FIINETE,SIMPLE AND UNDIRECTED GRAPH ENCODING

ADDIS ABABA UNIVERSITY

COLLEGE OF COMPUTATIONAL AND NATURAL SCIENCES
SCHOOL OF GRADUATE STUDIES
DEPARTMENT OF MATHEMATICS

m A project Submitted “in partial fulfillment of the

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By: Tsedeke Demissie Geberu
Advisor: Yrgalem (PhD)

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Abstract

In this project we will see the existence of periodic solution(s) to the second order ODE of the form:

$$-x''(t) + a(t)x'(t) = g(t, x) - f(t, x(t), x'(t))$$

by means of Schauders Fixed Point Theorem where a is a continuous ω-periodic function , $g(t, u)$, $f(t, u, v)$ are ω-periodic functions in t for $u = x(t)$, $v = x'(t)$ and $\omega > 0$. The method of proof is composed of two steps, the first step is to transform the original equation into integro-differential equation through a linear integral operator and the second step is an application of the Schauder's Fixed Point Theorem.

Keywords: Periodic solution; Schauder's fixed point theorem; Fundamental matrix.

Contents

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List of Notations and Abbreviations

◇ $\mathbb{N}$= The set of all natural numbers
◇ $\mathbb{R}$= The set of all real numbers.
◇ $\mathbb{R}^n$=n-dimensional Euclidean space
◇ $<$ = The usual "less than" symbol in the ordered set $\mathbb{R}$
◇ $\leq$ = The usual "less than or equal to" symbol in the ordered set $\mathbb{R}$
◇ lim=limit
◇ max=maximum
◇ min=minimum
◇ exp=$e \approx$ 2.73
◇ $\{x_n\} = \{x_n\}_{n=1}^{\infty}$
◇ $C^n(X,\mathbb{R})$=$\{f : X \to \mathbb{R} | f^{(n)}$ exists and is continuous on X for all $n \in \mathbb{N}\}$
◇ $C^n(X,X)$:=$C^n(X)$=$\{f : X \to X | f^{(n)}$ exists and is continuous on X for all $n \in \mathbb{N}\}$
◇ $C(X)$=$\{f : X \to \mathbb{R} | f$ is continuous on $X\}$ where X is non empty set.
◇ $\cap$ = symbol for intersection of sets
◇ $\cup$ = symbol for union of sets
◇ $conv(M) = \bigcap_{M \subseteq H} H$ whenever H is convex set.
◇ **B** (x_0)=open ball centered at x_0 with radius δ
◇ dim=dimension

Introduction

It is truism that nothing is permanent except changing, and the primary purpose of differential equation is to serve as tool for the general laws of nature by modeling numerous problems in science, engineering, economics and other areas in the language of Mathematics. It is the source of most of the ideas and theories which constitute higher analysis such as power series, Fourier series, integral equations, existence Theories and other special functions.

Ordinary differential equations serve as mathematical models for many exciting real-world problems, not only in science and technology, but also in such diverse fields as economics, psychology, defense, and demography. Particularly, second order ODE used to model problems such as in physics to model vibrational motions (Simple Harmonic motion , electrical circuits,damped motion,forced motion) and other areas like engineering,economics etc.

The study of existence and property of periodic solutions of ordinary differential equations has already attracted the attention of many researchers in the area. Many results are obtained by appealing to classical methods that range from upper and lower solutions techniques to fixed point theorems, e.g. Shauder's fixed point theorem[10].Existence and uniqueness of periodic solution of explicit second order ordinary differential equation

$$x''(t) + f(t, x(t), x'(t)) = 0$$

With the nonlinear function f is studied among others by J.Zu.In this recent paper,J.Zu discussed existence and uniqueness of periodic solutions of this problem constructing the so called upper and lower bounderies combined with the use of Lerar-Schauder degree theory.

In recent past, J.Robert et.al investigated the existence and non- existence of periodic solutions of nonlinear second order ODE with p-periodic boundary conditions

$$x'' + f(x(t)) = h(t)$$

$$x(0) = x(p); x'(0) = x'(p)$$

In their study , f is a nonlinear function of state variable x while h is a forcing function that depends solely on the temporal variable t.

In this particular Project we will investigate the existence of periodic solutions to

$$-x''(t) + a(t)x'(t) = g(t, x) - f(t, x(t), x'(t))$$

where a is a continuous ω-periodic function, $g(t,u)$, $f(t,u,v)$ are ω-periodic functions in t for $u = x(t)$, $v = x'(t)$ and $\omega > 0$. This equation includes many important models, for example

$$
\begin{gathered}
x''(t) + \mu \sin(x(t)) = h(t), \\
x''(t) + cx'(t) + \mu \sin(x(t)) = h(t), \\
x''(t) + f(x'(t)) + g(x(t)) = h(t), \\
x''(t) + f(x(t))x'(t) + g(x(t)) = h(t), \\
x''(t) = \frac{1}{x^{\lambda}} + h(t),\ \lambda > 0
\end{gathered}
$$

which arise in many fields such as physics, mechanics and engineering.

This project consists **four** main chapters and the first three chapters are **preliminaries** which are basic concepts for the main body of this Project. A brief description of the topics covered in this projec is as follows:
Periodic functions and properties of their definite integral are explained in **chapter one** .**Chapter two** is about Schauder's Fixed Point Theory .ODE With Periodic Data is described in **Chapter three**. **Chapter four** deals with the main body of this project which is existence of Periodic solutions for class of second order ODE with Periodic Data.Periodic solution(s) of linear system of first order ODE is also explained in this chapter.

Chapter 1

F-decomposition and F-cover of A Graph

1.1 F-decomposition

Definition 1.1. *A decomposition of a graph* G *is a set of subgraphs* $G_1, G_2, \ldots, G_K$ *that partition the edges of* G*,that is, for all* i *and* j *,*$E(G_i) \cap E(G_j) = \emptyset$ *and* $\cup_{1 \leq i \leq k} G_i = G$*.*

Note that a decomposition of a given graph G is not unique. **Example:**

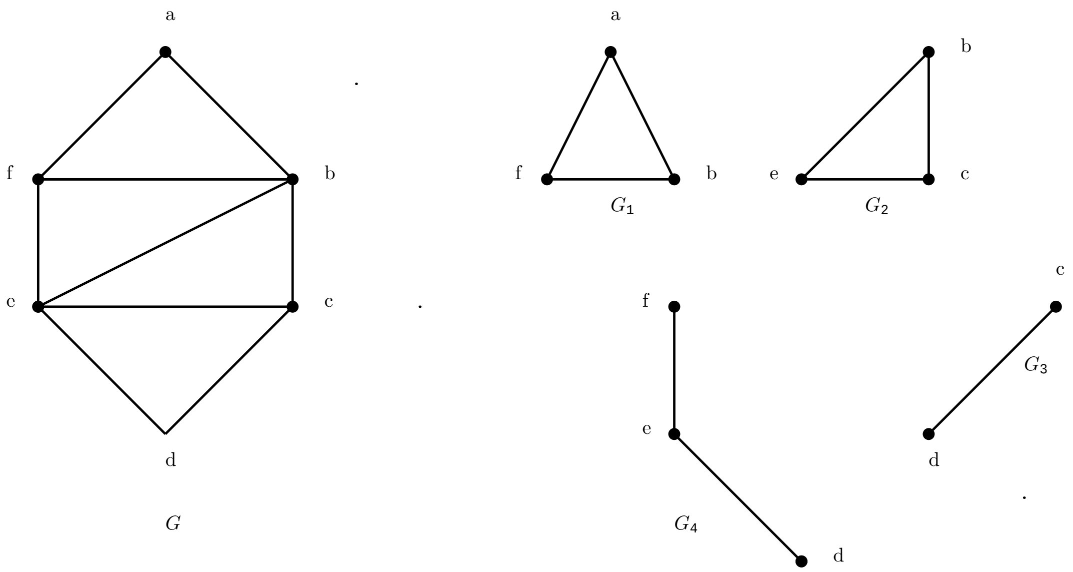


Fig.1.1 The graph G and its decomposition in subgraphs.

Hence the set of $\{G_1, G_2, G_3, G_4\}$ is a decomposition since $\cup_{1 \le i \le 4} G_i = G$ and its elements are pairwise edge disjoint. of G.

Definition 1.2. *Let H be a fixed family of graphs.An H-decomposition of a given graph G is a decompositions of G such that each subgraph G_i in the decomposition is isomorphic to elements of H*
Note that H can the set of paths,cycles,cliques,etc.

Example: Using the graph G at **Figure 1.1** the set of the following subgraphs can be a path decomposition of G.

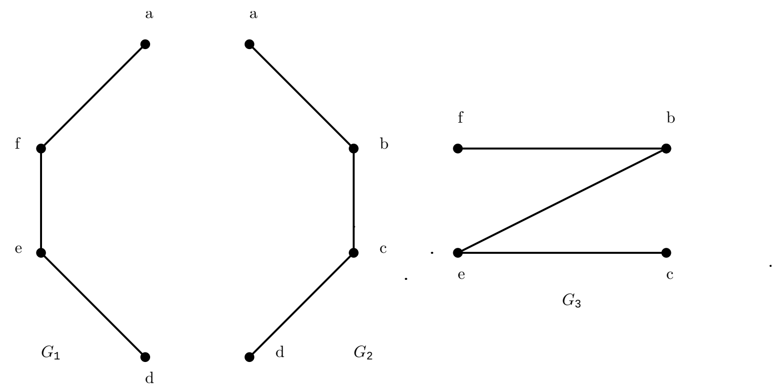


Fig.1.2 Path(P_4)decomposition of G.

$\{G_1, G_2, G_3\}$ is path(P_4) decomposition of G since $G_i \cong P_4$ for $1 \le i \le 3$.

Definition 1.3. A clique in a graph G is a subset C of vertex set $V(G)$,such that for every two vertices in C ,there exists an edge connecting the two.This is equivalent to saying the subgraph induced by C is complete.
Note that we may use the term clique to refer a complete subgraph of G.

Example: Let us consider the graph G at **figure 1.1** By looking this graph G ,the set of $C = \{a, b, f\}$ and $C' = \{c, d, e\}$ in $V(G)$ are a clique since there is an edge connecting every two vertices in C and C' .

Definition 1.4. *A maximal clique is a clique that cannot be extended by including one more adjacent vertex,that is ,a clique which does not exist with in the vertex set of a larger clique.*

Example:

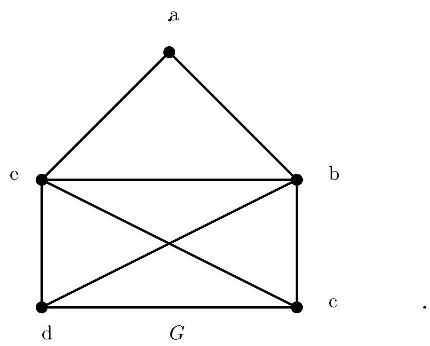


Fig 1.3 The graph G.

$\{a, b, e\}$,$\{b, c, d, e\}$,$\{b, c, d\}$ are cliques in G and since there is no larger clique containing $\{a, b, e\}$,$\{b, c, d, e\}$ then $\{a, b, e\}$,$\{b, c, d, e\}$ are maximal clique in G. and their induced complete subgraphs are maximal. But $\{b, c, d\} \subseteq \{b, c, d, e\}$ therefore $\{b, c, d\}$ is not maximal clique hence we can say the complete subgraph induced by $\{b, c, e\}$ is not maximal in G.

Definition 1.5. *A clique-decomposition of A graph G is a decomposition of G in to cliques,which together contains each edge of G exactly once.*

Example: The graph G at Figure 1.3 can be decomposed in to cliques as follows.

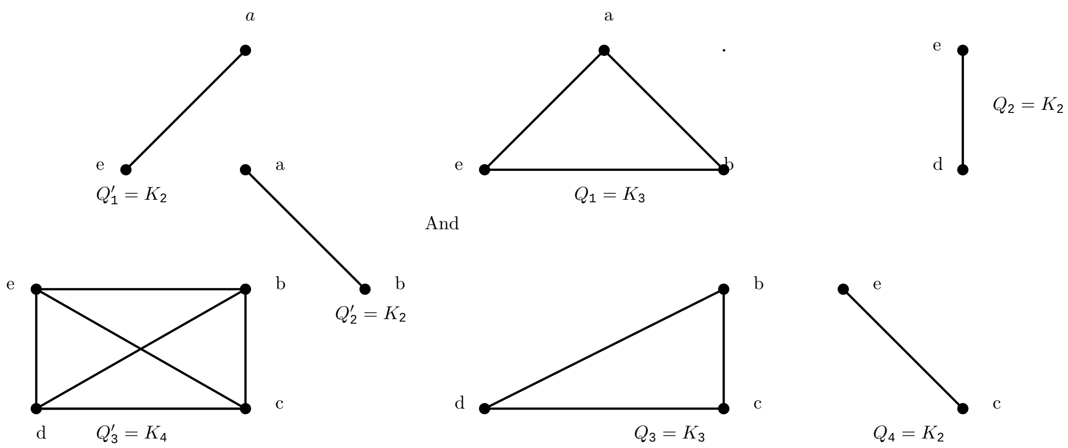
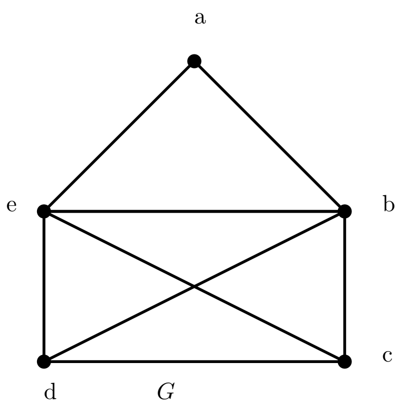


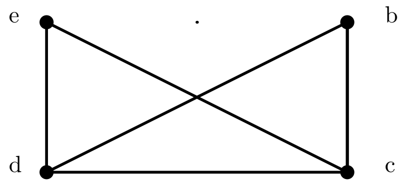
Fig 1.4 Clique(complete subgraph) decompositions of the graph G.

The sets $\{Q'_1, Q'_2, Q'_3\}$ and $\{Q_1, Q_2, Q_3, Q_4\}$ are clique decompositions of G.

Definition 1.6. *A Greedy clique decomposition of a graph is a clique decomposition obtained by removing maximal cliques from a graph one by one until the graph is empty.*

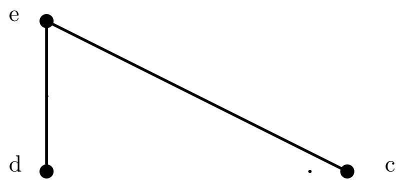
Let Q_1 be a maximal complete subgraph abe in G.Now,removing Q_1 from G leaves $G - E(Q_1)$.





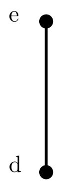
$G - E(Q_1)$

Since edc and bcd are maximal in $G - Q_1$ we can remove first either of them.
Let $Q_2 = bcd$.Thus removing Q_2 from $G - E(Q_1)$ leves $G - E(Q_1) - E(Q_2)$.



$G - E(Q_1) - E(Q_2)$

Let $Q_3 = ec$.Then removing $Q_3 = ec$ from $G - E(Q_1) - E(Q_2)$ leaves $G - E(Q_1) - E(Q_2) - E(Q_3)$



Finally asign Q_4 for a complete subgraph ed then removing Q_4 from $G - E(Q_1) - E(Q_2) - E(Q_3)$ leaves empty graph.

$G - E(Q_1) - E(Q_2) - E(Q_3)$

Fig 1.5 Greedy clique decompositions of the graph G.

Hence a set $\{Q_1, Q_2, Q_3, Q_4\}$ is a greedy clique decomposition of G. Note that the set of all edges of the graph G in figure 1.5 can be the decomposition of G; but it cannot be a greedy clique decomposition of G since each edge of G is not maximal in G.

1.2 F-cover

Definition 1.7. *A covering of G is a family of subgraphs $\{G_1, G_2, ..., G_t\}$ having the property that each edge of G is contained in at least one graph G_i,for some i ,that is $\cup_{1 \leq i \leq t} G_t = G$.*

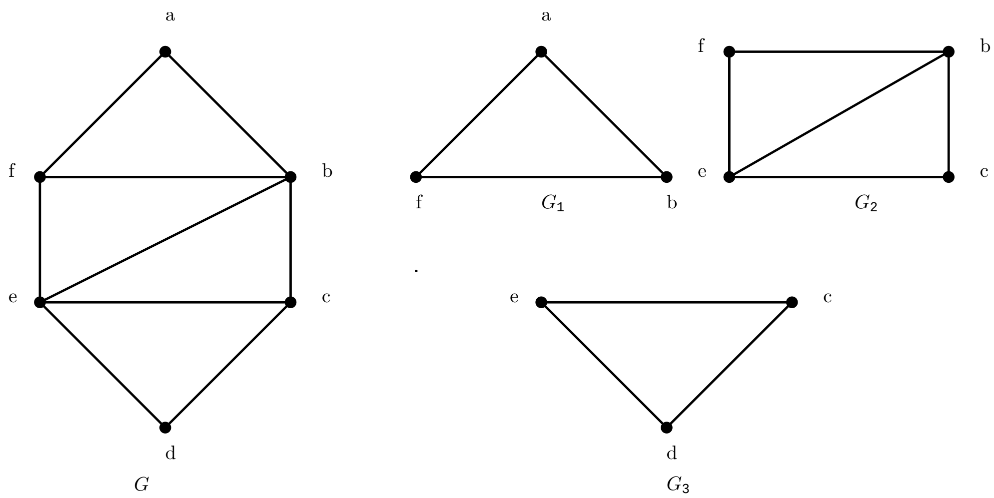


Fig 1.5 The graph G and its covering.

$\cup_{i \leq i \leq 3} G_i = G$ thus $\{G_1, G_2, G_3\}$ is a covering of G but not a decomposition of G since for example an edge fb is contained in both G_1 and G_2.

Definition 1.8. *Let H be a fixed family of graphs.An H-covering of a given graph G is a covering of G such that each subgraph G_i in the covering is isomorphic to elements of H .*

Note that like as an H-decomposition of a graph G, in this case H can be also a family of paths,cycles,clique,etc.

Example Using the graph at figure 1.5 the following subgraphs can be cycle-covering of G.

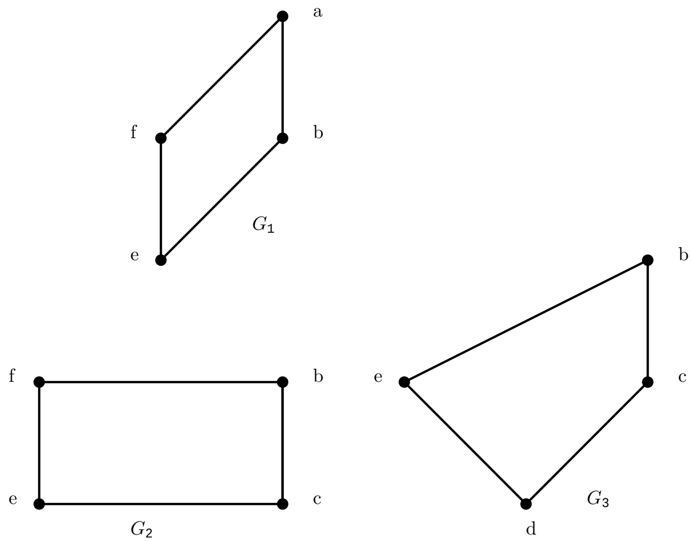


Fig 1.5 cycle (C_4) covering.

Since $G_i \cong C_4$ for $i \leq i \leq 3$ the set of $\{G_1, G_2, G_3\}$ is cycle-covering of G.

Definition 1.9. *A clique covering of a Graph G is any family* $\varepsilon = \{Q_1, Q_2, ...Q_k\}$ *of complete subgraphs of G such that every edge of G is in at leat one of* $Q_1, Q_2, ...Q_k$.

Using the graph at figure 1.5 the following subgraphs can be clique-covering of G

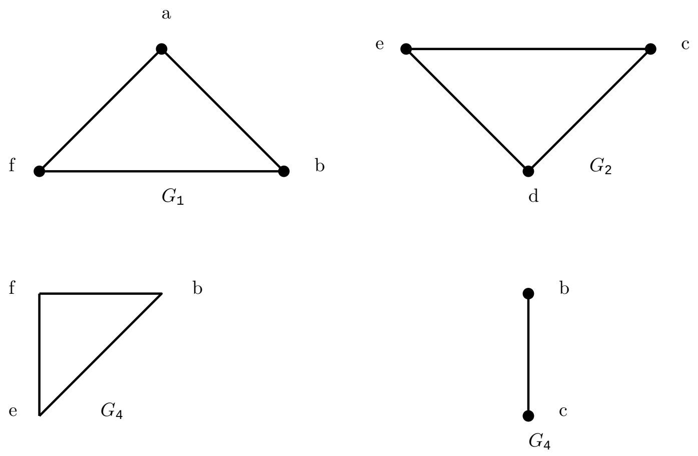


Fig 1.5 clique covering of G.

$\{G_1, G_2, G_3, G_4\}$ is a clique covering of G since $\cup_{i \leq i \leq 4} G_i = G$ and $G_i \cong K_3$ for $i = 1, 2, 4$ and $G_3 \cong K_2$.
Note that when F is not closed under talking subgraphs F-decompositions may require more subgraphs than F-covering. For instance we can cover the kite($K_4 - e$) with two complete subgraphs but three complete subgraphs are needed decompose it.

Example:

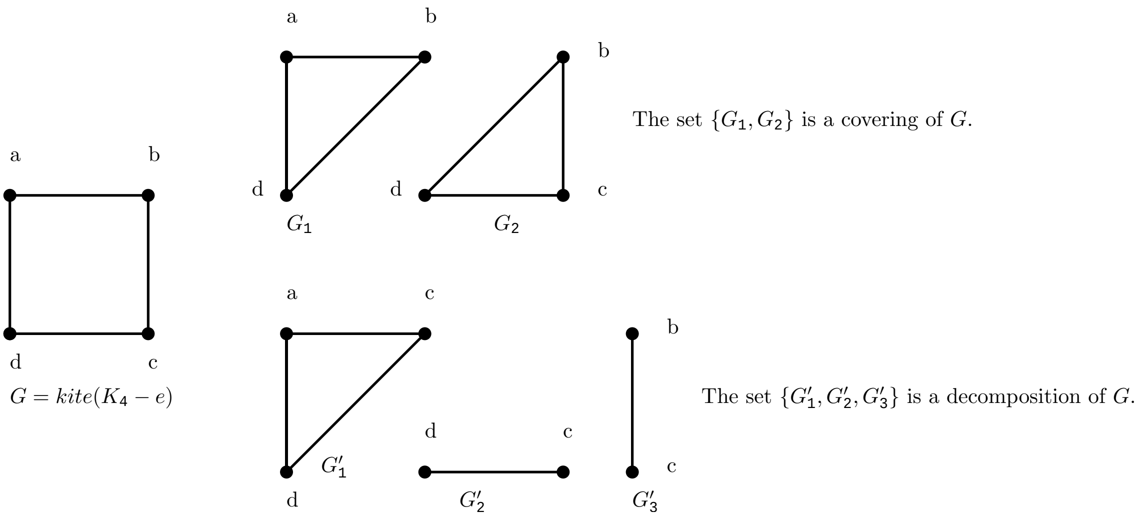


Fig 1.5 Clique covering and Decomposition of G.

We can generalize this statement by the following theorem.

Theorem 1.1. *If* $\{F_1, F_2, ...F_k\}$ *is a decomposition of a given graph* G *then it is a covering of* G.

Proof. Suppose that $\{F_1, F_2, ...F_k\}$ is not a covering of G this implies at least one of the edge of a graph G is not contained in one of $F_i's$ which leads us to conclude that $\cup_{i=1}^{k} F_i \neq G$ therefore $F_1, F_2, ...F_k$ is not a decomposition of G which contradicts.
Hence the proof □

Chapter 2

Encoding of Graphs

In this chapter we will deal with the definition of graph encoding and the two types of encoding graphs.

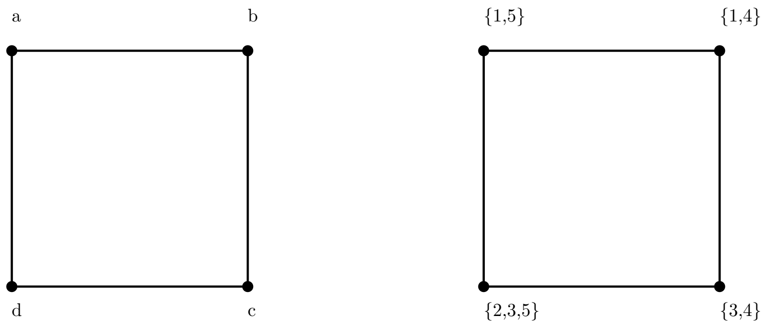
Definition 2.1. *For any given graph G encoding graph is a representing of this graph G into another form by assigning vectors to the vertices of G.*

The two types of Graph encoding are **An intersection representation** and **A product representation** ,the parameters that we are going to study which are related with the two types of encoding are ***an intersection number*** and ***a product dimension*** respectively.Note that the ways that we are going to assign vectors to the vertices of G are differ according to the types of encoding that we will discus indetail

2.1 Intersection representation of graphs

Definition 2.2. *An intersection representation of length t assigns each vertex a 0,1-vector of length t such that $u \leftrightarrow v$ if and only if their vectors have a 1 in a common position.Equivalently,it assigns each $x \epsilon V(G)$ a set $S_x \subseteq [t]$ such that $u \leftrightarrow v$ if and only if $S_u \bigcap S_v \neq \emptyset$. **The intersection number** $\theta'(G)$ is the minimum length of an intersection representation G.*

The graph G represented by an intersection representation is said to be **intersection graph**.
Example 1: In this example let us see the intersection representation of the graph $G = C_4$.

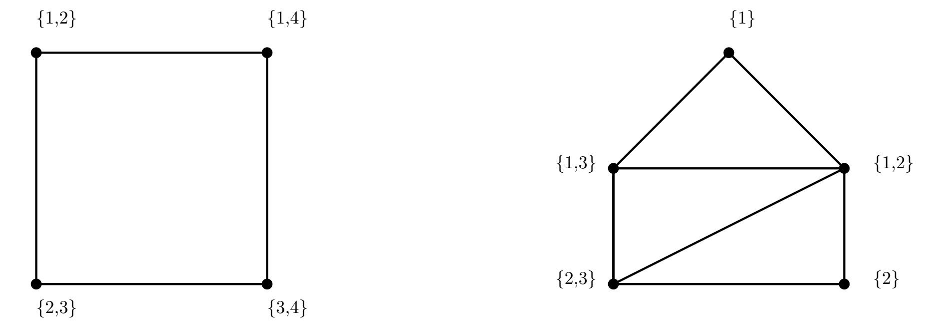


The graph G Intersection graph.

Fig 2.1 The graph G and its intersection representation.

The family of sets corresponds to $v \in V(G) = \{a, b, c, d\}$ are $S_a =$ $\{1, 5\}$,$S_b = \{1, 4\}$,$S_c = \{3, 4\}$,$S_d = \{2, 3, 5\}$.Usin.3definita 10()Tjta 10(w)J /etv 10(ha)J /waV

the following.
$S_a \cap S_b = \{2,3\} \Leftrightarrow a \leftrightarrow b$
$S_a \cap S_c = \emptyset \Leftrightarrow a \nleftrightarrow c$
$S_a \cap S_d = \emptyset \Leftrightarrow a \nleftrightarrow d$
$S_a \cap S_e = \{1,2\} \Leftrightarrow a \leftrightarrow e$
$S_b \cap S_c = \{5\} \Rightarrow b \leftrightarrow c$
$S_b \cap S_d = \{5\} \Leftrightarrow b \leftrightarrow d$
$S_b \cap S_e = \{2\} \Leftrightarrow b \leftrightarrow e$
$S_c \cap S_d = \{5\} \Leftrightarrow c \leftrightarrow d$
$S_c \cap S_e = \emptyset \Leftrightarrow c \nleftrightarrow e$
$S_d \cap S_e = \{4\} \Leftrightarrow d \leftrightarrow e$
Hence $F = \{S_a, S_b, S_c, S_d, S_e\}$ is an intersection representation of a graph H of length 6 since $\cup_{v \in V(H)} S_v = [6]$.
Note that we can represent those graphs G and H in above with minimum length of intersection representation than the length mentioned in the previous example.



Intersection representation of G with $\theta'(G) = 4$. Intersection representation of H with $\theta'(H) = 3$.

Fig 2.3 The intersection representation of G and H with minimum length.

The elements of [t] in a representation correspond to complete subgraphs that cover $E(G)$.

Proposition 2.1. (Erdos-Goodman-posa [1966]) The intersection number equals the minimum number of complete subgraphs needed cover E(G).

Proof. Given an intersection representation $F = \{S_1, S_2, ..., S_n\}$ of a graph G of vertex $V(G) = \{v_1, v_2, ..., v_n\}$ .Now,we need to construct a complete subgraphs of G which covers $E(G)$. Let $[t] = \bigcup_{a=1}^{n} S_a = \{1, 2, ..., t\}$.For each $i \in [t]$ we generate a clique $Q_F(i) = \{v \in V(G) : i \in S_v\}$.Let a vertices v_1 and v_2 be in $Q_F(i)$ thus $i \in S_{v_1}$ and $i \in S_{v_2}$.
$\Rightarrow i \in S_{v_1} \bigcap S_{v_2}$.
$\Rightarrow S_{v_1} \bigcap S_{v_2} \neq \emptyset$
$\Rightarrow v_1 \leftrightarrow v_2$.
Hence any two vertices in $Q_F(i)$ are adjacent since their corresponding sets have a non empty intersection containing i.Thus $Q_F(i)$ is a clique. So we obtain that $Q(F) = \{Q_F(1), Q_F(2), ..., Q_F(t)\}$ for for each $i \in [t]$.Now,the complete subgraphs induced by $Q_F(1), Q_F(2), ..., Q_F(t)$ are $G[Q_F(1)], G[Q_F(2)], ..., G[Q_F(t)]$ respectively but we need to show that the induced complete subgraphs covers $E(G)$. Let e_1 be an arbitrary element in $E(G)$ with end points u and v then $S_u \cap S_v \neq \emptyset$ there fore $u \leftrightarrow v$.This implies that the set of u and v forms a clique.Let $K = \{u, v\}$ then K belongs to either of the element in $Q(F)$.Hence $e_1 \in G[K]$ which is contained in at least one of the complete subgraphs induced by members $Q(F)$ and at the end we can conclude that the set of $\{G[Q_F(1)], G[Q_F(2)], ..., G[Q_F(t)]\}$ covers $E(G)$ such that $|\bigcup_{i=1}^{t} G[Q_F(i)]| = t = |\bigcup_{i=1}^{n} S_a|$
Conversely let $Q = \{Q_1, Q_2, ..., Q_t\}$ be a set of complete subgraphs of G which covers $E(G)$ and let $I = \{1, 2, .., t\} = [t]$ be a collection of index number of complete subgraphs in Q then we want to generate an intersection representation of G.Let $\{v_1, v_2, ..., v_n\}$ be the set of vertices of G.Now assigninig the set $\{i : v \in V(Q_i)\}$ to each vertex v yields an intersection representation.Let $S_Q(v) = \{i : v \in V(Q_i)\}$ which corresponds to v in G;that is each vertex v of G represented by the collection of index elements in I of a complete subgraphs in Q that contains v.Now for each $v \in V(G)$ we obtain $F(Q) = \{S_Q(v) : v \in V(G)\}$ $=\{S_Q(v_1), S_Q(v_1), ..., S_Q(v_1)\}$. For u and v in $V(G)$,let $S_Q(v)$ and $S_Q(u)$ be in $F(Q)$.Then $S_Q(v) \cap S_Q(u)$ is the set of complete subgraphs in Q simultaneously containing v and u.Now,
if $S_Q(v) \cap S_Q(u) \neq \emptyset$ (*i.e* if there exists at leat one complete subgraph contains u and v then by definition of intersection representation of a graph u and v are adjacent.Thus $uv \in E(G)$.conversily if $uv \in E(G)$ then u and v are adjacent.There fore there exists at least one complete subgraph which simultaneously contains u and v.Thus $S_Q(v) \cap S_Q(u) \neq \emptyset$.Hence $F(Q) =$

$\{S_Q(v_2), S_Q(v_2), ..., S_Q(v_2)\}$ is an intersection representation of G.
Since each $V(Q_i)$ contains at leat one vertex v of G then each complete subgraph contributes an index element $i \in [t]$ for $S_Q(v)$.Hence,
$|\bigcup_{v \in V(G)} S_Q(v)| = t = |Q|$. □

Example: Consider the graphs G and H at fig 2.1 and fig 2.2.
The set of complete subgraphs with minimum number of elements; induced by the vertex sets $\{a, b\}$, $\{b, c\}$,$\{c, d\}$,$\{d, a\}$ covers G thus the minimum number of complete subgraph which cover G is four.Similarly the set of complete subgraphs induced by the vertex sets $\{a, b, e\}$,$\{e, b, e\}$,$\{b, c, d\}$ covers H and its members are minimum in number.Hence the minimum number of complete subgraphs which covers H is three. Just now,from figure 2.3 and above proposition we conclude that $\theta'(H)$ = the minimum complete subgraphs which cover H=3 and $\theta'(G)$ ==the minimum number of complete subgraphs which cover G=4.

Lemma 2.1. *A graph G of vertex $n \geq 3$ is complete subgraph free if it is a triangle free.*

Proof. **Claim:**G is K_n free where is $n \geq 3$.
suppose not!.that is $\exists K_n \subseteq G$ for $n \geq 3$.
since $K_3 \subseteq K_n \subseteq G$ this implies $K_3 \subseteq G$.thus G is not triangle free which contradicts.
hence the proof. □

Corollary 2.1. *if a graph G is a triangle free then $\theta'(G) = ||G||$.*

Proof. From *lemma*2.1 we have that a triangle free graph is complete subgraph free.again from *proposition*2.1 we have showed that the intersection number equals the minimum number of complete subgraphs needed to cover the set of edges of G.Thus,for a complete free graph the only complete subgraphs of G which cover the set of edges of G is it self the set of edges of G.that is for a triangle free graph the only **optimal complete subgraph cover** has one complete subgraph per edge.
Hence,
From *proposition*2.1 $\theta'(G) = ||G||$. □

Definition 2.3. *A bipartite graph is one whose vertex set can be partitioned into two subsets X and Y,so that each edge has one end in X and one end in Y ; if each vertex of X joined to each vertex of Y the then it is said to be complete bipartite graph.if $|X| = m$ and $|Y| = n$, such a graph is denoted by $K_{n;m}$*

Corollary 2.2. *if a graph G is bipartite graph then $\theta'(G) = ||G||$.*

Proof. Using corollary 1 it is enough to show that a bipartite graph is triangle free graph.
suppose G is bipartite graph which contains a triangle. that is $V(G) = X \bigcup Y$ such that $X \bigcap Y = \emptyset$ for $X \subseteq V(G)$,$Y \subseteq V(G)$ just consider the following triangle.
a,b and c are in in the set of vertices of G.
$a \in X \Rightarrow b \in Y$
$a \in X \Rightarrow c \notin Y$
$b \in X \Rightarrow c \notin Y$
Thus $c \notin X \bigcup Y = V(G)$ it contradicts to the fact that G is bipartite.Hence G is triangle free. thus, using $corollary(2.1)$ $\theta'(G) = ||G||$

□

2.1.1 Upper bounds for intersection number

Proving $\theta'(G) \leq \lfloor n^2/4 \rfloor$ for n vertex graph means showing that every n-vertex graph can be covered with $\lfloor n^2/4 \rfloor$ complete subgraphs;we prove the stronger result that there is always a decomposition using at most this many complete subgraphs.in fact,we can find such a decomposition greedily.

Lemma 2.2. If a given graph G is an n-vertex simple graph,the $\varepsilon \leq \binom{n}{2}$.

Proof. since a graph G is a graph of order n then, $G \subseteq K_n$
$\Rightarrow ||G|| \leq ||K_n|| = \binom{n}{2}$,Using combinatorial approach.
$\Rightarrow ||G|| \leq \binom{n}{2}$ □

Theorem 2.1. [McGuinnes[1994]] Every greedy clique decomposition of an n vertex graph uses at most $\lfloor n^2/4 \rfloor$.

Proof. We use induction on n.A graph with n=1 is a trivial graph ($i.e$ the graph with one vertex) so here we have nothing to decompose.A graph with n=2 is a complete graph K_2($i.e$ a graph with two vertices and an edge connecting them) therefore the only maximal complete subgraph of K_2 is it self K_2 .Hence the theorem is obvious for $n \leq 2$; consider $n > 2$ .Supose the theorem is true for some $n' < n$.We want to show that the theorem holds for n.Let $\mathbf{Q} = \{Q_1, Q_2, ..., Q_m\}$ be a greedy clique decomposition of G,meaning each Q_i is a maximal complete subgraph in $G - \cup_{j<i} E(Q_j)$.Note that deleting Q_j from the list of of Q leaves a greedy decomposition of $G - E(Q_j)$.Suppose that each Q_i has at leat three edges; that is, $||Q_i|| \geq 3$ for $1 \leq i \leq m$. we know that $\cup_{i=1}^{m} ||Q_i|| = G$ since $\mathbf{Q}$ covers G.
Now,$3 \leq ||Q_i||$
$\Rightarrow \sum_{i=3}^{m} 3 \leq \sum_{i=1}^{m} ||Q_i|| = ||G|| \leq \binom{n}{2}$,Using Lemma 2.2

$\Rightarrow 3m \leq \binom{n}{2}$
$\Rightarrow m \leq \binom{n}{2}/3$
$\Rightarrow m \leq \frac{n^2-n}{6} < n^2/6$
$\Rightarrow m < n^2/6$
So we may assume that some Q_j is edge xy.Let R consist of the element of $Q - \{Q_j\}$ that are incident to x ,and let S consist of those incident to y.The set $Q' = Q - (R \cup S \cup \{Q_j\})$ is a greedy clique decomposition of a subgraph of $G - x - y$.By induction hypothesis $|Q'| \leq \frac{(n-2)^2}{4}$. Such that $n' = n - 2 < n$
Now , $Q' = Q - (R \cup S \cup \{Q_j\})$
$\Rightarrow Q = Q' + (R \cup S \cup \{Q_j\})$
$\Rightarrow |Q| = |Q' + (R \cup S \cup \{Q_j\})|$
$\Rightarrow |Q| = |Q' \cup (R \cup S \cup \{Q_j\})|$
$\Rightarrow |Q| = |Q'| + |R \bigcup S| + |Q_j|$ Since $R \bigcup S \bigcap \{Q_j\} = \emptyset$
This implies that $|Q| = |Q'| + |R \cup S| + |Q_j| \leq \frac{(n-2)^2}{4} + |R \cup S| + 1$.Thus,this indicates that $Q \leq n^2/4$ if $|R| + |S| \leq n - 2$.So,it is sufficient to prove that $|R| + |S| \leq n - 2$. We prove this by choosing distinct vertices in $V(G) - \{x, y\}$ from the vertex sets of the elements of $R \cup S$.Since each edge is deleted exactly once,each $v \notin \{x, y\}$ appears once in R if $v \in N(x)$ and once in S if $v \in N(y)$.Let $V(R)$ is the set of vertices in R and $V(S)$ is the set of vertices in S in such way that $V(R) \bigcup V(S) \subseteq V(G) - \{x, y\}$.Now consider $Q \in R$.
case 1.
If Q uses a vertex v not adjacent to y ,then we choose such a v for Q.let $R = \{Q_1, Q_2, ..., Q_k\}$ and $V(Q_i)$ be the vertex set in $Q_i \in R$ not adjacent to y.Then ,by our case we chose $V(Q_i)$ for Q_i.Since at least one vertex $v \in V(Q_i)$ is chosen for each Q_i then $K = |R| = |\{Q_1, ..., Q_k\}| \leq |\bigcup_{i=1}^{K} V(Q_i)|$.
case 2
If all vertices in Q are adjacent to y ,then we chose for Q a vertex $v \in Q$ such that vy belongs to the earliest element of $\mathbf{Q}$ that contains both y and some vertex of Q.Call this element Q';not that since each $v \notin \{x, y\}$ appears once in S then Q' is the only element of S containing v. .Now we have two cases ,that is Q precedes xy in $\mathbf{Q}$ or xy precedes Q in $\mathbf{Q}$. for the first case ,since Q and xy are maximal while chosen ,Q' must precedes Q in $\mathbf{Q}$ for otherwise from aforementioned hypothesis that all vertices in Q are adjacent to y and Q precedes xy in $\mathbf{Q}$,Q should have contained y and hence xy.for the second case ,since xy is maximal while chosen one of Q, Q' precedes xy but in this case xy precedes Q.therefore Q' must precede xy otherwise xy should have contained v.Note that in both cases we have that Q' precedes both Q and xy in Q.Similarly for elements S ,we chose vertices by reversing the roles of x and y.Consider $Q' \in S$.
case 1

If Q' uses a vertex v not adjacent to x ,then we chose such a v for Q'.Let $S = \{Q'_1, ..., Q'_p\}$ and $V(Q'_i)$ be a vertex set in Q'_i not adjacent to x,then by our case we chose $V(Q'_i)$ for Q'_i.since at least one vertex $v \in V(Q'_i)$ is chosen for Q'_i then $P = |S| = |\{Q'_1, ..., Q'_p\}| \leq | \cup_{i=1}^{p} V(Q'_i)|$
case 2
If all vertices in Q' are adjacent to x,then we chose for Q' a vertex $v \in Q'$ such that vx belongs to the earliest element of $\mathbf{Q}$ that contains both x and some vertex of Q'.Call this element Q ;note that since any $v \notin \{x, y\}$ appears once in R then Q is the only element of R containing v .Now we have two cases ,that is Q' precedes xy in $\mathbf{Q}$ or that xy precedes Q' in $\mathbf{Q}$.For the first case ,since Q' and xy are maximal while chosen ,Q must precedes Q' in $\mathbf{Q}$.For otherwise from aforementioned hypothesis that all vertices in Q' are adjacent to x and Q' precedes xy in Q ,Q' should have contained x and hence xy.For the second case ,since xy ia maximal while chosen,one of Q,Q' precedes xy but in this case xy precedes Q' therefore Q must precede xy otherwise xy should have contained v.Thus in this case Q precedes Q' in Q.Note that in both cases ,we have that Q precedes both Q' ,xy in Q.
In above from the two cases we have proved that $k = |R| = |\{Q_1, ..., Q_k\}| \leq | \cup_{i=1}^{k} V(Q_i)|$ and $P = |S| = |\{Q'_1, ..., Q'_p\}| \leq | \cup_{i=1}^{p} V(Q'_i)|$ This implies
$|R| + |S| \leq | \cup_{i=1}^{k} V(Q_i)| + | \cup_{i=1}^{p} V(Q'_i)|$.
Let $U(Q_i) = \cup_{i=1}^{k} V(Q_i)$ and $U'(Q'_i) = \cup_{i=1}^{k} V(Q'_i)$ note that U and U' have empty intersection.
Then $|R| + |S| \leq |U(Q_i)| + |U'(Q_i)| = |U(Q_i) \cup U'(Q_i)| \leq |V(R) \cup V(S)| \leq |V(G) - \{x, y\}| \Rightarrow |R| + |S| \leq |V(G) - \{x, y\}|$
$\Rightarrow |R| + |S| \leq n - 2$.
And,from the case's a above we have shown that if v belongs to some $Q \in R$ and to some $Q' \in S$,and v is chosen for one of them ,then the one for which it chosen occurs after the other one in the ordered set Q.Hence no vertex is chosen twice .Therefore the union of R and S does not exceed to the number of $V(G) - \{x, y\}$ in G.Hence from case's 1 and 2 of R and S we conclude that $R \cup S \leq n - 2$. □

Following the prove of the above theorem we can conclude that for any n-vertex graph G $\theta'(G) \leq \lfloor n^2/4 \rfloor$;the unique n-vertex graph maximizing $\theta'(G)$ is $K_{\lfloor n=2 \rfloor ; \lceil n=2 \rceil}$.$i.e$ $K_{\lfloor n=2 \rfloor ; \lceil n=2 \rceil} = \lfloor n^2/4 \rfloor$.
Trivially the number of complete subgraphs which covers any graph G does not exceed $||G||$ .Hence a graph with m number of edges has an intersection at number most m.

Theorem 2.2 (Chung[1981). *and Gyori-kostochka[1979]] Every n-vertex graph has a decomposition into complete subgraph whose orders sum to at most* $\lfloor n^2/2 \rfloor$.

Proof. let $Q_1, Q_2..., Q_m$ be a decompositions of a given graph G.which are complete.then for $n \geq 3$ the order of G is less than or equal to the size of G that is $|Q_i| \leq ||Q_i||$ let $k = \sum_{i=1}^{m} |Q_i| \leq \sum_{i=1}^{m} ||Q_i|| = ||G||$
$\Rightarrow K \leq ||G|| \leq \binom{n}{2}$
$\Rightarrow K \leq \binom{n}{2} = (n^2 - n)/2 \leq n^2/2$
$\Rightarrow k = \sum_{i=1}^{m} |Q_i| < \binom{n}{2}$
$\Rightarrow k < \lfloor n^2/2 \rfloor$ □

Now we consider the second encoding model.

Definition 2.4. ***A product representation of length t*** *assigns the vertices distinct vectors of length t so that u ↔ v if and only if their vectors differ in every position.The* ***product dimension*** *pdim G is the minimum length of such a representation of G.*

Example: Every complete graph has product dimension 1.For a graph $\overline{K_n}$ since each vertex in $\overline{K_n}$ are isolated vertex then each pair of vertices must agree in some coordinate ,but we cannot assign two vertices the same vector using the previous definition.Hence two coordinates are needed ,and assigning $(0, j)$ to v_j for each j suffices.See the following specific Examples.

Example: For a graph $K_1 + K_{n-1}$,the vectors for the clique must differ in each coordinate.The vector for the isolated vertex must agree with each of the others somewhere,but it cannot agree with more than one in a single coordinate.Hence at least $n - 1$ coordinates are needed.This suffices ,by using (1,2,...,n-1) for the isolated vertex and $(i, i, ..., i)$ for the ith vertex of the clique K_1 .

Definition 2.5. *A spanning subgraph of G is a subgraph H with $V(H) = V(G)$.*

Definition 2.6. *An equivalence on G is the spanning subgraph of G whose components are complete graphs.*

Proposition 2.2. *The product dimension of G is the minimum number of equivalences $E_1, E_2, ..., E_t$ such that $\cup_{i=1}^{t} E_i = \overline{G}$ and $\cap_{i=1}^{t} E_i = \emptyset$.*

Proof. □

The purpose of the present note is to draw attention of a fundamental theorem of infinite-dimensional Banach space to the topic of ordinary differential equations.

Recall that a complete normed space is called Banach space.

Theorem 2.3. *[10] Let* $X = \{x \in C(\mathbb{R}) : x(\omega + t) = x(t) \text{ for all } t \in \mathbb{R}\}$ *with norm*

$$\|x\| = \max_{t \in [0;\omega]} | x(t) |$$

Then X *is a Banach space.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in X. Then for each $\epsilon > 0$ there exists an index $N = N(\epsilon)$ such that

$$||x_n - x_m|| = \max_{t \in [0;\omega]} |x_n(t) - x_m(t)| < \epsilon \ \forall m, n \geq N(\epsilon)$$

Hence,

$$|x_n(t) - x_m(t)| < \epsilon \ \ \forall m, n \geq N(\epsilon) \ \forall t \in [0, \omega] \ (N \textit{ does not depend on } t) \quad (2.1)$$

This implies $\{x_n(t)\}$ is a Cauchy sequence in the complete space $\mathbb{R}$ for $\forall t \in [0, \omega]$. Therefore $\{x_n(t)\}$ is convergent, i.e. a limit exists say

$$x(t) = \lim_{n \to \infty} x_n(t)$$

Then from (2.1) for $\forall t \in [0, \omega]$ we get

$$\lim_{n \to \infty} |x_n(t) - x_m(t)| = |\lim_{n \to \infty} [x_n(t) - x_m(t)]| = |x(t) - x_m(t)| \leq \epsilon \ \forall m \geq N, \ \ \forall \, t \in [0, \omega] \quad (2.2)$$

It follows that the sequence $\{x_n(t)\}$ is uniformly convergent. Since $\{x_n\}$ is sequence of continuous functions which is uniformly convergent the limit function

$$x(t) = \lim_{n \to \infty} x_n(t)$$

is continuous. Moreover,

$$\begin{aligned} x(t + \omega) &= \lim_{n \to \infty} x_n(t + \omega) \\ &= \lim_{n \to \infty} x_n(t) \\ &= x(t) \end{aligned}$$

which implies that x is ω-periodic function.Then from (2.2) we get $|x(t) - x_m(t)| \leq \epsilon \ \forall m \geq N$ or

$$||x - x_m|| \leq \epsilon \ \forall m \geq N$$

So we've that $\{x_m\}$ converges to $x \in X$. Therefore, X is Banach space. □

Chapter 3

Overview of Fixed Point Theory

3.1 Fixed Point

The development of fixed point theory is very closely linked with the study of various problems in ordinary differential equations. Ordinary and partial differential equations are also at the source of fixed point theory in infinite-dimensional spaces. The method of successive approximations or iterations for finding solutions requires the equation to be put in a fixed form. Picards systematic application of this method to various differential equations problems led to Banach fixed point theorem for contractions ; boundary value problems for nonlinear ordinary differential equations were motivating Birkhoff-Kelloggs extension of Brouwer fixed point theorem to some function spaces.

Definition 3.1. *[5] Let* X *be nonempty set. A fixed point of a mapping* $T : X \to X$ *of a set* X *into itself is an* $x \in X$ *which is mapped onto itself, that is,* $T(x) = x$. *The image* $T(x)$ *coincides with* x.

.

Theorem 3.1. *Let* $M \subseteq \mathbb{R}^n$ *be homeomorphic to the closed unit ball* $\overline{B^n}$. *Then every continuous map* $f : M \to M$ *possesses a fixed point.*

Proof. Let $g : M \to M$ be a continuous map and $\phi : \overline{B^n} \subseteq \mathbb{R}^n \to M$ a homeomorphism, i.e. ϕ and ϕ^{-1} are continuous and bijective. Since the map defined by

$$f = \phi^{-1} \ o \ g \ o \ \phi : \overline{B^n} \to \overline{B^n}$$

is continuous, it possesses a fixed point $x_* \in \overline{B^n}$ by Brouwer's Fixed Point Theorem, i.e. $f(x_*) = x_*$.An application of ϕ to both sides of the equation yields
$g(\phi(x_*)) = \phi(x_*)$ which shows that $y_* = \phi(x_*) \in M$ is the fixed point sought after. □

3.2 Completely Continuous Operator

Definition 3.2. *[5]Let X be metric space and Y be topological space.Also,let $T : X \to Y$ be continuous operator.Then $T : X \to Y$ is called completely continuous operator if and only if the image of each bounded set in X is compact subset of Y.*

Example 3.1. *:The Fredholm Integral operator*
Let $I = [a, b]$ be any interval and suppose that k is continuous on $I \times I$. Then the operator $T : C[a, b] \to C[a, b]$ defined by

$$Tx(s) = \int_a^b k(s, t)x(t)dt$$

for $t \in [a, b]$, $x \in C[a, b]$ is completely continuous operator.

Proof. Obviously T is bounded linear operator . Let $\{x_n\}$ be any bounded sequence in $C[a, b]$ say $||x_n|| \leq \gamma \quad \forall n$. Let $y_n = Tx_n$. Then
$||y_n|| = ||Tx_n|| \leq ||T||\ ||x_n||$. Hence $\{y_n\}$ is also bounded. Next we want to show $\{y_n\}$ is equicontinuous. Since k is continuous on $I \times I$ and $I \times I$ is compact , k is uniformly continuous on $I \times I$.Hence, given any $\epsilon > 0$ there is a $\delta > 0$ such that for all $t \in I$ and $s_1, s_2 \in I$ satisfying $|s_2 - s_1| < \delta$ we have $|k(s_2, t) - k(s_1, t)| < \frac{}{(b-a)}$. Consequently, for $s_1, s_2 \in I$ and every $n \in \mathbb{N}$ we obtain

$$\begin{aligned}
|y_n(s_2) - y_n(s_1)| &\leq |\int_a^b (k(s_2, t) - k(s_1, t))x_n(t)dt| \\
&\leq \int_a^b |k(s_2, t) - k(s_1, t|\ |x_n(t)|dt \\
&< (b - a)\frac{\epsilon}{\gamma(b - a)}\gamma \\
&= \epsilon
\end{aligned}$$

which implies that the sequence $\{y_n\}$ is equicontinuous. Therfore, by Arzela Ascoli Theorem, T is completely continuous operator. □

Theorem 3.2. *Let X and Y be Banach spaces, $M \subseteq X$ a nonempty, bounded subset and $T : M \to Y$ an operator. Then the following are equivalent:*

(i) T is completely continuous operator

(ii) For every $n \in \mathbb{N}$, there exists a completely continuous operator $P_n : M \to Y$ such that

$$\sup_{x \in M} ||T(x) - P_n(x)|| \leq \frac{1}{n} \quad \textit{and} \quad \dim(spanP_n(M)) < \infty \tag{3.1}$$

Proof. Let T be completely continuous operator. Then $T(M)$ is compact, so for each $n \in \mathbb{N}$ there exists elements $y_i \in T(M)$, $i = 1, \ldots, N$ such that

$$T(M) \subseteq \bigcup_{i=1}^{N} \mathbf{B}_{\frac{1}{n}}(y_i) \Rightarrow \forall x \in M, \ \exists y_i \in T(M) : \min_i ||T(x) - y_i|| < \frac{1}{n}$$

We construct a partition of unity as follows:Define the functions $a_i : M \to \mathbb{R}$ by

$$a_i(x) = \max\{\frac{1}{n} - ||T(x) - y_i||, 0\} \quad for \quad i = 1, \ldots, N,$$

which have the following properties:

◇ The functions a_i are continuous, since T is continuous and the maximum of two continuous functions are continuous.

◇ $a_i(x) \geq 0 \quad \forall x \in M$

◇ Due to the covering property, $\sum_{i=1}^{N} a_i(x) > 0$ for all $x \in M$.

◇ If $x \in M$ such that $a_i(x) > 0$, then $||T(x) - y_i|| < \frac{1}{n}$.

Due to the 3^{rd} property,we may define the functions $\lambda_i : M \to \mathbb{R}$ as

$$\lambda_i(x) = \frac{a_i(x)}{\sum_{j=1}^{N} a_j(x)} \quad for \quad i = 1, \ldots, N,$$

which is the desired partition of unity, i.e.,

◇ The functions λ_i, for $i = 1, \ldots, N$, are continuous.

◇ $0 \leq \lambda_i \leq 1$ for all $x \in M$.

◇ $\sum_{i=1}^{N} \lambda_i(x) = 1$ for all $x \in M$.

◇ If $x \in M$ such that $\lambda_i(x) > 0$, then $||T(x) - y_i|| < \frac{1}{n}$.

Set $M_n = conv(y_1, \ldots, y_N) \subseteq span\{y_1, \ldots, y_N\}$ and define the operator $P_n : M \to M_n$ given by

$$P_n(x) = \sum_{i=1}^{N} \lambda_i(x) y_i \tag{3.2}$$

We show that P_n has the desired approximation property, i.e.,
$\sup_{x\in M} ||T(x) - P_n(x)|| \leq \frac{1}{n}$.
Indeed, due to the properties of the partition of unity mentioned above, we have

$$||T(x) - P_n(x)|| = ||T(x) - \sum_{i=1}^{N} \lambda_i(x) y_i|| \leq \sum_{i=1}^{N} \lambda_i(x) ||T(x) - y_i|| \leq \frac{1}{n}.$$

By definition $P_n(M) \subseteq span\{y_1, \dots, y_N\}$, so the image of P_n lies in a finite dimensional subspace of Y.
All that is left to show is that P_n is a completely continuous operator.The continuity of P_n follows from the continuity of the partition functions λ_i.Let $U \subseteq M$ be bounded, then

$$||P_n|| \leq ||T(x) - P_n(x)|| + ||T(x)|| \leq ||T(x)|| + \frac{1}{n} \quad \forall x \in U$$

which implies the boundedness of $P_n(U)$ and further compact.
We now prove the converse. $n \in \mathbb{N}$ be arbitrary but fixed.Then there exists a $\delta > 0$ such that for all $x, y \in M$ with $||x - y|| < \delta$:

$$\begin{aligned} ||T(x) - T(y)|| &\leq ||T(x) - P_n(x)|| + ||P_n(x) - P_n(y)|| + ||P_n(y) - T(y)|| \\ &\leq \frac{1}{n} + ||P_n(x) - P_n(y)|| + \frac{1}{n} \\ &\leq \frac{3}{n}, \end{aligned} \tag{3.3}$$

which implies the continuity of the operator T .To prove that $T(M)$ is compact, we show the existence of a finite covering for $T(M$. Since M is bounded and P_n is compeletely continuous, there exists $x_i, \dots, x_N \in M$ such that

$$P_n(M) \subseteq \bigcup_{i=1}^{N} \mathbf{B}_{\frac{1}{n}} P_n(x_i),$$

i.e., for all $y \in M$, there exists an index $i \in \{1, \dots, N\}$ with
$||P_n(x_i) - P_n(y)|| < \frac{1}{n}$. Together with (3.3), this shows that for any $y \in M$
, there exists an index $i \in \{1, \dots, N\}$ such that
$||T(x_i) - T(y)|| < \frac{3}{n}$,
i.e., $T(M)$ has a finite covering and hence T is completely continuous operator. □

3.3 Schauder's fixed Point Theorems

Shauder's Fixed Point Theorem is very important to proof existence of solution(s) for differential equations.

Theorem 3.3 (Schauder's fixed Point Theorem). *[8] Let M be nonempty compact and convex subset of the Banach space X and $T : M \to M$ be continuous. Then T has fixed point in M.*

Proof. The operator T is completely continuous operator since $T(M) \subseteq M$ is compact. Due to Theorem (3.2) there exists completely continuous operators $P_n : M \to M_n$ where $M_n = conv(y_1, \ldots, y_N)$, such that

$$||T(x) - P_n(x)|| \leq \frac{1}{n} \tag{3.4}$$

The convexity of M implies that $M_n \subseteq conv(T(M)) \subseteq M$. Therefore, $\widetilde{P_n} = P_n \mid_{M_n}: M_n \to M_n$ is continuous. The set M_n is closed and homeomorphic to the closed unit ball $\overline{B}$ in $\mathbb{R}^N$. By Theorem (3.1), there exists a fixed poin $x_n \in M_n \subseteq M$ for each $n \in \mathbb{N}$ with

$$x_n = \widetilde{P_n}(x_n) \tag{3.5}$$

Since M is compact, there exists an $x \in M$ and a convergent subsequence, again denoted by $\{x_n\} \subseteq M$ such that $x_n \to x$ as $n \to \infty$. We now show that this x is indeed the desired fixed point.From (3.5) we obtain

$$||T(x) - x_n|| \leq ||T(x) - T(x_n)|| + ||T(x_n) - \widetilde{P_n}(x_n)|| \to 0 \ for\ n \to \infty,$$

due to the continuity of T and the approximation property (3.4) , so $T(x) = x$.

□

Theorem 3.4 (generalization of Schauder's fixed point theorem). *[8] If D is nonempty closed and convex subset of the Banach space X and $T : D \to D$ is completely continuous operator, then T has fixed point in D.*

Proof. Assume that D is nonempty closed and convex subset of the Banach space X and $T : D \to D$ is completely continuous operator.Then by Mazur theorem, $conv(T(M))$ is compact.Furthermore, it is non-empty and convex.Note that $T(M) \subseteq M$, and since M is convex, $conv(T(M)) \subseteq M$. Thus, Schauders Fixed Point Theorem in (3.3) can be applied to the continuous map $T : conv(T(M)) \to conv(T(M))$, and T has a fixed point in $conv(T(M))$. □

Chapter 4

ODE With Periodic Data

Several models in Science and Engineering are described using boundary value problems.Here, we mention the Dirichlet problem as a prototype.In molecular dynamics whereby one simulates, bulk fluid(gasses,liquids)the pertinent constraint is periodic boundary condition

4.1 ODE with Periodic Coefficients

A boundary value problem for a given differential equation consists of finding a solution of the given differential equation subject to a given set of boundary conditions. A boundary condition is a prescription some combinations of values of the unknown solution and its derivatives at more than one point.A linear ODE with periodic coefficients, oftentimes admits periodic solution provided that some suitable condition is met on the boundary.
Let $I = (a, b) \subseteq \mathbb{R}$ be an interval. Let $p, q, f : I \to \mathbb{R}$ be continuous functions. Then the equations

$$y' + p(x)y = f(x) \tag{4.1}$$

with boundary condition $y(a) = y(b)$
And

$$y'' + p(x)y' + q(x)y = f(x) \tag{4.2}$$

with boundary conditions $y(a) = y(b), \quad y'(a) = y'(b)$
are respectively first and second order linear explicit ODEs with periodic boundary conditions.For the sake of demonstration,we take a closer look at (4.1).In this problem, if the coefficient $p(x)$ and the function $f(x)$ are periodic with the same period $\ell = b - a$, then the corresponding periodic boundary value problem(PBVP)admits periodic solution as stated in the following Theorem.

Theorem 4.1. *[13]If p and f of (4.1) are periodic functions with period $\ell = b - a$ and if u is a solution of (4.1) with $u(\ell) = u(0)$, then u is $\ell-$ periodic.*

Proof. The key ingredient is the uniqueness assertion:
If u and v both satisfy (4.1) with $u(0) = v(0)$, then $u(x) = v(x)$ for all x.
Since u is solution of (4.1) for all x, then

$$u'(x+\ell) + p(x+\ell)u(x+\ell) = f(x+\ell) \qquad \forall \ x.$$

Because p and f are $\ell-$ periodic:

$$u'(x+\ell) + p(x)u(x+\ell) = f(x) \qquad \forall \ x.$$

Thus, if we let $v(x) := u(x+\ell)$, then

$$v'(x) + p(x)v(x) = f(x)$$

But, since $u(0) = u(\ell)$, we have that $u(0) = v(0)$.
Threfore, by Picard -Lindelöf (Existence and uniqueness Theorem),
$u(x) = v(x)$, that is $u(x+\ell) = u(x)$ for all x.
This completes the Theorem. □

The related assertion for a solution of a second order in (4.2) is essentially identical except there we need to assume that both $u(0) = u(\ell)$, and $u'(0) = u'(\ell)$, since the corresponding uniqueness assertion for second order ODEs requires that.

Consider the Banach space $X = \{x \in C(\mathbb{R}) : x(\omega + t) = x(t)$ for all $t \in \mathbb{R}\}$ with norm

$$\|x\| = \max_{t\in[0;!]} \mid x(t) \mid$$

Let $p, q \in X$ and consider the following two differential equations

$$x'(t) = -p(t)x(t) + q(t) \tag{4.3}$$

$$x'(t) = p(t)x(t) - q(t) \tag{4.4}$$

Lemma 4.1. *[10] Assume that $\int_0^{!} p(t)dt \neq 0$, then (4.3) has ω- periodic solution*

$$x(t) = \int_t^{t+!} \frac{\exp(\int_t^s p(r)dr)}{\exp(\int_0^{!} p(r)dr) - 1} q(s)ds$$

and (4.4) *has* ω- *periodic solution*

$$y(t) = \int_t^{t+\omega} \frac{\exp(\int_s^{t+\omega} p(r)dr)}{\exp(\int_0^{\omega} p(r)dr) - 1} q(s)ds$$

Proof. Here we only consider (4.3) and we show that $x(t)$ is the periodic solution of (4.3) . Differentiating $x(t)$, we obtain that

$$\begin{aligned}
x'(t) &= \frac{\exp(\int_t^{t+\omega} p(r)dr)}{\exp(\int_0^{\omega} p(r)dr) - 1} q(t+\omega) - \frac{q(t)}{\exp(\int_0^{\omega} p(r)dr) - 1} - \int_t^{t+\omega} \frac{p(t)\exp(\int_t^{s} p(r)dr)}{\exp(\int_0^{\omega} p(r)dr) - 1} q(s)ds \\
&= \frac{\exp(\int_0^{\omega} p(r)dr)}{\exp(\int_0^{\omega} p(r)dr) - 1} q(t) - \frac{1}{\exp(\int_0^{\omega} p(r)dr) - 1} q(t) - \int_t^{t+\omega} \frac{p(t)\exp(\int_t^{s} p(r)dr)}{\exp(\int_0^{\omega} p(r)dr) - 1} q(s)ds \\
&= \frac{\exp(\int_0^{\omega} p(r)dr) - 1}{\exp(\int_0^{\omega} p(r)dr) - 1} q(t) - p(t) \int_t^{t+\omega} \frac{\exp(\int_t^{s} p(r)dr)}{\exp(\int_0^{\omega} p(r)dr) - 1} q(s)ds \\
&= q(t) - p(t) \int_t^{t+\omega} \frac{\exp(\int_t^{s} p(r)dr)}{\exp(\int_0^{\omega} p(r)dr) - 1} q(s)ds \\
&= q(t) - p(t)x(t) \\
&= -p(t)x(t) + q(t)
\end{aligned}$$

and

$$\begin{aligned}
x(t+\omega) &= \int_{t+\omega}^{t+2\omega} \frac{\exp(\int_{t+\omega}^{s} p(r)dr)}{\exp(\int_0^{\omega} p(r)dr) - 1} q(s)ds \\
&= \int_{t+\omega}^{t+2\omega} \frac{\exp(\int_{t+\omega}^{u+\omega} p(r)dr)}{\exp(\int_0^{\omega} p(r)dr) - 1} q(u+\omega)d(u+\omega) \\
&= \int_{t+\omega}^{t+2\omega} \frac{\exp(\int_t^{u} p(r)dr)}{\exp(\int_0^{\omega} p(r)dr) - 1} q(u)du \\
&= \int_t^{t+\omega} \frac{\exp(\int_t^{u} p(r)dr)}{\exp(\int_0^{\omega} p(r)dr) - 1} q(u)du \\
&= x(t)
\end{aligned}$$

Hence, $x(t)$ is ω- periodic solution of (4.3). □

4.2 ODE With Periodic Boundary Conditions(PBC)

Explicit ODE with periodic boundary condition may admits a non-trivial solution under some additional assumptions. Consider the explicit ODE of second order

$$u'' + f(x, u, u') = 0, \qquad 0 \le x \le T \tag{4.5}$$

where f is a Lipschitz continuous function with respect to the last two components. If f is P-periodic with respect to the first component, i.e $f(x + P, u, u') = f(x, u, u')$
and satisfies the inequality restriction
$f(x, \alpha, 0) \le f(x, \beta, 0)$
then the associated PBVP has a non-trivial solution as detailed in the next Theorem.

Theorem 4.2. *Suppose that there are constants α and β such that $\alpha \le \beta$ and*
$f(x, \alpha, 0) \le f(x, \beta, 0)$.
Then there exists $T_0 \in (0, T]$ such that fo every $x \in (0, T_0]$, the PBVP

$$\begin{cases} u'' + f(x, u, u') = 0 \\ u(0) = u(T_0) \\ u'(0) = u'(T_0) \end{cases} \tag{4.6}$$

has a solution.

Proof. Let $M = \max\{|\alpha|, |\beta|\}$, $N > 0$ be given
$Q = \max\{|f(x, u, u')|\}$: $0 \le x \le T$, $|u| \le 2M$, $|u'| \le N$
and $G(x, s)$ be the Green's function,

$$G(x, s) = \begin{cases} \frac{s(x-w)}{w}, & 0 \le s \le x \le w \\ \frac{x(s-w)}{w}, & 0 \le x \le s \le w \end{cases}$$

Set $\tilde{\mathbf{B}} = \{\varphi \in C^1[0, w] : |\varphi(t)| \le 2M,\ |\varphi'(t)| \le N\}$
Define, the operator S on $\tilde{\mathbf{B}}$ as follows,

$$(S\varphi)(x) = \int_0^w G(x, s) f(s, \varphi(s), \varphi'(s)) ds + y$$

where $\alpha \le y \le \beta$
Then

$$|S\varphi| \le \frac{w^2}{2} Q + |y| \le \frac{w^2}{2} Q + M\ , \quad |S\varphi'| \le \frac{w}{2} Q.$$

Hence, S maps $\tilde{\mathbf{B}}$ continuously into itself,provided that

$$w \le \min\{\sqrt{\frac{2M}{Q}}, \frac{2N}{Q}\} \tag{4.7}$$

If $w > 0$ is chosen so that (4.7) holds, it then follows from Schauder's Fixed Point Theorem that (4.6) has a solution $u(x)$ such that
$|u(x)| \leq 2M$ and $|u'(x)| \leq N$. $\square$

Chapter 5

Existence of periodic solutions for class of second order ODEs with periodic data

5.1 Motivation

The existence of periodic solutions is an important aspect in differential equations.Much work about periodic solutions for second order differential equations has been done by using various Theorems and methods of nonlinear functional analysis. In this chapter, we investigate the existence of periodic solutions of the following differential equation

$$-x''(t) + a(t)x'(t) = g(t,x) - f(t,x(t),x'(t)) \tag{5.1}$$

where a is a continuous ω-periodic function, $g(t,u)$, $f(t,u,v)$ are ω-periodic functions in t for $u = x(t)$, $v = x'(t)$ and $\omega > 0$.

Recall that the generalized Schauder's fixed point theorem that states if D is closed and convex subset of the Banach space X and $T : D \to D$ is completely continuous operator, then T has fixed point in D which is crucial in our arguments.
Remark: In this chapter, X stands for the Banach space $X = \{x \in C(\mathbb{R}) : x(\omega + t) = x(t)$ for all $t \in \mathbb{R}\}$ with norm

$$\|x\| = \max_{t\in[0;\omega]} \mid x(t) \mid$$

Define an operator J on X by

$$(Ju)(t) = \int_t^{t+\omega} \frac{e^{p(s-t)}}{e^{p\omega} - 1} u(s)ds, u \in X$$

where $p > 0$ is constant.
For any $u \in X$, $Ju \in X \cap C^1(\mathbb{R})$ and

$$(Ju)'(t) = -p(Ju)(t) + u(t) \tag{5.2}$$

If $u \in X \cap C^1(\mathbb{R})$ then $Ju \in X \cap C^2(\mathbb{R})$ and

$$(Ju)''(t) = -p(Ju)'(t) + u'(t) = p^2(Ju)(t) - pu(t) + u'(t) \tag{5.3}$$

We transform (5.1) to
$p^2(Ju)(t) - pu(t) + u'(t) - a(t)[-p(Ju)(t) + u(t)] = f(t, (Ju)(t), u(t) - p(Ju)(t)) - g(t, (Ju)(t))$ that is

$$u'(t) = [a(t)+p]u(t) - [p^2 Ju + pa(t)Ju + g(t, Ju) - f(t, Ju, u(t) - pJu)] \tag{5.4}$$

By Lemma 4.1 of equation (4.4) we obtain that if $u(t)$ is ω-periodic solution of (5.4), then $u(t)$ satisfies

$$u(t) = \int_t^{t+\omega} \frac{\exp(\int_s^{t+\omega}[p + a(r)]dr)}{\exp(\int_0^{\omega}[p + a(r)]dr) - 1}(Hu)(s)ds$$

where $\int_0^{\omega}[p + a(t)]dt \neq 0$ and

$$(Hu)(s) = p^2(Ju)(s) + pa(s)(Ju)(s) + g(t, Ju) - f(s, (Ju)(s), u(s) - p(Ju)(s))$$

In order to put more emphasis on the above facts, we summarize them in the following Lemma.

Lemma 5.1. *[10] Define an operator T on X by*

$$(Tu)(t) = \int_t^{t+\omega} \frac{\exp(\int_s^{t+\omega}[p + a(r)]dr)}{\exp(\int_0^{\omega}[p + a(r)]dr) - 1}(Hu)(s)ds \tag{5.5}$$

where $\int_0^{\omega}[p + a(t)]dt \neq 0$. Then the fixed point u of T on X is ω-periodic solution of (5.4) and Ju is ω-periodic solution of (5.1).

Proof. Since $(Tu)(t) = u(t)$ and

$$\begin{aligned}(Tu)'(t) &= u'(t)\\ &= [a(t) + p]u(t) - [p^2 Ju + pa(t)Ju + g(t, Ju) - f(t, Ju, u(t)]\end{aligned}$$

we obtain that u is ω-periodic solution of (5.4). In order to prove that Ju is ω- periodic solution of (5.1), we only show that Tu satisfies (5.1). Form (5.2)-(5.4), this result follows immediately. □

Definition 5.1 (Monotone functions). *[6] Let* $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. *Then*

- *We say that* $x \preceq y$ *if and only if* $x_i \leq y_i$ *for every* $i \in \{1, 2, 3, \ldots, n\}$
- *We say that* $x \prec y$ *if and only if* $x_i < y_i$ *for every* $i \in \{1, 2, 3, \ldots, n\}$
- *The function* $f : \mathbb{R}^n \to \mathbb{R}$ *is called* ***nondecreasing/increasing*** *if and only if for every* $x, y \in \mathbb{R}^n$
  $x \prec y$ *implies that* $f(x) \leq f(y)$
- *The function* $f : \mathbb{R}^n \to \mathbb{R}$ *is called* ***nonincreasing/decreasing*** *if and only if for every* $x, y \in \mathbb{R}^n$
  $x \prec y$ *implies that* $f(x) \geq f(y)$

5.2 Main Theorems And Corollaries

In this section, we will see the main Theorems and Corollaries of this project.

Theorem 5.1. *[10] Assume that there exist constants $m < M, p > 0$ such that*

(A_1) $g \in C(\mathbb{R} \times [m, M], \mathbb{R})$, *and* $p(p + a(t))u + g(t, u)$ *is nondecreasing in* $u \in [m, M]$.

(A_2) $f \in C(\mathbb{R} \times [m, M] \times [p(m - M), p(M - m)], \mathbb{R})$ *and*

$$g(t, M) \leq f(t, u, v) \leq g(t, m)$$

for any $(t, u, v) \in \mathbb{R} \times [m, M] \times [p(m - M), p(M - m)]$.

Then (5.1) has at least one ω*- periodic solution* x *with* $m \leq x \leq M$.

Proof. Let $\Omega = \{x \in X : mp \leq x \leq Mp \text{ for } t \in [0, \omega]\}$, then Ω is convex and closed set in X. For any $u \in \Omega$ we compute to obtain that $m \leq Ju \leq M$ and $pm - pM \leq u - pJu \leq pM - pm$.
Moreover, according to (A_1), we have
$[p^2+pa(t)]m+g(t, m) \leq [p^2+pa(t)]Ju+g(t, Ju) \leq [p^2+pa(t)]M+g(t, M)$ for $u \in \Omega$.
Using (A_2), we obtain that for any $u \in \Omega$

$$\begin{aligned}(Hu)(t) &= p^2(Ju)(t) + pa(t)(Ju)(t) + g(t, Ju) - f(t, (Ju)(t), u(t) - p(Ju)(t)) \\ &\leq [p^2 + pa(t)]M + g(t, M) - g(t, M) \\ &= [p^2 + pa(t)]M\end{aligned}$$

And

$$\begin{aligned}(Hu)(t) &= p^2(Ju)(t) + pa(t)(Ju)(t) + g(t, Ju) - f(t, (Ju)(t), u(t) - p(Ju)(t)) \\ &\geq [p^2 + pa(t)]m + g(t, m) - g(t, m) \\ &= [p^2 + pa(t)]m\end{aligned}$$

Which imply that

$$[p^2 + pa(t)]M \geq [p^2 + pa(t)]m$$

That is

$$p(p+a(t))(M-m) \geq 0$$

And hence

$$p+a(t) \geq 0$$

for all $t \in \mathbb{R}$. If $p+a(t))=0$, according (A_1) and (A_2), we easily to check that

$$g(t,u)=f(t,u,v)\ , \quad \forall\ (t,u,v) \in \mathbb{R} \times [m,M] \times [p(m-M), p(M-m)].$$

Thus, any constant $c \in [m,M]$ is the periodic solution of equation (5.1). We assume that $p+a(t)>0$. Now we show that T satisfies all conditions of Lemma 5.1.
Noting

$$\int_t^{t+!} \frac{\exp(\int_s^{t+!}[p+a(r)]dr)}{\exp(\int_0^{!}[p+a(r)]dr)-1}[p+a(s)]ds = 1$$

and

$$\frac{\exp(\int_s^{t+!}[p+a(r)]dr)}{\exp(\int_0^{!}[p+a(r)]dr)-1} > 0 \quad for \quad t \leq s \leq t+\omega$$

we obtain that for any $u \in \Omega$,

$$\begin{aligned}
(Tu)(t) &= \int_t^{t+!} \frac{\exp(\int_s^{t+!}[p+a(r)]dr)}{\exp(\int_0^{!}[p+a(r)]dr)-1}(Hu)(s)ds \\
&\leq \int_t^{t+!} \frac{\exp(\int_s^{t+!}[p+a(r)]dr)}{\exp(\int_0^{!}[p+a(r)]dr)-1}[p^2+Pa(s)]Mds \\
&= pM \int_t^{t+!} \frac{\exp(\int_s^{t+!}[p+a(r)]dr)}{\exp(\int_0^{!}[p+a(r)]dr)-1}[p+a(s)](s)ds \\
&= pM
\end{aligned}$$

Also,

$$\begin{aligned}
(Tu)(t) &= \int_t^{t+!} \frac{\exp(\int_s^{t+!}[p+a(r)]dr)}{\exp(\int_0^{!}[p+a(r)]dr)-1}(Hu)(s)ds \\
&\geq \int_t^{t+!} \frac{\exp(\int_s^{t+!}[p+a(r)]dr)}{\exp(\int_0^{!}[p+a(r)]dr)-1}[p^2+Pa(s)]Mds \\
&= pm \int_t^{t+!} \frac{\exp(\int_s^{t+!}[p+a(r)]dr)}{\exp(\int_0^{!}[p+a(r)]dr)-1}[p+a(s)](s)ds \\
&= pm
\end{aligned}$$

which imply that $Tu \in \Omega$, that is $T(\Omega) \subseteq \Omega$.
Next we show that $T : \Omega \to \Omega$ is completely continuous. Obviously, $T(\Omega)$ is uniformly bounded set and T is continuous on Ω, so it suffices to show that $T(\Omega)$ is equicontinuous by Ascoli-Arzela Theorem. For any $u \in \Omega$, we have that

$$(Tu)'(t) = [a(t)+p](Tu)(t) - [p^2 Ju + pa(t)Ju + g(t, Ju) - f(t, Ju, u(t) - pJu)]$$

Since, $T(\Omega)$ is bounded and f, g, a are continuous, there exists $\rho > 0$ such that

$$|(Tu)'(t)| \leq \rho, \quad u \in \Omega$$

which implies that $T(\Omega)$ is equicontinuous. Also, since $T(\Omega)$ is equicontinuous ,closed and bounded, by Ascoli-Arzela Theorem,we have that $T(\Omega)$ is compact set. So, $T : \Omega \to \Omega$ is completely continuous operator. By Theorem (3.4) , there exists $u \in \Omega$ with $Tu = u$. Moreover,
$Ju \in [m, M]$ is the periodic solution of (5.1). □

Analogously, we have the following Theorem.

Theorem 5.2. *[10] Assume that there exist constants $m < M, p > 0$ such that*

(A_3) $g \in C(\mathbb{R} \times [m, M], \mathbb{R})$, and $p(p + a(t))u + g(t, u)$ is noninecreasing in $u \in [m, M]$.

(A_4) $f \in C(\mathbb{R} \times [m, M] \times [p(m - M), p(M - m)], \mathbb{R})$ and

$$g(t, m) \leq f(t, u, v) \leq g(t, M)$$

for any $(t, u, v) \in \mathbb{R} \times [m, M] \times [p(m - M), p(M - m)]$.

Then (5.1) has at least one ω- periodic solution x with $m \leq x \leq M$.

Corollary 5.1. *[10] Let $f(t, u, v) = f(t, u)$. Assume that there exist constants $m < M$ such that $\frac{\partial}{\partial u} g, f \in C(\mathbb{R} \times [m, M], \mathbb{R})$ and*

$$g(t, M) \leq f(t, u) \leq g(t, m)$$

for any $(t, u,) \in \mathbb{R} \times [m, M]$.
Then (5.1) has at least one ω- periodic solution x with $m \leq x \leq M$.

Corollary 5.2. *[10] Assume that* c, μ *are constants and* h *is* ω*- periodic continuous function with* $\|h\| \leq |\mu|$*. Then*

$$x''(t) + cx'(t) + \mu \sin x(t) = h(t) \tag{5.6}$$

has at least one ω*- periodic solution . Further suppose that* $c \geq 2\sqrt{|\mu|}$ *and* $h \neq \pm\mu$*. Then (5.6) has at least two* ω*- periodic solutions .*

Proof. If $\mu = 0$ then $h = 0$ and any constant k is periodic solution. Now we assume that $\mu \neq 0$. Here we have

$$a(t) = -c, \quad g(u) = \mu \sin u, \quad f(t, u, v) = h(t).$$

Put $p_1 = (|c|+1)(|\mu|+1)$. Then $p_1(p_1 - c)u + g(u)$ is increasing in $\mathbb{R}$ and (A_1) is fulfilled. If $\mu > 0$, choosing

$$m_1 = 0.5\pi, \ M_1 = 1.5\pi;$$

If $\mu < 0$, choosing

$$m_1 = 1.5\pi, \ M_1 = 2.5\pi$$

we obtain that

$$g(M_1) \leq h(t) \leq g(m_1), \quad \forall t \in \mathbb{R}$$

Hence, (5.6) has at least one ω- periodic solution x_1 with $m_1 \leq x_1 \leq M_1$.

Further suppose that $c \geq 2\sqrt{|\mu|}$ and $h \neq \pm\mu$. Put $p_2 = \frac{c}{2}$.
Then $p_2(p_2 - c)u + g(u)$ is non-increasing in $\mathbb{R}$ and (A_3) is fulfilled.
If $\mu > 0$, choosing

$$m_2 = -0.5\pi, \quad M_2 = 0.5\pi;$$

If $\mu < 0$, choosing

$$m_2 = 0.5\pi, \quad M_2 = 1.5\pi$$

we obtain that

$$g(m_2) \leq h(t) \leq g(M_2) \ , \quad \forall t \in \mathbb{R}$$

and hence (5.6) has at least one ω- periodic solution x_2 with

$$m_2 \leq x_2 \leq M_2.$$

Since $h \neq \pm\mu$, $x_i \neq m_i$ and $x_i \neq M_i (i = 1, 2)$, we have that $x_1 \neq x_2$. □

5.3 Examples

Example 5.1. *[10] Consider the differential equation*

$$x''(t) + \frac{1}{8}(x'(t))^2 - (x(t))^2 = \sin t - 1 \tag{5.7}$$

We claim that (5.7) *has at least one* 2π*-periodic solution. In fact*

$$g(u) = -u^2, \quad f(t,u,v) = \sin t - 1 - \frac{1}{8}(v)^2, \quad a(t) = 0.$$

Put $p = 2$, $m = 0$, $M = 2$, *then since* $\sin t$ *is* 2π*-periodic function* , f *is* 2π*-periodic function in* t *and*

$$p(p + a(t))u + g(u) = 4u - u^2 := k(u).$$

Also,
$p(p+a(t))u+g(u)$ *is increasing in* $[0,2]$ *as* $k'(u) = 4-2u > 0$ *in* $(-\infty, 2)$.
For any $(t,u,v) \in [0,2] \times [0,2] \times [-4,4]$, *we have that*

$$g(M) = -4 \leq f(t,u,v) \leq g(m) = 0.$$

Therefore, by Theorem (5.1), (5.7) has at least one 2π*-periodic solution* x *with* $0 \leq x \leq 2$.

Example 5.2. *[10] Consider the differential equation*

$$-x''(t) + a(t)x'(t) = x \ \ \sin x(t) - f(t, x(t)) \tag{5.8}$$

where $\alpha > 0$, *a is continuous* ω*- periodic function,* $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous function of* u *and* t *And* ω*- periodic function in* t*. We claim that equation (5.8) has infinitely many* ω*- periodic solutions if the following condition is fulfilled:*

$$\lim_{u\to\infty} \frac{|f(t,u)|}{u} < 1$$

with respect to t*. In fact there exists* $\rho > 0$ *such that* $|f(t,u)| \leq u$, $u \geq \rho$ *if the above condition is fulfilled. Choose integer* n *such that* $2\pi n > \rho$.
Let $m = 2\pi n + 0.5$ *and* $M = 2\pi n + 1.5$*. Then* $g \in C^1([m,M], R)$
where

$$g(u) = u \ \ \sin u \ , \ \ g(M) = -M \ \ \leq f(t,u) \leq g(m) = -m \ \ , \ \ (t,u) \in \mathbb{R}\times[m,M].$$

By Corollary (5.1), (5.8) has at least one ω*- periodic solution* x *with* $m \leq x \leq M$*. Since* n *is arbitrary sufficiently large integer, (5.8) has infinitely many* ω*- periodic solutions.*

5.4 Existence of Periodic solution of linear first order system of ODEs

5.4.1 Motivation

A single differential equation on one unknown function is often not enough to describe certain physical problems. The description of a point particle moving in space under Newton's law of motion requires three functions of time; the space coordinates of the particle, to describe the motion together with three differential equations. To describe several proteins activating and deactivating each other inside a cell also requires as many unknown functions and equations as proteins in the system. Hence, system of first order ODE requires to model such problems.

Now we will discuss the sufficient and necessary conditions that guarantees the existence of periodic solutions of system of first order linear ODE having two equations with two unknowns.

A system of first order ODEs is a set of n equations with n unknowns and has the form of

$$\begin{cases} \frac{dx_1}{dt} = f_1(t, x_1, x_2, \ldots, x_n) \\ \frac{dx_2}{dt} = f_2(t, x_1, x_2, \ldots, x_n) \\ \frac{dx_3}{dt} = f_3(t, x_1, x_2, \ldots, x_n) \\ \qquad \vdots \\ \frac{dx_n}{dt} = f_n(t, x_1, x_2, \ldots, x_n) \end{cases} \tag{5.9}$$

Every n^{th} order ODE $\frac{d^n y}{dt^n} = F(t, y(t), \frac{dy}{dt}, \frac{d^2 y}{dt^2}, \ldots, \frac{d^{n-1} y}{dt^{n-1}})$ can be reduced into (5.9) as follows.
Let $x_1 = y,\ x_2 = \frac{dy}{dt},\ x_3 = \frac{d^2 y}{dt^2},\ \ldots,\ x_{n-1} = \frac{d^{n-2} y}{dt^{n-2}},\ x_n = \frac{d^{n-1} y}{dt^{n-1}}$. Then we have

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = x_3 \\ \qquad \vdots \\ \frac{dx_{n-1}}{dt} = x_n \\ \frac{dx_n}{dt} = F(t, x_1, x_2, \ldots, x_n) \end{cases}$$

Consider the special case where (5.9) is linear .If f_i is linear, that is $f_i(t, x_1, x_2, \ldots, x_n) = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n + b_i(t),\ 1 \le i \le n$ where

a_{ij} are functions of t for $1 \leq j \leq n$, then (5.9) becomes the following linear system of first order ODEs

$$\begin{cases} \frac{dx_1}{dt} = a_{11}x_1(t) + a_{12}x_2(t) + \ldots + a_{1n}x_n(t) + b_1(t) \\ \frac{dx_2}{dt} = a_{21}x_1(t) + a_{22}x_2(t) + \ldots + a_{2n}x_n(t) + b_2(t) \\ \quad \vdots \\ \frac{dx_n}{dt} = a_{n1}x_1(t) + a_{n2}x_2(t) + \ldots + a_{nn}x_n(t) + b_n(t) \end{cases} \tag{5.10}$$

Equivalently, (5.10) can be written as matrix form as follows

$$X'(t) = A(t)X(t) + b(t) \tag{5.11}$$

Where

$$\mathbf{X(t)} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \ \mathbf{A(t)} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \ldots a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots a_{2n}(t) \\ \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \ldots a_{nn}(t) \end{pmatrix}, \ \mathbf{b(t)} = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}$$

Remark: If $A(t) = (a_{ij}(t))_{m\times n}$, then

$$\frac{d}{dt}A(t) = (\frac{d}{dt}a_{ij}(t))_{m\times n} \ \ and \ \ \int A(t)dt = (\int a_{ij}(t)dt)_{m\times n}$$

where $A(t)$ is continuous on an interval I. If $b(t) = 0$, then (5.11) is called homogeneous system otherwise nonhomogeneous system. Let t_0 be any point on an interval I and

$$\mathbf{X(t_0)} = \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_O) \end{pmatrix}, \ \mathbf{X_O} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix}$$

where the γ_i, $i = 1, 2, \ldots, n$ are given constants.Then the problem

$$\begin{aligned} solve : X'(t) &= A(t)X(t) + b(t) \\ subject\ to : X(t_0) &= X_0 \end{aligned} \tag{5.12}$$

is an IVP on the interval.

5.4.2 Solution of Linear System

Definition 5.2. *[14] A solution of (5.11) on an interval I is any column matrix*

$$\mathbf{X(t)} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$ *whose entries are differentiable functions satisfying (5.11) on the interval.*

The methods of **undetermined coefficients** and **variation of parameters** can both be adapted to the solution of a nonhomogeneous linear system (5.11). Even if variation of parameters is the most power full technique, there are few instances when the method of undetermined coefficients gives a quick means of finding a particular solution X_p.
Consider the system of first order linear ODE

$$X'(t) = A(t)X(t) + b(t) \tag{5.13}$$

And the corresponding homogeneous system

$$X'(t) = A(t)X(t) \tag{5.14}$$

where

$$\mathbf{X(t)} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \ \mathbf{A(t)} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \dots a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots a_{2n}(t) \\ \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots a_{nn}(t) \end{pmatrix}, \ \mathbf{b(t)} = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}$$

Then (5.14) consists n fundamental set of solutions, that is n linearly independent solutions.If $X_1, \ X_2, \ \dots, \ X_n$ are fundamental set of solutions of (5.14), then by Supper Position Theorem,

$$c_1X_1(t) + c_2X_2(t) + \ \dots \ + c_nX_n(t)$$

is also solution which is general solution of (5.14) where c_i are constants for $1 \leq i \leq n$.If $X_p(t)$ is particular solution of (5.13), that is

$$X_p'(t) = A(t)X_p(t) + b(t) \ \textit{and} \ X_h(t) = c_1X_1(t) + c_2X_2(t) + \ \dots \ + c_nX_n(t)$$

is the general solution of (5.14), then

$$X(t) = X_p(t) + X_h(t)$$

is the general solution of (5.13).
Let

$$\mathbf{X_1(t)} = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \ \mathbf{X_2(t)} = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \\ \vdots \\ x_{n2}(t) \end{pmatrix}, \ \ldots, \ \mathbf{X_n(t)} = \begin{pmatrix} x_{1n}(t) \\ x_{2n}(t) \\ \vdots \\ x_{nn}(t) \end{pmatrix}$$

be n fundamental set of solution vectors of (5.14) on an interval I. Then the general solution of (5.14) is given by

$$\begin{aligned} X(t) &= c_1X_1(t) + c_2X_2(t) + \ \ldots \ + c_nX_n(t) \\ &= c_1 \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix} + c_2 \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \\ \vdots \\ x_{nn}(t) \end{pmatrix} + \ldots + c_n \begin{pmatrix} x_{1n}(t) \\ x_{2n}(t) \\ \vdots \\ x_{nn}(t) \end{pmatrix} \\ &= \begin{pmatrix} c_1 \ x_{11}(t) + c_2 \ x_{12}(t) + \ldots + c_n \ x_{1n}(t) \\ c_1 \ x_{21}(t) + c_2 \ x_{22}(t) + \ldots + c_n \ x_{2n}(t) \\ \vdots \\ c_1 \ x_{n1}(t) + c_2 \ x_{n2}(t) + \ldots + c_n \ x_{nn}(t) \end{pmatrix} \\ &= \begin{pmatrix} x_{11}(t) & x_{12}(t) & \ldots x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \ldots x_{2n}(t) \\ \vdots & \vdots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \ldots x_{nn}(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \\ &= \phi(t)c \end{aligned}$$

where

$$\phi(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) & \ldots x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \ldots x_{2n}(t) \\ \vdots & \vdots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \ldots x_{nn}(t) \end{pmatrix}, \ c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \tag{5.15}$$

Theorem 5.3. *[14] Let the entries of the matrix* $A(t)$, $b(t)$ *be continuous function on a common interval* I *that contains the point* t_0. *Then equation (5.12) has unique solution.*

5.4.3 Fundamental Matrix

Definition 5.3. *[14] The matrix* $\phi(t)$ *in (5.15) is called **fundamental matrix** of (5.14)on the interval* I.

From (5.15), we can notice that if $\phi(t)$ is fundamental matrix of (5.14) on the interval I and C is $n \times 1$ constant column matrix, then the general solution of (5.14) on the interval I can be written as

$$X_h(t) = \phi(t)C$$

Since by the Wronskian

$$\det \phi(t) = \det \begin{pmatrix} x_{11}(t) & x_{12}(t) & \dots x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \dots x_{2n}(t) \\ \vdots & \vdots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \dots x_{nn}(t) \end{pmatrix} \neq 0$$

for every $t \in I$, we have that $\phi^{-1}(t)$ exists.Furthermore, to say that

$$X(t) = \phi(t)c$$

is a solution of (5.14) means

$$\phi'(t)c = A(t)\phi(t)c \quad or \quad (\phi'(t) - A(t)\phi(t))c = 0$$

Since the last equation is to hold for every $t \in I$, and every $n \times 1$ constant column matrix c, we have that

$$\phi'(t) - \phi(t)A(t) = 0 \qquad or \qquad \phi'(t) = A(t)\phi(t)$$

If each component $x_i(t)$ of $X(t)$, each component $b_i(t)$ of $b(t)$ and each component $a_{ij}(t)$ of $A(t)$ are ω-periodic for $1 \leq i,\ j \leq n$, then $X(t),\ b(t)$ and $A(t)$ of (5.11) are said to be **ω-periodic**.

5.4.4 Necessary and Sufficient Conditions

In this section we will focus only on the necessary and sufficient conditions for the existence of periodic solutions of (5.14), when $A(t)$ is 2×2 matrix as every linear second order ODE

$$y''(t) + a_1(t)y'(t) + a_2(t)y(t) = f(t)$$

can equivalently written as

$$X'(t) = A(t)X(t) + b(t)$$

where

$$X(t)=\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},\ A(t)=\begin{pmatrix} 0 & 1 \\ -a_2(t) & -a_1(t) \end{pmatrix},\ b(t)=\begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

Consider the system of first order linear ODE

$$X'(t)=A(t)X(t)+b(t) \tag{5.16}$$

And the corresponding homogeneous system

$$X'(t)=A(t)X(t) \tag{5.17}$$

where

$$X(t)=\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},\ A(t)=\begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix},\ b(t)=\begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}$$

Theorem 5.4. *[14] Let $\phi(t)$ be fundamental matrix of (5.17). Then the general solution of (5.16) is given by $X(t)=\phi(t)C+\int_{t_0}^{t}\phi(t)\phi^{-1}(s)b(s)ds$ where C is constant 2×1 column matrix , $X(t)$ and $\phi(t)$ are defined on the interval I that contains the point t_0.*

Proof. This part will be proved using variation of parameters.
Let $X_h(t)=\phi(t)c$ be the general solution of (5.17) and $X_p(t)$ be the particular solution of (5.16) where $\phi(t)$ is fundamental matrix of (5.17) defined in an interval I and c is constant 2×1 column matrix. Then $X_p'(t)=A(t)X_p(t)+b(t)$ and the general solution of (5.16) is given by $X(t)=X_h(t)+X_p(t)$. Let $X_p(t)=\phi(t)U(t)$ where $U(t)$ is 2×1 column matrix of functions. Then

$$X_p'(t)=\phi'(t)U(t)+\phi(t)U'(t)=A(t)\phi(t)U(t)+b(t)$$

But, since $\phi(t)$ is fundamental matrix of (5.17) , we have that

$$\phi'(t)=A(t)\phi(t)$$

and it follows that

$$A(t)\phi(t)U(t)+\phi(t)U'(t)=A(t)\phi(t)U(t)+b(t)$$

which implies that

$$U'(t)=\phi^{-1}(t)b(t)$$

OR

$$U(t)=\int_{t_0}^{t}\phi^{-1}(s)b(s)ds$$

and hence

$$X_p(t) = \phi(t)\int_{t_0}^{t}\phi^{-1}(s)b(s)ds$$

Therefore,

$$X(t) = \phi(t)c + \int_{t_0}^{t}\phi(t)\phi^{-1}(s)b(s)ds$$

□

Theorem 5.5. *[1] Let the matrix $A(t)$ and the function $b(t)$ be continuous and ω -periodic in $\mathbb{R}$. Then the differential system (5.16) has ω -periodic solution $X(t)$ if and only if $X(0) = X(\omega)$.*

Proof. Let $X(t)$ be ω -periodic solution. Then by definition

$$X(0) = X(\omega + 0) = X(\omega).$$

It remains only to show that the converse.Let $X(t)$ be a solution of (5.16) satisfying $X(0) = X(\omega)$. If $V(t) = X(\omega + t)$, then

$$\begin{aligned} V'(t) &= X'(\omega + t) \\ &= A(t+\omega)X(t+\omega) + b(t+\omega) \\ &= A(t)X(t) + b(t) \\ &= A(t)V(t) + b(t) \end{aligned}$$

That is V(t) is solution of equation (5.16).
However, since $V(0) = X(\omega) = X(0)$, the uniqueness of initial value problem implies that
$X(t) = V(t) = X(\omega + t)$ and hence $X(t)$ is ω- periodic □

Corollary 5.3. *[1] Let the matrix $A(t)$ be continuous ω-periodic in $\mathbb{R}$. Further, let $\phi(t)$ is a fundamental matrix of the differential system (5.17). Then the differential system (5.17) has a nontrivial ω-periodic solution $X(t)$ if and only if $\det(\phi(0) - \phi(\omega)) = 0$.*

Proof. We know that the general solution of the differential (5.17) is

$$X(t) = \phi(t)c$$

where c is an arbitrary constant vector. Thus,by Theorem 5.5, $X(t)$ is ω-periodic solution if and only if $\phi(0)c = \phi(\omega)c$, that is

$$(\phi(0) - \phi(\omega))c = 0$$

has nontrivial solution c. But, this system has nontrivial solution if and only if

$$\det(\phi(0) - \phi(\omega)) = 0.$$

□

Corollary 5.4. *[1] Let the matrix $A(t)$ and the function $b(t)$ be continuous and ω-periodic in $\mathbb{R}$. Then the differential system (5.16) has a unique ω-periodic solution $X(t)$ if and only if the differential system (5.17) does n't have a ω-periodic solution $X(t)$ other than the trivial one.*

Proof. Let $\phi(t)$ be a fundamental matrix of the differential system (5.17). Then the general solution of (5.16) is given by

$$X(t) = \phi(t)c + \int_0^t \phi(t)\phi^{-1}(s)b(s)ds$$

where c is constant 2×1 column matrix , $X(t)$ and $\phi(t)$ are defined on the interval I. This $X(t)$ is ω-periodic solution if and only if

$$\phi(0)c = \phi(\omega)c + \int_0^{!} \phi(\omega)\phi^{-1}(s)b(s)ds$$

That is the system

$$(\phi(0) - \phi(\omega))c = \int_0^{!} \phi(\omega)\phi^{-1}(s)b(s)ds$$

has a unique solution vector c.But, this system has unique ω-periodic solution if and only if $\det(\phi(0) - \phi(\omega)) \neq 0$. Now the conclusion follows from corollary (5.3) □

Example 5.3. *Consider the first order linear system of ODEs*

$$X'(t) = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} X(t) + \begin{pmatrix} -8 \\ 3 \end{pmatrix}$$

Then show that

(*a*) $\phi(t) = \begin{pmatrix} \cos t + \sin t & \cos t - \sin t \\ \cos t & -\sin t \end{pmatrix}$ *is the fundamental matrix of the corresponding homogeneous part.*

(*b*) $X_p(t) = \begin{pmatrix} 14 \\ 11 \end{pmatrix}$ *is the particular solution for the first order linear system of ODEs.*

(*c*) $X(t) = \begin{pmatrix} \cos t + \sin t & \cos t - \sin t \\ \cos t & -\sin t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 14 \\ 11 \end{pmatrix}$ *is the general 2π- periodic solution of the nonhomogeneous first order linear system of ODEs.*

Solution

(a) Let $X_1(t) = \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix}$, $X_2(t) = \begin{pmatrix} \cos t - \sin t \\ -\sin t \end{pmatrix}$. Then since

$$X_1'(t) = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} X_1(t) \ , \quad X_2'(t) = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} X_2(t)$$

we have that $X_1(t)$ and $X_2(t)$ are solutions of the homogeneous ODE $X'(t) = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} X(t)$.Moreover,

$$\det \begin{pmatrix} \cos t + \sin t & \cos t - \sin t \\ \cos t & -\sin t \end{pmatrix} = -1 \neq 0$$

It follows that $X_1(t)$ and $X_2(t)$ are fundamental set of solutions and hence $\phi(t) = \begin{pmatrix} \cos t + \sin t & \cos t - \sin t \\ \cos t & -\sin t \end{pmatrix}$ is the fundamental matrix of the corresponding homogeneous ODE $X'(t) = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} X(t)$. Therefore

$$X_h(t) = \begin{pmatrix} \cos t + \sin t & \cos t - \sin t \\ \cos t & -\sin t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

is the general solution of $X'(t) = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} X(t)$.

(b) From $X_p(t) = \begin{pmatrix} 14 \\ 11 \end{pmatrix}$, we have that $X_p'(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
Also, $\begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} X_p(t) + \begin{pmatrix} -8 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ which implies that
$X_p(t) = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} X_p(t) + \begin{pmatrix} -8 \\ 3 \end{pmatrix}$.
Therefore, $X_p(t) = \begin{pmatrix} 14 \\ 11 \end{pmatrix}$ is the particular solution for the first order linear system of ODEs.

(c) The general solution of the nonhomogeneous system is given by

$$X(t) = X_h(t) + X_p(t) = \begin{pmatrix} \cos t + \sin t & \cos t - \sin t \\ \cos t & -\sin t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 14 \\ 11 \end{pmatrix}$$

Since $X(t+\pi) = \begin{pmatrix} \cos(t+2\pi) + \sin(t+2\pi) & \cos(t++2\pi) - \sin(t+2\pi) \\ \cos(t+2\pi) & -\sin(t+2\pi) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 14 \\ 11 \end{pmatrix} = \begin{pmatrix} \cos t + \sin t & \cos t - \sin t \\ \cos t & -\sin t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 14 \\ 11 \end{pmatrix} = X(t)$,we have that $X(t)$ is 2π- periodic solution of the system.

Example 5.4. *Consider the second linear order ODE* $y'' + y = \cos t$. *Then*

a) find the corresponding system of first order ODE.

b) Show that why the system of first order ODE in (a) has not unique periodic solution?

Solution

a) Let $y(t) = x_1(t)$, $y'(t) = x_2(t)$. Then we have that $x_1'(t) = x_(t)$, $x_2'(t) = -x_1(t) + \cos t$ and hence the corresponding first order linear system becomes

$$X'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X(t) + \begin{pmatrix} 0 \\ \cos t \end{pmatrix}$$

where $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

b) $\phi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ is the fundamental matrix of

$$X'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X(t)$$

which is 2π-periodic.Then

$$\phi(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \phi(2\pi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $\phi(2\pi) - \phi(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ which implies $\det(\phi(2\pi) - \phi(0)) = 0$. Thus, by corollary (5.3) $X'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X(t)$ admits nonntrivial 2π-periodic solution.It follows by corollary (5.17), that

$$X'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X(t) + \begin{pmatrix} 0 \\ \cos t \end{pmatrix}$$

where $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ has not unique 2π-periodic solution

Summary

The study of existence and property of periodic solutions of ordinary differential equations has already attracted the attention of many researchers in the area.In this particular project, conditions are obtained which are sufficient for existence of periodic solution(s) of (5.1).The periodic solution of (5.1) is highly linked with the periodicity and other conditions of a, f and g.The existence of fixed point for a given operator play a great role on the existence of periodic solution(s) and One can apply Shauder's Fixed Point Theorem to determine the existence of periodic solution(s) of (5.1) after transforming the original equation into integro-differential equation through a linear integral operator.

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Addis Ababa University
College Of Computational And Natural Sciences
School of Graduate Studies
Department of Mathematics

The undersigned hereby certify that they have read and recommend to the department of mathematics for acceptance of this project entitled "**Existence Of Periodic Solutions For Class Of Second Order Ordinary Differential Equations with Periodic Data**" by Sahle Weldemichael in partial fulfillment of the requirements for the degree of Master of Science in mathematics.

Advisor: Dr. Tadesse Abdi

Sign.________________ Date________________

Examiner1: ________________

Sign.________________ Date________________

Examiner2: ________________

Sign.________________ Date________________

By: Sahle Weldemichael Sign.________________

Date________________