) and $b\bar{b}$(dashed line) of axial vector meson.

Now, treating the coulomb term as a perturbation to the unperturbed mass spectral equation, we can write the complete mass spectra for the ground and excited states for equal mass heavy Axial vector (1^{++}) mesons using first order perturbation theory as:

$$\frac{1}{2\beta_s^2}\left[\frac{M^2}{4} - m^2 + \frac{\beta_s^4 C_0}{\omega_0^2}\sqrt{1 + 2A_0(N + \frac{3}{2})}\right] + \gamma_A < V_{coul}^A >= N + \frac{3}{2}, \qquad (3.132)$$

with $l = 1$, where $< V_{coul}^A >$ is the expectation value of V_{coul}^A between the unperturbed states of given quantum numbers n (with $l = 1$) for axial vector mesons, and has been weighted by a factor of $\gamma_A = C_0^2\beta_A^2$ to have the coulomb term dimensionally consistent with the harmonic term. Its expectation values for the 1P, 2P, 3P and 4P states are:

$$
\begin{aligned}
&< 1P \mid V^A_{coul} \mid 1P >= -\frac{64\pi\alpha_s}{9\beta_A^2},\\
&< 2P \mid V^A_{coul} \mid 2P >= -\frac{32\pi\alpha_s}{18\beta_A^2},\\
&< 3P \mid V^A_{coul} \mid 3P >= -\frac{1856\pi\alpha_s}{213\beta_A^2},\\
&< 4P \mid V_{coul^A} \mid 4P >= -\frac{576\pi\alpha_s}{25\beta_A^2}.
\end{aligned}
\tag{3.133}
$$

3.5 Numerical calculation of mass spectral for heavy Axial vector meson(1^{++})

From the mass spectral equation given by Eq. (3.132), one can obtain the mass spectral values for the axial vector meson composed of $c\bar{c}$ and is found to be as shown below. From the mass spectral equation one can see that, the mass spectral equation depends on not only by the principal quantum number N but also the orbital quantum number L. We are now in a position to calculate the numerical values for mass spectral of heavy Axial vector meson constituent from equal mass quark- antiquark with the same parameter set above for scalar meson.

3.6 Derivation of Bethe Salpeter wave function and the mass spectral equation of axial vector meson(1^{+-})

The general form for the relativistic Salpeter wave function of state $J^{pc} = 1^{+-}$ for equal mass system is given by

$$
\psi(\hat{q}) = \hat{q}\gamma_5[f_1(\hat{q}) + i\not{P}f_2(\hat{q}) - i\not{\hat{q}}f_3(\hat{q}) + 2\not{P}\not{\hat{q}}f_4(\hat{q})]. \tag{3.134}
$$

Here, $f_i(q^2, q.P, P^2)$ are the Lorentz scalar amplitudes multiplying the various Dirac structures in the BS wave function $\psi(P, q)$. By use of power counting rule, it can be

	M($\omega_0 = 0.120$)	M($\omega_0 = 0.150$)	M($\omega_0 = 0.180$)	Exp[7]
$M_{\chi_{c1}}(1P_1)$	3.4330	3.4782	3.5279	3.510±0.0007
$M_{\chi_{c1}}(2P_1)$	3.9279	3.9560	4.0333	3.871.69±0.0017
$M_{\chi_{c1}}(3P_1)$	4.3858	4.4650	4.6636	
$M_{\chi_{c0}}(4P_0)$	4.8155	4.9593	5.3151	

Table 3.5: Mass spectrum in BSE-CIA of ground and excited states of χ_{c0} with quantum numbers $J^{PC} = 1^{++}$ in GeV. units with a range of variations of parameter $\omega_0 \in [0.130 - 0.160]$ GeV. for the parameter ratio, $\frac{C_0}{\omega_0}$= 9.3333GeV^2 , and other parameters Λ = 0.250GeV.,A_0 = 0.01, and m_c = 1.490GeV. However for comparison with experiment, we take the value ω_0=0.145GeV. and C_0 = .186889

	BSE-CIA	Expt[7]	Pot.Model[28]	BSE[55]	RQM[29]
$M_{\chi_{c1}}(1^3p_1)$	3.4783	3.510±0.0007	3.440		3.413
$M_{\chi_{c1}}(2^3p_1)$	3.9560	3.871.69±0.0017	3.920	3.928	3.870
$M_{\chi_{c1}}(3^3p_1)$	4.4650			4.228	4.301
$M_{\chi_{c1}}(4^3p_1)$	4.9593				

Table 3.6: Mass spectrum of ground and excited states of χ_{c1} with quantum numbers $J^{PC} = 1^{++}$ in GeV. units with the set of parameters, ω_0 = 0.145GeV,C_0 = .186889,A_0 = 0.01 and m_c = 1.490GeV.

shown that The Dirac structures associated with amplitudes f_i(i=1, ..., 4) are suppressed by various powers of M and among the various Dirac structures those associated with amplitudes f_1 and f_2 have magnitudes $O(\frac{1}{M^0})$ and are leading order, while the Dirac structures associated with f_3 and f_4 have magnitudes $O(\frac{1}{M^1})$ and are sub leading and would contribute lessor to meson observable calculations in comparison to leading Dirac structures associated to f_1 and f_2. Among the two leading Dirac structures associated with f_1 and f_2, the structure associated with f_1 is dominant. In the center of mass frame, where $q_\mu = (\hat{q}, 0)$, we can then write the general decomposition of the instantanious BS wave function for Pseudo vector meson ($J^{PC} = 1^{+-}$), of dimensionality M as[39]

$$\psi(\hat{q}) = \hat{q}\gamma_5[f_1(\hat{q}) + \frac{i\not{P}}{M}f_2(\hat{q}) - \frac{i\not{q}}{M}f_3(\hat{q}) + \frac{2\not{P}\not{q}}{M^2}f_4(\hat{q})]. \tag{3.135}$$

Taking the leading order covariants, one can write

$$\psi(\hat{q}) = \hat{q}\gamma_5\left[f_1(\hat{q}) + i\frac{\not{P}}{M}f_2(\hat{q})\right]. \tag{3.136}$$

Put the wave function eq. (3.136) into the first two equations of eq. (2.123) and by taking various traces for γ-matrices to both sides of the equations and carrying out the integration, we obtain the independent coupled differential equations

$$\begin{aligned}(M - 2\omega)\big[f_1 + \frac{mf_2}{\omega}\big] &= -\Theta_{A^-}\overline{V}_c\big[f_1 + \frac{mf_2}{\omega}\big],\\ (M + 2\omega)\big[f_1 + \frac{mf_2}{\omega}\big] &= \Theta_{A^-}\overline{V}_c\big[f_1 - \frac{mf_2}{\omega}\big],\end{aligned} \tag{3.137}$$

where M is the mass of the meson.

Taking the sum of these two equation and solving for f_1 following by subtraction the second equation from the first and solving for f_2, we get

$$\begin{aligned}f_1(\hat{q}) &= \frac{1}{M}\left[2m - \frac{m\Theta_{A^-}\bar{V}_c}{\omega}\right]f_2(\hat{q}),\\ f_2(\hat{q}) &= \frac{\omega}{mM}\left[2\omega - \Theta_{A^-}\bar{V}_c\right]f_1(\hat{q}),\end{aligned} \tag{3.138}$$

where $\bar{V}_c = \omega_{q\bar{q}}^2\left[\frac{\vec{\nabla}_{\hat{q}}^2}{\sqrt{1+2A0(N+\frac{3}{2})}} + \frac{C_0}{\omega_0^2}\right]$. Substituting f_1 into the second equation of Eq. (3.137) and f_2 into the first Equation of Eq. (3.137), one can obtain

$$\begin{aligned}[\frac{M^2}{4} - \omega^2]f_2(\hat{q}) &= -\omega\Theta_{A^-}\bar{V}_c f_2(\hat{q}),\\ [\frac{M^2}{4} - \omega^2]f_1(\hat{q}) &= -\omega\Theta_{A^-}\bar{V}_c f_1(\hat{q}).\end{aligned} \tag{3.139}$$

In view of this equation, one can see that $f_1(\hat{q})$ and $f_2(\hat{q})$ satisfy the same equation and can be approximately written as $f_1(\hat{q}) \approx f_2(\hat{q}) \approx \phi_{A^-}(\hat{q})$. so that in heavy quark approximation, we write

$$\left[\frac{M^2}{4} - m^2 - \hat{q}^2\right]\phi_p(\hat{q}) = -4m\omega_{q\bar{q}}^2\left[\frac{\vec{\nabla}_{\hat{q}}^2}{\sqrt{1+2A_0(N+\frac{3}{2})}} + \frac{C_0}{\omega_0^2}\right]\phi_p(\hat{q}). \tag{3.140}$$

This can inturn be expressed as

$$E_p\phi_p = \left[-\beta_p^4\vec{\nabla}_{\hat{q}}^2 + \hat{q}^2\right]\phi_{A^-}, \tag{3.141}$$

where $\beta_{A^-} = \left(\frac{4m\omega_{q\bar{q}}^2}{\sqrt{1+2A_0(N+\frac{3}{2})}}\right)^{1/4}$ and total energy of the system expressed as

$$E_{A^-} = \frac{M^2}{4} - m^2 + \frac{\beta_{A^-}^4 C_0}{\omega_0^2}\sqrt{1+2A_0(N+\frac{3}{2})}. \tag{3.142}$$

Putting the expression for the laplacian operator in Eq. (3.141) in spherical coordinates, we get

$$\frac{d^2\phi(\hat{q})}{d\hat{q}^2} + \frac{2}{\hat{q}}\frac{d\phi(\hat{q})}{d\hat{q}} - \frac{l(l+1)\phi(\hat{q})}{\hat{q}^2} + \left(\frac{E}{\beta_p^4} - \frac{\hat{q}^2}{\beta_p^4}\right)\phi(\hat{q}) = 0. \tag{3.143}$$

This is 3D harmonic oscillator equation, whose solutions can be found by using power series method. Assuming the form of the solutions of this equation as $\phi(\hat{q}) = h(\hat{q})e^{\frac{-\hat{q}^2}{2\beta_A^2}}$, the above equation can be expressed as

$$h''(\hat{q}) + \left(\frac{2}{\hat{q}} - \frac{2\hat{q}}{\beta_{A^-}^2}\right)h'(\hat{q}) + \left(\frac{E}{\beta_{A^-}^4} - \frac{3}{\beta_{A^-}^2} - \frac{l(l+1)}{\hat{q}^2}\right)h(\hat{q}) = 0. \tag{3.144}$$

The eigen values of this equation can be obtained using the power series method as

$$E_N = 2\beta_{A^-}^2(N+\frac{3}{2}); N = 2n' + l, \tag{3.145}$$

with $l = 1$. Thus, to each value of $n = 0, 1, 2, 3, ...$ will correspond apolynomial $h(\hat{q})$ of order $2n+1$ in $\hat{q}$ that are obtained as solutions of Eq. (3.144). The odd parity normalized wave function $\phi(\hat{q})$ thus derived are

$$\phi(1p,\hat{q}) = (\frac{2}{3})^{\frac{1}{2}}\frac{1}{(\beta_{A-}^{\frac{5}{2}})(\pi^{\frac{3}{4}})}\hat{q}e^{-\frac{\hat{q}^2}{2\beta_{A-}^2}},$$
$$\phi(2p,\hat{q}) = (\frac{5}{3})^{\frac{1}{2}}\frac{1}{(\beta_{A-}^{\frac{5}{2}})(\pi^{\frac{3}{4}})}\hat{q}\left[1-\frac{2\hat{q}^2}{5\beta_{A-}^2}\right]e^{-\frac{\hat{q}^2}{2\beta_{A-}^2}},$$
$$\phi(3p,\hat{q}) = (\frac{35}{12})^{\frac{1}{2}}\frac{1}{(\beta_{A-}^{\frac{5}{2}})(\pi^{\frac{3}{4}})}\hat{q}\left[1-\frac{4\hat{q}^2}{5\beta_{A-}^2}+\frac{4\hat{q}^4}{35\beta_{A-}^4}\right]e^{-\frac{\hat{q}^2}{2\beta_{A-}^2}},$$
$$\phi(4p,\hat{q}) = (\frac{1890}{46359})^{\frac{1}{2}}\frac{1}{(\beta_{A-}^{\frac{5}{2}})(\pi^{\frac{3}{4}})}\hat{q}\left[1-\frac{6\hat{q}^2}{5\beta_{A-}^2}+\frac{12\hat{q}^4}{35\beta_{A-}^4}-\frac{24\hat{q}^6}{945\beta_{A-}^6}\right]e^{-\frac{\hat{q}^2}{2\beta_{A-}^2}} \quad (3.146)$$

The $b\bar{b}$ wave functions are drawn for acadamic interest.

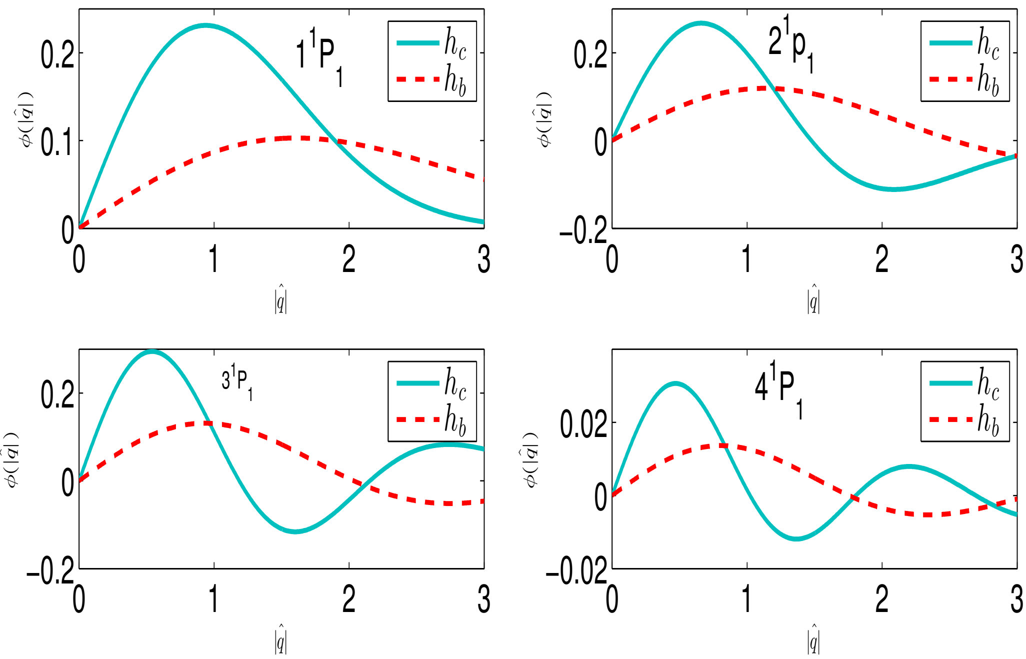

Figure 3.3: The graph of unperturbed wave functions of the ground, first and second excited states with quantum number $J^{pc} = 1^{+-}$ for $c\bar{c}$(solid line) and $b\bar{b}$(dashed line) of pseudo vector meson .

The mass spectrum of ground and excited states for equal mass heavy pseudovector 1^{+-} meson is written as

$$\frac{1}{2\beta_{A-}^2}\left[\frac{M^2}{4}-m^2+\frac{C_0\beta_p^4}{\omega_0^2}\sqrt{1+2A_0(N+\frac{3}{2})}\right] = N+\frac{3}{2} \quad (3.147)$$

with $N = 2n + l; n = 0, 1, 2, 3, ...,$.

Now treating the mass spectral equation Eq. (3.141) as the unperturbed equation, with the unperturbed wave function $\phi_p(nP, \hat{q})$ for pseudovector meson in Eq. (3.146), we now incorporate the OGE(coulomb) term in this equation. Then the above mass spectral equation can be written as

$$E_{A^-}\phi_{A^-}(\hat{q}) = \big[-\beta_{A^-}^4\vec{\nabla}_{\hat{q}}^2 + \hat{q}^2 + V_{coul}^{A^-}\big]\phi_{A^-}(\hat{q}). \tag{3.148}$$

Treating the Coulomb term as a perturbation to the unperturbed mass spectral equation, we can write the complete mass spectra for the ground and excited states for equal mass heavy pseudovector meson using first order perturbation theory as

$$\frac{1}{2\beta_{A^-}^2}\left[\frac{M^2}{4} - m^2 + \frac{C_0\beta_{A^-}^4}{\omega_0^2}\sqrt{1 + 2A_0(N + \frac{3}{2})}\right] + \gamma_A^- < V_{coul}^{A^-} >= N + \frac{3}{2}, \tag{3.149}$$

where $< V_{coul}^{A^-} >$ is the expectation value of $V_{Coul}^{A^-}$ between the unperturbed states of given quantum numbers n with $l = 1$ for pseudovector meson and has been weighted by a factor of $\gamma_A^- = C_0^2\beta_{A^-}^2$ to the Coulomb term dimensionally consistent with the harmonic term. Its expectation values for the 1P, 2P, 3P and 4P states are

$$\begin{aligned}
< 1P \mid V_{coul}^{A^-} \mid 1P >&= -\frac{128\pi\alpha_s}{9\beta_{A^-}^2},\\
< 2P \mid V_{coul}^{A^-} \mid 2P >&= -\frac{64\pi\alpha_s}{18\beta_{A^-}^2},\\
< 3P \mid V_{coul}^{A^-} \mid 3P >&= -\frac{3712\pi\alpha_s}{213\beta_{A^-}^2},\\
< 4P \mid V_{coul}^{A^-} \mid 4P >&= -\frac{1152\pi\alpha_s}{25\beta_{A^-}^2}.
\end{aligned} \tag{3.150}$$

3.7 Numerical calculation of mass spectral for axial vector meson(1^{+-})

Now the mass spectral results can be calculated from the above mass spectral equation and the result for 1P, 2P, 3P, 4P states of h_c are given in the table 3.5 below with the same parameter set above for 0^{++}.

	M$(\omega_0 = 0.130)$	M$(\omega_0 = 0.1450)$	M$(\omega_0 = 0.160)$	Exp[7]
$M_{h_c}(1P_1)$	3.19687	3.40382	3.5983	3.526±0.000
$M_{h_c}(2P_1)$	3.77493	4.10936	4.43295	
$M_{h_c}(3P_1)$	4.37008	4.82397	5.25924	
$M_{h_c}(4P_1)$	4.94834	5.51168	6.04837	

Table 3.7: Mass spectrum in BSE-CIA of ground and excited states of χ_{c0} with quantum numbers $J^{PC} = 1^{+-}$ in GeV. units with a range of variations of parameter $\omega_0 \in [0.120 - 0.180]$ GeV. for the parameter ratio, $\frac{C_0}{\omega_0}$= 9.3333GeV^2 , and other parameters Λ = 0.250GeV.,A_0 = 0.01, and m_c = 1.490GeV. However for comparison with experiment, we take the value ω_0=0.150GeV. and C_0 = 0.186889

	BSE-CIA	Expt[7]	Pot.Model[28]	BSE[55]	Quark Model[29]
$M_{h_c}(1^1p_1)$	3.469	3.526±0.000	3.520		3.539
$M_{h_c}(2^1p_1)$	4.03894		3.96	3.943	3.996
$M_{h_c}(3^1p_1)$	4.46445			4.242	
$M_{\chi c}(4^1p_1)$	5.08769				

Table 3.8: mass spectrum of ground and excited states of h_c systems with quantum numbers $J^{PC} = 1^{+-}$ in units of GeV.units with the above set of parameters parameters. Expt means experimental value from PDG.

4

Spectroscopy of pseudoscalar and vector meson

4.1 Mass Spectral Equation for Pseudoscalar Quarkonia

We start with the general decomposition of instantaneous BS wave function for pseudoscalar mesons, ($J^{pc} = 0^{-+}$) of dimensionality M, in the centre of mass frame where q_μ = ($\hat{q}$, 0) is given[69].

$$\psi(P,\hat{q}) = \gamma_5[M\phi_1(\hat{q}) + \not{P}\phi_2(\hat{q}) + \not{\hat{q}}\phi_3(\hat{q}) + \frac{\not{P}\not{\hat{q}}}{M}\phi_4(\hat{q})]. \tag{4.1}$$

Using the constraint equations that arise from the last two Salpeter equations, one can put $\phi_3 = 0; \phi_4 = \frac{-M\phi_2}{m}$. Then the wave function can be put in the form

$$\psi_p(\hat{q}) = \gamma_5[M\phi_1(\hat{q}) + \not{P}\phi_2(\hat{q}) + \frac{\not{P}\not{\hat{q}}\phi_2(\hat{q})}{m}] \tag{4.2}$$

which is determined by only two independent functions ϕ_1, and ϕ_2. Inserting the above wave function into the first two Salpeter equations in Eq.(2.123), and evaluating trace over the γ-matrices on both sides, we obtain two BS coupled integral equations, which with the use of only the confining part of the interaction kernel, Vc are:

$$\begin{aligned}(M-2\omega)\Big[\phi_1(\hat{q}) + \frac{\omega}{m}(1-\frac{\hat{q}^2}{m^2})\phi_2(\hat{q})\Big] &= \theta_p\int\frac{d^3\hat{q}'}{(2\pi)^3}V_c(\hat{q},\hat{q}')\big[\phi_1(\hat{q}') + \frac{\omega}{m}(1+\frac{\hat{q}.\hat{q}'}{m^2})\phi_2(\hat{q}')\big]\\ (M+2\omega)\Big[\phi_1(\hat{q}) - \frac{\omega}{m}(1-\frac{\hat{q}^2}{m^2})\phi_2(\hat{q})\Big] &= -\theta_p\int\frac{d^3\hat{q}'}{(2\pi)^3}V_c(\hat{q},\hat{q}')\big[\phi_1(\hat{q}') - \frac{\omega}{m}(1+\frac{\hat{q}.\hat{q}'}{m^2})\phi_2(\hat{q}')\big]\end{aligned} \tag{4.3}$$

Decoupling them under heavy quark approximation leads to equation

$$\left[\frac{M^2}{4} - m^2 - \hat{q}^2\right]\phi_p(\hat{q}) = \theta_c m\bar{V}_c\phi_p(\hat{q}), \tag{4.4}$$

this is due to the fact that the two equations have identical nature in $\phi_1(\hat{q})$, and $\phi_2(\hat{q})$, for pseudoscalar mesons, due to which their solutions were written as: $\phi_1(\hat{q})$ = $\phi_2(\hat{q})$ $\approx$ $\phi(\hat{q})$. These represent the eigenfunction of the pseudoscalar meson obtained by solving the full Salpeter equations, and we can write the wave function for J^{pc} = 0^{-+} state as:

$$\psi_p(\hat{q}) \approx \gamma_5\big[M + \not{P} + \frac{\not{P}\not{\hat{q}}}{m}\big]\phi_p(\hat{q}). \tag{4.5}$$

The wave functions ϕ_p satisfies the 3D BSE (without the coulomb term) for equal mass heavy pseudoscalar mesons and is given by

$$\left[\frac{M^2}{4} - m^2 - \hat{q}^2\right]\phi_p(\hat{q}) = \theta_c m\omega_{q\bar{q}}^2\left[\frac{\vec{\nabla}_{\hat{q}}^2}{\sqrt{1+2A_0(N+\frac{3}{2})}} + \frac{c_0}{\omega_0^2}\right]\phi_p(\hat{q}), \tag{4.6}$$

where θ_p is obtained by evaluating $\gamma_\mu\psi_p(\hat{q})\gamma_\mu$ which can be approximated by use of leading order Dirac structures in $\psi_p(\hat{q})$ after ignoring $\frac{\hat{q}^2}{m^2}$ terms, that are negligible under heavy-quark approximation on RHS of the corresponding equations as $\theta_p\psi_p(\hat{q})$, where,θ_p=-4 involves the spin-spin interactions alone. The above equation can be put in the form,

$$\left[\frac{M^2}{4} - m^2 - \hat{q}^2\right]\phi_p(\hat{q}) = -\beta_p^4\left[\kappa\vec{\nabla}_{\hat{q}}^2 + \frac{c_0}{\omega_0^2}\right]\phi_p(\hat{q}) \tag{4.7}$$

which can be inturn expressed as

$$E_p\phi_p(\hat{q}) = \left[-\beta_p^4\vec{\nabla}_{\hat{q}}^2 + \hat{q}^2\right]\phi_p(\hat{q}). \tag{4.8}$$

Putting the expression for Laplacian operator in sphereical coordinates, one can write

$$\frac{d^2\phi(\hat{q})}{d\hat{q}^2} + \frac{2}{\hat{q}}\frac{d\phi(\hat{q})}{d\hat{q}} - \frac{l(l+1)\phi(\hat{q})}{\hat{q}^2} + \left(\frac{E}{\beta^4} - \frac{\hat{q}^2}{\beta^4}\right)\phi(\hat{q}) = 0, \tag{4.9}$$

Let the solution of this equation be

$$\phi(\hat{q}) = h(\hat{q})e^{-\frac{\hat{q}^2}{2\beta^2}}. \tag{4.10}$$

Now by taking the first and second derivative and plugging these two equation into the above homogeneous differential equation, one can obtain

$$h''(\hat{q}) + (\frac{2}{\hat{q}} - \frac{2\hat{q}}{\beta^2})h'(\hat{q}) + \left(\frac{E}{\beta^4} - \frac{3}{\beta^2} - \frac{L(L+1)}{\hat{q}^2}\right)h(\hat{q}) = 0. \tag{4.11}$$

The solution of this equation can be put in the form

$$h(\hat{q}) = \hat{q}^k \sum_{n=0}^{\infty} a_n \hat{q}^n = \sum_{n=0}^{\infty} a_n \hat{q}^{k+n}. \tag{4.12}$$

Now taking the derivative of the series with respect to $\hat{q}$ and putting them into the above differential equation, one can derive the recursion relation for $l = 0$ to be

$$a_{2n+2} = \frac{\left[\frac{2(2n+l)+3}{\beta^2} - \frac{E}{\beta^4}\right]a_{2n}}{(2n+2)(2n+3)}, \tag{4.13}$$

where the principal quantum number n is given by $n = 0, 1, 2, 3, 4, ...$ Now the power series solution must terminate. For this to happen there must occur some "highest" n (call it n'), such that the recursion formula spits out $a_{2n'+2} = 0$. This will truncate the series $h(\hat{q})$ for physicaly acceptable solutions, then the recursion relation requires that

$$\left[\frac{2(2n')+3}{\beta^2} - \frac{E}{\beta^4}\right] = 0. \tag{4.14}$$

From this one can obtained the energy eigenvalue equation

$$E_N = 2\beta^2\left[N + \frac{3}{2}\right], \tag{4.15}$$

where $N = 2n'$ and N is known as the principal quantum number and n as the radial quantum number. $n = 0, 1, 2, 3....$ The allowed values of N are integers, $N = 0, 1, 2, 3, ...$.

To each value of $n = 0, 1, 2, 3, ...$ of N will then correspond to a polynomial of order $2n$ in $\hat{q}$ and an even physically acceptable wave function $\phi(\hat{q})$ given by the above equation for the orbitally ground states. For the allowed values of E, the recursion formula leads to

$$a_{2n+2} = \frac{\Big[- \frac{4}{\beta^2}(n' - n) \Big] a_{2n}}{(2n+2)(2n+3)}. \tag{4.16}$$

Now the power series solution can be written in the form

$$h_N(\hat{q}) = \sum_{n=0}^{n'} a_{2n}\hat{q}^{2n}. \tag{4.17}$$

In view of this the un normalized wave function(eigenfunction) can be put in the form

$$\phi_N(\hat{q}) = h_N(\hat{q}) e^{-\frac{\hat{q}^2}{2\beta^2}}. \tag{4.18}$$

In view of this, one can derive the normalized eigen function for pseudosclar meson as in the scalar meson above in an approximate harmonic oscillator basis to be

$$\begin{aligned}
\phi_p(1S, \hat{q}) &= \frac{1}{\pi^{3/4}\beta_p^{3/2}} e^{-\frac{\hat{q}^2}{2\beta_p^2}} \\
\phi_p(2S, \hat{q}) &= (\frac{3}{2})^{1/2} \frac{1}{\pi^{3/4}\beta_p^{3/2}} (1 - \frac{2\hat{q}^2}{3\beta_p^2}) e^{-\frac{\hat{q}^2}{2\beta_p^2}} \\
\phi_p(3S, \hat{q}) &= (\frac{15}{8})^1/2 \frac{1}{\pi^{3/4}\beta_p^{3/2}} (1 - \frac{20\hat{q}^2}{15\beta_p^2} + \frac{4\hat{q}^4}{15\beta_p^4}) e^{-\frac{\hat{q}^2}{2\beta_p^2}} \\
\phi_p(4S, \hat{q}) &= (\frac{35}{16})^1/2 \frac{1}{\pi^{3/4}\beta_p^{3/2}} (1 - \frac{210\hat{q}^2}{105\beta_p^2} + \frac{84\hat{q}^4}{105\beta_p^4} - \frac{8\hat{q}^6}{105\beta_p^6}) e^{-\frac{\hat{q}^2}{2\beta_p^2}},
\end{aligned} \tag{4.19}$$

where the inverse range parameter β_p is given by $(4\frac{m\omega_{q\bar{q}}^2}{\sqrt{1+2A_0(N+\frac{3}{2})}})^{\frac{1}{4}}$ with the total energy of the system expressed as

$$E_p = \frac{M^2}{4} - m^2 + \frac{\beta_p^4 c_0}{\omega_0^2}\sqrt{1 + 2A_0(N + \frac{3}{2})}. \tag{4.20}$$

This is the total energy without the coulomb potential. Now with the incorporation of the OGE (Coulomb) term, the mass spectral equation can be written as:

$$E_p\phi_p(\hat{q}) = \Big[- \beta_p^4 \vec{\nabla}_{\hat{q}}^2 + \hat{q}^2 + V_{coul}^P \Big] \phi_p(\hat{q}). \tag{4.21}$$

Now, treating the coulomb term as a perturbation to the unperturbed mass spectral equation, Eq.(4.20), and regarding the wave functions $\phi_p(\hat{q})$ in Eq.(4.19), as the unperturbed wave functions, we can write the complete mass spectra of ground (1S) and excited states for equal mass heavy pseudoscalar (0^{-+}) mesons using first order degenerate perturbation theory as:

$$\frac{1}{2\beta_p^2}\left[\frac{M^2}{4}-m^2+\frac{\beta_P^4 c_0}{\omega_0^2}\sqrt{1+2A_0(N+\frac{3}{2})}\right]+\gamma_p < V_{coul}^p >= N+\frac{3}{2}, N=2n+l, n=0,1,2,... \tag{4.22}$$

where $< V_{coul}^p >$ has been weighted by a factor of γ_p to keep the equation the dimensionalty consistent and used as a parameter. to have the coulomb term dimensionally consistent with the harmonic term is the matrix element of V_{coul}^p between unperturbed states in Eq.(4.19) of given quantum numbers n. It is to be noted that, V_{coul}^p connects only the equal parity states with the same values of quantum numbers n. The only non-vanishing matrix elements of the perturbation between states with the given quantum numbers n are listed below

$$< nS \mid V_{coul}^p \mid nS >= \frac{1}{\beta_p^2}\frac{\pi\alpha_s}{12}. \tag{4.23}$$

The plots for unpurturbed seudoscalar quarkonia is given in Figure 4.1 below.

4.2 Numerical values of masses of pseudoscalar $c\bar{c}$ states

We are now in a position to calculate the numerical values of mass spectrum for seudoscalar quarkonia. The non-zero values of $< Vcoul >$ given above leads to bringing the masses of different states of pseudoscalar quarkonia closer to data, as can be seen from the mass spectral results for pseodoscalar (0^{-+}), which are compared with the experimental data and other models for each state (where ever available), as given in Table 4.1 below with the above parameter set as:

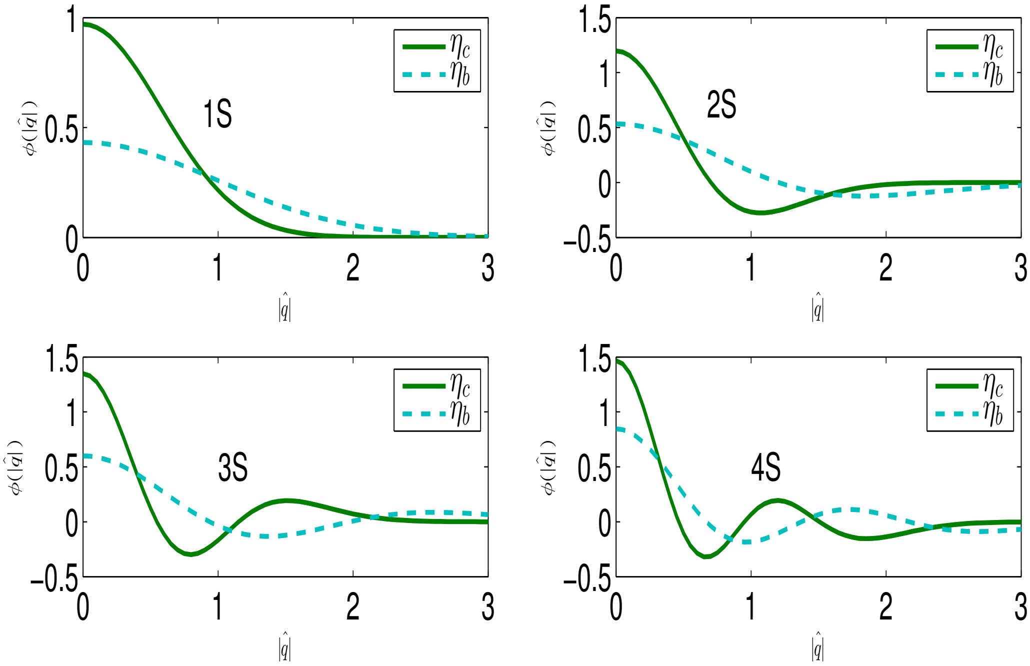

Figure 4.1: The graph of the unperturbed wave functions of the ground, first, second and the third excited states with quantum number $J^{pc} = 0^{-+}$ for $c\bar{c}$ meson(solide line). We also draw the graphs of the corresponding unperturbed wave function for $b\bar{b}$ states(with dashed lines), which are mainly for acadamic interest

	M($\omega_0 = 0.120$)	M($\omega_0 = 0.150$)	M($\omega_0 = 0.180$)	Exp[7]
$M_{\eta_c}(1S)$	3.0625	2.95062	2.9920	2.983$\pm$0.0007
$M_{\eta_c}(2S)$	3.7018	3.7351	3.8231	3.639$\pm$0.0013
$M_{\eta_c}(3S)$	4.2894	4.44842	4.5759	
$M_{\eta_c}(4S)$	4.8344	5.1046	5.9069	
$M_{\eta_c}(5S)$	5.3439	5.7142	5.9069	

Table 4.1: Mass spectrum in BSE-CIA of ground and excited states of η_c with quantum numbers $J^{PC} = 0^{-+}$ in GeV. units with a range of variations of parameter $\omega_0 \in [0.1300.160]$ GeV. for the parameter ratio, $\frac{C_0}{\omega_0}^2$= 8.8888GeV^2 , and other parameters Λ = 0.250GeV.,A_0 = 0.01, and m_c = 1.490GeV. However for comparison with experiment, we take the value ω_0=0.145GeV.

	BSE-CIA	Expt.[7]	Pot. Model[56]	QCD sum rule[57]	Lattice QCD[58]	[29]
$M_{\eta c}(1S)$	2.9759	2.983$\pm$ 0.0007	2.980	3.11$\pm$ 0.52	3.292	2.981
$M_{\eta c}(2S)$	3.7264	3.639$\pm$ 0.0013	3.600		4.240	3.635
$M_{\eta c}(3S)$	4.4812		4.060			3.989
$M_{\eta c}(4S)$	5.1368		4.4554			4.401
$M_{\eta c}(5S)$	5.7442					

Table 4.2: Masses of ground and radially excited states of η_c (in GeV.) in present calculation (BSE-CIA) along with experimental data, and their masses in other models.

4.3 Mass spectra of Vector quarkonia

The general decomposition of the wave function for vector mesons ($J^{pc} = 1^{--}$) is given by[69]

$$\psi(\hat{q}) = \not{\epsilon}\chi_1 + \not{\epsilon}\not{P}\chi_2 + [\hat{q}.\epsilon - \not{\epsilon}\not{\hat{q}}]\chi_3 + [2\hat{q}.\epsilon\not{P} + \not{\epsilon}(\not{P}\not{\hat{q}} - \not{\hat{q}}\not{P})]\chi_4$$
$$\hat{q}.\epsilon\chi_5 + (\hat{q}.\epsilon)\not{P}\chi_6 + (\hat{q}.\epsilon)\not{\hat{q}}\chi_7 + (\hat{q}.\epsilon)[\not{P}\not{\hat{q}} - \not{\hat{q}}\not{P}]\chi_8. \quad (4.24)$$

We again mention that, by use of a naive power counting rule, one can show that the Dirac structures associated with amplitudes χ_1 and χ_2 are of leading order. Those with amplitudes χ_3, . . ., χ_6 are sub-leading. It is also shown that, the structures associated with χ_7 and χ_8 are much more suppressed than even the sub-leading Dirac structures, and would contribute very little to meson observable (specially heavy mesons) calculations in comparison to Dirac structures associated with χ_3, . . ., χ_6. Thus we ignore the Dirac structures associated with χ_7 and χ_8. It is also shown that among the two leading Dirac structures associated with amplitudes χ_1, and χ_2, the structure associated with χ_1 i.e. $\gamma.\epsilon$ is most dominant. It is also shown that χ_1 is the dominant amplitude for not only ground state vector mesons, but also for their higher excitations.

We thus write the instantaneous BS wave function $\psi_v(\hat{q})$ with dimensionality M for vector quarkonia, up to sub-leading order (i.e. O($\frac{1}{M^1}$) as per the power counting scheme as in case of pseudoscalar mesons, in the center of mass frame, as:

$$\psi_v(\hat{q}) = M\not{\epsilon}\chi_1 + \not{\epsilon}\not{P}\chi_2 + [\not{\epsilon}\not{\hat{q}} - \hat{q}.\epsilon]\chi_3 + [\not{P}\not{\epsilon}\not{\hat{q}} - (\hat{q}.\epsilon)\not{P}]\chi_4$$
$$+(\hat{q}.\epsilon)\chi_5 + \frac{(\hat{q}.\epsilon)\not{P}}{M}\chi 6. \quad (4.25)$$

We now derive the coupled mass spectral equations from Salpeter equation. Putting Eq.(4.25) into the last two equations of Eq.(2.123), and evaluating the trace over the

Υ matrices on both sides of these equations, we obtain the independent constraints on the components for the Instantaneous BS wave function as in[40]

$$\chi_5 = \frac{M\chi_1}{m}; \chi_4 = \frac{-M\chi_2}{m}; \chi_6 = \chi_3 = 0. \tag{4.26}$$

So we can apply the obtained constraints to Eq.(4.26) and rewrite the relativistic wave function for state $J^{pc} = 1^{--}$ as

$$\psi_v(\hat{q}) = [M\not{\epsilon} + \hat{q}.\epsilon\frac{M}{m}]\chi_1(\hat{q}) + [\not{\epsilon}\not{P} + \frac{2\not{P}\hat{q}.\epsilon}{m} - \frac{\not{P}\not{\epsilon}\not{\hat{q}}}{m}]\chi_2(\hat{q}), \tag{4.27}$$

where, we have been able to express the instantaneous wave function $\psi_v(\hat{q})$ in terms of only the leading Dirac structures associated with amplitudes χ_1 and χ_2. Putting the wave function into the first two Salpeter equations and by evaluating trace over the γ-matrices on both sides, we can obtain two independent BS coupled integral equations, where using the same procedure as mentioned above in the pseudoscalar case, the potential on the right side is Vc:

$$\begin{aligned}(M-2\omega)\Big[\chi_1(\hat{q}) - \frac{\omega}{m}\chi_2(\hat{q})\Big] &= \theta_v \int \frac{d^3\hat{q}'}{(2\pi)^3} V_c(\hat{q},\hat{q}')\Big[\chi_1(\hat{q}') - \frac{\omega}{m}\chi_2(\hat{q}')\Big],\\ (M+2\omega)\Big[\chi_1(\hat{q}) + \frac{\omega}{m}\chi_2(\hat{q})\Big] &= \theta_v \int \frac{d^3\hat{q}'}{(2\pi)^3} V_c(\hat{q},\hat{q}')\Big[-\chi_1(\hat{q}') - \frac{\omega}{m}\chi_2(\hat{q}')\Big]\end{aligned} \tag{4.28}$$

To decouple these equations, we proceed as in the above case of pseudoscalar mesons. We first add these above equations. Then subtract the second equation from the first. And for a kernel expressed as $V(\hat{q},\hat{q}') = \overline{V}\delta^3(\hat{q}-\hat{q}')$, we get two algebraic coupled equations in χ_1 and χ_2. We Eliminate χ_1 from the first equation in terms of χ_2, and plug it in the second equation to get an equation entirely in terms of χ_2. Similarly, we eliminate χ_2 from the second equation in terms of χ_1, and plugging in the first equation to get an equation entirely in terms of χ_1. We thus get two identical decoupled equations, one in $\chi_1(\hat{q})$, and the other in $\chi_2(\hat{q})$, and in the approximation, $\omega \approx m$, (valid for heavy quarkonia), these equations can be expressed as:

$$[\frac{M^2}{4} - \hat{q}^2 - m^2]\chi_1(\hat{q}) = m\Theta_v\overline{V}(\hat{q}) + \frac{\Theta_v^2}{4}\overline{V}^2\chi_1(\hat{q})$$
$$[\frac{M^2}{4} - \hat{q}^2 - m^2]\chi_2(\hat{q}) = m\Theta_v\overline{V}(\hat{q}) + \frac{\Theta_v^2}{4}\overline{V}^2\chi_2(\hat{q}). \quad (4.29)$$

In the above approximation, we can express the solutions as, $\chi_1(\hat{q}) = \chi_2(\hat{q}) \approx \phi_v(\hat{q})$, and we can write the wave function for ($J^{PC} = 1^{--}$) S-wave bound state as:

$$\left[\frac{M^2}{4} - m^2 - \hat{q}^2\right]\phi_v(\hat{q}) = \theta_c m\bar{V}_c(\hat{q})\phi_v(\hat{q}). \quad (4.30)$$

which arises due to the identical nature of equations in χ_1, and χ_2 for vector mesons, due to which their solutions are written as, $\chi_1(\hat{q}) = \chi_2(\hat{q}) = \phi(\hat{q})$, and we can write the wave function for $J^{pc} = 1^{--}$

$$\psi_v(\hat{q}) = [M\not{\epsilon} + \hat{q}.\epsilon\frac{M}{m} + \not{\epsilon}\not{P} + \frac{2\not{P}\hat{q}.\epsilon}{m} - \frac{\not{P}\not{\epsilon}\not{\hat{q}}}{m}]\phi_(\hat{q}). \quad (4.31)$$

The wave functions $\phi_v(\hat{q})$ satisfies the 3D BSE (without the coulomb term) for equal mass heavy vector mesons as shown below

$$\left[\frac{M^2}{4} - m^2 - \hat{q}^2\right]\phi_v(\hat{q}) = \theta_v m\omega_{q\bar{q}}^2\left[\frac{\vec{\nabla}_{\hat{q}}^2}{\sqrt{1+2A_0(N+\frac{3}{2})}} + \frac{c_o}{\omega_o^2}\right]\phi_v(\hat{q}), \quad (4.32)$$

where θ_v is obtained by evaluating $\gamma_\mu\psi_v\gamma_\mu$ which can be approximated by use of leading order Dirac structures in $\psi_v(\hat{q})$ after ignoring $\frac{\hat{q}^2}{m^2}$ terms, that are negligible under heavy-quark approximation on RHS of the corresponding equations as $\theta_v\psi_v(\hat{q})$, where, $\theta_v = -2$ involves the spin-spin interactions alone. The above equation can be put in the form,

$$\left[\frac{M^2}{4} - m^2 - \hat{q}^2\right]\phi_v(\hat{q}) = -\beta_v^4\left[\kappa\vec{\nabla}_{\hat{q}}^2 + \frac{c_0}{\omega_0^2}\right]\phi_v(\hat{q}). \quad (4.33)$$

This can be rewritten in the form

$$E_v\phi_v(\hat{q}) = [-\beta_v^4\vec{\nabla}_{\hat{q}}^2 + \hat{q}^2]\phi_v(\hat{q}). \quad (4.34)$$

Using power series, the non-perturbative energy eigen functions for $l = 0$(S), and for $l = 2$(D) states obtained as solutions of above spectral equation in approximate harmonic oscillator basis for vector quarkonia (for states 1S, ..., 3D) are given by,

$$\begin{aligned}
\phi_{(v)}(1S,\hat{q}) &= \frac{1}{\pi^{3/4}\beta_v^{3/2}}e^{-\frac{\hat{q}^2}{2\beta_v^2}},\\
\phi_{(v)}(2S,\hat{q}) &= (\frac{3}{2})^{1/2}\frac{1}{\pi^{3/4}\beta_v^{3/2}}(1-\frac{2\hat{q}^2}{3\beta_v^2})e^{-\frac{\hat{q}^2}{2\beta_v^2}},\\
\phi_v(1D,\hat{q}) &= (\frac{4}{15})^1/2\frac{1}{\pi^{3/4}\beta_v^{7/2}}\hat{q}^2e^{-\frac{\hat{q}^2}{2\beta_v^2}},\\
\phi_{(v)}(3S,\hat{q}) &= (\frac{15}{8})^1/2\frac{1}{\pi^{3/4}\beta_v^{3/2}}(1-\frac{20\hat{q}^2}{15\beta_v^2}+\frac{4\hat{q}^4}{15\beta_v^4})e^{-\frac{\hat{q}^2}{2\beta_v^2}},\\
\phi_v(2D,\hat{q}) &= (\frac{14}{15})^1/2\frac{1}{\pi^{3/4}\beta_v^{7/2}}(1-\frac{2\hat{q}^2}{7\beta_v^2})\hat{q}^2e^{-\frac{\hat{q}^2}{2\beta_v^2}},\\
\phi_{(v)}(4S,\hat{q}) &= (\frac{35}{16})^1/2\frac{1}{\pi^{3/4}\beta_v^{3/2}}(1-\frac{210\hat{q}^2}{105\beta_v^2}+\frac{84\hat{q}^4}{105\beta_v^4}-\frac{8\hat{q}^6}{105\beta_v^6})e^{-\frac{\hat{q}^2}{2\beta_v^2}},\\
\phi_v(3D,\hat{q}) &= (\frac{21}{10})^1/2\frac{1}{\pi^{3/4}\beta_v^{7/2}}(1-\frac{36\hat{q}^2}{63\beta_v^2}+\frac{4\hat{q}^4}{63\beta_v^4})\hat{q}^2e^{-\frac{\hat{q}^2}{2\beta_v^2}},
\end{aligned} \tag{4.35}$$

where the inverse range parametrs is given by $\beta_v = (2\frac{m\omega_{q\bar{q}}^2}{\sqrt{1+2A_o(N+\frac{3}{2})}})^{\frac{1}{4}}$, with the total energy of the system

$$E_v = \frac{M^2}{4} - m^2 + \frac{\beta_v^4 c_0}{\omega_0^2}\sqrt{1+2A_0(N+\frac{3}{2}}). \tag{4.36}$$

This is the total energy with out the coulomb potential. The mass spectra of vector charmonium and bottomonium states using the above spectral equation was found to have degenerate S and D states.

Now with the incorporation of the OGE (Coulomb) term, the mass spectral equation can be written as:

$$E_v\phi_v(\hat{q}) = \Big[-\beta_v^4\vec{\nabla}_{\hat{q}}^2 + \hat{q}^2 + V_{coul}^v\Big]\phi_v(\hat{q}). \tag{4.37}$$

Now, treating the coulomb term as a perturbation to the unperturbed mass spectral equation, Eq.(4.34), and regarding the wave functions $\phi_v(\hat{q})$ in Eq.(4.35), as the unperturbed wave functions, we can write the complete mass spectra of ground

(1S) and excited states for equal mass heavy vector (1^{--}) mesons using first order degenerate perturbation theory as:

$$\frac{1}{2\beta_v^2}\left[\frac{M^2}{4}-m^2+\frac{\beta_v^4 c_0}{\omega_0^2}\sqrt{1+2A_0(N+\frac{3}{2})}\right]+\gamma_v < V_{coul}^v >= N+\frac{3}{2}, N = 2n+l, n = 0, 1, 2, ... \tag{4.38}$$

where $< V_{coul}^v >$ (has been weighted by a factor of γ. To have the coulomb term dimensionally consistent with the harmonic term is the matrix element of V_{coul}^v between unperturbed states in Eq.(4.35) of given quantum numbers n, and l. It is to be noted that, V_{coul}^v connects only the equal parity states with the same values of quantum numbers n. The only non-vanishing matrix elements of the perturbation between states with the given quantum numbers n and l are listed below

$$\begin{aligned} < nS \mid V_{coul}^v \mid nS >&= \frac{1}{\beta_v^2}\frac{\pi\alpha_s}{24}, \\ < nD \mid V_{coul}^v \mid nD >&= \frac{1}{5\beta^2 v}\frac{\pi\alpha_s}{24}. \end{aligned} \tag{4.39}$$

The plots of unperturbed wave functions for vector quarkonia is given below

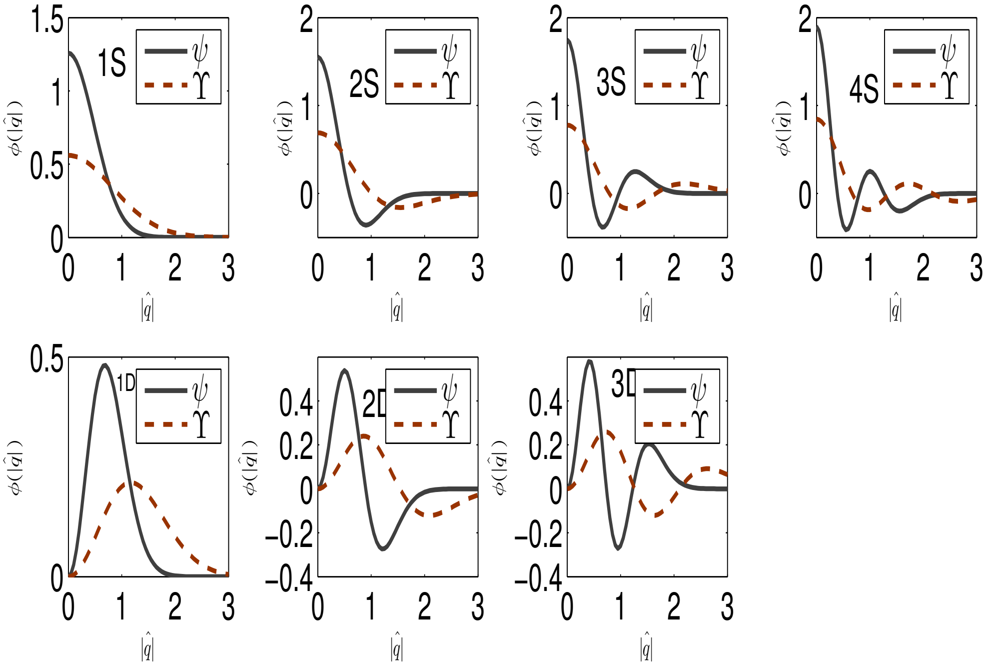

Figure 4.2: The graphs of the unperturbed wave functions of the ground, first, second and the third excited states with quantum number $J^{pc} = 1^{--}$ for $c\bar{c}$ vector meson(solid line) . We also draw the graphs of the corresponding unperturbed wave function for $b\bar{b}$ states (with dashed line), which are shown for acadamic interest

4.4 Numerical values of Vector quarkonia

We next calculate the numerical values for vector quarkonia. For this, the non-zero values of $< Vcoul >$ given above not only lead to the lifting up of the degeneracy between the S and D levels with the same principal quantum number N in vector quarkonia, but also leads to bringing the masses of different states of vector quarkonia closer to data, as can be seen from the mass spectral results for vector (1^{--}) quarkonia, which are compared with the experimental data, and other models for each state (where ever available), as given in the Table below with the same parameter set as above.

	M($\omega_0 = 0.120$)	M($\omega_0 = 0.150$)	M($\omega_0 = 0.180$)	Exp[6]
$M_{J/\psi}(1S)$	3.1291	3.0974	3.0396	3.0969±0.000011
$M_{\psi}(2S)$	3.5917	3.6675	3.7146	3.6861±0.00034
$M_{\psi}(1D)$	3.6088	3.6817	3.7235	3.773±0.00033
$M_{\psi}(3S)$	4.0235	4.1945	4.3333	4.03±0.001
$M_{\psi}(2D)$	4.0595	4.2314	4.3684	4.191±0.005
$M_{\psi}(4S)$	4.4289	4.6856	4.9059	4.421±0.004
$M_{\psi}(3D)$	4.4820	4.7427	4.9642	
$M_{\psi}(5S)$	4.8116	5.1463	5.4404	
$M_{\psi}(4D)$	4.8803	5.2216	5.5196	

Table 4.3: Mass spectrum in BSE-CIA of ground and excited states of ψ with quantum numbers $J^{PC} = 1^{--}$ in GeV. units with a range of variations of parameter $\omega_0 \in [0.1300.160]$ GeV. for the parameter ratio, $\frac{C_0}{\omega_0}^2$= 8.8888GeV^2 , and other parameters Λ = 0.250GeV.,A_0 = 0.01, and m_c = 1.490GeV. However for comparison with experiment, we take the value ω_0=0.145GeV.

	BSE-CIA	Expt.[7]	Rel. pot. Model[29]	Pot. Model[52]	BSE[21]	Lattice QCD[53]
$M_J/\psi(1S)$	3.1017	3.0969±0.000011	3.096	3.0969		3.099
$M_\psi(2S)$	3.6854	3.6861±0.00034	3.685	3.6890	3.686	3.653
$M_\psi(1D)$	3.7011	3.773±0.00033	3.783		3.759	
$M_\psi(3S)$	4.2154	4.03±0.001	4.039	4.1407	4.065	4.099
$M_\psi(2D)$	4.2518	4.191±0.005	4.150		4.108	
$M_\psi(4S)$	4.7044	4.421±0.004	4.427	4.5320	4.344	
$M_\psi(3D)$	4.7628		4.507		4.371	
$M_\psi(5S)$	5.1642		4.837	4.8841	4.567	
$M_\psi(4D)$	5.2409		4.857			

Table 4.4: Masses of ground, radially and orbitally excited states of heavy vector quarkonium,J/ψ in BSE-CIA along with their masses in other models and experimental data (all units are in GeV).

5

Decays of heavy mesons

5.1 Two photon decay of scalar quarkonia

Apart from describing the mass spectrum of mesons, any realistic model must also be able to describe mesonic transitions and decays.The important question arises whether a good description of the extremely deep bound states can be combined with a reasonable description of confinement. We now study the two-photon decay width (decay rate) of a scalar quarkonium (0^{++}), which proceeds through the quark-triangle diagrams (triangle loop) as shown in the Figure blow.

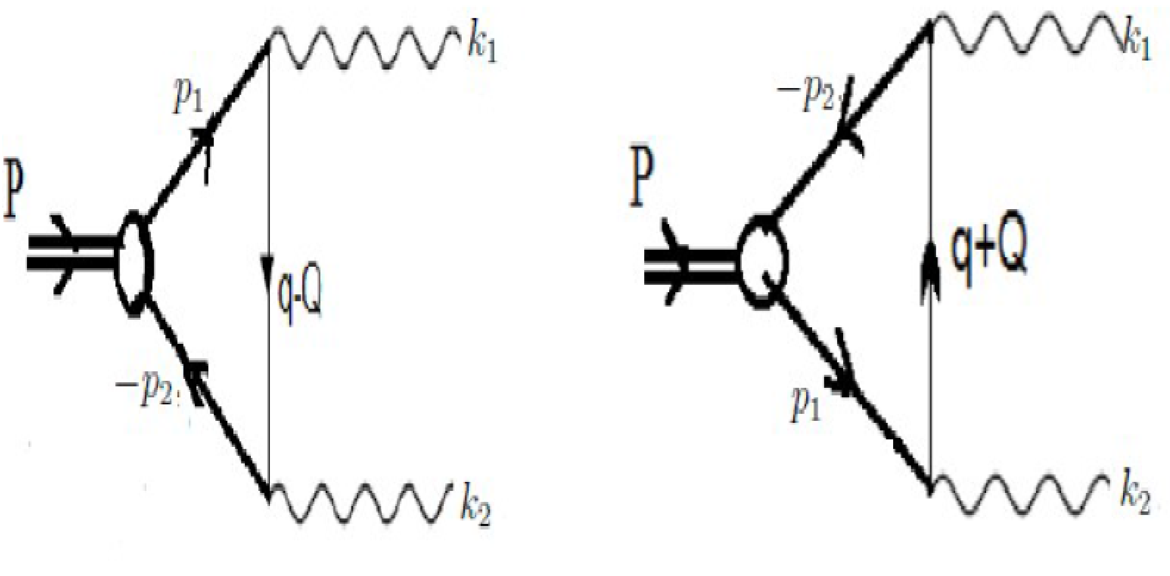

Figure 5.1: Diagrams contributing to the two photon decays of scalar (0^{++}) quarkonia .

Here we note that P is the total momentum of the scalar quarkonia and $k_{1,2}$ is the momenta of the two emitted photons with polarizations $\epsilon_{1,2}$ respectively. Then one can write $P = k_1 + k_2$, and let $2Q = k_1 - k_2$, where Q is the relative momentum of the two photons.The transition or decay of bound states are calculated from BS amplitudes. The invariant amplitude for this process can be written as:

$$M_{fi}(S \rightarrow \gamma\gamma) = i\sqrt{3}(ie_Q)^2 \int \frac{d^4q}{(2\pi)^4} Tr\left[\psi(P,q)\left[\not{\epsilon_1} S_F(q-Q)\not{\epsilon_2} + \not{\epsilon_1} S_F(q-Q)\not{\epsilon_2}\right]\right]. \quad (5.1)$$

where $e_Q = +\frac{2}{3}e$ for c$\bar{c}$, and $\frac{1}{2}e$ for b$\bar{b}$ systems.This can be put in the form

$$M_{fi}(S \rightarrow \gamma\gamma) = \frac{i\sqrt{3}(ie_Q)^2}{m^2 + \frac{M^2}{4}} \int \frac{d^3\hat{q}}{(2\pi)^3} Tr\left[\psi(\hat{q})\left[\not{\epsilon_1}(m + i\not{Q})\not{\epsilon_2} + \not{\epsilon_2}(m + i\not{Q})\not{\epsilon_1}\right]\right]. \quad (5.2)$$

The 3D structure of Dirac wave function,$\psi(\hat{q})$ is given in Eq.(3.47). The propagators for the third quark in the two diagrams is expressed as, $S_F(q \mp Q) = \frac{-i(\not{q} \mp \not{Q}) + m}{(q \mp Q)^2 + m^2}$. Now for heavy hadrons, where the system can be regarded as non-relativistic, it is a good approximation to take the internal momentum $q << M$, and hence $q^2 << Q^2$, where it can be seen that $Q^2 = \frac{M^2}{4}$. Using the propagator expressions given above, and evaluating trace over the gamma matrices, we can write the invariant amplitude given above as:

$$M_{fi}(S \rightarrow \gamma\gamma) = \epsilon_1.\epsilon_2.F_s, \quad (5.3)$$

where

$$F_s = \frac{(16N_s\alpha_{em})}{3\sqrt{3}\pi^2} \frac{mM}{m^2 + \frac{M^2}{4}} \int d^3\hat{q}\phi_s(\hat{q}) \quad (5.4)$$

is the decay constant for the scalar quarkonium, χ_{co}. The decay width(decay rate) for the process is then given by

$$\Gamma(S \rightarrow \gamma\gamma) = \frac{1}{32\pi M}|F_s|^2. \quad (5.5)$$

	BSE-CIA	Expt[7]	Pot.Model[28]	BSE[54]	RQM[55]
$\Gamma_{\chi_{c0}}(1p_0)$	2567.06	2341.50	1290.00±3.45	1390.00	2900.00
$\Gamma_{\chi_{c0}}(2p_0)$	1376.20		950.00±3.88	1110.00±130.00	1900.00

Table 5.1: Two-photon decay widths of ground and first excited states of $\chi_{c0}(1P_0)$ and $\chi_{c0}(2P_0)$ in eV with the input set of parameters given above. Expt means experimental value from PDG.

6

conclusion

We have employed a 3D reduction of the BSE [with a 4×4 representation for the two-body ($q\bar{q}$) BS amplitude] under the CIA to derive the algebraic forms of the mass spectral equations for scalar, pseudoscalar, vector, and axial-vector quarkonia using the full BS kernel comprised of the one-gluon-exchange and the confining part in an approximate harmonic oscillator basis, which led to analytic solutions (both eigenfunctions and eigenvalues). We thus obtained the mass spectra of $c\bar{c}$ quarkonia for the ground and excited states of 0^{++}, 0^{-+}, 1^{--}, and 1^{++} states. The mass spectral results for all of these states which were compared with the experimental data [6] and other models for each state (where available)are given in Tables IIIX respectively. The masses and the algebraic forms of the eigenfunctions for each quarkonium state so obtained will be used to calculate their various transitions. As mentioned in Sec. II, in this work the partitioning of relativistic internal momentum q comes from the Wightman-Garding definitions $\hat{m}_{1,2}$ of the masses of individual quarks (explained in detail in Sec. II). Hence, for equal-mass quarkonia (such as $c\hat{c}$ studied in this work) we get $p_{1,2} = \frac{1}{2}P \pm q$ (as in Refs. [37,38]), and the momentum is shared equally between the two quarks, which is a logical choice. However, for unequal-mass mesons (where $m_1 \neq m_2$) the $\hat{m}_{1,2}$ allocate most of the momentum to the heavy quark, while a smaller part of the momentum is allocated to the lighter quark (such that the relation $\hat{m}_1$+ $\hat{m}_2$ =1 is always satisfied), which is a reasonable choice in a heavy-light meson. But this choice is not needed in a Poincare-covariant formulation of the amplitudes [39,40], where one could use, in general, $p_1 = q + \alpha P$ and $p_2 = q + (1\alpha P)$, where the momentum partitioning parameter $\alpha\epsilon[0, 1]$ and is arbitrary, and the numerical results for the amplitudes and masses are independent of α. Its arbitrariness is equivalent to a freedom in the definition of the quark-antiquark relative momentum, since in a Poincare covariant formulation no physical observable should depend on the choice of momentum partitioning. Hence, even though the 3D reduction through the CIA employed by us makes our formulation relativistically covariant, it may not be Poincare covariant since our results depend on the choice of internal momentum. To have both Lorentz and Poincare covariance, the full 4D structure of the BS amplitudes has to be used. With the omission of the full structure of amplitudes comes the complication that the results can be sensitive to the definition of the relativistic internal momentum (see Ref. [42]). This may also be due to the numerical approximations used. We wish to mention that, in principle, in $Q\bar{Q}$ quarkonia the constituents are close enough to each other to warrant a more accurate treatment of the Coulomb term. Though for $b\bar{b}$ systems the Coulomb term will be extremely dominant in comparison to the confining term, it may not be so unreasonable to treat the Coulomb term perturbatively for $c\bar{c}$ systems. Here we wish to point out that if we get reasonable results for orbital $c\bar{c}$ excitations, it is mainly due to centrifugal effects [43] which ensure that the $c\bar{c}$ separation is large enough to feel the effect of the confining term more strongly than the Coulomb term. Further, the present approach is in line with some earlier works [4345] where the OGE (Coulomb) term was treated perturbatively for charmed mesons and baryons. Similarly, in a recent work [46] the 3D harmonic oscillator wave functions were used as trial wave functions for obtaining the $c\bar{c}$ spectrum and their decays, where these trial wave functions did not take into account the importance of the Coulomb potential for heavy quarkonium systems. A similar treatment was followed earlier in Ref. [47]. Also, in a recent work [48] in which one of us was involved, we did not take into account the Coulomb interactions between $c\bar{c}$ and $b\bar{b}$ states, and used only the confining interaction to study their spectra and decays, in line with other works [49-51]. However, in the present approach (using only the confining interactions) we first analytically derived the algebraic forms of the wave functions for various $c\bar{c}$ states, which are in the form of harmonic oscillators. Though we then introduced the Coulomb term perturbatively for $c\bar{c}$ (and not for $b\bar{b}$), this present calculation is a substantial improvement over our previous work in Ref. [48] (where we did not treat the Coulomb interaction at all). However, for $b\bar{b}$ states, a more exact nonperturbative treatment of the Coulomb interaction would be needed. Further, our results for the $c\bar{c}$ mass spectra suggest that the perturbative incorporation of OGE with the use of analytically derived harmonic oscillator wave functions is closer to reality than some of the previous approaches [46,47], where they used harmonic oscillator wave functions only as trial wave functions, which did not take into account the importance of the Coulomb potential for heavy quarkonium systems. All numerical calculations have been done using MATHEMATICA. We selected the best set of five input parameters that gave reasonable matching with data for the masses of the ground and excited states of $c\bar{c}$ quarkonia for 0^{++}, 0^{-+}, 1^{--}, and 1^{++} states. It was seen that this set of input parameters is not unique. It was observed that the mass spectra of mesons of various J^{PC} is some what insensitive (as can be seen from Tables IIVIII) to a range of variations of parameter $\omega_0\epsilon[0.1300.160]$ GeV as long as $\frac{C_0}{\omega_0^2}$ is a constant, and reasonably good fits are obtained for the parameter ratio $\frac{C_0}{\omega_0^2} = 8.8888GeV^{-2}$, with the other parameters $\Lambda = 0.250GeV$, $A_0 = 0.01$, and $m_c = 1.490GeV$. Thus,for $\omega_0 = 0.130$ GeV we have $C_0 = 0.15022$, while for $\omega_0 = 0.160$ GeV we have $C_0 = 0.227556$. (Such a relationship between two parameters that seem to be independent was also observed in Refs. [33,38].) However, for comparison with experiment, we took the values $\omega_0 = 0.145$ GeV, and $C_0 = 0.186889,$ along with $\Lambda = 0.250$ GeV, $A_0 = 0.01$, and $m_c = 1.490$ GeV. It was seen that only the lowest states (ground and one or two excited states) have reasonable fits. However, the disagreement with experiment increases as one goes to higher excited states. The analytic forms of the wave functions derived for 0^{++} states of $c\bar{c}$ were employed to calculate their two-photon decays.We wish to also point out that the present calculation with perturbative incorporation of the one-gluon-exchange interaction lifts up the degeneracy between the S and D states of vector quarkonia, and also gives a better agreement with the data for these states. The results obtained for the ground (1P) state of χ_{c0}, as well as the ground and first excited states of χ_{c1}, are in reasonable agreement with data. However, there are open questions about the quantum number assignments of the state $\chi(3915)$, which are available in Particle Data Group tables. Some authors [7,8] have argued that it is difficult to assign $\chi(3915)$ to $\chi_{c0}(2P)$ and that could be χ_{c2}, and that it could be χ_{c2} (2P). The possible mass of $\chi_{c0}(2P)$ was recently predicted in Ref. [7] as 3.837 ± 0.0115 MeV. We then worked out the decay widths of χ_{c0} for the ground (1P) and excited (2P) states, and compared our results with data and other models in Table VI. Though our results are smaller than data [6], they compare roughly other models [56,57]. Further, a large variation in the decay widths can be seen in other models. However, as mentioned earlier, our main emphasis in this paper was to show that this problem of the 4 x 4 BSE under the heavy-quark approximation can indeed be handled analytically for both the masses and wave functions of radially and orbitally excited states. In this framework, from the beginning, we employed a 4 x 4 representation for the two-body ($q\bar{q}$) BS amplitude to calculate both the mass spectra and the transition amplitudes. However, the price we had to pay was to solve a coupled set of equations for scalar, pseudoscalar, vector, and axial-vector quarkonia, which we have explicitly shown get decoupled in the heavy-quark approximation ($\omega \approx$ m), leading to a mass spectral equation with analytical solutions for both massesas well as eigenfunctions for all of the above statesin an approximate harmonic oscillator basis. The analytical forms of the eigenfunctions for ground and excited states so obtained can be used to calculate various transitions of these quarkonia. The heavy-quark approximation for quarkonia is valid because not only is the relative momentum between heavy quarks in the bound states considered small, but also these quarks are treated as almost on mass shell [53], which is justified for the calculation of low-energy properties like the mass spectrum and the decays of quarkonia ($c\bar{c}$ systems). This approximation, is well under control, in the context of heavy quark systems. We further wish to point out that this analytic approach (giving an explicit dependence of the spectra on the principal quantum number N) under the heavy-quark approximation gives much deeper insight into the spectral problem than the purely numerical approaches prevalent in the literature. The plots of the analytical forms of wave functions for various J^{PC} states (obtained as solutions of their mass spectral equations) are also given in Figs. 1 and 2 for scalar and axial-vector quarkonia. The correctness of our approach can be gauged by the fact that these plots are very similar to the corresponding plots of amplitudes for these quarkonia in Ref. [22] obtained by purely numerical methods. The analytical forms of the eigenfunctions for ground and excited states so obtained can be used to evaluate the various other processes involving scalar and axial-vector quarkonia, which we intend to do next.

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Publication

1. Shashank Bhatnagar, Lmenew Alemu, Physical review D 97, 034021 (2018)