ADDIS ABABA UNIVERSITY
COLLEGE OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS

Transfer Matrices

A Project Submitted to the Department of Mathematics of Addis Ababa University in Partial Fulfillment of the Requirements of the Master of Science Degree in Mathematics

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We,the undersigned, hereby certify that we have read and examined this project, a project on ***Transfer Matrices****, which is done by Wossenu Abate Mekonnen in partial fulfillment of the requirements for the degree of master of science and recommend to the school of graduate studies for acceptance of a project.*

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Contents

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Abstract

The transfer function of a linear time-invariant system describes the input output behavior of the system when all initial conditions are zero. Hence, it should be independent of the state variables used in a state space description of the system. A state space description of an input-output system can be regarded as a time-domain description. For linear time-invariant systems, transfer functions provide another way of describing input-output systems; this description is sometimes called a frequency domain description. To discuss transfer functions of continuous-time systems, we need the Laplace transform.

Chapter 1

PRELIMINARIES

1.1 The Laplace Transform

This section is devoted to the discussion of Laplace Transform,which converts certain types of initial value problems in to algebra problem.We solve the algebra problem and then apply the inverse transform to produce the solution of the IVP. The process may be diagrammed as follows :
IVP→algebra problem → solutions of algebra problem → solution of the IVP

Definition 1.1.1. *(The Laplace Transform) Suppose f(t) is defined for* $t \geq 0$ *. The Laplace Transform,*$\mathfrak{L}[f]$ *is the function defined by*

$$\mathfrak{L}[f](s) = \int_0^\infty e^{-st} f(t)dt$$

for all s such that this improper integral converges. Sometimes ,the Laplace Transform of f is denoted by $F(S) = \mathfrak{L}[f](s)$

Example 1.1.1. *For any real number a, we have*

$$\mathfrak{L}[e^{at}](s) = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{(a-s)t} dt = \lim_{k\to\infty} \int_0^k e^{(a-s)t} dt = \lim_{k\to\infty} [\frac{1}{a-s} e^{(a-s)t}|_0^k] = \frac{1}{s-a}$$

if $a - s < 0$ *or equivalently,*$s > a$ *Hence the Laplace transform of* $f(t) = e^{at}$ *is* $F(s) = \frac{1}{s-a}$ *for* $s > a$ *.*

Example 1.1.2. *Determine the Laplace Transform of* $f(t) = 1$,$t \geq 0$
solution:

$$F(s) = \int_0^\infty e^{-st} dt = \lim_{k\to\infty} \int_0^{\ k} e^{(-st)} dt$$

$$= \lim_{k\to\infty} [\frac{e^{-st}}{-s}|_0^k]$$

$$= \lim_{k\to\infty} [\frac{e^{-sk}}{-s} + \frac{1}{s}]$$

$$= \frac{1}{s}$$

for $s > 0$

Thus, $\mathfrak{L}[1](s) = \frac{1}{s}$ *for* $s > 0$

Theorem 1.1.1. *(Linearity of the transform) Let* f_1 *and* f_2 *be functions whose laplace transform exist for* $s > \alpha$ *and let* c *be constant.Then*

$$1.\mathfrak{L}[f_1 + f_2] = \mathfrak{L}[f_1] + \mathfrak{L}[f_2]$$

$$2.\mathfrak{L}[cf_1] = c\mathfrak{L}[f_1]$$

Proof. 1.

$$\mathfrak{L}[f_1 + f_2] = \int_0^\infty e^{-st}(f_1 + f_2)dt$$

$$= \int_0^\infty e^{-st}(f_1)dt + \int_0^\infty e^{-st}(f_2)dt$$

$$= \mathfrak{L}[f_1] + \mathfrak{L}[f_2]$$

2. similar □

Definition 1.1.2. *(piecewise continuity) 1.A function* f *is piecewise continuous on a finite interval* $[a, b]$ *if* f *is continues at every point in* $[a, b]$ *except possibly for finite number of points at which* f *has a jump discontinuity.*
2.A function f *is piecewise continuous on* $[0, \infty)$ *if* f *is piecewise continuous on* $[0, N]$ *for all* $N > 0$.

Definition 1.1.3. *A function* f *is said to be of exponential order* α *if there exist positive constants* T *and* M *such that*
$|f(t)| \le Me^{\alpha t}$ *for all* $t \ge T$

Theorem 1.1.2. *(Conditions for existence of the transform) If* f *is piece wise continuous on* $[0, \infty)$ *and of exponential order* α*,then* $\mathfrak{L}[f](s)$ *exists for* $s > \alpha$

1.1.1 properties of Laplace Transform

Theorem 1.1.3. *If the laplace transform* $\mathfrak{L}[f](s) = F(s)$ *exists for* $s > \alpha$ *then* $\mathfrak{L}[e^{at}f(t)](s) = F(s - a)$

Proof.

$$F(s - a) = \int_0^\infty e^{(a-s)t}f(t)dt$$

$$= \int_0^\infty e^{-st}e^{at}f(t)dt$$

$$= \int_0^\infty e^{at}e^{-st}f(t)dt$$

$$= \mathfrak{L}[e^{at} f(t)](s)$$

□

Theorem 1.1.4. *Let f(t)be continuous on* $[0, \infty)$ *and* $\dot{f}$*(t)be piecewise continuous on* $[0, \infty)$ *with both exponential order* α*,Then,for* $s > \alpha$*,*

$$\mathfrak{L}[\dot{f}(t)](s) = s\mathfrak{L}[f(t)](s) - f(0)$$

Theorem 1.1.5. *Let* $f(t), \dot{f}(t)...f^{(n-1)}(t)$*be continuous on* $[0, \infty]$ $f^{(n)}(t)$ *be piecewise continuous on* $[0, \infty]$ *,with all these function of exponential order* α*.Then for* $s > \alpha$

$$\mathfrak{L}[f^n(t)](s) = s^n \mathfrak{L}[f(t)](s) - s^{n-1} f(0) - s^{n-2} \dot{f}(0)...f^{n-1}(0)$$

1.1.2 Inverse Laplace Transform

If $F(s) = \mathfrak{L}[f](s)$ then we write $\mathfrak{L}^{-1}[F](t) = f(t)$ we call $\mathfrak{L}^{-1}$ the inverse laplace transform.In words,the inverse laplace transform of $F(s)$ is $f(t)$ if the laplace transform of $f(t)$ is $F(s)$. For instance

$$\mathfrak{L}^{-1}[\frac{1}{s-a}](t) = e^{at}$$

1.2 Taylor and Maclaurin series

Theorem 1.2.1. *If f has a power series representation (expansion) at a , that is, if*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Proof. We start by supposing that f is any function that can be represented by a power series

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4... \qquad (1.1)$$

Lets try to determine what the coefficients c_n must be in terms of f . To begin, notice that if we put $x = a$ in Equation 1.1, then all terms after the first one are 0 and we get

$$f(a) = c_0$$

we can differentiate the series in Equation 1.1 term by term:

$$\dot{f}(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3... \qquad (1.2)$$

and substitution of$x = a$ in Equation 1.2 gives

$$\dot{f}(a) = c_1$$

Now we differentiate both sides of Equation 1.2 and obtain

$$\ddot{f}(x) = 2c_2 + 2 \times 3c_3(x-a) + 3 \times 4c_4(x-a)^2... \tag{1.3}$$

Again we put $x = a$ in Equation 1.3. The result is

$$\ddot{f}(a) = 2c_2$$

Lets apply the procedure one more time. Differentiation of the series in Equation 1.3 gives

$$f^{(3)}(x) = 2 \times 3c_3 + 2 \times 3 \times 4c_4(x-a)... \tag{1.4}$$

and substitution of $x = a$ in Equation 1.4 gives

$$f^{(3)}(a) = 2 \times 3c_3 = 3!c_3$$

By now you can see the pattern. If we continue to differentiate and substitute $x = a$, we obtain

$$f^{(n)}(a) = 2 \times 3 \times 4 \times ... \times nc_n = n!c_n$$

Solving this equation for the nth coefficient c_n,we get

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This formula remains valid even for $n = 0$ if we adopt the conventions that$0! = 1$ and $f^{(0)} = f$. □

Substituting this formula for c_n back into the series, we see that if f has a power series expansion at a , then it must be of the following form.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{1.5}$$

The series in Equation 1.5 is called the Taylor series of the function f at a (or about a or centered at a). For the special case $a = 0$the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n \tag{1.6}$$

This case arises frequently enough that it is given the special name Maclaurin series.

Example 1.2.1. *Find the Maclaurin series of the function $f(x) = e^x$*

If $f(x) = e^x$, then $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = e^0 = 1$ for all . Therefore the Taylor series for f at 0 (that is, the Maclaurin series) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n = \sum_{n=0}^{\infty} \frac{x^{(n)}}{n!} = 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3!} + ...$$

1.3 Laurent Series

1 What is a Laurent series?

The Laurent series is a representation of a complex function f(z) as a series. Unlike the Taylor series which expresses f(z) as a series of terms with non-negative powers of z, a Laurent series includes terms with negative powers. A consequence of this is that a Laurent series may be used in cases where a Taylor expansion is not possible.

2. Calculating the Laurent series expansion

To calculate the Laurent series we use the standard and modified geometric series which are

$$\frac{1}{1-z} = \begin{cases} \sum_{n=0}^{\infty} z^n & , \quad |z| < 1; \\ -\sum_{n=1}^{\infty} \frac{1}{z^n} & , \quad |z| > 1 \end{cases}$$

Here $f(z) = \frac{1}{1-z}$ is analytic everywhere apart from the singularity at $z = 1$. Above are the expansions for f in the regions inside and outside the circle of radius 1, centred on $z = 0$, where $|z| < 1$ is the region inside the circle and $|z| > 1$ is the region outside the circle.

Example 1.3.1. *Determine the Laurent series for*

$$f(z) = \frac{1}{z+5}$$

that are valid in the regions (i) $z : |z| < 5$, and (ii) $z : |z| > 5$.

Solution

The region (i) is an open disk inside a circle of radius 5, centred on $z = 0$, and the region (ii) is an open annulus outside a circle of radius 5, centred on $z = 0$. To make the series expansion easier to calculate we can manipulate our f(z) into a form similar to the series expansion shown in equation (1). So

$$f(z) = \frac{1}{5(1+\frac{z}{5})} = \frac{1}{5(1-\frac{-z}{5})}$$

Now using the standard and modified geometric series, equation (1), we can calculate that

$$f(z) = \frac{1}{5(1-\frac{-z}{5})} = \begin{cases} \frac{1}{5}\sum_{n=0}^{\infty}(\frac{-z}{5})^n & , \quad |z| < 5; \\ -\frac{1}{5}\sum_{n=1}^{\infty}\frac{1}{(\frac{-z}{5})^n} & , \quad |z| > 5 \end{cases}$$

1.4 Introduction to systems and their representations

A system is a collection of parts that interact with each other to function as a whole. We are interested in dynamical systems. The components of a dynamical system evolve in time.Mathematically speaking,they are function of time,they will be called "signals".

Definition 1.4.1. *A dynamical system $\sum$ is determined by the following data:*
∗ a set T,called the time set;
∗ a set W,called the signal value set;
∗ a set $\beta \subseteq W^T$,called the behavior.

The set T is our mathematical model of time. We will deal with following cases:
$T = \mathbb{Z}$ or a subinterval, especially $\mathbb{N} = \{0, 1, 2, ...\}$(discrete time);
$T = \mathbb{R}$ or a subinterval, especially $\mathbb{R}_+ := [0, \infty)$ (continuous time).

A signal w is a function of time, taking its values in the signal value set W.
We write $w : T \to W$,$t \to w(t)$.
The set W^T is the set of all functions from T to W, therefore it is the set of all signals.

Typically, not all signals in W^T can occur in our system (or at least, a system in which anything can happen would not be very interesting from the mathematical point of view). Usually, there will be a system law which is satisfied only by some signals.

The subset β of W^T formalizes this law which governs the system. The signals $w \in \beta$ are precisely those which are compatible with the system law. We also call them admissible signals and we write

$\beta = \{w \in W^T | w \quad satisfies \quad the \quad system \quad law\}$. A signal has several, say q, components coming from the same set K (usually, $K = \mathbb{R}$). Then $W = K^q$, and a signal has the form $w : T \to K^q$,

$$t \to w(t) = \begin{bmatrix} w_1 \\ \vdots \\ w_q \end{bmatrix}$$

Definition 1.4.2. *A dynamical system $\sum = (T, W, \beta)$ is called a differential (difference) system if its time set is continuous (discrete) and its system law is given by differential (difference) equations.*

Differential systems: Systems of linear differential equations with constant coefficients. These can be put in the form

$$(R_d \frac{d^d}{dt^2} + ... + R_1 \frac{d}{dt} + R_0)w = 0 \tag{1.7}$$

where $R_i \in \mathbb{R}^{p \times q}$ are real matrices. We define

$$R := R_d s^d + ... + R_1 s + R_0$$

Then R is a polynomial $p \times q$ matrix, and we may rewrite (1.7) in the concise form

$$R(\frac{d}{dt})w = 0$$

. Difference systems: Systems of linear difference equations with constant coefficients. These can be put in the form

$$(R_d \sigma^d + ... + R_1 \sigma + R_0)w = 0$$

.

$$R(\sigma)w = 0$$

. In the following, let R be a $p \times q$ polynomial matrix in the variable s, with real coefficients. We write

$$R \in \mathbb{R}[s]^{p \times q}$$

. $*$ The continuous standard model is

$$\beta = \{w \in A^q | R(\frac{d}{dt})w = 0\}$$

where $A = D\prime(T)$ with $T = \mathbb{R}$ or $T = \mathbb{R}_+$. $*$ The discrete standard model is

$$\beta = \{w \in A^q | R(\sigma)w = 0\}$$

where $A = \mathbb{R}^T$ with $T = \mathbb{N}$ or $T = \mathbb{Z}$. The polynomial matrix R is called a representation of β.

Remark 1.1. *The behavior β is uniquely determined by R. The equation $RW = 0$ is called kernel representation of the system.*

Definition 1.4.3. *A square polynomial matrix U is called unimodular if its determinant is a non-zero constant. This is equivalent to the existence of a polynomial matrix V such that*

$$UV = VU = I$$

.

We observe that pre-multiplication by a unimodular matrix U does not change the behavior represented by R. More precisely, R and $\hat{R} = UR$ represent the same behavior.

Definition 1.4.4. *A representation R of the behavior β is called a minimal if there exists no polynomial matrix which represents the same behavior and has a smaller number of rows.*

Theorem 1.4.1. *(Smith form) For every polynomial matrix $R \in \mathbb{R}[s]^{p\times q}$ there exist unimodular matrices $U \in \mathbb{R}[s]^{p\times p}$ and $V \in \mathbb{R}[s]^{q\times q}$ such that*

$URV = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$

where $D = diag(d_1, ., ., d_r)$ is a non singular diagonal matrix with$d_1|d_2|...|d_r$ and $r = rank(R)$ As a direct consequence of the Smith form we have:

Corollary 1.1. *For every polynomial matrix $R \in \mathbb{R}[s]^{p\times q}$ there exists a unimodular matrix $U \in \mathbb{R}[s]^{p\times p}$ such that*

$UR = \begin{bmatrix} R1 \\ 0 \end{bmatrix}$ *where $R_1 \in \mathbb{R}[s]^{r\times q}$ has full row rank, that is, $rank(R_1) = r$.*

Since R and R_1 represent the same behavior, we have the following conclusion.

Theorem 1.4.2. *Any β possesses a representation matrix with full row rank. If R is a minimal representation of β, then R has full row rank.*

Now, given the kernel representation

$$Rw = 0$$

of a system. We may assume that $R \in \mathbb{R}[s]^{p\times q}$ has full row rank.Then there exists a $p \times p$ non singular submatrix of R.We can permute the columns of R such that $R = \begin{bmatrix} -Q & P \end{bmatrix}$ where P is such a non singular $p \times p$ matrix.Writing $w = \begin{bmatrix} u \\ y \end{bmatrix}$ correspondingly,our system law $Rw = 0$ takes the form:

$$Py = Qu \tag{1.8}$$

Definition 1.4.5. *A system law in the form of equation (1.8), where P is square and non-singular, is called an input-output representation of β. One calls $p = rank(R)$ the number of outputs (or output-dimension), and $m = q - p$ the number of inputs.*

Remark 1.2. *The input output representation of β is not unique.*

For every input-output representation $P(\frac{d}{dt})y = Q(\frac{d}{dt})u$, of a behavior β we can find a first order representation of the form:

$$P(\frac{d}{dt})y = Q(\frac{d}{dt})u \Leftrightarrow \exists x \in A^n : \begin{cases} \frac{d}{dt}x = & \text{Ax+Bu;} \\ y = & \text{Cx+Du.} \end{cases}$$

and similarly for σ instead of $\frac{d}{dt}$. The explicit equations

$\dot{x} = Ax + Bu$
$y = Cx + Du$
or
$\sigma x = Ax + Bu$
$y = Cx + Du$
are called state space representations.

1.5 Controllability and Observability of a system

consider the state space equations

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

or

$$\sigma x = Ax + Bu$$

$$y = Cx + Du$$

where $A \in \mathbb{R}^{n\times n}$,$B \in \mathbb{R}^{n\times m}$,$C \in \mathbb{R}^{p\times n}$,$D \in \mathbb{R}^{p\times m}$.

1.5.1 Controllability

Does there exist an input $u(.)$that drives the state $x(.)$ to zero is finite time?($x(.)$ is arbitrary)

Theorem 1.5.1. *The following are equivalent: a).$\dot{x} = Ax + Bu$ or $\sigma x = Ax + Bu$ is controllable.*

b).$K = [BAB...A^{n-1}B]$ has full row rank.

In this case we say that the matrix pair (A, B) is Controllable. Equivalently,(A,B) is controllable if and only if

$$rank[\lambda I - A \qquad B] = n$$

for all $\lambda \in \mathbb{C}$

1.5.2 Observability

In the state space representations given above u and y are called manifest variables and x is called latent variable. Observability is concerned with the following problem : If we know the manifest variables of a system,what can we conclude about the latent variables.Thus ,an observabile system is one in which the latent variables can be reconstructed from the manifest variables.

Theorem 1.5.2. *The following are equivalent:*

a).$\dot{x} = Ax$. $y = Cx$ is observable.

b)

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

In this case we say that the matrix pair (A,C) is observable.

Equivalently,(A,C) is observable if and only if

$$rank \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$$

for all $\lambda \in \mathbb{C}$

Chapter 2

TRANSFER MATRICES

An input-output representation has the form:

$$P(\tfrac{d}{dt})y = Q(\tfrac{d}{dt})u \qquad \text{or} \qquad P(\sigma)y = Q(\sigma)u$$

where $P \in \mathbb{R}[s]^{p\times p}$ is non-singular, and $Q \in \mathbb{R}[s]^{p\times m}$.Non singularity of P is not a restriction because we can achieve this by removing rows that are linear combinations of the other rows. The rational matrix

$$H := P^{-1}Q \in \mathbb{R}(s)^{p\times m}$$

is called transfer matrix (or: transfer function) of the input-output representation.

Lemma 2.0.1. *The transfer matrix of a state space system $\dot{x} = Ax + Bu$,$y = Cx + Du$ is*

$$H = C(sI - A)^{-1}B + D \tag{2.1}$$

Proof. Consider the linear time invariant system system

$$\dot{x} = Ax(t) + Bu(t)$$
$$y = Cx(t) + Du(t)$$

along with the initial conditions $x(0) = 0$. Taking the laplace transform of the state equation results in

$$s\hat{x}(s) = A\hat{x}(s) + B\hat{u}(s)$$

which can be written as

$$(sI - A)\hat{x}(s) = B\hat{u}(s)$$

$\hat{x}$ and $\hat{u}$ are the laplace transform of x and u,respectively.
The matrix $sI - A$ is invertible

$$(sI - A)\hat{x}(s) = B\hat{u}(s)$$

$$\hat{x}(s) = (sI - A)^{-1}B\hat{u}(s)$$

and the above output equation is

$$\hat{y}(s) = C\hat{x}(s) + D\hat{u}(s),$$

$\hat{y}$ is the laplace transform of y. Using the expression of $\hat{x}$.

$$\begin{aligned} \hat{y}(s) &= C(sI - A)^{-1}B\hat{u}(s) + D\hat{u}(s) \\ &= C(sI - A)^{-1}B + D)\hat{u}(s) \\ &= \hat{H}(s)\hat{u}(s) \end{aligned}$$

Here,

$\hat{H}(s) = C(sI - A)^{-1}B + D$ Where the transfer matrix $\hat{H}(s)$ is defined by $\hat{H}(s) = C(sI - A)^{-1}B + D$, which relates the laplace transforms of the input and the output.

□

Example 2.0.1.

$$\dot{x}(t) = -x(t) + u(t)$$

$$y(t) = x(t)$$

Here

$$G(s) = C(sI - A)^{-1} + D = \frac{1}{s+1}$$

Example 2.0.2.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u$$

$$y = x_1$$

Here $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$ *Hence*

$$G(s) = C(sI - A)^{-1}B = \frac{1}{s^2}$$

Example 2.0.3. *consider the system* $\dot{x} = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, y = x$ *find the transfer matrix.*

solution:

We have

$$H(s) = c(sI - A)^{-1}B$$

$$= \begin{bmatrix} s-1 & 2 \\ 1 & s-2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{s} & (\frac{\frac{2}{3}}{s} + \frac{\frac{1}{3}}{s-3}) & \frac{1}{s-3} \\ \frac{1}{s} & (\frac{\frac{1}{3}}{s} - \frac{\frac{1}{3}}{s-3}) & \frac{-1}{s-3} \end{bmatrix}$$

2.1 Realization theory

By the previous section ,we have seen that, it is easy to compute H if A,B,C,D are known. However, one often faces the inverse problem: Given H, find matrices A,B,C,D such that (2.1) holds. This is known as the realization problem. If (2.1) is satisfied, the matrix quadruple (A,B,C,D) is called a realization of H, and H is called realizable if it possesses a realization.

We first observe that any H according to (2.1) will be a proper rational matrix, that is, if we write $H = \frac{N}{d}$, where $N \in \mathbb{R}[s]^{p \times m}$ is a polynomial matrix, and $0 \neq$ d $\in \mathbf{R}[s]$ is a scalar polynomial, then

$$deg(N_{ij}) \leq deg(d) \qquad for \qquad all \qquad i, j. \tag{2.2}$$

This follows from $(sI - A)^{-1} = \frac{adj(sI-A)}{det(sI-A)}$ using Cramers rule. In (2.2), strict inequality holds for all i, j if and only if $D = 0$. In this case, one says that H is strictly proper. It turns out that properness is not only necessary but also sufficient for realizability.

Theorem 2.1.1. *A rational matrix is realizable if and only if it is proper.*

Proof. Let$H \in \mathbb{R}(s)^{p \times m}$ be proper, then we can write$H = D + H_1$ with $D \in \mathbb{R}^{p \times m}$ andH_1 strictly proper. Thus$H_1 = \frac{N}{d}$ with $d = s^v + d_{v-1}s^{v-1} + ... + d_1 s + d_0$ and $N = N_{v-1}s^{v-1} + ... + N_1 s + N_0$ for some $v = deg(d)$,$d_i \in \mathbb{R}$,$N_i \in \mathbb{R}^{p \times m}$. Put $n = vm$ and □

$$A = \begin{bmatrix} 0 & I & & & \\ . & & . & & \\ . & & & . & \\ . & & & & . \\ 0 & & & & I \\ -d_0 I & -d_1 I & . & . & . & -d_{v-1} I \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ I \end{bmatrix}$$

$$C = \begin{bmatrix} N_0 & N_1 & . & . & . & N_{v-1} \end{bmatrix}$$

then (A,B,C,D) is a realization of H. To see this, note that

$$[sI - A] \begin{bmatrix} I \\ SI \\ . \\ . \\ . \\ s^{v-1}I \end{bmatrix}$$

$=$

$$\begin{bmatrix} sI & -I & & & & \\ 0 & sI & -I & & & \\ . & & . & . & & \\ . & & & . & . & \\ 0 & & & & sI & -I \\ d_0I & d_1I & . & . & . & sI + d_{v-1}I \end{bmatrix} \begin{bmatrix} I \\ SI \\ \vdots \\ s^{v-1}I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ dI \end{bmatrix}$$

and hence

$$\begin{bmatrix} I \\ SI \\ . \\ . \\ . \\ s^{v-1}I \end{bmatrix} = (sI - A)^{-1} \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ dI \end{bmatrix} = (sI - A)^{-1}Bd$$

Pre-multiplying this by C, we obtain

$$C \begin{bmatrix} I \\ SI \\ . \\ . \\ . \\ s^{v-1}I \end{bmatrix} = N = C(sI - A)^{-1}Bd$$

$$\Rightarrow H_1 = \frac{N}{d} = C(sI - A)^{-1}B$$

which yields the desired result, after division by d.

Thus, any proper rational matrix H is realizable. Let (A,B,C,D) be a realization of H, with $A \in \mathbb{R}^{n \times n}$. We call the number n the size of the realization. Of course, it is desirable to have small realizations. We say that a realization of H is minimal if there exists no realization of H with a smaller size.

Definition 2.1.1. *The realization (A,B,C,D) of H with $A \in \mathbb{R}^{n \times n}$ is said to be minimal if there exists no other realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ of H such that $\tilde{A} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ and $\tilde{n} < n$. The next lemma gives an important relation between two realizations of transfer function*

Lemma 2.1.1. *If (A, B, C, D) and $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ are two realizations of the same transfer matrix, then$D = \tilde{D}$ and*

$$CA^iB = \tilde{C}\tilde{A}^i\tilde{B}$$

for all $i \in \mathbb{N}$.

Proof. If$H = C(sI - A)^{-1}B + D$, then

$$\lim_{s \to \infty} H(s) = D$$

This shows that we must have$D = \tilde{D}$. Moreover, we can expand $H - D$ into a Laurent series

$$H - D = C(sI - A)^{-1}B = \sum_{n=0}^{\infty} CA^iBs^{-i-1}$$

and this is convergent on $|s| > \rho(A)$, where$\rho(A)$ is the spectral radius of A. Therefore, by comparing coefficients,$CA^iB = \tilde{C}\tilde{A}^i\tilde{B}$ for all i. □

Now we can give a sufficient condition for minimality which will soon turn out to be also necessary.For the proof we need the folllowing inquality from linear algebra called Sylvesters inequality.

Lemma 2.1.2. *(Sylevesters law of nullity) Let A and B be $n \times k$ and $k \times m$ matrices respectively.Then* 1) $rank(AB) \leq min\{rank(A), rank(B)\}$

2) $rank(AB) \geq rank(A) + rank(B) - K$

Lemma 2.1.3. *Let (A, B, C, D) be such that (A, B) is controllable and (A, C) is observable. Then (A, B, C, D) is a minimal realization of* $H = C(sI - A)^{-1}B + D$.

Proof. Suppose that (A, B, C, D) is a realization of$H$, with size n, in which (A, B) is controllable and (A, C) is observable. Let$(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ be another realization of H, with size $\tilde{n}$. We need to prove that $n \leq \tilde{n}$. Define:

$$O = \begin{bmatrix} C \\ CA \\ . \\ . \\ . \\ CA^{n-1} \end{bmatrix} K = \begin{bmatrix} B & AB & . & . & . & A^{n-1}B \end{bmatrix} and$$

$$\tilde{O} = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ . \\ . \\ . \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix} \tilde{K} = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & . & . & . & \tilde{A}^{n-1}\tilde{B} \end{bmatrix}$$

Note that whereas O,K are precisely the Kalman matrices associated to (A, B) and (A, C), this is not true for$\tilde{O}$, $\tilde{K}$ (we have n instead of $\tilde{n}$ in the highest power of A). Then, because $CA^iB = \tilde{C}\tilde{A}^i\tilde{B}$for all i,

$$OK = \tilde{O}\tilde{K}$$

We have

$$rank(K) \geq rank(OK) \geq rank(O) + rank(K) - n.$$

By assumption, K and O both have rank n.Since the system is controllable and observable. Therefore,

$$n = rank(OK) = rank(\tilde{O}\tilde{K}) \leq rank(\tilde{O}) \leq \tilde{n}$$

□

as desired.

Theorem 2.1.2. *(Reduction to minimality) Let (A, B, C, D) be a realization of H.*

1. Consider a Kalman controllability decomposition $T^{-1}AT = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, T^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, CT = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$

Then $H = C(sI - A)^{-1}B + D = C1(sI - A_1)^{-1}B_1 + D$*, that is,* (A_1, B_1, C_1, D) *is another realization of*H*, with size* $r = rank(K)$*, where* K *is the Kalman controllability matrix.*

2. Consider a Kalman observability decomposition $T^{-1}AT = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}, T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, CT = \begin{bmatrix} C_1 & 0 \end{bmatrix}$

Then$H = C(sI-A)^{-1}B + D = C1(sI-A_1)^{-1}B_1 + D$, *that is,* (A_1, B_1, C_1, D) *is another realization of H, with size* $r = rank(O)$, *where O is the Kalman observability matrix.*

3. If the two reduction steps are done successively, one ends up with a minimal realization of H.

Proof. One can easily check that a similarity transform does not change the transfer function. Therefore, assume that a Kalman controllability decomposition has already been performed. Then

$$H = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} sI-A_1 & -A_2 \\ 0 & sI-A_3 \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + D$$

$$= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} (sI-A_1)^{-1} & * \\ 0 & (sI-A_3)^{-1} \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + D$$

$$= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} (sI-A_1)^{-1}B_1 \\ 0 \end{bmatrix} + D = C_1(sI-A_1)^{-1}B_1 + D.$$

The second statement is analogous. Recall that after a Kalman controllability decomposition, the matrix pair (A_1, B_1) is controllable. Now if one performs a Kalman observability decomposition with the already reduced system (A_1, B_1, C_1, D), then one obtains

$$T_1^{-1}A_1T_1 = \begin{bmatrix} A_{11} & 0 \\ A_{12} & A_{13} \end{bmatrix}, T_1^{-1}B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, C_1T_1 = \begin{bmatrix} C_{11} & 0 \end{bmatrix}$$

in which (A_{11}, C_{11}) is observable. We only need to convince ourselves that the controllability of (A1,B1) implies the controllability of(A_{11}, B_{11}) (i.e., the controllability established in the first reduction step is not destroyed by the second reduction step in which we achieve observability).

Let $K = [B_1, A_1B_1, ..., A_1^{r-1}B_1]$ be Kalman controllable matrix for (A_1, B_1).Then,

$T^{-1}K = [T^{-1}B_1T^{-1}A_1TT^{-1}B_1...T^{-1}A_1^{r-1}TT^{-1}B_1]$

$$= \begin{bmatrix} B_{11} & A_{11}B_{11} & . & . & . & A_{11}^{r-1}B_{11} \\ B_{12} & A_{12}B_{11} + A_{13}B_{12} & . & . & . & * \end{bmatrix}$$

since T^{-1} is invertible,$rank(T^{-1}K) = rank(K)$

$[B_{11}, A_{11}B_{11}, ..., A_{11}^{r-1}B_{11}]$ has full row rank

$[B_{11}, A_{11}B_{11}, ..., A_{11}^{s-1}B_{11}]$ has full row rank where $s = rank(O)$and O is Kalman observability matrix of (A_1, C_1) . Thus , (A_{11}, B_{11}) is controllable. Therefore, the resulting realization $(A_{11}, B_{11}, C_{11}, D)$ is both controllable and observable, and hence minimal according to Lemma 2.1.3.

In particular, this theorem shows that if in a realization (A, B) is not controllable or (A, C) is not observable, then the realization can be reduced in size. Moreover, this can be done constructively, using a Kalman decomposition. In other words, a minimal realization will always be both controllable and observable. Combining this result with Lemma 2.2.2, we obtain the following theorem as a summary.

□

Theorem 2.1.3. *The matrix quadruple (A, B, C, D) is a minimal realization of $H = C(sI - A)^{-1}B + D$ if and only if (A, B) is controllable and (A, C) is observable.*

The next theorem says that minimal realizations are essentially unique (up to similarity transforms).

Proof. ($\Leftarrow$)suppose (A, B, C, D) is minimal realization of H.Suppose also that either (A, B) is not controllable or (A, C) is not observable.If (A,B)is not controllable then by Kalman controllability decomposition there exist a realization (A_1, B_1, C_1, D) with realization size less than the size of the realization (A,B,C,D) which contradicts the minimality of (A,B,C,D). Thus,(A,B) is controllable and (A,C) is observable. □

Theorem 2.1.4. *Any two minimal realizations of a transfer matrix are similar, that is, if (A, B, C, D) and $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ are two minimal realizations of H, then there exists a non-singular matrix T such that*

$$\tilde{A} = T^{-1}AT,\ \tilde{B} = T^{-1}B,\ \tilde{C} = CT, D = \tilde{D}.$$

Proof. Since both realizations are minimal, their size must be the same, that is,$n = \tilde{n}$. Moreover,$D = \tilde{D}$, and$CA^iB = \tilde{C}\tilde{A}^i\tilde{B}$ for all i. Let$O, K, \tilde{O}, \tilde{K}$ be the observability and controllability matrices of the two realizations. Then $OK = \tilde{O}\tilde{K}$ and

$$OAK = \tilde{O}\tilde{A}\tilde{K}, OB = \tilde{O}\tilde{B}, CK = \tilde{C}\tilde{K}\ .$$

By assumption,$\tilde{K}$ has full row rank, and$\tilde{O}$ has full column rank. Therefore, there exist matrices L,N such that $\tilde{K}L = I$ and$N\tilde{O} = I$.

Put$T := KL \in \mathbb{R}^{n\times n}$, then$T^{-1} = NO$, because $NOKL = N\tilde{O}\tilde{K}L = I$. We have

$$\begin{aligned} T^{-1}AT = NOAKL = N\tilde{O}\tilde{A}\tilde{K}L = \tilde{A} \\ T^{-1}B = NOB = N\tilde{O}\tilde{B} = \tilde{B} \\ CT = CKL = \tilde{C}\tilde{K}L = \tilde{C} \end{aligned}$$

which completes the proof.

Starting with an arbitrary realization of H, one can determine the size of a minimal realization of H by successively computing two Kalman decompositions, as outlined in Theorem 2.1.2. However, there is also a direct way to determine the size of a minimal realization. This will be discussed in the next section. □

Example 2.1.1. *show that any minimal realization of* $H(s) = \frac{1}{s+1}\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ *must have a state space dimension n at least two.*

solution:

Assume there exists a realization with dimension $n = 1$,Then we have $(x(t) : \mathbb{R}+ \to \mathbb{R})$ such that

$\dot{x}(t) = ax(t) + \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} x(t)$

Then to achieve the required transfer matrix,we need

$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \frac{1}{s-a} \begin{bmatrix} b_1 & b_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{s+1} & \frac{-1}{s+1} \end{bmatrix}$ *Thus,*$a = -1$,$c1b1 = 1$,$c1b2 = 0$,$c2b1 = 1$ *and* $c2b2 = -1$ *which is impossible.This rules out any realization with* $n = 1$*. We now show that there exists a realization with n=2 that is*

$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ *including that a minimal realization must have* $n = 2$

Example 2.1.2. *show that there is a realization of* $H(s) = \frac{1}{s+1} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ *with state space dimension* $n = 1$

solution:

observe that $A = -1$,$B = (1, -1)$ *and* $C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *works.*

Example 2.1.3. *Find a realization of a)*

$$H(s) = \frac{s^3 + 2s^2 + 3s - 1}{s^3 - s^2 + s + 1}$$

b)

$$H(s) = \frac{s^2 + s - 2}{s^3 + 3s^2 + 2s}$$

in controller form.Check the resulting system for observability and minimality.Explain the difference between (a) and (b)

solution: a) we have

$$H = \frac{3s^2 + 2s - 2}{s^3 - s^2 + s + 1} + 1$$

Thus,D=1 and $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, C = \begin{pmatrix} -2 & 2 & 3 \end{pmatrix}$

(A,B,C,D)is a realization of H clearly,(A,B) is controllable and since

$O = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 3 \\ -3 & -5 & 5 \\ -5 & -8 & 0 \end{bmatrix}$ *has full column rank,the system is observable. therefore,The realization is minimal.*

b)

$$H(s) = \frac{s^2 + s - 2}{s^3 + 3s^2 + 2s}$$

$D = 0$,$N = s^2+s-2$,$d = s^3+3s^2+2s+0$ *Thus,* $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{pmatrix}$,$B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$,$C = \begin{pmatrix} -2 & 1 & 1 \end{pmatrix}$

Here the ranks of the observable matrix

$$O = \begin{pmatrix} -2 & 1 & 1 \\ 0 & -4 & -2 \\ 0 & 4 & 2 \end{pmatrix}$$

is 2 *instead of* 3 *that is the system is not observable hence not minimal.(1 uncontrollable state can be eliminated). Reason:* s^2+s-2 *and* s^3+3s^2+2s *are not coprime*

$$H(s) = \frac{(s+2)(s-1)}{s(s+2)(s+1)} = \frac{s-1}{s(s+1)}$$

Example 2.1.4. *compute a minimal realization of* $H(s) = \frac{1}{(s+1)^2}\begin{bmatrix} 3s+4 & -4s-5 \\ 4s+7 & -7s-10 \end{bmatrix}$

solution: we have $H(s) = \frac{1}{s^2+2s+1}(\begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix} s + \begin{bmatrix} 4 & -5 \\ 7 & -10 \end{bmatrix})$

Realizing the transfer function(or matrix) in block controller form yields

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\ C = \begin{bmatrix} 4 & -5 & 3 & -4 \\ 7 & 10 & 4 & -7 \end{bmatrix}\ D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, $K = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix}$ *and* $O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 4 & -5 & 3 & -4 \\ 7 & 10 & 4 & -7 \\ -3 & 4 & -2 & 3 \\ -4 & 7 & -1 & 4 \end{bmatrix}$

Then, $rank(K) = 4$ *and* $rank(O) = 2$.

(A,B) is controllable and (A,C) is not observable.

Thus, two unobservable states can be eliminated,and we obtain a minimal realization given by:

$$A = \frac{1}{5}\begin{pmatrix} -2 & -1 \\ 9 & -8 \end{pmatrix},B = \begin{pmatrix} 3 & -4 \\ 4 & 7 \end{pmatrix},C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

2.2 Matrix fraction descriptions

Let $H \in \mathbb{R}(s)^{p\times m}$ be given. If $H = P^{-1}Q$ for some $Q \in \mathbb{R}[s]^{p\times m}$,$P \in \mathbb{R}[s]^{p\times p}$ with $det(P) \neq 0$, we call (P,Q) a left factorization (or: left matrix fraction description) of H.

Similarly, if$H = QP^{-1}$ for some polynomial matrices$Q \in \mathbb{R}[s]^{p\times m}$,$P \in R[s]^{m\times m}$ with $det(P) \neq 0$, we call (Q,P) a right factorization of H.

For example, we have already used the representation$H = \frac{N}{d}$ several times. In other words, (dI_p, N) is a left and (N, dI_m) is a right factorization of H.

In the scalar case, it is desirable to write a rational function $h \in \mathbb{R}(s)$ as the ratio (fraction) of two coprime polynomials. We wish to do the same with polynomial matrices.
We say that a left factorization (P, Q) is left coprime if the matrix $\begin{bmatrix} P & Q \end{bmatrix}$ is left irreducible . Similarly, a right factorization (Q, P) is called right coprime if the matrix $\begin{bmatrix} Q \\ P \end{bmatrix}$ is right irreducible, which means, by definition, that its transpose is left irreducible.

Definition 2.2.1. *The polynomial matrix $R \in \mathbb{R}(s)^{p\times q}$ is said to be left irreducible left coprime (respectively right irreducible) if there exists a matrix $S \in \mathbb{R}(s)^{q\times p}$ such that $RS = I$(res. $SR = I$) Equivalently,R is left coprime if $R = UR_1$ for some $U \in \mathbb{R}(s)^{p\times p}$,$R_1 \in \mathbb{R}(s)^{p\times q}$,then U must be unimodular.*

Lemma 2.2.1. *1. Let (P,Q) be a left coprime factorization of H. The degree of the determinant of P is independent of the specific choice of the coprime factorization and therefore*

$$d(H) := deg\ det(P) \tag{2.3}$$

is well-defined. If (P, Q) is an arbitrary (not necessarily coprime) left factorization of H, then

$$d(H) \leq degdet(P).$$

2. Let (Q, P) be a right coprime factorization of H. The degree of the determinant of P is independent of the specific choice of the coprime factorization and moreover,

$$d(H) = degdet(P)$$

where d(H) is the number defined in (2.3). If (Q, P) is an arbitrary (not necessarily coprime) right factorization of H, then

$$d(H) \leq degdet(P).$$

Remark 2.1. *The degree of the determinant of the denominator matrix P is as small as possible if the factorization is coprime. This generalizes the well known fact that the degree of d is minimal if we write a scalar rational function $h = \frac{n}{d} \in \mathbb{R}(s)$ as the ratio of two coprime polynomials. The choice of the coprime factorization does not influence this minimal degree, it does not even matter whether we take a right or left matrix fraction description.*

In the proof, we use the formula

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D-CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}$$

which holds provided that A is invertible. Thus

$$det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = det(A)det(D-CA^{-1}B).$$

This shows that the block matrix is invertible if and only if $D-CA^{-1}B$ *is invertible, and we have*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & (D-CA^{-1}B)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} = \begin{bmatrix} * & * \\ * & (D-CA^{-1}B)^{-} \end{bmatrix}$$

The matrix $D-CA^{-1}B$ *is called Schur complement of A.*

Proof. 1. Let (P,Q) and (P_1,Q_1) be two left factorizations of H, that is,

$$H = P^{-1}Q = P_1^{-1}Q_1$$

which implies that $Q_1 = UQ$ and $P_1 = UP$, where $U := P_1P^{-1}$. We show that U is polynomial if (P,Q) is left coprime; and even unimodular if both factorizations are coprime. If (P,Q) is left coprime, then there exist, polynomial matrices R, S such that

$$PR + QS = I.$$

Then

$$P^{-1} = R + P^{-1}QS = R + HS = R + P_1^{-1}Q_1S$$

and thus

$$U = P_1P^{-1} = P_1R + Q_1S.$$

This shows that U is polynomial. Then

$$det(P_1) = det(UP) = det(U)det(P)$$

means that $det(P)$divides $det(P_1)$, in particular,

$$degdet(P1) \geq degdet(P).$$

Similarly, if also (P_1,Q_1) is coprime, there exist polynomial matrices R_1, S_1 such that $P_1R_1 + Q_1S_1 = I$ and then

$$P_1^{-1} = R_1 + P_1^{-1}Q_1S_1 = R_1 + HS_1 = R_1 + P^{-1}QS_1$$

and thus

$$U^{-1} = PP_1^{-1} = PR_1 + QS_1$$

which shows that U^{-1} is polynomial and hence, U is unimodular. Thus we have

$$det(P_1) = det(U)det(P)$$

where det(U) is a non-zero constant, and hence

$$degdet(P_1) = degdet(P).$$

2. In view of part 1, it suffices to show that if(P, Q) is a left coprime, and (Q_1, P_1) is a right coprime factorization of H, then

$$degdet(P) = degdet(P1)$$

. We have $H = P^{-1}Q = Q_1 P_1^{-1}$ and thus

$$QP_1 = PQ_1.$$

There exist polynomial matricesR, S, R_1, S_1 such that (because (P, Q) left coprime and (Q_1, P_1)right coprime)

$$PR + QS = I$$

and

$$R_1 P_1 + S_1 Q_1 = I$$

Thus
$\begin{bmatrix} P & -Q \\ S_1 & R_1 \end{bmatrix} \begin{bmatrix} R & Q_1 \\ -S & P_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$ where$X = S_1 R - R_1 S$. Post-multiplying this by $\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}$ yields
$\begin{bmatrix} P & -Q \\ S_1 & R_1 \end{bmatrix} \begin{bmatrix} \tilde{R} & Q_1 \\ -\tilde{S} & P_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

This shows that the matrices are unimodular, and inverse to each other,
$\begin{bmatrix} P & -Q \\ S_1 & R_1 \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{R} & Q_1 \\ -\tilde{S} & P_1 \end{bmatrix}$
ThusP^{-1} is the Schur complement of P that is $P_1^{-1} = R_1 + s_1 P^{-1} Q$,and hence
$det(P)det(P_1^{-1})$ =$det[\begin{matrix} P & -Q \\ S_1 & R_1 \end{matrix}] = c$

where

$0 \neq c \in \mathbb{R}$ because $\begin{bmatrix} P & -Q \\ S_1 & R_1 \end{bmatrix}$. Thus $det(P) = cdet(P1)$, in particular,$degdet(P) = degdet(P_1)$.

It turns out below that the integer d(H) from (2.3) equals the size of a minimal realization of a proper rational matrix H. For the proof, we need the concept of a row-reduced polynomial matrix. □

Definition 2.2.2. *The degree of a non-zero polynomial row vector is defined to be the highest power of s appearing in it with a non-zero coefficient, and the degree of the zero row is set to be* $-\infty$*. For a non-singular polynomial matrix*$P \in \mathbb{R}[s]^{p \times p}$*, let*$\delta_i(P)$ *be the degree of the* i^{th}*row of* P*, for* $i = 1, ..., p$*. Then* P *has a unique representation*

$$P = SP_{hr} + L$$

where$S = diag(s^{\delta_1(P)}, ..., s^{\delta_p(p)})$, $P_{hr} \in \mathbb{R}^{p\times p}$, *and* $L \in \mathbb{R}[s]^{p\times p}$ *is such that* $\delta_i(L) < \delta_i(P)$. *One calls*$P_{hr}$ *the highest row coefficient matrix. If*P_{hr} *is non-singular, we say that* P *is row-reduced (or row-proper). For*$R \in \mathbb{R}[s]^{p\times q}$ *with full row rank, the highest row coefficient matrix*$R_{hr} \in \mathbb{R}^{p\times q}$ *is defined analogously, and* R *is called row-reduced if*R_{hr} *has full row rank.*

Example 2.2.1. *Let* $P = \begin{bmatrix} s^3+s & s^2+s+1 \\ s+2 & 1 \end{bmatrix}$, *then* $\delta_1(P) = 3$ *and* $\delta_2(P) = 1$.*Thus we can write* P *as :*

$P = \begin{pmatrix} s^3 & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} s & s^2+s+1 \\ 2 & 1 \end{pmatrix}$

where $S = diag(s^3, s)$,$P_{hr} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$,$L = \begin{pmatrix} s & s^2+s+1 \\ 2 & 1 \end{pmatrix}$ *Observe that* $\delta_i(L) < \delta_i(P)$ *for* $i = 1, 2$ *and* P *is not row proper because* P_{hr} *is singular.*

Example 2.2.2. *Let* $R = \begin{pmatrix} s^2+1 & s^2-1 & s^2+2s+3 \\ s-1 & s-1 & s+1 \end{pmatrix}$

Then R *can be written as:*

$R = \begin{pmatrix} s^2 & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 2s+3 \\ -1 & -1 & 1 \end{pmatrix}$

where $S = diag(s^2, s)$,$R_{hr} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$,$L = \begin{pmatrix} 1 & -1 & 2s+3 \\ -1 & -1 & 1 \end{pmatrix}$ *Observe that* R *is not row proper because* R_{hr} *has rank 1.*

Lemma 2.2.2. *Let* $P \in \mathbb{R}[s]^{p\times p}$ *be non-singular.*

1. We have

$$\sum_{i=1}^{p} \delta_i(P) \geq degdet(P)$$

The matrix P *is row-reduced if and only if*

$$\sum_{i=1}^{p} \delta_i(P) = degdet(P)$$

2. If $H = P^{-1}Q$ *is strictly proper, then* $\delta_i(Q) < \delta_i(P)$ *for* $i = 1, ..., p$*; if* H *is proper, then* $\delta_i(Q) \leq \delta_i(P)$ *for all* i*. The converse is also true, provided that* P *is row-proper.*

3. For every non-singular $P \in \mathbb{R}[s]^{p\times p}$ *there exists a unimodular matrix* $U \in \mathbb{R}[s]^{p\times p}$ *such that* UP *is row-proper.*

4. For every $R \in \mathbb{R}[s]^{p\times q}$ *with full row rank, there exists a unimodular matrix* $U \in \mathbb{R}[s]^{p\times p}$ *such that* UR*is row-proper.*

Proof. 1. Rewrite$P = SP_{hr} + L$ as$P_{hr} = S^{-1}P - S^{-1}L$. Consider the limit as $s \to \infty$. Then we have, since

$$\lim_{s\to\infty} S^{-1}L = 0$$

,

$$Phr = \lim_{s\to\infty} S^{-1}P$$

and, putting

$$\delta(P) := \sum_{i=1}^{p} \delta_i(P) \ ,$$

$$det(P_{hr}) = \lim_{s\to\infty} \frac{detP}{s^{\delta(P)}}$$

This shows that $degdet(P) \leq \delta(P)$ and

$$det(P_{hr}) = 0 \Leftrightarrow degdet(P) < \delta(P)$$

2. Let$H = P^{-1}Q$, then

$$Q_{ij} = \sum_{k=1}^{p} P_{ik}H_{kj}$$

and

$$\frac{Q_{ij}}{s^{\delta_i(p)}} = \sum_{k=1}^{p} \frac{P_{ik}}{s^{\delta_i(p)}} H_{kj}$$

Consider again the limit as $s \to \infty$. If H is strictly proper, the right hand side tends to zero, and hence all the powers of s appearing in Q_{ij} must be strictly less than $\delta_i(P)$. Since this holds for all j, we obtain$\delta_i(Q) < \delta_i(P)$. For the converse, write$P = SP_{hr} + L$, and assume that $P_{hr}$is invertible. Then

$$P^{-1}Q = (SP_{hr} + L)^{-1}Q = (I + P_{hr}^{-1}S^{-1}L)^{-1}P_{hr}^{-1}S^{-1}Q$$

and thus, since

$$\lim_{s\to\infty} S^{-1}L = 0$$

and

$$\lim_{s\to\infty} S^{-1}Q = 0,$$

we have

$$\lim_{s\to\infty} P^{-1}Q = 0$$

that is,$P^{-1}Q$ is strictly proper. The argument for proper is similar.

3. If P is row-proper, we are finished. Therefore, assume otherwise, that is, let $det(P_{hr}) = 0$. We show that there exists a unimodular matrix U such that

$$degdet(P) = degdet(UP) \leq \sum_{j=1}^{p} \delta_j(UP) < \sum_{j=1}^{p} \delta j(P)$$

. Iteratively, this yields the result. We write $\delta_j := \delta_j(P)$ for simplicity. Since $det(P_{hr}) = 0$, there exists $0 \neq \alpha = (\alpha_1, ..., \alpha_p) \in \mathbb{R}^{1\times p}$ with

$$\sum_{j=1}^{p} \alpha_j P_{hr}^{(j)} = 0$$

where$P_{hr}^{(j)}$denotes the$j-th$ row ofP_{hr}. Among the j with$\alpha_j \neq 0$, select j^* with $\delta_j^* \leq \delta_j$ for all j with$\alpha_j \neq 0$. Without loss of generality, let $\alpha_j^* = 1$. Then we have

$$P_{hr}^{(j^*)} + \sum_{j\neq j^*} \alpha_j P_{hr}^{(j)} = 0$$

Now perform the elementary operation:P^{j^*} plus

$$\sum_{j\neq j^*} \alpha_j s^{\delta_j * - \delta_j} P^j$$

The new matrix$P^| = UP$ satisfies $\delta_{j^*}(P^|) < \delta_{j^*}(P)$ and $\delta_j(P^|) = \delta_j(P)$ for all $j \neq j^*$. This establishes the claim.

4. The final statement is analogous to the previous one.

Example 2.2.3. *a).compute a unimodular matrix U such that UPis row proper,*

where $P = \begin{bmatrix} s^3+s & s^2+s+1 \\ s+2 & 1 \end{bmatrix}$

b). Let $\beta = \{w \in A^3 | R(\frac{d}{dt})w = 0\}$ *with*

$R = \begin{bmatrix} s^2+1 & s^2-1 & s^2+2s+3 \\ s-1 & s-1 & s+1 \end{bmatrix}$

Determine all i/o structures of β *with proper transfer matrix.*

solution: a). we have $P_{hr} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

choose $\alpha = [1, -1]$ *and* $j^* = 1$*.perform:1st row minus* s^2 *times second row:*

$P\prime = \begin{bmatrix} -2s^2+s & s+1 \\ s+2 & 1 \end{bmatrix}$,$P_{hr}\prime = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}$ *choose* $\alpha = [1, 2]$ *and* $j^* = 1$*.perform:1st row plus 2s times 2nd row.*

$P\prime\prime = \begin{pmatrix} 5s & 3s+1 \\ s+2 & 1 \end{pmatrix}$,$P_{hr}\prime\prime = \begin{pmatrix} 5 & 3 \\ 1 & 0 \end{pmatrix}$

$P\prime\prime$ *is row proper and* $P\prime\prime = UP$ *with* $U = \begin{pmatrix} 1 & 2s-s^2 \\ 0 & 1 \end{pmatrix}$ *we did* $P\prime = \begin{pmatrix} 1 & -s^2 \\ 0 & 1 \end{pmatrix} P$

$P\prime\prime = \begin{pmatrix} 1 & 2s \\ 0 & 1 \end{pmatrix} P\prime$

$P\prime\prime = UP$

b) $rank(R) = 2$ *and since all* 2×2 *sub determinants are nonzero. there are three possible i/o structures:each variable can play the role of input(and the other two variables are output).To find out which transfer functions are proper,one may use brute force,that is* *compute the three transfer functions and check whether they are proper.A more systematic way involves transforming R to row properness.*

We have
$R_{hr} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ *choose* $\alpha = [1, -1]$ *and* $j^* = 1$*.perform:1st row minus s times 2nd row:* $R\prime = \begin{pmatrix} s+1 & s-1 & s+3 \\ s-1 & s-1 & s+1 \end{pmatrix}$,$R_{hr}\prime = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ *choose* $\alpha = [1, -1]$ *and* $j^* = 1$*.perform:1st row minus 2nd row:* $R\prime\prime = \begin{pmatrix} 2 & 0 & 2 \\ s-1 & s-1 & s+1 \end{pmatrix}$,$R = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ $R\prime\prime$ *is row proper and* $R\prime\prime = UR$ *with* $U = \begin{pmatrix} 1 & -s-1 \\ 0 & 1 \end{pmatrix}$

Moreover,if $w_1(w_3)$ *is chosen as input,the* 2×2 *sub matrix of* $R_{hr}\prime\prime$ *resulting from deletion of the first and 3rd column,is non singular. Therefore, the corresponding transfer function will be proper.However,if* w_2 *is chosen as input,the resulting sub matrix of* $R_{hr}\prime\prime$ *is singular.That is we may expect some problems.Indeed,this i/o structure yields a non proper transfer function*

$$H = P^{-1}Q = P\prime\prime^{-1}Q\prime\prime = \begin{pmatrix} 2 & 2 \\ S-1 & S+1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -s+1 \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} s-1 \\ -s+1 \end{pmatrix}.$$

□

Given a full-row-rank representation R of β, we can thus assume without loss of generality that$R$ is row-proper. ThenR_{hr} has full row rank, and thus it contains a sub-matrix that is square and invertible. If we choose an input-output decomposition of β such that the output corresponds to such a choice of the columns of R, then one can show that in the resulting representation (2.6), the matrixN_2 is invertible (recall that this assumption was needed to transform (2.6) into state space form). This shows that any$\mathcal{B} = \{w \in A^q | R(\frac{d}{dt}w = 0\}$ admits a partition of its signal components into inputs and outputs such that it has a state space representation, in particular, the resulting transfer matrix is proper.

Theorem 2.2.1. *Let H be a proper rational matrix. The size of a minimal realization of H is given by the integer* $d(H)$ *from (2.3).*

Proof. There is no loss of generality in assuming that H is strictly proper. Let $(A, B, C, 0)$ be a realization of H, that is,

$$H = C(sI_n - A)^{-1}B.$$

Define$G := (sI_n - A)^{-1}B$. Then $(sI_n - A, B)$ is a left factorization of G. Thus

$$degdet(sI_n - A) = n \geq d(G)$$

On the other hand, if (Q, P) is a right factorization of G, that is,$G = QP^{-1}$, then$H = CG = CQP^{-1}$, that is, (CQ, P) is a right factorization of H. We conclude that

$$d(G) \geq d(H).$$

This shows that$n \geq d(H)$, that is, the size of any realization of H must be at least $d(H)$. Conversely, we show that a strictly proper H possesses a realization of size $d(H)$. Let $H = P^{-1}Q$ be a left coprime factorization. Without loss of generality, let P be row-proper. Now consider the i-th row of$P$ and$Q$, and denote them by P_i and Q_i, respectively. According to Theorem 2.22, there exist matrices$K_i, L_{ij}, M_i, N_{ij}(j = 1, 2)$ such that

$$P_i(\tfrac{d}{dt})y = Q_i(\tfrac{d}{dt})u \Leftrightarrow \exists x : \{\tfrac{d}{dt}xi = K_i x_i + L_{i1}u + L_{i2}y$$

$$\{0 = M_i x_i + N_{i1}u + N_{i2}y.$$

Here$K_i$can be chosen to be a $\delta_i(P) \times \delta_i(P)$ matrix. Combining these representations via

$$X = \begin{bmatrix} X_1 \\ X_2 \\ . \\ . \\ . \\ X_p \end{bmatrix} K = \begin{bmatrix} K_1 & & & & \\ & K_2 & & & \\ & & . & & \\ & & & . & \\ & & & & . \\ & & & & & K_p \end{bmatrix} L_j = \begin{bmatrix} L_{1j} \\ L_{2j} \\ . \\ . \\ . \\ L_{pj} \end{bmatrix}$$

$$M = \begin{bmatrix} M_1 & & & & \\ & M_2 & & & \\ & & . & & \\ & & & . & \\ & & & & . \\ & & & & & M_p \end{bmatrix} N_j = \begin{bmatrix} N_{1j} \\ N_{2j} \\ . \\ . \\ . \\ N_{pj} \end{bmatrix}$$

We obtain

$$P\tfrac{d}{dt}y = Q\tfrac{d}{dt}u \Leftrightarrow \exists x :\{\tfrac{d}{dt}x = Kx + L_1u + L_2y;$$

$$0 = Mx + N_1u + N_2y$$

where the size of K is $\sum \delta_i(P) = degdet(P) = d(H)$. Moreover,$N_1 = 0$ (because$P^{-1}Q$ is strictly proper) and $N_2 = P_{hr}$, which is invertible. Thus we can find, as in Section 2.7, matrices A, B, C, D with A of the same size as K, such that

$$P(\tfrac{d}{dt})y = Q(\tfrac{d}{dt})u \Leftrightarrow \exists x : \{\tfrac{d}{dt}x = Ax + Bu$$

$$\{Y = Cx.$$

Then $H = P^{-1}Q = C(sI - A)^{-1}B$ which shows that H has a realization of size $d(H)$. □

Remark 2.2. *In the final step of the proof, we have used the fact that equivalent representations have the same transfer function. This statement has not been proven. Alternatively, one may give a direct proof of*

$$H = P^{-1}Q = C(sI - A)^{-1}B$$

using the special form of A,B,C that results from Section 2.7. The details are omitted. The question now is how to get left coprime factorization to get d(H).This is another problem.We use the following theorem to determine the size of minimal realization.

Theorem 2.2.2. *(McMillan form) For each rational matrix* $H \in \mathbb{R}(s)^{p\times m}$ *there exist unimodular matrices* $U \in \mathbb{R}[s]^{p\times p}$ *and* $V \in \mathbb{R}[s]^{m\times m}$ *such that*

$$UHV = [\begin{matrix} D & 0 \\ 0 & 0 \end{matrix}] \qquad (2.4)$$

where $D \in \mathbb{R}(s)^{r\times r}$ is a diagonal matrix

$$D = \begin{bmatrix} \frac{\gamma_1}{\delta_1} & & & & \\ & . & & & \\ & & . & & \\ & & & . & \\ & & & & \frac{\gamma_r}{\delta_r} \end{bmatrix}$$

with polynomials γ_i, $\delta_i \neq 0$such that each pair (γ_i, δ_i) is coprime and $\gamma_1|...|\gamma_r$ and $\delta_r|...|\delta_1$. Clearly, $r = rank(H)$. The matrix on the right hand side of (2.4) is called McMillan form of H and the integer

$n := \sum_{i=1}^{r} deg(\delta_i)$ is called McMillan-degree of H.

Proof. Write $H = \frac{N}{d}$,where $N \in \mathbb{R}[s]^{p\times m}$ and $0 \neq d \in \mathbb{R}[s]$. compute the Smith form of N,say

$$UNV = \begin{bmatrix} \tilde{D} & 0 \\ 0 & 0 \end{bmatrix}$$

Where U,V are unimodular and $\tilde{D} = diag(d_1, ..., d_r)$ for some non zero polynomials di.We divide UNV by d and put

$$D := \frac{\tilde{D}}{d} = diag(\frac{d_1}{d}, ..., \frac{d_r}{d})$$

Let γ_i,δ_i be coprime polynomials with

$$\frac{d_i}{d} = \frac{\gamma_i}{\delta_i}$$

Since $d_1|d_2|...|d_r$,we have $d_{i+1} = d_i e_i$ for some polynomials e_i,where $i = 1, ..., r-1$.This implies

$$\gamma_{i+1}\delta_i = \gamma_i\delta_{i+1}e_i$$

for all i.
because

$$\frac{\gamma_{i+1}}{\delta_{i+1}} = \frac{d_{i+1}}{d} = \frac{d_i e_i}{d} = \frac{\gamma_i e_i}{\delta_i}$$

which implies $\gamma_i|\gamma_{i+1}\delta_i$ and $\delta_{i+1}|\gamma_{i+1}\delta_i$ since δ_i and γ_i are coprime,γ_{i+1} must be a multiple of γ_i. similarly,coprimeness of δ_{i+1} and γ_{i+1} implies δ_i must be a multiple of δ_{i+1}. □

Theorem 2.2.3. *The size of a minimal realization of H equals the McMillan-degree of H.*

Proof. We show that $d(H)$ from (2.3) coincides with the McMillan-degree of H. Let
$UHV = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$
be the McMillan form of H, with $D = diag(\frac{\gamma_1}{\delta_1}...\frac{\gamma_r}{\delta_r})$ Define
$\Gamma = \begin{bmatrix} diag(\gamma_1...\gamma_r) & 0 \\ 0 & 0 \end{bmatrix}$ and $\Delta = \begin{bmatrix} diag(\delta_1...\delta_r) & 0 \\ 0 & I \end{bmatrix}$ Then $UHV = \Gamma\Delta^{-1}$, that is, (Γ, Δ) is a right factorization of UHV . Since each pair (γ_i, δ_i) is coprime, it is even a right coprime factorization of UHV . Then $(U^{-1}\Gamma, V\Delta)$ is a right coprime factorization of H. Therefore

$$d(H) = degdet(V\Delta) = degdet(\Delta) = deg \prod_{i=1}^{r} \delta_i = \sum_{i=1}^{r} deg(\delta_i)$$

which is precisely the McMillan-degree of H. □

2.3 Poles

A complex number λ is called a pole of a rational matrix H if it is a pole of one of its entries. Equivalently, the poles of H are the zeros of the polynomials δ_i in the McMillan form of H. Still equivalently, they are the zeros of $det(P)$, where P is the denominator matrix in a coprime factorization of H.

Theorem 2.3.1. *Let (A, B, C, D) be a realization ofH. Any pole ofH is an eigenvalue of A. Conversely, an eigenvalue of A which is not a pole of H must be an uncontrollable mode of(A, B) or an unobservable mode of (A, C). In particular, if (A, B, C, D) is a minimal realization ofH, then the eigenvalues of A are precisely the poles of H.*

Remark 2.3. *This theorem shows that there is a close relation between the eigenvalues of A and the poles of $H = C(sI - A)^{-1}B + D$. This is the reason why one speaks of pole shifting in Section 5.3, although eigenvalue shifting would probably be more appropriate.*

Proof. Without loss of generality, let H be strictly proper. Since

$$H = C(sI - A)^{-1}B = \frac{Cadj(sI-A)B}{det(sI-A)}$$

any pole of H must be a zero of $det(sI - A)$, that is, it must be an eigenvalue of A. Now letλ be an eigenvalue of A which is not a pole of H. Assume that λ is not an unobservable mode, that is,

$$rank \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n \tag{2.5}$$

We need to show that λ is an uncontrollable mode. Let$G = (sI - A)^{-1}B = QP^{-1}$, where (Q, P) is right coprime. Then we have

$$BP = (sI - A)Q \tag{2.6}$$

and $H = CG = CQP^{-1}$, with

$$rank \begin{bmatrix} CQ(\lambda) \\ P(\lambda) \end{bmatrix} = m \tag{2.7}$$

To see this, let v be such that$CQ(\lambda)v = 0$ and $P(\lambda)v = 0$. We need to show that this implies $v = 0$. From (2.7),

$$0 = BP(\lambda)v = (\lambda I - A)Q(\lambda)v$$

and thus

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} Q(\lambda)v = 0$$

which implies that$Q(\lambda)v = 0$because of (2.5). But then both $P(\lambda)v = 0$ and $Q(\lambda)v = 0$, which implies that $v = 0$ due to the right coprimeness of (Q, P).
Now we must have $det(P(\lambda)) \neq 0$. If conversely, $det(P(\lambda)) = 0$, then there would exist a $v \neq 0$ such that $P(\lambda)v = 0$. Since λ is not a pole of H, the complex matrix $H(\lambda)$ is well-defined, and thus

$$0 = H(\lambda)P(\lambda)v = CQ(\lambda)v$$

and this would be a contradiction to (8.9).
Sinceλ is an eigenvalue of A, there exists $0 \neq z \in \mathbb{C}^{1\times n}$ such that $z(\lambda I - A) = 0$. Then (8.8) implies $zBP(\lambda) = 0$ and hence, since $P(\lambda)$ is non-singular, we must have $zB = 0$. Then

$$z \begin{bmatrix} \lambda I - A & B \end{bmatrix} = 0$$

which shows that

$$rank \begin{bmatrix} \lambda I - A & B \end{bmatrix} < n$$

that is, λ is an uncontrollable mode of (A, B). □

2.4 Zeros

A complex number λ is called a zero of a rational matrix H if it is not a pole of H and

$$rank(H(\lambda)) < rank(H).$$

Equivalently, the zeros of H are the zeros of the polynomials $\gamma_i$in the McMillan form of H provided that they are not also zeros of the δ_i. Equivalently, they are the λ with $rank(Q(\lambda)) < rank(Q)$ and $det(P(\lambda)) \neq 0$ in a coprime factorization of H.
Let (A, B, C, D) be a realization of H. A complex number λ is called a zero of (A, B, C, D) if

$$rank \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} < rank \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$$

Theorem 2.4.1. *Let (A, B, C, D) be a realization of H. Any zero of H must be a zero of (A, B, C, D). Conversely, a zero of (A, B, C, D) which is not a zero of H must be a pole of H or an uncontrollable mode of (A, B) or an unobservable mode of (A, C). In particular, let (A, B, C, D) be a minimal realization of H and let λ be not a pole of H. Then λ is a zero of H if and only if it is a zero of (A, B, C, D).*

Proof. Over R(s), we have the Schur complement formula

$$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ C(A - sI)^{-1} & I \end{bmatrix} \begin{bmatrix} A - sI & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} I & (A - sI)^{-1}B \\ 0 & I \end{bmatrix}$$

Thus we get

$$rank \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = rank(A - sI) + rank(H).$$

Without loss of generality, let (A, B, C, D) be such that

$$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 - sI & A_2 & B_1 \\ 0 & A_3 - sI & 0 \\ C_1 & C_2 & D \end{bmatrix} \sim \begin{bmatrix} A_1 - sI & B_1 & A_2 \\ C_1 & D & C_2 \\ 0 & 0 & A_3 - sI \end{bmatrix}$$

where the equality comes from a Kalman decomposition, and $\sim$ denotes unimodular equivalence of matrices. By construction, we have $H = C(sI - A)^{-1}B + D = C_1(sI - A_1)^{-1}B_1 + D$, and (A_1, B_1) is controllable. Let $G := (sI - A_1)^{-1}B_1$ and let$G = QP^{-1}$ be a right coprime factorization. Then we have

$$B_1P = (sI - A_1)Q, RP + SQ = I, (A_1 - sI)R_1 + B_1S_1 = I$$

for some polynomial matrices R, S, R_1, S_1. In matrix notation,

$$\begin{bmatrix} A_1 - sI & B_1 \\ S & R \end{bmatrix} \begin{bmatrix} R_1 & Q \\ S_1 & P \end{bmatrix} = \begin{bmatrix} I & 0 \\ * & I \end{bmatrix}$$

which yields

$$\begin{bmatrix} A_1 - sI & B_1 \\ S & R \end{bmatrix} \begin{bmatrix} \tilde{R}_1 & Q \\ \tilde{S}_1 & P \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

similarly as in an earlier proof. We may conclude that the matrices on the left hand side are unimodular. Since

$$\begin{bmatrix} A_1 - sI & B_1 \\ C_1 & D \end{bmatrix} \begin{bmatrix} \tilde{R}_1 & Q \\ \tilde{S}_1 & P \end{bmatrix} = \begin{bmatrix} I & 0 \\ * & C_1 Q + DP \end{bmatrix}$$

we get

$$\begin{bmatrix} A_1 - sI & B_1 \\ C_1 & D \end{bmatrix} \sim \begin{bmatrix} I & 0 \\ 0 & C_1 Q + DP \end{bmatrix}$$

Coming back to the original state space system, we obtain

$$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \sim \begin{bmatrix} I & 0 & 0 \\ 0 & C_1 Q + DP & * \\ 0 & 0 & A_3 - sI \end{bmatrix}$$

Recalling that $G = QP^{-1}$, we have $H = C_1 G + D = (C_1 Q + DP)P^{-1}$. Hence the zeros of H are zeros of $C_1 Q + DP$, which implies that they are also zeros of (A, B, C, D).
Now let λ be a zero of (A, B, C, D).
Case 1: λ is an eigenvalue of A and neither uncontrollable nor unobservable. Then λ is a pole of H.
Case 2: λ is an eigenvalue of A and uncontrollable or unobservable.
Case 3: λ is not an eigenvalue of A. Then it is not a pole of H and we may use the Schur complement formula from above with $s = \lambda$. Noting that $rank(A\lambda I) = n = rank(A - sI)$, we see that $rank(H(\lambda)) < rank(H)$. Thus λ is a zero of H. □

Remark 2.4. *Interpretation of zeros: Assume that* $rank(H) = m$. *Then*

$$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$$

has full column rank. Thus λ *is a zero of* (A, B, C, D) *if and only if there exists* $(x_0, u_0) \neq (0, 0)$ *such that*

$$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then the input $u(t) = u_0 e^{\lambda t}$ *and the initial condition* $x(0) = x_0$ *lead to* $x(t) = x_0 e^{\lambda t}$, *and the corresponding output* $y(t) = Cx(t) + Du(t)$ *is identically zero. (We admit complex-valued signals, for simplicity; but the same argument holds for the real parts of* u, x, y*.) Thus the system annihilates the input* u *of frequency* λ, *and this is why* λ *can be seen as a zero of the system. If* $u_0 = 0$, *then* $x_0 \neq 0$ *is indistinguishable from zero. In particular, any unobservable eigenvalue of* A *w.r.t.* C *is a zero of* (A, B, C, D). *A similar interpretation is possible for* $rank(H) = p$. *Then any uncontrollable eigenvalue of* A *w.r.t.* B *is a zero of* (A, B, C, D).

Example 2.4.1. *Let*

$$A = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 1 \\ , & & \end{bmatrix}, D = 1$$

Then $H = \frac{s}{s-1}$ *, which has a pole at 1 and a zero at 0. We have* $spec(A) = \{1, 2\}$*. The eigenvalue 2 is not a pole, and indeed one can check that it is uncontrollable (but not unobservable). Note that the eigenvalue 1 is both uncontrollable and unobservable, but still a pole of H. The zeros of* (A, B, C, D) *are 0, 1, 2: 0 is a zero of H, 1 is a pole, and 2 is an uncontrollable eigenvalue. A minimal realization of H is given by* $A_1 = B_1 = C_1 = D = 1$*. Then 1 is the only eigenvalue of* A_1 *and* 0 *is the only zero of* (A_1, B_1, C_1, D)*. If we modify this example by taking* A, B *as above and*

$$C^{|} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, D^{|} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

then we have

$$H^{|} = \begin{bmatrix} \frac{s}{s-1} \\ 0 \end{bmatrix}$$

and $(A, B, C^{|}, D^{|})$ *has zeros at 0 and 1, but not at 2. This shows that an uncontrollable mode is not necessarily a zero of the realization if* $rank(H) < p$*.*

Remark 2.5. *In the tutorial, we have seen that the series interconnection of two state space systems is controllable if both systems are controllable and additionally, for all* $\lambda \in spec(A_2)$*, we have that*

$$\begin{bmatrix} A1 - \lambda I & B_1 \\ C_1 & D_1 \end{bmatrix}$$

has full row rank. This requirement amounts (more or less) to saying that no pole of H_2 *should be a zero of* H_1 *(to avoid cancellation effects). In terms of the system trajectories, this means that no characteristic frequency of the second system should be blocked by the first system. A similar fact can be shown for* $H = H_2H_1$ *with*$H_1 = P^{-1}Q_1$ *and* $H_2 = P^{-1}Q_2$ *where the requirement was that for all* λ *with* $det(P_2(\lambda)) = 0$*, the matrix* $Q_1(\lambda)$ *has full row rank. Again, this can be interpreted (roughly) as meaning that no pole of* H_2 *should be a zero of* H_1*.*

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