

# **ADDIS ABABA UNIVERSITY SCHOOL OF GRADUATE STUDIES**



**Faculty of Natural science**

**Department of Mathematics**

**Graduate Project Paper**

**On**

## **Boundary Value Problems for Laplace's and Poisson's Equations in Space**

**In Partial fulfillment of the requirements for the Degree of Master of  
Science in Mathematics**

**Compiled by: Zeleke Amare Workie**

**Advisor : Dr. Tsegaye Gedif Ayele**

**January 2011**

**AA, Ethiopia**



## **Declaration**

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any degree, Diploma, Associate ship, Fellowship or any other similar title to me.

Signature: \_\_\_\_\_

## **Permission**

This is to certify that this project is compiled by Mr. Zeleke Amare Workie in the Department of Mathematics, Addis Ababa University, under my supervision.

Signature: \_\_\_\_\_

## **Abstract**

A differential equation is the most important part of mathematics for understanding many of the basic laws of physical sciences as well as other sciences. Some of the laws are formulated in terms of mathematical equations involving certain known and unknown quantities and their derivatives.

In this project paper we give a brief introduction to distributions, formulate constant and variable coefficient boundary value problems and analyse their solution methods.

# Acknowledgment

I would like to express my heart felt thanks to my advisor and teacher Dr. Tsegaye Gedif Ayele for his invaluable comments, suggestions, encouragement and guidance on the whole work of my project paper.

I am deeply grateful to Ato Yibeltal Nigussie, who has thoroughly read the paper, made a lot of valuable remarks, and suggested quite a few improvements.

I would like to thank my wife W/ro ke lemua Mekonnen and my daughters Eyerusalem and Dagmawit for their supporting and encouraging me to pursue my study and without their encouragement; I would not have completed my study.

Special thanks go to my brother Terfie Workie and his wife W/ro Muluemebet Enkusellassie with their children Abraham and Kidist for their continuous moral encouragement and financial assistance throughout my University study.

Finally, my thanks go to all my colleagues, Ato Waltenigus Bekele with his families and my mother W/ro Yemenzwork Bualew for their encouragement, material and financial supports during my study.

# Contents

<b>Declaration</b> -----	<b>i</b>
<b>Permission</b> -----	<b>ii</b>
<b>Abstract</b> -----	<b>iii</b>
<b>Acknowledgement</b> -----	<b>iv</b>
<b>Introduction</b> -----	<b>1</b>
<b>Preliminaries</b> -----	<b>3</b>
<b>I Notations and Symbols</b> -----	<b>3</b>
<b>II Definitions and Concepts</b> -----	<b>4</b>
<b>CHAPTER1. Distributions and Basic Operations</b> -----	<b>6</b>
1.1 Linear Spaces and Functional-----	6
1.2 The Space of Test Functions, $D$ -----	6
1.3 The Space of Generalized Functions, $D'$ -----	7
1.4 Derivatives of Generalized Functions and their Properties-----	8
1.5 The convolution of Generalized Functions and its Properties-----	10
1.6 Generalized Functions of Slow Growth-----	11
1.7 The Structure of Generalized Functions with point Support-----	11
1.8 The Fourier Transform of Generalized Functions of slow growth-----	11
<b>CHAPTER 2. Boundary Value Problems for Equations of Elliptic Type</b> -----	<b>13</b>
2.1 The Basic Equations of Typical Physical Problems-----	13
2.1.1 The Vibration Equation-----	13
2.1.2 The Diffusion Equation-----	14
2.1.3 The Steady State Equation-----	14

2.2 The <b>Boundary Value Problems for Equations of Elliptic Type in <math>\mathbb{R}^2</math></b> -----	14
2.2.1 “Mixed” Boundary Value Problems -----	18
2.2.2 Green’s Formula-----	23
2.2.3 Harmonic Functions-----	24
<b>CHAPTER 3 Boundary Value Problems for Elliptic Equations with Constant Coefficients in <math>\mathbb{R}^3</math></b> -----	<b>30</b>
3.1 Statement of the Main Boundary Value Problems-----	30
3.2 Uniqueness Theorems for the Solution of the Boundary Value Problems -----	31
3.3 Integral Equations-----	34
3.4 The Newtonian Potential -----	36
3.5 Reducing Boundary Value Problems to Boundary Integral Equations-----	43
3.6 Investigation of the Boundary Integral Equations of Potential Theory -----	44
<b>CHAPTER 4 Boundary Value Problems for Elliptic Equations with Variable Coefficients in <math>\mathbb{R}^3</math></b> -----	<b>47</b>
4.1 Parametrix, Potential Type Operators and one-Operator Green’s Identities-----	47
4.2 Formulation of Boundary Value Problem-----	48
4.3 Parametrix-----	49
4.4 Potential Type Operators-----	50
4.5 One-Operator Green’s Identities-----	53
<b>References</b> -----	<b>56</b>





## Introduction

In the study of physical phenomena one is frequently unable to find directly the laws relating the quantities that characterize a phenomenon, whereas a relationship between the quantities and their derivatives or differentials can readily be established. One then obtains equations containing the unknown functions or vector functions under the sign of the derivative or differential.

Equations in which the unknown function or the vector function appears under the sign of the derivative or the differential are called *differential equations*. The finding of unknown functions defined by differential equations is the principal task of the theory of differential equations. If in a differential equation the unknown functions or the vector functions are functions of one variable, then the differential equation is called *ordinary*. But if the unknown function appearing in the differential equation is a function of two or more independent variables, the differential equation is called a *partial differential equation*.

Since a solution of a differential equation is not unique, for a full description of a physical process we must give not only the equation that describes this process but also the initial state of this process (initial conditions) and the behavior on the boundary of the region where the process is taking place (boundary conditions). Therefore, to isolate a particular solution describing a real physical process we must impose additional conditions. These additional conditions are the initial and boundary conditions. The corresponding problem is called a *boundary value problem*.

In this project paper, boundary value problems and potential theory for Laplace's and Poisson's equations in space are considered.

In the paper, the interior or exterior Dirichlet problems and the interior or exterior Neumann problems are treated with enough examples. The mixed boundary value problems are also discussed with illustrative examples.

In addition to solving different types of boundary value problems, it is also included the proofs of different uniqueness Theorems for the solution of the boundary value problems.

The theories of Newtonian potentials are included in the paper to reduce the Dirichlet and Neumann problems for Laplace's equation to Fredholm integral equations with a polar kernel.

The first chapter deals with distributions and basic operations. In this chapter linear spaces, the space of test functions, the space of generalized functions, derivatives and convolution of generalized functions and the Fourier transform of generalized functions of slow growth are discussed. The second chapter deals with boundary value problems for equations of elliptic type. In this chapter the Dirichlet, Neumann and mixed boundary value problems with different homogeneous and inhomogeneous boundary value problems are treated. The third deals with boundary value problems for Laplace's and Poisson's equations in space. The uniqueness Theorems for the solution of the boundary value problems and the Newtonian potentials are treated in detail. Finally, in the fourth chapter parametrix, potential type operators, one operator Green's identities and formulation of boundary value problems are discussed in detail.

## PRELIMINARIES

### I. NOTATIONS AND SYMBOLS

- ❖  $\mathbb{R}^n$ ---An n-dimensional space and its points by  $x = (x_1, x_2, x_3, \dots, x_n)$
- ❖ Domain  $\Omega$ ---An open connected region.
- ❖ A domain  $\Omega$  in  $\mathbb{R}^n$  --- An open connected subset  $\Omega \subseteq \mathbb{R}^n$ .
- ❖  $S = \partial\Omega$  =Boundary of region  $\Omega$ .
- ❖  $\bar{\Omega}$ =The closure of region  $\Omega$ .
- ❖  $\langle x, y \rangle = x_1y_1 + x_2y_2 + \dots + x_ny_n$ ---The scalar product in  $\mathbb{R}^n$ .
- ❖  $|x| = \sqrt{\langle x, x \rangle} = \sqrt{x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2}$ ---The length (norm) in  $\mathbb{R}^n$ .
- ❖  $|x - y|$ =The Euclidean distance between points x and y in  $\mathbb{R}^n$ .
- ❖  $u_{x_i} = \frac{\partial u}{\partial x_i} = D_i u$ ---The first partial derivatives of u with respect to  $x_i$ .
- ❖  $u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j} = D_{ij} u$ ---The second partial derivatives of u with respect to  $x_i$  and  $x_j$ .
- ❖  $D^\alpha u(x)$ =The  $\alpha^{th}$  derivative of the function u(x) of order  $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$  where  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$  is a multi-indices,

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u(x_1, x_2, \dots, x_n)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3} \dots \partial x_n^{\alpha_n}}, \quad D^0 u(x) = u(x); \quad D = (D_1, D_2, \dots, D_n),$$

$$D_j = \frac{\partial}{\partial x_j}, \quad j=1, 2, \dots, n.$$

- ❖  $Du = \sum_{i=1}^n \frac{\partial u}{\partial x_i}$
- ❖  $Dv \cdot Du = \sum_{i=1}^n \frac{\partial v}{\partial x_i} \cdot \frac{\partial u}{\partial x_i}$
- ❖  $\Delta u = u_{x_1 x_1} + u_{x_2 x_2} + u_{x_3 x_3} + \dots + u_{x_n x_n}$  ---The Laplacian of u.
- ❖  $C^0(\Omega) = C(\Omega)$ ---The set of all continuous functions in  $\Omega$ .
- ❖  $C^0(\bar{\Omega})=C(\bar{\Omega})$ --- The set of all continuous functions on  $\bar{\Omega}$ .
- ❖  $C^1(\Omega)$ =The set of all once continuously differentiable functions in  $\Omega$
- ❖  $C^2(\Omega)$ =The set of all twice continuously differentiable functions in  $\Omega$ .

- ❖  $C^p(\Omega)$  = A set of (complex-valued) functions  $u$  continuous together with the derivatives  $D^\alpha u(x)$ ,  $|\alpha| \leq p$  ( $0 \leq p \leq \infty$ ).
- ❖  $\Omega^+$  = An open three-dimensional region of  $\mathbb{R}^3$  and  $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$ .
- ❖  $S = \partial\Omega^+$  is simply connected, closed infinitely smooth surface such that  $S = S_D^- \cup S_N^-$  where  $S_D^-$ ,  $S_N^-$  are non-empty, non-intersecting simply connected sub-manifolds of  $S$  with infinitely smooth boundary curve  $\ell := \partial S_D^- = \partial S_N^-$ .

## II. DEFINITIONS AND CONCEPTS

- **Open Ball.** The set of points  $x$  belonging to  $\mathbb{R}^n$  and satisfying the inequality  $|x - x_0| < R$  is said to be an *open ball of radius  $R$*  centered at the point  $x_0$  and denoted by  $U(x_0, R)$ . An open ball of radius  $R$  centered at the origin is denoted by  $U_R = U(0, R)$ .
- A set is said to be *bounded* in  $\mathbb{R}^n$  if there is a ball that contains the set.
- A point  $x_0$  is called an *interior point* of a set if there is a ball  $U(x_0, \varepsilon)$  contained in this set.
- A set is said to be *open* if all its points are interior.
- A set is called *simply connected* if any two of its points can be joined by a piecewise smooth curve lying in this set.
- A point  $x_0$  is called a *limit point* of a set  $A$  if there is a sequence of  $x_k$ ,  $k=1, 2, \dots$  such that  $x_k \in A$ ,  $x_k \rightarrow x_0$ ,  $x_k \neq x_0$ , as  $k \rightarrow \infty$ .
- If to set  $A$  we add all its limit points, we arrive at a set  $\overline{A}$  that is the *closure* of  $A$ . Clearly,  $A \subset \overline{A}$ .
- A set  $A$  is said to be a *linear set* if for any elements  $f, g$  belong to  $A$  and for any complex numbers  $\alpha$  and  $\beta$ , the linear combination  $\alpha f + \beta g$  belongs to  $A$ .
- If a set coincides with its closure, we call it closed. That is,  $A$  is said to be a closed set if  $A = \overline{A}$ .
- Each open set containing  $A$  is called a *neighborhood* of  $A$ .

- Let  $\Omega$  be a region. The points of the closure  $\overline{\Omega}$  that do not belong to  $\Omega$  form a closed set  $S$  called the *boundary* of  $\Omega$ , so that  $S = \overline{\Omega} \setminus \Omega$ . For instance, the sphere  $|x - x_0| = R$  is the boundary of the open ball  $U(x_0, R)$ .
- A bounded region  $\Omega'$  is called a *sub region lying strictly* in the region  $\Omega$  if  $\overline{\Omega'} \subset \Omega$  and we write  $\Omega' \subset\subset \Omega$ .
- A function  $f$  is said to be *piecewise continuous* in  $\mathbb{R}^n$  if there exist a finite or denumerable number of regions  $\Omega_k, k=1,2,3,\dots$ , without common points and with piecewise smooth boundaries such that each ball is covered by a finite number of closed regions  $\{\overline{\Omega}_k\}$  and  $f \in C(\overline{\Omega}_k), k=1,2,3,\dots$ . A piecewise continuous function is called finite if it vanishes outside a certain ball.
- Let  $\varphi \in C(\mathbb{R}^n)$ . The *support* of the continuous function  $\varphi$  is the closure of the set of all points  $x$  where  $\varphi(x) \neq 0$  and denoted by  $\text{supp}\varphi$ . Hence, a function  $\varphi(x)$  is finite if and only if  $\text{supp}\varphi$  is bounded.
- **The delta function  $\delta(x)$  (Dirac delta function):** Strongly peaked function defined by  $\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$  and  $\int_{-\infty}^{\infty} \delta(x) dx = 1$
- **The Fourier Sine Series:** Let  $f(x)$  be a piecewise continuous function,  $x \in [0, l]$  and  $f(x)$  be expressed as  $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ . Then, the coefficients  $b_n$  of the Fourier sine series is given by  $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}, n \in \mathbb{N}$ .
- Let the functions  $f(x)$  and  $w(x)$  be defined in a neighborhood of a point  $x_0$  (finite or at infinity). We will write  $f(x) = O[w(x)]$  or  $f(x) = o[w(x)], x \rightarrow x_0$ , depending on whether the ratio  $\frac{f(x)}{w(x)}$  is bounded or tends to 0 as  $x \rightarrow x_0$ , respectively
  - For a linear operator  $L_*$  we introduce the subspace  $H^1(\Omega), H^{1,0}(\Omega, L_*) := \{g \in H^1(\Omega) : L_*g \in L_2(\Omega)\}$  endowed with the norm,  $\|g\|_{H^{1,0}(\Omega, L_*)} := \|g\|_{H^1} + \|L_*g\|_{L_2(\Omega)}$ .
- $u \in H^1(\Omega)$  implies  $u|_S^\pm = r_S^\pm u \in H^{1/2}(S)$  and  $r_S^\pm$  is the trace operator on  $S$  from  $\Omega^\pm$  and  $u^\pm := [u]^\pm = u|_S^\pm$ .

## CHAPTER 1: DISTRIBUTIONS AND BASIC OPERATIONS.

### 1.1 Linear spaces and functional

**Definition.** Let M and N be linear sets. An operator L transforming the elements of set M into elements of set N is called *linear* if for any elements f and g belonging to M and any complex numbers  $\lambda$  and  $\mu$ ,  $L(\lambda f + \mu g) = \lambda Lf + \mu Lg$ .

**Example.**  $Lf = \int_{\Omega} K(x, y)f(y)dy$  is linear integral operator where  $K(x, y)$  is its kernel.

**Proof.** Let f and g belong to M and  $\lambda, \mu$  are complex numbers. Then,

$$\begin{aligned}L(\lambda f + \mu g) &= \int_{\Omega} K(x, y)(\lambda f + \mu g)(y)dy, x \in \Omega \\&= \int_{\Omega} K(x, y)\lambda f(y)dy + \int_{\Omega} K(x, y)\mu g(y)dy, x \in \Omega \\&= \lambda \int_{\Omega} K(x, y)f(y)dy + \mu \int_{\Omega} K(x, y)g(y)dy, x \in \Omega \\&= \lambda Lf + \mu Lg\end{aligned}$$

Hence, L is a linear integral operator.

The set  $M = M_L$  is said to be the *domain of definition* of the operator. For a linear operator L to be continuous from M to N is necessary and sufficient that  $Lf_k \rightarrow 0$  as  $k \rightarrow \infty$  in N follow from  $f_k \rightarrow 0$  as  $k \rightarrow \infty$  in M.

**Definition.** If a linear operator  $l$  transforms a set of elements M into a set of complex numbers  $lf$  with  $f \in M$ , then  $l$  is called a *linear functional* on M. The value of the functional  $l$  on the element f, the complex number  $lf$ , is denoted by  $(l, f)$ .

**Remark** Linear functionals are a particular case of linear operators.

### 1.2 The space of test functions, D.

In the case of the delta function we know that  $\int_{-\infty}^{\infty} \delta(x)f(x)dx = f(0)$  and the delta function is defined by means of continuous functions as a linear continuous functional. Continuous functions are said to be *test functions* for the delta function.

A sequence of functions  $\varphi_k \in D$  is said to *converge* to the function  $\varphi \in D$  if  $\lim_{k \rightarrow \infty} \varphi_k = \varphi$  in  $D$ .

**Definition.** A linear set  $D$  equipped with the above convergence is called the *space of test functions*,  $D$ .

### 1.3 The space of generalized functions, $D'$

**Definition.** A linear continuous functional on the space of test functions  $D$  is called a *generalized function/distribution*. The set of all generalized functions is denoted by  $D' = D'(\mathbb{R}^n)$ .

The value of the functional (generalized function)  $f$  on the test function  $\varphi$  is written as  $(f, \varphi)$ .

More explanation on the definition of generalized function:

- 1) A generalized function  $f$  is a *functional* on  $D$ . That means for each  $\varphi \in D$  there is associated a (complex-valued) number  $(f, \varphi)$ .
- 2) A generalized function  $f$  is a *linear functional* on  $D$ . That means for  $\varphi, \psi \in D$  and  $\lambda, \mu$  are complex numbers, then  $(f, \lambda\varphi + \mu\psi) = \lambda(f, \varphi) + \mu(f, \psi)$ .
- 3) A generalized function  $f$  is said to be a *continuous functional* on  $D$ , if  $\varphi_k \rightarrow 0$  ( $k \rightarrow \infty$ ) in  $D$ , then  $(f, \varphi_k) \rightarrow 0$  as  $k \rightarrow \infty$ .

**Note.** The set  $D'$  is *linear* if we define the linear combination  $\lambda f + \mu g$  of the generalized functions  $f$  and  $g$  as a functional acting via the formula

$$(\lambda f + \mu g, \varphi) = \lambda(f, \varphi) + \mu(g, \varphi), \varphi \in D.$$

**Proof.** We want to show that  $\lambda f + \mu g$  is a linear and continuous functional on  $D$ , i.e. belongs to  $D'$ .

Let  $\varphi \in D$  and  $\psi \in D$  and  $\alpha, \beta$  are any complex numbers.

$$\begin{aligned} \text{Then, } (\lambda f + \mu g, \alpha\varphi + \beta\psi) &= \lambda(f, \alpha\varphi + \beta\psi) + \mu(g, \alpha\varphi + \beta\psi) \\ &= \alpha[\lambda(f, \varphi) + \mu(g, \varphi)] + \beta[\lambda(f, \psi) + \mu(g, \psi)] \\ &= \alpha(\lambda f + \mu g, \varphi) + \beta(\lambda f + \mu g, \psi) \end{aligned}$$

Hence,  $\lambda f + \mu g$  is linear.



Since the functional  $f$  and  $g$  are continuous, if  $\varphi_k \rightarrow 0$  as  $k \rightarrow \infty$  in  $D$ , then

$$(\lambda f + \mu g, \varphi_k) = \lambda(f, \varphi_k) + \mu(g, \varphi_k) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Hence,  $\lambda f + \mu g$  is continuous. Therefore,  $\lambda f + \mu g$  is a linear and continuous on  $D$ . i.e.  $\lambda f + \mu g \in D'$ .

#### **1.4 Derivatives of generalized functions and their properties.**

Let  $f \in C^p(\mathbb{R}^n)$ . Then for all  $\alpha$ , with  $|\alpha| \leq p$ , and  $\varphi \in D$  we have the following integration- by-parts formula:

$$(D^\alpha f, \varphi) = \int D^\alpha f(x) \varphi(x) dx = (-1)^{|\alpha|} \int f(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} (f, D^\alpha \varphi)$$

**Proof by induction.** Let  $|\alpha| = 1$  and  $\varphi \in D$ .  $(Df, \varphi) = \int_{-\infty}^{\infty} Df(x) \varphi(x) dx$  ;

$$\text{Let } u = \varphi(x), \quad Du = D\varphi(x) dx \text{ and } Dv = Df(x) dx, \quad v = f(x).$$

$$\begin{aligned} (Df, \varphi) &= \underbrace{f(x)\varphi(x) \Big|_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} f(x) D\varphi(x) dx \\ &= - \int_{-\infty}^{\infty} f(x) D\varphi(x) dx = (-1)^{|1|} (f, D\varphi) \end{aligned}$$

$$\text{Let } |\alpha| = 2: (D^2 f, \varphi) = (D(Df), \varphi) = (-1)^{|1|} (Df, D\varphi)$$

$$= (-1)^{|1|} (-1)^{|1|} (f, D(D\varphi)) = (-1)^{|2|} (f, D^2 \varphi)$$

Assume it is true for  $|\alpha| = k$ :  $(D^k f, \varphi) = (-1)^{|k|} (f, D^k \varphi)$ , by induction assumption.

We want to show that it is true for  $|\alpha| = k+1$ .

$$\text{Hence, } (D^{k+1} f, \varphi) = (D(D^k f), \varphi) = (-1)^{|1|} (D^k f, D\varphi) = (-1)^{|1|} ((-1)^{|k|} (f, D^k(D\varphi)))$$

$$= (-1)^{|k+1|} (f, D^{k+1} \varphi). \text{ Then the assertion is true.}$$

We have this equation for the definition of a (generalized) derivative  $D^\alpha f$  of the generalized function  $f \in D'$ :  $(D^\alpha f, \varphi) = (-1)^{|\alpha|} (f, D^\alpha \varphi), \varphi \in D$  (\*)

Let us see whether  $D^\alpha f \in D'$ .

Indeed, since  $f \in D'$ , the functional  $D^\alpha f$  defined by the right-hand side of (\*) is linear,

$$\begin{aligned} (D^\alpha f, \lambda\varphi + \mu\psi) &= (-1)^{|\alpha|} (f, D^\alpha(\lambda\varphi + \mu\psi)) \\ &= (-1)^{|\alpha|} (f, \lambda D^\alpha \varphi + \mu D^\alpha \psi) = \lambda (-1)^{|\alpha|} (f, D^\alpha \varphi) + \mu (-1)^{|\alpha|} (f, D^\alpha \psi) \\ &= \lambda (D^\alpha f, \varphi) + \mu (D^\alpha f, \psi) \end{aligned}$$

and continuous,  $(D^\alpha f, \varphi_k) = (-1)^{|\alpha|}(f, D^\alpha \varphi_k) \rightarrow 0, k \rightarrow \infty$ , for  $\varphi_k \rightarrow 0$  as  $k \rightarrow \infty$  in  $D$ , it follows that  $D^\alpha \varphi_k \rightarrow 0, k \rightarrow \infty$  in  $D$ .

In particular, when  $f = \delta$ , equation (\*) takes the form  $(D^\alpha \delta, \varphi) = (-1)^{|\alpha|} D^\alpha \varphi(0), \varphi \in D$ . The following properties of the operation of differentiation of generalized functions hold true.

a) The operation of differentiation  $D^\alpha$  is linear and continuous from  $D'$  into  $D'$ :

$D^\alpha(\lambda f + \mu g) = \lambda D^\alpha f + \mu D^\alpha g, f, g \in D'$ ;  $D^\alpha f_k \rightarrow 0, k \rightarrow \infty$  in  $D'$  if  $f_k \rightarrow 0, k \rightarrow \infty$  in  $D'$ .

**Proof.** By the definition of a derivative, for all  $\varphi \in D$  we have

$$(D^\alpha f_k, \varphi) = (-1)^{|\alpha|}(f_k, D^\alpha \varphi) \rightarrow 0, k \rightarrow \infty$$

This signifies that  $D^\alpha f_k \rightarrow 0, k \rightarrow \infty$  in  $D'$ . Hence,  $D^\alpha$  is continuous.

Let  $\lambda, \mu$  be complex numbers,  $\varphi \in D$  and  $f, g \in D'$ .

Then,  $(D^\alpha(\lambda f + \mu g), \varphi) = (-1)^{|\alpha|}(\lambda f + \mu g, D^\alpha \varphi)$ , for some  $\alpha$

$$\begin{aligned} &= (-1)^{|\alpha|}[(\lambda f, D^\alpha \varphi) + (\mu g, D^\alpha \varphi)] \\ &= (-1)^{|\alpha|}[\lambda(f, D^\alpha \varphi) + \mu(g, D^\alpha \varphi)] \\ &= \lambda(-1)^{|\alpha|}(f, D^\alpha \varphi) + \mu(-1)^{|\alpha|}(g, D^\alpha \varphi) \\ &= \lambda(D^\alpha f, \varphi) + \mu(D^\alpha g, \varphi) \end{aligned}$$

Hence,  $D^\alpha$  is linear.

b) Any generalized function is infinitely differentiable.

Indeed, if  $f \in D'$ , then  $\frac{\partial f}{\partial x_i} \in D'$ ; in turn,  $\frac{\partial(\partial f / \partial x_i)}{\partial x_j} \in D'$ , and so on.

c) The result of differentiation does not depend on the order of differentiation.

That is,  $D_1(D_2 f) = D_2(D_1 f) = D^{(1,1)} f, f \in D'$ .

In general,  $D^{\alpha+\beta} f = D^\alpha(D^\beta f) = D^\beta(D^\alpha f)$ .

d) If the generalized function  $f = 0, x \in \Omega$ , then also  $D^\alpha f = 0$  for  $x \in \Omega$ , so that  $\text{supp} D^\alpha f \subset \text{supp} f$ .

Indeed, if  $\varphi \in D(\Omega)$ , then  $D^\alpha \varphi \in D(\Omega)$ .

Thus,  $(D^\alpha f, \varphi) = (-1)^{|\alpha|}(f, D^\alpha \varphi) = 0, \varphi \in D(\Omega)$ . Hence,  $D^\alpha f = 0, x \in \Omega$ .

### 1.5 The convolution of Generalized Functions and its properties.

**Definition.** Let  $f$  and  $g$  be locally integrable functions in  $\mathbb{R}^n$ . If the integral  $\int f(y)g(x-y)dy$  exists for almost all  $x \in \mathbb{R}^n$  and defines a locally summable function in  $\mathbb{R}^n$ , then it is called the *convolution* of the functions  $f$  and  $g$  and is denoted by  $f * g$  so that  $h(x) = (f * g)(x) = \int f(y)g(x-y)dy$ .

Note that  $f * g = g * f$

**Proof.** From the definition we have  $(f * g)(x) = \int f(y)g(x-y)dy$

Then, let  $t=x-y$ . This implies  $y=x-t$  and  $dy=-dt$ .

$$\therefore (f * g)(x) = \int f(x-t)g(t)(-dt) = \int g(t)f(x-t)(-dt) = (g * f)(x)$$

The following are some properties of convolution.

a) Linearity of the convolution.

The convolution  $f * g$  is a linear operation from  $D'$  into  $D'$  with respect to  $f$  and  $g$  separately, e.g.  $(\lambda f + \mu f_1) * g = \lambda(f * g) + \mu(f_1 * g)$ ,  $f, f_1, g \in D'$ , provided that the convolutions  $f * g$  and  $f_1 * g$  exist.

b) Differentiating the convolution.

If the convolution  $f * g$  exists, then so also does the convolutions  $D^\alpha f * g$  and  $f * D^\alpha g$ , with  $D^\alpha f * g = D^\alpha (f * g) = f * D^\alpha g$ .

c) Convolution with the delta function.

The convolution of any generalized function  $f$  in  $D'$  with the  $\delta$  function exists and is equal to  $f$ :  $f * \delta = \delta * f = f$ .

**Proof.**  $(f * \delta)(x) = \int f(y)\delta(x-y)dy = \int \delta(y)f(x-y)dy = f(x)$ .

Note that the meaning of  $f = f * \delta$  is that any generalized function  $f$  may be expanded in terms of delta functions and written as  $f(x) = \int f(y)\delta(x-y)dy$ .

**Remark.** The first and basic operation to which Dirac sought to subject  $\delta(x)$  is the integral  $\int_{-\infty}^{\infty} \delta(x)f(x)dx$ , where  $f(x)$  is any continuous function. This integral can be “evaluated” by the following argument:

Since  $\delta(x)$  is zero for  $x \neq 0$ , the limit of integration may be changed to  $-\varepsilon$  to  $+\varepsilon$ , where  $\varepsilon$  is a small positive number. Moreover, since  $f(x)$  is continuous at  $x=0$ , its value within the interval  $(-\varepsilon, +\varepsilon)$  will not differ much from  $f(0)$  and we can claim, approximately, that  $\int_{-\infty}^{\infty} \delta(x)f(x)dx = \int_{-\varepsilon}^{\varepsilon} \delta(x)f(x)dx$   
 $\Rightarrow \int_{-\infty}^{\infty} \delta(x)f(x)dx \cong f(0) \int_{-\varepsilon}^{\varepsilon} \delta(x)dx$ , with the approximation improving as  $\varepsilon$  approaches to zero.

However,  $\int_{-\varepsilon}^{\varepsilon} \delta(x)dx = 1$ ...for all values of  $\varepsilon$ , because  $\delta(x) = 0$  for  $x \neq 0$ , and  $\delta(x)$  is normalized. Now, letting  $\varepsilon \rightarrow \infty$ , we have exactly  $\int_{-\infty}^{\infty} \delta(x)f(x)dx = f(0)$ .

### 1.6. Generalized Functions of Slow Growth.

#### The space of Basic Functions and Generalized Functions of Slow Growth

**Definition.** The set of all functions infinitely differentiable in  $\mathbb{R}^n$  that decrease together with all their derivatives, as  $|x| \rightarrow \infty$ , faster than any power of  $|x|^{-1}$  is called the set of *basic function*, denoted by  $S = S(\mathbb{R}^n)$ . This set together with the convergence/as we defined in test function/ is the space of basic function.

**Definition.** A generalized function of slow growth is any continuous linear functional on the space  $S$  of basic functions and denoted by  $S' = S'(\mathbb{R}^n)$ .

#### 1.7 The Structure of Generalized Functions with point support.

**Theorem** If the support of a generalized function  $f$  is the point  $\{0\}$ , then it is uniquely representable in the form  $f(x) = \sum_{|\alpha|=0}^m C_{\alpha} D^{\alpha} \delta(x)$ . (1)

### 1.8 The Fourier Transform of Generalized Functions of Slow Growth.

#### The Fourier Transform of Test Functions belonging to S.

Since test functions  $\varphi(x)$  belonging to  $S$  are absolutely integrable on  $\mathbb{R}^n$ , the operation  $\mathcal{F}$  of the Fourier transform is defined by  $\mathcal{F}[\varphi](\xi) = \int \varphi(x) e^{i(\xi, x)} dx$ ,  $\varphi \in S$ .

Note that, the function  $\mathcal{F}[\varphi](\xi)$  is bounded and continuous in  $\mathbb{R}^n$ . A basic function  $\varphi(x)$  decreases at infinity faster than any power of  $|x|^{-1}$ . Therefore, its Fourier transform is differentiable under the integral sign to any number of times:

$$D^{\alpha} \mathcal{F}[\varphi](\xi) = \int (ix)^{\alpha} \varphi(x) e^{i(\xi, x)} dx = \mathcal{F}[(ix)^{\alpha} \varphi](\xi) \quad (2)$$

Whence it follows that  $\mathcal{F}[\varphi] \in C^\infty(\mathbb{R}^n)$ . Further, every derivative  $D^\alpha \varphi$  has the same property, and so  $\mathcal{F}[D^\alpha \varphi](\xi) = \int D^\alpha \varphi(x) e^{i(\xi, x)} dx = (-i\xi)^\alpha \mathcal{F}[\varphi](\xi)$  (3)

Finally, from Equations (2) and (3) it follows that

$$\xi^\beta D^\alpha \mathcal{F}[\varphi](\xi) = \xi^\beta \mathcal{F}[(ix)^\alpha \varphi](\xi) = i^{|\alpha|+|\beta|} \mathcal{F}[D^\beta(x^\alpha \varphi)](\xi) \quad (4)$$

### The Fourier Transform of Generalized Functions belonging to $S'$

First let  $f(x)$  be an (absolutely) integrable function on  $\mathbb{R}^n$ . Then its Fourier transform is defined by  $\mathcal{F}[f](\xi) = \int f(x) e^{i(\xi, x)} dx$ . Observe that  $|\mathcal{F}[f](\xi)| \leq \int |f(x)| < \infty$ . Hence  $\mathcal{F}[f]$  is a continuous bounded function in  $\mathbb{R}^n$  and, consequently defines a generalized function from  $S'$ :  $(\mathcal{F}[f], \varphi) = \int \mathcal{F}[f](\xi) \varphi(\xi) d\xi, \varphi \in S$ .

Using Fubini's Theorem on the change of order of integration, we transform the last integral thus:

$$\int \mathcal{F}[f](\xi) \varphi(\xi) d\xi = \int \left[ \int f(x) e^{i(\xi, x)} dx \right] \varphi(\xi) d\xi = \int f(x) \int \varphi(\xi) e^{i(\xi, x)} d\xi dx = \int f(x) \mathcal{F}[\varphi](x) dx, \text{ i.e. } (\mathcal{F}[f], \varphi) = (f, \mathcal{F}[\varphi]), \varphi \in S.$$

We will take this equation as the definition of the Fourier transform  $\mathcal{F}[f]$  of any generalized function of slow growth  $f$ :  $(\mathcal{F}[f], \varphi) = (f, \mathcal{F}[\varphi]), f \in S', \varphi \in S$  (5)

### Properties of the Fourier Transform

a) Differentiating the Fourier Transform.

$$\text{If } f \in S', \text{ then } D^\alpha \mathcal{F}[f] = \mathcal{F}[(ix)^\alpha f] \quad (6)$$

Indeed, using (3), we obtain for all  $\varphi \in S$ ,

$$\begin{aligned} (D^\alpha \mathcal{F}[f], \varphi) &= (-1)^{|\alpha|} (\mathcal{F}[f], D^\alpha \varphi) = (-1)^{|\alpha|} (f, \mathcal{F}[D^\alpha \varphi]) = (-1)^{|\alpha|} (f, (-ix)^\alpha [\varphi]) \\ &= ((ix)^\alpha f, \mathcal{F}[\varphi]) \end{aligned}$$

$$= (\mathcal{F}[(ix)^\alpha f], \varphi), \text{ from which Equation(6) follows.}$$

b) The Fourier transform of a derivative.

$$\text{If } f \in S', \text{ then } \mathcal{F}[D^\alpha f] = (-i\xi)^\alpha \mathcal{F}[f] \quad (7)$$

Indeed, using (2), we obtain for all  $\varphi \in S$  the following:

$$\begin{aligned} (\mathcal{F}[D^\alpha f], \varphi) &= (D^\alpha f, \mathcal{F}[\varphi]) = (-1)^{|\alpha|} (f, D^\alpha \mathcal{F}[\varphi]) = (-1)^{|\alpha|} (f, \mathcal{F}[(i\xi)^\alpha \varphi]) \\ &= (-1)^{|\alpha|} [(\mathcal{F}[f], (i\xi)^\alpha \varphi)] \\ &= ((-i\xi)^\alpha \mathcal{F}[f], \varphi), \text{ from which Equation(7) follows.} \end{aligned}$$

## **CHAPTER-2 BOUNDARY VALUE PROBLEMS FOR EQUATIONS OF ELLIPTIC TYPE.**

In this chapter we will study boundary value problems for equations of elliptic type; in particular we will study potential theory for Laplace's and Poisson's equations in space.

**Note.** If not stated otherwise, the region  $\Omega$  is considered bounded and its boundary  $S$  a piecewise smooth surface. We will denote the exterior of  $\bar{\Omega}$  by  $\Omega_1$  (i.e.  $\Omega_1 = \mathbb{R}^n \setminus \bar{\Omega}$ )

### **2.1 The Basic Equations of Typical physical Problems.**

A mathematical description of many physical processes leads to linear differential and integral equations or even integrodifferential equations. A broad class of physical problems is described by linear second-order differential equations of the form

$$\sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = F(x) \quad (8)$$

In this section we will consider typical physical problems that lead to the various boundary value problems (for short form, BVP) for differential equations.

#### **2.1.1 The Vibration Equation**

Many problems of mechanics (vibration of strings, rods, membranes, and three dimensional volumes) and of physics (electromagnetic waves) leads to an equation of the form

$$\rho \frac{\partial^2 u}{\partial t^2} = \text{div}(p \text{grad} u) - qu + F(x,t) \quad (9)$$

where the unknown function  $u(x,t)$  depends on  $n$  spatial coordinates  $x=(x_1, x_2, x_3)$  and time  $t$ , the coefficients  $\rho, p,$  and  $q$  are determined by the properties of the medium where the vibration takes place, and the forcing function  $F(x,t)$  gives the intensity of the external perturbation, disturbance. In equation (9), in agreement with the definitions of the operators divergence (div) and gradient (grad),  $\text{div}(p \text{grad} u) = \sum_{i=1}^n \frac{\partial}{\partial x_i} (p \frac{\partial u}{\partial x_i})$ .

### 2.1.2 The Diffusion Equation

Heat diffusion or the diffusion of particles in a medium is described by the following diffusion equation:  $\rho \frac{\partial u}{\partial t} = \text{div}(p \text{grad} u) - qu + F(x,t)$  (10)

### 2.1.3 The Steady State Equation

For a steady state process  $F(x,t)=F(x), u(x,t)=u(x)$ , and both the wave Equation (9) and the diffusion Equation (10) assume the form  $-\text{div}(p \text{grad} u) + qu = F(x)$  (11)

When  $p=\text{const}$  and  $q=0$ , Equation (11) is called *Poisson's equation*

$$\Delta u = -f, \text{ where } f = F/p \quad (12)$$

Verification:  $-\text{div}(p \text{grad} u) + qu = F(x) \Rightarrow -p \text{div}(\text{grad} u) = F(x)$ , since  $p=\text{const}$  and  $q=0$

$$\Rightarrow \text{div}(\text{grad} u) = -F/p \Rightarrow \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \frac{\partial}{\partial x_i} u \right) = -F/p \Rightarrow \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} u = -F/p$$

$$\therefore \Delta u = -F/p. \text{ Hence, } \Delta u = -f \text{ where } f = F/p.$$

When  $f=0$ , Equation (12) is called Laplace's equation

## 2.2 The Boundary Value Problem for Equations of Elliptic Type in $\mathbb{R}^2$

The boundary value problem for Equation (11) (elliptic type) consists of finding a function  $u(x)$  of class  $C^2(\Omega) \cap C^1(\bar{\Omega})$  satisfying, in the region  $\Omega$ , Equation (11) and a

$$\text{boundary condition on } S \text{ of the form } \alpha u + \beta \frac{\partial u}{\partial \mathbf{n}} \Big|_S = v \quad (13)$$

where  $\alpha, \beta$ , and  $v$  are given continuous functions on  $S$ , with  $\alpha$  and  $\beta$  nonnegative and  $\alpha + \beta$  positive.

$$\text{Boundary condition of the first kind ( } \alpha = 1 \text{ and } \beta = 0 \text{): } u \Big|_S = u_0 \quad (14)$$

$$\text{Boundary condition of the second kind ( } \alpha = 0 \text{ and } \beta = 1 \text{): } \frac{\partial u}{\partial \mathbf{n}} \Big|_S = u_1 \quad (15)$$

$$\text{Boundary condition of the third kind ( } \alpha \geq 0 \text{ and } \beta = 1 \text{): } \frac{\partial u}{\partial \mathbf{n}} + \alpha u \Big|_S = u_2 \quad (16)$$

The corresponding boundary value problems are called *boundary value problems* of the first, second, and third kind.

$$\text{For Laplace's equation the boundary value problem of the first kind } \begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = u_0 \text{ on } S \end{cases} \quad (17)$$

is called Dirichlet's problem ;

the boundary value problem of the second kind 
$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ \frac{\partial u}{\partial n} = u_1 \text{ on } S \end{cases} \quad (18)$$

is called Neumann's problem.

For Poisson's equation the boundary value problem of the first kind 
$$\begin{cases} \Delta u = -f \text{ in } \Omega \\ u = u_0 \text{ on } S \end{cases} \quad (19)$$

is called Dirichlet's problem ;

the boundary value problem of the second kind 
$$\begin{cases} \Delta u = -f \text{ in } \Omega \\ \frac{\partial u}{\partial n} = u_1 \text{ on } S \end{cases} \quad (20)$$

is called Neumann's problem.

### Homogeneous equation and Boundary conditions

Consider the boundary-value problem 
$$\begin{cases} u_t - \alpha^2 u_{xx} = 0, 0 < x < l, t > 0 \\ u(x, 0) = \varphi(x), 0 \leq x \leq l, \\ u(0, t) = u(l, t) = 0, t \geq 0 \end{cases} \quad (21)$$

**Solution.** Our goal is to find the solution of (21) using the method of separation of variables or Fourier method.

A separable solution is a solution of the form  $u(x,t)=X(x)T(t)$  to the problem

$$\begin{cases} u_t - \alpha^2 u_{xx} = 0, 0 < x < l, t > 0 \\ u(0, t) = u(l, t) = 0, t \geq 0 \end{cases} \quad (22)$$

Substituting  $u(x,t)$  into equation (22), we get

$$X(x)T'(t) - \alpha^2 X''(x)T(t) = 0 \text{ or } \frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}. \quad (23)$$

In order for the relation (23) to be an equality each side must be identically equal to a

constant: 
$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = \lambda \quad (24)$$

By the boundary conditions  $u(0,t)=X(0)T(t)=0$ ,  $u(l,t)=X(l)T(t)=0$  it follows

$X(0)=X(l)=0$  so that  $X(x)$  satisfies the following eigenvalue problem

$$\begin{cases} X''(x) = \lambda X(x), 0 < x < l, \\ X(0) = X(l) = 0, \end{cases} \quad (25)$$

while  $T(t)$  satisfies the equation  $T'(t)-\lambda\alpha^2 T(t)=0 \quad (26)$

We are looking for the values of  $\lambda$  which lead to nontrivial solutions. Consider the following three cases.



i) Let  $\lambda = \beta^2 > 0, \beta > 0$ . Then the equation (25) has the general solution

$$X(x) = c_1 e^{\beta x} + c_2 e^{-\beta x}.$$

By the boundary conditions it follows  $\begin{cases} c_1 + c_2 = 0 \\ c_1 e^{\beta l} + c_2 e^{-\beta l} = 0 \end{cases}$  and  $c_1 = c_2 = 0$  because

$$\Delta = \begin{vmatrix} 1 & 1 \\ e^{\beta l} & e^{-\beta l} \end{vmatrix} = e^{-\beta l} - e^{\beta l} \neq 0.$$

ii) If  $\lambda = 0$ ,  $X(x)$  has the form  $X(x) = c_1 x + c_2$ . It follows again  $c_1 = c_2 = 0$ .

So in the first two cases the problem (25) admits the trivial solution only.

iii) If  $\lambda = -\beta^2 < 0, \beta > 0$ , then the equation (25) has the general solution

$$X(x) = c_1 \cos \beta x + c_2 \sin \beta x.$$

By the boundary conditions it follows that for a nontrivial solution

$$c_1 = 0 \text{ and } \sin \beta l = 0.$$

Then,  $\beta l = n\pi, n \in \mathbb{Z}$ . So the only nontrivial solution of (25) appears when

$$\lambda = \lambda_n = -\left(\frac{n\pi}{l}\right)^2, n \in \mathbb{N} \text{ and has the form } X_n(x) = a_n \sin \frac{n\pi x}{l}, n \in \mathbb{N}.$$

Solving (26) with  $\lambda = \lambda_n$ , we obtain  $T_n(t) = b_n e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t}$ .

$$\text{Therefore functions of the form } u_n(x, t) = A_n e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t} \sin \frac{n\pi x}{l}, n \in \mathbb{N}, \quad (27)$$

are solutions of the problem (22).

In order to find the solution of (21) we take a superposition of  $u_n(x, t)$ . Then,

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t} \sin \frac{n\pi x}{l}, \quad (28)$$

is the solution of (21) provided that  $\varphi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}, 0 \leq x \leq l$ .

Note that  $A_n$  are the Fourier sine coefficients of  $\varphi(x)$ , i.e.

$$A_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi x}{l} dx, n=1,2,\dots \quad (29)$$

So the formal solution of (21) is the function (28) with coefficients  $A_n$  determined by (29)

**Remark.** The values  $\lambda_n$  are called eigenvalues and the functions  $X_n(x)$  eigenfunctions.

### Inhomogeneous Equation and Boundary Conditions

Let us consider the boundary-value problem for the inhomogeneous diffusion equation

$$\begin{cases} u_t - \alpha^2 u_{xx} = f(x, t), 0 < x < l, t > 0 \\ u(x, 0) = \varphi(x), 0 \leq x \leq l, \\ u(0, t) = u(l, t) = 0, t > 0 \end{cases} \quad (30)$$

**Proof.** To solve (30) we apply the method of variation of constants (parameters) looking

for a solution of the form  $u(x, t) = \sum_{n=1}^{\infty} A_n(t) e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t} \sin \frac{n\pi x}{l}$ .

Substituting formally  $u(x, t)$  into the equation  $u_t - \alpha^2 u_{xx} = f(x, t)$ ,

$$\text{we obtain } \sum_{n=1}^{\infty} A'_n(t) e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t} \sin \frac{n\pi x}{l} = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{l}, \quad (31)$$

where  $f(x, t) = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{l}$ ;  $f_n(t) = \frac{2}{l} \int_0^l f(x, t) \sin \frac{n\pi x}{l} dx$ .

Then equating coefficients in (31), we obtain

$$A'_n(t) = e^{\left(\frac{n\pi\alpha}{l}\right)^2 t} f_n(t) \text{ or } A_n(t) = A_n(0) + \int_0^t e^{\left(\frac{n\pi\alpha}{l}\right)^2 s} f_n(s) ds.$$

In order to calculate  $A_n(0)$  observe that  $u(x, 0) = \sum_{n=1}^{\infty} A_n(0) \sin \frac{n\pi x}{l} = \varphi(x)$  and therefore  $A_n(0) = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi x}{l} dx$ .

Then the solution of (30) is

$$u(x, t) = \sum_{n=1}^{\infty} (a_n e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t} \sin \frac{n\pi x}{l} + \int_0^t e^{\left(\frac{n\pi\alpha}{l}\right)^2 (s-t)} f_n(s) ds) \sin \frac{n\pi x}{l}, \text{ where}$$

$$f_n(s) = \frac{2}{l} \int_0^l f(x, s) \sin \frac{n\pi x}{l} dx; a_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi x}{l} dx.$$

Let us consider the case of inhomogeneous boundary conditions for the diffusion equation with sources at both endpoints

$$\begin{cases} u_t - \alpha^2 u_{xx} = 0, 0 < x < l, t > 0 \\ u(x, 0) = 0, 0 \leq x \leq l, \\ u(0, t) = p(t), u(l, t) = q(t), t \geq 0 \end{cases} \quad (32)$$

To solve the problem, we use the method of shifting the data.

Problem (32) can be reduced to a problem (30) by subtracting from  $u$  any known function satisfying the boundary conditions  $u(0, t) = p(t), u(l, t) = q(t)$ .

The linear combination  $S(x, t) = (1 - \frac{x}{l})p(t) + \frac{x}{l}q(t), 0 \leq x \leq l$  satisfies the boundary

conditions. Consider now  $v(x,t)=u(x,t)-S(x,t)$ .

Then, as  $S_t = (1 - \frac{x}{l})p'(t) + \frac{x}{l}q'(t)$ , and  $S_{xx} = 0$ , the function  $v(x,t)$  satisfies the problem (30)

$$\begin{cases} v_t - \alpha^2 v_{xx} = -\left(1 - \frac{x}{l}\right)p'(t) - \frac{x}{l}q'(t), 0 < x < l, t > 0 \\ v(x, 0) = -\left(1 - \frac{x}{l}\right)p(0) - \frac{x}{l}q(0), 0 \leq x \leq l, \\ v(0, t) = 0, v(l, t) = 0, t > 0 \end{cases} \quad \text{which has been}$$

considered before.

Finally, if we have an inhomogeneous equation and inhomogeneous boundary conditions

$$\begin{cases} u_t - \alpha^2 u_{xx} = f(x, t), 0 < x < l, t > 0 \\ u(x, 0) = \varphi(x), 0 \leq x \leq l, \\ u(0, t) = p(t), u(l, t) = q(t), t \geq 0 \end{cases} \quad (33)$$

We can split it into two problems  $\begin{cases} v_t - \alpha^2 v_{xx} = f(x, t), 0 < x < l, t > 0 \\ v(x, 0) = \varphi(x), 0 \leq x \leq l, \\ v(0, t) = v(l, t) = 0, t \geq 0 \end{cases} \quad (34)$

and  $\begin{cases} w_t - \alpha^2 w_{xx} = 0, 0 < x < l, t > 0 \\ w(x, 0) = 0, 0 \leq x \leq l, \\ w(0, t) = p(t), w(l, t) = q(t), t \geq 0 \end{cases} \quad (35)$

Solving (34) and (35) by previous procedures we obtain that  $u(x,t)=v(x,t)+w(x,t)$  is a solution of (33).

### 2.2.1 "Mixed" Boundary Value Problems

In this problem of Dirichlet and Neumann the function  $u$  or its normal derivative  $\frac{\partial u}{\partial \mathbf{n}}$  or a linear combination of them is prescribed over the entire surface  $S$  bounding the region  $\Omega$  in which  $\Delta u = 0$ . In "mixed" boundary value problems conditions of different types are satisfied at various regions of  $S$ .

In this problem we have to determine a function  $u$  which satisfies  $\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = f \text{ on } S_1 \\ \frac{\partial u}{\partial \mathbf{n}} = g \text{ on } S_2 \end{cases}$

where  $S_1+S_2=S$ , the boundary of  $S$ , and the functions  $f$  and  $g$  are prescribed.

### Homogeneous equation and Boundary conditions

Let us consider the Dirichlet boundary value problem for the homogeneous wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, 0 < x < l, t > 0 \\ u(x, 0) = \varphi(x), 0 \leq x \leq l, \\ u_t(x, 0) = \psi(x), 0 \leq x \leq l, \\ u(0, t) = u(l, t) = 0, t > 0 \end{cases} \quad (36)$$

which describes the motion of the vibrating string.

**Solution.** Our goal is to find the solution of (36) using the Fourier method. A separable solution of the form  $u(x,t)=X(x)T(t)$  to the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, 0 < x < l, t > 0 \\ u(0, t) = u(l, t) = 0, t > 0 \end{cases} \quad (37)$$

Substituting  $u(x,t)$  into equation (37), we get

$$X(x)T''(t) - c^2 X''(x)T(t) = 0 \text{ or } \frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = \lambda \quad (38)$$

By the boundary conditions  $u(0,t)=X(0)T(t)=0$ ,  $u(l,t)=X(l)T(t)=0$  it follows

$$X(0)=X(l)=0 \quad (39)$$

So  $X(x)$  satisfies the problem  $\begin{cases} X''(x) - \lambda X(x) = 0, 0 < x < l, \\ X(0) = X(l) = 0, \end{cases} \quad (40)$

Then, problem (40) has nontrivial solutions  $X_n(x) = a_n \sin \frac{n\pi x}{l}$ ,  $n=1,2,3,\dots$

corresponding to  $\lambda = \lambda_n = -\left(\frac{n\pi}{l}\right)^2$ ,  $n=1, 2, 3, \dots \quad (41)$

Substituting (41) into (38), we obtain the ODE  $T''(t) + \left(\frac{n\pi c}{l}\right)^2 T(t)=0$  with general solution

$T_n(t) = b_n \cos \frac{n\pi c}{l} t + c_n \sin \frac{n\pi c}{l} t$ ,  $n \in \mathbb{N}$ . Therefore functions of the form

$u_n(x, t) = (A_n \cos \frac{n\pi c}{l} t + B_n \sin \frac{n\pi c}{l} t) \sin \frac{n\pi x}{l}$ ,  $n \in \mathbb{N}$  are solutions of the problem (37). In order to find a solution of (36) we take a superposition of  $u_n(x, t)$ . Then, the

solution has the form  $u(x, t) = \sum_{n=1}^{\infty} (A_n \cos \frac{n\pi c}{l} t + B_n \sin \frac{n\pi c}{l} t) \sin \frac{n\pi x}{l}$ ,  $(42)$

Formally the function  $u(x,t)$  satisfies the initial conditions if

$$u(x, 0) = \varphi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}, \quad u_t(x, 0) = \psi(x) = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi c}{l}\right) \sin \frac{n\pi x}{l}$$

Using the Fourier sine series for  $\varphi(x)$  and  $\psi(x)$  we obtain

$$A_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi x}{l} dx, n \in \mathbb{N} \quad \text{and} \quad B_n = \frac{2}{n\pi c} \int_0^l \psi(x) \sin \frac{n\pi x}{l} dx, n \in \mathbb{N} \quad (43)$$

Then, the function (42), where the coefficients  $A_n$  and  $B_n$  are determined by (43), is the unique solution of the Problem (36).

### **Inhomogeneous equation and Boundary conditions**

Let us consider the mixed boundary value problem for the homogeneous wave equation:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x, t), 0 < x < l, t > 0 \\ u(x, 0) = \varphi(x), 0 \leq x \leq l, \\ u_t(x, 0) = \psi(x), 0 \leq x \leq l, \\ u(0, t) = u(l, t) = 0, t > 0 \end{cases} \quad (44)$$

**Solution.** The solution of (44) can be constructed by superposing the unique solution of boundary value problem for the homogeneous wave equation with the unique solution of

$$\text{the problem} \quad \begin{cases} v_{tt} - c^2 v_{xx} = f(x, t), 0 < x < l, t > 0 \\ v(x, 0) = 0, 0 \leq x \leq l, \\ v_t(x, 0) = 0, 0 \leq x \leq l, \\ v(0, t) = v(l, t) = 0, t > 0 \end{cases} \quad (45)$$

This problem can be solved by expanding  $f(x, t)$  in the Fourier sine series

$$f(x, t) = \sum_{n=1}^{\infty} f_n \sin \frac{n\pi x}{l}, 0 < x < l, \text{ where}$$

$$f_n(t) = \frac{2}{l} \int_0^l f(x, t) \sin \frac{n\pi x}{l} dx, \quad n \in \mathbb{N} \quad (46)$$

$$\text{Let us try to find a solution } v(x, t) \text{ of the form } v(x, t) = \sum_{n=1}^{\infty} v_n \sin \frac{n\pi x}{l}, \quad (47)$$

$$\text{where } v_n(0) = v'_n(0) = 0 \quad (48)$$

Formally, substituting (47) into equation (45), we get  $v''_n(t) + \left(\frac{n\pi c}{l}\right)^2 v_n(t) = f_n(t)$ .

This linear second-order ODE with initial conditions (48) has the unique solution

$$v_n(t) = \frac{l}{n\pi c} \int_0^t f_n(\tau) \sin\left(\frac{n\pi c}{l}(t - \tau)\right) d\tau. \quad (49)$$

The solution of (45) is (47), where  $f_n(t)$  and  $v_n(t)$  are determined by (46) and (49).

Finally, let us consider the case of inhomogeneous boundary conditions

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x, t), 0 < x < l, t > 0 \\ u(x, 0) = \varphi(x), 0 \leq x \leq l, \\ u_t(x, 0) = \psi(x), 0 \leq x \leq l, \\ u(0, t) = g(t), u(l, t) = h(t), t > 0 \end{cases} \quad (50)$$

The solution of problem (50) can be found by superposing the solution of the problem (44) with the solution  $w$  of the problem with zero initial data and source

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0, 0 < x < l, t > 0 \\ w(x, 0) = 0, 0 \leq x \leq l, \\ w_t(x, 0) = 0, 0 \leq x \leq l, \\ w(0, t) = g(t), w(l, t) = h(t), t > 0 \end{cases} \quad (51)$$

In order to solve (51), we use method of shifting the data.

Now, considering  $w(x,t) = l w(x,t) - ((l-x)g(t) + xh(t))$  we reduce (51) again to a problem of the type (44):

$$\begin{cases} w_{tt} - c^2 w_{xx} = -((l-x)g''(t) + xh''(t)), 0 < x < l, t > 0 \\ w(x, 0) = -((l-x)g(0) + xh(0)), 0 \leq x \leq l, \\ w_t(x, 0) = ((l-x)g'(t) + xh'(t)), 0 \leq x \leq l, \\ w(0, t) = w(l, t) = 0, t > 0 \end{cases}$$

which can be solved by the previous methods.

### **Boundary value problems for the Laplace equation in a Rectangle**

Let us consider the Laplace equation  $u_{xx} + u_{yy} = 0$  in  $\Omega$  (52)

where  $\Omega = \{(x, y): 0 < x < a, 0 < y < b\}$  is a rectangle in a  $p$  plane. On each side of  $\Omega$  we assume that either Dirichlet or Neumann boundary conditions are prescribed. These problems can be solved by the method of separation of variables.

Now, solve  $u_{xx} + u_{yy} = 0$  in  $\Omega$ , with the boundary conditions

$$u(x,0)=0, u(x,b)=0, 0 \leq x \leq a; u(0,y)=g(y), u_x(a,y)=h(y), 0 \leq y \leq b.$$

**Proof.** The solution of the problem has a form  $u = u_1 + u_2$ , where  $u_1$  and  $u_2$  satisfy (52) respectively with the boundary conditions

$$\begin{cases} u_1(x, 0) = u_1(x, b) = 0, 0 \leq x \leq a, \\ u_1(0, y) = g(y), u_{1x}(a, y) = 0, 0 \leq y \leq b \end{cases} \quad (53)$$

$$\begin{cases} u_2(x, 0) = u_2(x, b) = 0, 0 \leq x \leq a, \\ u_2(0, y) = g(y), u_{2x}(a, y) = h(y), 0 \leq y \leq b. \end{cases} \quad (54)$$

We find each one of  $u_1$  and  $u_2$  by the Fourier method. Separating variables for

$$u_1(x, y) = X(x)Y(y) \text{ we have } \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0 \text{ in } \Omega.$$

$$\text{This implies that } X''(x) + \lambda X(x) = 0, 0 < x < a, \quad (55)$$

$$Y''(y) - \lambda Y(y) = 0, 0 < y < b, \quad (56)$$

for a constant  $\lambda$ . Since the function  $u_1$  satisfies (53) we should have  $Y(0)=Y(b)=0$  (57)

$$X'(a)=0 \quad (58)$$

Nontrivial solutions of (56), (57) are  $Y_n(y) = \sin \frac{n\pi y}{b}$  corresponding to

$$\lambda = \lambda_n = -\left(\frac{n\pi}{b}\right)^2, n \in \mathbb{N}.$$

The differential equation for  $X(x)$ ,  $X''(x) - \left(\frac{n\pi}{b}\right)^2 X(x)=0$  implies that

$$X(x) = c_1 \cosh \frac{n\pi}{b} x + c_2 \sinh \frac{n\pi}{b} x.$$

The condition (58) is satisfied if  $\frac{c_2}{c_1} = -\tanh \frac{n\pi a}{b}$ .

Then  $X(x)$  has the form  $X_n(x) = a_n \left( \cosh \frac{n\pi x}{b} - \tanh \frac{n\pi a}{b} \sinh \frac{n\pi x}{b} \right)$ .

We are looking for a solution  $u_1$  of the form

$$u_1(x, y) = \sum_{n=1}^{\infty} a_n \left( \cosh \frac{n\pi x}{b} - \tanh \frac{n\pi a}{b} \sinh \frac{n\pi x}{b} \right) \sin \frac{n\pi y}{b} \quad (59)$$

It satisfies the boundary condition  $u_1(0, y) = g(y)$ ,  $0 \leq y \leq b$ , when

$$\sum_{n=1}^{\infty} a_n \sin \frac{n\pi y}{b} = g(y), 0 \leq y \leq b, \text{ which implies that } a_n =$$

$$\frac{2}{b} \int_0^b g(y) \sin \frac{n\pi y}{b} dy \quad (60)$$

Suppose now  $u_2(x, y) = X(x)Y(y)$  satisfies (52) and boundary condition (54). As before, we have the equations (55) and (56) for  $X(x)$  and  $Y(y)$  with the boundary conditions  $Y(0)=Y(b)=0$  and  $X(0)=0$ .

Then,  $Y_n(y) = \sin \frac{n\pi y}{b}$  corresponding to  $\lambda = \lambda_n = -\left(\frac{n\pi}{b}\right)^2, n \in \mathbb{N}$ .

For  $X(x)$ ,  $X''(x) - \left(\frac{n\pi}{b}\right)^2 X(x)=0, X(0)=0$ , which implies  $X_n(x) = b_n \sinh \frac{n\pi x}{b}, n \in \mathbb{N}$ .

Looking for  $u_2(x, y)$  in the form  $u_2(x, y) = \sum_{n=1}^{\infty} b_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}$  the condition  $u_{2x}(a, y) = h(y)$ , should be satisfied, which yields

$$b_n = \frac{2}{n\pi} \frac{1}{\cosh \frac{n\pi a}{b}} \int_0^b h(y) \sin \frac{n\pi y}{b} dy \quad (61)$$

Finally the solution is  $u(x, y) = u_1(x, y) + u_2(x, y)$ , where  $a_n$  and  $b_n$  are determined by (60) and (61).

## 2.2.2 Green's Formulas

### Green's First Formula

If  $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ , then Green's First Formula is valid, namely

$$\int_{\Omega} vLudx = - \int_{\Omega} p \sum_{i=1}^n \frac{\partial v}{\partial x_i} \cdot \frac{\partial u}{\partial x_i} dx + \int_S pv \frac{\partial u}{\partial \mathbf{n}} dS + \int_{\Omega} quvdx \quad (62)$$

where  $L = \text{div}(p\text{grad}) + q$  is a differential operator of linear homogeneous boundary value equations of elliptic type  $-\text{div}(p\text{grad}u) + qu = F(x)$

$$\alpha u + \beta \frac{\partial u}{\partial \mathbf{n}} \Big|_S = 0$$

$\frac{\partial u}{\partial \mathbf{n}} =$  The regular normal derivative on  $S$ .

**Proof.** Take an arbitrary region  $\Omega'$  with a piecewise boundary  $S'$  lying strictly inside  $\Omega$ .

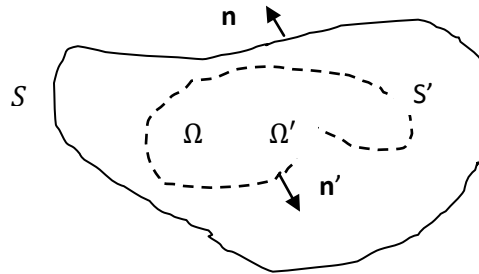


Fig. 1

Since  $u \in C^2(\Omega)$ , we conclude that  $u \in C^2(\bar{\Omega}')$  and consequently,

$$\begin{aligned} \int_{\Omega'} vLudx &= \int_{\Omega'} v[-\text{div}(p\text{grad}u) + qu]dx \\ &= - \int_{\Omega'} \text{div}(pv\text{grad}u)dx + \int_{\Omega'} p \sum_{i=1}^n \frac{\partial v}{\partial x_i} \cdot \frac{\partial u}{\partial x_i} dx + \int_{\Omega'} quvdx. \end{aligned}$$

Employing the divergence theorem, we obtain

$$\int_{\Omega'} vLudx = - \int_{\Omega'} p \sum_{i=1}^n \frac{\partial v}{\partial x_i} \cdot \frac{\partial u}{\partial x_i} dx + \int_{S'} pv \frac{\partial u}{\partial \mathbf{n}'} dS' + \int_{\Omega'} quvdx.$$

Allowing  $\Omega'$  to tend to  $\Omega$  in this formula and using the fact that both  $u$  and  $v$  belong to  $C^1(\bar{\Omega})$ , we conclude that the limit of the right-hand side exists and, consequently, so does the limit of the left-hand side, which proves the validity of the formula.

**Remark.** The integral on the left-hand side of (62) must be understood to be improper.

### Green's Second Formula

If both  $u$  and  $v$  belong to  $C^2(\Omega) \cap C^1(\bar{\Omega})$ , then Green's Second formula is valid:



$$\int_{\Omega} (vLu - uLv)dx = \int_S p(u \frac{\partial v}{\partial \mathbf{n}} - v \frac{\partial u}{\partial \mathbf{n}})dS \quad (63)$$

**Proof.** Let us interchange u and v in Green's first formula (62),

$$\int_{\Omega} vLudx = - \int_{\Omega} p \sum_{i=1}^n \frac{\partial v}{\partial x_i} \cdot \frac{\partial u}{\partial x_i} dx + \int_S pv \frac{\partial u}{\partial \mathbf{n}} dS + \int_{\Omega} quvdx ,$$

$$\int_{\Omega} uLvdx = - \int_{\Omega} p \sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx + \int_S pu \frac{\partial v}{\partial \mathbf{n}} dS + \int_{\Omega} quvdx$$

Then, subtracting the second equation from the first equation, we get

$$\int_{\Omega} (vLu - uLv)dx = \int_S p(u \frac{\partial v}{\partial \mathbf{n}} - v \frac{\partial u}{\partial \mathbf{n}})dS.$$

**Remark:** In particular, when p=1 and q=0 , Green's formula (62) and (63) transform into the following formulas:

$$\int_{\Omega} v\Delta u dx = - \int_{\Omega} \sum_{i=1}^n \frac{\partial v}{\partial x_i} \cdot \frac{\partial u}{\partial x_i} dx + \int_S v \frac{\partial u}{\partial \mathbf{n}} dS \quad (64)$$

$$\int_{\Omega} (v\Delta u - u\Delta v)dx = \int_S (v \frac{\partial u}{\partial \mathbf{n}} - u \frac{\partial v}{\partial \mathbf{n}})dS \quad (65)$$

### 2.2.3 Harmonic Functions

**Definition.** A real-valued function u(x) of the class  $C^2(\Omega)$  is called *harmonic* in a region  $\Omega$  if it satisfies the Laplace equation  $\Delta u=0$  in  $\Omega$ .

For n=1 harmonic functions are simply linear functions and are of no interest to the theory. For this reason we will always assume that  $n \geq 2$ .

#### **Examples**

1) Let  $a \in \mathbb{R}^n$  and  $u(x) = a \cdot x$  , for  $x \in \mathbb{R}^n$ .

**Solution.**  $u(x) = a \cdot x = a_1x_1 + a_2x_2 + a_3x_3 + \dots + \dots + a_nx_n$ .

Then ,  $u_{x_i}(x) = a_i$ , and  $u_{x_ix_i}(x) = 0$ .

Therefore,  $\Delta u(x) = \sum_{i=1}^n u_{x_ix_i}(x) = 0$ . Hence u is a harmonic function.

2) Let  $u(x) = e^{x_1 + x_2 + \dots + x_{n-1}} \sin(\sqrt{n-1} x_n)$  . Show that u is harmonic.

**Proof.**  $u_{x_i} = e^{x_1 + x_2 + \dots + x_{n-1}} \sin(\sqrt{n-1} x_n)$  , for  $i=1,2,\dots,n-1$

$$u_{x_ix_i} = e^{x_1 + x_2 + \dots + x_{n-1}} \sin(\sqrt{n-1} x_n) , \text{ for } i=1,2,\dots,n-1$$

$$u_{x_n} = \sqrt{n-1} e^{x_1 + x_2 + \dots + x_{n-1}} \cos(\sqrt{n-1} x_n)$$

$$u_{x_n x_n} = -(n-1)e^{x_1 + x_2 + \dots + x_{n-1}} \sin(\sqrt{n-1} x_n)$$

$$\text{Then, } \Delta u(x) = \sum_{i=1}^n u_{x_i x_i} = \sum_{i=1}^{n-1} u_{x_i x_i} + u_{x_n x_n}$$

$$= (n-1)e^{x_1 + x_2 + \dots + x_{n-1}} \sin(\sqrt{n-1} x_n) - (n-1)e^{x_1 + x_2 + \dots + x_{n-1}} \sin(\sqrt{n-1} x_n) = 0$$

Hence,  $u$  is harmonic in  $\mathbb{R}^n$ .

**Remark.** Let an elliptic type of boundary value problem is given. Then, the solution of the problem can be stated as shown below.

If  $u \in C^2(\bar{\Omega})$  and  $u(x) = 0, x \notin \bar{\Omega}$ , then for  $x \notin S$  we have Green's formula :

$$u(x) = \frac{1}{(n-2)\sigma_n} \int_{\Omega} \frac{\Delta u(y)}{|x-y|^{n-2}} dy + \frac{1}{(n-2)\sigma_n} \int_S \left[ \frac{1}{|x-y|^{n-2}} \frac{\partial u(y)}{\partial \mathbf{n}} - u(y) \frac{\partial}{\partial \mathbf{n}_y} \frac{1}{|x-y|^{n-2}} \right] dy, n \geq 3;$$

$$u(x) = \frac{-1}{2\pi} \int_{\Omega} \Delta u(y) \ln \frac{1}{|x-y|} dy + \frac{1}{2\pi} \int_S \left[ \ln \frac{1}{|x-y|} \frac{\partial u(y)}{\partial \mathbf{n}} - u(y) \frac{\partial}{\partial \mathbf{n}_y} \ln \frac{1}{|x-y|} \right] dS_y, n = 2 \quad (66)$$

where  $\sigma_n$  is the surface area of a unit sphere in  $\mathbb{R}^n$ :  $\sigma_n = \int_{S_1} dS = \frac{2\pi^{n/2}}{\Gamma(n/2)}$  is an

Eulerian integral of first kind and  $\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt$  is the gamma function.

For a function  $u$  belonging to  $C^1(\bar{\Omega})$  that is harmonic in  $\Omega$ , Green's formula (66) transforms to

$$u(x) = \frac{1}{(n-2)\sigma_n} \int_S \left[ \frac{1}{|x-y|^{n-2}} \frac{\partial u(y)}{\partial \mathbf{n}} - u(y) \frac{\partial}{\partial \mathbf{n}_y} \frac{1}{|x-y|^{n-2}} \right] dS_y, n \geq 3;$$

$$u(x) = \frac{1}{2\pi} \int_S \left[ \ln \frac{1}{|x-y|} \frac{\partial u(y)}{\partial \mathbf{n}} - u(y) \frac{\partial}{\partial \mathbf{n}_y} \ln \frac{1}{|x-y|} \right] dS_y, n = 2 \quad (67)$$

### The Theorem of the Mean Value

We will first prove the following proposition:

**Proposition.** If a function  $u$  is harmonic in a region  $\Omega$  belongs to  $C^1(\bar{\Omega})$ , then

$$\int_S \frac{\partial u}{\partial \mathbf{n}} dS = 0 \quad (68)$$

**Proof.** Consider the Green's First Formula (64):

$$\int_{\Omega} v \Delta u dx = - \int_{\Omega} \sum_{i=1}^n \frac{\partial v}{\partial x_i} \cdot \frac{\partial u}{\partial x_i} dx + \int_S v \frac{\partial u}{\partial \mathbf{n}} dS \text{ at } v=1.$$

$$\text{Then, } \int_{\Omega} 1 \cdot \Delta u dx = - \int_{\Omega} \sum_{i=1}^n \frac{\partial 1}{\partial x_i} \cdot \frac{\partial u}{\partial x_i} dx + \int_S 1 \cdot \frac{\partial u}{\partial \mathbf{n}} dS$$

$$\Rightarrow 0 = \int_S \frac{\partial u}{\partial \mathbf{n}} dS, \text{ since } \Delta u = 0 \text{ in } \Omega \text{ and } \frac{\partial 1}{\partial x_i} = 0.$$

$$\text{Therefore, } \int_S \frac{\partial u}{\partial \mathbf{n}} dS = 0.$$

**Mean Value Theorem** If a function  $u(x)$  is harmonic in the ball  $U_R$  and continuous on  $\overline{U_R}$ , then its value at the center of the ball is its mean value over the sphere  $S_R$ , given by

$$u(0) = \frac{1}{\sigma_n R^{n-1}} \int_{S_R} u(y) dS \quad (69)$$

**Proof.** Let us apply Green's formula (67) at the point  $x=0$  to an arbitrary ball  $|x| < \rho, \rho < R$ , for  $n \geq 3$ .

$$u(x) = \frac{1}{(n-2)\sigma_n} \int_S \left[ \frac{1}{|x-y|^{n-2}} \frac{\partial u(y)}{\partial \mathbf{n}} - u(y) \frac{\partial}{\partial \mathbf{n}_y} \frac{1}{|x-y|^{n-2}} \right] dS_y, n \geq 3.$$

$$\text{Then, } u(0) = \frac{1}{(n-2)\sigma_n} \int_{S_\rho} \left[ \frac{1}{\rho^{n-2}} \frac{\partial u(y)}{\partial \mathbf{n}} - u(y) \frac{\partial}{\partial \mathbf{n}_y} \frac{1}{|\mathbf{y}|^{n-2}} \right] dS_y, n \geq 3, \text{ since } |\mathbf{y}| = \rho < R.$$

$$= \frac{1}{(n-2)\sigma_n} \left[ \frac{1}{\rho^{n-2}} \int_{S_\rho} \frac{\partial u(y)}{\partial \mathbf{n}} dS_y - \int_{S_\rho} u(y) \frac{\partial}{\partial \mathbf{n}_y} \frac{1}{|\mathbf{y}|^{n-2}} dS_y \right]$$

$$= \frac{-1}{(n-2)\sigma_n} \int_{S_\rho} u(y) \frac{\partial}{\partial \mathbf{n}_y} \frac{1}{|\mathbf{y}|^{n-2}} dS_y, \text{ by equation (68)}$$

$$= \frac{-1}{(n-2)\sigma_n} \int_{S_\rho} u(y) \cdot D_y |\mathbf{y}|^{2-n} \cdot \mathbf{n}_y dS_y, \text{ by divergence Theorem.}$$

$$= \frac{-1}{(n-2)\sigma_n} \int_{S_\rho} u(y) \cdot (2-n) |\mathbf{y}|^{1-n} \cdot \frac{\mathbf{y}}{|\mathbf{y}|} \cdot \frac{\mathbf{y}}{|\mathbf{y}|} dS_y, \text{ since } \mathbf{n}_y = \frac{\mathbf{y}}{|\mathbf{y}|}$$

$$= \frac{-1}{(n-2)\sigma_n} \int_{S_\rho} u(y) \cdot (2-n) \frac{|\mathbf{y}|^{1-n}}{|\mathbf{y}|^2} \cdot |\mathbf{y}|^2 dS_y$$

$$= \frac{1}{\sigma_n} \int_{S_\rho} u(y) \cdot \frac{1}{|\mathbf{y}|^{n-1}} dS_y$$

$$= \frac{1}{\sigma_n \rho^{n-1}} \int_{S_\rho} u(y) dS_y, \text{ since } |\mathbf{y}| = \rho$$

Since  $u(x)$  is continuous on the closed ball  $\overline{U_R}$ , the equation is valid as  $\rho \rightarrow R$ . Hence,

$$u(0) = \frac{1}{\sigma_n R^{n-1}} \int_{S_R} u(y) dS. \text{ The case with } n=2 \text{ can be considered in a similar manner.}$$

**The Maximum Principle**

**Theorem** If  $u(x)$  is no n-constant harmonic function in a bounded region  $\Omega$  and continuous on  $\bar{\Omega}$ , then it cannot assume its minimum and maximum in  $\Omega$ , i.e.  $\min_{x \in S} u(x) < u(x) < \max_{x \in S} u(x), x \in \Omega$  (70)

**Proof**

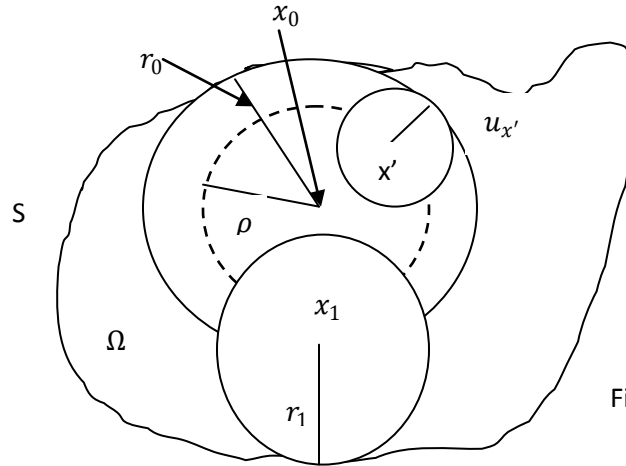


Fig. 2

Assume  $u(x)$  assumes its maximum value at a certain point  $x_0 \in \Omega$ :

$$M = u(x_0) = \max_{x \in \bar{\Omega}} u(x) \tag{71}$$

Since  $x_0$  is an interior point of  $\Omega$ , there is a ball  $U(x_0, r_0)$  of largest radius  $r_0$  contained in  $\Omega$ . We will prove that  $u(x) \equiv M, x \in \bar{U}(x_0, r_0)$  (72)

It follows from (71) that  $u(x) \leq M = u(x_0), x \in U(x_0, r_0)$  (73)

If at a point  $x' \in \bar{U}(x_0, r_0)$  we were to find that  $u(x') < M$ , then due to continuity, we would have  $u(x) < M$  in a neighborhood  $u_{x'}$  of point  $x'$ . But then, applying the mean value formula (69) to the sphere  $S(x_0, \rho)$ , with  $\rho = |x' - x_0|$ , and using the inequality (73) and the fact that  $u(x) < M$  when  $x \in u_{x'}$ , we obtain

$$u(x_0) = \frac{1}{\sigma_n \rho^{n-1}} \int_{S(x_0, \rho)} u(x) dS < \frac{M}{\sigma_n \rho^{n-1}} \int_{S(x_0, \rho)} dS = M, \text{ which contradicts (71).}$$

This proves the validity of (72).

Let us now take an arbitrary point  $x_1 \in \Omega$  lying on the boundary of the ball  $\bar{U}(x_0, r_0)$ . By what has been proved,  $u(x_1) = M$ . Applying the previous agreements to the  $x_1$ , we conclude that  $u(x) \equiv M$  in the largest ball  $\bar{U}(x_1, r_1) \subset \bar{\Omega}$ , and so on. By virtue of the Heine-Borel lemma, the entire region  $\Omega$  will be thus exhausted in no more than a finite number of steps and, hence,  $u(x) \equiv M, x \in \Omega$ , contrary to the assumption.

This contradiction shows that our original supposition is invalid –the function  $u(x)$  cannot assume its maximum value in  $\Omega$ . Replacing  $u$  by  $-u$ , we conclude that  $u(x)$  cannot assume its minimal value in  $\Omega$ .

**Remark.** It follows from this Theorem that a harmonic function can have no local maxima or minima inside a region.

**Corollaries of the Maximum**

**Principle** \_\_\_\_\_ a) If the function  $u \in C(\bar{\Omega})$  is harmonic in  $\Omega$ , then  $|u(x)| \leq \max_{x \in S} |u(x)|$ ,  $x \in \bar{\Omega}$  (74)

For one, if  $u=0$  on  $S$ , then  $u(x) \equiv 0$ ,  $x \in \Omega$ .

**Proof.** From (70), we have  $\min_{x \in S} u(x) < u(x) < \max_{x \in S} u(x)$ ,  $x \in \Omega$ .

Then,  $\pm u(x) \leq \max_{x \in S} \pm u(x) \leq \max_{x \in S} |u(x)|$ ,  $x \in \bar{\Omega}$ .

Hence,  $|u(x)| \leq \max_{x \in S} |u(x)|$ ,  $x \in \bar{\Omega}$ .

Note that we say that a (generalized) function  $u(x)$  is continuous at infinity and assumes the value  $a$  there,  $u(\infty) = a$ , if it is continuous outside a ball and tends to  $a$  as  $|x| \rightarrow \infty$ .

b) If the function  $u \in C(\bar{\Omega}_1)$  is harmonic in  $\Omega_1 = \mathbb{R}^n \setminus \bar{\Omega}$  and  $u(\infty) = 0$ , then  $|u(x)| \leq \max_{x \in S} |u(x)|$ ,  $x \in \bar{\Omega}_1$  (75)

For one, if  $u=0$  on  $S$  and  $u(\infty) = 0$ , then  $u(x) \equiv 0$ ,  $x \in \Omega_1$

**Proof**

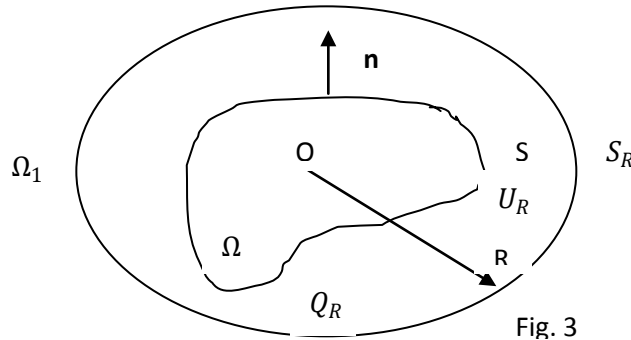


Fig. 3

Let the ball  $U_R$  contain  $\bar{\Omega}$ . Then  $S \cup S_R$  is the boundary of the region  $Q_R = \Omega_1 \cap U_R$ .

Applying (74) to this, we find that

$$|u(x)| \leq \max_{x \in S \cup S_R} |u(x)| \leq \max_{x \in S} |u(x)| + \max_{x \in S_R} |u(x)|, x \in \bar{Q}_R.$$

Since by assumption  $u(\infty) = 0$ , we find that  $\max_{x \in S_R} |u(x)| \rightarrow 0$  as  $R \rightarrow \infty$ .

Therefore,  $|u(x)| \leq \max_{x \in S} |u(x)|$ ,  $x \in \bar{\Omega}_1$ .

c) If the functions  $u_1, u_2, \dots$  that are harmonic in a region  $\Omega$  and continuous on  $\bar{\Omega}$

constitute a sequence that is uniformly convergent on the boundary  $S$ , it also converges uniformly on  $\bar{\Omega}$ .

**Proof** From (74) we have  $|u(x)| \leq \max_{x \in S} |u(x)|$ ,  $x \in \bar{\Omega}$ .

$$\text{Then, } |u_p(x) - u_q(x)| \leq \max_{x \in S} |u_p(x) - u_q(x)| \rightarrow 0, p, q \rightarrow \infty, x \in \bar{\Omega} \quad (76)$$

### Generalized Harmonic Functions

**Definition.** The real-valued continuous function  $u(x)$  is said to be *generalized harmonic* in a region  $\Omega$  if it satisfies Laplace's equation in the region, i.e.

$$(\Delta u, \varphi) = \int u(x) \Delta \varphi dx = 0, \varphi \in D(\Omega) \quad (77)$$

### An Analog of Liouville's Theorem

The following Theorem, which is similar to Liouville's Theorem for analytic functions, is valid for harmonic functions in the entire space  $\mathbb{R}^n$ .

**Theorem:** If  $u \in S'$  satisfies Laplace's equation in the entire space  $\mathbb{R}^n$ , then  $u$  is a polynomial.

**Proof.** Applying the Fourier transform of a derivative (7) to the equation  $\Delta u = 0$ , we obtain the following:

$$\mathcal{F}[\Delta u](\xi) = (-i\xi)^2 \mathcal{F}[u](\xi) = i^2(\xi)^2 \mathcal{F}[u](\xi) = -|\xi|^2 \mathcal{F}[u](\xi) = 0.$$

$$\Rightarrow \mathcal{F}[u] = 0, \xi \neq 0, \text{ i.e. either } \mathcal{F}[u] = 0 \text{ or } \text{supp } \mathcal{F}[u] \text{ is point } \{0\}.$$

But, by using equation (1),  $\mathcal{F}[u]$  can be represented by the expansion

$$\mathcal{F}[u](\xi) = \sum_{|\alpha|=0}^m C_\alpha D^\alpha \delta(\xi) \quad (78)$$

from which it follows that  $u$  is a polynomial.

### The Behavior of a Harmonic function

**at Infinity** **Theorem** Suppose that the function  $u(x)$  is harmonic outside the ball  $\bar{U}_R$  and, as  $|x| \rightarrow \infty$ ,  $u(x) = o(1)$  ( $n \geq 3$ ) or  $u(x) = O(1)$  ( $n=2$ ). Then

$$D^\alpha u(x) = O\left(\frac{1}{|x|^{n-2+|\alpha|}}\right), |x| \rightarrow \infty, \text{ if } n \geq 3$$

$$(79) \quad \text{But, if } n=2, \text{ then } \lim_{|x| \rightarrow \infty} u(x) = a, \quad D^\alpha u(x) =$$

$$O\left(\frac{1}{|x|^{1+|\alpha|}}\right), |x| \rightarrow \infty, (|\alpha| \geq 1).$$

## CHAPTER-3: BOUNDARY VALUE PROBLEMS FOR ELLIPTIC EQUATION WITH CONSTANT COEFFICIENT IN $\mathbb{R}^3$

### 3.1: STATEMENT OF THE MAIN BOUNDARY VALUE PROBLEMS

Laplace's equation,  $\Delta u = 0$ , which is the elliptic equation occurring most frequently in physical problem. It is also called as the *potential equation*.

We will study the following four boundary value problems of the first and second kinds for the three-dimensional Laplace's equation and Poisson's equation and we will take a region  $\Omega$  such that  $\Omega_1 = \mathbb{R}^3 \setminus \overline{\Omega}$  is a region.

- a) **The interior Dirichlet problem:** To find a function  $u \in C(\overline{\Omega})$  that is harmonic in  $\Omega$  and assumes prescribed (continuous) values  $u_0^-$  on  $S$ , we denote as

$$(IDP): \begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = u_0^- \text{ on } S \end{cases}$$

- b) **The exterior Dirichlet problem:** To find a function  $u \in C(\overline{\Omega_1})$  that is harmonic in  $\Omega_1$  and assumes prescribed (continuous) values  $u_0^+$  on  $S$  and vanishes at infinity, we denote as (EDP):  $\begin{cases} \Delta u = 0 \text{ in } \Omega_1 \\ u = u_0^+ \text{ on } S \end{cases}$

#### **Example**

The problem of finding the distribution of temperature within a body in the steady state when each point of its surface is kept at a prescribed steady temperature is an interior Dirichlet problem, while that of determining the distribution of potential outside a body whose surface potential is prescribed is an exterior Dirichlet problem.

- c) **The interior Neumann problem:** To find a function  $u \in C(\overline{\Omega})$  that is harmonic in  $\Omega$  and which has a prescribed (continuous) regular normal derivative  $u_1^-$  on  $S$ ,

we denote by (INP):  $\begin{cases} \Delta u = 0 \text{ in } \Omega \\ \frac{\partial u}{\partial \mathbf{n}} = u_1^- \text{ on } S \end{cases}$

- d) **The exterior Neumann problem:** To find a function  $u \in C(\overline{\Omega_1})$  that is harmonic in  $\Omega_1$  and which has a prescribed (continuous) regular normal derivative  $u_1^+$

(an inward normal) and vanishes at infinity, we denote by

$$(ENP): \begin{cases} \Delta u = 0 \text{ in } \Omega_1 \\ \frac{\partial u}{\partial \mathbf{n}} = u_1^+ \text{ on } S \end{cases}$$

In a similar way, we can formulate the following four boundary value problems of the first and second kinds for the three-dimensional Poisson's equation  $\Delta u = -f$ .

- a) **The interior Dirichlet problem:** To find a function  $u \in C^2(\Omega) \cap C(\overline{\Omega})$  that is  $\Delta u = -f$  in  $\Omega$  and which assumes prescribed (continuous) values  $u_0^-$  on  $S$  (the boundary of  $\Omega$ ).
- b) **The exterior Dirichlet problem:** To find a function  $u \in C^2(\Omega_1) \cap C(\overline{\Omega_1})$  that is  $\Delta u = -f$  in  $\Omega_1$  and which assumes prescribed (continuous) values  $u_0^+$  on  $S$  and vanishes at infinity. i.e.  $u(\infty) = 0$ .
- c) **The interior Neumann problem:** To find a function  $u \in C^2(\Omega) \cap C(\overline{\Omega})$  that is  $\Delta u = -f$  in  $\Omega$  and which has a prescribed (continuous) regular normal derivative  $u_1^-$  on  $S$ .
- d) **The exterior Neumann problem:** To find a function  $u \in C^2(\Omega_1) \cap C(\overline{\Omega_1})$  that is  $\Delta u = -f$  in  $\Omega_1$  and which has a prescribed (continuous) regular normal derivative  $u_1^+$  on  $S$  (an inward normal) and vanishes at infinity.

### 3.2: Uniqueness Theorems for the Solution of the Boundary Value Problems.

Let us prove our uniqueness theorems for the solution of the boundary value problems formulated in sec.3.1.

**Theorem 1** The solution of Poisson's equation is unique in the class of generalized functions that vanish at infinity.

**Proof.** Let  $u$  be the solution of the Poisson's equation. It is sufficient to establish that Laplace's equation has a zero solution in the class of generalized functions that vanish at infinity. By the analog of Liouville's Theorem (78), if  $u \in S'$  satisfies Laplace's equation in the entire space  $\mathbb{R}^n$ , then  $u$  is a polynomial. And, since the generalized function vanishes at infinity, the Poisson's equation becomes the Laplace's equation  $\Delta u = 0$ . Then,  $u$  is unique, since a polynomial has a unique solution.



**Theorem 2** The solution of the interior ( exterior) Dirichlet problem is unique and depends continuously on the boundary value  $u_0^-(u_0^+)$  in the following sense:

if  $|u_0^\pm - \widetilde{u}_0^\pm| \leq \varepsilon$  on  $S$ , then the corresponding solutions  $u$  and  $\tilde{u}$  satisfy the estimate

$$|u(x) - \tilde{u}(x)| \leq \varepsilon, x \in \bar{\Omega} \quad (x \in \bar{\Omega}_1) \quad (80)$$

**Proof.** Applying the inequality (74):  $|u(x)| \leq \max_{x \in S} |u(x)|, x \in \bar{\Omega}$  and (75):

$|u(x)| \leq \max_{x \in S} |u(x)|, x \in \bar{\Omega}_1$ , to the harmonic function  $u - \tilde{u}$ , we obtain the following

$$|u(x) - \tilde{u}(x)| \leq \max_{x \in S} |u_0^\mp(x) - \widetilde{u}_0^\mp(x)|, x \in \bar{\Omega} \quad (x \in \bar{\Omega}_1). \text{ That is,}$$

since if  $|u_0^\pm - \widetilde{u}_0^\pm| \leq \varepsilon$  on  $S$ , we have

$$|u(x) - \tilde{u}(x)| \leq \max_{x \in S} |u_0^\mp(x) - \widetilde{u}_0^\mp(x)| \leq \varepsilon, x \in \bar{\Omega} \quad (x \in \bar{\Omega}_1).$$

Hence,  $|u(x) - \tilde{u}(x)| \leq \varepsilon, x \in \bar{\Omega} \quad (x \in \bar{\Omega}_1)$ .

**Theorem 3** If  $S$  is a sufficiently smooth surface, the solution of the interior Neumann problem is defined to within an arbitrary constant term. The necessary condition for the solvability of this problem is  $\int_S u^-(x) dS + \int_\Omega f(x) dx = 0$  (81)

**Proof.** If  $u$  and  $\tilde{u}$  are two solutions of the interior Neumann problem, its difference  $\eta \in C(\bar{\Omega})$  is a harmonic function in  $\Omega$  and has a zero regular normal derivative on  $S$ .

Applying Green's formula (64):  $\int_\Omega v \Delta u dx = - \int_\Omega \sum_{i=1}^n \frac{\partial v}{\partial x_i} \cdot \frac{\partial u}{\partial x_i} dx + \int_S v \frac{\partial u}{\partial \mathbf{n}} dS$  for

$$u=v=\eta, \text{ we obtain } \int_\Omega \eta \Delta \eta dx = - \int_\Omega \sum_{i=1}^n \frac{\partial \eta}{\partial x_i} \cdot \frac{\partial \eta}{\partial x_i} dx + \int_S \eta \frac{\partial \eta}{\partial \mathbf{n}} dS$$

$$\Rightarrow 0 = - \int_\Omega \sum_{i=1}^n \left( \frac{\partial \eta}{\partial x_i} \right)^2 dx + \int_S \eta \frac{\partial \eta}{\partial \mathbf{n}} dS, \text{ since } \eta \text{ is harmonic in } \Omega \text{ i.e. } \Delta \eta = 0$$

$$\Rightarrow \int_\Omega |\text{grad} \eta|^2 dx = \int_S \eta \frac{\partial \eta}{\partial \mathbf{n}} dS, \text{ since } \sum_{i=1}^n \frac{\partial \eta}{\partial x_i} = \text{grad} \eta$$

$$\Rightarrow \int_\Omega |\text{grad} \eta|^2 dx = 0, \text{ since } \eta \text{ has a zero regular normal derivative on } S.$$

$$\Rightarrow \text{grad} \eta = 0, x \in \Omega. \text{ Hence, } \eta = u - \tilde{u} = \text{const.}$$

The necessity condition for the solvability of the interior Neumann problem follows

$$\text{from the formula (65): } \int_\Omega (v \Delta u - u \Delta v) dx = \int_S \left( v \frac{\partial u}{\partial \mathbf{n}} - u \frac{\partial v}{\partial \mathbf{n}} \right) dS \text{ for } v=1.$$

$$\text{Thus, } \int_\Omega \Delta u dx = \int_S \frac{\partial u}{\partial \mathbf{n}} dS.$$

If  $u$  is the solution of this problem, we have  $\int_{\Omega} -f dx = \int_S u_1^- dS$  by equation (20) and definition of the interior Neumann problem.

Hence,  $\int_S u^-(x) dS + \int_{\Omega} f(x) dx = 0$ , (since  $u^-(x) = u_1^-(x)$  on  $S$ .)

**Theorem 4** If  $S$  is a sufficiently smooth surface the solution of exterior Neumann problem is unique.

**Proof.** Let  $u$  and  $\tilde{u}$  be two solutions of the exterior Neumann problem. Then their difference  $\eta \in C(\overline{\Omega_1})$  is a harmonic function in  $\overline{\Omega_1}$ , has a zero regular normal derivative on  $S$ , and vanishes at infinity.

By Theorem (79)  $\eta$  satisfies the following inequalities:

$$|\eta(x)| < \frac{c}{|x|}, |grad\eta(x)| < \frac{c_1}{|x|^2}, x \rightarrow \infty \quad (82)$$

Applying Green's formula (64):  $\int_{\Omega} v \Delta u dx = - \int_{\Omega} \sum_{i=1}^n \frac{\partial v}{\partial x_i} \cdot \frac{\partial u}{\partial x_i} dx + \int_S v \frac{\partial u}{\partial \mathbf{n}} dS$  for  $u=v=\eta$  to the region  $Q_R = \Omega_1 \cap U_R$ , we obtain

$\int_{Q_R} \eta \Delta \eta dx = - \int_{Q_R} \sum_{i=1}^n \frac{\partial \eta}{\partial x_i} \cdot \frac{\partial \eta}{\partial x_i} dx + \int_T \eta \frac{\partial \eta}{\partial \mathbf{n}} dT$ , where  $T = S \cup S_R$ , is the boundary of region  $Q_R$

$\Rightarrow 0 = - \int_{Q_R} \sum_{i=1}^n \left( \frac{\partial \eta}{\partial x_i} \right)^2 dx + \int_T \eta \frac{\partial \eta}{\partial \mathbf{n}} dT$ , since  $\eta$  is harmonic in  $\Omega_1$  i.e.  $\Delta \eta = 0$ .

$\Rightarrow \int_{Q_R} |grad\eta|^2 dx = \int_S \eta \frac{\partial \eta}{\partial \mathbf{n}} dS + \int_{S_R} \eta \frac{\partial \eta}{\partial \mathbf{n}} dS$ , since  $\sum_{i=1}^n \frac{\partial \eta}{\partial x_i} = grad\eta$  and  $T = S \cup S_R$ .

$\Rightarrow \int_{Q_R} |grad\eta|^2 dx = \int_{S_R} \eta \frac{\partial \eta}{\partial \mathbf{n}} dS$ , since  $\eta$  has a zero regular normal derivative on  $S$  (83)

But,  $\left| \int_{S_R} \eta \frac{\partial \eta}{\partial \mathbf{n}} dS \right| \leq \int_{S_R} |\eta| \left| \frac{\partial \eta}{\partial \mathbf{n}} \right| dS \leq \int_{S_R} |\eta| |grad\eta| dS$ .

$$\Rightarrow \left| \int_{S_R} \eta \frac{\partial \eta}{\partial \mathbf{n}} dS \right| \leq \int_{S_R} |\eta| |grad\eta| dS \quad (84)$$

From (82) it follows that, as  $R \rightarrow \infty$ ,  $|\eta(x)| |grad\eta(x)| < \frac{c}{|x|} \cdot \frac{c_1}{|x|^2} = \frac{cc_1}{|x|^3}$

$\Rightarrow \int_{S_R} |\eta| |grad\eta| dS < \int_{S_R} \frac{cc_1}{|x|^3} dS$  ---integrating both sides of the inequality.

$\Rightarrow \int_{S_R} |\eta| |grad\eta| dS < \frac{cc_1}{|x|^3} \int_{S_R} dS$

$$\Rightarrow \int_{S_R} |\eta| |grad\eta| dS < \frac{cc_1}{R^3} \int_{S_R} dS, R \rightarrow \infty \quad (85)$$

From (84) and (85), we get:

$$\Rightarrow \left| \int_{S_R} \eta \frac{\partial \eta}{\partial \mathbf{n}} dS \right| \leq \int_{S_R} |\eta| |grad\eta| dS < \frac{cc_1}{R^3} \int_{S_R} dS = \frac{cc_1}{R^3} \cdot 4\pi R^2 = \frac{4\pi cc_1}{R}, \text{ since } \int_{S_R} dS = 4\pi R^2 \text{---surface area of a sphere } S_R.$$

$$\Rightarrow \left| \int_{S_R} \eta \frac{\partial \eta}{\partial \mathbf{n}} dS \right| < \frac{4\pi cc_1}{R} \quad (86)$$

Therefore, sending  $R \rightarrow \infty$  in (83), we obtain  $\int_{\Omega_1} |grad\eta|^2 dx = 0$ .

Hence,  $grad \eta = 0$  i.e.  $\eta(x) = const, x \in \Omega_1$ . Since  $\eta(\infty) = 0$ , we conclude

$$\eta = u - \tilde{u} \equiv 0, x \in \Omega_1.$$

That is,  $u = \tilde{u}$  showing that the solution of the exterior Neumann problem is unique.

Before discussing Reducing Boundary Value Problems to integral equations and Investigation of the integral equations of potential type, we shall study the following basic concepts about Integral equations in relation to Fredholm's Theorems and the Newtonian Potentials in detail as follows:

### **3.3 Integral Equations**

**Definition.** Equations that contain the unknown function under the integral sign are known as *Integral Equations*.

Many problems can be reduced to linear integral equations of the type:

$$\int_{\Omega} \mathcal{K}(x, y) \varphi(y) dy = f(x) \quad (87)$$

$$\varphi(x) = \lambda \int_{\Omega} \mathcal{K}(x, y) \varphi(y) dy + f(x) \quad (88)$$

w.r.t the unknown function  $\varphi(x)$  in a region  $\Omega \subset \mathbb{R}^n$ . Equations (87) and (88) are commonly known as *Fredholm's integral equations* of the first and the second kind, respectively. The known functions  $\mathcal{K}(x, y)$  and  $f(x)$  are called the *kernel* and the *inhomogeneous term* of the integral equation;  $\lambda$  is a complex parameter.

$$\text{The integral equations (88) with } f=0 \text{ i.e. } \varphi(x) = \lambda \int_{\Omega} \mathcal{K}(x, y) \varphi(y) dy \quad (89)$$

is known as the *homogeneous* Fredholm integral equation of the second kind corresponding to equation (88).

The Fredholm integral equations of the second kind

$$\psi(x) = \bar{\lambda} \int_{\Omega} \mathcal{K}^*(x, y)\psi(y)dy + g(x), \quad (90)$$

$$\psi(x) = \bar{\lambda} \int_{\Omega} \mathcal{K}^*(x, y)\psi(y)dy \quad (91)$$

where  $\mathcal{K}^*(x, y) = \overline{\mathcal{K}(y, x)}$ , are said to be *adjoint* to equations (88) and (89), respectively. The kernel  $\mathcal{K}^*(x, y)$  is known as the *hermitian conjugate* (adjoint) kernel of the kernel  $\mathcal{K}(x, y)$ .

We will write the integral equations (88) through (91) in abbreviated fashion by using the operator notation:  $\varphi = \lambda K\varphi + f$ ,  $\varphi = \lambda K\varphi$ ,  $\psi = \bar{\lambda}K^*\psi + g$ ,  $\psi = \bar{\lambda}K^*\psi$ , where the integral operators  $K$  and  $K^*$  are determined via the kernels  $\mathcal{K}(x, y)$  and  $\mathcal{K}^*(x, y)$ , respectively :

$$(Kf)(x) = \int_{\Omega} \mathcal{K}(x, y)f(y)dy, \text{ and } ((K^*f)(x) = \int_{\Omega} \mathcal{K}^*(x, y)f(y)dy .$$

**Remarks.**

1) A complex value of  $\lambda$  for which the homogeneous integral equation (89) has non-zero solutions belonging to  $L_2(\Omega)$  is said to be eigenvalue(characteristic number) of the kernel  $\mathcal{K}(x, y)$ , and the corresponding solutions are called eigenfunctions(characteristic functions) of this kernel. Thus, the eigenvalues of the kernel  $\mathcal{K}(x, y)$  and those of the operator  $K$  are mutually inverse and the eigenfunctions of the kernel  $\mathcal{K}$  and the operator  $K$  coincide.

2) Let us assume that the region  $\Omega$  in the integral equation (88) is bounded in  $\mathbb{R}^n$ , the function  $f$  is continuous on the closed region  $\overline{\Omega}$ , and the kernel  $\mathcal{K}(x, y)$  is continuous on  $\overline{\Omega} \times$

$\overline{\Omega}$  **Fr**

**Fredholm's Theorems**

In this section we will state Fredholm's solvability theorems for the Fredholm integral equation  $\varphi = \lambda K\varphi + f$  (92)

with a continuous kernel  $\mathcal{K}(x, y)$  and the adjoint equation  $\psi = \lambda K^*\psi + g$  (93)

**Fredholm's First Theorem:** If the integral equation (92) with a continuous kernel has a solution in  $C(\bar{\Omega})$  for any inhomogeneous term  $f \in C(\bar{\Omega})$ , the adjoint equation (93) has a solution in  $C(\bar{\Omega})$  for any inhomogeneous term  $g \in C(\bar{\Omega})$ , and these solutions are unique.

**Fredholm's Second Theorem:** If the integral equation (92) has a solution in  $C(\bar{\Omega})$  only for some inhomogeneous term  $f$ , then the homogeneous equations corresponding to equation (92) and (93) have an equal (finite) number of linearly independent solutions.

**Fredholm's Third Theorem:** If the integral equation (92) has a solution in  $C(\bar{\Omega})$  only for some inhomogeneous term  $f$ , then for equation (92) to have a solution it is necessary and sufficient that the inhomogeneous term  $f$  be orthogonal to each solution of the adjoint homogeneous equation corresponding to equation (93).

### 3.4 The Newtonian Potential

In this section we will study the properties of the Newtonian Potential in three-dimensional space.

**Definition.** The Newtonian Potential is defined as the convolution of a generalized function  $\rho$  (called the density) with the function  $|x|^{-1}$ :  $V = \frac{1}{|x|} * \rho = -4\pi\epsilon_3 * \rho$

#### The Volume Potential

If  $\rho$  is an (absolutely) integrable function on  $\Omega$  and  $\rho(x) = 0$  when  $x \in \Omega_1 = \mathbb{R}^3 \setminus \bar{\Omega}$ , then the Newtonian potential  $V$ , known as the *volume potential*, is defined by

$$V(x) = \int_{\Omega} \frac{\rho(y)}{|x-y|} dy \tag{94}$$

and is a function locally integrable in  $\mathbb{R}^n$ .

**Remark.** Let  $\rho(y)$  be (absolutely) integrable on a bounded region  $\Omega \subset \mathbb{R}^n$  and vanish outside  $\Omega$ . The integral  $I(x) = \int_{\Omega} \frac{\rho(y)}{|x-y|^\alpha} dy$ ,  $0 < \alpha < n$  is called an integral of the potential type.

If  $\rho$  belongs to  $C(\Omega)$  and  $\Omega$  is a bounded region, then the volume potential  $V$  belongs to the class  $C^1(\mathbb{R}^3)$ , is harmonic in  $\Omega_1$ , and  $V(x) = O\left(\frac{1}{|x|}\right)$ ,  $|x| \rightarrow \infty$ .

**Remark** A simple layer on a surface is a generalization of the delta function.

Suppose that  $S$  is a piecewise smooth surface and  $\mu(x)$  is a continuous function

defined on  $S$ . We introduce a generalized function  $\mu\delta_S$  acting according to the rule  $(\mu\delta_S, \varphi) = \int_S \mu(x)\varphi(x)dS, \varphi \in D$ . The generalized function  $\mu\delta_S$  is a simple layer on the surface  $S$  with density  $\mu$ .

**Remark:** A generalization of  $-\delta'(x)$  is the double layer on a surface.

Let  $S$  be a piecewise smooth two-sided surface,  $\mathbf{n}$  a normal to  $S$  (see fig.), and  $v(x)$  a continuous function on  $S$ .

We introduce the generalized function  $-\frac{\partial(v\delta_S)}{\partial\mathbf{n}}$ ,

which operates via the rule

$$\left(-\frac{\partial(v\delta_S)}{\partial\mathbf{n}}, \varphi\right) = \int_S v(x) \frac{\partial\varphi(x)}{\partial\mathbf{n}} dS, \varphi \in D.$$

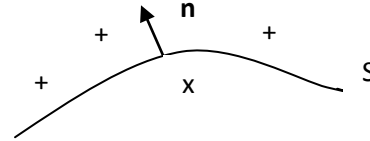


Fig.4

The generalized function  $-\frac{\partial(v\delta_S)}{\partial\mathbf{n}}$  is called the double layer on  $S$  with a density  $v(x)$ .

If  $\rho$  is a finite (absolutely) integrable function on  $\mathbb{R}^n$ , the corresponding Newtonian (logarithmic) potential  $V_n$  is called the volume (surface) potential.

The volume potential  $V_n$ , a locally integrable function in  $\mathbb{R}^n$ , is expressed thus:

$$V_n(x) = \int \frac{\rho(y)}{|x-y|^{n-2}} dy, n \geq 3 \quad ; \quad \text{and} \quad V_2(x) = \int \rho(y) \ln \frac{1}{|x-y|} dy, n = 2.$$

The Newtonian potentials generated by the layers  $\mu\delta_S$  and  $-\frac{\partial(v\delta_S)}{\partial\mathbf{n}}$  on  $S$  with surface densities  $\mu$  and  $v$ , respectively, are

$$V_n^{(0)} = \begin{cases} \frac{1}{|x|^{n-2}} * \mu\delta_S, n \geq 3 \\ \ln \frac{1}{|x|} \mu\delta_S, n = 2 \end{cases} ; V_n^{(1)} = \begin{cases} -\frac{1}{|x|^{n-2}} * \frac{\partial}{\partial\mathbf{n}}(v\delta_S), n \geq 3 \\ -\ln \frac{1}{|x|} * \frac{\partial}{\partial\mathbf{n}}(v\delta_S), n = 2 \end{cases}$$

and are called, respectively, the surface potentials of a simple layer and a double layer with densities  $\mu$  and  $v$ .

The surface potentials  $V_n^{(0)}$  and  $V_n^{(1)}$  are functions that are locally integrable in  $\mathbb{R}^n$  and

$$\text{are expressed by the formulas: } V_n^{(0)}(x) = \begin{cases} \int_S \frac{\mu(y)}{|x-y|^{n-2}} dS_y, n \geq 3 \\ \int_S \mu(y) \ln \frac{1}{|x-y|} dS_y, n = 2 \end{cases};$$

$$V_n^{(1)}(x) = \begin{cases} \int_S v(y) \frac{\partial}{\partial \mathbf{n}_y} \frac{1}{|x-y|^{n-2}} dS_y, n \geq 3 \\ \int_S v(y) \frac{\partial}{\partial \mathbf{n}_y} \ln \frac{1}{|x-y|} dS_y, n = 2 \end{cases}$$

### The Potentials of a Simple and a Double layer

Let  $S$  be a bounded two-sided piecewise smooth surface,  $\mathbf{n}$  its outward normal, and  $\mu$  and  $v$  functions continuous on  $S$ . The Newtonian potentials  $V^{(0)} = \frac{1}{|x|} * \mu \delta_S$  and

$V^{(1)} = -\frac{1}{|x|} * \frac{\partial}{\partial \mathbf{n}} (v \delta_S)$ , which are said to be the potentials of a simple and a double layer, respectively, are expressed in terms of integrals:

$$V^{(0)}(x) = \int_S \frac{\mu(y)}{|x-y|} dS_y \quad (95)$$

$$V^{(1)}(x) = \int_S v(y) \frac{\partial}{\partial \mathbf{n}_y} \frac{1}{|x-y|} dS_y \quad (96)$$

and are locally integrable functions in  $\mathbb{R}^3$ .

We will fix a point  $x_0$  on  $S$  and let  $\mathbf{n}_0$  be the outward normal to  $S$  at this point.

Differentiating (95) for  $x \notin S$  along  $\mathbf{n}_0$  and using the equation

$$\frac{\partial}{\partial \mathbf{n}_0} \frac{1}{|x-y|} = \sum_{i=1}^3 \cos(\mathbf{n}_0, \mathbf{x}_i) \frac{y_i - x_i}{|x-y|^3} = \frac{\cos \psi_{xy}}{|x-y|^2} \quad (97)$$

Where  $\psi_{xy}$  is the angle between the vector  $y-x$  and  $\mathbf{n}_0$ , (fig. 5)

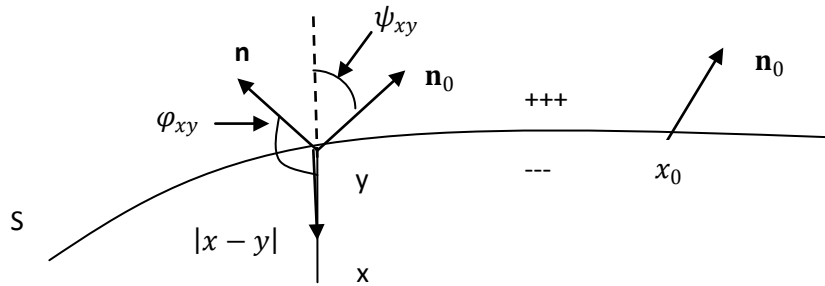


Fig. 5

we obtain an expression for the normal derivative of the simple layer potential:

$$\frac{\partial V^{(0)}(x)}{\partial \mathbf{n}_0} = \int_S \mu(y) \frac{\partial}{\partial \mathbf{n}_0} \frac{1}{|x-y|} dS_y = \int_S \mu(y) \frac{\cos \psi_{xy}}{|x-y|^2} dS_y, x \notin S \quad (98)$$

$$\text{Similarly, by virtue of the equation } \frac{\partial}{\partial \mathbf{n}_y} \frac{1}{|x-y|} = \sum_{i=1}^3 \cos(\mathbf{n}_y, \mathbf{x}_i) \frac{x_i - y_i}{|x-y|^3} = \frac{\cos \varphi_{xy}}{|x-y|^2}, \quad (99)$$

where  $\varphi_{xy}$  is the angle between the vector  $x-y$  and  $\mathbf{n}$ , (figure 5), equation (96) for the

$$\text{double layer potential } V^{(1)} \text{ takes the form } V^{(1)}(x) = \int_S \nu(y) \frac{\cos \varphi_{xy}}{|x-y|^2} dS_y \quad (100)$$

The potentials  $V^{(0)}$  and  $V^{(1)}$  are harmonic functions outside the surface  $S$ ,  $V^{(0)}$  belongs to  $C(\mathbb{R}^3)$ , and  $V^{(0)}(x) = O\left(\frac{1}{|x|}\right)$ ,  $V^{(1)}(x) = O\left(\frac{1}{|x|^2}\right)$ ,  $x \rightarrow \infty$ .

**Proposition:** Prove that the double layer potential  $V^{(1)}(x)$ , with a density  $\nu=1$  is equal to

$$\int_S \frac{\cos \varphi_{xy}}{|x-y|^2} dS_y = \begin{cases} -4\pi, & x \in \Omega; \\ 0, & x \in \Omega_1 = \mathbb{R}^3 \setminus \bar{\Omega}, \end{cases}$$

provided that  $S$  is the boundary of  $\Omega$ .

### The Physical Meaning of

#### Newtonian Potentials

The potential  $V = \frac{1}{|x|} * \rho$  with an arbitrary (finite) density  $\rho$  satisfies Poisson's equation  $\Delta V = -4\pi\rho$ . Therefore,  $V$  is the Newtonian or Coulomb potential created by the masses or charges distributed in space with the density  $\rho$ . For one, a continuous distribution of masses or charges creates a volume potential; but if the masses or charges are concentrated on a surface, they create a (Newtonian or Coulomb) potential of a simple layer, while if electric dipoles are concentrated on a surface, they produce a Coulomb potential that is the potential of a double layer.

#### Lyapunov's Surfaces

**Definition.** A closed bounded surface  $S$  is said to be a *Lyapunov surface* if at each point  $x \in S$  there is a normal  $\mathbf{n}_x$  that is Hölder continuous on  $S$ , i.e. there are numbers  $c > 0$  and  $0 < \alpha \leq 1$  such that  $|\mathbf{n}_x - \mathbf{n}_y| \leq c|x - y|^\alpha, x, y \in S$  (101)

**Remark** Lyapunov's surface belong to the class of surfaces  $C^1$ ; on the other hand, each bounded closed surface of the class  $C^2$  is a Lyapunov surface (at  $\alpha = 1$ ).

**Lemma 1** If  $S$  is a Lyapunov surface, then  $|\cos \varphi_{xy}| \leq 3c|x - y|^\alpha, x, y \in S$  (102)



$$|\cos\varphi_{x'y} + \cos\psi_{x'y}| \leq 3c|x' - y|^\alpha, x, y \in S, x' \in \mathbf{n}_x, \quad (103)$$

where c is the constant of (101).

**Lemma 2** If S is a Lyapunov surface, then there is a constant k such that

$$\int_S \frac{\cos\varphi_{x'y}}{|x'-y|^2} ds_y \leq k, x' \in \mathbb{R}^3 \quad (104)$$

**Properties of the Potentials of a Simple and a Double Layer on a Surface**

Assuming the boundary S of a region  $\Omega$  to be a Lyapunov surface, we can establish some properties of the potentials  $V^{(0)}$  and  $V^{(1)}$  on S.

**Proposition.** Prove that 
$$\int_S \frac{\cos\varphi_{xy}}{|x-y|^2} ds_y = \begin{cases} -4\Pi, & x \in \Omega; \\ -2\Pi, & x \in S; \\ 0, & x \in \Omega_1 \end{cases} \quad (105)$$

**Remarks** 1) The potential of a double layer,  $V^{(1)}(x)$ , is a continuous function on S.

2) The function  $\frac{\partial V^{(0)}(x)}{\partial \mathbf{n}}$  is said to be the direct value of the normal derivative of the simple layer potential on a surface S and it is continuous on S.

3) The simple layer potential  $V^{(0)}(x)$  is continuous function on S, since 
$$V^{(0)} \in C(\mathbb{R}^3)$$

**Discontinuity of**

**the Double Layer Potential** **Theorem** If S is a

Lyapunov surface and  $v \in C(S)$ , then the double layer potential  $V^{(1)}$  belongs to  $C(\bar{\Omega})$  and to  $C(\bar{\Omega}_1)$ , and its limiting values  $V_+^{(1)}$  and  $V_-^{(1)}$  on S from outside and inside of S are

$$V_+^{(1)}(x) = 2\pi v(x) + V^{(1)}(x) = 2\pi v(x) + \int_S v(y) \frac{\cos\varphi_{xy}}{|x-y|^2} ds_y, \quad (106)$$

$$V_-^{(1)}(x) = -2\pi v(x) + V^{(1)}(x) = -2\pi v(x) + \int_S v(y) \frac{\cos\varphi_{xy}}{|x-y|^2} ds_y, \quad (107)$$

**Proof** Let us introduce the function

$$w(x', x) = \int_S [v(y) - v(x)] \frac{\cos\varphi_{x'y}}{|x'-y|^2} ds_y, x \in \mathbb{R}^3, x \in S.$$

In view of (105), this function for  $x' = x \in S$  is

$$w(x, x) = \int_S v(y) \frac{\cos \varphi_{xy}}{|x-y|^2} ds_y + 2\pi v(x) = 2\pi v(x) + V_+^{(1)}(x) \quad (108)$$

The latter is continuous on  $S$ , in view of the fact that both the density  $v$  and the potential  $V^{(1)}$  are continuous on  $S$ .

$$\text{Let us prove that } w(x', x) \xrightarrow{x \in S} w(x, x), x' \rightarrow x \in S \quad (109)$$

We take a positive  $\varepsilon$ . Since the function  $v$  is uniformly continuous on  $S$ , there exists a number  $\delta = \delta_\varepsilon > 0$  such that for all  $x \in S$  we have

$$|v(y) - v(x)| < \frac{\varepsilon}{4k}, y \in U_x = S \cap U(x, \delta) \quad (110)$$

where  $k$  is the number in (104).

Let us estimate

$$|w(x', x) - w(x, x)| \leq \left( \int_{U_x} + \int_{S \setminus U_x} \right) |v(y) - v(x)| \left| \frac{\cos \varphi_{x'y}}{|x'-y|^2} - \frac{\cos \varphi_{xy}}{|x-y|^2} \right| ds_y \quad (111)$$

In view of (110) and (104), the first integral on the right-hand side of (111) does not exceed  $\varepsilon/2$ :

$$\int_{U_x} |v(y) - v(x)| \left| \frac{\cos \varphi_{x'y}}{|x'-y|^2} - \frac{\cos \varphi_{xy}}{|x-y|^2} \right| ds_y \leq \frac{\varepsilon}{4k} \int_S \left( \frac{|\cos \varphi_{x'y}|}{|x'-y|^2} + \frac{|\cos \varphi_{xy}|}{|x-y|^2} \right) ds_y \leq \frac{\varepsilon}{4k} 2k = \varepsilon/2.$$

Next, the integrand in (111) is uniformly continuous function in  $(x, x', y)$  for  $|x - x'| \leq \delta/2$ ,  $x \in S$ , and  $y \in S \setminus U_x$  and vanishes at  $x' = x$ . Therefore, there is a number  $\delta' \leq \delta/2$  such that for all  $x' \in U(x, \delta')$  the second integral on the right-hand side of (111) is less than  $\varepsilon/2$ . Consequently,

$$|w(x', x) - w(x, x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon, x' \in U(x, \delta'), x \in S, \text{ which proves the limiting relation (108).}$$

Taking  $x' \in \Omega_1$  and using (105), we can write the potential  $V^{(1)}(x')$  in the following:  $V^{(1)}(x') = \int_S [v(y) - v(x)] \frac{\cos \varphi_{x'y}}{|x'-y|^2} ds_y + w(x', x) \quad (112)$

Proceeding to the limit in last equation as  $x' \rightarrow x \in S, x' \in \Omega_1$ , and allowing for the limiting relation (109), we obtain  $V^{(1)}(x') \xrightarrow{x \in S} w(x, x) = V_+^{(1)}(x), x \in S$ , which

implies that  $V^{(1)}$  belongs to  $C(\overline{\Omega_1})$  and that, in view of (108), equation (106) is valid. The other case can be proved along similar lines.

From (106) and (107) it follows that  $4\pi v(x) = V_+^{(1)}(x) - V_-^{(1)}(x), x \in S$

**(113). Discontinuity of the Normal Derivative of the Simple Layer Potential**

**Theorem** If  $S$  is a Lyapunov surface and  $\mu \in C(S)$ , the simple layer potential  $V^{(0)}$  has regular normal derivative  $\left(\frac{\partial V^{(0)}}{\partial \mathbf{n}}\right)_+$  and  $\left(\frac{\partial V^{(0)}}{\partial \mathbf{n}}\right)_-$  on  $S$  from the outside and inside of  $S$ , with

$$\left(\frac{\partial V^{(0)}}{\partial \mathbf{n}}\right)_+(x) = -2\pi\mu(x) + \frac{\partial V^{(0)}(x)}{\partial \mathbf{n}} = -2\pi\mu(x) + \int_S \mu(y) \frac{\cos \psi_{xy}}{|x-y|^2} dS_y \quad (114)$$

$$\left(\frac{\partial V^{(0)}}{\partial \mathbf{n}}\right)_-(x) = 2\pi\mu(x) + \frac{\partial V^{(0)}(x)}{\partial \mathbf{n}} = 2\pi\mu(x) + \int_S \mu(y) \frac{\cos \psi_{xy}}{|x-y|^2} dS_y \quad (115)$$

**Proof.** Let  $V^{(1)}$  be the double layer potential on  $S$  with a density  $\mu$ .

We introduce the function  $w_1(x', x) = \frac{\partial V^{(0)}(x')}{\partial \mathbf{n}_x} + V^{(1)}(x'), x' \notin S, x \in S$ , and

wish to prove that, as  $x' \rightarrow x \in S$  and  $x' \in \mathbf{n}_x$ ,

$$w_1(x', x) \xrightarrow{x \in S} W_1(x, x) = \frac{\partial V^{(0)}(x)}{\partial \mathbf{n}} + V^{(1)}(x) \quad (116)$$

By properties of the potentials of a simple and a double layer on a surface,  $w_1(x', x)$  is a continuous function on  $S$ . Employing (98) and (100), we can write  $w_1(x', x)$  in the form of an integral:  $w_1(x', x) = \int_S \mu(y) \frac{\cos \psi_{x'y} + \cos \varphi_{x'y}}{|x'-y|^2} dS_y$ .

Let us take an  $\varepsilon > 0$ . We estimate the difference  $|w_1(x', x) - w_1(x, x)| \leq$

$$\left( \int_{U_x} + \int_{S \setminus U_x} \right) |\mu(y)| \left| \frac{\cos \psi_{x'y} + \cos \varphi_{x'y}}{|x'-y|^2} - \frac{\cos \psi_{xy} + \cos \varphi_{xy}}{|x-y|^2} \right| dS_y, U_x = S \cap U(x, \delta) \quad (117)$$

In view of the estimate (102) and (103), the first integral on the right-hand side of (117) does not exceed the (absolutely) convergent integral

$\int_{U_x} |\mu(y)| \left( \frac{3c}{|x'-y|^{2-\alpha}} + \frac{6c}{|x-y|^{2-\alpha}} \right) dS_y$ , and therefore can be made less than  $\varepsilon/2$  for a sufficiently small  $\delta = \delta_x$ . Moreover, the integrand in (117) is a uniformly continuous function in the variables  $(x', x, y)$  for  $|x' - x| \leq \frac{\delta}{3}, x \in S$ , and  $y \in S \setminus U_x$  and vanishes at  $x' = x$ . Therefore, there is a number  $\delta' \leq \frac{\delta}{3}$  such that for all  $x' \in U(x, \delta')$  the second

integral on the right-hand side of (117) is always less than  $\varepsilon/2$ .

Hence,  $|w_1(x', x) - w(x, x)| < \varepsilon$ ,  $x' \in U(x, \delta')$ ,  $x' \in \mathbf{n}_x$ ,  $x \in S$ , which proves the limiting relation (116).

By the Theorem of discontinuity of the double layer potential,  $V^{(1)}$  belongs to  $C(\bar{\Omega})$  and  $V_+^{(1)}(x) = 2\pi\mu(x) + V^{(1)}(x)$ . Whence, the limiting relation (116), as  $x' \rightarrow x \in S$  and  $x' \in \mathbf{n}_x$ , takes the form

$$\frac{\partial V^{(0)}(x')}{\partial \mathbf{n}_x} \xrightarrow{x \in S} -V_+^{(1)}(x) + w_1(x, x) = -2\pi\mu(x) + \frac{\partial V^{(0)}(x)}{\partial \mathbf{n}_x},$$

from which we conclude that the regular normal derivative  $\left(\frac{\partial V^{(0)}}{\partial \mathbf{n}}\right)_+$  on  $S$  from the outside exists and, since

$$\frac{\partial V^{(0)}}{\partial \mathbf{n}} = \int_S \mu(y) \frac{\cos \psi_{xy}}{|x-y|^2} dS_y = \int_S \mu(y) \frac{\partial}{\partial \mathbf{n}_x} \frac{1}{|x-y|} dS_y, \quad x \in S,$$

is expressed by equations (114). The other case can be considered similarly.

Combining (114) and (115), we find that

$$4\pi\mu(x) = \left(\frac{\partial V^{(0)}}{\partial \mathbf{n}}\right)_-(x) - \left(\frac{\partial V^{(0)}}{\partial \mathbf{n}}\right)_+(x), \quad x \in S \quad (118)$$

### **3.5 Reducing Boundary Value Problems to Boundary Integral Equations**

If we use the theory of the Newtonian potential, we can reduce the Dirichlet and Neumann problems for Laplace's equation to Fredholm integral equations with a polar kernel.

Let  $S$  be a sufficiently smooth surface. Now, let us consider the interior Dirichlet problem

$$(IDP): \begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = u_0^- \text{ on } S \end{cases} \quad \text{and the exterior Dirichlet problem (EDP):} \quad \begin{cases} \Delta u = 0 \text{ in } \Omega_1 \\ u = u_0^+ \text{ on } S \end{cases}.$$

Then, we will find the solutions of these Dirichlet problems in the form of the double

layer potential  $V^{(1)}(x) = \int_S v(y) \frac{\cos \varphi_{xy}}{|x-y|^2} dS_y$ , where  $v$  is an unknown density continuous

on  $S$ . The function  $V^{(1)}$  is harmonic in both  $\Omega$  and  $\Omega_1$ , belongs to the classes

$C(\bar{\Omega})$ ,  $C(\bar{\Omega}_1)$ , and  $C(S)$ , and vanishes at infinity. Therefore,  $V^{(1)}$  is the solution of the

$$\text{interior or exterior Dirichlet problem iff } V_{\mp}^{(1)}(x) = u_0^{\mp}(x), \quad x \in S, \quad (119)$$

where  $V_{\mp}^{(1)}(x)$  are the limiting values of  $V^{(1)}$  from the inside and outside of  $S$ .

By the Theorem of the discontinuity of the double Layer potential, equations (119)

$$\text{take the form } \mp 2\pi v(x) + \int_S v(y) \frac{\cos \varphi_{xy}}{|x-y|^2} ds_y = u_0^\mp(x), x \in S, \quad (120)$$

These are Fredholm integral equations in the unknown density  $v$ .

Hence, the interior or exterior Dirichlet problems, respectively, are given by

$$u_0^- = -2\pi v(x) + \int_S v(y) \frac{\cos \varphi_{xy}}{|x-y|^2} ds_y, x \in S \text{ and } u_0^+ = 2\pi v(x) + \int_S v(y) \frac{\cos \varphi_{xy}}{|x-y|^2} ds_y, x \in S. \text{ If we introduce a real parameter } \lambda \text{ and the kernel } \mathcal{K}(x, y) = \frac{\cos \varphi_{xy}}{2\pi|x-y|^2} \quad (121)$$

we can write the integral equations (120) in the single form

$$v(x) = \lambda \int_S \mathcal{K}(x, y)v(y)ds_y + f(x), x \in S \quad (122)$$

For the interior Dirichlet problem  $\lambda = 1$  and  $f = -\frac{u_0^-}{2\pi}$ , while for the exterior Dirichlet problem  $\lambda = -1$  and  $f = \frac{u_0^+}{2\pi}$ .

Again, let us consider the interior Neumann problem

$$\text{(INP): } \begin{cases} \Delta u = 0 \text{ in } \Omega \\ \frac{\partial u}{\partial \mathbf{n}} = u_1^- \text{ on } S \end{cases} \text{ and the exterior Neumann problem (ENP): } \begin{cases} \Delta u = 0 \text{ in } \Omega_1 \\ \frac{\partial u}{\partial \mathbf{n}} = u_1^+ \text{ on } S \end{cases}$$

Then, we will find the solutions of these Neumann problems in the form of the simple layer potential  $V^{(0)}(x) = \int_S \frac{\mu(y)}{|x-y|} ds_y$ , where  $\mu$  is an unknown density continuous on  $S$ . The function  $V^{(0)}$  is harmonic in both  $\Omega$  and  $\Omega_1$  and continuous in  $\mathbb{R}^3$ , has regular normal derivatives  $\left(\frac{\partial V^{(0)}}{\partial \mathbf{n}^\mp}\right)$  on  $S$  from the inside and outside of  $S$ , and vanishes at infinity. Therefore,  $V^{(0)}$  is the solution of the interior or exterior Neumann problem iff  $\left(\frac{\partial V^{(0)}}{\partial \mathbf{n}^\mp}\right)(x) = u_1^\mp(x) \quad x \in S \quad (123)$

By the Theorem of the discontinuity of the normal derivative of the simple Layer Potential, equations (123) transform into Fredholm integral equations

$$\pm 2\pi\mu(x) + \int_S \mu(y) \frac{\cos \psi_{xy}}{|x-y|^2} ds_y = u_1^\mp(x), x \in S \quad (124)$$

in the unknown density  $\mu$ .

Hence, the interior or exterior Neumann problems, respectively, are given by

$$u_1^- = 2\pi\mu(x) + \int_S \mu(y) \frac{\cos \psi_{xy}}{|x-y|^2} ds_y, x \in S \text{ and}$$

$u_1^+ = -2\pi\mu(x) + \int_S \mu(y) \frac{\cos\psi_{xy}}{|x-y|^2} ds_y$ ,  $x \in S$ . From the fact that  $\psi_{xy} = \varphi_{yx}$  when  $x, y \in S$  and from (121) it follows that the kernel of the integral equations (124) is  $\mathcal{K}(y, x) = \mathcal{K}^*(x, y)$ , so that equations (120) and (124) are adjoint to each other.

Introducing the parameter  $\lambda$ , we can write the integral equations (124) in the single form  $\mu(x) = \lambda \int_S \mathcal{K}^*(x, y)\mu(y)ds_y + g(x)$ ,  $x \in S$ , (125)

For the interior Neumann problem  $\lambda = -1$  and  $g = -\frac{u_1^-}{2\pi}$ , while for the exterior Neumann problem  $\lambda = 1$  and  $g = -\frac{u_1^+}{2\pi}$ .

### **3.6 Investigation of the Boundary Integral Equations of Potential Theory**

**Theorem 1** The interior Dirichlet problem and exterior Neumann problems are solvable for any continuous data  $u_0^-$  and  $u_1^+$ , and their solutions are the double and simple layer potentials, respectively.

**Proof.** First we will prove that  $\lambda = 1$  is not an eigenvalue of the kernel  $\mathcal{K}^*(x, y)$ . Assume the contrary, i.e. let  $\lambda = 1$  be an eigenvalue of this kernel and the corresponding eigenfunction  $\mu^*(x) = \int_S \mathcal{K}^*(x, y)\mu^*(y)ds_y = \frac{1}{2\pi} \int_S \mu^*(y) \frac{\cos\psi_{xy}}{|x-y|^2} ds_y$ , (126)

The eigenfunction  $\mu^*$  belongs to  $C(S)$ . We construct the simple layer potential  $V^{(0)}$  with a density  $\mu^*$ . The function  $V^{(0)}$  is harmonic outside  $S$  and continuous on  $\mathbb{R}^3$  and vanishes at infinity.

Moreover, by formula (114):

$$\left(\frac{\partial V^{(0)}}{\partial \mathbf{n}}\right)_+(x) = -2\pi\mu(x) + \frac{\partial V^{(0)}(x)}{\partial \mathbf{n}} = -2\pi\mu(x) + \int_S \mu(y) \frac{\cos\psi_{xy}}{|x-y|^2} ds_y$$
 and equation

(126), its regular normal derivative on  $S$  from the outside of  $S$  is zero. That is,

$$\begin{aligned} \left(\frac{\partial V^{(0)}}{\partial \mathbf{n}}\right)_+(x) &= -2\pi \left(\frac{1}{2\pi} \int_S \mu^*(y) \frac{\cos\psi_{xy}}{|x-y|^2} ds_y\right) + \int_S \mu^*(y) \frac{\cos\psi_{xy}}{|x-y|^2} ds_y \\ &= - \int_S \mu^*(y) \frac{\cos\psi_{xy}}{|x-y|^2} ds_y + \int_S \mu^*(y) \frac{\cos\psi_{xy}}{|x-y|^2} ds_y = 0 \\ \therefore \left(\frac{\partial V^{(0)}}{\partial \mathbf{n}}\right)_+(x) &= 0 \end{aligned}$$

From this, by Theorem 4 of sec.3.2 concerning the uniqueness of the solution of the exterior Neumann problem, we conclude that  $V^{(0)}(x) \equiv 0$  when  $x \in \overline{\Omega}_1$  and, for one,  $V^{(0)} = 0$  on  $S$ .

But then, by Theorem 2 of sec.3.2 on the uniqueness of the solution of the interior Dirichlet problem,  $V^{(0)}(x) \equiv 0$ ,  $x \in \overline{\Omega}$ . Thus,  $V^{(0)}(x) \equiv 0$  when  $x \in \mathbb{R}^3$ .

Therefore, using the formula (118):  $4\pi\mu(x) = \left(\frac{\partial V^{(0)}}{\partial \mathbf{n}}\right)_-(x) - \left(\frac{\partial V^{(0)}}{\partial \mathbf{n}}\right)_+(x)$ ,  $x \in S$ , we conclude that  $\mu^*(x) \equiv 0$  when  $x \in S$ .

Thus,  $\lambda = 1$ , is not an eigenvalue of the kernel  $\mathcal{K}^*(x, y)$ . From this, by Fredholm's second theorem,  $\lambda = 1$  is neither an eigenvalue of  $\mathcal{K}^*(x, y)$ . But then, according to Fredholm's third and first Theorems, the integral equations (122) and (125) with  $\lambda = 1$  are uniquely solvable for all continuous  $u_0^-$  and  $u_1^+$ , and hence the assertion follows.

**Theorem 2** The exterior Dirichlet problem is solvable for any continuous function  $u_0^+$ , and its solution is given by the sum of the double layer potential and the potential

$$\frac{1}{|x|} \int_S u_0^+ \mu_0(x) dS$$

**Proof.** We will assume that  $0 \in \Omega$ . We seek the solution of the exterior Dirichlet problem in the form of the double layer potential  $V^{(1)}$  with an unknown density  $v$  on  $S$  and the Newtonian potential  $\frac{\alpha}{|x|}$  produced by a charge at the point  $x=0$  of unknown

$$\text{magnitude } \alpha : u(x) = V^{(1)}(x) + \frac{\alpha}{|x|} = \int_S v(y) \frac{\cos \varphi_{xy}}{|x-y|^2} dS_y + \frac{\alpha}{|x|} \quad (127)$$

The corresponding integral equation (122) is

$$v(x) = - \int_S \mathcal{K}(x, y) v(y) dS_y + \frac{u_0^+(x)}{2\pi} - \frac{\alpha}{2\pi|x|} \quad (128)$$

By what has been proved, the integral equation (128) is solvable iff

$$\frac{1}{2\pi} \int_S [u_0^+(x) - \frac{\alpha}{|x|}] \mu_0(x) dS = 0 \quad (129)$$

Since  $0 \in \Omega$ , and normalizing the eigenfunction  $\mu_0$  we have

$$\int_S \frac{\mu_0(y)}{|y|} dS = V^{(0)}(0) = 1, x \in \overline{\Omega}, \text{ due to which the solvability condition (129) takes the}$$

$$\text{form } \alpha = \int_S u_0^+(x) \mu_0(x) dS \quad (130)$$

That is,  $\int_S u_0^+(x) \mu_0(x) dS = \int_S \frac{\alpha}{|x|} \mu_0(x) dS$ , from (129)

$$= \int_S \alpha \cdot \frac{\mu_0(x)}{|x|} dS = \alpha \underbrace{\int_S \frac{\mu_0(x)}{|x|} dS}_{=1} = \alpha$$

Hence,  $\alpha = \int_S u_0^+(x) \mu_0(x) dS$ .

Therefore, the exterior Dirichlet problem is solvable for any continuous function  $u_0^+$ , and

$$\begin{aligned} \text{its solution is given by } u(x) &= V^{(1)}(x) + \frac{\alpha}{|x|} = \int_S v(y) \frac{\cos \varphi_{xy}}{|x-y|^2} dS_y + \frac{\int_S u_0^+(x) \mu_0(x) dS}{|x|} \\ &= \int_S v(y) \frac{\cos \varphi_{xy}}{|x-y|^2} dS_y + \frac{1}{|x|} \int_S u_0^+(x) \mu_0(x) dS. \end{aligned}$$

## **CHAPTER 4: BOUNDARY VALUE PROBLEMS FOR ELLIPTIC**

### **EQUATIONS WITH VARIABLE COEFFICIENT IN $\mathbb{R}^3$**

#### **4.1 Parametrix, Potential Type Operators and One-Operator Green's Identities**

A fundamental solution is generally not available if the coefficients of the original partial differential are not constants. Then we use a parametrix (Levi function) in Green formulae allows the reduction of a boundary value problem to one operator boundary domain integral equations.

#### **Scalar Elliptic Differential Operator**

$$L_a := L_a(x, \partial_x) := \sum_{i=1}^3 \frac{\partial}{\partial x_i} (a(x) \frac{\partial}{\partial x_i}) \quad (131)$$

$$E_a(u, v) := \sum_{i=1}^3 a(x) \frac{\partial u(x)}{\partial x_i} \cdot \frac{\partial v(x)}{\partial x_i} = a(x) \nabla u(x) \nabla v(x) \quad (132)$$

**Verification:** Since  $\sum_{i=1}^3 a(x) \frac{\partial u(x)}{\partial x_i} \cdot \frac{\partial v(x)}{\partial x_i} = a(x) \left\{ \frac{\partial}{\partial x_1} u(x) \frac{\partial}{\partial x_1} v(x) + \frac{\partial}{\partial x_2} u(x) \frac{\partial}{\partial x_2} v(x) + \frac{\partial}{\partial x_3} u(x) \frac{\partial}{\partial x_3} v(x) \right\}$

$$= a(x) \left( \frac{\partial}{\partial x_1} u(x), \frac{\partial}{\partial x_2} u(x), \frac{\partial}{\partial x_3} u(x) \right) \cdot \left( \frac{\partial}{\partial x_1} v(x), \frac{\partial}{\partial x_2} v(x), \frac{\partial}{\partial x_3} v(x) \right)$$

$$= a(x) \nabla u(x) \nabla v(x)$$

**Classical Co-normal**

#### **Derivative Operator**



$$T_a^\pm(x, \mathbf{n}(x), \partial_x) := \sum_{i=1}^3 a(x) \mathbf{n}_i(x) \left( \frac{\partial}{\partial x_i} \right)^\pm = a(x) \left( \frac{\partial}{\partial \mathbf{n}(x)} \right)^\pm \quad (133)$$

**Verification:** Let  $T_a^\pm(x, \mathbf{n}(x), \partial_x)u(x) := \sum_{i=1}^3 a(x) \mathbf{n}_i(x) \left( \frac{\partial}{\partial x_i} u(x) \right)$

$$= a(x) \left\{ \mathbf{n}_1(x) \frac{\partial}{\partial x_1} u(x) + \mathbf{n}_2(x) \frac{\partial}{\partial x_2} u(x) + \mathbf{n}_3(x) \frac{\partial}{\partial x_3} u(x) \right\}$$

$$= a(x) (\mathbf{n}_1(x), \mathbf{n}_2(x), \mathbf{n}_3(x)) \cdot \left( \frac{\partial}{\partial x_1} u(x), \frac{\partial}{\partial x_2} u(x), \frac{\partial}{\partial x_3} u(x) \right)$$

$$= a(x) \frac{\partial}{\partial \mathbf{n}(x)} u(x), \text{ (which is the directional derivative of } u(x) \text{ in the}$$

direction of the unit normal vector  $\mathbf{n}(x)$ .) where  $\mathbf{n}(x)$  is the exterior to  $\Omega$  unit normal vector at the point  $x \in S$

**Weak Co-**

**normal Derivative**

$$\langle T_a^\pm u, v \rangle_S := \pm \int_{\Omega^\pm} [v L_a u + E_a(u, v)] d\Omega(x), \forall v \in H^1(\Omega) \quad (134)$$

For simplicity:

$$\int_S v(x) T_a u(x) d\Gamma(x) = \int_\Omega \{v(x) L_a u(x) + a(x) \nabla v(x) \nabla u(x)\} d\Omega(x)$$

**Verification:**  $\int_\Omega [v L_a u + E_a(u, v)] d\Omega(x) = \int_\Omega \{v(x) [\nabla a(x) \cdot \nabla u(x) + a(x) \Delta u(x)] + a(x) \nabla v(x) \nabla u(x)\} d\Omega(x).$

$$= \int_\Omega \{a(x) v(x) \Delta u(x) + [\nabla a(x) v(x) + a(x) \nabla v(x)] \nabla u(x)\} d\Omega(x)$$

$$= \int_\Omega \nabla [a(x) v(x) \nabla u(x)] d\Omega$$

$$= \int_\Omega v(x) a(x) \nabla u(x) \cdot \mathbf{n}(x) d\Gamma, \text{ by divergence Theorem.}$$

$$\langle T_a u, v \rangle_S := \int_S v(x) T_a u(x) d\Gamma(x) = \int_\Omega \{v(x) L_a u(x) + a(x) \nabla v(x) \nabla u(x)\} d\Omega(x)$$

which is the First Green Identity.

**4.2 Formulation of Boundary Value Problems**

In many instances, solutions of differential equations must be found which satisfy certain conditions. These conditions which may arise quite naturally as in physical problems are often referred to as initial or boundary conditions depending on whether they are specified at one or more than one point. The corresponding problem of finding such a

solution are then collectively called boundary value problem.

The equation that represents those boundary conditions may involve  $u$  itself at points on the boundary, in addition, some conditions on the continuity of  $u$  and derivative within the domain and on the boundary may be required. Such a set of requirement constitute a boundary value problem in the function  $u$ . We use the terminology whenever the differential equation is accompanied by some boundary condition.

Let  $u$  denotes the dependent variable in a boundary problem. A condition that prescribes the value of  $u$  itself along a portion of the boundary is known as a Dirichlet condition, and also a Neumann condition prescribes the value of normal derivatives  $\frac{\partial u}{\partial n}$  on a part of the boundary.

Now, let us formulate the following mixed boundary value problem as shown below.

$$L_a u(x) = f(x) \text{ in } \Omega^+ \quad (135)$$

$$u^+ = \varphi_0 \text{ on } S_D \quad (136)$$

$$T_a u(x) = \psi_0 \text{ on } S_N \quad (137)$$

where  $\varphi_0 \in H^{1/2}(S_D)$ ,  $\psi_0 \in H^{-1/2}(S_N)$  and  $f \in L_2(\Omega^+)$ .

Equation (135) is understood in distributional sense, condition (136) is understood in the trace sense while condition (137) is understood in the functional sense.

### 4.3 Parametrix

**Definition.** A function  $P_a(x, y)$  of two variables  $x, y \in \Omega$  is a parametrix (Levi function) for the operator  $L_a(x, \partial_x)$  in  $\mathbb{R}^3$  if  $L_a(x, \partial_x)P_a(x, y) = \delta(x - y) + R_a(x, y)$  where  $\delta(x - y)$  is Dirac delta function defined by  $\delta(x - y) = 0$  if  $x \neq y$  and  $\int \delta(x - y)dx = 1$  and  $R_a(x, y)$  is the remainder which posses a weak (integrable) singularity at  $x=y$ .

In particular, the function  $P_a(x, y) = \frac{-1}{4\pi a(y)|x-y|}$ ,  $x, y \in \mathbb{R}^3$  is a parametrix and the corresponding weakly singular remainder is  $R_a(x, y) =$

$$\sum_{i=1}^3 \frac{x_i - y_i}{4\pi a(y)|x-y|^3} \cdot \frac{\partial a(x)}{\partial x_i}, x, y \in \mathbb{R}^3$$

**Definition.** A function  $F_a(x, y)$  is a fundamental solution for the operator  $L_a(x, \partial_x)$  in  $\mathbb{R}^3$  if  $L_a(x, \partial_x)F_a(x, y) = \delta(x - y)$ .

**Remark.** If the parametrix is a fundamental solution of the linear operator  $L_a$ , that is,  $P_a(x, y) = F_a(x, y)$ , then  $R_a(x, y) = 0$

**Proof**  $L_a(x, \partial_x)P_a(x, y) = \delta(x - y) + R_a(x, y)$ , since  $P_a(x, y)$  is a parametrix.

$$L_a(x, \partial_x)F_a(x, y) = \delta(x - y) \quad , \text{ since } F_a(x, y) \text{ is a fundamental solution.}$$

$$\Rightarrow L_a(x, \partial_x)P_a(x, y) - L_a(x, \partial_x)F_a(x, y) = R_a(x, y) \quad .$$

$$\text{But, } P_a(x, y) = F_a(x, y). \quad \Rightarrow R_a(x, y) = 0.$$

#### 4.4 Potential Type Operators

Let us first introduce the single and double layer surface potential operators and remainder volume potential operators.

##### Volume Potentials

The parametrix-based Newtonian and the remainder volume potentials are given, respectively, by:

$$\mathcal{P}_a g(y) := \int_{\Omega} P_a(x, y)g(x)dx \quad \text{and} \quad \mathcal{R}_a g(y) := \int_{\Omega} R_a(x, y)g(x)dx \quad (138)$$

**Remark.**  $\mathcal{P}_a g = \frac{1}{a} \mathcal{P}_{\Delta} g$  and  $\mathcal{R}_a g = \frac{-1}{a} \sum_{i=1}^3 \partial_j \mathcal{P}_{\Delta} [g(\partial_j a)]$  where

$\mathcal{P}_{\Delta} h(y) := \frac{-1}{4\pi} \int_{\Omega} \frac{h(x)}{|x-y|} dx$  is the Newtonian potential corresponding to the fundamental solution of the Laplacian  $\Delta$ .

**Proof.**  $\mathcal{P}_a g := \int_{\Omega} P_a(x, y)g(x)dx$

$$= \int_{\Omega} \frac{-1}{4\pi a(y)|x-y|} g(x)dx \quad , \text{ since } P_a(x, y) = \frac{-1}{4\pi a(y)|x-y|}$$

$$= \frac{1}{a(y)} \left[ \frac{-1}{4\pi} \int_{\Omega} \frac{g(x)}{|x-y|} dx \right]$$

$$= \frac{1}{a} \mathcal{P}_{\Delta} g \quad , \text{ by the given condition above.}$$

$$\mathcal{R}_a g := \int_{\Omega} R_a(x, y)g(x)dx$$

$$\begin{aligned}
&= \int_{\Omega} \sum_{i=1}^3 \frac{x_i - y_i}{4\pi a(y)|x-y|^3} \cdot \frac{\partial a(x)}{\partial x_i} g(x) dx, \text{ since } R_a(x, y) \text{ is given above.} \\
&= \frac{-1}{a(y)} \left[ \frac{-1}{4\pi} \int_{\Omega} \sum_{i=1}^3 \left[ \frac{x_i - y_i}{|x-y|^3} \cdot \frac{\partial a(x)}{\partial x_i} \right] g(x) dx \right] \\
&= \frac{-1}{a(y)} \left[ \frac{-1}{4\pi} \int_{\Omega} \sum_{i=1}^3 \left[ \frac{\partial}{\partial y_i} \left( \frac{1}{|x-y|} \right) \frac{\partial a(x)}{\partial x_i} \right] g(x) dx \right] \\
&= \frac{-1}{a(y)} \sum_{j=1}^3 \partial_j \mathcal{P}_{\Delta} [g(\partial_j a)]
\end{aligned}$$

### **Surface Potentials**

Single and double layer surface potential operators are, respectively, given by:

$$V_a g(y) := - \int_S P_a(x, y) g(x) dS_x, y \notin S \quad (139)$$

$$W_a g(y) := - \int_S [T_a(x, \mathbf{n}(x), \partial_x) P_a(x, y)] g(x) dS_x, y \notin S \quad (140)$$

### **Pseudo-Differential Operators**

The boundary integral (pseudo-differential) operators of direct value simple layer potential  $\mathcal{V}_a$  and the double layer potential  $\mathcal{W}_a$  are, respectively, given by:

$$\mathcal{V}_a g(y) := - \int_S P_a(x, y) g(x) dS_x, y \in S \quad (141)$$

$$\mathcal{W}_a g(y) := - \int_S [T_a(x, \mathbf{n}(x), \partial_x) P_a(x, y)] g(x) dS_x, y \in S \quad (142)$$

$$\mathcal{W}'_a g(y) := - \int_S [T_a(y, \mathbf{n}(y), \partial_y) P_a(x, y)] g(x) dS_x, y \in S \quad (143)$$

$$\mathcal{L}_a^{\pm} g(y) := [T_a^{\pm}(y, \mathbf{n}(y), \partial_y) P_a(x, y)] W_a g(x) dS_x, y \in S \quad (144)$$

where  $\mathcal{L}_a^{\pm} g(y)$  is the co-normal derivative of double layer and  $\mathcal{W}'_a g(y)$  is the direct value associated with the co-normal derivative of single layer.

From equations 138-144 we deduce the representation of the parametric based surface potentials boundary operators in-terms of their counter parts for  $a=1$ .

$$V_a g = \frac{1}{a} V_\Delta g \quad , \quad W_a g = \frac{1}{a} W_\Delta (ag) \quad (145)$$

$$\mathcal{V}_a g = \frac{1}{a} \mathcal{V}_\Delta g \quad , \quad \mathcal{W}_a g = \frac{1}{a} \mathcal{W}_\Delta (ag) \quad (146)$$

$$\mathcal{W}'_a g = \mathcal{W}'_\Delta g + \left[ a \frac{\partial}{\partial \mathbf{n}} \left( \frac{1}{a} \right) \right] \mathcal{V}_\Delta g \quad (147)$$

$$\mathcal{L}_a^\pm g = \mathcal{L}_\Delta^\pm (ag) + \left[ a \frac{\partial}{\partial \mathbf{n}} \left( \frac{1}{a} \right) \right] W_\Delta^\pm (ag) \quad (148)$$

**Proof.** Consider  $P_a(x, y) = \frac{-1}{4\pi a(y)|x-y|} = \frac{1}{a(y)} \left( \frac{-1}{4\pi|x-y|} \right) = \frac{P_\Delta(x, y)}{a(y)}$

where  $P_\Delta(x, y) = \frac{-1}{4\pi|x-y|}$ , to verify the relations as follows.

**Proof (145).**

$$\begin{aligned} \diamond V_a g(y) &:= - \int_S P_a(x, y) g(x) dS_x, \quad y \notin S \\ &= - \int_S \frac{P_\Delta(x, y)}{a(y)} g(x) dS_x \\ &= \frac{-1}{a(y)} \int_S P_\Delta(x, y) g(x) dS_x = \frac{1}{a} V_\Delta g \quad \therefore V_a g = \frac{1}{a} V_\Delta g \end{aligned}$$

$$\begin{aligned} \diamond W_a g(y) &:= - \int_S [T_a(x, \mathbf{n}(x), \partial_x) P_a(x, y)] g(x) dS_x, \quad y \notin S \\ &= \int_S \left[ \sum_{i=1}^3 a(x) \mathbf{n}_i(x) \frac{\partial}{\partial x_i} P_a(x, y) \right] g(x) dS_x, \quad y \notin S \\ &= \frac{-1}{a(y)} \int_S \sum_{i=1}^3 \mathbf{n}_i(x) \frac{\partial P_\Delta(x, y)}{\partial x_i} (ag)(x) dS_x = \frac{1}{a} W_\Delta (ag) \end{aligned}$$

**Proof (146).**

$$\begin{aligned} \diamond \mathcal{V}_a g(y) &:= - \int_S P_a(x, y) g(x) dS_x, \quad y \in S \\ &= - \int_S \frac{P_\Delta(x, y)}{a(y)} g(x) dS_x \\ &= \frac{-1}{a(y)} \int_S P_\Delta(x, y) g(x) dS_x = \frac{1}{a} \mathcal{V}_\Delta g \end{aligned}$$

$$\begin{aligned} \diamond \mathcal{W}_a g(y) &:= - \int_S [T_a(x, \mathbf{n}(x), \partial_x) P_a(x, y)] g(x) dS_x, \quad y \in S \\ &= - \int_S \left[ \sum_{i=1}^3 a(x) \mathbf{n}_i(x) \frac{\partial}{\partial x_i} P_a(x, y) \right] g(x) dS_x \end{aligned}$$

$$= \frac{-1}{a(y)} \int_S \sum_{i=1}^3 \mathbf{n}_i(x) \frac{\partial P_\Delta(x,y)}{\partial x_i} (ag)(x) dS_x = \frac{1}{a} \mathcal{W}_\Delta(ag)$$

**Proof (147).**

$$\begin{aligned} \diamond \mathcal{W}'_a g &:= - \int_S [T_a(y, \mathbf{n}(y), \partial_y) P_a(x, y)] g(x) dS_x, y \in S \\ &= - \int_S [T_a(y, \mathbf{n}(y), \partial_y) \frac{-1}{4\pi a(y)|x-y|}] g(x) dS_x, y \in S \\ &= - \int_S [\sum_{i=1}^3 a(y) \mathbf{n}_i(y) \frac{\partial}{\partial y} [\frac{-1}{4\pi a(y)|x-y|}]] g(x) dS_x \\ &= - \int_S [\sum_{i=1}^3 a^2(y) \mathbf{n}_i(y) \frac{\partial}{\partial y} \frac{P_\Delta(x,y)}{a^2(y)}] g(x) dS_x + \\ &\quad \int_S \sum_{i=1}^3 \frac{a(y) \mathbf{n}_i(y) P_\Delta(x,y) \frac{\partial a(y)}{\partial y}}{a^2(y)} g(x) dS_x \\ &= \mathcal{W}'_\Delta g + \sum_{i=1}^3 \frac{\mathbf{n}_i(y) \frac{\partial a(y)}{\partial y}}{a(y)} \int_S P_\Delta(x, y) g(x) dS_x \\ &= \mathcal{W}'_\Delta g + [-a(y) \frac{\partial}{\partial y} (\frac{1}{a})] (-\mathcal{V}_\Delta g) \\ &= \mathcal{W}'_\Delta g + [a \frac{\partial}{\partial \mathbf{n}} (\frac{1}{a})] \mathcal{V}_\Delta g \end{aligned}$$

**Proof (148).**

$$\begin{aligned} \diamond \mathcal{L}_a^+ g &:= [T_a(y, \mathbf{n}(y), \partial_y) W_a g(y)]^+ \\ &= \sum_{i=1}^3 [a(y) \mathbf{n}_i(y) \frac{\partial}{\partial y} W_a g(y)] \\ &= - \sum_{i=1}^3 [a(y) \mathbf{n}_i(y) \frac{\partial}{\partial y} \int_S [T_a(x, \mathbf{n}(x), \partial_x) P_a(x, y)] g(x) dS_x] \\ &= \sum_{i=1}^3 a(y) \mathbf{n}_i(y) \frac{\partial}{\partial y} \cdot [- \int_S [\sum_{i=1}^3 a(x) \mathbf{n}_i(x) \frac{\partial}{\partial x_i} P_a(x, y)] g(x) dS_x] \\ &= - \sum_{i=1}^3 a(y) \mathbf{n}_i(y) \int_S [\sum_{i=1}^3 a(x) \mathbf{n}_i(x) \frac{\partial}{\partial y_i} (\frac{\frac{\partial}{\partial x_i} P_\Delta(x,y)}{a(y)})] g(x) dS_x \\ &= - \sum_{i=1}^3 a(y) \mathbf{n}_i(y) \int_S [\sum_{i=1}^3 a(x) \mathbf{n}_i(x) [\frac{\partial}{\partial y_i} (\frac{\partial}{\partial x_i} \frac{P_\Delta(x,y)}{a(y)})] a(y) - \\ &\quad \frac{\partial}{\partial x_i} \frac{P_\Delta(x,y)}{a(y)} \frac{\partial}{\partial y_i} a(y)] g(x) dS_x \end{aligned}$$

$$= \mathcal{L}_\Delta^+(ag) + \left[ a \frac{\partial}{\partial \mathbf{n}} \left( \frac{1}{a} \right) \right] W_\Delta^+(ag)$$

#### 4.5 One Operator Green Identities

By considering the weak co-normal derivative (134), we write the one operator first Green's identities for operator  $L_a$  as:

$$\int_\Omega v(x) L_a u(x) d\Omega(x) = \int_S v(x) T_a u(x) d\Gamma(x) - \int_\Omega E_a(u, v) d\Omega \quad (149)$$

And one-operator second Green identities as:

$$\int_\Omega [u(x) L_a v(x) - v(x) L_a u(x)] d\Omega(x) = \int_S [u(x) T_a v(x) - v(x) T_a u(x)] d\Gamma(x) \quad (150)$$

**Proof.** Let  $\int_\Omega v(x) L_a u(x) d\Omega(x) = \int_S v(x) T_a u(x) d\Gamma(x) - \int_\Omega E_a(u, v) d\Omega$  and

$$\int_\Omega u(x) L_a v(x) d\Omega(x) = \int_S u(x) T_a v(x) d\Gamma(x) - \int_\Omega E_a(u, v) d\Omega$$

from first Green identity and take their difference, then we get

$$\int_\Omega [u(x) L_a v(x) - v(x) L_a u(x)] d\Omega(x) = \int_S [u(x) T_a v(x) - v(x) T_a u(x)] d\Gamma(x).$$

Assume  $u$  is a solution of partial differential equation (135) and use the parametrix  $P_a(x, y)$  as  $v(x)$  in the one-operator second Green identity (150).

Then, we obtain linear one-operator third Green identity

$$\begin{aligned} u(y) - \int_S u(x) T_a P_a(x, y) d\Gamma(x) + \int_S P_a(x, y) T_a u(x) d\Gamma(x) + \\ \int_\Omega R_a(x, y) u(x) d\Omega(x) = \int_\Omega P_a(x, y) f(x) d\Omega(x) \end{aligned} \quad (151)$$

**Proof.** From one-operator second Green identity, we have

$$\int_\Omega \{u(x) L_a v(x) - v(x) L_a u(x)\} d\Omega(x) = \int_S \{u(x) T_a v(x) - v(x) T_a u(x)\} d\Gamma(x)$$

$$\text{Let } v(x) = P_a(x, y).$$

$$\begin{aligned} \Rightarrow \int_\Omega \{u(x) L_a P_a(x, y) - P_a(x, y) L_a u(x)\} d\Omega(x) = \int_S \{u(x) T_a P_a(x, y) - \\ P_a(x, y) T_a u(x)\} d\Gamma(x) \end{aligned}$$

$$\Rightarrow \int_\Omega u(x) L_a P_a(x, y) d\Omega(x) - \int_\Omega P_a(x, y) L_a u(x) d\Omega(x) =$$

$$\int_S u(x) T_a P_a(x, y) d\Gamma(x) - \int_S P_a(x, y) T_a u(x) d\Gamma(x)$$

Since  $u$  is a solution and  $P_a(x, y)$  is a parametrix of (135),

$$\begin{aligned}
&\Rightarrow \int_{\Omega} u(x)\{\delta(x-y) + R_a(x, y)\} d\Omega(x) - \int_{\Omega} P_a(x, y)f(x)d\Omega(x) = \\
&\quad \int_S u(x)T_a P_a(x, y) d\Gamma(x) - \int_S P_a(x, y)T_a u(x)d\Gamma(x) \\
&\Rightarrow \int_{\Omega} u(x)\delta(x-y) d\Omega(x) + \int_{\Omega} u(x)R_a(x, y) d\Omega(x) - \int_{\Omega} P_a(x, y)f(x)d\Omega(x) = \\
&\quad \int_S u(x)T_a P_a(x, y) d\Gamma(x) - \int_S P_a(x, y)T_a u(x)d\Gamma(x) \\
&\Rightarrow u(y) + \int_{\Omega} u(x)R_a(x, y) d\Omega(x) - \int_{\Omega} P_a(x, y)f(x)d\Omega(x) = \\
&\quad \int_S u(x)T_a P_a(x, y) d\Gamma(x) - \int_S P_a(x, y)T_a u(x)d\Gamma(x) \\
&\Rightarrow u(y) - \int_S u(x)T_a P_a(x, y) d\Gamma(x) + \int_S P_a(x, y)T_a u(x)d\Gamma(x) + \\
&\quad \int_{\Omega} R_a(x, y)u(x) d\Omega(x) = \int_{\Omega} P_a(x, y)f(x)d\Omega(x)
\end{aligned}$$

**Remark.** If the parametrix is a fundamental solution of the linear operator  $L_a$ , (i.e.

if  $P_a(x, y) = F_a(x, y)$ , then  $R_a(x, y) = 0$ .), then

$$\begin{aligned}
&\Rightarrow u(y) - \int_S u(x)T_a P_a(x, y) d\Gamma(x) + \int_S P_a(x, y)T_a u(x)d\Gamma(x) = \\
&\quad \int_{\Omega} P_a(x, y)f(x)d\Omega(x)
\end{aligned}$$

From the one-operator third Green identity (151), we can formulate the boundary domain integral equation as:  $u + R_a u - V_a T_a^+ u + W_a u^+ = P_a L_a u$  in  $\Omega$ .



## REFERENCES

- [1]. V.S.Vladimirov, Equations of Mathematical physics, Mir Publishers, Moscow (1984).
- [2]. V.S.Vladimirov, Generalized Functions in Mathematical physics, Mir Publishers, Moscow (1984).
- [3]. IAN N.S NEDDON, Elements of Partial Differential Equations, McGraw-Hill International Editions, (1988).
- [4]. R.Deumlich, Functional Analysis III, Fourier Transformation and Distribution, AAU (2001).
- [5]. O.chkadua, S.E. Mikhailov, and D.natroshvili: Analysis of direct boundary-domain integral equations for a mixed Boundary Value Problem with variable coefficient.
- [6]. Murray R .Spiegel, Applied Differential Equations, Prentice-Hall, inc. Englewood Cliffs, New Jersey 07632(1981).

[7]. Ioannis P. Stavroulakis and Stephan A. Tersian, Partial Differential Equations (2<sup>nd</sup> edition): An Introduction with Mathematica and MAPLE, World Scientific Publishing Co.pte.Ltd(2004).