LIFETIME ESTIMATE OF A NEUTRON STAR DUE TO GRAVITATIONAL RADIATION REACTION

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To my son: Natay.

Table of Contents

List of Figures

Abstract

A rotating neutron star/pulsar can emit gravitational radiation.The third time derivative of quadrupole moment of of an isolated systems must be non-zero in order for it to emit gravitational radiation. We follow the emission of gravitational radiation using quadrupole approximation. The energy and angular momentum of a rotating neutron star can slowly decrease as it get older.

We estimate the average lifetime of a gravitational wave damping for a neutron star wobble.Our calculation is based on the well known idea that energy loss within an isolated wobbling neutron star causes the axis of inertia of the star to align with its angular momentum vector. We model the neutron star as a rigid body with quadrupled deformation of its moment of inertia tensor.We find out the numerical lifetime estimate of a neutron star with two different dervations of the result: one is by adding the Burke-Thorne radiation reaction potential to the newtonian equation of motion and the other is based on energy and angular momentum balance.

Key words: neutron star, quadrupole moment, gravitational radiation, angular momentum,lifetime of neutron star

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Addis Ababa, Ethiopia Tadele Shiferaw

Introduction

During the past century, astronomy has been revolutionized by the use of new methods for observing the universe. The great progress that astronomy has made since 1960 is largely due to the fact that technology has permitted astronomers to begin to observe in many different parts of the electromagnetic spectrum. Because they were restricted to observing visible lights, the astronomers of the 1940s could have no inkling of such diverse and exiting phenomena,neutron stars, giant radio galaxies, quasars, compact X-ray binaries, molecular -line masses in dense clouds, the cosmic microwaves background radiation. Since technology has progressed, since then spectral region has revealed unexpected and important information. There are still regions of the electromagnetic spectrum that are largely unexplored, but there is another spectrum which is as yet complectly untouched: the gravitational wave spectrum.The question of gravitational radiation has always been the central issue in the theory of general relativity[34].

Gravitational waves, i.e. small deformations of space-time travelling at the speed of light,first pointed out by Einstein (1918), are fundamental of Einstein's General Theory of Relativity(GTR).There has been no distinct observation of gravitational waves so far, although an indirect evidence was found in the observed spiral history of the binary pulsar PSR 1913+16, which agrees perfectly with the prediction of General

Relativity (Weinberg and Taylor(1984)). Gravitational waves are purely transverse, characterized by two polarization states(referred as to '+' and 'x' polarizations). The two polarizations differ by a rotation of 45 degrees around the the polarization axis, corresponding to quadrupolar(spin 2) nature of the gravitational field.

The signals of gravitational waves from the astrophysical sources reaching the earth have very small amplitudes and are nearly plane waves. A linearized version of Einstein field equation (Misner et al. 1973) can therefore be used to describe gravitational waves in terms of a small metric perturbation $h_{\mu v}$. The emission of gravitational waves is generally well described by the so called quadrupole formula. Also the energy emission rate in gravitational waves can be expressed in quadrupole formalism. The quadrupole formalism shows that time-varying mass distribution generally emit gravitational waves.

Nearly all astrophysical phenomena emit gravitational waves and the most violent ones give off radiations in copious amounts(as we shall see in chapter 1). In some situations, gravitational radiation carries information that no electromagnetic radiations can give us.

Neutron star rotate extremely rapidly starting from their formation due to the conservation of angular momentum. Like a spinning ice skaters pulling in her arms, the slow rotation of the original star's core speeds up as it shrinks.A newborn neutron star can rotate several hundred times a second, turning itself into an oblate spheroid against its own immerse gravity. In wobbling neutron star, energy will be dissipated within the star,converting the kinetic energy into thermal energy. Also the gravitational wave energy and angular momentum will be radiated to infinity which must be subtracted from the star's energy and momentum[25]. The emission of gravitational radiation determines the evolution of the emitting source due to radiation reaction. A precessing neutron star is a possible source of gravitational radiation.The first clear observation of free precession in a pulsar signal was very recently[I1]. In this thesis we will be concerned on the gravitational radiation reaction torque which affect the neutron star wobble. We are aimed the analytical study of the effect of the radiation torque on a neutron star undergoing precessional motion. The effect of gravitational radiation reaction on precessing axisymmetric rigid bodies was first derived 35 years ago by Bertotti and Anile[I2] .

The structure of this thesis is as follows. In chapter 1 we briefly describe the neutron star formation, structure and the rotating neutron star as a source of gravitational waves and the generating mechanisms. In chapter 2 we deal about Einstein field equations and the weak field approximation. In chapter 3 we discuss gravitational wave and its mathematical description. In chapter 4 we deal about the gravitational radiation reaction torque and alignment timescale of wobbling neutron star. Our conclusion are given in chapter 5 with some discussion.

Chapter 1

The Physics of Neutron Star

This chapter gives a general overview about neutron star formation and structure, pulsar,astrophysical sources of gravitational wave and the emission mechanisms.

1.1 Neutron Star Formation

Compact objects such as neutron stars, white dwarfs and ultimately black holes represents the final states of stellar evolution[26]. When very massive stars (up to $25M_\odot$,where $M_\odot$ is solar mass,die, after they have finished their nuclear fuel, they spew their outer layers into space in a violent explosion called *supernova*.The cores of such stars remain as neutron stars. So, *a neutron star* is a type of remnant that can result from the core collapse of a massive star during supernova event[27]. A *supernova* occurs when the iron core of a giant star collapses to the density of the nucleus. At such events high densities,protons and electrons fuse together to form neutrons[1]. Hence the name 'neutron star'. Neutron stars are very hot and are supported against further collapse because of degenerate neutron pressure resulting from Pauli exclusion principle. This principle states that no two neutrons can occupy the same quantum state simultaneously.

In general, compact stars with mass $< 1.4\ M_{\odot}$ are white dwarfs.On the other hand,a neutron star is about 20 km in diameter and has mass of about $1.4\ M_{\odot}$. This means that a neutron star is so dense that one teaspoonful of neutron star matter would weigh a billion tons! The result is a surface gravitational field strength about 2×10^{11} times that of the earth. A neutron star can also have magnetic fields a million times strongest magnetic field produced on the earth[5]. As the core of a massive star is compressed during supernova, and collapse into a neutron star it retains most of its angular momentum. Since it has only a tiny fraction of its parent's radius (its moment of inertia is reduced sharply) so that a neutron star is formed with very high rotation speed.It gradually slows down due to mostly gravitational radiation initially and then electromagnetic radiation.

The number of neutron stars in the Galaxy has been estimated to be of the order 10^9[3,4].The number of observed neutron stars is much lower, about 800 are observed as radio pulsar[31],and about 150 as X-ray binaries.The population of neutron stars is concentrated along the disc of the milkway although the spread perpendicular to the disc is fairly large. The reason for this spread is that neutron stars born with high speeds (400 km/s) as a result of an imparted momentum kick from an asymmetry during the supernova explosion.

1.2 The Structure of Neutron Star

The internal structure of neutron stars are less well known because of uncertainties in the equation of state of degenerate nuclear matter. The problems involved in determining the equation of states are elegantly presented by Shapiro and Teukolsky (1983). Our proposed model for the neutron star is shown in fig. 1.1[1].

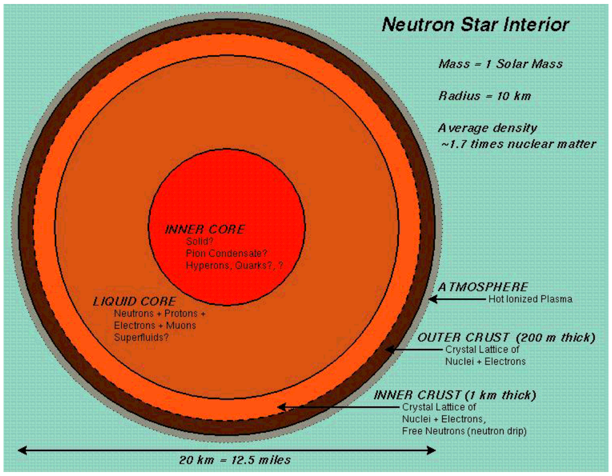

Figure 1.1: Structure of neutron star

Following Shapiro and Teukolsky, the various regions in the model may be described as: The *surface layers*; are taken to be the regions with density less than about $10^9 kgm^{-3}$. At these large density, the matter consists iron in the form of a closed packed solid. *The outer crust* is taken to be the regions with density in the range $(10^9 - 4.3 \times 10^{14}) kgm^{-3}$ and consists of a solid region composed of heavy nuclei.The *inner crust* has densities about between $(4.3 \times 10^{14} - 2 \times 10^{17}) kgm^{-3}$.The neutron liquid has densities greater than about $2 \times 10^{17} kgm^{-3}$ and consists mainly of neutrons with a small concentration of protons and electrons.A *core region* , the very center of neutron star,has very high densities greater than $3X10^{18} kgm^{-3}$.

1.3 Pulsar: Spinning Neutron Star

The first pulsar was are discovered in 1967 by Jocelyn Bell Burnell, a radio source that blinks on and off at a constant frequency. Pulsars are spinning neutron stars that have jets of particles moving almost at the speed of light streaming out above

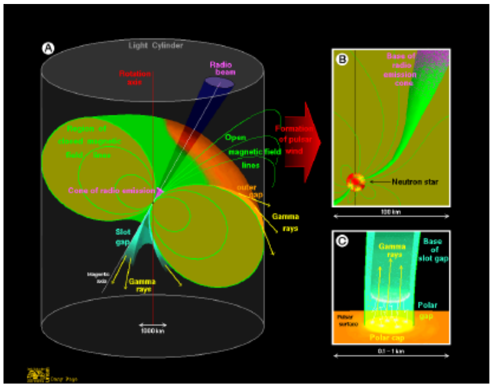

Figure 1.2: The Pulsar Model

their magnetic poles[11] (Fig 1.2). As the neutron star settles down into its final state, its crust begins to solidify (crystalize). The solid crust will assume nearly the oblate axisymmetric shape that centrifugal forces are trying to maintain with poloidal ellipticity, $\varepsilon_p \alpha$ (α is angular velocity of rotation). However the principal axis of the star's moment of inertia tensor may deviate from its spin axis by some small "wobble angle", and the star may deviate slightly from axisymmetry about its principal axis .As this slightly imperfect crust spins, it will radiate gravitational wave[8]. So a pulsar must be axisymmetric in order to radiate gravitationally. A wobbling pulsar may thus radiate gravitational wave.

A pulsar produces beams of radio emission above its magnetic poles and these sweep like lighthouse beams across the sky. The Jodrell Bank scientists (Ingrid Stairs, Andrew Hyne and Setnam Sheman) have been studying the pulsar PSR 131828-11 for 13 years. This pulsar rotates 2.5 times per second , but, unlike any other, wobbles regularly with a period of about 1000days. The motion is very much similar like the wobble of a top or gyroscope. This wobble or precession causes the observed pulse to change its shape and causes the time between pulses to vary, becoming sometimes shorter and sometimes longer. The Manchester astronomers argued these variation imply that the neutron star instead of being perfectly spherical is slightly oblate[12]. Stairs explains " The bulge in neutron star causes the angle between the pulsar's rotation axis and its radio beam to change with time creating the wobbling effect that we measure. " Current theories predict that the interaction between the superfluid and the crust should cause any precession to die out extremely quickly. " But this pulsar is 100,000 years old, and it's still wobbling!" exclaims Lyne.

1.4 Sources of Gravitational Waves

Gravitational waves are one of the features of General Relativity. The brightest gravitational wave sources are objects with strong gravitational fields, so the study of gravitational waves is a study of General Relativity[10] and the detection of gravitational waves can certainly test General Theory of Relativity and opens a window for us to observe astrophysical phenomena in the universe.

Sources of gravitational waves are best understood by drawing analogies to the case of electromagnetic radiation emission. Waves are formed by time change in the position and distribution of the 'charges' in the systems, whether those charges are electric or gravitational. The most obvious way to produce electromagnetic waves should be to change the total amount of charge in the system with time (called a monopole moment). However this violates the law of conservation of charges, and is not physically acceptable. The next best way to accomplish electromagnetic radiation is to vary the distribution of charge or to have a time-varying dipole moment. The *time-varying* *dipole moment* is the dominant contributor to *electromagnetic waves.*

Now consider gravitational waves. There can not be monopole in gravitational system for the same reason for that of there are no magnetic monopoles. The next solution is to have a time-varying gravitational dipole moment. This, however, is found to be impossible because of violation of conservation of angular momentum. We therefore have to look to the next higher moment of mass distribution, the *quadrupole moment*, for the possible *emission of gravitational radiation*, and, this is found to be physically acceptable.

However, this presents another challenge that the gravitational wave produced by 'everyday' matter, moving with time-varying quadrupole moment are so extraordinary small that they are not worth considering. The waves only become significant in systems that move at near relativistic speeds and are very massive. The only known sources of gravitational waves strong enough to be detected are those involved in astrophysical phenomena. Only in deep space are very massive bodies found moving fast enough to produce gravitational wave signals and among these are:

- rotating neutron stars
- binary star systems
- black holes
- supernovas
- colliding galaxies etc.

Here we will describe only about rotating neutron star, although further mathematical details on it will be given in chapter 3 and 4 of this thesis.Neutron stars may appear as:

- isolated objects or

- binary systems.

Isolated rotating neutron stars emit gravitational radiation if their configuration is not axisymmetric with respect to the rotation axis. The assymmetry can be the consequence of deformation of the stars' shape or of a misalignment of the symmetry axis with respect to the rotation axis. Thus there are two basic configuration of a rotating neutron star: rotation around a principal axis of inertia and rotation around an axis differen from the principal axis of inertia. As a result of stars quakes, strong internal magnetic fields or interactions with other stars, the rotation of a neutron star can be different from one of its principal axis of inertia and thus two different cases have been analyzed in the literature

- *wobbling or precession of a rigid neutron star* and
- *rotation of a distorted fluid neutron star.*

The gravitational waves emitted by a wobbling neutron star will carry energy and momentum away to infinity. So, over time, the wobbling neutron star slows down and tends to decrease the wobble angle. That means gravitational radiation reaction torque damp out the as consequently align the axes. We will calculate the time for the alignment.

Chapter 2

Linearized Einstein Field Equation

In general relativity laws of nature remain invariant with respect to any space time coordinate system while in special relativity this will be true in inertial frame. So we must express all laws of nature in terms of covariant equations that make no use of particular coordinate system.Such equations are referred to as "covariant". In this chapter we derive the so called Einstein's equations for gravity, which are general covariant in nature, and linearize them.

2.1 Einstein Field Equations

The *Einstein field equations* (EFE) or *Einstein's equations* are a set of ten equations in the theory of general relativity which describes the fundamental force of gravitation as a curved space time caused by matter and energy[7].They collectively form a tensor equation and equate the curvature of space time with the energy and momentum. The EFE are used to determine the curvature of space time resulting from the presence of mass and energy. Because of the relationship between the metric tensor of space time and the Einstein tensor, the EFE becomes a set of coupled, *non-linear* differential equations.Unlike Maxwell equation, which are linear, the non linearity of

EFE represents the effect of the gravitation on it self.

We now derive the field equations. Newtonian gravitation can be written as the theory of a scalar field, Φ:

$$\nabla^2\Phi(\vec{x},t) = 4\Pi G\rho(\vec{x},t) \tag{2.1.1}$$

where G is universal gravitational constant and ρ is mass density. The orbit of a free falling test particle satisfies

$$\ddot{x}(t) = -\nabla\Phi(\vec{x},t) \tag{2.1.2}$$

In tensor notation (2.1.1) and (2.1.2) become:

$$\Phi_{,ii} = 4\Pi G\rho \tag{2.1.3}$$

$$x,00 = -\Phi_{,i} \tag{2.1.4}$$

where the comma (,) indicates ordinary partial differentiation and in general relativity, these equations are replaced by,

$$R_{\mu\upsilon} = C(T_{\mu\upsilon} - \frac{1}{2}g_{\mu\upsilon}T) \tag{2.1.5}$$

for some constant C and the geodesic equations:

$$\frac{d^2x^\lambda}{d\tau^2} = -\Gamma^\lambda_{\mu\upsilon}\frac{dx^\mu}{d\tau}\frac{dx^\upsilon}{d\tau} \tag{2.1.6}$$

where $R_{\mu\upsilon}$ is the Ricci tensor, $T_{\mu\upsilon}$ is the energy-momentum tensor, $g_{\mu\upsilon}$ is the metric tensor and $\Gamma^\lambda_{\mu\upsilon}$ is the Christoffel symbol defined by

$$\Gamma^\lambda_{\mu\upsilon} = \frac{1}{2}g^{\lambda\rho}(g_{\rho\mu,\upsilon} + g_{\rho\upsilon,\mu} - g_{\mu\upsilon,\rho}) \tag{2.1.7}$$

To see how (2.1.6) reduces to (2.1.4)actually using the Post Newtonian approximation we have

$$(\frac{d\tau}{dt})^2 = -g_{00} - 2g_{i0}v^i - g_{ij}v^iv^j \tag{2.1.8}$$

Assuming $\vec{v} = 0$

$$\frac{d\tau}{dt})^2 = -g_{00}$$

or

$$(\frac{dt}{d\tau})^2 = -g_{00}^{-1}$$

To the order of v^{-4} this gives

$$(\frac{dt}{d\tau})^2 = 1 - [v^2 + ...]$$

where $\vec{v} = GM/r$. So, for $\vec{v} = 0$

$$\frac{dt}{d\tau} \simeq 1$$

$$\frac{d}{dt}(\frac{dt}{d\tau}) \simeq 0 \tag{2.1.9}$$

and the metric and its derivative are approximately static and that the square of derivatives from the Minkowiski metric are negligible.Applying these simplifying assumptions the special components of the geodesic equations (2.1.6) gives:

$$\frac{d^2x^i}{d\tau^2} \simeq -\Gamma^i_{00}$$

Hence,

$$\frac{d^2x^i}{dt^2} \simeq -\Gamma^i_{00} \tag{2.1.10}$$

where two factors of $\frac{dt}{d\tau}$ have been divided out.Then (2.1.10) will reduce to its newtonian counterpart,provided

$$\Phi_{,i} \cong \Gamma^i_{00} = \frac{1}{2}g^{i\rho}(g_{\rho 0,0} + g_{0\rho,0} - g_{00,\rho}) \tag{2.1.11}$$

The time derivatives to be zero, this simplifies to

$$2\Phi_{,i} \cong g^{ij}(-g_{00,j}) \cong -g_{00,i} \tag{2.1.12}$$

Since $g^{i\rho} \rightarrow g^{ij}$ and $g^{i0} \rightsquigarrow \overline{v}^3 << 1$ which is satisfied by letting the time-time component of the metric tensor as:

$$g_{00} \cong -(1 + 2\Phi) \tag{2.1.13}$$

where Φ is the Newtonian potential.

Further more the energy density T_{00} for *non relativistic matter* is equal to its *mass density.*That is

$$T_{00} \approx \rho \tag{2.1.14}$$

Combining this with (2.1.3) and (2.1.13), we have then,

$$\nabla^2 g_{00} = -8\Pi G T_{00} \tag{2.1.15}$$

This field equation is only supposed to hold for weak static fields generated by non-relativistic matter and is not Lorentz invariant as it stands.However (2.1.15) leads us to guess the field equation for a general distribution $T_{\mu\upsilon}$ of energy and momentum as:

$$G_{\mu\upsilon} = -8\Pi G T_{\mu\upsilon} \tag{2.1.16}$$

where $G_{\mu\upsilon}$ is a tensor which is a linear combination of the metric $g_{\mu\nu}$and its first and second derivative.The nature of $G_{\mu\upsilon}$

- by definition *it is a tensor*

- by assumption *it contains only terms that are either linear in the second derivatives or quadratic in the first derivative of the metric.*

- *it is symmetric*, since $T_{\mu\upsilon}$ is symmetric

- *it is conserved*, since $T_{\mu\upsilon}$ is conserved. That is,

$$G^{\mu}_{\upsilon;\mu} = 0 \qquad (2.1.17)$$

- *for a weak static field limit*

$$G_{00} \cong \nabla^2 g_{00} \qquad (2.1.18)$$

To find $G_{\mu\upsilon}$,we construct the so called Riemann- Christofell curvature tensor $R^{\lambda}_{\mu\upsilon k}$, given as

$$R^{\lambda}_{\mu\upsilon k} = \Gamma^{\lambda}_{\mu\upsilon,k} - \Gamma^{\lambda}_{\mu k,\upsilon} + \Gamma^{\eta}_{\mu\upsilon}\Gamma^{\lambda}_{k\eta} - \Gamma^{\eta}_{\mu k}\Gamma^{\lambda}_{\upsilon\eta} \qquad (2.1.19)$$

which may also be rewritten as:

$$R_{\lambda\mu\upsilon k} = g_{\lambda\sigma}R^{\sigma}_{\mu\upsilon k} \qquad (2.1.20)$$

The curvature tensor satisfies the Bianchi identities written as

$$R_{\lambda\mu\upsilon k;\eta} + R_{\lambda\mu\eta\upsilon;k} + R_{\lambda\mu k\eta;\upsilon} = 0 \qquad (2.1.21)$$

where ";" represents the covariant differentiation which can be defined as

$$T_{\mu\nu;\alpha} = T_{\mu\nu,\alpha} - \Gamma^{\lambda}_{\mu\alpha}T_{\lambda\nu} - \Gamma^{\lambda}_{\nu\alpha}T_{\mu\lambda} \qquad (2.1.22)$$

where $T_{\mu\nu}$ is a tensor of rank two. From (2.1.19) we may read off the following *algebraic properties of the curvature tensor:*

- *symmetry*:

$$R_{\lambda\mu\upsilon k} = R_{\upsilon k\lambda\mu} \qquad (2.1.23)$$

- *antisymmetry*:

$$R_{\lambda\mu\upsilon k} = -R_{\mu\lambda\upsilon k} = -R_{\lambda\mu k\upsilon} = +R_{\mu\lambda k\upsilon} \tag{2.1.24}$$

- *cyclicity*:

$$R_{\lambda\mu\upsilon k} + R_{\lambda k\mu\upsilon} + R_{\lambda\mu k\upsilon} = 0 \tag{2.1.25}$$

Then (2.1.21) shows that there are only two tensors that can be formed by contracting $R_{\lambda\mu\upsilon k}$. These are the Ricci tensor:

$$R_{\mu k} = g^{\lambda\upsilon} R_{\lambda\mu\upsilon k} = R^{\lambda}_{\mu\lambda k}$$

or

$$R_{\mu k} = \Gamma^{\lambda}_{\mu\lambda,k} - \Gamma^{\lambda}_{\mu k,\lambda} + \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{k\eta} - \Gamma^{\eta}_{\mu k}\Gamma^{\lambda}_{\eta\lambda} \tag{2.1.26}$$

and the curvature scalar:

$$R = g^{\mu k} R_{\mu k} = R^{\mu}_{\mu} \tag{2.1.27}$$

Hence the first and the second properties of $G_{\mu\upsilon}$ require it to take the form:

$$G_{\mu\upsilon} = C R_{\mu\upsilon} + C' g_{\mu\upsilon} R \tag{2.1.28}$$

where C and C' are constants.

and on contraction of λ with υ, (2.1.27) becomes

$$R_{\mu k;\eta} - R_{\mu\eta;k} + R^{\upsilon}_{\mu k\eta;\upsilon} = 0 \tag{2.1.29}$$

Contracting this again gives

$$R_{;\eta} - R^{\mu}_{\eta;\mu} - R^{\upsilon}_{\eta;\upsilon} = 0$$

or

$$(R^{\upsilon}_{\eta} - \frac{1}{2}\delta^{\mu}_{\eta} R)_{;\mu} = 0 \tag{2.1.30}$$

The covariant divergence of $G_{\mu v}$ then gives us

$$G^{\mu}_{v;\mu} = (\frac{C}{2} + C')R_{;v} \tag{2.1.31}$$

So the conservation of $G_{\mu v}$ allows two possibilities: either $C' = -\frac{C}{2}$ or $R_{;v}$ vanishes every where. Ignoring the second possibility, because (2.1.24) and (2.1.5) give

$$G^{\mu}_{\mu} = (C + 4C')R = -8\Pi G T^{\mu}_{\mu} \tag{2.1.32}$$

So (2.1.26) becomes

$$G_{\mu v} = C(G_{\mu v} - \frac{1}{2}g_{\mu v}R) \tag{2.1.33}$$

Finally we use the last property of $G_{\mu v}$ (2.1.5) which is

$$\nabla^2 g_{00} = 8\Pi G T_{00}$$

or (2.1.18)

$$G_{00} = \nabla^2 g_{00} \tag{2.1.34}$$

to fix C. A non-relativistic system always has

$$|T_{ij}| << |T_{00}|$$

So

$$|G_{ij}| << |G_{00}|$$

$$G_{ij} \simeq 0 \tag{2.1.35}$$

Then equation (2.1.32) becomes

$$G_{ij} = C(R_{ij} - \frac{1}{2}g_{ij}R) \tag{2.1.36}$$

Substitution of (2.1.34) follows

$$0 \simeq C(R_{ij} - \frac{1}{2}g_{ij}R)$$

$$R_{ij} \simeq \frac{1}{2} g_{ij} R \tag{2.1.37}$$

Since we deal with a weak field:

$$g_{\mu\upsilon} \simeq \eta_{\mu\upsilon} \tag{2.1.38}$$

Thus using (2.1.36) and (2.1.37)

$$\begin{aligned} \eta^{ij} R_{ij} &= \frac{1}{2} \eta^{ij} \eta_{ij} R \\ R_{ii} &= \frac{3}{2} R \end{aligned} \tag{2.1.39}$$

and

$$\begin{gathered} R_{ii} - R_{00} \simeq \eta^{\mu\upsilon} R_{\mu\upsilon} \\ \frac{3}{2} R - R_{00} \simeq R \\ R \simeq 2R_{00} \end{gathered} \tag{2.1.40}$$

Thus using equations(2.1.37) and (2.1.39) in (2.1.32) we obtain

$$G_{00} = C(R_{00} - \frac{1}{2} \eta_{00} R)$$

$$G_{00} = 2CR_{00} \tag{2.1.41}$$

To calculate R_{00} for weak field we may use the linear part of $R_{\lambda\mu\upsilon k}$:

$$R_{\lambda\mu\upsilon k} \simeq \frac{1}{2} (g_{\lambda\upsilon,k\mu} - g_{\mu\upsilon,k\lambda} - g_{\lambda k,\mu\upsilon} + g_{\mu k,\upsilon\lambda}$$

When the field is static all time derivatives vanish, and the component we need become

$$R_{0000} \simeq 0, R_{i0j0} \simeq \frac{1}{2} R_{00,ij}$$

But from (2.1.24),we have

$$\begin{aligned} R_{00} &= g^{\lambda\nu} R_{\lambda 0 \nu 0} \\ &= \eta^{\lambda\nu} R_{\lambda 0 \nu 0} \\ &= R_{ioio} - R_{0000} \end{aligned}$$

Hence,

$$G_{00} \simeq 2C(R_{ioio} - R_{0000})$$

$$G_{00} \simeq 2C(\frac{1}{2}\nabla^2 g_{00} - 0)$$

$$G_{00} \simeq C_1 \nabla^2 g_{00} \qquad (2.1.42)$$

Comparing this with (2.1.18) we find that the fourth properties of $G_{\mu\nu}$ is satisfied if and only if $C = 1$ in (2.1.32),and we get:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \qquad (2.1.43)$$

with (2.1.16), this gives the *Einstein field equations*:

$$(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = -8\Pi G T_{\mu\nu} \qquad (2.1.44)$$

Contracting this with $g^{\mu\nu}$ gives:

$$R - 2R = -8\Pi G T^{\mu}_{\mu}$$

or

$$R = 8\Pi G T^{\mu}_{\mu} \qquad (2.1.45)$$

plugging this in to (2.1.43),the Einstein field equations can be rewritten in the equivalent form as

$$R_{\mu\nu} = -8\Pi G (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\lambda}_{\lambda}) \qquad (2.1.46)$$

In a vacuum $T_{\mu\nu}$ vanishes, so from (2.1.43) the Einstein field equation is empty space are

$$R_{\mu\nu} \;=\; 0$$

The Einstein field equation may be written in the form:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{1}{2}g_{\mu\nu}\Lambda = -8\Pi G T_{\mu\nu} \tag{2.1.47}$$

The term $\Lambda g_{\mu\nu}$ was originally introduced by Einstein for cosmological reasons; for this reason, Λ is called cosmological constant.Through out this thesis Λ is considered as zero.

2.2 Weak Field Approximation

Einstein field equations, discussed in the previous section, are highly non-linear and hence difficult to handle [2]. However, if the field is weak (in the Newtonian sense) then it is natural to expect that the field equations can be adequately approximated by a set of linear equations which are much easier to work with.We will demonstrate this approximation in this section.

We estimate the metric $g_{\mu\nu}$ as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.2.1}$$

where $|h_{\mu\nu}| << 1$ and

$$\eta_{\mu\nu} = dia(-1, +1, +1, +1)$$

is the usual metric of flat space, Minkowski metric tensor. The assumption that $|h_{\mu\nu}| << 1$ allows us to ignore higher order of $h_{\mu\nu}$, from which we obtain

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{2.2.2}$$

where

$$h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$$

Using equations (2.2.2) and (2.2.3) in (2.1.7) we get:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}(\eta^{\lambda\rho} - h^{\lambda\rho})[(\eta_{\rho\mu} + h_{\rho\mu})_{,\nu} + (\eta_{\rho\nu} + h_{\rho\nu})_{,\mu} - (\eta_{\mu\nu} + h_{\mu\nu})_{,\rho}]$$

$$\Gamma^{\lambda}_{\mu\nu} \cong \frac{1}{2}\eta^{\rho\lambda}(h_{\rho\nu,\mu} + h_{\rho\mu,\nu} - h_{\mu\nu,\rho}) \tag{2.2.3}$$

Now we need to solve for the Riccii curvature tensor (2.1.24):

$$R_{\mu k} = \Gamma^{\lambda}_{\mu\lambda,k} - \Gamma^{\lambda}_{\mu k,\lambda} + \Gamma^{\lambda}_{k\eta}\Gamma^{\eta}_{\mu\lambda} - \Gamma^{\lambda}_{\lambda\eta}\Gamma^{\eta}_{\mu k}$$

Then under this approximation,the terms that are products of two Γ are order of h^2 and taking this terms to be zero,we obtain:

$$R_{\mu k} \cong \Gamma^{\lambda}_{\mu\lambda,k} - \Gamma^{\lambda}_{\mu k,\lambda}$$

or with $k \longrightarrow \nu$

$$R_{\mu\nu} \cong \Gamma^{\lambda}_{\mu\lambda,\nu} - \Gamma^{\lambda}_{\mu\nu,\lambda} \tag{2.2.4}$$

Substituting (2.2.4) in to (2.2.5), we find

$$R_{\mu\nu} \cong [\frac{1}{2}\eta^{\rho\lambda}(h_{\rho\lambda,\mu} + h_{\rho\mu,\lambda} - h_{\mu\lambda,\rho})]_{,\nu} - [\frac{1}{2}\eta^{\rho\lambda}(h_{\rho\mu,\nu} + h_{\rho\nu,\mu} - h_{\mu\nu,\rho})]_{,\lambda}$$

$$R_{\mu\nu} \cong [\frac{1}{2}(h^{\lambda}_{\lambda,\mu} + h^{\lambda}_{\mu,\lambda} - h^{\rho}_{\mu,\rho})]_{,\nu} - [\frac{1}{2}(h^{\lambda}_{\nu,\mu} + h^{\lambda}_{\mu,\nu} - \eta^{\rho\lambda}h_{\mu\nu,\rho})]_{,\lambda}$$

$$R_{\mu\nu} \cong [\frac{1}{2}(\eta^{\rho\lambda}h_{\mu\nu,\rho\lambda} + h^{\lambda}_{\lambda,\mu\nu} - h^{\lambda}_{\nu,\mu\lambda} - h^{\rho}_{\mu,\rho\nu})$$

or

$$R_{\mu\nu} \cong [\frac{1}{2}(\Box h_{\mu\nu} + h^{\lambda}_{\lambda,\mu\nu} - h^{\lambda}_{\nu,\mu\lambda} - h^{\rho}_{\mu,\rho\nu}) \tag{2.2.5}$$

Where $\Box$ is the D'Alembertian operator defined by

$$\Box h_{\mu\nu} = \eta^{\rho\lambda}h_{\mu\nu,\rho\lambda} \tag{2.2.6}$$

Plugging (2.2.6) in to Einstein field equation (2.1.43) gives:

$$\frac{1}{2}(\Box h_{\mu\nu} + h^{\lambda}_{\lambda,\mu\nu} - h^{\lambda}_{\nu,\mu\lambda} - h^{\rho}_{\mu,\rho\mu}) - \frac{1}{2}(\eta_{\mu\nu} + h_{\mu\nu})R = -8\Pi G T_{\mu\nu} \tag{2.2.7}$$

Compared to $\eta_{\mu\nu}R$, $h_{\mu\nu}R$ is small and using $R = 8\Pi G T^{\mu}_{\mu}$ from (2.1.44), then equation (2.2.8) becomes

$$\frac{1}{2}(\Box h_{\mu\nu} + h^{\lambda}_{\lambda,\mu\nu} - h^{\lambda}_{\nu,\mu\lambda} - h^{\rho}_{\mu,\rho\mu}) - \frac{1}{2}(\eta_{\mu\nu}(8\Pi G T^{\mu}_{\mu}) = -8\Pi G T_{\mu\nu} \tag{2.2.8}$$

or

$$\Box h_{\mu\nu} + h^{\lambda}_{\lambda,\mu\nu} - h^{\lambda}_{\nu,\mu\lambda} - h^{\lambda}_{\mu,\lambda\nu} = -16\Pi G S_{\mu\nu} \tag{2.2.9}$$

where $S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{\mu}_{\mu}$

Let us choose some particular convenient choice of a coordinate system which is represented by the *harmonic coordinate condition:*

$$\Gamma^{\lambda} = g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0 \tag{2.2.10}$$

Plugging Eqns.(2.2.3) and (2.2.4) into (2.2.11) we find

$$(\eta^{\mu\lambda} - h^{\mu\lambda})[\frac{1}{2}(\eta^{\mu\rho} + h^{\mu\rho})[(\eta_{\rho\nu} + h_{\rho\nu})_{,\lambda} + (\eta_{\rho\lambda} + h_{\rho\lambda})_{,\nu} - (\eta_{\nu\lambda} + h_{\nu\lambda})_{,\rho}] = 0 \tag{2.2.11}$$

Neglecting higher order terms of h, this becomes

$$\begin{aligned} \frac{1}{2}\eta^{\mu\rho}\eta^{\nu\lambda}(h_{\rho\nu,\lambda} + h_{\rho\lambda,\nu} - h_{\nu\lambda,\rho}) &= 0 \\ \frac{1}{2}\eta^{\mu\rho}(h^{\lambda}_{\rho,\lambda} + h^{\lambda}_{\rho,\lambda} - h^{\lambda}_{\lambda,\rho}) &= 0 \\ \frac{1}{2}\eta^{\mu\rho}(2h^{\lambda}_{\rho,\lambda} - h^{\lambda}_{\lambda,\rho}) &= 0 \end{aligned}$$

Letting $\mu = \rho$ and hence

$$h^{\lambda}_{\rho,\lambda} - \frac{1}{2}h^{\lambda}_{\lambda,\rho} = 0$$

$$h^{\lambda}_{\rho,\lambda} = \frac{1}{2}h^{\lambda}_{\lambda,\rho} \tag{2.2.12}$$

From this we can get

$$h^{\lambda}_{\rho,\lambda\nu} = \frac{1}{2}h^{\lambda}_{\lambda,\rho\nu} \tag{2.2.13}$$

and

$$h^{\lambda}_{\nu,\lambda\rho} = \frac{1}{2}h^{\lambda}_{\lambda,\rho\nu} \tag{2.2.14}$$

Adding the expressions (2.2.13) and (2.2.14) gives

$$h^{\lambda}_{\rho,\lambda\nu} + h^{\lambda}_{\nu,\lambda\rho} = h^{\lambda}_{\lambda,\rho\nu}$$

or rearranging and with $\rho \to \mu$, this becomes

$$h^{\lambda}_{\lambda,\mu\nu} - h^{\lambda}_{\nu,\mu\lambda} - h^{\lambda}_{\mu,\lambda\nu} = 0 \tag{2.2.15}$$

Now plugging (2.2.15) in to (2.2.10),we get

$$\Box h_{\mu\nu} = -16\Pi G S_{\mu\nu} \tag{2.2.16}$$

Alternatively plugging (2.2.16) in to the linearized Einstein field equations (2.2.10) it simplifies to:

$$\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h = -16\Pi G T_{\mu\nu} \tag{2.2.17}$$

which represents an inhomogeneous wave equation establishing the fact that in this approximation (far field approximation) the field equations have radiative solution.

Chapter 3

Gravitational Wave

In the previous chapter we saw that the linearized Einstein field equations of general relativity could be written in the form of a wave equation (2.2.17):

$$\Box h_{\mu\nu} = -16\pi G S_{\mu\nu}$$

This suggests that the *existence of gravitational wave* in analogue manner to that in which Maxwell's equations predict electromagnetic waves.In this chapter we discuss the propagation, generation of gravitational wave and derive the expression for quadrupole emission in gravitational radiation,the lowest multipole emission in general theory of relativity and the quadrupole formula.

3.1 Plane Gravitational Wave Solution

Here we can take the Einstein field equations in a weak field limit (2.1.18)as

$$\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h = -16\pi G T_{\mu\nu} \tag{3.1.1}$$

It is often convenient to work with a slightly different description of the metric perturbation $h_{\mu\nu}$ defined by

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{3.1.2}$$

This metric tensor is called *'trace-reversed' perturbation* because:

$$\overline{h}^{\mu}_{\mu} = -h^{\mu}_{\mu} \tag{3.1.3}$$

The *harmonic gauge condition* (that is ,$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0$) further reduces to

$$\partial_{\lambda}\overline{h}^{\lambda}_{\mu} = 0 \tag{3.1.4}$$

The Einstein field equations are then

$$\Box\overline{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \tag{3.1.5}$$

From this we obtain the *linearized field equations in vacuum* :

$$\Box\overline{h}_{\mu\nu} = 0 \tag{3.1.6}$$

The vacuum equations for $\overline{h}_{\mu\nu}$ are similar to the wave equations in electromagnetism.These equations admits the plane wave solutions,

$$\overline{h}_{\mu\nu} = \epsilon_{\mu\nu} exp(ik_{\alpha}x^{\alpha}) \tag{3.1.7}$$

where $\epsilon_{\mu\nu}$ is a constant, symmetric rank two tensor and k_{α} is a constant four vector known as wave vector. Of course, in physical application, one uses only the real part of this expression, allowing $\epsilon_{\mu\nu}$ to be complex. By the theorems of Fourier analysis, any solutions of the equation (3.1.6) is a superposition of plane wave solutions. Plugging the solutions in (3.1.7) in to the wave equation (3.1.6), we obtain

$$\Box\overline{h}_{\mu\nu} \equiv 0$$

$$\eta^{\alpha\beta}\overline{h}_{\mu\nu,\alpha\beta} = 0$$

$$\overline{h}_{\mu\nu,\beta} = k_\alpha \overline{h}_{\mu\nu}$$

$$\eta^{\alpha\beta}\overline{h}_{\mu\nu,\alpha\beta} = \eta^{\alpha\beta} k_\alpha k_\beta \overline{h}_{\mu\nu} = 0$$

$$k_\alpha k^\alpha \overline{h}_{\mu\nu} = 0$$

Since $\overline{h}_{\mu\nu} \neq 0$, we obtain the condition

$$k_\alpha k^\alpha = 0 \tag{3.1.8}$$

This implies that (3.1.7) gives a solution to the wave equation (3.1.6) if the k_α is null; that is, the tangent to the world line of the photon. This shows that *gravitational waves propagate at the speed of light* . The time -like component of the wave vector is often refers to as the frequency of the wave(k_o or ω).The four-vector ,k_α is usually written as $k_\alpha = (\omega, \overrightarrow{k})$. Since k_α is null it means that

$$|\overrightarrow{k}|^2 - \frac{\omega^2}{c^2} = 0$$

If c=1 ,

$$\omega^2 = |\overrightarrow{k}|^2 \tag{3.1.9}$$

where $\overrightarrow{k}$ refers to k_i.

This is often referred to as the dispersion relation for the gravitational wave. We can specify the plane wave with a number of independent parameter; ten from the coefficients , $\epsilon_{\mu\nu}$ and three from the null vector, k_α. The Einstein equations only assume the simple form, if we impose the harmonic condition (3.1.4). Then we find

$$\partial_\mu \overline{h}_{\mu\nu} = 0$$

$$\partial_\mu(\epsilon^{\mu\nu}e^{ik_\alpha x^\alpha}) = 0$$

$$i\epsilon^{\mu\nu}k_\mu e^{ik_\alpha x^\alpha} = 0$$

Or

$$k_\mu\epsilon^{\mu\nu} = 0 \tag{3.1.10}$$

This imposes a restriction on $\epsilon^{\mu\nu}$; it is orthogonal(transverse) to k_μ. The number of independent components of $\epsilon_{\mu\nu}$ is thus reduced to six. We have to impose a gauge condition too as any coordinate transformation which leaves (3.1.1) unchanged; is very small change in the coordinate of the form

$$x'^\mu = x^\mu + \xi^\mu(x) \tag{3.1.11}$$

where ξ^μ are functions of the coordinate.This will leave the harmonic coordinate condition

$$\Box x^\mu = 0 \tag{3.1.12}$$

satisfied as long as

$$\Box\xi^\mu = 0 \tag{3.1.13}$$

Let's choose a solution to the wave equation (3.1.13) for ξ^μ as

$$\xi_\mu = B_\mu exp(ik_\alpha x^\alpha) \tag{3.1.14}$$

where B_μ are constants and k_α is the same null vector as for our wave solution .This produces of the coordinate change in our metric perturbation $h_{\mu\nu}$ as

$$h_{\mu\nu}^{(New)} \to h_{\mu\nu}^{(Old)} - \epsilon_{\nu,\mu} - \epsilon_{\mu,\nu} \tag{3.1.15}$$

where the new $h_{\mu\nu}$ is still small and we are still in acceptable coordinate system.This change is called a *gauge transformation*,which is a term used because of the analogies between (3.1.15) and gauge transformation of electromagnetism.Then the gauge change in (3.1.15) induces a change in the trace-reversed perturbation.

$$\overline{h}^{(New)}_{\mu\nu} = h^{(New)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{(New)}$$

$$\overline{h}^{(New)}_{\mu\nu} = \overline{h}^{(Old)}_{\mu\nu} - \epsilon_{\nu,\mu} - \epsilon_{\mu,\nu} - \frac{1}{2}\eta_{\mu\nu}(h^{(old)} - 2\xi^{\lambda})$$

$$\overline{h}^{(New)}_{\mu\nu} = \overline{h}^{(Old)}_{\mu\nu} - \epsilon_{\nu,\mu} - \epsilon_{\mu,\nu} + \eta_{\mu\nu}\xi^{\lambda}{}_{,\lambda} \tag{3.1.16}$$

Using equation(3.1.14) and dividing out the exponential factor common to all terms gives;

$$\varepsilon^{New}_{\mu\nu} = \varepsilon^{Old}_{\mu\nu} - ik_{\mu}B_{\nu} - ik_{\nu}B_{\mu} + i\eta_{\mu\nu}k_{\lambda}B^{\lambda} \tag{3.1.17}$$

Then B_{μ} can be chosen to impose to two further restriction on $\varepsilon^{New}_{\mu\nu}$;

$$\varepsilon^{(New)\mu}_{\mu} = 0 \quad \text{(traceless)} \tag{3.1.18}$$

and

$$\varepsilon^{(New)}_{o\nu} = 0 \tag{3.1.19}$$

The equations (3.3.10),(3.1.18) and (3.1.19) together are called the '*transverse - traceless* ' (**TT**) *gauge* (or sometimes the radiative gauge). The name comes from the fact that the *metric perturbation is traceless and perpendicular to the wave vector*. Therefore, applying (3.1.18) means

$$\varepsilon^{(old)\mu}_{\mu} + 2ik_{\lambda}B^{\lambda} = 0 \quad or \tag{3.1.20}$$

$$k_{\lambda}B^{\lambda} = \frac{i}{2}\varepsilon^{(old)\mu}_{\mu} \tag{3.1.21}$$

Then we can impose (3.1.20), first for $\nu = 0$;

$$\varepsilon^{(old)}_{oo} - 2ik_{o}B_{o} + \frac{1}{2}ik_{\lambda}B^{\lambda} = 0$$

$$\varepsilon_{oo}^{(old)} - 2ik_o B_o + \frac{1}{2}\varepsilon_{\mu}^{(old)\mu} = 0 \qquad (3.1.22)$$

or

$$B_o = \frac{-i}{2k_o}\left(\varepsilon_{oo}^{(old)} + \frac{1}{2}\varepsilon_{\mu}^{(old)\mu}\right) \qquad (3.1.23)$$

Then impose (3.1.20) for $\nu = j$

$$\varepsilon_{oj}^{(old)} - ik_o B_j - ik_j B_o = 0$$

$$\varepsilon_{oj}^{(old)} - ik_o B_j - ik_j[\frac{-i}{2k_o}(\varepsilon_{oo}^{(old)} + 1/2\varepsilon_{\mu}^{(old)\mu})] = 0 \qquad (3.1.24)$$

$$B_j = \frac{i}{2(k_o)^2}\left[-2k_o\varepsilon_{oj}^{(old)} + k_j\left(\varepsilon_{oo}^{(old)} + \frac{1}{2}\varepsilon_{\mu}^{(old)\mu}\right)\right] \qquad (3.1.25)$$

To show that these choices are mutually consistent we should plug (3.1.25)and (3.1.23) back in to (3.1.18) and shows that

$$\varepsilon_{\mu}^{(New)\mu} = \varepsilon^{New} = 0$$

$$\varepsilon^{(New)} = \varepsilon - 2c_o k_o + 2c_i k^i = 0$$

Therefore, choosing the harmonic gauge implied the four conditions(3.1.10) brought the ten independent components in $\varepsilon_{\mu\nu}$ down to six and the remaining gauge freedom led to the one condition (3.1.18) and the four conditions (3.1.20) bring us to two independent components. We have used up all of our possible freedom so that these two independent elements represents the physical information characterizing our plane wave in this gauge.By replacing new components $\varepsilon_{\mu\nu}^{(New)}$ simply as $\varepsilon_{\mu\nu}$,this can be seen more explicitly by choosing our spatial coordinates such that the wave is travelling in the x^3 direction ;that is,

$$k^{\mu} = (\omega, 0, 0, k^3) = (\omega, 0, 0, \omega) \qquad (3.1.26)$$

Where $k^3 = \omega$ because the wave vector is null.In this case,

$$k^\mu \varepsilon_{\mu\nu} = 0 \quad \text{and} \quad \varepsilon_{o\nu} = 0$$

together imply

$$\varepsilon_{3\nu} = 0 \tag{3.1.27}$$

The only non-zero component of $\varepsilon_{\mu\nu}$ are therefore $\epsilon_{11}, \epsilon_{12}, \epsilon_{21}and\epsilon_{22}$. But $\epsilon_{\mu\nu}$ is traceless and symmetric ,so we can write it in matrix form as

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_{11} & \epsilon_{12} & 0 \\ 0 & \epsilon_{12} & -\epsilon_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus ,for a plane wave in this gauge travelling in the $\hat{x}^3$ direction, the two components ϵ_{11}andϵ_{12} completely characterize the wave.Ofcourse ,we have been working with trace-reversed perturbation $\bar{h}_{\mu\nu}$ rather than the perturbation $h_{\mu\nu}$ it self ; but since $\bar{h}_{\mu\nu}$ (because $\epsilon_{\mu\nu}$ is),and is equal to the trace-reverse of $h_{\mu\nu}$,in trace condition Eq(3.1.18) we have

$$\bar{h}^{TT}_{\mu\nu} = h^{TT}_{\mu\nu} \tag{3.1.28}$$

So we can drop the bars over $h_{\mu\nu}$ as long as we are in this gauge.It follows that $h_{\mu\nu}$ comprises two degree of freedom associated with the two polarization states of gravitational radiation:

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.1.29}$$

or

$$h_{\mu\nu} = h_+ \epsilon^+_{\mu\nu} + h_\times \epsilon^\times_{\mu\nu} \tag{3.1.30}$$

in terms of the two polarization tensors $\epsilon^{+}_{\mu\nu}$ and $\epsilon^{\times}_{\mu\nu}$,show for propagation along the z-direction.

3.2 Generation of Gravitational Wave

In section 3.1, we obtained the plane wave solution to linearized Einstein's field equations.In this section we discuss the generation of gravitational radiation by sources.For this purpose it is necessary to consider the equation coupled with matter,

$$\Box\bar{h}_{\mu\nu} = -16\pi G S_{\mu\nu} \tag{3.2.1}$$

Here we will make some simplifying ,but realistic ,assumptions.The assumptions are

• the time dependent part of $S_{\mu\nu}(\overrightarrow{x}, t)$ is in sinusoidal oscillation with frequency ω ,that is the real part of

$$S_{\mu\nu}(\overrightarrow{x}, t) = \int s^{\omega}_{\mu\nu}(x) e^{-i\omega t} d\omega \tag{3.2.2}$$

and that the region of space in which $S_{\mu\nu} \neq 0$ is small compared with the wavelength $\left(\frac{2\pi}{\omega}\right)$ of gravitational wave of frequency.This assumption is not much of restriction,since a general time dependence can be reduced to sum over sinusoidal motion by Fourier analysis.In addition ,many astrophysical sources are roughly periodic;pulsating stars,pulsars ,binary system.

• the typical velocity inside the source region should be much less than one ($\Omega\times$size of the region).That is slow motion assumption.We now consider the gravitational radiation emitted by an isolated far away source .The fourier transformation of the metric perturbation $\bar{h}_{\mu\nu}$ is

$$\tilde{\bar{h}}_{\mu\nu}(\omega, \overrightarrow{x}) \;\; = \;\; \frac{1}{\sqrt{2\pi}} \int dt exp(i\omega t) \bar{h}_{\mu\nu}(t, \vec{x}) = 0$$

or its inverse

$$\bar{h}_{\mu\nu}(t, \overrightarrow{x}) = \frac{1}{\sqrt{2\pi}} \int d\omega exp(-i\omega t)\tilde{\bar{h}}_(\omega, \vec{x}) \tag{3.2.3}$$

Then plugging Eq.(3.2.2) and the second equation of Eq.(3.2.3)in to Eq.(3.2.1) the wave equation take the form :

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\left[\frac{1}{\sqrt{2\pi}} \int d\omega exp(-i\omega t)\tilde{\bar{h}}(\omega, \vec{x})\right] = -16\pi G_o \int \bar{T}(\omega, \vec{x}) exp(-i\omega t) d\omega$$

$$(\nabla^2 + \omega^2)\tilde{\bar{h}}(\omega, \vec{x}) = -16\pi G_o \bar{T}_{\mu\nu}(\omega, \vec{x}) \tag{3.2.4}$$

If the sources is considered as a superposition of point sources at $\vec{x}'$,then each source potential $G(\vec{x}, \vec{x}')$ should satisfy the relation:

$$(\nabla^2 + \omega^2)G(\vec{x}, \vec{x}') = -16\pi G_o \delta|\vec{x} - \vec{x}'| \tag{3.2.5}$$

where $\delta|\vec{x} - \vec{x}'|$ is the delta function source.Multipying Eq.(3.2.5)by $\tilde{T}(\omega, \vec{x}')$ and integrating it to get;

$$(\nabla^2 + \omega^2) \int \bar{T}_{\mu\nu}(\omega, \vec{x}) G(\vec{x}, \vec{x}') d^3x' = -16\pi G_o \int \bar{T}_{\mu o}(\omega, \vec{x}')\delta|\vec{x} - \vec{x'}| d^3x' \tag{3.2.6}$$

Comparing this with the left hand left hand side of the Eq.(3.2.4) we get,

$$\tilde{\bar{h}}_{\mu\nu}(\omega, \vec{x}) = \int \bar{T}_{\mu\nu}(\vec{x'}) d^3x' \tag{3.2.7}$$

To find the solution $\tilde{\bar{h}}_{\mu\nu}(\omega, \vec{x})$, first let us determine $G(\vec{x}, \vec{x'})$. Let r be the spherical polar radial coordinate whose origin is chosen inside the source.So ,far points out side the source ,we,have

$$\frac{1}{r^2}\frac{d^2}{dr^2}(rG) + \omega^2 G = 0$$

Integrating this gives :

$$G = \frac{A}{r} e^{\pm i\omega r} \tag{3.2.8}$$

Thus ,combining these results follow:

$$\tilde{\bar{h}}_{\mu\nu}(\omega,\vec{x}) = A\int \frac{\bar{T}_{\mu\nu}(\omega,\vec{x}')}{r(\vec{x},\vec{x}')} e^{\pm i\omega r(\vec{x},\vec{x}')} d^3\vec{x}' \tag{3.2.9}$$

or

$$\bar{h}_{\mu\nu}(t,\vec{x}) = A\int\int \frac{\bar{T}_{\mu\nu}(\omega,\vec{x}')}{r(\vec{x},\vec{x}')} e^{-i\omega t} e^{\pm i\omega r(\vec{x},\vec{x}')} d\omega d^3x' \tag{3.2.10}$$

Next to determine the value of the constant A ,taking (3.2.4) the singularity point $r = 0$,we find

$$A\nabla^2(\frac{1}{r}) + A\omega^2\left(\frac{1}{r}\right) \simeq -16\Pi G_o \delta|\vec{x} - \vec{x}'|$$

Integrating this over $\vec{x}'$ gives:

$$\begin{aligned}
A\int \nabla^2(\frac{1}{r}) d^3\vec{x'} + A\omega^2 \int \frac{d^3\vec{x}'}{r} &= -16\Pi G_o \int \delta|\vec{x} - \vec{x'}| d^3\vec{x'} \\
A\int \nabla(\frac{1}{r}) \cdot d^2\vec{x'} + A\omega^2 \int \frac{d^3\vec{x}'}{r} &= -16\Pi G_o \\
-A\int \frac{\vec{r}.d^2\vec{x}'}{r^3} + A\omega^2 \int \frac{d^3\vec{x}'}{r} &= -16\Pi G_o
\end{aligned}$$

Neglecting terms of order r^{-1} term and higher order ,this becomes

$$-A\int d\Omega = -16\Pi G_o$$

$$4\Pi A = 16\Pi G_o$$

$$A = 4G_o \tag{3.2.11}$$

Plugging (3.2.11) in to (3.2.10),we obtain

$$\bar{h}(\vec{x},t) = 4G_o \int\int \frac{\bar{T}_{\mu\nu}(\omega,\vec{x'})e^{-i\omega(t\mp r)}}{r(\vec{x},\vec{x'})} d\omega d^3x' \tag{3.2.12}$$

But

$$\int \bar{T}_{\mu\nu}(\omega,\vec{x}')e^{-i\omega(t\mp r)} d\omega = \int \tilde{T}_{\mu\nu}(\omega,\vec{x}')e^{-i\omega t}$$

$$\int \tilde{T}_{\mu\nu}(\omega, \vec{x}')e^{-i\omega(t\mp r)}d\omega = \int \tilde{T}_{\mu\nu}(\vec{x}', t \mp r(\vec{x}, \vec{x}'))$$

where $t' = t \mp r$

Thus Eq.(3.2.12)leads to

$$\bar{h}_{\mu\nu}(\vec{x}, t) = 4G_o \int \frac{T_{\mu\nu}(\vec{x}', t \mp r(\vec{x}, \vec{x}'))}{|\vec{x} - \vec{x}'|} d^3\vec{x}' \tag{3.2.13}$$

There fore the time retarded solution is

$$\bar{h}_{\mu\nu}(\vec{x}, t) = 4G_o \int \frac{T_{\mu\nu}(\vec{x}', t - |\vec{x}, \vec{x}')|}{|\vec{x} - \vec{x}'|} d^3\vec{x}' \tag{3.2.14}$$

This is the *expression for the gravitational waves generated by the source.*From the expression(3.2.14) ,we observe that the disturbance in the gravitational field at $(t, \vec{x})$ is the sum of the influences from the energy and momentum sources at the point $(t_r, \vec{x}')$ on the past light cone . As for plane waves we studied in sec .3.1 we have here the freedom to make further restriction of the gauge ,so that in the **TT** gauge we have the simplest form of the wave. Therefore using (3.1.28) (because $\bar{h}_{jk}$ and h_{jk} differ only in the trace, they have the same TT parts)(3.2.14)becomes:

$$h_{jk}^{TT} = \left[4G_o \int \frac{d^3\vec{x}' T_{jk}(\vec{x}', t - |\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|}\right]^{TT} \tag{3.2.15}$$

which may be approximately written as

$$h_{jk}^{TT} = 4G_o \left[\frac{1}{r} \int d^3\vec{x}' T_{jk}(\vec{x}', t - r)\right]^{TT} \tag{3.2.16}$$

Expanding this to

$$\begin{aligned} h_{jk}^{TT} &= 4\frac{G_o}{r}(\int d^3\vec{x}[\frac{1}{2}T^{oo}_{,oo}(x', t - r)x'_j x'_k \\ &+ [x'_j T_{kl}(\vec{x}', t - r) + x'_k T_{jl}(x', t - r)]_{,l} \\ &+ [x_{kj} T^{lm}(\vec{x}', t - r)]_{,lm})^{TT} \end{aligned} \tag{3.2.17}$$

But

$$\bullet(x^i T^{jk}),k = \delta^i_k T^{jk} + x^i T^{jk}_{,k} = T^{ij} - x^i T^{jo}_{,o} \tag{3.2.18}$$

$$\begin{aligned}
\bullet(x^i x^j T^{kl})_{,lk} &= [(\delta^i_l x^j + x^i \delta^j_l)T^{kl} + x^i x^j T^{kl}_{,l}]_{,k} \\
&= [T^{ki} x^j + T^{kj} x^i - x^i x^j T^{ko}_{,o}],k \\
&= x^i T^{ki}_k + \delta^j_k T^{ki} + T^{ki}_{,k} x^i + T^{kj}\delta^i_k \\
&- \delta^i_k x^j T^{ko}_{,o} - x^i \delta^j_k T^{ko}_{,o} - x^i x^j T^{ko}_{,ok} \\
&= -x^j T^{io}_{,o} + T^{ij} - x^i T^{jo}_{,o} + T^{ij} \\
&- x^j T^{io}_{,o} - x^i T^{jo}_{,o} + x^i x^j T^{oo}_{,oo}
\end{aligned} \tag{3.2.19}$$

Plugging (3.2.7) in to this equation gives

$$\begin{aligned}
(x^i x^j T^{kl})_{,lk} &= 2T^{ij} - 2(x^i T^{jo}_{,o} + x^j T^{io}_{,o}) + x^i x^j T^{oo} \\
(x^i x^j T^{kl})_{,lk} &= -2T^{ij} - (x^i T^{jk} + x^j T^{ik})_{,k} + x^i x^j T^{oo}_{,oo}
\end{aligned}$$

Rearranging this gives :

$$(x^i x^j T^{oo}_{,oo}) = 2T^{ij} - 2(x^i T^{jk} + x^j T^{ik})_{,k} + (x^i x^j T^{kl})_{,l} \tag{3.2.20}$$

and the last two terms in (3.2.17) vanishes so it becomes

$$\begin{aligned}
h^{TT}_{jk} &= \frac{2}{r}\left[d^3\vec{x}' T^{oo}_{,oo}(\vec{x}, t-r) x'_j x'_k\right]^{TT} \\
h^{TT}_{jk} &= \frac{2}{r}\frac{d^2}{dt^2}\left[d^3\vec{x}' T^{oo}(\vec{x'}, t-r) x'_j x'_k\right]^{TT}
\end{aligned}$$

$$h^{TT}_{jk} = \frac{2}{r}\left[\ddot{I}_{jk}(t-r)\right]^{TT} \tag{3.2.21}$$

Where

$$I_{jk} = \int d^3\vec{x}' T^{oo} x'_j x'_k \tag{3.2.22}$$

is referred to as the moment of mass distribution and dots represent the time derivatives.It will in fact ,be more convenient to work in terms of the trace-free or reduced quadrupole moment tensor of the source distribution which is defined by

$$\mp = \int d^3\vec{x}' T^{oo}(t-r)(x'_j x'_k - \frac{1}{3}\delta_{jk} x'^2) \tag{3.2.23}$$

or

$$\mp = I_{jk} - \frac{1}{3}\delta_{jk} I \tag{3.2.24}$$

Where $I = I^j_j$ is the trace of the original tensor.One immediately see that $\mp$ is simply the traceless version of I_{jk}. As a result we may write the transverse - traceless gravitational field tensor or the gravitational wave amplitude as

$$h^{TT}_{jk} = \bar{h}^{TT}_{jk} = \frac{2}{r}\left[\ddot{\mp}(t-r)\right]^{TT} \tag{3.2.25}$$

Thus the gravitational wave produced by an isolated non relativistic source is proportional to the second derivative of the quadrupole moment of the matter density distribution .By contrast, the leading contribution to electromagnetic radiation comes from the changing dipole moment of the charge density.

3.3 The Energy of Gravitational Wave

3.3.1 Average Energy -Momentum Tensor

Let us consider Einstein's field equations (in vacuum) to second order ,and see how the result can be interpreted in terms of energy momentum tensor for the gravitational field.If we write the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

and the Ricci tensor that is linear in $h_{\mu\nu}$ is given by

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left[h^{\lambda}_{\lambda,\mu k} - h^{\lambda}_{\mu,\lambda k} - h^{\lambda}_{k,\lambda\mu} + h_{\mu k,\lambda\lambda}\right] \tag{3.3.1}$$

So the exact Einstein equations in this approximation can be rewritten as

$$R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)}_{\lambda\lambda} = -8\Pi G(T_{\mu\nu} + t_{\mu\nu}) \tag{3.3.2}$$

where

$$t_{\mu\nu} = \left(\frac{1}{8\Pi G}\right)\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^{\lambda}_{\lambda} - R^{(1)}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}R^{(1)\lambda}_{\lambda}\right] \tag{3.3.3}$$

and the tensor $t_{\mu\nu}$ is the energy momentum tensor of gravitational field (in weak field).Agian $t_{\mu\nu}$ is conserved in vacuum:

$$t^{\mu\nu}_{,\mu} = 0 \tag{3.3.4}$$

We can compute $t_{\mu\nu}$ as a power of in h and find that the highest term is quadratic :

$$t_{\mu\nu} \cong \frac{1}{8\Pi G}\left[\frac{1}{2}h_{\mu\nu}R^{(1)}_{\lambda\lambda} + \frac{1}{2}\eta_{\mu\nu}h^{\rho\alpha}R^{(1)}_{\rho\alpha} + R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{\rho\alpha}R^{(2)}_{\rho\alpha}\right] \tag{3.3.5}$$

where

$$\begin{aligned} R^{(2)}_{\mu\nu} &= \frac{1}{2}\left(h_{\mu\lambda,\nu k} - h_{k\lambda,\mu\nu} - h_{\mu\nu,k\lambda} + h_{\mu k,\lambda\nu}\right) \\ &+ \frac{1}{4}\left(2h^{\nu}_{\alpha,\nu} - h^{\nu}_{\nu,\alpha}\right)\left(h^{\alpha}_{\mu,k} - h_{\mu k,\alpha}\right) \\ &- \frac{1}{2}\left(h_{\alpha\lambda,k} + h_{\alpha k,\lambda} - h_{k\lambda,\alpha}\right)\left(h^{\alpha}_{\mu,\lambda} + h^{\alpha\alpha}_{,\mu} - h^{\alpha}_{\mu,\alpha}\right) \end{aligned} \tag{3.3.6}$$

Then dropping terms having $R^{(1)}_{\mu\nu}$ (since $R^{(1)}_{\mu\nu} = 0$ in vacuum) (3.3.5) become:

$$t_{\mu\nu} \cong \frac{1}{8\Pi G}\left[R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{\rho\alpha}R^{(2)}_{\rho\alpha}\right] \tag{3.3.7}$$

Plugging the plane wave solution in to $R^{(2)}_{\mu\nu}$ to calculate :

$$\begin{aligned} \langle R^{(2)}_{\mu\nu}\rangle &= Re(\varepsilon^{\lambda\rho *}(k_\mu k_\lambda \varepsilon_{\nu\rho} - k_\lambda k_\rho \varepsilon_{\mu\nu})) \\ &+ (k_\lambda \varepsilon^{\lambda}_{\rho})^{\star}(k_\mu \varepsilon^{\rho}_{\nu} + k_\nu \varepsilon^{\rho}_{\mu} - k^{\rho}\varepsilon_{\mu\nu}) \\ &- \frac{1}{2}(k_\lambda \varepsilon_{\nu\rho} + k_\nu \varepsilon_{\rho\lambda} - k_\rho \varepsilon_{\lambda\nu})^{\star}(k^{\lambda}\varepsilon^{\rho}_{\mu} + k_\mu \varepsilon^{\rho\lambda} - k^{\rho}\varepsilon^{\lambda}_{\mu}) \end{aligned}$$

Using the harmonic condition ,this will be reduced to

$$\langle R^{(2)}_{\mu\nu}\rangle = \frac{k_\nu k_\nu}{2}\left(\varepsilon^{\lambda\rho*} - \frac{1}{2}|\varepsilon^{\lambda}_{\lambda}|^2\right) \tag{3.3.8}$$

But

$$\eta^{\mu\nu}\langle R^{(2)}_{\mu\nu}\rangle = k^\nu k_\nu\left(\varepsilon^{\lambda\rho*} - \frac{1}{2}|\varepsilon^{\lambda}_{\lambda}|^2\right) \tag{3.3.9}$$

Since $k^\nu k_\nu = 0$.Therefore , the average energy momentum tensor of the plane wave will be

$$\langle t_{\mu\nu}\rangle = \frac{1}{8\Pi G}\langle R^{(2)}_{\mu\nu}\rangle$$

or

$$\langle t_{\mu\nu}\rangle = \frac{k_\mu k_\nu}{16\Pi G}\left(\varepsilon^{\lambda\rho*} - \frac{1}{2}|\varepsilon^{\lambda}_{\lambda}|^2\right) \tag{3.3.10}$$

in particular for the wave travelling along z-axis ,$\hat{x}_3$ the average energy-momentum tensor takes the form

$$\langle t_{\mu\nu}\rangle = \frac{k_\mu k_\nu}{8\Pi G}\left[|\varepsilon_{11}|^2 + |\varepsilon_{12}|^2\right] \tag{3.3.11}$$

3.3.2 Derivation of Quadrupole Formula

Having the average energy- momentum tensor of a gravitational wave,in sec 3.3.1,an explicit calculation on far from the source gives[29]

$$t_{\mu\nu} = \frac{c^2}{32\Pi G}\langle\partial_\mu h_{\alpha\beta}\partial_\nu h_{\alpha\beta}\rangle \tag{3.3.12}$$

Therefore,for a plane wave ,using TT gauge ,the energy density in gravitational wave is

$$t_{oo} = \frac{c^2}{32\Pi G}\langle h^{TT}_{jk,o} h^{TT}_{jk,o}\rangle$$

$$t_{oo} = \frac{c^2}{32\Pi G}\langle \dot{h}^{TT}_{jk}\dot{h}^{TT}_{jk}\rangle \tag{3.3.13}$$

or

$$t_{oo} = \frac{c^2}{16\Pi G}\langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle \tag{3.3.14}$$

To calculate the rate at which the physical system loses energy,we equate it to the energy flux of the emitted gravitational radiation evaluated on a sphere of large radius r centered at the origin .Thus ,if E is the energy of the physical system ,we have

$$\frac{dE}{dt} = -r^2 \int t^{or}(\vec{e}_r) d\Omega \tag{3.3.15}$$

Where t^{or} is the gravitational wave energy flux at a radius r in the radial direction $\hat{e}_r$ and $d\Omega$ is an element solid angle.

In general ,using (3.3.12) we may write the gravitational energy flux in a unit spatial direction $\vec{n}$ as

$$\begin{aligned} t^{oi}(\vec{n}) &= t^{oi} n_i \\ &= \frac{c^4}{32\Pi G}\langle \partial_t h_{jk}^{TT} \partial_i h_{jk}^{TT} \rangle n_i \end{aligned}$$

$$t^{oi}(\vec{n}) = \frac{c^4}{32\Pi G}\langle \partial_t h_{jk}^{TT} (\hat{n}\cdot\vec{\nabla}) h_{jk}^{TT} \rangle \tag{3.3.16}$$

where $\partial_t = \frac{\partial}{\partial t}$ Thus taking $\hat{n}$ to lie in the radial direction and writing $\partial_r = \frac{\partial}{\partial r}$,we have

$$t^{oi}(\hat{e}_r) = \frac{c^4}{32\Pi G}\langle \partial_t h_{jk}^{TT} \partial_r h_{jk}^{TT} \rangle \tag{3.3.17}$$

Using the expression in (3.2.24),the derivations in (3.3.17) for the energy flux are given by

$$\partial_t h_{jk}^{TT} = \frac{2G}{c^5 r}\left[\dddot{\mathcal{I}}_{jk}(t-r)\right]^{TT}, \tag{3.3.18}$$

$$\partial_r h_{jk}^{TT} = \frac{2G}{c^5 r^2}\left[\ddot{\mathcal{I}}_{jk}(t-r)\right]^{TT} + \frac{2G}{c^5 r}\left[\dddot{\mathcal{I}}_{jk}(t-r)\right]^{TT}$$

$$\partial_r h_{jk}^{TT} \simeq \frac{2G}{c^5 r}\left[\ddddot{\mathcal{I}}_{jk}(t-r)\right]^{TT} \tag{3.3.19}$$

where ,in the second equation ,we have retained only the term in $1/r$ which dominates for large r . Plugging these expression in to (3.3.17), we obtain

$$t^{oi}(\hat{e}_r) = \frac{G}{8\Pi r^2 c^5}\langle \dddot{\mp}^{TT}_{jk}(t-r)\dddot{\mp}^{TT}_{jk}(t-r)\rangle \tag{3.3.20}$$

Since at any point on the sphere the direction of the gravitational wave propagation is radial, the transverse-traceless,$\dddot{\mp}^{TT}_{jk}$ is related to $\mp_{ab}$ as

$$\mp^{TT}_{jk} = \left(p^a_j p^b_k - \frac{1}{2}p_{jk}p_{ab}\right)\mp_{ab} \tag{3.3.21}$$

where

$$p^{jk} = \delta^{jk} - \hat{e}^j_r \hat{e}^k_r$$

is the *spatial projection tensor* which projects tensor components on to the spatial surface orthogonal to the radial direction at any point .For convenience we now use (3.3.21) to rewrite the product of transverse traceless quadrapole moment in terms of reduced unit radial vector by x_j:

$$\begin{aligned}
\dddot{\mp}^{TT}_{jk}\dddot{\mp}^{TT}_{jk} &= [p_{ja}\dddot{\mp}_{ab}p_{bk} - \frac{1}{2}p_{jk}\dddot{\mp}_{ab}p_{ab}][p_{jc}\dddot{\mp}_{cd}p_{dk} - \frac{1}{2}p_{jk}\dddot{\mp}_{cd}p_{cd}] \\
&= p_{ac}p_{bd}\dddot{\mp}_{ab}\dddot{\mp}_{cd} - \frac{1}{2}p_{ab}\dddot{\mp}_{ab}p_{cd}\dddot{\mp}_{cd}\frac{1}{4}p_{jk}p_{jk}\dddot{\mp}_{ab}p_{ab}\dddot{\mp}_{cd} \\
&= p_{ab}p_{bd}\dddot{\mp}_{ab}\dddot{\mp}_{cd} - \frac{1}{2}p_{ab}\dddot{\mp}_{cd}p_{cd}\dddot{\mp}_{cd}
\end{aligned}$$

Since the last two terms cancels ,it becomes

$$\dddot{\mp}^{TT}_{jk}\dddot{\mp}^{TT}_{jk} = (\delta_{ac} - x_a x_c)(\delta_{bd} - x_{bd})\dddot{\mp}_{ab}\dddot{\mp}_{cd} - \frac{1}{2}\left[x_a x_b \dddot{\mp}_{ab}\right]^2 \tag{3.3.22}$$

But

$$(\delta_{ab} - x_a x_b)\dddot{\mp}_{ab} \equiv (\delta_{cd} - x_c x_d)\dddot{\mp}_{cd} \tag{3.3.23}$$

for gauge moving along $\hat{e}_3$.Therefore

$$\dddot{\mp}^{TT}_{jk}\dddot{\mp}^{TT}_{jk} = \dddot{\mp}_{ab}\dddot{\mp}_{ab} - 2x_a x_c \dddot{\mp}_{ab}\dddot{\mp}_{cd} + \frac{1}{2}(x_a \dddot{\mp}_(ab)x_b) \tag{3.3.24}$$

Thus,using (3.3.20), we have

$$r^2 t^{or} = \frac{1}{8\Pi}\frac{G}{c^5}\langle \dddot{\mp}_{jk}\dddot{\mp}_{jk} - 2x_i\dddot{\mp}_{ij}\dddot{\mp}_{jk}x_k + \frac{1}{2}x_j x_k \dddot{\mp}_{jk} x_l x_m \dddot{\mp}_{lm}\rangle \tag{3.3.25}$$

and the total power radiated become

$$\frac{dE}{dt} = -r^2 \int t^{or} d\Omega$$

$$\frac{dE}{dt} = -\frac{1}{8\Pi}\frac{G}{c^5}\int \langle \dddot{\mp}_{jk}\dddot{\mp}_{jk} - 2x_i\dddot{\mp}_{ij}\dddot{\mp}_{jk}x_k + \frac{1}{2}x_j x_k \dddot{\mp}_{jk} x_l x_m \dddot{\mp}_{lm}\rangle \tag{3.3.26}$$

Since the reduced quadrupole moments $\mp$ is defined as an integral over all space ,it does depend on the angular coordinate and so may be taken outside the integral .The remaining angular integrals are easily evaluated as follows.

$$\int d\Omega = 4\Pi$$

$$A_{jk} = \int d\Omega x_j x_k = A\delta_{ij}$$

and

$$B_{jklm} = \int d\Omega x_j x_k x_l x_m = B(\delta_{jk}\delta_{lm} + \delta_{jm}\delta_{kl} + \delta_{jl}\delta_{km})$$

where A and B are mutual coefficients.To evaluate the last two integrals we proceed to contract all indices in the expressions.That is

$$A^j_j = \int d\Omega = 3A$$

The result is $A = \frac{4\Pi}{3}$ In the same way find $B = \frac{4\Pi}{15}$

Now

$$\int d\Omega x_j x_k = \frac{4\Pi}{3}\delta_{jk} \tag{3.3.27}$$

$$\int d\Omega x_j x_k x_l x_m = \frac{4\Pi}{15}(\delta_{jk}\delta_{lm} + \delta_{jm}\delta_{kl} + \delta_{jl}\delta_{km}) \tag{3.3.28}$$

Using these equations and (3.2.24) becomes

$$\begin{aligned}\frac{dE}{dt} &= -\frac{4\Pi}{8\Pi}\frac{G}{c^5}[\langle \dddot{\text{Ɨ}}_{jk}\dddot{\text{Ɨ}}_{jk}\rangle - \frac{2}{3}\langle \dddot{\text{Ɨ}}_{jk}\dddot{\text{Ɨ}}_{jk} \\ &+ \frac{1}{30}\langle \dddot{\text{Ɨ}}_{jk}\dddot{\text{Ɨ}}_{lm}\rangle(\delta_{jk}\delta_{lm} + \delta_{jm}\delta_{kl} + \delta_{jl}\delta_{km})]\end{aligned}$$

$$\frac{dE}{dt} = -\frac{1}{2}\frac{G}{c^5}(1 - \frac{2}{3} + \frac{2}{30}) < \dddot{\text{Ɨ}}_{jk}\dddot{\text{Ɨ}}_{jk} >$$

Then the total power radiated becomes

$$\frac{dE}{dt} = -\frac{1}{5}\frac{G}{c^5} < \dddot{\text{Ɨ}}_{jk}\dddot{\text{Ɨ}}_{jk} > \tag{3.3.29}$$

In the literature this equation is generally denoted as *quadrable formula*

3.3.3 Gravitational Wave Back-Reaction

In the preceding subsection, we have discussed the gravitational wave energy loss in terms of the the radiation that reaches a distant observer. Here it is desirable to model a direct back reaction that the wave have on the sources. It is a common to model the radiation reaction acting on a body of mass m as local force

$$F^{rr} = -m\nabla\phi^{rr} \tag{3.3.30}$$

and the Burke-Thorne radiation reaction potential

$$\phi^{rr} = \frac{1}{5}\frac{G}{c^5}x_i x_j \text{Ɨ}^{(5)}_{jk} \tag{3.3.31}$$

In order to be consistent this formula must lead to the energy loss we predict from the quadrupole formula(4.3.29). Let us consider for a system of N particles , the rate of energy radiated is given as

$$\frac{dE}{dt} = \sum_N \frac{d\vec{x}_j^N}{dt}\vec{F}^{N_{rr}} \tag{3.3.32}$$

Plugging (3.3.30) and (3.3.31) into (3.3.32) gives

$$\frac{dE}{dt} = \sum_N \frac{dx_j^N}{dt} - (m_N \frac{2}{5}\frac{G}{c^5} x_k^N) \mp_{jk}^{(5)}$$

$$\frac{dE}{dt} = -\frac{2}{5}\frac{G}{c^5} \mp_{jk}^{(5)} \sum_N m_N \frac{1}{2}\frac{d}{dt}(x_j^N x_k^N)$$

$$\frac{dE}{dt} = -\frac{1}{5}\frac{G}{c^5} \mp_{jk}^{(5)} \frac{d}{dt}(\sum_N m_N x_j^N x_k^N) \tag{3.3.33}$$

But from (3.2.21) the moment of inertia for the system of particles is given as

$$I_{jk} = \sum_N m_N x_j^N x_k^N \tag{3.3.34}$$

and under 'TT' gauge condition, this is simply given as

$$I_{jk} = \mp_{jk} = \sum_N m_N x_j^N x_k^N \tag{3.3.35}$$

Therefore, using this equation the energy flux becomes

$$\frac{dE}{dt} = -\frac{1}{5}\frac{G}{c^5} \mp_{jk}^{(1)} \mp_{jk}^{(5)} \tag{3.3.36}$$

where $\mp_{jk}^{(1)} = \frac{d}{dt}(\sum_N m_N x_j^N x_k^N)$. At this point we recall that the rate of energy loss we deduced qudrupole formula required averaging over at least one orbit. Carrying out this averaging essentially corresponds to integrating in time over an entire period and we can readily use integration by part as

$$\begin{aligned}
\langle \int (\frac{dE}{dt}) dt \rangle &= -\frac{G}{c^5} \langle \int (\frac{1}{5} \mp_{jk}^{(1)} \mp_{jk}^{(5)}) dt \rangle \\
&= -\frac{G}{c^5} \langle \frac{1}{5} \mp_{jk}^{(1)} \mp_{jk}^{(4)} - \frac{1}{5} \int \frac{d}{dt} [\mp_{jk}^{(1)} \mp_{jk}^{(5)}] dt \rangle \\
&= -\frac{G}{c^5} \langle \frac{1}{5} \mp_{jk}^{(1)} \mp_{jk}^{(4)} - \frac{1}{5} \mp_{jk}^{(2)} \mp_{jk}^{(3)} + \int \mp_{jk}^{(3)} \mp_{jk}^{(3)} \rangle \qquad (3.3.37) \\
&= -\frac{G}{c^5} \langle \int [\mp_{jk}^{(3)} \mp_{jk}^{(3)}] dt \rangle \qquad (3.3.38)
\end{aligned}$$

Since the average of the first two terms are cancels out , the average rate of energy flux is given as

$$\langle\frac{dE}{dt}\rangle = -\frac{1}{5}\frac{G}{c^5}\langle\dddot{\bar{I}}_{jk}\dddot{\bar{I}}_{jk}\rangle \tag{3.3.39}$$

Finally if we account for this formula represents the energy change in the system while our previous result described the energy carried away by the waves we see that the two pictures are consistent. Further let ua show that there is no gravitational waves radiated from monopole and dipole moments. Using the energy momentum conservation law

$$T^{\mu\nu}{}_{;\nu} = T^{\mu\nu}{}_{,\nu} = 0 \tag{3.3.40}$$

This may be rewritten as

$$T^{\mu 0}{}_{,0} + T^{\mu i}{}_{,i} = 0 \tag{3.3.41}$$

which can also be computed as

$$T^{00}{}_{,0} + T^{0i}{}_{,i} = 0 \tag{3.3.42}$$

$$T^{j0}{}_{,0} + T^{ji}{}_{,i} = 0 \tag{3.3.43}$$

Multiplying (3.4.43) x^k, integrate over all space, and neglecting the surface terms on the assumption that $T^{\mu\nu}$ goes to zero sufficiently at infinity, we obtain

$$\begin{aligned} \int d^3x x^k T^{jo}{}_{,0} &= \partial_t \int d^3x x^k T^{jo} \\ &= \int d^3x x^k T^{ji}{}_{,i} \\ &= -\left(x^k T^{ji} - \int d^3x x^k{}_{,i} T^{ji}\right) \end{aligned} \tag{3.3.44}$$

or

$$\int d^3x T^{jk} = \partial_t \int d^3x x^k T^{jo} \tag{3.3.45}$$

where we have used also

$$x^k{}_{,i} = \delta_i^k \tag{3.3.46}$$

Now multiplying (3.4.42) by $x^j x^k$ and integrate in the same manner to get

$$\begin{aligned}\int d^3x x^j x^k T^{00}{}_{,0} &= \partial_t \int d^3x x^j x^k T^{00} \\ &= -\int d^3x x^j x^k T^{oi}{}_{,i} \\ &= -\left[x^j x^k T^{oi} - \int d^3x \left(x^j{}_{,i} x^k + x^j x^k{}_{,i}\right) T^{oi}\right]\end{aligned}$$

or

$$\int d^3x \left(x^k T^{oj} + x^j T^{ok}\right) = \partial_t \int d^3x x^j x^k T^{00} \tag{3.3.47}$$

Since T^{jk} is symmetric in jk we may write (3.4.45) as

$$\int d^3x T^{jk} = \frac{1}{2}\partial_t \int d^3x \left(x^k T^{oj} + x^j T^{ok}\right) \tag{3.3.48}$$

Plugging (3.4.47) into (3.4.48) gives

$$\int d^3x T^{jk} = \frac{1}{2}\partial_t^2 \int d^3x x^j x^k T^{00} \tag{3.3.49}$$

If we multiply (3.4.43) by x^k and integrate, we obtain

$$\begin{aligned}\int d^3x x^k T^{00}{}_{,k} &= \partial_t \int d^3x x^k T^{00} \\ &= -\int d^3x x^k T^{0i}{}_{,i} \\ &= -\left(x^k T60i - \int d^3x x^k{}_{,i} T60i\right)\end{aligned}$$

or

$$\int d^3x T^{0k} = \partial_t \int d^3x x^k T^{00} \tag{3.3.50}$$

Defining:

$$M = \int d^3xT^{00} \tag{3.3.51}$$

$$\mp^k = \int d^3xx^kT^{00} \tag{3.3.52}$$

$$\mp^{jk} = \int d^3xx^jx^kT^{00} \tag{3.3.53}$$

as mass, monopole and dipole moment of the mass-energy distributions,then we have from (3.4.49) and (3.4.50)

$$\int d^3xT^{jk} = \frac{1}{2}\ddot{\mp}^{jk} \tag{3.3.54}$$

and

$$\int d^3xT^{0k} = \dot{\mp}^k \tag{3.3.55}$$

where the dot represents ∂_t.Then from (3.4.51) and (3.4.43) we have

$$\dot{M} = \int d^3xT^{00}{}_{,0} = -\int d^3xT^{0i}{}_{,i} \tag{3.3.56}$$

This can be expressed as surface integral using Gauss law.So with the above assumptions

$$\dot{M} = 0 \tag{3.3.57}$$

Likewise by the same argument

$$\ddot{\mp}^k = \int d^3xT^{0k}{}_{,0} = -\int d^3xT^{ik}{}_{,i} = 0 \tag{3.3.58}$$

This indicates there is no gravitational wave emitted from the monopole and dipole moments. Then from equation (2.4.39) it follows that the total luminosity of the source is given as

$$L_{GW} = \frac{1}{5}\frac{G}{c^5} < \dddot{\mp}_{jk}\dddot{\mp}_{jk} >$$

gravitational waves not only carry away the energy, but also angular momentum which will be derived in the next chapter.

Chapter 4

Lifetime Estimate of Neutron Star Wobble

In this chapter, before solving the problem, we discuss the wobbling of neutron star, derive the gravitational radiation reaction torque using the radiation reaction potential and using the mass quadrupole expression for energy and angular momentum balance.Then we find out the numerical lifetime estimate of a neutron star.

4.1 Wobbling of a Neutron Star

A rotating neutron star, such as pulsar, will emit gravitational waves as a result of small deviations from symmetry around its rotation axis or if it remains axisymmetric so long as it has a time-varying quadrupole moment.A precessing neutron star was first discussed as a source of gravitational radiation by Zimmermann(1978)and Zimmermann and Szedenits(1979), who showed that the mass quadrupole gravitational radiation was produced at frequencies $\dot{\phi}$ and $2\dot{\phi}$[10]. So one class of emission mechanisms for gravitational waves from rotating neutron star is free precession i.e. the wobble of neutron star which has misaligned rotation axis with respect to its symmetry axis[1].Here, we describe the dynamics of a rigid body rotation i.e. the classical problem of of free Eulerian motion (Landau and Lifshitz 1960; Greenwood 1980).

Consider an axisymmetric rigid body with principal axes lie along $\hat{e}_1, \hat{e}_2$ and $\hat{e}_3$ and principal moment of inertia $I_1 = I_2 \neq I_3$. $\mathbf{J}$ is the total angular momentum of the body, misaligned from $\hat{e}_3$, which is fixed in inertial space because no external torques act.fig.1.3

An inertial coordinate system is partly aligned by specifying that the unit vector $\hat{k}$ be parallel to $\mathbf{J}$.The moment of inertia tensor of a rigid body as given in(3.2.22) as,

$$I_{ij} = \int \rho(x_i x_j - 1/3\delta_{ij} r^2) dv \tag{4.1.1}$$

where i,j equal 1,2 or 3 for x,y and z respectively , $r = \mathbf{x}$ and δ_{ij} is the kronecker delta.

The set of body coordinate system x' is related to an inertial system x in terms of Euler angles as

$$x' = Rx \tag{4.1.2}$$

where

$$R(\phi, \theta, \psi) = B(\psi)C(\theta)D(\phi) \tag{4.1.3}$$

and (ϕ, θ, ψ) are Euler angles which describe the orientation of the rigid body (Landau and Lifshitz 1976). The three separate transformations are given in three separate coordinate system as[2]

$$B(\psi) = \begin{pmatrix} cos\psi & sin\psi & 0 \\ -sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.1.4}$$

$$C(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{4.1.5}$$

$$D(\phi) = \begin{pmatrix} cos\phi & sin\phi & 0 \\ -sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.1.6}$$

Therefore, using these matrices, the transformation matrix (4.1.3)can be obtained as

$$R(\phi,\theta,\psi) = \begin{pmatrix} cos\psi & sin\psi & 0 \\ -sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} cos\phi & sin\phi & 0 \\ -sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.1.7}$$

which is equal to

$$R(\phi,\theta,\psi) = \begin{pmatrix} cos\phi cos\psi - sin\phi cos\theta sin\psi & sin\phi cos\psi + cos\phi cos\theta sin\psi & sin\theta sin\psi \\ -cos\phi sin\psi - sin\phi cos\theta cos\psi & -sin\phi sin\psi + cos\phi cos\theta cos\psi & sin\theta cos\psi \\ sin\theta sin\phi & -sin\theta cos\phi & cos\theta \end{pmatrix} \tag{4.1.8}$$

Then the components of the moment of inertia tensor in the inertial coordinate system is given as

$$I_{ij} = R_{ij}^T I R_{ij} \tag{4.1.9}$$

and by spectral theorem, it is possible to find a cartesian system in which it is diagonal, have the form;

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{4.1.10}$$

where the coordinate axes are called the principal axes and the constants I_1, I_2 and I_3 are called moment of inertia.

Thus using (4.1.8) and 4.1.10) the moment of inertia tensor (4.1.9) is explicitly given as

$$\begin{aligned}
I_{11} &= I_1(cos\psi cos\phi - cos\theta sin\phi sin\psi)^2 + I_2(-sin\psi cos\phi - cos\theta sin\phi cos\psi)^2 \\
&+ I_3(sin\theta sin\psi)^2 \\
&= I_1(cos^2\phi cos^2\psi + cos^2\theta sin^2\phi sin^2\psi - 2cos\theta cos\phi sin\phi cos\phi sin\psi \\
&+ sin^2\psi cos^2\phi + cos^2\theta cos^2\psi sin^2\phi + 2cos\theta cos\phi sin\phi cos\phi sin\psi) \\
&+ I_3(sin\theta sin\psi)^2 \\
&= I_1(cos^2\phi cos^2\psi + sin^2\psi cos^2\phi + cos^2\theta sin^2\phi sin^2\psi + cos^2\theta cos^2\psi sin^2\phi) \\
&+ I_3 sin^2\theta sin^2\psi \\
&= I_1(cos^2\phi + cos^2\theta sin^2\phi) + I_3 sin^2\theta sin^2\phi \\
&= I_1(cos^2\phi + I_1(1 - sin^2\theta)sin^2\phi + I_3 sin^2\theta sin^2\phi \\
&= I_1(cos^2\phi + sin^2\phi) - \frac{1}{2}(I_1 - I_3)sin^2\theta(1 - cos2\phi \\
&= \underbrace{I_1 - \frac{1}{2}(I_1 - I_3)sin^2\theta} + \frac{1}{2}(I_1 - I_3)sin^2\theta cos2\phi \\
I_{11} &= \frac{1}{2}(I_1 - I_3)sin^2\theta cos2\phi + constants \qquad (4.1.11)
\end{aligned}$$

$$
\begin{aligned}
I_{12} &= I_1[(cos\phi cos\psi - cos\phi cos\theta sin\psi)(sin\phi cos\psi - cos\phi cos\theta sin\psi) \\
&+ (-cos\phi sin\psi - sin\phi cos\theta cos\psi)(-sin\phi sin\psi \\
&+ -cos\phi cos\theta cos\psi)] + I_3(sin\theta sin\psi)(-sin\theta cos\psi) \\
&= I_1(sin\phi cos\phi cos^2\psi + cos\theta cos^2\phi sin\psi cos\psi - cos\theta sin^2\phi sin\psi cos\psi \qquad (4.1.12) \\
&- cos^2\theta sin\phi cos\phi sin^2\psi + sin\phi cos\phi sin^2\psi - cos\theta cos^2\phi sin\psi cos\psi \\
&+ cos\theta sin^2\phi sin\psi cos\psi - cos^2\theta sin\phi cos\phi cos^2\psi - I_3 sin^2\theta sin\phi cos\phi) \\
&= I_1(sin\phi cos\phi cos^2\psi - cos^2\theta sin\phi cos\phi sin^2\psi + sin\phi cos\phi sin^2\psi \\
&- cos^2\theta sin\phi cos\phi cos^2\psi - I_3 sin^2\theta sin\phi cos\phi \\
&= I_1[sin\phi cos\phi(cos^2\psi + sin^2\psi) - cos^2\theta sin\phi cos\phi(sin^2\psi + cos^2\psi)] - I_3 sin^2\theta sin\phi cos\phi \\
&= I_1 sin\phi cos\phi - I_1 sin\phi cos\phi cos^2\theta - I_3 sin\phi cos\phi sin^2\theta \\
&= \frac{1}{2}I_1 sin2\phi - \frac{1}{2}I_1 sin2\phi(1 - sin^2\theta) - \frac{1}{2}I_3 sin^2\theta sin2\phi \\
&= \frac{1}{2}I_1 sin2\phi - \frac{1}{2}I_1 sin2\phi + \frac{1}{2}I_1 sin^2\theta sin2\phi - \frac{1}{2}I_1 sin^2\theta sin2\phi \\
&= \frac{1}{2}(I_1 - I_3)sin^2\theta sin2\phi \qquad (4.1.13)
\end{aligned}
$$

$$
\begin{aligned}
I_{13} &= I_1[(sin\theta sin\psi)(cos\phi cos\psi - cos\theta sin\phi sin\psi) + (sin\theta cos\psi) \\
&\quad (-cos\phi sin\psi - cos\theta sin\phi cos\psi)] + I_3 sin\phi sin\theta cos\theta \\
&= I_1(sin\theta cos\phi sin\psi cos\psi - sin\theta cos\theta sin\phi sin^2\psi - sin\theta cos\phi \\
&\quad sin\psi cos\psi - sin\theta cos\theta sin\phi cos^2\psi) + I_3 sin\theta cos\theta sin\phi \\
&= -I_1 sin\theta cos\theta sin\phi(sin^2\psi + cos^2\psi) + I_3 sin\theta cos\theta sin\phi \\
&= -I_1 sin\theta cos\theta sin\phi + I_3 sin\theta cos\theta sin\phi \\
&= -(I_1 - I_3)sin\theta cos\theta sin\phi \qquad (4.1.14)
\end{aligned}
$$

$$
\begin{aligned}
I_{22} &= I_1[(sin\phi cos\psi + cos\phi cos\theta sin\psi)^2 + (-sin\phi sin\psi + cos\phi cos\theta cos\psi)^2] + I_3(-sin\theta cos\phi)^2 \\
&= I_1[sin^2\phi cos^2\psi + cos^2\phi cos^2\theta sin^2\psi + sin^2\phi sin^2\psi + cos^2\phi cos^2\theta cos^2\psi \\
&+ 2sin\phi cos\phi cos\theta sin\psi cos\psi - 2sin\phi cos\phi cos\theta sin\psi cos\psi] + I_3 sin^2\theta cos^2\phi \\
&= I_1 sin^2\phi + I_1 cos^2\theta cos^2\phi + I_3 sin^2\theta cos^2\phi \\
&= I_1 sin^2\phi + I_1(1 - sin^2\theta)cos^{\phi} + I_3 sin^2\theta cos^2\phi \\
&= I_1 sin^2\phi + I_1 cos^2\phi - I_1 sin^2\theta cos^2\phi + I_3 sin^2\theta cos^2\phi \\
&= I_1 - (I_1 - I_3)sin^2\theta cos^2\phi \\
&= I_1 - \frac{1}{2}(I_1 - I_3)sin^2\theta(1 + cos2\phi) \\
&= \underbrace{I_1 - \frac{1}{2}(I_1 - I_3)sin^2\theta} - \frac{1}{2}(I_1 - I_3)cos2\phi) \\
&= -\frac{1}{2}(I_1 - I_3)sin^2\theta Cos2\phi + constants \qquad (4.1.15)
\end{aligned}
$$

$$
\begin{aligned}
I_{23} &= I_1[(sin\phi cos\psi + cos\phi cos\theta sin\psi) + (-sin\phi sin\psi\psi + cos\phi cos\theta sin\psi) \\
&\quad (sin\theta cos\phi)] + I_3(-sin\theta cos\phi)cos\theta \\
&= I_1[sin\phi sin\theta sin\psi cos\psi + cos\phi sin\theta cos\theta sin^2\psi \\
&- sin\phi sin\theta sin\psi cos\psi + cos\phi sin\theta cos\theta cos^2\psi] - I_3 sin\theta cos\theta cos\phi \\
&= I_1 sin\theta cos\theta cos\phi - I_3 sin\theta cos\theta cos\phi \\
&= (I_1 - I_3)sin\theta cos\theta cos\phi \qquad (4.1.16)
\end{aligned}
$$

$$
\begin{aligned}
I_{33} &= I_1[(sin\theta sin\psi)^2 + (sin\theta cos\psi)^2] + I_3(cos\theta)^2 \\
&= I_1 sin^2\theta + I_3 cos^2\theta \\
&= I_1 sin^2\theta + I_3(1 - sin^2\theta) \\
&= \underbrace{I_3 + (I_1 - I_3)sin^2\theta} \\
&= constants \qquad (4.1.17)
\end{aligned}
$$

Then the power radiated (3.3.29) can be expressed in terms of the wobble angle θ as

$$\frac{dE}{dt} = -\frac{1}{5}\frac{G}{c^5} < \dddot{\mathcal{I}}_{11}\dddot{\mathcal{I}}_{11} + 2\dddot{\mathcal{I}}_{12}\dddot{\mathcal{I}}_{12} + 2\dddot{\mathcal{I}}_{13}\dddot{\mathcal{I}}_{13} + \dddot{\mathcal{I}}_{22}\dddot{\mathcal{I}}_{22} + 2\dddot{\mathcal{I}}_{23}\dddot{\mathcal{I}}_{23} > \qquad (4.1.18)$$

Using equations (4.1.11)-(4.1.17),(3.2.24) and taking the third time derivatives these equations give:

$$\begin{aligned} \frac{dE}{dt} &= -\frac{1}{5}\frac{G}{c^5} < 16 \bigtriangleup I^2\dot{\phi}^6 sin^4\theta sin^2 2\phi + 32 \bigtriangleup I^2\dot{\phi}^6 sin^4\theta cos^2 2\phi \\ &+ 2 \bigtriangleup I^2\dot{\phi}^6 sin^2\theta cos^2\theta cos^2\phi + 16 \bigtriangleup I^2\dot{\phi}^6 sin^4\theta sin^2 2\phi \\ &+ 2 \bigtriangleup I^2\dot{\phi}^6 sin^2\theta cos^2\theta sin^2\phi > \end{aligned}$$

$$\frac{dE}{dt} = -\frac{1}{5}\frac{G}{c^5} \bigtriangleup I^2\dot{\phi}^6 < 32 sin^4\theta + 2 sin^2\theta cos^{\theta} > \qquad (4.1.19)$$

or

$$\frac{dE}{dt} = -\frac{1}{5}\frac{G}{c^5} \bigtriangleup I^2\dot{\phi}^6 sin^2\theta(16 sin^2\theta - cos^2\theta) \qquad (4.1.20)$$

Now it is standard result from the classical mechanics that the body axis $\hat{e}_3$ precesses around $\vec{J}$ with angular/precession frequency:

$$\vec{\dot{\phi}} = \frac{J\hat{k}}{I_1} \qquad (4.1.21)$$

with $\hat{e}_3$ maintaining a constant angle with respect to $\hat{k}$. In addition, the angular velocity vector precesses about $\hat{e}_3$ with angular frequency:

$$\vec{\dot{\psi}} = \dot{\psi}\hat{e}_3 = \frac{(1 - I_3/I_1)J cos\theta \hat{e}_3}{I_3} \qquad (4.1.22)$$

where θ is the wobble angle and $\dot{\psi}$ is also called the intrinsic spin frequency of the star or body frame precession frequency.

The total angular frequency $\vec{\omega}$ can be written as the sum of the two terms as

$$\vec{\omega} = \vec{\dot{\psi}} + \vec{\dot{\phi}} \qquad (4.1.23)$$

or

$$\vec{\omega} = \dot{\psi}\hat{e}_3 + \dot{\phi}\hat{k} \quad (4.1.24)$$

We write the kinetic energy,E stored in the body as

$$E = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2) \quad (4.1.25)$$

Lastly we obtain

$$E = \frac{J^2(1 - \triangle I/I_3 cos^2\theta)}{2I_1} \quad (4.1.26)$$

where $\triangle I = I_3 - I_1$. When $\triangle I > 0$,the star is said to be oblate and when is $\triangle I > 0$,it is problate.Ofcourse, the oblate case is the more physically plausible.

4.2 Gravitational Radiation reaction Torque

Here we derive the reaction torque due to the emission of gravitational radiation from a wobbling neutron star by adding the Burke-Thorne loccal radiation reaction force to the equation of motion.The gravitational radiation reaction problem was first addressed by[7],using local formulation.To examine how the emission of gravitational radiation leads to a loss of angular momentum, the concept of the gravitational radiation reaction potential Φ^{rr} (Misner et al.,1973) is used.

The Burke-Thorne radiation reaction potential at a point x is given by[6]:

$$\Phi^{rr} = \frac{G}{5c^5} x^j x^k \overline{+}_{jk}^{(5)} \quad (4.2.1)$$

where $\overline{+}_{jk}$ denotes the trace-reduced quadrupole moment tensor and the superscript 5 represents the 5^{th} time derivatives and the cartesian coordinates x_j are centered on spinning mass. Note that $\overline{+}_{jk}$ is related to the moment of inertia tensor, I_{jk} by as referred in section 3.2 as:

$$\overline{+}_{jk} = I_{jk} - \frac{2}{3}\delta_{jk} I \quad (4.2.2)$$

The radiation-reaction force,F^{rr} corresponding the energy loss given in (3.3.27) can be written as a gradient of the Burke-Thorne radiation-reaction potential,Φ^{rr} as

$$F^{rr} = -m\overrightarrow{\nabla}\Phi^{rr} \tag{4.2.3}$$

where $m = \sum_N m_N = \int \rho d^3x$ is the total mass of point particles of the body.

Thus in the absence of any dissipative mechanism other than gravitatiional waves emission, the angular momentum loss rate/the radiation-reaction torque can be calculated as

$$\frac{dJ_i}{dt} = \int \epsilon_{ijk}x_j F_k^{rr} d^3x \tag{4.2.4}$$

where $\epsilon_{ijk}$is the Levi-Civita symbol and F_k^{rr} is the radiation-reaction force on a source is given as

$$F_k^{N_{rr}} = -m\overrightarrow{\nabla}\Phi^{rr} = -m\phi^{rr}{}_{,l} \tag{4.2.5}$$

and

$$\overrightarrow{\nabla}\Phi^{rr} = \overrightarrow{\nabla}(\frac{1G}{5c^5}x_k x_l \text{I}_{kl}^{(5)}) \tag{4.2.6}$$

$$\overrightarrow{\nabla}\Phi^{rr} = \frac{2G}{5c^5}x_l \text{I}_{kl}^{(5)} \tag{4.2.7}$$

Thus using (4.2.6),the total angular momentum loss rate becomes

$$\frac{dJ_i}{dt} = \int \epsilon_{ijk}x_j(-\frac{2G}{5c^5}\rho x_l \text{I}_{kl}^{(5)})d^3x \tag{4.2.8}$$

$$\frac{dJ_i}{dt} = -\frac{2G}{5c^5}\epsilon_{ijk}\text{I}_{kl}^{(5)} \int \rho x_j x_l d^3x \tag{4.2.9}$$

But the quadrupole moment of inertia tensor for a system is given as

$$\text{I}_{jl} = \int \rho[x_j x_l - \frac{1}{3}\delta_{jl}(x)^2]d^3x \tag{4.2.10}$$

and using TT-guage condition, as used in the previous section, this equation becomes

$$\text{I}_{jl} = \int \rho x_j x_l d^3x \tag{4.2.11}$$

Now using (4.2.11)in (4.2.8) it becomes

$$\frac{dJ_i}{dt} = -\frac{2G}{5c^5}\epsilon_{ijk}\bar{I}_{jl}\bar{I}^{(5)}_{kl} \tag{4.2.12}$$

Taking two integrations of this equation by parts, we find:

$$\begin{aligned}
\int(\frac{dJ_i}{dt})dt &= -\frac{2G}{5c^5}\epsilon_{ijk}\int \bar{I}_{jl}\frac{d}{dt}(\frac{d^4}{dt^4}\bar{I}_{kl})dt \\
&= -\frac{2G}{5c^5}\epsilon_{ijk}[\bar{I}_{jl}\frac{d^4}{dt^4}\bar{I}_{kl} - \int(\frac{d}{dt}\bar{I}_{jl})(\frac{d^4}{dt^4}\bar{I}_{kl})dt] \\
&= -\frac{2G}{5c^5}\epsilon_{ijk}[\bar{I}_{jl}\frac{d^4}{dt^4}\bar{I}_{kl} - [\frac{d^2}{dt^2}\bar{I}_{jl}\frac{d^3}{dt^3}\bar{I}_{kl} \\
&- \int(\frac{d^2}{dt^2}\bar{I}_{jl})(\frac{d^3}{dt^3}\bar{I}_{kl})dt]]
\end{aligned} \tag{4.2.13}$$

After averaging (4.2.13) over several periods,since the first two terms on the right hand sides cancel each other,the gravitational radiation-reaction torque become

$$\begin{aligned}
\langle\int(\frac{dJ_i}{dt})dt\rangle &= \langle -\frac{2G}{5c^5}\epsilon_{ijk}\int(\frac{d^2}{dt^2}\bar{I}_{jl})(\frac{d^3}{dt^3}\bar{I}_{kl})dt]\rangle \\
&= -\frac{2G}{5c^5}\epsilon_{ijk}\langle\int(\frac{d^2}{dt^2}\bar{I}_{jl})(\frac{d^3}{dt^3}\bar{I}_{kl})dt\rangle
\end{aligned}$$

and lastly dropping the integration

$$\begin{aligned}
\frac{dJ_i}{dt} &= -\frac{2G}{5c^5}\epsilon_{ijk}\langle(\ddot{\bar{I}}_{jl})(\dddot{\bar{I}}_{kl})\rangle \\
\frac{dJ_i}{dt} &= -\frac{2G}{5c^5}\epsilon_{ijk}\langle\ddot{\bar{I}}_{jl}\dddot{\bar{I}}_{kl}\rangle
\end{aligned} \tag{4.2.14}$$

where the dots indicates the time derivatives. Now, it is necessary to calculate the radiation-reaction torque, (4.2.14) for a precessing neutron star in terms of the wobble angle,θ.Hence we find

$$\begin{aligned}
\frac{dJ}{dt} &= -\frac{2G}{5c^5}[\epsilon_{312}\langle\ddot{\pm}_{1l}\dddot{\pm}_{2l}\rangle + \epsilon_{321}\langle\ddot{\pm}_{2l}\dddot{\pm}_{1l}\rangle] \\
&= -\frac{2G}{5c^5}[\langle\ddot{\pm}_{1l}\dddot{\pm}_{2l}\rangle - \langle\ddot{\pm}_{2l}\dddot{\pm}_{1l}\rangle] \\
&= -\frac{2G}{5c^5}[\langle\ddot{\pm}_{11}\dddot{\pm}_{21}\rangle - \langle\ddot{\pm}_{21}\dddot{\pm}_{11}\rangle + \langle\ddot{\pm}_{12}\dddot{\pm}_{22}\rangle \\
&- \langle\ddot{\pm}_{22}\dddot{\pm}_{12}\rangle + \langle\ddot{\pm}_{13}\dddot{\pm}_{23}\rangle - \langle\ddot{\pm}_{23}\dddot{\pm}_{13}\rangle]
\end{aligned} \tag{4.2.15}$$

By making use of equations (4.1.11)-(4.1.17) and finding the second and third time derivatives of the corresponding moment of inertia components, we can find each term in equation (4.2.15) as

$$\begin{aligned}
\ddot{\pm}_{1l}\dddot{\pm}_{2l} &= \ddot{\pm}_{11}\dddot{\pm}_{21} \\
&= (-2\dot{\phi}^2 \triangle Isin^2\theta cos2\phi)(-4\phi^3) \triangle Isin^2\theta cos2\phi) \\
&= 8\phi^5 \triangle I^2 sin^4\theta cos^2 2\phi
\end{aligned}$$

In the same way:

$$\begin{aligned}
\ddot{\pm}_{21}\dddot{\pm}_{11} &= -8\dot{\phi}^5 \triangle I^2 sin^4\theta sin^2 2\phi \\
\ddot{\pm}_{12}\dddot{\pm}_{22} &= 8\dot{\phi}^5 \triangle I^2 sin^4\theta sin^2 2\phi \\
\ddot{\pm}_{22}\dddot{\pm}_{12} &= -8\dot{\phi}^5 \triangle I^2 sin^4\theta cos^2 2\phi \\
\ddot{\pm}_{13}\dddot{\pm}_{23} &= \dot{\phi}^5 \triangle I^2 sin^2\theta cos^2\theta sin^2\phi \\
\ddot{\pm}_{23}\dddot{\pm}_{13} &= -\dot{\phi}^5 \triangle I^2 sin^2\theta cos^2\theta cos^2\phi
\end{aligned}$$

Plugging these expressions into (4.2.15),we obtain

$$\begin{aligned}
\frac{dJ}{dt} &= -\frac{2G}{5c^5}[\langle 8\dot{\phi}^5 \vartriangle I^2 sin^4\theta cos^2 2\phi\rangle - \langle -8\dot{\phi}^5 \vartriangle I^2 sin^4\theta sin^2 2\phi\rangle \\
&\langle 8\dot{\phi}^5 \vartriangle I^2 sin^4\theta sin^2 2\phi\rangle - \langle -8\dot{\phi}^5 \vartriangle I^2 sin^4\theta cos^2 2\phi\rangle \\
&\langle \dot{\phi}^5 \vartriangle I^2 sin^2\theta cos^2\theta sin^2\phi\rangle - \langle -\dot{\phi}^5 \vartriangle I^2 sin^2\theta cos^2\theta cos^2\phi\rangle] \\
&= -\frac{2G}{5c^5}[\langle 8\dot{\phi}^5 \vartriangle I^2 sin^4\theta\rangle(\langle cos^2 2\phi\rangle + \langle sin^2 2\phi\rangle) + \langle 8\dot{\phi}^5 \vartriangle I^2 sin^4\theta\rangle \\
&(\langle sin^2 2\phi\rangle + \langle cos^2 2\phi\rangle) + \langle \vartriangle I^2 sin^2\theta cos^\theta\rangle(\langle sin^2\theta\rangle + \langle cos^2\theta\rangle)] \\
&= -\frac{2G}{5c^5}[16\dot{\phi}^5 \vartriangle I^2\langle sin^4\theta\rangle + \dot{\phi}^5 \vartriangle I^2\langle sin^2\theta cos^2\theta\rangle] \\
&= -\frac{2G}{5c^5} \vartriangle I^2\dot{\phi}^5\langle sin^2\theta\rangle(16\langle sin^2\theta\rangle + \langle cos^2\theta\rangle)
\end{aligned}$$

Therefore, the rate of angular momentum loss in terms of the wobble angle,θ, becomes

$$\frac{dJ}{dt} = -\frac{2G}{5c^5} \vartriangle I^2\dot{\phi}^5 sin^2\theta(16sin^2\theta + cos^2\theta) \tag{4.2.16}$$

Alternatively, the radiation torque can be obtained from the power radiated (4.1.20) from a gravitational waves emitting sources as[35]

$$\frac{dJ}{dt} = \frac{dE/dt}{d\phi/dt} \tag{4.2.17}$$

Plugging the expression for $\frac{dE}{dt}$ into this , it gives

$$\frac{dJ}{dt} = \frac{-\frac{2G}{5c^5} \vartriangle I^2\dot{\phi}^6 sin^2\theta(16sin^2\theta + cos^2\theta)}{\dot{\phi}} \tag{4.2.18}$$

which is equal to

$$\frac{dJ}{dt} = -\frac{2G}{5c^5} \vartriangle I^2\dot{\phi}^5 sin^2\theta(16sin^2\theta + cos^2\theta) \tag{4.2.19}$$

4.3 Wobble Evolution Equation

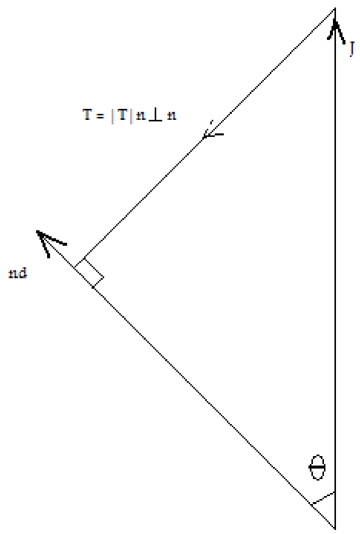

Figure 4.1: For the rigid body the gravitational radiation reaction torque **T** lies in the reference plane.It acts perpendicular to the symmetry axis.

4.3.1 By Using Radiation-Reaction Torque

Considering for axisymmetric rigid body, the symmetry axis of the deformation lies along **n** which moves in a cone half angle θ around the angular momentum vector $\vec{J}$ (fig) with the precession frequency $\dot{\phi} = \vec{J}/I_1$.

The precessional motion of the body is then specified by the pair of parameters $(\theta, \dot{\phi})$.We need to find the effect of the torque on the parameter θ which is the wobble angle.

The action of the torque has two parts;the component along $\vec{J}$ acts to change the inertial precession frequency. If the torque causes the magnitude of the angular momentum to change at a rate of $\dot{\vec{J}}$, then the precessional evolution equation (i.e.following from the differentiation of $\dot{\phi} = \vec{J}/I_1)$ can be given as

$$\ddot{\phi} = \frac{\dot{J}}{I_1} \tag{4.3.1}$$

or substitution of $\dot{J}$ in (4.2.16) gives

$$\ddot{\phi} = -\frac{2G}{5c^5}\frac{\triangle I^2}{I_1}\dot{\phi}^5 sin^2\theta(16sin^2\theta + cos^2\theta) \tag{4.3.2}$$

and the component of the torque projected into the reference plane which lies perpendicular to $\vec{J}$ acts to change the wobble angle.If $J_{\perp\mathbf{n}}$ is the component of the angular momentum perpendicular to the symmetry axis $\mathbf{n}$ then differentiation of the relation, i.e using fig 4.1 :

$$sin\theta = \frac{J_{\perp\mathbf{n}}}{J} \tag{4.3.3}$$

according[28] this leads to

$$\dot{\theta} = -\frac{T_{\perp\mathbf{J}}}{J} = -\frac{T cos\theta}{J} \tag{4.3.4}$$

where $T_{\perp\mathbf{J}}$ is the component of the torque perpendicular to $\vec{J}$ and T is the torque for free precession,that is from (4.2.16) we have

$$\vec{T} = \frac{2G}{5c^5} \triangle I^2\dot{\phi}^5 sin\theta(16sin^2\theta + cos^2\theta) \tag{4.3.5}$$

Substitution of (4.3.5) in (4.3.4) reproduces the the wobble damping equation as:

$$\dot{\theta} = -\frac{2G}{5c^5} \triangle I^2\dot{\phi}^4 sin\theta cos\theta(16sin^2\theta + cos^2\theta) \tag{4.3.6}$$

4.3.2 By Energy and Angular Momentum Balance

Here we re-derive the wobble damping rate for neutron star using energy and angular momentum balance.To calculate the rate wobble angle, recall the rate of energy radiated,that is by differentiation of $E = E(J, \theta)$, as[11]

$$\frac{dE}{dt} = \frac{\partial E}{\partial J}\frac{dJ}{dt} + \frac{\partial E}{\partial \theta}\frac{d\theta}{dt} \tag{4.3.7}$$

Substitution of $\dot{J} = \dot{E}/\dot{\phi}$ to this equation and rearranging it, the rate of change of the wobble angle become

$$\dot{\theta} = \frac{\dot{J}[\dot{\phi} - \frac{\partial E}{\partial J}]}{\partial E/\partial \theta} \tag{4.3.8}$$

where the dot represents the time derivative.But the energy of the source(star) is its kinetic energy as indicated in (4.1.23) i.e.

$$E = \frac{J^2}{2I_1}[1 - cos^2\theta\frac{\triangle\ I}{I_3}] \tag{4.3.9}$$

Thus

$$\frac{\partial E}{\partial J} = \frac{J}{I_1}[1 - cos^2\theta\frac{\triangle\ I}{I_3}] \tag{4.3.10}$$

and

$$\frac{\partial E}{\partial \theta} = \frac{J^2}{I_1}sin\theta cos\theta\frac{\triangle\ I}{I_3} \tag{4.3.11}$$

Plugging equations (4.3.10) and (4.3.11) into (4.3.8) we find

$$\dot{\theta} = -\frac{[\dot{\phi} - \frac{J}{I_1}(1 - cos^2\theta\frac{\triangle I}{I_3})]\frac{2G}{5c^5} \triangle\ I^2\dot{\phi}^6 sin^2\theta(16sin^2\theta + cos^2\theta)}{\dot{\phi}\frac{J^2}{I_1}sin\theta cos\theta\frac{\triangle I}{I_3}} \tag{4.3.12}$$

This gives the wobble damping equation as

$$\dot{\theta} = -\frac{2G}{5c^5} \triangle\ I^2\dot{\phi}^4 sin\theta cos\theta(16sin^2\theta + cos^2\theta) \tag{4.3.13}$$

Comparing (4.3.6) and (4.3.13),so the two methods of calculation agree.

4.4 Lifetime Estimate

We can calculate the time on which the alignment occur as

$$\tau_\theta = -\frac{sin\theta}{\frac{d}{dt}sin\theta} = \frac{5c^5}{2G}\frac{1}{\dot{\phi}^4}\frac{I_1}{\triangle I^2}\frac{1}{cos\theta(16sin^2\theta + cos^2\theta} \tag{4.4.1}$$

Radiation-reaction causes $sin\theta$ to decrease, regardless of whether the body is oblate or prolate.In the limit of small wobble angle,θ decreases exponentially0n the timescale

$$\tau_{\theta<<1} = \frac{5c^5}{2G}\frac{1}{\dot{\phi}^4}\frac{I_1}{\triangle I^2} \tag{4.4.2}$$

Then we have the following parameters.

$$\begin{aligned} c &= 3\times 10^{10} cm.sec^{-1} \\ G &= 6.67\times 10^{-8} cm^3/g.sec^2 \\ \dot{\phi} &= 2\pi\nu \end{aligned} \tag{4.4.3}$$

Up on substituting these values into (4.4.2),the lifetime value is

$$\tau_\theta = 0.058557\times 10^{58}\frac{gcm^2}{s^3}\frac{1}{\nu}\frac{I_1}{\triangle I^2} \tag{4.4.4}$$

Parameterizing this equation gives

$$\tau_\theta = 1.8\times 10^6 yr\left(\frac{10^{-7}}{\triangle I/I_1}\right)^2\left(\frac{kHz}{\nu}\right)^4\left(\frac{10^{45}gcm^2}{I_1}\right) \tag{4.4.5}$$

This is the numerical lifetime estimate of a neutron star wobble modelling it as a rigid body.

Chapter 5

Discussion and Conclusion

5.1 Discussion

As a newborn neutron star settles down into its final state ,its solid crust has preferred shape oblate axisymmetric about some preferred axis. If the star's angular momentum $\vec{J}$ deviates from the crust's preferred symmetry axis, the neutron star will wobble as it spins, with small 'wobble angle'[1].So a rotating neutron star emit gravitational wave by means of time-dependent quadrupole moment ,generated either by the lack of body symmetry on the equatorial plane or by precession caused by misalignment of the spin and the symmetry axis.In the later case, wobbling neutron star emit at frequencies close to the rotation one if the wobble angle is small[2]. As discussed in chapter four a large mass with a quadrupole moment,rotating about some axis, generates gravitational waves. The quadrupole radiation approximation says the gravitational radiation is generated when,not the traceless mass quadrupole moment tensor, but its second time derivative is nonzero[29].Infact there is an isolated massive objects which has nonzero but constant traceless quadrupole moment and then no gravitational radiation results (at least no at mass quadrupole level approximation).Gravitational radiation is characterized by polarization, amplitude, frequancy just like electromagnetic radiation where the polarization modes of gravitational radiation are those appropriate to a rank two tensor field.

The averaged energy and momentum fluxes as well as the instantaneous torque, in equations (3.3.29),(4.2.16)and(4.3.5) respectively, depend only on the orientation of the mass quadrupole of the source. As described in the preceding chapter the radiation torque acting on a neutron star has two components: the braking torque, responsible for the secular spin down of the star and the component associated with the inertia of the radiation fields whose effect is to make the star to wobble. The alignment rate of the body due to the gravitational radiation reaction is calculated using two methods; by energy and momentum balance and by radiation reaction torque in which the resulting expression is consistent.

5.2 Conclusion

Finally in this thesis,based on the time-varying quadrupole moment model, the evolution of gravitational radiation of a rotating neutron star is discussed. In particular the resulting gravitational radiation torque produce the alignment of the axis of rotation with the angular momentum vector by damping the wobble angle.We find the wobble evolution or the gravitational wave damping takes place over a time of [33]

$$\tau_\theta = 1.8X10^6 yr \left(\frac{10^{-7}}{\triangle I/I_1}\right)^2 \left(\frac{kHz}{\nu}\right)^4 \left(\frac{10^{45} gcm^2}{I_1}\right) \tag{5.2.1}$$

This is a good agreement with recently accepted theoretical results found by on the alignment timescale of neutron star due to gravitational radiation. Further studies on the wobble evolution in precessing neutron star due to internal torques and others (especially considering the superfluid model) seem necessary.

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Declaration

I hereby declare that this thesis is my original work and has not been presented for a degree in any other university. All sources of materials used for the thesis have been duly acknowledged.

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