Addis Ababa University

Faculty of Computer and Mathematical Sciences

Department of Mathematics

A Graduate Project Report

On

The Motzkin Numbers and their Combinatorial Interpretations

Submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

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Declaration

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other similar title to me.

………………..

Shibru Ararsa

Permission

This is to certify that this project is compiled by Shibru Ararsa in the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiners and eventual presentation ⁄ defense.

…………………

Prof. Melkamu Zeleke

Acknowledgements

I would like to thank my advisor Prof. Melkamu Zeleke grate fully for his following every thing which is included in this project by giving suggestions for each proof, explanations etc. starting from the designing of the Table of contents up to the end of the project.

And also I would like to thank my School Administration and my department in the school, for their making me free from the work and covering my classes.

Table of contents

Project Summary

This project paper is divided into four sections. we discuss the origin of Motzkin numbers using the division of finite number of points on a circle by non-intersecting chords. The idea of division of finite number of points on a circle which leads us to the origin of Motzkin numbers after getting the Catalan numbers is raised by Th. Motzkin in his paper "Relations between Hyper Surface Cross ratios and the Combinatorial formula for partitions of a polygon…" .

In addition to this the relation between the Motzkin and Catalan numbers are proved. Here we have the two important relations which are given by

1. $C_{n+1} = \sum_{i=0}^{n} \binom{n}{i} M_i$, and
2. $M_n = \sum_{i=0}^{n} \binom{n}{2i} C_i$, where M_n is the nth Motzkin number and C_{n+1} is the $(n+1)^{th}$ Catalan number.

Combinatorial settings which are enumerated by the Motzkin numbers are stated and proved using diagrams for different values of n. The Combinatorial objects through which the sequence of Motzkin numbers can be constructed are the division of points on the circle with non-intersecting chords, the Motzkin paths, the Dyck paths etc.

In this paper we also showed that the Generating Function of the sequence of Motzkin numbers can be constructed from the enumeration of ordered- trees and the recursion formula is found by using Wen- jin Woan's general idea of lattice paths. Using the recursion formula one can easily show that the asymptotic property of the regular Motzkin sequence which says the ratio between two consecutive terms of the Motzkin numbers approaches 3 for large values of n.

Finally the generalization of Motzkin numbers using k- trees which is the main purpose of this project is explained. Here k- tree is the generalization of ordered trees which are counted by the Catalan numbers. The general Motzkin numbers (k- Motzkin numbers) for k >2 are defined using k- trees and we show that these numbers agree with Baxter's generalization of the Temperley – Lieb operators for k=3. As M. Jani and Melkamu Zeleke showed in [3] there is a one to one correspondence between the permissible modes of connections of X_{2n} and the 3- Motzkin numbers.

SECTION I

1.1. **The Origin of Motzkin Numbers**

Motzkin numbers originated after Israeli Mathematician Th. Motzkin published his paper about "The relations between Hyper Surface Cross ratios and a Combinatorial formula for partitions of a polygon …" in the Bulletin of the American Mathematical Society in 1948. Consider the divisions of 2n-points on a circle into n-pairs. Let [2n] be the set of 2n points where [2n] = {1, 2, … , n, n+1, …, 2n}. The division D can now be symbolized by n-chords. D has a number C of crossings (pairs of chords intersecting with in a circle). Each chord of division consist 2-points at a time.

Here for n = 2, we have 3- divisions.

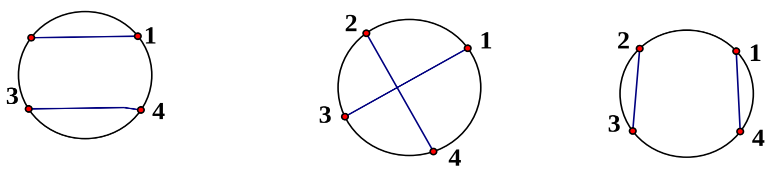


Fig.1.

3 divisions of pairs out of these, 2 of them are non-crossing divisions

For n=3, there are 15 divisions of pairs out of these, 5 of them are non-crossing divisions.

Let the total number of divisions D be denoted by N. Then N is the total number of arrangements of 2n points on a circle into non-crossings with each chord consists of 2-points at a time. Then such type of arrangement of of 2n-points is given by a permutation of 2n-points, where n of them are taken at a time.

Therefore, $N = \frac{P_{2n,n}}{2^n}$, where 2^n is because of each chord consists 2-points at a time.

Now, $N = \frac{2n!}{n!2^n}$

$$= \frac{2n!}{n!2^n}\ \frac{(n+1)!}{(n+1)!}$$

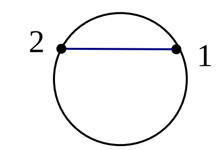
$$= \frac{2n!}{n!n!}.\frac{(n+1)!}{(n+1)!2^n}$$

$$\Longrightarrow \frac{N.2^n}{(n+1)!} = \frac{2\text{n}!}{n!n!}\left(\frac{1}{\text{n}+1}\right)$$

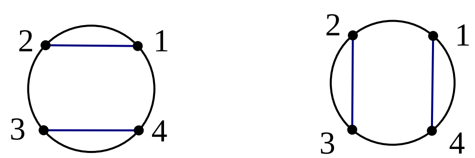
$$= C_n \ \text{(the } n^{th} \text{ Catalan number)}$$

Note: division by (n+1)! is because of complete chording.

Here, for n = 1 (C_1 = 1)



for n = 2, (C_2 = 2)



for n=3, (C_3 = 5)

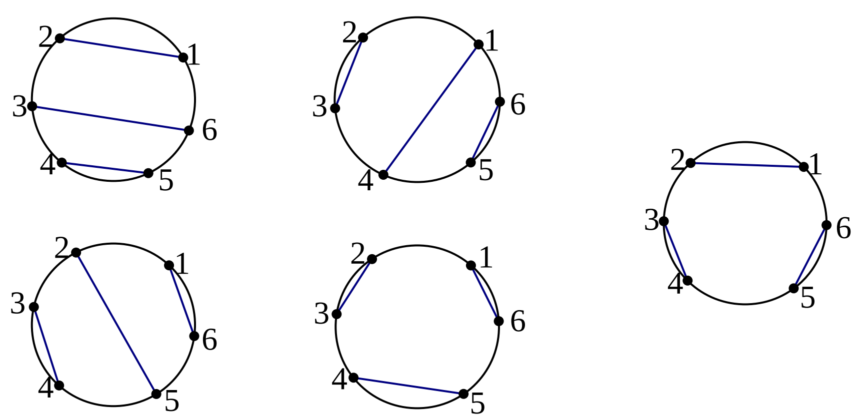


Fig.2.

Hence C_n = # of divisions of 2n points on a circle into n-pairs with non- crossing chords in a circle. We have complete chordings for n points.

For example,

C_3 = # of divisions of 6-points on a circle into 3-pairs using non crossing

chords in a circle.

As we have seen from the above diagram,

$$C_3 = \frac{1}{3+1}\binom{6}{3} = 5$$

Let A be a point on the circle which is not in [2n]. Here A must be paired with a point B for which the number of points on either sides of AB are even. B is not in [2n] also. Consider that chord AB will never intersect with any of the other chords. Now we have 2n+2 points. Then we have to search for C_{n+1} rather than C_n.

If the arcs on both sides of chord AB consist 2j and 2k points in addition to A and B, then

$2j + 2k = 2(j + k) = 2n \Rightarrow j + k = n.$

<u>Example</u>

For n=3,

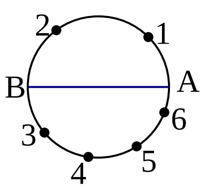


Fig.3.

On arc A2B , we have 2-points i.e 2 = 2.1. On arc A4B, we have 4 points, i.e 4 = 2.2.

Therefore, 2j + 2k = 2.1+2.2

= 2(1+2)

= 2(3)

= 6

Hence, j+ k = 1+2 = 3.

Now, there are $\sum C_j C_k$ divisions, without crossings containing the pair AB.

Let (C_0 = C_1 = 1). Hence, the recursion formula C_{n+1} = $\sum C_j C_k$ holds, and the points A and B contribute as the initial values for the recursion formula. C_{n+1} is because of the additional points A and B to 2n points . That means 2n+2 = 2(n+1).

For the recursion formula $C_{n+1} = \sum C_j C_k$, the series $C(x) = \sum C_n x^n$ satisfies the functional equation, $C^2(x) = \frac{C(x)-1}{x}$

Now, $C^2(x) = \frac{C(x)-1}{x}$

$\Longrightarrow xC^2(x) = C(x) - 1$

$\Longrightarrow xC^2(x) - C(x) + 1 = 0$

Using the general form of quadratic formula, $C(x) = \frac{1-\sqrt{1-4x}}{2x}$

By using the binomial theorem or Mathematica,

$$C(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^{\mathbf{n}}$$

Hence, $[x^n]C(x) = \frac{1}{n+1}\binom{2n}{n} = C_n$, the n^{th} Catalan number.

$\frac{1+\sqrt{1-4x}}{2x} \neq C(x)$ using the binomial expansion or using Mathematica.

Note: We can get the same recursion formula for the number of partitions of a convex polygon of n+2 sides, by non- intersecting diagonals into triangles.

After dividing a polygon by a triangle there remain two polygons of i+2 and k+2 sides with j+k = n to be divided in the same manner. For j = 0,1,2,…,n and C_n = 1 . Hence the value of C_n is $\frac{1}{n+1}\binom{2n}{n}$.

Generally, to determine the number of divisions of nk points on a circle into nk -tuples without crossing , choose a fixed point from (n+1)k points, the k -tuple containing the point. The remaining arcs are with j_1k, j_2k, …,j_kk points such that j_1+ j_2+,…+ j_k = n. For the first arc, there are C_{j1} divisions (C_0 = 1) and for the second arc there are C_{j2} divisions and so on.

Hence the recursion formula will be $C_{n+1} = \sum C_{j1} \ldots . C_{jk}$ corresponds to a function

$C(x) = \sum C_n x^n$, with $C^2(x) = \frac{C(x)-1}{x}$, using the Snake Oil Method.

$$\Longrightarrow\ xC^k(x) = C(x) - 1$$

Applying the Lagrange's inversion formula (LIF) implies that,

$$xC^k(x) = C(x) - 1$$

Let w (x) = $C(x)\ -\ 1$

$$\Longrightarrow C(x)\ =\ \text{w}(x) + 1$$

$$\Longrightarrow \text{w}(x) = x(1+\text{w}(x))^k$$

Let $\Phi(x) = (1+x)^{k}$

Now LIF implies that

$$w_n = [x^n]w(x) = \frac{1}{n}[x^{n-1}]\Phi(x)^n$$

$$= \frac{1}{n}[x^{n-1}]((1+x)^{k})^{n}$$

$$= \frac{1}{n}[x^{n-1}](1+x)^{nk}$$

$$\Longrightarrow\ C_{n,k}\ = \frac{1}{n}\binom{kn}{n}$$

$$= \frac{1}{n}\left[\frac{nk!}{((k-1)n+1)!(n-1)!}\right]$$

$$= \frac{1}{n}\left[\frac{nk!}{(k-1)n+1)\ (\ (k-1)n)!(n-1)!}\right]$$

$$= \frac{1}{(k-1)n+1}\left[\frac{nk!}{((k-1)n)!n(n-1)!}\right]$$

$$= \frac{1}{(k-1)n+1}\left[\frac{nk!}{(k-1)n)!\ n!}\right]$$

$$= \frac{1}{(k-1)n+1}\binom{nk}{n}$$

Therefore , $C_{n,k} = \frac{1}{(k-1)n+1}\binom{nk}{n}$, the general catalan number

For k = 2 , $C_{n,2} = \frac{1}{n+1}\binom{2n}{n}$, the regular Catalan number,

Here, k is not restricted to a single value.

Note: The same recursion formula can be obtained for the number of divisions of a convex polygon of n-sides by non-intersecting diagonals into polygons of k sides, where k is not restricted to a single value as mentioned above.

The number C_n of division of n-points on a circle into sets of k points without crossing can be generalized to different values of k .

For k = 2 or 3, $C(x) - 1 = x^2C^2(x) + x^3C^3(x)^2$

For k =1 or 2, $C(x) - 1 = xC(x) + x^2C^2(x)$

$C(x) = \sum_{k=0}^{\infty} C^k(x)\, x^k$, using the Snake Oil Method.

$\Longrightarrow x^2C^2(x) + (x-1)C(x) + 1 = 0$

Using the general form of quadratic formula,

$$C(x) = \frac{-(x-1)-\sqrt{(x-1)^2-4(1)x^2}}{2x^2}$$
$$= \frac{1-x-\sqrt{1-2\text{x}-3x^2}}{2x^2}$$
$$= 1 + x + 2x^2 + 4x^3 + 9x^4 + 21x^5 + 51x^6 + \ldots$$

Hence the sequence 1,1,2,4,9,12,51,… is the family of Motzkin numbers.

1.2. **Algebraic Relations Between Motzkin and Catalan numbers**

Since some modifications of the ways of Combinatorial Settings which give the Catalan numbers yield the Motzkin numbers, there is a close relation between the Catalan numbers and Motzkin numbers. By constructing the difference triangles and using the Euler's transformation which is stated by R. Donaghey and Louis Shapiro in [4], we can find the formula of the Catalan numbers in terms of Motzkin numbers, and the formula of the Motzkin numbers in terms of the Catalan numbers. We can prove in Combinatorial way.

Combinatorial proof

i) $M_n = \sum_{i=o}^{n} \binom{n}{2i} C_i$

Proof

In chording of points on the circle , the sequence $\{M_n\}_{n=0}^{\infty}$ enumerates the number of ways of connecting n points on a circle by non-intersecting chords where

$\{M_n\}_{n=0}^{\infty}$ =1,1,2,4,9,21,51,127, … .

Consider n points on a circle to be connected by non-intersecting chords. Let i be the number of chords used in connection. Then we have to group 2i points through i chords. Hence we have $\binom{n}{2i}$choices for each i where 2i ≤ n.

Hence, such type of choices can be done in C_i ways, where C_i is the i[th] Catalan number.

Therefore $M_n = \sum_{i-0}^{n(\infty)} \binom{n}{2i} C_i$

ii. $C_{n+1} = \sum_{i=o}^{n} \binom{n}{i} M_i$

Proof

For Catalan numbers we have to consider 2n points to be connected on a circle using non-intersecting chords in a circle.

Consider a point A on the circle which is not in [2n]. Then A must be connected with the point B which is also not in [2n]. Now we divide the points into two groups by chord AB which is never intersecting with any of the other chords. Each group must contain an even number of points to be connected by different chords.

Let such groups contain 2j and 2k points where 2j+2k=2(j+k) = 2n⇒ j+k= n.

Then we are going to have 2n+2 points, where 2n+2= 2(n+1). This leads us to find C_{n+1} instead of C_n.

To express C_{n+1} in terms of Motzkin numbers, we have n chords to connect 2n points and then choose i chords out of n chords which are non intersecting. Here we have $\binom{n}{i}$ choices and such choices can be done in M_i ways where M_i is the i[th] Motzkin number.

Therefore the total number of ways of connecting such 2n+2 points on the circle with non-intersecting chords in a circle by taking the sum over all i is,

$$C_{n+1} = \sum_{i=0}^{n} \binom{n}{i} M_i$$

Now, from the given two relations we can derive the second formula for C_{n+1}.

That is , $C_{n+1} = \sum_{i=0}^{n} \binom{n}{i} M_i$

$$= \sum_{i=0}^{n} \binom{n}{i} \sum_{j=0}^{i} \binom{i}{2j} C_j$$

$$= \sum_{i,j} \binom{n}{i} \binom{i}{2j} C_j$$

$$= \sum_{i,j} \binom{n}{2j} \binom{n-2j}{i-2j} C_j$$

$$= \sum_{j=0}^{n} \binom{n}{2j} C_j \sum_{i-2j}^{n-2j} \binom{n-2j}{i-2j}$$

$$= \sum_{j=0}^{n} \binom{n}{2j} C_j 2^{n-2j} \text{ from } \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Hence, $\boldsymbol{C_{n+1} = \sum_{j=0}^{n} \binom{n}{2j} C_j 2^{n-2j}}$

SECTION II

Combinatorial Settings enumerated by the Motzkin numbers

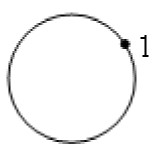
Situations where the Motzkin numbers occur

2.1 **Circle Chording settings**

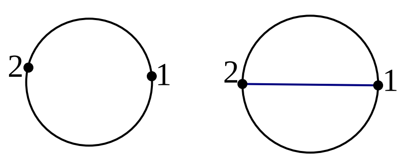
The Motzkin numbers were first originated by a circle chording settings. Here the connection of n points on a circle using non-intersecting chords has been operated. Then the sequence of numbers $\{M_n\}_{n=0}^{\infty}$= 1, 1, 2, 4, 9, 21, 51, 127, … is shown to enumerate the number of ways of connecting a subset of n-points on a circle by non-intersecting chords. The general case for any Motzkin number can be found from the generating function given in SECTION I.

Example

For n = 1, $M_1 = 1$ (only one point on the surface of a circle)



For n=2, $M_2 = 2$ (2-points)



For n=3, $M_3 = 4$ (3-points)

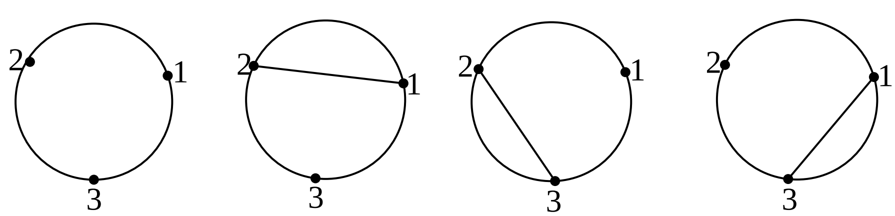


Fig.4.

Continuing in this way we can find M_n such that,

$M_n = \#$ of ways of connecting n-points on a circle with out intersecting chords or with non-intersecting chords, where M_n is the n^{th} Motzkin number. The general formula of M_n can be found using the relations between the Motzkin and Catalan numbers which we proved in SECTION I.

2.2.1 **Motzkin paths**

The most popular combinatorial objects counted by Motzkin numbers are the Motzkin paths. A Motzkin path on n-steps is a lattice path in the (x, y) coordinate plane from(0,0) to (n,0) using only steps of the type (1,1) - Up ,(1,- 1) - Down , (1,0) - Level and never falling below the x-axis .

Example

Fig.5.

Hence $M_n = \#$ of Motzkin paths from (0,0) to (n,0) using only the steps (1,1) -Up ,

(1,1) - Down , (1,0) - Level and never cross the x-axis.

Here we can construct the Motzkin triangle using the Motzkin paths specified in the above diagram.

n/k	0	1	2	3	4	5
0	1					
1	1	1				
2	2	2	1			
3	4	5	3	1		
4	9	12	9	4	1	
5	21	30	25	14	5	1

Now the first column consists of the Motzkin numbers which count the number of paths from (0,0) to (n,0) down the triangle in the 1^{st} column.

For example , we have 5 paths to reach the point (3,1) on the Motzkin path. From this we can construct the recursion formula ,

M(n+1,k) = M(n,k-1) + M(n,k) + M(n,k+1) .

For M(3,1) = M(2,0) + M(2,1) + M(2,2) = 2+2+1= 5 ,

where M(n,k) counts the number of paths from (0,0) to (n,k) in n-steps.

Example ; we can go from (0,0) to (2,1) in two paths having length 2.

For n = 3, we have the point (3,0) with 4 Motzkin paths.

That is M(3,0) = M(2,-1) + M(2,0) + M(2,1)

= 0 + 2 + 2

= 4

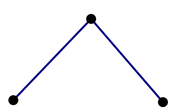
Hence, $M_3 = 4$

2.2.2. **Dyck paths**

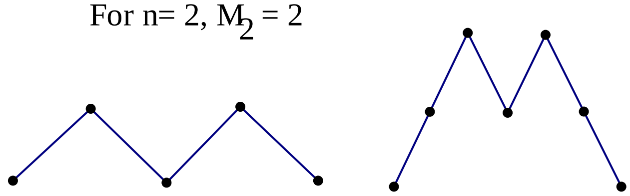
Definition: A Dyck path is a lattice path starting from (0,0) and ending at (n,0) using the steps (1,1) - Up, (1,-1) - Down , no Level step and never cross the x- axis. A Dyck path is enumerated by the Motzkin numbers if it is containing neither long slopes and lonely peaks. A peak of a Dyck path is said to be a lonely peak if both the two slopes joined at a peak have length more than one step. Here a peak is a point at which an Up-step is immediately followed by the Down-step. A long slope is a slope of length more than 2-steps.

Example

For n = 1, $M_1 = 1$



For n= 2, $M_2 = 2$



For n = 3, $M_3 = 4$

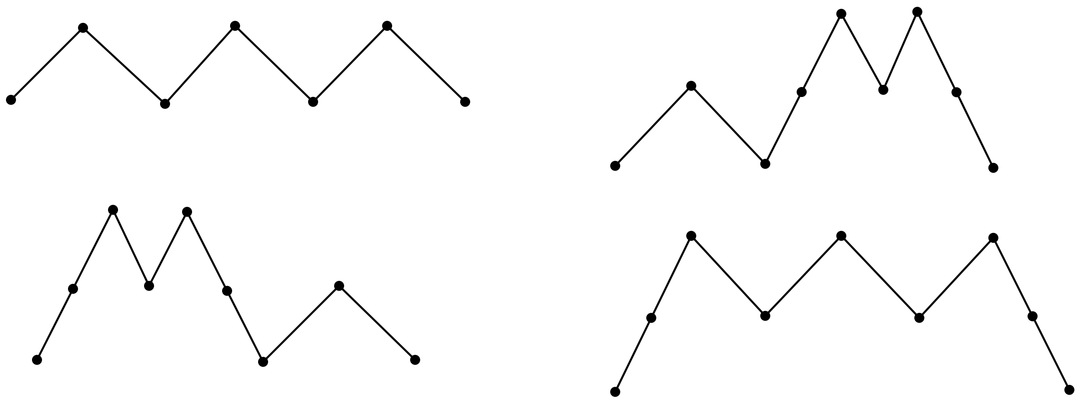


Fig.6

Therefore, $M_n = \#$ of Dyck paths with n peaks those containing neither long slopes nor lonely peaks and using the steps (1,1) - Up, (1,-1) - Down , no Level step and never falling below the x- axis.

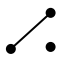
2.3**. The ballot problem**

The ballot problem is another way of the occurrence of Motzkin numbers. It is one of the combinatorial settings and it is enumerated by the Motzkin numbers. Here it is equivalent to the competitions in which each game yields one point for the victor and the game can proceed between two or more competitors. In this case we use also the triangle similar to the above one.

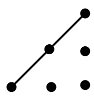
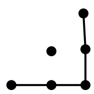
The game is proceeding as follows. Each game yields one point to the victor, in which player A is never behind player B in point. Let a tie T be an event where the two players score the same point. When A leads B, the tie T will not change. So A can be k points ahead of B after n-games in M(n,k), similar to the above Motzkin path. If we allow ties with the point split half - half in this event the Motzkin sequence arises. On the grid of half integers in the first quadrant up to the main diagonal, let a win by A be represented by an edge from (p,q) to (p+1,q) , a win by B be represented by an edge from (p,q) to (p,q+1), and a tie T be represented by an edge from (p,q) to (p+$\frac{1}{2}, q + \frac{1}{2}$) in the (x,y) coordinate plane up to the main diagonal in the first quadrant of the (x,y) co-ordinate plane.

Example

For n = 1, $M_1 = 1$

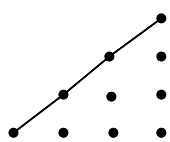

T

For n = 2, $M_2 = 2$

TT


AB

$M_3 = 4$

TTT



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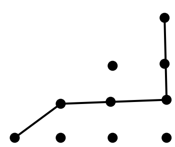

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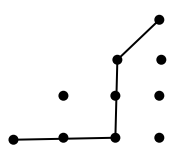
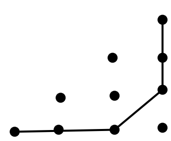




ATB

Fig.7.

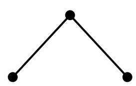
Here $M_n = \#$ of possible situations for n-games ending in tie T.

2.4. **Counting the rooted plane trees of n+1 edges or bushes**

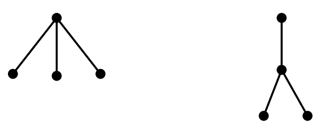
Consider the plane trees in which no vertex except the root has out degree one. These trees are called the branch reduced trees or bushes and they are enumerated by the Motzkin numbers.

Example

For n = 1, $M_1 = 1$ (2 edges)



For n = 2, $M_2 = 2$ (3 edges)



For n = 3, $M_3 = 4$ (4 edges)

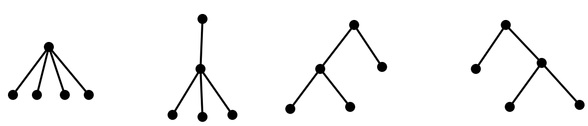


Fig.8.

Hence $M_n = \#$ of rooted plane trees or bushes with n+1 edges whose no vertex except the root has out degree one.

2.5. **Counting Bipartite Graphs**

Counting the bipartite graphs is another application of the Motzkin family. It is one of the Combinatorial setting which is enumerated by the Motzkin numbers.

consider $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$, with $a_1 < a_2 < \ldots < a_k$ and

$b_1 < b_2 < \ldots < b_k$, where $a_i \leq b_i$, for all i = 1, 2,…, k.

Here the way of counting is that, the edges meet every row of the graph. No two points parallel to each other will left disconnect. That means if a_i is disconnected then b_i must be connected with any of the rest of a_i's group. Here we have 2n points for the bi-partite graph given below.

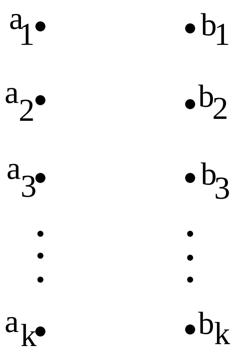


Fig.9.

Example

Now for n = 3, we will have 6-points where there are 3 points for each group .

Here $M_3 = 4$

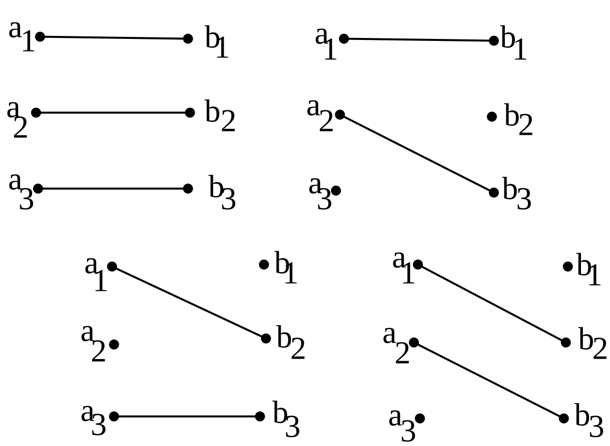


Fig.10.

Here, $a_1 < a_2 < a_3$ and $b_1 < b_2 < b_3$

Where, $a_i \leq b_i$ for each i =1, 2, 3 .

The ordered pairs must be (a_1,b_1) , (a_2,b_2) (a_3,b_3) and ,(a_i,b_j),for $a_i<a_j$,$i \neq j$ and $i \leq j$.

SECTION III

3.1. **Counting ordered trees**

Definition: An ordered tree is an unlabeled rooted tree where the subtrees are linearly ordered from left to right.The number of ordered trees with n-edges is the Catalan number,

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The out degree of a vertex is called its degree. A vertex of degree zero is called a leaf. An ordered tree is partitioned into three disjoint sets; the root, the nodes and the leaves. A node is an internal vertex which is neither the root nor the leaf as stated in [2].

<u>Example</u>

For n=3, where n is the number of edges of an ordered tree, we have $C_3 = 5$ rooted ordered trees which are known as the Catalan family.

For n = 3

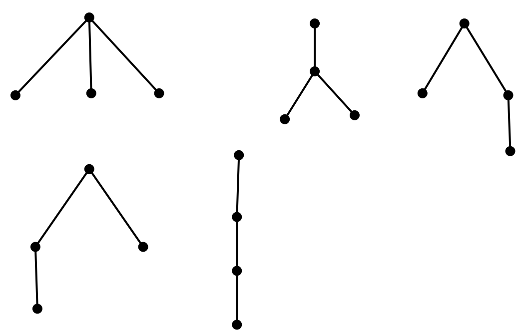


Fig.11.

$C_3 = \#$ of rooted ordered trees with 3 edges .

Therefore, $C_n = \#$ of rooted ordered trees with n-edges.

Consider the rooted ordered trees with all vertices of out degree at most 2.(i.e 0,1,2). Such trees are said to be {0,1,2} – trees and they are enumerated by the Motzkin numbers. That means, the number of {0,1,2}-trees with a given number of edges is a Motzkin number.

<u>**Example**</u>

For n = 3 , where n is the number of edges as above, we have the following {0,1,2}- trees.

For n =3

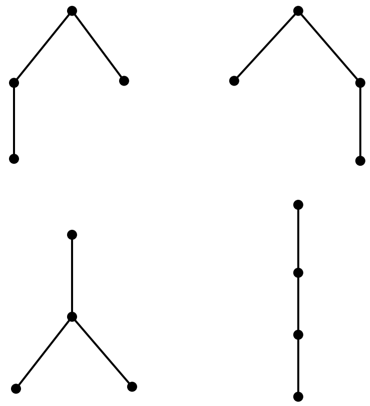


Fig.12

Hence M_3 = ⋕ of rooted ordered trees with 3- edges whose all vertices have out degree at most 2.

Therefore, M_n = ⋕ of ordered trees with n-edges whose all vertices have out degree at most 2.

3.2. **The Generating function of Motzkin numbers**

The generating function of Motzkin numbers is from the sequence $\{M_n\}_{n=0}^{\infty} = 1, 1, 2, 4, 9, 21, 51, \dots$, can be given as $M(x) = \sum_{n=0}^{\infty} M_n x^n$.

$$M(x) = 1 + x + 2x^2 + 4x^3 + 9x^4 + \cdots$$

Then it has the functional equation,

$$\Longrightarrow x^2M^2(x) + xM(x) - M(x) + 1 = 0$$

As we have seen in SECTION I, from the series $C(x) = \sum C^k(x)x^n$ the general case, for k≤ 2.

Using the quadratic formula,

$M(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$, x ≠ 0 is the closed form.

Using Mathematica for the expansion of $\sqrt{1-2x-3x^2}$ or using binomial expansion, $\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$ is exactly equal to $M(x)$,but $\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$ is not equal to $M(x)$.

3.3. **Finding the n^{th} motzkin number M_n using the generating function**

As we have seen in 4.2. , the generating function of Motzkin numbers has a functional equation $M(x) = 1 + xM(x) + x^2\, M^2(x)$.

Let $z = xM(x)$

$\Longrightarrow M(x) = 1 + z + z^2$

$\Longrightarrow z = x(1 + z + z^2)$

By Combinatorial interpretation, using Lagrange's Inversion Formula (LIF),

Let $f(z) \ = \ 1 + z + z^2$

Then $M_n \ = [z^{n+1}]f(z)^{n+1}$

$= \frac{1}{n+1} [z^n](1 + z{+}z^2)^{n+1}$

$\Longrightarrow M_n = \frac{1}{n+1} \sum_{i=0}^{n} \binom{n+1}{i} \binom{n+1-i}{i+1}$

Algebraically, if we apply the binomial expansion two times, then we can getM_n.

Let us use x for z.

Then $(1+x+x^2)^{n+1} = [(1+x)+x^2]^{n+1}$

$= \sum_{i=0}^{n+1} \binom{n+1}{i} (1+x)^{n+1-i} x^{2k}$

$= (1+x)^{n+1} + \binom{n+1}{1}(1+x)^n x^2 + \binom{n+1}{2}(1+x)^{n-1}x^4 + \cdots$

$= (1+x)^{n+1} + \frac{(n+1)!}{n!}(1+x)^n x^2 + \frac{(n+1)(n)}{2!}(1+x)^{n-1}x^4 + \ldots$

Now,

1) $(1+x)^{n+1} = 1 + \frac{n+1}{1!}x + \frac{(n+1)(n)}{2!}x^2 + \frac{(n+1)(n)(n-1)x^3}{3!} + \ldots + \frac{(n+1)!}{n!}x^n + x^{n+1}$

Here the coefficient of x^n is $\frac{(n+1)}{n!} = n+1$

Where, $n+1 = \frac{(n+1)!}{(n+1)!0!}\frac{(n+1)!}{n!1!}$ (1)

2) $\frac{(n+1)}{1!}(1+x)^n x^2 = \frac{n+1}{1!} x^2[(1+x)^n]$

$= \frac{(n+1)}{1!}[1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots]$

$$= (n+1)[x^2 + \frac{n}{1}x^3 + \frac{n(n-1)}{2!}x^4 + \cdots + \frac{n(n-1)\ldots(3)x^n}{(n-2)!} + \cdots.]$$

Here the coefficient of x^n is

$$\frac{(n+1)(n)(n-1)\ldots(3)}{(n-2)!} = \frac{(n+1)(n)\ldots3.2.1}{2!(n-2)!}$$

$$= \frac{(n+1)!}{2!(n-2)!}.\frac{n!}{n!}$$

$$= \frac{(n+1)!}{n!}.\frac{n!}{(n-2)!(2)}$$

$$= \binom{n+1}{2}\binom{n}{2}$$

$$= \binom{n+1}{1}\binom{n+1-1}{1+1} \qquad (2)$$

3) $\frac{(n+1)(n)}{2!}(1+x)^{n-1}x^4 = \frac{(n+1)(n)}{2!}x^4 [(1+x)^{n-1}]$

$$= \frac{(n+1)(n)}{2!}x^4 [1+ \frac{n-1)}{1!}x + \frac{(n-1)(n-2)}{2!}x^2 + \ldots \frac{(n-1)\ldots(4)}{(n-4)!}x^{n-1}]$$

$$= \frac{(n+1)(n)}{2!}[x^4 + \frac{(n-1)}{1!}x^5 + \cdots + \frac{(n-1)(n-2)\ldots(4)}{(n-4)!}x^{n-1} + \cdots$$

Hence the coefficient of x^n is, $\frac{(n+1)!}{2!(n-4)!3!} = \frac{(n+1)!(n+1)!}{2!(n-1)!(n-4)!3!}$

$$= \binom{n+2}{2}\binom{n-1}{3}$$

$$= \binom{n+1}{2}\binom{n+1-2}{2+1} \qquad (3)$$

Continuing in this procedure for each term we will arrive at the general formula of M_n such that , for z = $\sum_{n=0}^{\infty} M_n\ x^{n+1}$, the coefficient of x^n is (n+1)M_n since (n+1)M_n x^n is the derivative of $M_n x^{n+1}$.

Therefore, $(n+1)M_n = \sum_{i=0}^{n}\binom{n+1}{i}\binom{n+1-i}{i+1}$

$$= \frac{1}{n+1}\sum_{i=0}^{n}\binom{n+1}{i}\binom{n+1-i}{i+1}$$

Example

Calculate M_5 using the given formula

$M_5 = \frac{1}{6}\sum_{i=0}^{5}\binom{6}{i}\binom{6-i}{i+1}$

$= \frac{1}{6}[\binom{6}{0}\binom{6}{1} + \binom{6}{1}\binom{5}{2} + \binom{6}{2}\binom{4}{3} + \binom{6}{3}\binom{3}{4}]$

$= \frac{1}{6}[6+60+60+0]$

= 21, the 5th regular Motzkin number

3.4. **Recursion formula of Motzkin numbers**

Wen-jin Woan considered the family of lattice paths with unit steps NE, E or SE (Up, Level, or Down) without restriction and obtained the three term recurrence relation for the (1,d,c) - Motzkin sequence,

$$\mathbf{(n+2)\,M_n = d(2n+1)M_{n-1} + (4c - d^2)(n-1)M_{n-2}}$$

Now to prove such formula for the regular Motzkin numbers, we have to consider the general case first.

Consider those lattice paths in the Cartesian co-ordinate plane starting from (0,0) that use steps U, L and D , where U = (1,1); D = (1,1); L = (1,0). Now color the L steps with d colors and the D steps with c colors.

Let A(n,k) be the set of all colored paths ending at the point (n,k).

Let M(n,k) be the set of lattice paths in A(n,k) that never go below the x-axis.

Let B(n,k) denote the set of lattice paths in A(n,k) that never return back to the x-axis.

Now let $a_{n,k} = |A(n,k)|$

$m_{n,k} = |M(n,k)|$ and $b_{n,k} = |B(n,k)|$, $|M(n,o)| = m_n$

Note that $a_{n,k} = a_{n-1,k-1} + da_{n-1,k} + ca_{n-1,k+1}$

Then we have to prove in Combinatorial way that, the three term recursion formula for the (1,d,c) Motzkin sequence. Here U has no color for normal paths that is to normalize the weighted path to enumerate combinatorially.

If $\frac{\sqrt{c}}{2} \leq$ d, then $\lim_{n\to\infty} \frac{M_n}{M_{n-1}}$ = k = d+2$\sqrt{c}$

Proof **(combinatorial)**

1) $m_n = b_{n+1,1}$

Let P ∈ B(n+1,1). Remove the 1st step (U) and not that the remaining is in M(n,o).

For example, the path P = UULDLUUUDDLD ∈ B(12,1)

Becomes Q = ULDLUUUDDLD ∈ M(1,1).

P =

Q =

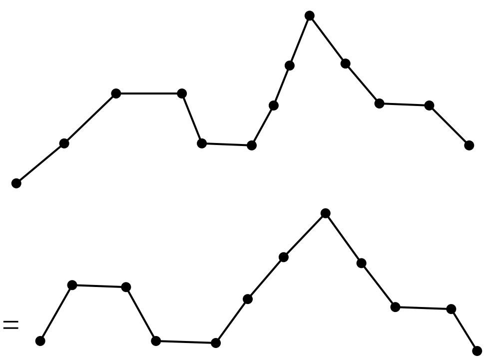


Fig.13

Q is starting from the origin and ending at the point on the x-axis.

There is a correspondence between P and Q since P∈B(n+1,1) and Q∈ M(n,o).

then $|M(n.o)| = m_n = |B(n+1,1)| = b_{n+1,1}$

Therefore , $m_n = b_{n+1,1}$ (1)

(2) $a_{n+1,1} = (n+1)b_{n+1,1}$

Let S(n+1) ={ P*: P ∈ B (n+1,1), where P* is P with one vertex marked }.

Hence $|S(n+1)| = (n+1)\, b_{n+1,1.}$

Let P*∈ S(n+1) . The marked vertex divides the path into P = FB where F is the front section and B is the back section.

Then Q = BF∈A(n+1,1) ,where Q is the inverse of P.

Conversely, starting with any path we can find the right most lowest point of Q graphically, and reverse the procedure to create a marked path P* in B(n+1,1). This shows a bijection between P* and Q.

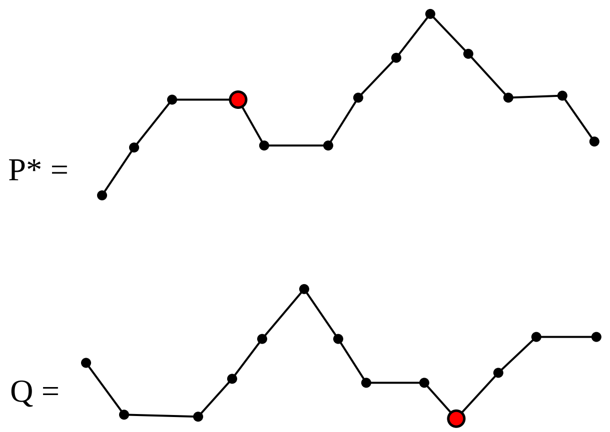


Fig.14

Therefore, |A(n+1,1)| = |S(n+1)|

$\Rightarrow a_{n+1,1} = (n+1)b_{n+1,1}$ (2)

Note: here $|S(n+1)| = (n+1)b_{n+1,1}$ is because of the marked vertex can be put in n+1 positions when attached to each n+1 points. For 11 points it has 12 ways to be in between any two arbitrary 12 points (ie. 12=11+1, n=11)

(3) The total number of L steps in M(n) is the same as that in B(n+1,1) and is d $a_{n,1}$

Proof **(Combinatorial)**

From (1) , there is a bijection between M(n) and B(n+1,1). Let P = FLB ∈ B(n+1,1)

with an L step where L divides the path in FB as above and when we remove L we remove 1 vertex. Then Q = BF∈ *A*(n,1).

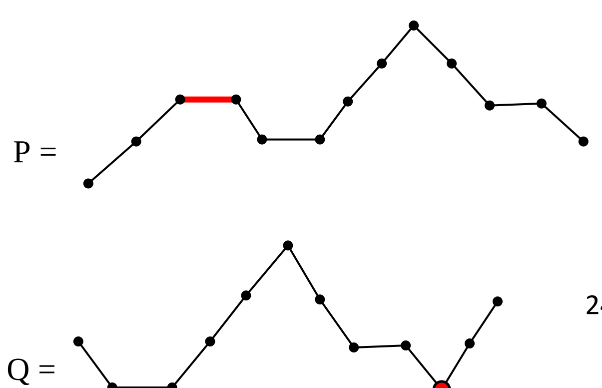


Fig.15

Since P∈B(n+1,1), the attachment point is the right most lowest point of Q . Now the inverse mapping will give us the path P (i.e Q is is the inverse of P and viceversal).

There are d colors for an L step.

Here, Q is starting at the origion and ending at one step above the x-axis.

Now, there is a one - to one correspondence between P∈B (n+1,1) and Q∈ A (n,1).

=>d|A(n,1)| = B(n+1,1)|

$\Longrightarrow d a_{n,1} = b_{n+1,1}$

But there are d colors for an L step. Then to reserve the bijection between A(n,1) and B(n+1,1), we have to multiply the left side by d.

Therefore, $b_{n+1,1} = da_{n,1}$ (3)

(4) $a_{n,o} = m_n + \frac{1}{2}(nm_n - da_{n,1}) = b_{n+1,1} + \frac{1}{2}(nb_{n+1,1} - dnb_{n-1})$

Proof **(Combinatorial)**

Let T(n) = { P^e : P ∈ M(n), where P^e is P with a U step marked }. By (2) and (3) , the number of L steps among all paths in M(n) is $da_{n,1} = dnb_{n,1}$.

The total number of steps among all paths in M(n) is $nm_n = nb_{n+1,1}$ from (1)

where m_n = |M(n,0)|⟹ |M(n)| = n.|M(n,0)| = nm_n

Hence the total number of U steps among all paths in M(n) is

$\frac{1}{2}\left(nb_{n+1,1} - dnb_{n,1}\right)$ = |T(n)|

Let P^e = FUB∈ T(n) with a U step marked.

Then Q = BUF∈ A(n,0) – M(n,0). Then the inverse mapping starts with the right most lowest point of Q.

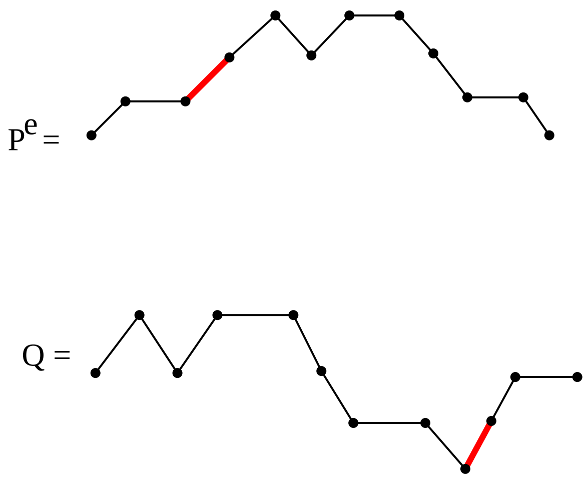


Fig.16

Now $|A(n,o)| - |M(n,o)| = a_{n,0} - m_n = \frac{1}{2}(nm_n - da_{n,1})$

(dividing by 2 is because of half of the number is that of D steps.)

Therefore, $m_n = b_{n+1,1}$ from(1)

$\Longrightarrow nm_n = nb_{n+1,1}$ and $da_{n,1} = dnb_{n,1}$ from (2)

$$a_{n,0} = m_n + \frac{1}{2}(nm_n - da_{n,1}) = b_{n+1,1} + \frac{1}{2}(nb_{n+1,1} - d_n b_{n,1}) \qquad (4)$$

There is a combinatorial proof for the equation,

$a_{n,o} = a_{n-1,-1} + d\, a_{n-1,0} + ca_{n-1,1} = 2ca_{n-1,1} + da_{n-1,0}$

$= 2c(n-1)b_{n-1,1} + d(b_{n,1} + \frac{1}{2}[(n-1)b_{n,1} - d(n-1)b_{n-1,1}])$

$a_{n,0} = a_{n-1,-1} + da_{n-1,0} + ca_{n-1.1}$ is the partition of A(n,o) by the last step

(U, L or D) as the entry in the next row in the matrix is partitioned by the three terms in the row immediately above it .

$2ca_{n-1,1} + da_{n-1,0}$ is from the fact that

$a_{n-1,-1} = ca_{n-1,1}$, since the elements in A(n-1,-1) have one more D step than those in

A(n-1,1).

Therefore , $a_{n-1,-1} + d\, a_{n-1,0} + c\, a_{n-1,1}$

$= ca_{n-1,-1} + d\, a_{n-1,0} + ca_{n-1,1} = 2ca_{n-1,1} + da_{n-1,0}$

Hence, $a_{n,0} = a_{n-1,-1} + da_{n-1,0} + ca_{n-1,1} = 2ca_{n-1,1} + da_{n-1,0}$

Now, $a_{n,o} = 2c(n-1)b_{n-1,1} + d(b_{n,1} + \frac{1}{2}[(n-1)b_{n,1} - d(n-1)\, b_{n-1,1}])$

From (2) and (4)

$2ca_{n-1,1} = 2c(n-1)\, b_{n-1,1}$ from (2)

and $a_{n+1,1} = (n+1)b_{n+1,1}$

$da_{n-1,0} = d[b_{n,1} + \frac{1}{2}((n-1)b_{n,1} - d(n-1)b_{n-1,1})]$

Therefore, $a_{n,o} = 2c(n-1)b_{n-1,1} + d(b_{n,1} + \frac{1}{2}[(n-1)b_{n,1} - d(n-1)\, b_{n-1},1)]$ (5)

Combining (2),(3) and (4) we can prove the recursion relation of Motzkin sequence ,

$(n+2)m_n = (2d_n+d)\, m_{n-1} + (4c-d^2)(n-1)\, m_{n-2}$

Now , $b_{n+1,1} + \frac{1}{2}(nb_{n+1,1} - dnb_{n,1}) = 2c(n-1)b_{n-1,1} + dnb_{n,1} + \frac{1}{2}((n-1)b_{n,1} - d(n-1)b_{n-1,1}))$.

From(4) and (5) ,

$m_n + \frac{1}{2}(nm_n - dnm_{n-1}) = 2c(n-1)m_{n-2} + dm_{n-1} + d(\frac{1}{2}((n-1)m_{n-1} - d(n-1)b_{n-2}))$. (6)

Since , $m_n = b_{n+1} \Longrightarrow m_{n-1} = b_{n,1}$ and $m_{n-2} = b_{n-1,1}$.

Therefore (6) becomes

$2m_n + nm_n - dnm_{n-1} = 4c(n-1)\, m_{n-2} + 2dm_{n-1} + d(n-1)m_{n-1} - d^2\, (n-1)\, m_{n-2}$

$(n+2)m_n = 4c(n-1)m_{n-2} + 2dm_{n-1} + 2dnm_{n-1} - dm_{n-1} - d^{2(n-1)}M_{n-2}$

$= (2dn+2d - d)m_{n-1} + (4c-d^2)(n-1)m_{n-2} = d(2n+1)m_{n-1} + (4c-d^2)(n-1)m_{n-2}$

Hence, $(n+2)\, m_n = d(2n+1)m_{n-1} + (4c-d^2)(n-1)m_{n-2}$

Using M_n for m_n ; M_{n-1} for m_{n-1} and M_{n-2} for m_{n-2} we can write the recursion formula as:

$\mathbf{(n+2)\, M_n = d(2n+1)\, M_{n-1} + (4c - d^2)\, (n-1)\, M_{n-2}}$

Now for c = d= 1, such that for (1,1,1) Motzkin sequence,

$\mathbf{(n+2)\, M_n = (2n+1)\, M_{n-1} + 3(n-1)\, M_{n-2}}$

Here Wen-jin Woan has proved the general case where the Level step and Down step are colored. When we make the colors c and d equal to one we will get the sequence (1,1,1) of regular Motzkin numbers. We can now find the asymptotic property of the Motzkin numbers which says the ratio between two consecutive terms of the Motzkin numbers approaches 3 for large values of n.

proof

Let $S_n = \frac{M_n}{M_{n-1}} = \frac{d(2n+1)}{n+2} + \frac{(4c-d^2)n-1}{n+2}\frac{M_{n-2}}{M_{n-1}}$

$$\Longrightarrow S_{n-1} = \frac{M_{n-1}}{M_{n-2}} \Longrightarrow \frac{M_{n-2}}{M_{n-1}} = \frac{1}{S_{n-1}}$$

Let $a_n = \frac{d(2n+1)}{n+2}$ and $b_n = \frac{(4c-d^2)(n-1)}{n+2}$

$$\Longrightarrow S_n = a_n + \frac{b_n}{s_{n-1}}$$

$$= \left(2d - \frac{3d}{n+2}\right) + (4c-d^2)\left(1-\frac{3}{n+2}\right)\frac{1}{s_{n-1}}$$

If S_n has a limit k, then

$$\lim_{n\to\infty}(S_n) = \lim_{n\to\infty}\left[\left(2d - \frac{3d}{n+2}\right) + (4c-d^2)\left(1-\frac{3}{n+2}\right)\left(\frac{1}{S_{n-1}}\right)\right]$$

Here $\lim_{n\to\infty}(S_{n-1})$ is almost k

$$\Longrightarrow k = 2d + (4c-d^2)\left(\frac{1}{k}\right)$$

$$\Longrightarrow k^2 = 2dk + (4c-d^2)$$

$$\Longrightarrow k = \frac{2d\sqrt{4d^2+4(4c-4d^2)}}{2}$$, using quadratic formula

$$\Longrightarrow k = d+2\sqrt{c}$$

Hence the $\lim_{n\to\infty} \frac{M_n}{M_{n-1}}$ for the (1,1,1)- Motzkin sequence will be 3,

(when c = d= 1) *i.e.* $\lim_{n\to\infty} \frac{M_n}{M_{n-1}} = 1+2\sqrt{1} = 1+2 = 3$.

SECTION IV

5.1 **Generalizing Motzkin numbers Using k-trees**

Using the idea of enumeration of rooted ordered trees in which every vertex has out degree of at most two , we can obtain k-Motzkin numbers or the generalized Motzkin numbers. This can be done by using k-trees where k-tree is a generalization of ordered trees.

As Mahendra Jani and Melkamu Zeleke showed in [3], a k-tree is constructed from a single distinguished k-cycle by repeatedly gluing other k-cycles to existing ones along an edge. More than one cycle can be glued to a non-terminal or the internal edge. For two 3-cycles we have 3 alternative ways to glue.

Example

For k=3 and n=3 we have 11 3-cycles.

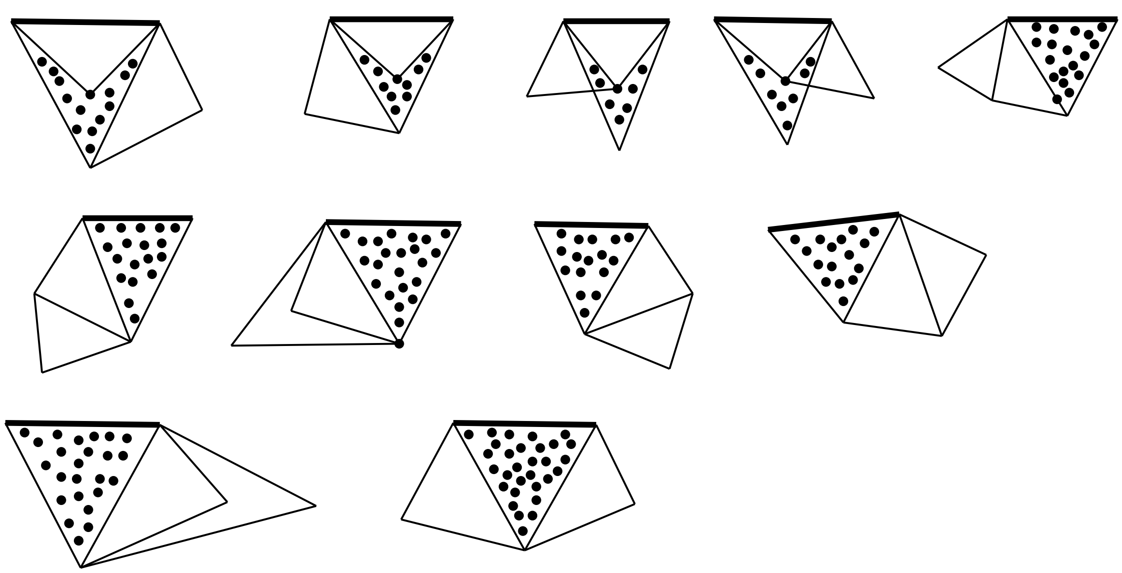


Fig.17

$$M_3^3 = 11$$

In each tree which is 3-tree, the degree of all edges is at most 2.

Therefore M_3^3 = # of 3-trees consisting of 3 3-cycls (three 3-cycles) in which every edge has out degree of at most two.

Generally M_n^k = # of k-trees consisting of n k-cycles in which every edge has out degree of at most two.

Ordered trees are 2-trees in which every edge between two vertices is drawn as a 2-cycle. This is why k-trees generalize ordered trees (rooted plane trees).

As we have seen in section I , the number of k-trees consisting of $n_i k_i$-cycles , for

i = 1, 2, 3, …, m is $C_n^k = \frac{1}{\sum n_i k_i + 1} \binom{\sum n_i k_i + 1}{n} \binom{n}{n_1, n_2 \ldots, n_m}$

where $n_1 + n_2 + \ldots + n_m = n$.

If all the cycles are of the same size say k, then

$$C_n^k = \frac{1}{kn+1} \binom{kn+1}{n}$$

$$= \frac{1}{kn+1} \left[\frac{(kn+1)!}{n!((kn+1)-n)!}\right]$$

$$= \frac{1}{(k-1)n+1} \binom{kn}{n}$$

Therefore, $C_k^n = \frac{1}{(k-1)n+1} \binom{kn}{n}$, the generalized Catalan number

Consider M_n^k as defined above, which is the generalized Motzkin number. The sequence $\{M_n^k\}$ enumerates the number of k-trees consisting of n k cycles for k≥2.

To find the formula for M_n^k , we have to use the generating function.

Let z be the edge of the tree. Then the generating function of k-Motzkin numbers which is denoted by $M(x)$ is defined as $M(z) = \sum_{n=0}^{\infty} M_n^k z^n$, for k≥2.

For k = 2, $\{M_n^k\}$ = 1, 1, 2, 4, 9, 21, 51, … which is the sequence of the regular Motzkin numbers, and its generating function is $M(z) = 1 + zM(z) + z^2 M^2(z)$.

This is from,

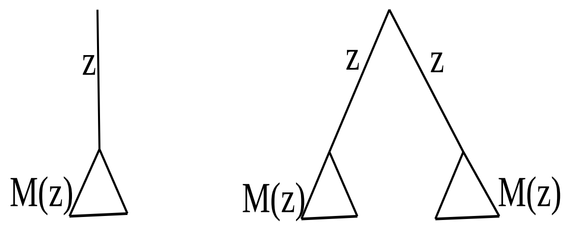


Fig.18

$M(z) = 1 + zM(z) + z^2M^2(z)$

(1 is for empty tree)

For the general case,

, and it satisfies the functional equation

$M(z) = 1 + zM^{k-1}(z) + z^2M^{2(k-1)}(z)$ where,

1 stands for the empty tree (a tree without an edge).

For regular Motzkin numbers, k = 2 and then $M(z) = 1 + zM(z) + z^2M^2(z)$.

For k=3,$M(z) = 1 + zM^2(z) + z^2M^4(z)$. Hence the generalized Motzkin number M_n^k can be found by using the general functional equation. Now we have to apply the Lagrange's Inversion Formula (LIE) here.

Let $M(z) = 1 + zM^{k-1}(z) + z^2M^{2(k-1)}(z)$

let $u(z) = zM^{k-1}(z)$

$\Longrightarrow M(z) = 1+u(z)+u^2(z)$

$\Longrightarrow u(z) = z(1+u+u^2)^{k-1}$

let $f(u) = 1+u+u^2$ and $\phi(u) = [f(u)]^{k-1}$.

Now applying the LIF, $M_n^k = [z^n] \{f(u(z))\}$

$$= \frac{1}{n} [u^{n-1}] \{f'(u)\, \phi^n(u)\}$$

$$= \frac{1}{n} [u^{n-1}] \{1+2u(1+u+u^2)^{(k-1)n})\}$$

Here we have two rules.

Rule 1. When $(1+u+u^2)^{(k-1)n}$ is multiplied by 1, we can get $\frac{1}{n}[u^{n-1}]\{1+2u(1+u+u^2)^{(k-1)n}\}$.

Rule 2. When $(1+u+u^2)^{(k-1)n}$ is multiplied by 2u, we can get $\frac{2}{n}[u^{n-2}]\{(1+u+u^2)^{(k-1)n}\}$.

These give us the disjoint cases and then we can use the summation formula.

That means,

$$M_n^k = [z^n]\{(f(u(z))\}$$

$$= \frac{2}{n}\sum_{n=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{(k-1)n}{i}\binom{(k-1)n-i}{n-1-2i} + \frac{2}{n}\sum_{i=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{(k-1)n}{i}\binom{(k-1)n-i}{n-2-2i}$$

For k = 2, $M_n^k = \frac{1}{n}\sum_{n=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{i}\binom{n-i}{n-1-2i} + \frac{2}{n}\sum_{i=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n}{i}\binom{n-i}{n-2-2i}$

which is equivalent to ,

$M_n = \frac{1}{n+1}\sum_{i=0}^{n}\binom{n+1}{i}\binom{n+1-i}{i+1}$, the regular n^{th} Motzkin number.

<u>Example</u>

Find M_4^3

Here using the general formula of Motzkin numbers,

$$M_4^3 = \frac{1}{4}\sum_{i=0}^{\lfloor\frac{3}{2}\rfloor}\binom{8}{i}\binom{8-i}{3-2i} + \frac{1}{2}\sum_{i=0}^{1}\binom{8}{i}\binom{8-i}{2-2i}$$

$$= \frac{1}{4}\left[\binom{8}{0}\binom{8}{3} + \binom{8}{1}\binom{7}{1}\right] + \frac{1}{2}\left[\binom{8}{0}\binom{8}{2} + \binom{8}{1}\binom{7}{0}\right]$$

$$= \frac{1}{4}[1.56 + 8.7] + \frac{1}{2}[1.28 + 8.1]$$

$$= \frac{1}{4}[56 + 56] + \frac{1}{2}[28 + 8]$$

$$= 28+18$$

$$= 46$$

Therefore, $M_4^3 = 46$, the 4^{th} term of 3 - Motzkin numbers.

We have the following table of generalized Motzkin numbers for different values of k and n. One can get such sequences of Motzkin numbers for various values of k on the OEIS (On Line

Encyclopedia of Integer Sequences) and can discover that 3–Motzkin numbers are the same as the sequence A006605 in OEIS. The entries of the sequence A006605 enumerate the number of modes of connections of 2n points as proposed by

R. Baxter.

n/k	0	1	2	3	4	5	6	7
2	1	1	2	4	9	21	51	127
3	1	1	3	11	46	207	979	4797
4	1	1	4	21	127	833	5763	41401
5	1	1	5	34	268	2299	20838	196326
6	1	1	6	50	485	5130	57391	667777

2. Motzkin Numbers and Baxter's generalization of Temperley - Lieb operators

Temperley and Lieb through their operators of wave functions, connecting 2n points in n disjoint pairs. For $i<s<j<t$, whenever x_i and x_j are connected for $i<j$, s and t can't be connected . This is a planarity condition which avoids the crossings of edges in graphing.

R. Baxter proposed a generalization of the Temperley - Lieb operators by allowing crossing and taking points from $X_{2n} = [2n] = \{1,2,\ldots,n,n+1,\ldots,2n\}$ in groups of four disjoint pairs as described in planarity condition, but whenever i and k are connected and j and l are connected for $i < j < k < l$, then s and t are connected for $s<t$ only if both s and t are in one of the four disjoint subsets.

Baxter divides the 2n points of X_{2n} in four disjoint subsets as

1, 2, 3,…, i, i+1,…, j, j+1,…, k, k+1,…, l, l+1,…,2n.

Then $S_1=\{x\in X_{2n} / x < i \text{ or } x > l\}$

$S_2=\{x\in X_{2n} / i< x < k\}$

$S_3 =\{x\in X_{2n} / j<x < k \}$

$S_4=\{x\in X_{2n} / k < x < l \}$

Let b_n be the number of modes of connections of X_{2n} which are permissible.

Such sequence $\{b_n\}$=1,1,3,11,46,… agrees with the sequence $\{M_3^3\}$ of the n 3-cycles of

3- trees for n = 0,1,2,3,… .

Theoreom: There is a one to one correspondence between the permissible modes of connections of X_{2n} and 3- Motzkin numbers.

Proof

Given a 3- Motzkin tree with n 3- cycles, label the edges 1,2,…,n,n+1,…,2n (excluding the distinguished edge) using the post order traversal. Here the distinguished edge is not in X_{2n}.

As R. Baxter stated, we can form the four disjoint subsets of X_{2n} as above.

$S_1=\{x\in X_{2n} / x < i \text{ or } x > l\}$

$S_2=\{x\in X_{2n} / i < x < k\}$

$S_3=\{x\in X_{2n} / j < x < k\}$

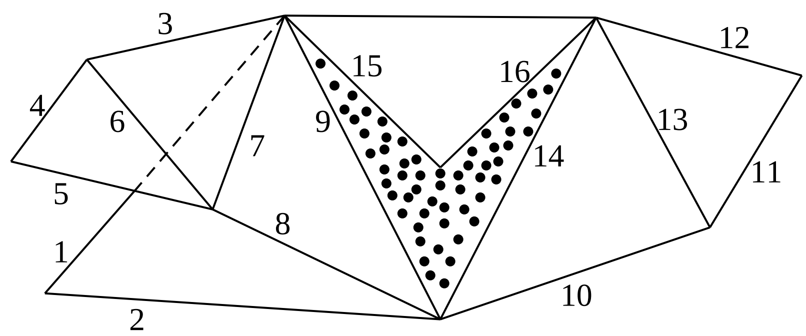
$S_4=\{x\in X_{2n} / k < x < l\}$

Then we can construct a labeled tree (say 3- tree) using the edges in each set. After the construction we have to split such constructed labeled tree into different sub trees using the following two important rules.

Rule1. Mappings of two cycles with common edge (the two cycles share common edge).

Rule2. Mappings of single cycles i.e. a single cycle is glued to an edge.

We can see the following labeled tree as an example.



After splitting into sub trees, we can form Baxter's generalization of the Temperley – Lieb operators by joining the labeled edges which are related in the original labeled tree

(i.e by writing the labels in order).

To obtain the inverse, consider the B To obtain the inverse, consider the Baxter's generalization given above and determine the corresponding sub trees consisting of a single cycle or two cycles. Here not more than two cycles to be glued on the distinguished edge, since the degree is at most 2 in case of Motzkin trees (trees which are enumerated by the Motzkin numbers). Here, first connect the distinguished edge of a sub tree with highest label i to a sub tree consisting of the edge labeled i+1.

Repeat the process until we obtain a connected k-tree containing all the n-cycles as the original labeled tree. Then we will get a labeled k-tree corresponding to the given labeled tree. Hence this shows the bijection between a permissible modes of connections of X_{2n} and the 3-Motzkin numbers.

Example

Consider a labeled 3-tree with X_{16} (n=8).

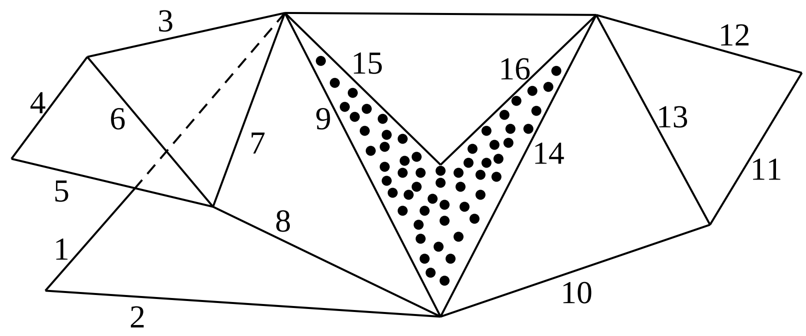


Fig.20

Using the above two rules we can carry out the mapping.

1. Consider the case of two cycles sharing a common edge. Group the edges depending on whether they are to the left or right of the common edge and allow crossing.

Fig.21

Mappings of two cycles with common edge.

Fig.22

Mappings of single cycles.

Putting the results in cases (1) and (2) we obtain Baxter's generalization of the Temperly –Lieb operators, which is corresponding to the labeled 3-tree given in Fig.20.

Fig.23

Baxter's generalization of Temperley-Lieb operators.

To obtain the inverse mapping, start with Baxter's generalization of Temperley –Lieb operators which are shown above in Fig.23. , and determine all the corresponding sub trees, consisting of a single cycle or two cycles.

Here are the sub trees.

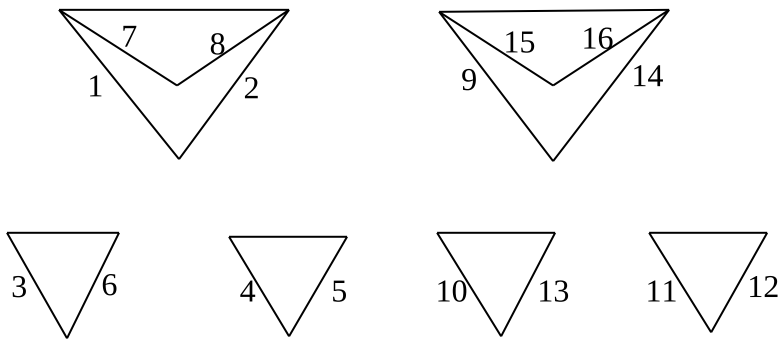


Fig.24.

Now, connect the distinguished edge of a sub tree with highest label i+1. Repeat the process until a connected labeled k-tree (3- tree) containing all the n cycles is obtained.

Here we have 8 cycles.

Fig.25

A labeled 3-tree corresponding to the labeled 3-tree in Fig. 20.

It is obtained by joining the two sub trees which are constructed in Fig.24.

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