ADDIS ABABA UNIVERSITY
COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES
DEPARTMENT OF MATHEMATICS

Quadratic optimal control

A Project Submitted to the Department of Mathematics in Partial Fulfillment of the Requirements for the degree of Master of Science in Mathematics

By: Kibremarkos Aklilu
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Advisor: Berhanu Guta(Ph.D)
Addis Ababa, Ethiopia

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Addis Ababa University
Department of Mathematics

The undersigned, hereby certify that they have read, recommend and examined this project, entitled **"Quadratic Optimal control "by Kibremarkos Aklilu** to the department of mathematics in partial fulfillment of the requirements for the degree of master of science in mathematics.

Advisor: Dr. Berhanu Guta

Signature:_____________________________
Date_____________________________

Examiners

1. Dr. __________________________

   Signature:_____________________________
   Date_____________________________

2. Dr. __________________________

   Signature:_____________________________
   Date_____________________________

ADDIS ABABA UNIVERSITY

Author : Kibremaros Aklilu

Title : Quadratic Optimal Control

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Kibremarkos Aklilu

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Abstract

Control theory is an area of applied mathematics that deals with principles,laws, and desire of dynamic systems.
Optimal control problems are generalized form of variation problems. A very important tool in variational calculus is the notion of Gateaux-differentiabilty.It is the basis of the development of necessary optimality conditions.
Euler lagrange differential equation(ELDE) is a necessary optimality condition to solve variational problems. The solution of Euler lagrange differential equation is an extremal function of a variational problem.
Characterizing theorem of convex optimization is the necessary and sufficient condition of many convex problems. i.e
Let (P) be given, S is convex set, f is convex function and $x_0 \in$ S. Then $x_0 \in$ M(f,S) if and only if $f'(x_0, x-x_0) \geq 0$, $\forall$ x$\in$ S.
In an optimal control problem our aim is to find the optimal state function $x^*(t)$ and the optimal control function $u^*(t)$ which optimize the objective functional in t$\in$[a,b] by using necessary and sufficient optimality conditions.
The necessary optimality conditions for (x^*, u^*) to be extremal solutions of optimal control problem is the validity of :- Pontryagin minimum principle, of ELDE with TR,and ODE conditions.
To determine whether the extremals are optimal solutions of OCP or not; sufficient optimality conditions are required; (e.g checking the convexity of the objective functional and the convexity of the feasible set).
Quadratic optimal control problem is a non linear optimization where the cost function is quadratic but the differential equation is linear.In quadratic control problem since the objective function is convex then the extremals are the optimal solution of the problem.
Linear-Quadratic optimal control problem is an important type of quadratic control problem that simplifies the work of feed back control system. Optimal control problem can be solved by different methods depending on the type of the problem.This paper mainly considers solving quadratic optimal control problem by using the method of lagrange multiplier.

Key words : Variational problem with fixed end points,Variational problem with free right end points,Euler Lagrange Differential Equation,optimal control problem,Quadratic control problem

List of notations

$\mathbb{R}^n$: the set of n dimensional Euclidean space

∇ f : gradient of real valued function f

∇^2f :The second gradient of real valued function f

$L_{\dot{x}}$: The partial derivative of L with respect to $\dot{x}$

$L_{\dot{x}\dot{x}}$: The second partial derivative of L with respect to $\dot{x}$

$\frac{\partial f}{\partial t}$: partial derivative of f with respect to t

$\frac{\partial^2 f}{\partial t}$: the second partial derivative of f with respect to t

C[a,b] The set of all continuous functions on [a,b].

$C^{(1)}$[a,b] : The set of all continuously differentiable functions on [a,b]

S[a,b] : The space of all piece wise continuous functions on [a,b]

S[a,b]^n :The space of all piece wise continuous functions on [a,b]^n

$RS[a,b]^n$: The space of all right sided piece wise continuous functions on $[a,b]^n$

$RCS^{(1)}$[a,b]^n : The space of all right sided piece wise differentiable functions on $[a,b]^n$

$\mathbb{R}^{n\times m}$: The set of n× m real matrices.

$RS[a,b]^{n\times m}$: The space of all right sided piece wise continuous fuctions on[a,b]$^{n\times m}$

M(f,s) : Minimizer of the functional f on the set S.

Contents

Introduction

Calculus of variation is a field of mathematical analysis that deals with maximizing or minimizing functionals,which are mapping from a set of functions to the real numbers. Optimal Control theory is an extension of Calculus of Variation.It is an area of applied mathematics that deals with principles,laws, and desire of dynamic systems. This theory is largely due to the work of Lev Pontryagin,after and Richard Bellman in 1950s after contribution to calculus of variation by Edward J.Msshane.

Optimal control problem (OCP) basically includes: objective functional and constraints.Objective functional expresses costs or benefits that can be optimized.Constraint are search spaces of decision(optimization) variables.

Optimal control problem can be the following type

Let $a, b \in \mathbb{R}$ and $g, \phi : \mathbb{R} \times \mathbb{R} \times [a, b] \to \mathbb{R}$ are continuous functions

$f(x, u) = \int_a^b g(x, u, t)dt \to Min$

$R = \{(x, u) : \dot{x} = \phi(x, u, t), t \in [a, b]\}$

$K = \{x \in C^{(1)}[a, b] : x(a) = \alpha\}$

$Q = \{u \in C[a, b] : u(t) \in U(t); t \in [a, b]\}$

In an optimal control problem our aim is to find optimal state function x$^*(t)$ and the optimal control function u$^*(t)$ which optimize the objective functional in t∈[a,b].

There are three kinds of necessary optimality conditions for(x^*, u^*) to be extremal solutions of an optimal control problem;these are Pontryagin minimum principle,validity of ELDE with TR,and ODE conditions.

To determine whether the extremals are optimal solutions of OCP or not sufficient optimality conditions are required; (e.g checking the convexity of the objective functional and the convexity of the feasible set).

This paper briefly tries to see how to solve **Linear optimal control problems** and **Quadratic optimal control problems**

Linear optimal control problem is an optimal problem where both the objective function and the differential equation are linear.

Quadratic optimal control problem is an optimal problem where the objective function is quadratic and the differential equation is linear.

Linear optimal control is widely applied in manufacturing firms and quadratic optimal control is applied in space science. This paper also focuses on a special case of the general non linear optimal control problem called **Linear Quadratic(LQ) optimal control problem**

This paper consists of three chapters.

chapter-1:-consists of definiteness of matrices, convex Analyses, necessary and sufficient optimality conditions to solve optimization problems.

Chapter-2:-includes variational problems with fixed end points,variational problem with free right end point.The chapter particularly focuses on how to find optimal solution of a Variational problem, using optimality conditions.And it also contains applications of variational calculus.

Chapter-3:- It is the main chapter of this paper which includes definitions and theorems on optimal control problems.This chapter focuses on how to find optimal control function and optimal state function for a given separated linear and quadratic optimal control problem by using Lagrange multiplier function.

Chapter 1

preliminary

1.1 Definiteness of a matrix

Definition 1.1.1. *A symmetric n×n real matrix M is said to be*
i)Positive semi definite if, $x^t Mx \geq 0$*, for every column vector* $x \in \mathbb{R}^n$*.*
ii)Positive definite if, $x^t Mx > 0$*,for every non zero column vector* $x \in \mathbb{R}^n$
OR
If all the eigen values of M are
i) positive then M is positive definite.
ii) Non negative then M is positive semi definite. Note: The converse of the above statements is also true.

1.2 Convexity of sets and functions

Definition 1.2.1. ***(Convex set)****: A subset K of* $\mathbb{R}^n$ *is said to be convex set*
if it satisfies $\lambda x + (1-\lambda)y \in K$*,* $\forall x,y \in K$*, for each* $\lambda \in [0,1]$
In other words : A set K in $\mathbb{R}^n$ *is said to be convex if every line*
intersects K in an interval containing K.

Example : Let A be m×n matrix and b$\in\mathbb{R}^n$,then the set
K = $\{X \in \mathbb{R}^n : AX \leq b\}$ is a convex set.

Definition 1.2.2. ***(Convex function)****: Let S be a non empty set in* $\mathbb{R}^n$*.*
The function $f:S\rightarrow \mathbb{R}$ *is said to be convex if*
$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \quad \forall x.y \in S, \quad$ *for each* $\lambda \in [0,1]$

In other words :If its epigraph(the set of points on or above the graph of the function) is a convex set. Example : f(x) = x^2 ,x$\in \mathbb{R}$ is a convex function.

1.3 Review on some notations of derivative

Let L : $\mathbb{R}^2 \to \mathbb{R}$ be partially differentiable with respect to x and y. Where x=x(t) and y=y(t).
then **total derivative** of L(x,y) with respect to t is

$$\frac{dL}{dt} = \frac{\partial L}{\partial x}\frac{dx}{dt} + \frac{\partial L}{\partial y}\frac{dy}{dt}$$

Proof: Let (x,y) be a point on the x-y plane,let $\triangle x$ and $\triangle y$ are the horizontal and vertical changes to x+△ x and y+△ y respectively ; and △ L be the change between (x,y) and (x+△ x, y+△ y).
△ L = L(x+△x,y+△y)-L(x,y)
$\triangle L = L(x + \Delta x, y + \Delta y) - L(x, y + \Delta y) + L(x, y + \Delta y) - L(x, y)$

$$\triangle L = L(x + \triangle x, y) - L(x, y) + L(x, y + \triangle y) - L(x, y)$$

$$dL = \lim_{\triangle x \to 0} \frac{L(x + \triangle x, y) - L(x, y)}{\triangle x}\triangle x + \lim_{\triangle y \to 0} \frac{L(x, y + \triangle y) - L(x, y)}{\triangle y}\triangle y$$

$$\Rightarrow: dL = L_x dx + L_y dy$$

$$Hence : \frac{dL}{dt} = \frac{\partial L}{\partial x}\frac{dx}{dt} + \frac{\partial L}{\partial y}\frac{dy}{dt}$$

Remark: If $L : \mathbb{R}^3 \to \mathbb{R}$,the total differential of L(x,y,z) with respect to t is

$$Hence : \frac{dL}{dt} = \frac{\partial L}{\partial x}\frac{dx}{dt} + \frac{\partial L}{\partial y}\frac{dy}{dt} + \frac{\partial L}{\partial z}\frac{dz}{dt}$$

Definition 1.3.1. ***Gateaux differentiability***
Let X and Y be vector spaces ,U be a normed space and U$\subseteq$ X .
More over let F:U$\to$ Y,$x_0 \in$ U and $z \in X$.The function F
is said to be Gateaux differentiable at x_0 in the direction of z
if and only if there exists an $\epsilon > 0$ such that [x_0-ϵz,x_0+ϵz]$\subset$ U and the limit

$$F'(x_0, z) = lim_{t\to 0}\frac{F(x_0 + tz) - F(x_0)}{t} exists$$

In this case F'(x_0,z) is called Gateaux differential of F at x_0 in the direction z

Definition 1.3.2. *F is said to be Gâteaux differentiable at x_0 if and only if F is Gateaux differentiable at x_0 in each direction of z i.e*
i) Right side directional derivative of F

$$F'_+(x_0, z) = \lim_{t\to 0^+}\frac{F(x_0 + tz) - F(x_0)}{t} \quad [x_0, x_0 + \epsilon z]$$

ii) Left side directional derivative of F

$$F'_-(x_0, z) = \lim_{t\to 0^-}\frac{F(x_0 + tz) - F(x_0)}{t} \quad [x_0 - \epsilon z, x_0]$$

Definition 1.3.3. *Consider the functional, $f : Y \to \mathbb{R}$, where $Y = C^1[a,b]$.*

The Gateaux derivative of f at $y_0 \in Y$ in the direction of $v \in Y$ is given by:

$$f'(y_0, v) = lim_{t\to 0}\frac{f(y_0 + tv) - f(y_0)}{t} = \frac{d}{dt}f(y_0 + tv)|_{t=0}$$

If f'(y_0, v) exists for all $v \in Y$, then the functional f((.)) is Gateaux differentiable at y_0 in the direction of v.

1.4 Optimality conditions

Definition 1.4.1. *Consider* ***unconstrained*** *optimization problem*
f(x)$\to$ Min subject to $x \in \mathbb{R}^n$ where f: $\mathbb{R}^n \to \mathbb{R}$.
A point x_0 is said to be a local minimizer of f, if there is a neighbourhood U of x_0 such that f(x_0) $\leq$ f(x) $\forall$ $x \in U$.
A point x_0 is said to be a global minimizer of f, if f(x_0) $\leq$ f(x), $\forall x \in \mathbb{R}^n$

Theorem 1.4.1. *(**Necessary optimality conditions**)*
i)First order necessary optimality condition
Let f be differentiable function over U and assume that x_0 is a local minimizer of f over U,then ∇ f(x_0)=0.
ii)Second order necessary optimality condition
Let f be twice differentiable function over U and assume that x_0 is a local minimizer of f over U. Then ∇ f(x_0)=0 and ∇^2f(x_0) $\geq$ 0.

Theorem 1.4.2. *(**Sufficient optimality conditions)** Let f be twice continuously differentiable function over U ,and let x_0 is a local minimizer of f over U and if, ∇ f(x_0)=0 and ∇^2f(x_0) $>$ 0 ,then x_0 is strict local minimizer of f.i.e f is locally convex over U*

Corollary 1.1. *If f is convex function then any local minimizer is a global minimizer.*

Definition 1.4.2. *Consider* ***constrained*** *optimization problem*
(P) f(x)$\to$ Min $x \in S$.
$x_0 \in S$ is said to be local minimizer of f on S,if there is a neighborhood U of x_0 such that $f(x_0) \leq f(x)$ for every $x \in U \cap S$

The following theorem gives a necessary and sufficient condition for a point to be an optimal point in the case of convex optimization.

Theorem 1.4.3. *(characterizing theorem of convex optimization)*
Let (P) be given ,S be convex set ,f be convex function and $x_0 \in S$. Then $x_0 \in$ M(f,S) $\Leftrightarrow$ f_+'(x_0,x - x_0) $\geq$ 0, $\forall$ $x \in S$.(It is a necessary and sufficient condition of convex optimization).

Chapter 2

Introduction to Variational calculus

Calculus of variation is a field of mathematical analysis that deals with maximizing or minimizing functionals,which are mapping from a set of functions to the real numbers.Functionals are often expressed as definite integrals involving functions and their derivatives.The aim is to find an extremal function at which the functional attain a maximum or minimum value,where the rate of change of the functional is zero.

2.1 Variational problems with fixed end points

Definition 2.1.1. *Let* $a, b, \alpha, \beta, \in \mathbb{R}$ *. Let,* $a < b, and, L : \mathbb{R} \times \mathbb{R} \times [a, b] \to \mathbb{R}$ *be continuous and continuously partially differentiable with regard to the first two components. Furthermore , Let f be a functional such that f :* $C^{(1)}[a, b] \to \mathbb{R}$*,where:*

$$f(y) = \int_a^b L(y(t), \dot{y}(t), t)dt \to \quad min \quad , y \in S$$

$$s.t : S = \{y \in C^{(1)}[a, b] : y(a) = \alpha, y(b) = \beta\}$$

then this type of optimization problem is called variational problem with fixed end points.

In the vector space C^1[a,b] ,we consider the linear sub space

$$V = \{v \in C^1[a, b] : v(a) = 0, v(b) = 0\}$$

And consider the function

$$i.e \quad x_0(t) = \frac{\beta - \alpha}{b - a}(t - a) + \alpha$$

Obviously we have S=x_0+V i.e the set S is linear manifold in C^1[a,b]

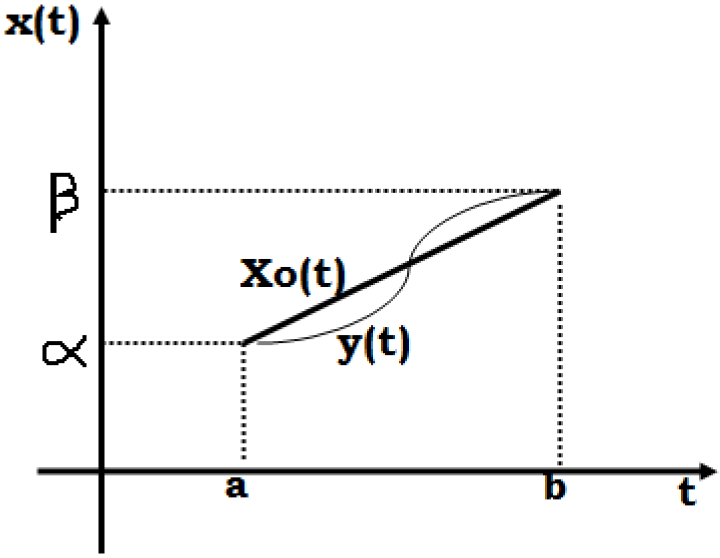

fig 2.1

The approach with respect to Gateaux derivative

Let S = $x_0 + V$ where S is a linear manifold in C^1[a,b].
A necessary condition for $y_0 \in$ S to be a minimum point of f on the set S is given on Theorem 1.4.3 as:

$$f'(y_0, v) = 0, \forall v \in V$$

Theorem 2.1.1. *Let a, b, α, β, ∈ ℝ . Let, a < b, and, L : ℝ × ℝ × [a, b] → ℝ be continuous and continuously partially differentiable with respect to the first two components.*

$$Let : f(y) = \int_a^b L(y(t), \dot{y}(t), t)dt \rightarrow \quad min \quad y \in S$$

$$S = \{y \in C^{(1)}[a, b] : y(a) = \alpha, y(b) = \beta\}$$

And let v be an element of V then the Gateaux derivative of f at $y_0 \in S$ in the direction of v is

$$f'(y_0, v) = \int_a^b (L_y v + L_{\dot{y}} \dot{v})dt$$

where $L_y = L_y(y_0, \dot{y}_0, t)$ and $L_{\dot{y}} = L_{\dot{y}}(y_0, \dot{y}_0, t)$

proof:Let y∈ S,such that; $\phi(\epsilon)$=f(y + ϵ v).

$$\phi'(\epsilon) = f'(y + \epsilon v)v = f'(y + \epsilon v, v)$$

$$\phi'(0) = f'(y, v)$$

$$\phi'(0) = \frac{d\phi}{d\epsilon}|_{\epsilon=0} = \int_a^b \frac{dL}{d\epsilon}|_{\epsilon=0} \tag{2.1}$$

$$f'(y, v) = \int_a^b \frac{d}{d\epsilon} L(y + \epsilon v, \dot{y} + \epsilon \dot{v}, t)dt.$$

$$f'(y,v) = \int_a^b (\frac{\partial L}{\partial y}\frac{d}{d\epsilon}(y+\epsilon v) + \frac{\partial L}{\partial \dot{y}}\frac{d}{d\epsilon}(\dot{y}+\epsilon\dot{v}) + \frac{\partial L}{\partial t}\frac{dt}{d\epsilon})dt.$$

$$f'(y,v) = \int_a^b (L_y v + L_{\dot{y}}\dot{v})dt$$

$$f'(y,v) = \int_a^b (L_y(y,\dot{y},t)v + L_{\dot{y}}(y,\dot{y},t)\dot{v})dt$$

If $y_0 \in$ S then

$$f'(y_0,v) = \int_a^b (L_y(y_0,\dot{y}_0,t)v + L_{\dot{y}}(y_0,\dot{y}_0,t)\dot{v})dt$$

$$f'(y_0,v) = \int_a^b (L_y v + L_{\dot{y}}\dot{v})dt$$

where $L_y = L_y(y_0,\dot{y}_0,t)$ and $L_{\dot{y}} = L_{\dot{y}}(y_0,\dot{y}_0,t)$

Corollary 2.1. *$Leta, b, \alpha, \beta, \in \mathbb{R}$. $Let, a < b, and, L : \mathbb{R} \times \mathbb{R} \times [a,b] \to \mathbb{R}$ be continuous and continuously partially differentiable with regard to the first two components.*

$$Let : f(y) = \int_a^b L(y(t),\dot{y}(t),t)dt \to \quad min \quad y \in s$$

$$S = \{y \in C^{(1)}[a,b] : y(a) = \alpha, y(b) = \beta\}$$

Let v be any vector in V,then the Gateaux derivative of f at $y_0 \in$ S, in the direction of v is

$$f'(y_0,v) = \int_a^b (L_y - \frac{d}{dt}L_{\dot{y}})v dt + L_{\dot{y}}v|_a^b, \forall v \in V.$$

where $L_y = L_y(y_0,\dot{y}_0,t)$ and $L_{\dot{y}} = L_{\dot{y}}(y_0,\dot{y}_0,t)$
Proof;

$$From, thm, 2.1.1, wehave; f'(y_0,v) = \int_a^b (L_y v + L_{\dot{y}}\dot{v})dt = \int_a^b L_y v dt + \int_a^b L_{\dot{y}}\dot{v}dt$$

Using integration by parts on the integral $\int_a^b L_{\dot{x}}\dot{v}dt$ we get

$$\int_a^b L_{\dot{y}}\dot{v} = L_{\dot{y}}v|_a^b - \int_a^b \frac{d}{dt}L_{\dot{y}}v dt$$

$$Hence : f'(y_0,v) = \int_a^b L_x v dt + L_{\dot{y}}v|_a^b - \int_a^b \frac{d}{dt}L_{\dot{y}}v dt$$

$$Finally : f'(y_0,v) = \int_a^b (L_y - \frac{d}{dt}L_{\dot{y}})v dt + L_{\dot{y}}v|_a^b$$

$$f'(y_0,v) = \int_a^b (L_y - \frac{d}{dt}L_{\dot{y}})v dt + L_{\dot{y}}v|_a^b, \forall v \in V$$

where $L_y = L_y(y_0,\dot{y}_0,t)$ and $L_{\dot{y}} = L_{\dot{y}}(y_0,\dot{y}_0,t)$.

Fundamental Lemma of Variational Problems

Lemma 2.1.1. *If f is contiuous function on an open interval (a,b) and h is a smooth function on (a,b) which compactly supports f,such that*

$$\int_a^b f(x)h(x)dx = 0, then, f(x) = 0.$$

2.1.1 Optimality condition for VP-with fixed end points

Theorem 2.1.2. *If $y_0 \in S$ is a solution of variational problem with fixed end points, and If f is Gateaux-differentiable at y_0 in the direction of v, ∀ v∈ V*
Then $f^{'}(y_0, v)$ = 0,for all v∈ V.

proof:
Since V is a sub space of $C^1[a, b]$, then v∈ V ⇒ -v∈ V.

$$f'(y_0, -v) = lim_{t\to 0}\frac{f(y_0 + t(-v)) - f(y_0)}{t}.$$

$$f'(y_0, -v) = lim_{t\to 0}\frac{f(y_0 - tv) - f(y_0)}{t}.$$

$$f'(y_0, -v) = lim_{t\to 0}\frac{f(y_0 + \lambda v) - f(y_0)}{-\lambda}.(where - t = \lambda)$$

$$f'(y_0, -v) = -lim_{t\to 0}\frac{f(y_0 + \lambda v) - f(y_0)}{\lambda}.$$

$$f'(y_0, -v) = -f'(y_0, v)$$

If $y_0 \in$ S is a minimizer of f over S;then
⇒ $f(y_0) \leq f(y_0 + tv), \forall v \in V$; where $y_0 + tv$ is admissible variation of y_0
⇒ $f(y_0 + tv) - f(y_0) \geq 0$
⇒ $\frac{f(y_0+tv)-f(y_0)}{t} \geq 0$
⇒ $\lim_{t\to 0} \frac{f(y_0+tv)-f(y_0)}{t} \geq 0$
⇒ $f'(y_0, v) \geq 0$ --→ (∗)
Since V is a subspace of $C^1[a, b]$,then v∈ V ⇒ -v∈ V
⇒ $f'(y_0, v) \geq 0$
⇒ $f'(y_0, -v) \geq 0$
⇒ $-f'(y_0, v) \geq 0$
⇒ $f'(y_0, v) \leq 0$ --→ (∗∗)
From (*) and (**); we have $f'(y_0, v) = 0$, $\forall v \in V$

2.2 The Euler Lagrange Differential Equation

If $y_0 \in$ S is a solution of a VP-fixed end points,then

$$f'(y_0, v) = 0, \forall v \in V$$

$$\Rightarrow \quad \int_a^b (L_y v + L_{\dot{y}}\dot{v})dt = 0 \quad \forall t \in [a,b]$$

$$\Rightarrow \int_a^b (L_y - \frac{d}{dt}L_{\dot{y}})vdt + L_{\dot{y}}v|_a^b = 0$$

$$\Rightarrow \int_a^b (L_y - \frac{d}{dt}L_{\dot{y}})vdt + L_{\dot{y}}(y(t),\dot{y}(t),t)v(t)|_a^b = 0$$

Since v(a)=0 and v(b)=0 we have

$$\int_a^b (L_y - \frac{d}{dt}L_{\dot{y}})vdt = 0$$

$$\Rightarrow (L_y - \frac{d}{dt}L_{\dot{y}})v = 0$$

Let v(t)=h(t)(t-a)(b-t) $>$ 0 where h $\in$ $C^{(1)}$[a,b]

$$(L_y - \frac{d}{dt}L_{\dot{y}}) = 0 \dashrightarrow (ELDE).$$

A solution of ELDE is the extremal function of the variational problem.

Theorem 2.2.1. ***(Necessary optimality condition***
Let a,b,α,β $\in$ R ,a<b and L be continuously partially
differentiable with regard to the first two variables
Consider the variational problem

$$f(y) = \int_a^b L(y,\dot{y},t)dt \to \quad min$$

$$S = \{y \in C^{(1)}[a,b] : y(a) = \alpha, y(b) = \beta\}$$

Let y_0 be a solution of the variational problem.
The necessary condition for y_0 to be extremal solution of the variational problem is that y_0 should satisfy the ELDE given by

$$\frac{d}{dt}L_{\dot{y}}(y_0(t),\dot{y}_0(t),t) = L_y(y_0(t),\dot{y}_0(t),t) \quad \forall t \in [a,b] \quad (ELDE)$$

Definition 2.2.1. *(Local minimum points) Let $S = x_0 + v$ and $y_0 \in S$. A point y_0 is called algebraically local minimum point of f on S if and only if for each v$\in$ V there exists an $\epsilon > 0$ such that y_0 is a minimum point of f on $[y_0 - \epsilon v, y_0 + \epsilon v]$.Then we have the following theorem*

Theorem 2.2.2. *Let a,b,α,β $\in$ R ,a<b,L : $\mathbb{R} \times \mathbb{R} \times$ [a,b]$\to$ $\mathbb{R}$ and L be continuously partially differentiable with regard to the first two variables. Further more Let*

$$f(y) = \int_a^b L(y(t),\dot{y}(t),t)dt \to \quad min \quad y \in s$$

$$S = \{y \in C^{(1)}[a,b] : y(a) = \alpha, y(b) = \beta\}$$

If y_0 is algebraically a local minimum point f on S then y_0 satisfies

$$\frac{d}{dt}L_{\dot{y}}(y_0(t), \dot{y}_0(t), t) = L_y(y_0(t), \dot{y}_0(t), t) \quad \forall t \in [a, b] \quad (ELDE)$$

2.3 Convex variational problems

In the case of convexity we are able to formulate also a sufficient condition for y_0 to be a minimum point. Let L:$\mathbb{R}\times\mathbb{R}\times$[a,b]$\rightarrow\mathbb{R}$,
be continuous and continuously partially differentiable with regard to the first two components. Let

$$f(y) = \int_a^b L(y, \dot{y}, t)dt$$

$$S = \{y \in C^{(1)}[a, b] : y(a) = \alpha, y(b) = \beta\}$$

Let the function L(y,$\dot{y}$,t) : $\mathbb{R}\times\mathbb{R}\times$[a,b] $\rightarrow \mathbb{R}$ be convex for all t $\in$ [a,b]. Then we will say that L is convex with regard to the first two components.Now let's show that f is convex . Since we have x,y $\in$ $C^{(1)}$[a,b] and
Let :

$$f(x) = \int_a^b L(x, \dot{x}, t)dt$$

And :

$$f(y) = \int_a^b L(y, \dot{y}, t)dt$$

$$L(\lambda x + (1-\lambda)y, \lambda\dot{x} + (1-\lambda)\dot{y}, t) \leq \lambda L(x, \dot{x}, t) + (1-\lambda)L(y, \dot{y}, t)$$

Taking the integrals on both sides

$$\int_a^b L(\lambda x + (1-\lambda)y, \lambda\dot{x} + (1-\lambda)\dot{y}, t) \leq \lambda \int_a^b L(x, \dot{x}, t) + (1-\lambda) \int_a^b L(y, \dot{y}, t)$$

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

Which follows that f is convex.

Proposition 2.1. ***(Necessary and sufficient optimality conditions)***
Let L be continuous and continuously partially differentiable with regard to the first two components and y_0 be a solution of the (VP)

$$f(y) = \int_a^b L(y, \dot{y}, t)dt \dashrightarrow Min$$

$$S = \{y \in C^{(1)}[a, b] : y(a) = \alpha, y(b) = \beta\}$$

i)If the ELDE condition holds and
ii)L is convex with regard to the first two components for all t in [a,b](i.e If the Heissiean matrix of L with regard to the first two components is positive semi definite for all t in [a,b]).

Then y_0 is optimal solution of the problem.

Theorem 2.3.1. *Let L be convex with regard to the first two components for all $t \in [a,b]$ .Then $y_0 \in S$ is a minimum point of f on S if and only if y_0 is a solution of ELDE.*

proof(i): Let y_0 be a minimum point of f on S . Then by the Thorem 2.2.1 ELDE holds

Proof(ii) : Let now the ELDE be true for y_0.

$$Let, L_y = L_y(y_0, \dot{y_0}, t), and$$

$$L_{\dot{y}} = L_{\dot{y}}(y_0, y_0, t) \quad t \in [a, b]$$

Then from the validty of ELDE we have

$$L_y - \frac{d}{dt}L_{\dot{y}} = 0$$

$$For \quad v \in V = \{v \in C^{(1)}[a, b] : v(a) = 0 \quad , v(b) = 0\}$$

$$0 = \int_a^b (L_y - \frac{d}{dt}L_{\dot{y})})vdt$$

$$\Rightarrow 0 = \int_a^b (L_y - \frac{d}{dt}L_{\dot{y}})vdt + L_{\dot{y}}v|_a^b$$

$$\Rightarrow \quad 0 = \int_a^b (L_x v + L_{\dot{x}}\dot{v})dt$$

$$\Rightarrow \quad 0 = f'(y_0, v)$$

So we have $f'(y_0,v) = 0 \geq 0 \quad \forall\ v \in V$

If we set $y = y_0 + v$ then $v=y-y_0$ there fore we get

$f'(y_0,y-y_0) \geq 0 \quad \forall\ y \in S$

By the characterizing theorem of convex optimization ,we have that y_0 is a minimum point of f on S.

Example : Find an extremal of the following variational problem.And check that whether the extremal is an optimal solution or not

$$Let f(x) = \int_0^1 (x^2(t) + \dot{x}^2(t))dt \rightarrow min \quad x \in S$$

$$s.t \quad S = \{x \in C^{(1)}[0, 1] : x(0) = 0 \quad x(1) = 1\}$$

$$Solution : \quad L(x, \dot{x}, t) = x^2 + \dot{x}^2 \quad then \quad L_x = 2x, and, L_{\dot{x}} = 2\dot{x}$$

Let $x \in S$ be an extremal of the given VP ,then x should satisfy the ELDE i.e

$\frac{d}{dt}L_{\dot{x}} = L_x$

$\Leftrightarrow \frac{d}{dt}\ (2\dot{x}) = 2x$

$\Leftrightarrow$ 2$\ddot{x}$ = 2x

$\Leftrightarrow$ $\ddot{x}$ - x = 0

$$Let \quad x = e^{\lambda t}, \quad \dot{x} = \lambda e^{\lambda t}, \quad \ddot{x} = \lambda^2 e^{\lambda t}$$

$$\lambda^2 e^{\lambda t} - e^{\lambda t} = 0$$

$$e^{\lambda t}(\lambda^2 - 1) = 0$$

$$Since \quad e^{\lambda t} > 0, \quad \lambda^2 - 1 = 0$$

$$Hence \quad \lambda = -1 \quad or \quad \lambda = 1$$

Therefore the general solution of $\ddot{x}$ - x = 0 is

x(t) = c_1e^t + c_2e^{-t}

From the initial conditions x(0) = c_1 + c_2 = 0 and x(1) = c_1e + c_2e^{-1} = 1 we get the only solution of ELDE to be

$$x(t) = \frac{e}{e^2-1}e^t - \frac{e}{e^2-1}e^{-t}$$

$$Moreover \quad from \quad L(x, \dot{x}, t) = x^2 + \dot{x}^2$$

the Heissian matrix H of L with respect to (x, $\dot{x}$)

$$\nabla^2_{x,}L = \begin{pmatrix} L_{xx} & L_{x\dot{x}} \\ L_{\dot{x}x} & L_{\dot{x}\dot{x}} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

So for v $\in$ X and y$\in$ $\begin{pmatrix} v \\ \dot{v} \end{pmatrix}$

$$y^T(\nabla^2_x L)y = \begin{pmatrix} v & \dot{v} \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} v \\ \dot{v} \end{pmatrix} = 2(v^2 + \dot{v}^2) > 0$$

So the Hessian matrix of L with respect to x and $\dot{x}$ is positive definite

Hence the objective function is convex with regard to x,$\dot{x}$,$\forall$ t $\in$ [a,b].

There fore the extremal x(t) = $\frac{e}{e^2-1}$ (e^t - e^{-t}) of

the given variational problem is the optimal solution.

2.4 Some applications of variational problems with fixed end points

1) **Arc length of a curve**

In x-t plane suppose X = x(t) is the curve joining two points p_1 and p_2

where $a \leq t \leq b$ then the arc length of the curve is given as:

s = $\int_a^b(\sqrt{1 + \dot{x}^2(t)})dt$

Proof; Take a very small portion $\triangle$ S from a curve as a hypothenuse of a right angled triangle so that Pythagoras theorem holds i.e.

$(\triangle x)^2 + (\triangle t)^2 = (\triangle s)^2$
$\Rightarrow \frac{(\triangle x)^2}{(\triangle t)^2} + \frac{(\triangle t)^2}{(\triangle t)^2} = \frac{(\triangle s)^2}{(\triangle t)^2}$
$\Rightarrow \frac{(\triangle x)^2}{(\triangle t)^2} + 1 = \frac{(\triangle s)^2}{(\triangle t)^2}$
As $\triangle t \to$ o,then
$\Rightarrow (\frac{ds}{dt})^2 = (\frac{dx}{dt})^2 + 1$
$\Rightarrow \frac{ds}{dt} = \sqrt{(\frac{dx}{dt})^2 + 1}$
$\Rightarrow \frac{ds}{dt} = \sqrt{\dot{x}^2 + 1}$
$\Rightarrow ds = \sqrt{(\dot{x}^2 + 1)}dt$
$\Rightarrow s = \int_a^b \sqrt{(\dot{x}^2 + 1)}dt$

Example(1) Let S $=\{x \in C^{(1)}[a,b] : x(a) = \alpha, x(b) = \beta\}$ be the set of all smooth curves joining p_1 to p_2 in x-t plane among all curves joining p_1 to $p_2$in; find a curve x^* whose length is minimal.
Solution: Let's formulate a variational problem:
f(x) $= \int_a^b \sqrt{(\dot{x}^2 + 1)}dt \to min, x \in S$
S.t : S $=\{x \in C^{(1)}[a,b] : x(a) = \alpha, x(b) = \beta\}$
Let x^* be a solution of the variational problem,then it satisfies ELDE
But,From :$L(x, \dot{x}, t) = \sqrt{\dot{x}^2 + 1}$
$\Rightarrow L_x = 0$
$\Rightarrow L_{\dot{x}} = \frac{1}{2\sqrt{(\dot{x}^2+1)}}2\dot{x}$
$\Rightarrow L_{\dot{x}} = \frac{\dot{x}}{\sqrt{(\dot{x}^2+1)}}$
And $L_{\dot{x}\dot{x}} = \frac{1}{(\dot{x}^2+1)^{3/2}}$
From ELDE we have : $\frac{d}{dt}L_{\dot{x}} = L_x$
i.e:$\frac{d}{dt}L_{\dot{x}}(x, \dot{x}, t) = L_x(x, \dot{x}, t)$
But; $\frac{d}{dt}L_{\dot{x}}(x, \dot{x}, t) = \frac{\partial L_{\dot{x}}}{\partial x}\frac{dx}{dt} + \frac{\partial L_{\dot{x}}}{\partial \dot{x}}\frac{d\dot{x}}{dt} + \frac{\partial L_{\dot{x}}}{\partial t}\frac{dt}{dt}$;using total differential
$\Rightarrow \frac{d}{dt}L_{\dot{x}}(x, \dot{x}, t) = (L_{\dot{x}x})\dot{x} + (L_{\dot{x}\dot{x}})\ddot{x} + L_{\dot{x}t}$
So we have :$(L_{\dot{x}x})\dot{x} + (L_{\dot{x}\dot{x}})\ddot{x} + L_{\dot{x}t} = L_x$ (it is second order ODE)
This implies:$(0)\dot{x} + \frac{1}{(\dot{x}^2+1)^{3/2}}\ddot{x} + 0 = 0$
$\Rightarrow \ddot{x} = 0$
$\Rightarrow \frac{d^2x}{dt^2} = 0$
$\Rightarrow \frac{d}{dt}(\frac{dx}{dt}) = 0$
$\Rightarrow \int d(\frac{dx}{dt}) = \int 0dt$
$\Rightarrow \frac{dx}{dt} = c_1$
$\Rightarrow \int dx = \int c_1dt$
$\Rightarrow x(t) = c_1x + c_2$
From,initial conditions we have:
$x(a) = c_1a + c_2 = \alpha$ and $x(b) = c_1b + c_2 = \beta \Rightarrow c_1 = \frac{\beta-\alpha}{b-a}$ --→(slope of a line)
From equation of a line :$\frac{x-\alpha}{t-a} = \frac{\beta-\alpha}{b-a}$
$\Rightarrow x^*(t) = \frac{\beta-\alpha}{b-a}(t-a) + \alpha$ is the only solution of the variational problem
Example(**Solid revolution where the surface area is minimal)**
Given the real numbers a,b,$\alpha, \beta, where, a > b, \beta > \alpha$
Let Let S $=\{x \in C^{(1)}[a,b] : x(a) = \alpha, x(b) = \beta\}$ be the set of all smooth curves joining p_1

and p_2

Let f(x) be the lateral surface area of a solid obtained by rotating a region bounded by the curve X = x(t) and the t axis,about the t-axis in the interval $a \leq t \leq b$.

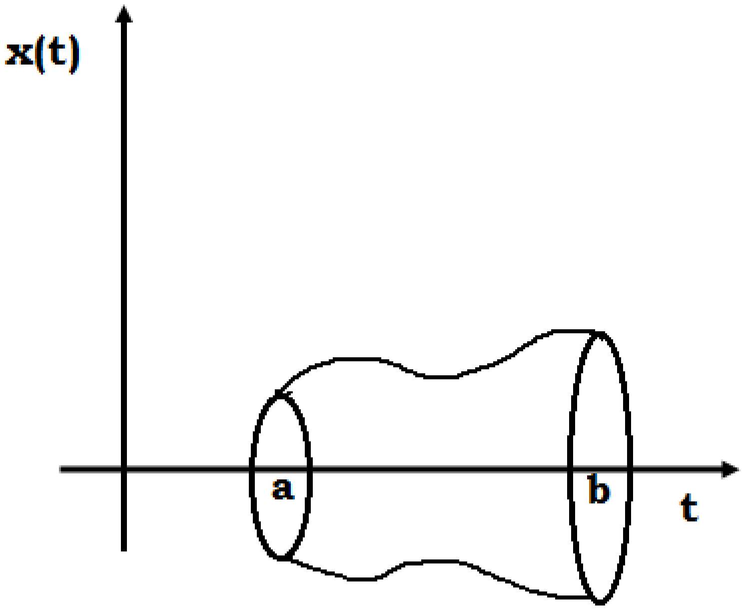

fig 2.2

So among all curves joining p_1 to p_2 to find the curve x^* so that the resulting surface area is minimal

. Let's consider a very small cylindrical portion from the surface

Such that : $\triangle A = 2\Pi r \triangle$h

$\Rightarrow \triangle A = 2\Pi x \triangle s$

$\Rightarrow dA = 2\Pi x ds$

$\Rightarrow dA = (2\Pi x)\sqrt{(1+\dot{x}^2)}$dt

So the total lateral surface area of the solid is

$\Rightarrow A = \int_a^b (2\Pi x)\sqrt{(1+\dot{x}^2)}$dt

So let's formulate the variational problem

$f(x) = \int_a^b (2\Pi x)\sqrt{(1+\dot{x}^2)}$dt $\dashrightarrow Min, x \in$ S

S.t S $=\{x \in C^{(1)}[a,b] : x(a) = \alpha, x(b) = \beta\}$

Solution : Let $L(x,\dot{x},t) = (2\Pi x)\sqrt{(1+\dot{x}^2)}$

$L_x = 2\Pi\sqrt{1+\dot{x}^2}$

$L_{\dot{x}} = \frac{2\Pi x\dot{x}}{\sqrt{1+\dot{x}^2}}$

$L_{\dot{x}\dot{x}} = \frac{2\Pi x}{(1+\dot{x}^2)^{3/2}}$

$L_{\dot{x}x} = \frac{2\Pi\dot{x}}{\sqrt{1+\dot{x}^2}}$

Let x^* be the extremal of the variational problem,then x^* should satisfy the 2^{nd} order ODE : i.e$(L_{\dot{x}x})\dot{x} + (L_{\dot{x}\dot{x}})\ddot{x} + L_{\dot{x}t} = L_x$

$\Rightarrow \frac{2\Pi x}{(1+\dot{x}^2)^{3/2}}\ddot{x} + \frac{2\Pi\dot{x}}{\sqrt{1+\dot{x}^2}}\dot{x} + 0 = 2\Pi\sqrt{1+\dot{x}^2}$

$\Rightarrow (x)(\ddot{x}) = 1 + \dot{x}^2$

$\Rightarrow \frac{d^2x}{dt^2}(x) - (\frac{dx}{dt})^2 - 1 = 0$

So, the solution is x(t)=$c_1 cosh(\frac{t}{c_1} + c_2), t \in [a,b]$,with,x(a)=$\alpha$,and,x(b)=$\beta$

2.5 Variational problems with free end points

Let L be continuous and continuously partially differentiable with regard to the first two components .then we conceder the following Variational Problem With one free end point:

$$f(y) = \int_a^b L(y(t), \dot{y}(t), t)dt \rightarrow \quad min \quad y \in R$$

$$R = \{y \in C^{(1)}[a, b] : y(a) = \alpha\}$$

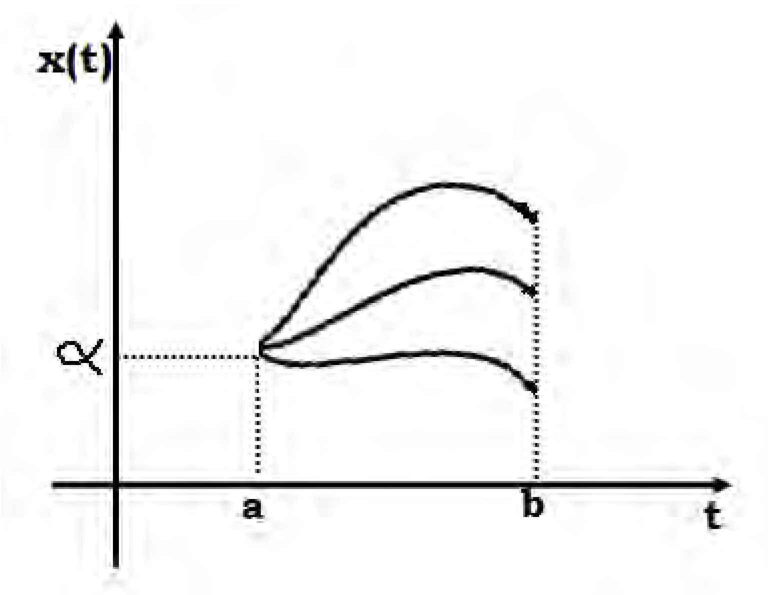

fig 2.3

The **value** of left boundary point is determined,but the **value** of the right boundary point is free. Trivially R is a linear manifold in the vector space $C^{(1)}$[a,b]
In order to find necessary conditions for the solution of the problem.We suppose that we have already found a solution $y_0(t)$ of (P)

$$Let \quad S = \{y \in R : y(b) = y_0(b)\} \subseteq R$$

i.e :R $=\{y \in C^{(1)}[a, b] : y(a) = \alpha, y(b) = y_0(b)\}$
If y_0 is a minimum point of f on S (as in the VP with fixed end points)
Then ELDE holds
i.e$\frac{d}{dt} L_{\dot{y}}(y_0, \dot{y}_0, t) = L_y(y_0, \dot{y}_0, t)$.
is a necessary condition for y_0 to be a minimum point of f on S. More over the continuity of L_y implies the continuity of $\frac{d}{dt}$ $L_{\dot{y}}$ .We consider the subspace

$$W = \{w \in C^{(1)}[a, b] : w(a) = 0\}$$

As in section 2.1 for y_0 to be a minimum point of f on R then following equation must hold

$$0 = f'(y_0, w) = \int_a^b (L_y w + L_{\dot{y}} \dot{w})dt = \int_a^b (L_y - \frac{d}{dt} L_{\dot{y}})wdt + L_{\dot{y}} w|_a^b$$

$$Since \quad L_y - \frac{d}{dt} L_{\dot{y}} = 0$$

$$\Rightarrow L_{\dot{x}}w|_a^b = 0$$

$$\Rightarrow L_{\dot{x}}w|_a^b = 0$$

$$\Rightarrow \quad L_{\dot{y}}(y_0(b), \dot{y}_0(b), b)w(b) - L_{\dot{y}}(y_0(a), \dot{y}_0(a), a)w(a) = 0$$

$$since \quad w(a) = 0 \Rightarrow L_{\dot{y}}(y_0(b), \dot{y}_0(b), b)w(b) = 0$$

$$Since \quad w(b) > 0, \forall w \in W \quad then, L_{\dot{y}}(y_0(b), \dot{y}_0(b), b) = 0 \quad (TR)$$

We call this condition transversality condition(**TR**)

Theorem 2.5.1. ***(Necessary optimality Conditions)***
Let L be continuous and continuously partially differentiable with regard to the first two components and y_0 be a solution of the (VP).

$$f(y) = \int_a^b L(y(t), \dot{y}(t), t)dt \rightarrow \quad min \quad y \in R$$

$$R = \{y \in C^{(1)}[a, b] : y(a) = \alpha\}$$

$$Then \quad 1)\frac{d}{dt}L_{\dot{y}}(y_0(t), \dot{y}_0(t), t) = L_y(y_0(t), \dot{y}_0(t), t) \quad \forall t \in [a, b] \quad (ELDE)$$

$$2)L_{\dot{y}}(y_0(b), \dot{y}_0(b), b)w(b) = 0 \qquad (TR)$$

Theorem 2.5.2. ***Necessary and sufficient optimality conditions)***
Let L be continuous and continuously partially differentiable with regard to the first two components and y_0 be a solution of the (VP)

$$f(y) = \int_a^b L(y(t), \dot{y}(t), t)dt \rightarrow \quad min \quad y \in R$$

$$R = \{y \in C^{(1)}[a, b] : y(a) = \alpha\}$$

i)If the ELDE and TR conditions hold and
ii)L is convex with regard to the first two components for all t in [a,b](i.e If the Heissiean Matrix of L with regard to the first two components is positive semi definite for all t in [a,b]).
Then y_0 is optimal solution of the problem.

Remark:Let L is convex with regard to the first two components for all t in [a,b].Then y_0 is a minimum point of f on R if and only if the above ELDE and TR conditions are satisfied.

$$Example : f(x) = \int_0^1 (x^2(t) + \dot{x}^2(t) + 2x(t))dt \dashrightarrow min \quad , x \in R$$

$$R = \{x \in C^{(1)}[0, 1] : x(0) = 0\}$$

Then we have ; L(x,$\dot{x}$,t) = $x^2 + \dot{x}^2 + 2x$, then; L_x=2x+2 and $L_{\dot{x}} = 2\dot{x}$
For x in R to be an extremal of the given VP then x should satisfy the ELDE
i.e $\frac{d}{dt}$ $L_{\dot{x}} = L_x \Leftrightarrow \frac{d}{dt} 2\dot{x} = 2x + 2 \Leftrightarrow 2\ddot{x} = 2x + 2 \Leftrightarrow \ddot{x} - x = 1$

To solve the homogeneous part $\ddot{x}$ - x = 0, Let x = $e^{\lambda t}$ be an assumed solution.So we have λ^2 $e^{\lambda t}$ - $e^{\lambda t}$ =0 $\Leftrightarrow$ λ^2 - 1 = 0, so λ = -1 or λ = 1
And since the particular solution is x_p(t)=-1 then the general solution of $\ddot{x}$ - x = 1 is
x(t) = c_1 e^t + c_2 e^{-t} - 1
From Transversality condition:$L_{\dot{x}}$ (x(b),$\dot{x}$(b),b)=0
For b=1 we have $L_{\dot{x}}$ (x(1),$\dot{x}$(1),1)=0
This implies $2\dot{x}(1) = 0$
So $\dot{x}(1) = 0$
From the initial condition x(0) = -1 + c_1 + c_2 = 0
From TR condition $\dot{x}$ (1) = c_1 e + c_2 e^{-1} = 0
We have $c_1 = \frac{1}{1+e^2}$ and $c_2 = \frac{1}{1+e^{-2}}$
There fore the extremal of the variational problem is
x(t) = -1 + $\frac{1}{1+e^2}$ e^t + $\frac{1}{1+e^{-2}}$ e^{-t}

$$More \quad over \quad from \quad L(x,\dot{x},t) = x^2 + \dot{x}^2 + 2x$$

the Heissian matrix H of L with respect to x $\dot{x}$

$$\nabla_x^2 L = \begin{pmatrix} L_{xx} & L_{x\dot{x}} \\ L_{\dot{x}x} & L_{\ddot{x}} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

So for v $\in$ X and y $\in$ $\begin{pmatrix} v \\ \dot{v} \end{pmatrix}$

$$y^T(\nabla_x^2 L)y = \begin{pmatrix} v & \dot{v} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} v \\ \dot{v} \end{pmatrix} = 2(v^2 + \dot{v}^2) > 0$$

So the Hessian matrix of L with respect to x and $\dot{x}$ is positive definite
Hence the objective function L with regard to x,$\dot{x}$ is convex ,$\forall$ t $\in$ [a,b].
There fore the extremal x(t) =-1 + $\frac{1}{1+e^2}$ e^t + $\frac{1}{1+e^{-2}}$ e^{-t} of
the given varational problem is th optimal solution.

2.6 Special Bolza and Mayer problem

Let G,H: $\mathbb{R}^n \to \mathbb{R}$ be frechet differentiable functions. Let L: $\mathbb{R}^n \times \mathbb{R}^n \times [a,b] \to \mathbb{R}$ be continuous function.Then the following class of variational problem is called special Bolza Problem.
f(x) = G(x(a)) + H(x(b)) + $\int_a^b$ L(x,$\dot{x}$,t) dt--→ Min, x is in S $\subseteq$ $C^{(1)}$ $[a,b]^n$
where S = { x $\in$ $C^{(1)}[a,b]^n$: x(a) = α,x(b) = β }

Theorem 2.6.1. *Let L be continuous and continuously partially differentiable with regard to the first two components and Let G and H be Frechet differentiable functions*
If $x_0 \in C^{(1)}$ *[a,b]* ^{n} *is a minimum point of*

f(x) $= G(x(a)) + H(x(b)) + \int_a^b L(x,\dot{x},t)\ dt$, *on* $C^{(1)}[a,b]^n$.
$S = \{\ x \in C^{(1)}[a,b]^n : x(a) = \alpha, x(b) = \beta\ \}$ *Then*

$$i)\frac{d}{dt}L_{\dot{x}}(x_0(t),\dot{x}_0(t),t) = L_x(x_0(t),\dot{x}_0(t),t) \quad \forall t \in [a,b] \quad (ELDE)$$

$$ii)L_{\dot{x}}(x_0(a),\dot{x}_0(a),a) = G'(x_0(a)) \quad and \quad L_{\dot{x}}(x_0(b),\dot{x}_0(b),b) = -H'(x_0(b)) \quad (TR)$$

Theorem 2.6.2. *Let L be continuous and continuously partially differentiable with regard to the first two components .Let* x_0 *be a solution of the problem:*
$g(x) = H(x(b)) + \int_a^b L(x(t),\dot{x}(t),t)dt \dashrightarrow$ *Min x is in R*
Where $R = \{\ x \in C^{(1)}[a,b]^n : x(a) = \alpha\ \}$,*then*

$$i)\frac{d}{dt}L_{\dot{x}}(x_0(t),\dot{x}_0(t),t) = L_x(x_0(t),\dot{x}_0(t),t) \quad \forall t \in [a,b] \quad (ELDE)$$

$$ii)L_{\dot{x}}(x_0(b),\dot{x}_0(b),b) = -H'(x_0(a)) \quad (TR)$$

Theorem 2.6.3. *(Necessary and sufficient condition of optimality of VP with fixed end points) Consider the variational problem*
f(x) $= G(x(a)) + H(x(b)) + \int_a^b L(x,\dot{x},t)\ dt$, *on* $C^{(1)}[a,b]^n$.
Let G and H be convex, K be convex set,L be convex with regard to the first two components for each $t \in [a,b]$ *and G and H be frechet differentiable.If for each* $x_0 \in K$ *the conditions*

$$i)\frac{d}{dt}L_{\dot{x}}(x_0(t),\dot{x}_0(t),t) = L_x(x_0(t),\dot{x}_0(t),t) \quad \forall t \in [a,b] \quad (ELDE)$$

$$ii)L_{\dot{x}}(x_0(a),\dot{x}_0(a),a) = G'(x_0(a)) \quad and \quad L_{\dot{x}}(x_0(b),\dot{x}_0(b),b) = -H'(x_0(a)) \quad (TR)$$

are fulfilled then x_0 *is the minimum point of f on K.*

Proof : We set g(x) = G(x(a)), h(x) = H(x(b)). Since ELDE and TR are fulfilled we get by g'(x_0,v) = v(a) G'(x_0(a)) and h'(x_0,v) = v(b) H'(x_0(b)).
From the relationship
f'(x_0,v) = g'(x_0,v) + h'(x_0,v) + $\int_a^b (L_x - \frac{d}{dt} L_{\dot{x}})dt + L_{\dot{x}}\, v(t)]_a^b$, Since $(L_x - \frac{d}{dt} L_{\dot{x}}) = 0$, then we have
f'(x_0,v) =v(a) G'(x_0(a)) + v(b) H'(x_0(b)) + $L_{\dot{x}}$ v(b) -$L_{\dot{x}}$ v(a) = 0 $\geq$ 0
This implies (by characterizing theorem of convex optimization) this x_0 is a minimum point of f on K.

Theorem 2.6.4. *(Necessary and sufficient optimality condition of VP free end point) Let H be convex,K be a convex set,L be convex with regard to the first two components for each* $t \in [a,b]$ *and H be Frechet differentiable .If for* x_0*in* $K \subseteq R$ *the condition*

$$i)\frac{d}{dt}L_{\dot{x}}(x_0(t),\dot{x}_0(t),t) = L_x(x_0(t),\dot{x}_0(t),t) \quad \forall t \in [a,b] \quad (ELDE)$$

$$ii)L_{\dot{x}}(x_0(b),\dot{x}_0(b),b) = -H'(x_0(a)) \quad (TR)$$

are fulfilled,then x_0 is a minimum point of
$g(x) = H(x(b)) + \int_a^b L(x,\dot{x},t)dt$ on K

2.7 Variational problem for piece wise differential functions

Definition 2.7.1. *A function f is in S[a,b] if there exist a partition of interval [a,b]. i.e* $a = t_0 < t_1 < t_2 < .... < t_{m-1} < t_m = b$. *such that the restriction* $f/(_{t_i}, _{t_{i+1}})$ *is continuous for all i=$\{0, 1, ..., m-1\}$*

Definition 2.7.2.

$$RS[a, b] = \{x \in S[a, b]^n : x(t_i) = \lim_{t \to t_i^+} x(t), \forall t_i \in [a, b], x(b) = \lim_{t \to b^-} x(t)\}$$

Let for a partition a $= t_0 < t_1 < t_2 < .... < t_{m-1} < t_m =$ b of interval [a,b] and for a function x:[a,b]$\to \mathbb{R}^n$ and if the following statements be valid,for i $\in \{0, 1, 2, ...m-1\}$
a) x is continuously differentiable on the interval(t_i,t_{i+1})
. b) $\lim_{t \to t_i^+} \frac{x(t)-x(t_i)}{t-t_i} \in \mathbb{R}^n$ and $\lim_{t \to b^-} \frac{x(t)-x(b)}{t-b} \in \mathbb{R}^n$
c) $\dot{x}(t_i) = \lim_{t \to t_i} \frac{x(t)-x(t_i)}{t-t_i}$ and $\dot{x}$(b) $= \lim_{t \to b^-} \frac{x(t)-x(b)}{t-b}$ then we define R$\subset$ $S^{(1)}[a,b]^n$ as a set of all x $\in$ C[a,b] for which there is a partition (a) such that (b) and (c) are valid. Correspondingly, we define $RS^{(1)}[a, b]^n$ as set of all functions x $\in$ $RS[a,b]^n$ for which there is a partition (a)such that(b) and (c) are valid.
Now we consider only Lagrange - functions L such that
d) $\lim_{t \to t_i^+} L_i(x(t), \dot{x}(t), t)$ exists for a given x $\in$ $RCS^{(1)}[a, b]$, i $\in \{0, 1, 2, ...m-1\}$

$$e) L_i^*(x(t), \dot{x}(t), t) = \begin{cases} L_i^*(x(t), \dot{x}(t), t) & if \quad t \in [t_{i+1}) \\ \lim_{t \to t_i^+} L_i(x(t), \dot{x}(t), t) & if \quad t = t_{i+1} \end{cases}$$

is defined on the closed interval $[t_{i+1}]$,i $\in \{0, 1, 2, ...m-1\}$ and
f)$L_i^*(x(t), \dot{x}(t)$,t) is continuous on $[t_{i+1}]$ for i $\in \{0, 1, 2, ...m-1\}$
for such function L we have the following theorem

Theorem 2.7.1. *Let L satisfy(d)(e)(f).Let $x_0 \in RCS^{(1)}[a, b]^n$ be a minimum point of*

$f(x) = \int_a^b L(x, \dot{x}, t)dt$
$S = \{x \in RCS^{(1)}[a, b] : x(a) = \alpha, x(b) = \beta\}$
Then i) $\frac{d}{dt} L_{\dot{x}}(x_0(t), \dot{x}_0(t), t) = L_x(x_0(t), \dot{x}_0(t), t), \forall t \in [a, b]$
And ii) $If t_0 \in [a, b], t_0 \in \{t_1, t_2, ...t_m\}$,then we have :
$L^*_{x(t_0)} = \lim_{t \to t_0^-} L_{\dot{x}}(x_0(t), \dot{x}_0(t), t) = \lim_{t \to t_0^+} L_x(x_0(t), \dot{x}_0(t), t) = L^*_{x(t_0)}$

Remark : The convexity of L and the validity of ELDE are not sufficient conditions for optimality,when x is an element of $RCS^{(1)}[a, b]$.But in connection with characterizing theorem of convex optimization we have the following theorem :

Theorem 2.7.2. *Let L and f be given in the above Theorem 2.7.1 ,L be convex with regard to the first*
two components for each t∈ [a,b] and K⊆ $RCS^{(1)}[a,b]^n$ *be convex set.*
Then $x_0 \in K$ *is a minimum point of f on K if and only if*
$\int_a^b [L_x(x_0, \dot{x_0}, t)h(t) + L_{\dot{x}}(x_0, \dot{x_0}, t)\dot{h}(t)]dt \geq 0 \ \forall x \in K and \quad h = x - x_0$

Theorem 2.7.3. *Let L and f be given as Theorem 2.7.1 and H is convex*
Let $R = \{x \in RCS^{(1)}[a,b]^n : x(a) = \alpha\}$
Let L be Convex with regard to the first two components $\forall t \in [a,b], K \subseteq RCS^{(1)}[a,b]^n$ *be convex set*
and $x_0 \in K$ *if*
i)$\frac{d}{dt} L_{\dot{x}}(x_0(t), \dot{x}_0(t), t) = L_x(x_0(t), \dot{x}_0(t), t), \forall t \in [a,b], \quad ELDE$
ii) $L_{\dot{x}}(x_0(b), \dot{x}_0(b), b) = -H'(x_0(b)), \quad TR$
iii)$\lim_{t \to t_0^+} L_{\dot{x}}(x_0(t), \dot{x}_0(t), t) = \lim_{t \to t_0^-} L_x(x_0(t), \dot{x}_0(t), t), \quad WE$
Then x_0 *is a minimum point of f on K.*

Chapter 3

Introduction to Mathematical optimal control theory

Definition 3.0.3. *:Control theory is an area of applied mathematics that deals with principles,laws, and desire of dynamic systems.*

System :is anything(entity) which respond to relevant inputs according to certain laws(physical laws,economic laws etc).
Dynamic System : is a system that evolve over time (i.e continuously changing over time). It is characterized by notion of state ,inputs,outputs on time horizon(duration);say [a,b] where a is the initial time and b is the terminal time.
Example(1): Consider the dynamic system involving **Moving Object** :- the states(trajectories) are x(t) = position at a time t and $\dot{x}$(t) = velocity at a time t
and the input(control) is u(t) : force(thrust) at a time t.
Example(2) :Consider the dynamic system involving **Manufacturing Firm** :-the states(trajectories) are x(t)=inventory level at a time t and $\dot{x}(t)$=rate of change of inventory level at a time t.
and the input(control) is u(t) = production level at a time t.
To control a dynamic system means to influence its state(behaviour) by exerting necessary control(inputs) so as to achieve the desired goal.
In control theory one needs to find (characterize) a control u(t) which steers(forces) the state x(t) from initial state x_a to the desired terminal state x_b where x(t) = state variable and u(t) = control variable which are related(governed) by a certain differential equation.
$\dot{x}(t) = \phi(x(t), u(t), t)$, t$\in$ [a,b] where u(t) $\in$ U, for some non empty set U called the set of admissible control.
A typical control problem(CP) is used to find (characterize) a control u(t) and the corresponding trajectory x(t) over [a,b] such that (x(t),u(t)) is a solution of a differential equation
$\dot{x}(t) = \phi(x(t), u(t), t)$ t$\in$ [a,b]
x(a) = x_a
x(b) = x_b
u(t) $\in$ U, t $\in$ [a,b]

Given a control problem(CP)
$\dot{x}(t) = \phi(x(t), u(t), t)$ t$\in$ [a,b]
x(a) = x_a
x(b) = x_b
u(t) $\in$ U, t $\in$ [a,b]
One often interested not merely in finding arbitrary control u(t) and the corresponding trajectory x(t) that solves the control problem but wishes to do so in a "best" possible manner say minimize the cost associated with the operation
Let the cost of exerting a control u(t) while the system is in state x(t) at a time t be g(x,u,t) where g:$\mathbb{R}\times\mathbb{R}\times$ [a,b] $\to \mathbb{R}$, then total cost over [a,b] is given by $\int_a^b$ g(x,u,t)dt.
Thus the typical optimal control problem that solves an optimal control u(t) and the corresponding trajectory x(t) is given as:
f(x,u)=$\int_a^b g(x,u,t)dt \dashrightarrow$ min
S.t
$\dot{x}(t) = \phi(x(t), u(t), t)$ t$\in$ [a,b]
x(a) = x_a
x(b) = x_b
u(t) $\in$ U, t $\in$ [a,b] is called optimal control problem
EXAMPLE :**Production Inventory Problem**
Suppose that a manufacturing firm produces a certain item and wants to plan the inventory level and production level of its item in order to meet a known demand over a planning period of time t$\in [a, b]$.

Let x(t) = inventory level(amount of items in stock) at a time t in [a,b]
Let u(t) = production level at a time t in [a,b]

If the selling rate = rx(t),for some r $\in$ [0,1];
Then the rate of change of inventory level can be approximated by :
$\dot{x}$(t) = r x(t) + u(t), t $\in$ [a,b].
There fore the control problem (CP) is given by

$$\dot{x}(t) = r\,x(t) + u(t) \quad t \in [a,b], r \in [0,1]$$
$$x(a) = x_a$$
$$u(t) \in [0,M], M > 0$$

Hence the associated cost function of production-inventory problem is equal to the sum of inventory(holding) cost and production cost.
i.e g(x,u,t) = C_1(x(t)) + C_2(u(t)) where C_1(x(t)) is the cost of holding x(t) items at a time t, and C_2(u(t)) is the cost of producing u(t) items at a time t.
Hence the total running cost during t $\in$ [a,b] is
$\int_a^b$ g(x,u,t) = $\int_a^b$ (C_1(x(t)) + C_2(u(t))) dt
So the optimal control problem(OCP) for production-inventory problem is stated

$$f(x,u) = \int_a^b (C_1(x(t)) + C_2(u(t)))\,dt \dashrightarrow \text{Min}$$

$$\text{S.t } \dot{x}(t) = r\, x(t) + u(t),\ t \in [a,b]$$
$$x(a) = x_a$$
$$u(t) \in [0,M]$$

It is a type of linear optimal control problem.
So the optimal state function x*(t) and the optimal control function u*(t) which minimize the given cost function over the given set can be obtianed.

EXAMPLE: **Rocket Launching Problem**

Consider the problem of a rocket that is to be launched from the ground level to the height y_b in time b (where it is to be at rest)
Let y(t) = The height from the ground at a time t in [a,b]
Let u(t) = The force(thrust) exerted in a vertical direction at a time t in [a,b].
Let m = The mass of a rocket.
From Newton's law of motion we have

$$F - mg = ma$$
$$u - mgy = m\,\ddot{y}$$
$$m\,\ddot{y} = -mgy + u$$
$$\ddot{y} = -gy + \frac{1}{m}u$$

$Let, y = x_1 \Rightarrow \dot{y} = \dot{x}_1$
$Let, \dot{y} = x_2 \Rightarrow \ddot{y} = \dot{x}_2$
So the above equation can be written as

$\dot{x}_2 = -gx_1 + \frac{1}{m}u$
$\dot{x_1} = x_2$

This implies
$\dot{x_1} = 0x_1 + 1x_2 + 0$
$\dot{x}_2 = -gx_1 + 0x_2 + \frac{1}{m}u$

So the control problem is given as

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -g & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u$$

It is subject to
:

$$x(a) = \begin{pmatrix} x_1(a) \\ x_2(a) \end{pmatrix} = \begin{pmatrix} y(a) \\ \dot{y}(a) \end{pmatrix} = x_a$$
$$x(b) = \begin{pmatrix} x_1(b) \\ x_2(b) \end{pmatrix} = \begin{pmatrix} y(b) \\ \dot{y}(b) \end{pmatrix} = x_b$$

Hence the above control problem (CP) for launching a rocket is re written as

$$\dot{x} = \text{Ax} + \text{Bu},\ t \in [a,b]$$
$$x(a) = x_a$$
$$x(b) = x_b$$
$$u(t) \in [-M,M], M \geq 0$$

Where $A = \begin{pmatrix} 0 & 1 \\ -g & 0 \end{pmatrix}$ And $B = \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}$

And suppose the cost(energy/fuel) used of exerting a control u(t) at a time t is $g(x,u,t) = u^2(t)$.

Hence the total cost over [a,b] is

$\int_a^b g(x,u,t) = \int_a^b u^2(t)$

Thus the optimal control problem (OCP) for rocket launching is stated as:

$$f(x,u) = \int_a^b u^2(t)dt \dashrightarrow \text{Min}$$
$$\text{S.t } \dot{x} = \text{Ax} + \text{Bu},\ t \in [a,b]$$
$$x(a) = x_a$$
$$x(b) = x_b$$
$$u(t) \in [-M,M],\ t \in [a,b]$$

It is a type of Quadratic optimal control problem. So that the minimal control function u*(t) and and the minimal state function x*(t) which minimize the given cost function over the given set can be obtained.

3.1 Optimal Control problems

Definition 3.1.1. *Let a,b $\in \mathbb{R}$,a<b,Let $g : \mathbb{R}^n \times \mathbb{R}^m \times [a,b] \to \mathbb{R}$ and $\phi = \mathbb{R}^n \times \mathbb{R}^m \times [a,b] \to \mathbb{R}$ be continuous functions.Then we consider the following optimal control problem,which is called optimal control problem with free right end point*

$f(x,u) = \int_a^b g(x(t), u(t), t)dt \dashrightarrow Min,\ (x,u) \in R$

$R = \{(x, u) \in K \times Q : \dot{x}(t) = \phi(x(t), u(t), t), t \in [a, b]$

$K = \{x \in RCS^{(1)}[a, b]^n : x(a) = \alpha\}$

$Q = \{u \in RS[a, b]^m : u(t) \in U(t), t \in [a, b], U(t) \subseteq \mathbb{R}^m\}\}$

3.1.1 Variational problem as problem of optimal control

It is easy to see that optimal control problems are generalized variational problems. Let the following variational problem be given

$f(x) = \int_a^b L(x, \dot{x}, t)dt \dashrightarrow \text{Min } x \in S$

$S = \{x \in RCS^{(1)}[a, b]^n : x(a) = \alpha, x(b) = \beta\}$.

Then by setting $u(t) = \dot{x}(t)$, we get optimal control problem: i.e

$f(x,u) = \int_a^b g(x(t), u(t), t)dt \dashrightarrow \text{Min}, (x,u) \in R$

$R = \{(x, u) \in K \times Q : \dot{x}(t) = u(t), t \in [a, b]$

$K = \{x \in RCS^{(1)}[a, b]^n : x(a) = \alpha, x(b) = \beta\}$

$Q = \{u \in RS[a, b]^m : u(t) \in U(t), t \in [a, b], U(t) \subseteq \mathbb{R}^m\}\}$

3.1.2 Elementary Lagrange approach

In this section we try to get necessary (and also sufficient) condition for
(x^*, u^*) to be a solution of the following optimal control problem
f(x,u) = $\int_a^b g(x(t), u(t), t)dt$ ⇢ Min, (x,u) ∈ R
R = $\{(x, u) \in K \times Q : \dot{x}(t) = \phi(x(t), u(t), t), t \in [a, b]$
K = $\{x \in RCS^{(1)}[a, b]^n\}$
Q = $\{u \in RS[a, b]^m : u(t) \in U(t), t \in [a, b], U(t) \subseteq \mathbb{R}^m\}$
General Approach: We want to apply the lagrange method for optimization problem with equation constraint for this we try to find a function,
$\lambda = K \times Q \to \mathbb{R}$,where $\lambda(x, u) = 0$ for all (x,u) ∈ R, then we conceder the Lagrange function
$f_\lambda(x, u) = f(x, u) + \lambda(x, u)$, (x,u) ∈ $K\times$ Q

Lemma 3.1.1. *Lagrange lemma*
Let $(x^*, u^*) \in R$ be a minimizer of $f_\lambda(x, u)$ on $K\times$ Q,for some $\lambda : [a.b]^n \to \mathbb{R}^n$. Then (x^*, u^*) is a minimizer of f(x,u) on R.

Proof: Let $(x^*, u^*) \in$ R be a minimizer of $f_\lambda(x, u)$ on K $\times Q$
we get :f(x^*, u^*) + $\lambda(x^*, u^*) \leq$ f(x,u) + $\lambda(x, u)$, ∀ (x,u) ∈ K × Q
Because R ⊆ K × Q weget
f(x^*, u^*) + $\lambda(x^*, u^*) \leq$ f(x,u) + $\lambda(x, u)$
, (since:$\lambda(x^*, u^*)$ =0 and $\lambda(x, u)$ =0);we have
⇒ f(x*,u*) + 0 ≤ f(x,u) + 0
⇒ f(x*,u*) ≤ f(x,u).i.e (x^*, u^*) is a minimum point of f(x,u) on R.
Hence if $(x^*, u^*) \in$ R is a minimum point of $f_\lambda(x, u)$,then $(x^*, u^*) \in$ R is a minimum point of $f(x^*, u^*)$

3.1.3 Lagrange approach to solve optimal control problems

Consider he Optimal control problem:
f(x,u) = $\int_a^b g(x(t), u(t), t)dt$ ⇢ Min, (x,u) ∈ R
R = $\{(x, u) \in K \times Q : \dot{x}(t) = \phi(x(t), u(t), t), t \in [a, b]$
K = $\{x \in RCS^{(1)}[a, b]\}$
Q = $\{u \in RS[a, b] : u(t) \in U(t), t \in [a, b], U(t) \subseteq \mathbb{R}^m\}$
In Lagrange approach,by considering ODE constraint $\dot{x} - \phi(x, u, t) = 0$,and we take Lagrange multiplier function λ,where $\lambda : [a, b] \to \mathbb{R}$.
We define the Lagrange optimal control problem as
f_λ(x,u) = $\int_a^b g(x, u, t) + \lambda[\dot{x} - \phi(x, u, t)]dt$ Min
K = $\{x \in RCS^{(1)}[a, b]\}$
Q = $\{u \in RS[a, b] :$ u(t) ∈ U(t),t ∈ [a,b];U(t) ⊆ $\mathbb{R}^m\}\}$
The integrand is called Lagrange fnctional,denoted by;
$L(x, \dot{x}, u, \lambda, t) = g(x, u, t) + \lambda[\dot{x} - \phi(x, u, t)]dt$

In order to solve optimal control problem, we need a necessary (and sufficient) conditions.For
this we have three kinds of necessary conditions;
1) Let x = x^* and $\lambda = \lambda^*$ are fixed functions,then we try
to find the conditions for minimizing $f_{\lambda^*}(x^*,u)$ with regard to u.
2) Let u = u^* and $\lambda = \lambda^*$ are fixed functions,then we try to
find the conditions for minimizing $f_{\lambda^*}(x, u^*)$ with respect to x
3) Let x = x^* and u = u^* are optimal (and therefore fixed) functions.Then by $(x^*, u^*) \in$ R
we have trivially
$\dot{x}^* = \phi(x^*, u^*, t)$, where $(x^*, u^*) \in$ K× Q

Theorem 3.1.1. *Let $\lambda \in RS[a,b]^n$,$x \in K$ be fixed and u^* be a minimizer of the function $f_\lambda(x,u)$ on Q,then*
$L(x, \dot{x}, u^*, \lambda, t) = min_{u\in Q}L(x, \dot{x}, u, \lambda, t)$,*for all t in [a,b]*

3.1.4 Optimal control problem with fixed end points

Let a,b $\in \mathbb{R}$ and $\alpha,\beta \in \mathbb{R}^n$,a<b,
Let $g : \mathbb{R}^n \times \mathbb{R}^m \times [a, b] \to \mathbb{R}$
and $\phi : \mathbb{R}^n \times \mathbb{R}^m \times [a, b] \to \mathbb{R}$ be
continuous functions,then we consider the following OCP with fixed end points :
f(x,u) = $\int_a^b g(x(t), u(t), t)dt \dashrightarrow$ Min, (x,u) $\in$ R
R = $\{(x, u) \in K \times Q : \dot{x}(t) = \phi(x(t), u(t), t), t \in [a, b]\}$
S = $\{x \in RCS^{(1)}[a, b]^n : x(a) = \alpha, x(b) = \beta\}$
Q = $\{u \in RS[a, b]^m : u(t) \in U(t), t \in [a, b], U(t) \subseteq \mathbb{R}^m\}$
Further more Let $L(x, \dot{x}, u, \lambda, t)$ be piece wise partially differentiable with regard to x,$\dot{x}$
then we have the following theorem.

Theorem 3.1.2. *Let $u \in RS[a, b]^m$, $\lambda \in RCS^{(1)}[a, b]^n$ be fixed functions and x^* be a minimizer of $f_\lambda(x,u)$ on S,then*
$\frac{d}{dt}L_{\dot{x}}(x^*, \dot{x}^*, u, \lambda, t) = L_x(x^*, \dot{x}^*, u, \lambda, t)$ *(ELDE)*

3.1.5 Optimal control problem with free right end point

Let a,b $\in \mathbb{R}$ and $\alpha \in \mathbb{R}^n$,a<b,Let
$g : \mathbb{R}^n \times \mathbb{R}^m \times [a, b] \to \mathbb{R}$ and $\phi : \mathbb{R}^n \times \mathbb{R}^m \times [a, b] \to \mathbb{R}$ be continuous functions.Then we consider the following OCP with free right end point
f(x,u) = $\int_a^b g(x(t), u(t), t)dt \to$ Min, (x,u) $\in$ R
R = $\{(x, u) \in K \times Q : \dot{x}(t) = \phi(x(t), u(t), t), t \in [a, b]\}$
W = $\{x \in RCS^{(1)}[a, b]^n : x(a) = \alpha\}$
Q = $\{u \in RS[a, b]^m : u(t) \in U(t), t \in [a, b], U(t) \subseteq \mathbb{R}^m\}$

Further more Let $L(x, \dot{x}, u, \lambda, t)$ be piece wise partially differentiable with regard to x,$\dot{x}$
then we have the following theorem.

Theorem 3.1.3. *Let $u \in RS[a,b]^m$, $\lambda \in RCS^{(1)}[a,b]^n$ be fixed functions and x^* be a minimizer of $f_\lambda(x,u)$ on W, Then*
$\frac{d}{dt}L_{\dot{x}}(x^*, \dot{x}^*, u, \lambda, t) = L_x(x^*, \dot{x}^*, u, \lambda, t)$ *(ELDE)*
And $\lambda(b) = 0$ *(TR)*

Theorem 3.1.4. *If (x^*, u^*) is optimal solution of the Lagrange optimal control problem*
$f_\lambda(x, u)$
for some λ that satisfies the (ODE) $\dot{x}(t) = \phi(x(t), u(t), t)$then, $u^*(t)$is the optimal control with optimal trajectory $x^*(t)$ for the optimal control problem f(x,u).

3.2 Solving Optimal Control problems

Steps and Necessary optimality conditions to solve optimal control problems
f_λ(x,u) = $\int_a^b g(x,u,t) + \lambda[\dot{x} - \phi(x,u,t)]dt$ Min
Let L(x,$\dot{x}$,u,λ,t) = g(x,u,t) + λ $\dot{x}$ - $\lambda\phi$ (x,u,t) be the Lagrange functional.
And H(x,u,λ,t) = g(x,u,t) + $\lambda\phi$ (x,u,t be Hamilonian functional.
1)**Pontryagin Minimum Condition**: Let x = x* and $\lambda = \lambda^*$ are fixed functions.If we minimize f_λ (x,u) with respect to u on Q then
L(x,$\dot{x}$,u*,λ,t) = $\text{Min}_{u\in Q}$ L(x,$\dot{x}$,u,λ,t).
Since $\lambda\dot{x}$ does not depend on u the above MIN condition of L is equivalent to
H(x,u*,λ,t) = $\text{Min}_{u\in Q}$ H(x,u,λ,t.(Minimum principle of pontryagin)
(Here we solve u* interims of λ)
2) **Validity of ELDE and TR** : Let u =u* and $\lambda = \lambda^*$ are fixed functions
If we minimize f_λ (x,u) with respect to x,on K,then the minimizer x satisfies :
$\frac{d}{dt}$ $L_{\dot{x}}$ = L_x --→ (ELDE)
. $\Rightarrow \dot{\lambda} = g_x - \lambda\phi_x$
And λ (b) = 0 --→ (TR)
(Here we solve λ^* interims of t and when we substitute λ^* in u*
then we write u* interims of t.
3) **Validity of ODE** : Let x=x* and u=u* are optimal(and therefore fixed) functions.Then by $(x^*, u^*) \in$ R,then x* and u* trivially satisfy the ODE given by ;
$\dot{x}^*(t) = \phi(x^*(t), u^*(t), t)$; $(x^*, u^*) \in K \times Q$
And we solve the ODE using (IC) :x(a) = α; t$\in$ [a,b]
(Here x* is solve interims of t)

3.2.1 Solving Optimal Control problems with free right end points

EXAMPLE: Find the optimal control u* and the corresponding optimal trajectory x* for the following **optimal control problem**
$f(x,u) = \int_0^1 x(t)dt$ --→ $Min \quad (x,u) \in R$
$R = \{(x,u) \in K \times Q : \dot{x}(t) = u(t)\}$

$K = \{x \in RCS^{(1)}[0,1] : x(0) = 0\}$
$Q = \{u \in RS[0,1] = -1 \leq u(t) \leq 1, t \in [0,1]\}$
Solution :The Lagrange OCP is given by:
$f_\lambda(x,u) = \int_0^1 x(t) + \lambda[\dot{x}(t) - u(t)]dt \dashrightarrow Min, (x,u) \in K \times Q$
Then we have$L(x,\dot{x},u,\lambda,t) = x(t) + \lambda[\dot{x}(t) - u(t)]$
This implies $L(x,\dot{x},u,\lambda,t) = x(t) + \lambda\dot{x}(t) - \lambda u(t)$
1)**Minimum condition**(let x=x* and λ=λ^* are fixed functions) : In this case, if we minimize f$_\lambda$(x,u) with respect to u on Q,then u satisfies;
$\Rightarrow L(x,\dot{x},u^*,\lambda,t) = Min_{u\in Q}\{L(x,\dot{x},u,\lambda,t)\}$
$\Rightarrow L(x,\dot{x},u^*,\lambda,t) = Min_{u\in Q}\{x(t) + \lambda\dot{x}(t) - \lambda u(t)\}$

Since u does not depend on $\lambda\dot{x}$ the condition MIN of L can be written using Hamiltonian function H as
$H(x,u^*,\lambda,t) = Min_{u\in Q}\{x - \lambda u\}$
Let $H(u) = -\lambda u + x$.
Since H is linear with respect to u,then the minimizer u^* of H over [-1,1] occurs at the end points.
From the function $H(u) = -\lambda u + x, \quad -1 \leq u \leq 1$ can be expressed as follows
i) If λ <0 the graph of H is increasing,hence the minimizer of H occurs at u = -1
ii)If λ >0 the graph of H is decreasing,hence the minimizer of H occurs at u =1
i.e

$$u^*(t) = \begin{cases} -1 & if, \lambda(t) < 0 \\ 1 & if, \lambda(t) > 0 \end{cases}$$

2)**Validity of ELDE and TR conditions** (let u =u*,$\lambda = \lambda^*$ are fixed functions).If we minimize $f_\lambda(x,u)$ with respect to x on K,then x satisfies ELDE and TR.
i)From ; $L(x,\dot{x},u,\lambda,t) = x + \lambda\dot{x} - \lambda u$
$L_x = 1$ and $L_{\dot{x}} = \lambda$
$\Rightarrow \frac{d}{dt}L_{\dot{x}} = L_x$
$\Rightarrow \frac{d}{dt}\lambda = 1$
$\Rightarrow \dot{\lambda} = 1$
$\Rightarrow \frac{d\lambda}{dt} = 1$
$\Rightarrow \int d\lambda = \int dt$
$\Rightarrow \lambda(t) = t + c$
i)And from TR condition:
i.e $L_{\dot{x}}(x(b),\dot{x}(b),u(b),\lambda(b),b) = 0$
We have $\lambda(1) = 0$
From, $\lambda(1) = 0$,we get:$\lambda^*(t)$ = t - 1,if $0 \leq t \leq 1$
So,From the function $\lambda(t) = t - 1, if, 0 \leq t \leq 1, then - 1 \leq \lambda(t) \leq 0$
Hence :$u^*(t) = -1, if, \lambda(t) < 0$ is a solution
$\Rightarrow u^*(t) = -1, if, \lambda(t) < 0$
So the optimal control is of the problem is u*(t) = -1,if, $0 \leq$ t $\leq$ 1
3)**Validity of ODE**:since the solution,u = u* satisfies ODE,

$\Rightarrow \dot{x} = u^*, x(0) = 0 \quad$ (IC)
$\frac{dx}{dt} = -1,$
$\Rightarrow \int dx = -\int dt$
$\Rightarrow x(t) = -t + c$

$\Rightarrow$ from (IC) x(0)=0,we get x(t) = -t,for, t $\in [0,1]$
so the optimal trajectory of the problem is x*(t) = -t,for, t $\in [0,1]$

EXAMPLE(2):Find the optimal control u*(t) and the corresponding optimal trajectory x*(t) for the following quadratic of optimal control problem:
f(x,u) = $\int_0^2 [u^2(t) + 3u(t) - 2x(t)]dt \dashrightarrow Min, (x,u) \in R$
R =$\{(x,u) \in K \times Q : \dot{x}(t) = x(t) + u(t), t \in [0,2]\}$
K =$\{x \in RCS^{(1)}[0,2] : x(0) = 5\}$
Q =$\{u \in RS[0,2] : 0 \leq u \leq 2\}$
Solution : $f_\lambda(x,u) = \int_0^2 [u^2(t) + 3u(t) - 2x(t) + \lambda(\dot{x}(t) - x(t) - u(t))]dt \dashrightarrow Min, (x,u) \in K \times Q$
We have $L(x, \dot{x}, u, \lambda, t) = u^2 + 3u - 2x + \lambda\dot{x} - \lambda x - \lambda u$
1)**Minimum condition**(let x=x* and $\lambda = \lambda^*$ are fixed functions) : If we minimize f$_\lambda$(x,u) with respect to u on Q,then u satisfies;
$\Rightarrow L(x, \dot{x}, u^*, \lambda, t) = Min_{u \in [0,2]} L(x, \dot{x}, u, \lambda, t)$
$\Rightarrow L(x, \dot{x}, u^*, \lambda, t) = Min_{u \in [0,2]} \{u^2 + 3u - 2x + \lambda\dot{x} - \lambda x - \lambda u\}$
Since u does not depend on $\lambda\dot{x}$, the condition MIN of L can be written using Haimltonian function H,i.e
$H(x, u^*, \lambda, t) = Min_{u \in [0,2]} \{u^2 + 3u - 2x - \lambda x - \lambda u\}$
$\Rightarrow \frac{\partial H}{\partial u} = 0$
$\Rightarrow H_u(u^2 + 3u - 2x - \lambda x - \lambda u) = 0$
$\Rightarrow 2u + 3 - \lambda = 0$
$\Rightarrow u(t) = \frac{1}{2}\lambda(t) - \frac{3}{2}, t \in [0,2]$
2) **Validity of ELDE and TR conditions**(let $u = u^*, \lambda = \lambda^*$ are fixed functions).If we minimize f_λ(x,u) with respect to x on K then x satisfies ELDE and TR.
i)From,$L(x, \dot{x}, u, \lambda, t) = u^2 + 3u - 2x + \lambda\dot{x} - \lambda x - \lambda u$
We have $L_x = -2 - \lambda$ and $L_{\dot{x}} = \lambda$
$\Rightarrow \frac{d}{dt} L_{\dot{x}} = L_x$
$\Rightarrow \frac{d}{dt}\lambda = -2 - \lambda$
$\Rightarrow \dot{\lambda} = -2 - \lambda$
$\Rightarrow -\int \frac{d\lambda}{\lambda+2} = \int dt$
$\Rightarrow \ln|\lambda + 2| = -t + c$
$\Rightarrow \lambda(t) = ce^{-t} - 2$
ii)From TR condition:
i.e $L_{\dot{x}}(x(b), \dot{x}(b), u(b), \lambda(b), b) = 0$
We have $\lambda(2) = 0$
From, $\lambda(2) = 0$,we get:$\lambda^*(t) = 2e^{2-t} - 2, for, 0 \leq t \leq 2$
Next Substitute $\lambda^*(t) = 2e^{2-t} - 2, t \in [0,2], in, u^*(t) = \frac{1}{2}\lambda^*(t) - \frac{3}{2}, t \in [0,2]$
Then we have :$u^*(t) = e^{2-t} - \frac{5}{2}, t \in [0,2]$

Our aim is to minimize u in the given interval of time :
But if $0 \leq t \leq 2 \Rightarrow -1.5 \leq u \leq 4.9$(out side the interval $0 \leq u \leq 2$)
So when $0 \leq u \leq 2 \Rightarrow 0.5 \leq t \leq 1.1$ (in side the interval $0 \leq t \leq 2$)

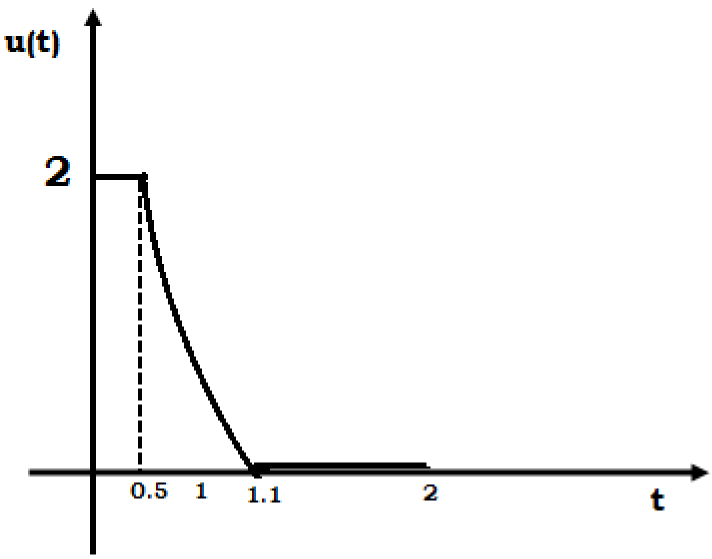

fig 3.1

And the end points are (0.5,2) and (1.1,0)
So the optimal control function of the problem is given by

$$u^*(t) = \begin{cases} 2 & if, 0 \leq t < 0.5 \\ e^{2-t} - \frac{5}{2} & if, 0.5 \leq t < 1.1 \\ 0 & if, 1.1 \leq t \leq 2 \end{cases}$$

3)**Validity of ODE** :since the solution,u = u* satisfies ODE,
given by $\dot{x} = x + u^*$ with (IC), x(0) = 5
Then,let's solve the ODE given below

$$\dot{x^*} = \begin{cases} x + 2 & if, 0 \leq t < 0.5 \\ x + e^{2-t} - 5/2 & if, 0.5 \leq t < 1.1 \\ x + 0 & if, 1.1 \leq t \leq 2 \end{cases}$$

Case(1) Solving $\dot{x}$ =x + 2,in $0 \leq t < 0.5$ gives
$x(t) = ce^t - 2$
Taking the (IC),x(0) = 5 $\Rightarrow x(t) = 7e^t - 2$
Case(2) Solving $\dot{x} = x + e^{2-t} - 5/2, in, 0.5 \leq t < 1.1$ gives
$x(t) = ce^t - \frac{1}{2}e^{2-x} + 5/2$
Since $7e^t - 2 = ce^t - \frac{1}{2}e^{2-x} + 5/2, at, t = 0.5, then$
$c = 7 + \frac{1}{2}e - \frac{9}{2}e^{-1/2}$
Case(3) Solving $\dot{x} = x, in, 1.1 \leq t \leq 2, gives$
$x(t) = c_1e^t$
Since $ce^t - \frac{1}{2}e^{2-x} + 5/2 = c_1e^t, at, t = 1.1$,then
$c_1 = 7 + \frac{1}{2}e - \frac{9}{2}e^{-1/2} - \frac{1}{2}e^{-0.2} + \frac{5}{2}e^{-1.1}$

So the optimal state function of he problem is given by

$$x^*(t) = \begin{cases} 7e^t - 2, & if, 0 \leq t < 0.5 \\ ce^t - \frac{1}{2}e^{2-t} + 5/2, & if, 0.5 \leq t < 1.1 \\ c_1 e^t, & if, 1.1 \leq t \leq 2 \end{cases}$$

where :$c = 7 + \frac{1}{2}e - \frac{9}{2}e^{-1/2}$
And :$c_1 = 7 + \frac{1}{2}e - \frac{9}{2}e^{-1/2} - \frac{1}{2}e^{-0.2} + \frac{5}{2}e^{-1.1}$

3.3 Sufficient conditions,separated optimal control problems

For convex(or partially convex)problems the Characterizing theorem of convex optimization can be used in order to get sufficient conditions.The following special case is particularly easy and many practical OCP have this structure.
Let $L(x, \dot{x}, u, \lambda, t) = G(x, \dot{x}, \lambda, t) + W(u, \lambda, t)$
We will denote this types of problems as separated problems.
Now consider the following optimization problem :
f(x,u)=$\int_a^b$ g(x,u,t)dt $\rightarrow$ min, (x,u) $\in R$
And the corresponding Lagrange problem is
$f_\lambda(x, u) = \int_a^b [g(x, u, t) + \lambda(\dot{x} - \phi(x, u, t)]dt \rightarrow Min, (x, u) \in K \times Q$
$f_\lambda(x, u) = \int_a^b [g(x, u, t) + \lambda\dot{x} - \lambda\phi(x, u, t)]dt \rightarrow Min, (x, u) \in K \times Q$
So we have the following types of optimization problems:
i)$f_\lambda(x, u) = \int_a^b [G(x, \dot{x}, \lambda, t) + W(u, \lambda, t)]$dt --→ Min, (x,u) $\in$ K$\times Q$ (Separated Problem)
ii) g(x) = $\int_a^b G(x, \dot{x}, \lambda, t)$dt --→ min, x $\in$ K (Variational problem)
iii) w(u) = $\int_a^b W(u, \lambda, t)$dt --→ min, u $\in$ Q (Normal optimization problem)

Theorem 3.3.1. *Let $f_\lambda(x, u) = \int_a^b [G(x, \dot{x}, \lambda, t) + W(u, \lambda, t)]dt \rightarrow$ Min is separable control problem.*
If x^* is a solution of g(x) $= \int_a^b G(x, \dot{x}, \lambda, t)dt \rightarrow$ min.
And u^* is a solution of w(u) $= \int_a^b W(u, \lambda, t)dt \rightarrow$ min.
then (x^*, u^*) is a solution of $f_\lambda(x, u)$.

Let x^* be a solution of g(x) $\Rightarrow g(x^*) \leq g(x), \forall x \in K$
Let u^* be a solution of w(u) $\Rightarrow w(u^*) \leq w(u), \forall u \in Q$
So $f_\lambda(x^*, u^*) = g(x^*) + w(u^*) \leq g(x) + w(u) = \int_a^b G(x, \dot{x}, \lambda, t)dt + \int_a^b W(u, \lambda, t)dt = \int_a^b [G(x, \dot{x}, \lambda, t)dt + W(u, \lambda, t)]dt = \int_a^b g(x(t), u(t), t) + \lambda[\dot{x} - \phi(x(t), u(t), t)]dt = f_\lambda(x, u)$
i.e $f_\lambda(x^*, u^*) \leq f_\lambda(x, u)$ $\forall$ (x,u) $\in$ K $\times$ Q
Hence (x^*, u^*) is a minimum point of $f_\lambda(x, u)$.
Note:A sufficient optimality condition for x^* and u^* to be the optimal solutions of separated

Optimal control problem is the convexity of the separated problems.(i.e the convexity of G with regard to x and $\dot{x}$,and the convexity of W with regard to u.)

Theorem 3.3.2. *Let* $\lambda \in [a, b], K \subseteq \{x \in RCS^{(1)}[a, b]^n : x(a) = \alpha, x(b) = \beta\}$ *be convex set and let the problem is separated furthermore,let G be convex with regard to x and be piece wise continuously differentiable.If for* $(x^*, u^*) \in K\times\ Q$,
1)$\frac{d}{dt}G_{\dot{x}}(x^*(t), \dot{x}^*(t), \lambda(t), t) = G_x(x^*(t), \dot{x}^*(t), \lambda(t), t),\ t \in [a, b]$
2)$W(u^*(t), \lambda(t), t) = MinW_{u\in Q}(u(t), \lambda(t), t), t \in [a, b]$
3)$\dot{x}^*(t) = \phi(x^*(t), u^*(t), t), t \in [a, b]$
then (x^*, u^*) *is a solution of (OP)*

3.3.1 Solving separated optimal control problems with free right end points

Example: Find the optimal control $u^*(t)$ and the corresponding optimal
trajectory x$^*(t)$ of the following optimal optimal control problem:
$f(x, u) = \int_0^1 (u + 2xt)dt \dashrightarrow Min, (x, u) \in R$
$R = \{(x, u) \in K \times Q : \dot{x}(t) = u(t)\}$
$K = \{x \in RCS^{(1)}[0, 1] : x(0) = 0]\}$
$Q = \{u \in RS[0, 1] : -1 \leq u(t) \leq 1, t \in [0, 1]\}$
So :$f_\lambda(x, u) = \int_0^1 (u + 2xt + \lambda[\dot{x} + u])dt$
$f_\lambda(x, u) = \int_0^1 [(\underbrace{2xt + \lambda\dot{x}}) + (\underbrace{u + \lambda u})]dt$
Where; $G(x, \dot{x}, \lambda, t) = 2xt + \lambda\dot{x}$ And; $W(u, \lambda, t) = u + \lambda u$
Obviously,G is convex with respect to x and $\dot{x}$. And W is also convex
with respect to u,then we have a sufficient condition for (x^*, u^*) to be
an optimal solution of the given problem
. 1) **Min Condition**
If u is a minimizer of f$_\lambda$(x,u) of on Q then ;
$\Rightarrow Min_{\in[0,1]}W(u, \lambda, t) = Min_{u\in[0,1]}\{\lambda u + u\}$
Let W(u) = $\lambda u + u, if, -1 \leq u \leq 1$
Since W is linear function with respect to u then,the minimizer u* of W over u$\in$ [-1,1] occurs at the end points.
From the function $W(u) = (1 + \lambda)u, -1 \leq u \leq 1,$
i)W is Increasing, if $\lambda >$ -1 hence the minimizer of W occurs at u =-1
ii)W is decreasing, if $\lambda <$ -1 hence the minimizer of W occurs at u = 1

$$So \quad u^*(t) = \begin{cases} -1 & if, \lambda(t) > -1 \\ 1 & if, \lambda(t) < -1 \end{cases}$$

2)**Validity of ELDE and TR conditions**.
If x is the minimizer of f$_\lambda$(x,u) on K then x satisfies;ELDE and TR
i) From $G(x, \dot{x}, \lambda, t) = 2xt + \lambda\dot{x}$
$\Rightarrow G_x = 2t, and, G_{\dot{x}} = \lambda$
$\Rightarrow \frac{d}{dt}G_{\dot{x}} = G_x$

$\Rightarrow \frac{d}{dt}\lambda = 2t$
$\Rightarrow \int d\lambda = \int 2tdt$
$\Rightarrow \lambda(t) = t^2 + c$
ii) From TR condition: i.e $G_{\dot{x}}(x(b), \dot{x}(b), \lambda(t), b) = 0$
We have $\lambda(1) = 0$
From $\lambda(1) = 0$,we get:$\lambda^*(t) = t^2 - 1, for, 0 \leq t \leq 1$
From the graph of a function $\lambda(t) = t^2 - 1, if, 0 \leq t \leq 1, then - 1 \leq \lambda(t) \leq 0$
Hence :$u^*(t) = -1, if, \lambda(t) > -1$
Then the optimal control u*(t) of the problem is $\Rightarrow u^*(t) = -1, if, 0 \leq t \leq 1$
3)**Validity of ODE**:
since the solution,u = u* satisfies ODE
$\Rightarrow \dot{x} = -u^*, x(0) = 0$
$\frac{dx}{dt} = -(-1),$
$\Rightarrow \int dx = \int dt$
$\Rightarrow x(t) = t + c$
From,(IC), x(0) = 0,we get x*(t) = t
So the optimal trajectory x*(t) of the problem is:
x*(t) = t

3.4 Linear optimal control problems

3.4.1 Linear optimal control problem with free right end point

In this section we consider optimal control problems ,where the functions
H and g are linear with respect to x and u,and the differential equation is also linear.
That is optimal control problem of the following type
Let a,b $\in \mathbb{R}$,a<b,$\alpha \in \mathbb{R}^n$, Let $H : \mathbb{R}^n \to \mathbb{R}$, g : $\mathbb{R}^n \times \mathbb{R}^m \times [a, b] \to \mathbb{R}$ and
$\phi : \mathbb{R}^n \times \mathbb{R}^m \times [a, b] \to \mathbb{R}$
be continuous functions and
f(x,u) = $\gamma^T x(b) + \int_a^b (c^T x + d^T u)dt \to$ Min, (x,u) $\in$ R
R = $\{(x, u) \in K \times Q : \dot{x} = Ax + Bu, t \in [a, b]\}$
K = $\{x \in RCS^{(1)}[a, b]^n : x(a) = \alpha\}$
Q = $\{u \in RS[a, b]^m : -1 \leq u_i \leq 1, i = 1, 2, ...m\}$
Where c $\in RS[a, b]^n$,d $\in RS[a, b]^m$,A $\in RS[a, b]^{n\times n}$, B $\in RS[a, b]^{n\times m}$
are continuous functions. And $\gamma \in \mathbb{R}^n$,
This type of problem is called Linear Optimal Control problem.
Then we get the lagrange function
$L(x, \dot{x}, u, \lambda, t) = c^T x + d^T u + \lambda^T(\dot{x} - Ax - Bu)$
$L(x, \dot{x}, u, \lambda, t) = c^T x + d^T u + \lambda^T \dot{x} - \lambda^T Ax - \lambda^T Bu$
$L(x, \dot{x}, u, \lambda, t) = \underbrace{c^T x + \lambda^T \dot{x} - \lambda^T Ax} + \underbrace{d^T u - \lambda^T Bu.} \quad where, \lambda \in RCS^{(1)}[a, b]^n$
So,we have $\quad G(x, \dot{x}, \lambda, t) = c^T x + \lambda^T \dot{x} - \lambda^T Ax$
And,also,$W(u, \lambda, t) = d^T u - \lambda^T Bu$

Obviously G is convex(since it is linear) with respect to the pair of variables (x,$\dot{x}$) and W is also convex (since it is linear) with respect to u.

a)**Validity of ELDE and TR**:

If x* is the minimizer of $f_\lambda(x,u)$ on K then x* satisfies ELDE and TR condition.

i) From the variational problem:$G(x,\dot{x},\lambda,t) = c^T x + \lambda^T \dot{x} - \lambda^T Ax$

we have: $G_x = c^T - \lambda^T$A and $G_{\dot{x}} = \lambda^T$,then

$\frac{d}{dt}G_{\dot{x}} = G_x$

$\frac{d}{dt}\lambda^T = c^T - \lambda^T$A

$\dot{\lambda}^T = c^T - \lambda^T A$

$\dot{\lambda} = -A^T\lambda + c$

ii)From Transversality condition(TR),we have

$G_{\dot{x}}(x^*(b), \dot{x^*}(b), \lambda(b), b) = -H'(x^*(b))$

$\Rightarrow \lambda^T(b) = -\gamma^T$

$\Rightarrow \lambda(b) = -\gamma$

b) By using the MIN condition : If u is a minimizer of f_λ(x,u) then u satisfies;

$Min_{u_i(t)\in[-1,1]}W(u,\lambda,t) = Min_{u_i(t)\in[-1,1]}(d^T u - \lambda^T Bu)$

$Min_{u_i(t)\in[-1,1]}W(u,\lambda,t) = Min_{u_i(t)\in[-1,1]}(d^T - \lambda^T B)u$

Setting:$h^T = d^T - \lambda^T B$,we have:

$Min_{u_i(t)\in[-1,1]}W(u,\lambda,t) = Min_{u_i(t)\in[-1,1]}h^T u$

$Min_{u_i(t)\in[-1,1]}W(u,\lambda,t) = h^T u^*$

$$Let \quad h = \begin{pmatrix} h_1 \\ h_2 \\ . \\ . \\ . \\ h_m \end{pmatrix} .Then \quad h_i, is, continuous, \forall i \in \{1,2,.....m\}, since \quad d, \lambda, and, B, are, continuous.$$

So,since$W(u,\lambda,t) = (d^T - \lambda^T B)u = h^T u$

$$W(u,\lambda,t) = \begin{pmatrix} h_1 & h_2 & . & . & h_m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ . \\ . \\ u_m \end{pmatrix}$$

Therefore $W(u,\lambda,t) = h_1u_1 + h_2u_2 + ... + h_iu_i + ... + h_mu_m$

In order to minimize W we have to minimize each summand h_iu_i

.Let h_i have a finite number of zeros $\{t_1^i, t_2^i, ...t_{k_i}^i\}$ in which the functional values changes.

Let t_0^i = a,and $t_{k_{i+1}}^i$ = b for all i = $\{1,2,3,......m\}$

$$Let \quad \xi_i = \begin{cases} 1 & if \quad h_i(t) \geq 0, for \quad t \in [a, t_1^{(i)}) \\ -1 & if \quad h_i(t) \leq 0, for \quad t \in [a, t_1^{(i)}) \end{cases}$$

Now we define ;

$$u_i(t) = \begin{cases} \xi_i(-1)^j \quad if \quad t \in [t_{j-1}^{(i)}, t_j^{(i)}), j \in \{1, 2...k\} \\ \xi_i(-1)^{k_{i+1}} \quad if \quad t \in [t_{k_i}^{(i)}, t_{k_{i+1}}^{(i)}] = [t_{k_{i+1}}^{(i)}, b] \end{cases}$$

In the drawing we have $\xi_i = 1$ because h$_i(t) \geq 0$ for t $\in [a, t_1^{(i)})$
. So we have:
$u_i(t) = \xi_i(-1)^1 = 1(-1) = -1, for \quad t \in [a, t_1^{(i)})$
$u_i(t) = \xi_i(-1)^2 = 1(1) = 1, for \quad t \in [t_1^{(i)}, t_2^{(i)})$
$u_i(t) = \xi_i(-1)^3 = 1(-1) = -1, for \quad t \in [t_2^{(i)}, t_3^{(i)})$
$u_i(t) = \xi_i(-1)^4 = 1(1) = 1, for \quad t \in [t_3^{(i)}, t_4^{(i)})$
etc
Similarly for h$_i(t) \leq 0$,we have ξ_i = -1 for t $\in [a, t_1^{(i)})$
So we have
$u_i(t) = \xi_i(-1)^1 = -1(-1) = 1, for \quad t \in [a, t_1^{(i)})$
$u_i(t) = \xi_i(-1)^2 = -1(1) = -1, for \quad t \in [t_1^{(i)}, t_2^{(i)})$
$u_i(t) = \xi_i(-1)^3 = -1(-1) = 1, for \quad t \in [t_2^{(i)}, t_3^{(i)})$
$u_i(t) = \xi_i(-1)^4 = -1(1) = -1, for \quad t \in [t_3^{(i)}, t_4^{(i)})$
etc
So for the solution we get : u^{*T}(t) = $\{\xi_1(-1)^{j_1}, \xi_2(-1)^{j_2}, \xi_3(-1)^{j_3}, ...\xi_m(-1)^{j_m}\}$
For e.g: if $x \in RCS^{(1)}[a, b]^2$,and $u \in RS[a, b]^2$;we have ;

$$u^{*T}(t) = \begin{cases} (-1, 1) \quad if \quad t \in [a, t_1) \\ (1, -1) \quad if \quad t \in [t_1, t_2) \\ (-1, 1) \quad if \quad t \in [t_2, t_3) \\ (1, -1) \quad if \quad t \in [t_3, t_4) \quad etc \end{cases}$$

c) **Validity of ODE**:
Let u* be an optimal control of the above Min.condiion.Then we can solve the differential equation;
$\dot{x}$ = Ax + Bu* , (since u* satisfy the ODE).
on piecewise for ward starting with the first interval
Let x* be a solution of the differential equation,then (x^*, u^*) is a solution of the given linear optimal control problem.

Example: Solve the following Linear control problem:
f(x,u) = $\gamma^T x(4) + \int_0^4 (c^T x + d^T u)dt \dashrightarrow Min, (x, u) \in R$
R = $\{(x, u) \in K \times Q : \dot{x}(t) = Ax(t) + B(t)u(t)\}$
K = $\{x \in RCS^{(1)}[0, 4]^2 : x(0) = (-1, 3)^T\}$
Q = $\{u \in RS[0, 4]^2 : -1 \leq u_i \leq 1, t \in [0, 4], i = \{1, 2\}\}$

Where:A=$\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$,B=$\begin{pmatrix} t & o \\ o & t \end{pmatrix}$,

$c^T = (-t+4, -1), d^T = (-3, 2), \gamma^T = (0, -1)$
In order to calculate the optimal solution(x^*, u^*),we concider the necessary (and sufficient) conditions:
From:$L(x, \dot{x}, u, \lambda, t) = G(x, \dot{x}, \lambda, t) + W(u, \lambda, t)$
$\Rightarrow L(x, \dot{x}, \lambda, t) = \underbrace{c^T x + \lambda^T \dot{x} - \lambda^T Ax} + \underbrace{d^T u - \lambda^T Bu}$
1)**Validity of ELDE and TR**:
If x is a minimizer of f_λ(x,u) on K then x satisfies ELDE and TR. i)From the variational problem:
$G(x, \dot{x}, \lambda, t) = c^T x + \lambda^T \dot{x} - \lambda^T Ax$
we have: $G_x = c^T - \lambda^T A, and, \quad G_{\dot{x}} = \lambda^T$
Let x be the extremum of the variational problem G on K, then it satisfy ELDE
$\frac{d}{dt} G_{\dot{x}} = G_x$
$\frac{d}{dt}\lambda^T = c^T - \lambda^T \text{A}$
$\dot{\lambda}^T = c^T - \lambda^T A$
$\dot{\lambda} = -A^T \lambda + c$

$$\Rightarrow \begin{pmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + \begin{pmatrix} -t+4 \\ -1 \end{pmatrix}$$

To solve the homogenous part of the system of ODE;

$$\begin{pmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

Let $\lambda_1 = k_1 e^{\alpha t}$, and, $\lambda_2 = k_2 e^{\alpha t} are assumed solutions$

$$\begin{pmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} \alpha k_1 e^{\alpha t} \\ \alpha k_2 e^{\alpha t} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k_1 e^{\alpha t} \\ k_2 e^{\alpha t} \end{pmatrix}$$

$\Rightarrow \alpha k_1 = k_1 + k_2$
$\text{and}, \alpha k_2 = 0 + k_2$

$$\Rightarrow \begin{pmatrix} 1-\alpha & 1 \\ 0 & 1-\alpha \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The homogeneous system given above has non zero solution ,if the determinant of the coefficient matrix is zero.,i.e

$$\text{Det} \begin{pmatrix} 1-\alpha & 1 \\ 0 & 1-\alpha \end{pmatrix} = 0$$

$\Rightarrow (1-\alpha)^2 = 0$
$\Rightarrow \alpha_1 = 1, and, \alpha_2 = 1$

Now considering the system $\Rightarrow \begin{pmatrix} 1-\alpha & 1 \\ 0 & 1-\alpha \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

i) For $\alpha_1 = 1, we have, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow k_1 = 1, k_2 = 0$

So $:S_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t$

ii) For $\alpha_2 = 1, we have, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow k_1 = 0, k_2 = 1$

So $:S_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t$

Hence the general solution of the homogeneous system is

$\lambda(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2[\begin{pmatrix} 1 \\ 0 \end{pmatrix} te^t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t]$

$\Rightarrow \lambda_1(t) = c_1e^t + c_2te^t$

and,$\lambda_2(t) = c_2e^t$

To find the particular solution

Let $\lambda = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is assumed solution.

$\Rightarrow \lambda' = A\lambda + F$

$$\Rightarrow (\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix})' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} [\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}] + \begin{pmatrix} -t+4 \\ -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1t + b_1 \\ a_2t + b_2 \end{pmatrix} + \begin{pmatrix} -t+4 \\ -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 - 1 \\ a_2 \end{pmatrix} t + \begin{pmatrix} b_1 + b_2 + 4 \\ b_2 - 1 \end{pmatrix}$$

So, $a_1 = 1, a_2 = 0, b_1 = -4, b_2 = 1$

$$\Rightarrow \lambda = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\Rightarrow \lambda = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

$$\Rightarrow \lambda = \begin{pmatrix} t-4 \\ 1 \end{pmatrix}$$

Hence the general solution of the system is

$\lambda_1 = c_1e^t + c_2te^t + t - 4$

$\lambda_2 = c_2e^t + 1$

ii)**From, Transversality condition(TR)**,we have

$G_{\dot{x}}(x(b), \dot{x}(b), \lambda(b), b) = -H'(x(b))$

$\Rightarrow \lambda^T(b) = -\gamma^T$

$\Rightarrow \lambda(b) = -\gamma$

$\Rightarrow \lambda(4) = -\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$\Rightarrow \lambda(4) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda_1(4) = c_1e^4 + c_2 4e^4 + 4 - 4 = 0$

$\lambda_2(4) = c_2e^4 + 1 = 1$

$\Rightarrow c_1 = 0, and, c_2 = 0$

Therefore : $\begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} = \begin{pmatrix} t-4 \\ 1 \end{pmatrix}$

2)**From Minimum Condition**:

If u is a minimizer of f_λ(x,u) on Q then u satisfies:

$Min_{u\in[-1,1]}W(u,\lambda,t) = Min_{u\in[-1,1]}(d^T - \lambda^T B)u$

$Min_{u\in[-1,1]}W(u,\lambda,t) = (d^T - \lambda^T B)u^*$

$Min_{u\in[-1,1]}W(u,\lambda,t) = h^T u^*, (where h^T = d^T - \lambda^T B)$

But,$h^T = d^T - \lambda^T B \Rightarrow h = d - B^T\lambda$

From:$h(t) = d - B^T\lambda$

$\Rightarrow h(t) = \begin{pmatrix} -3 \\ 2 \end{pmatrix} - \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} t-4 \\ 1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \begin{pmatrix} -3 - t^2 + 4t \\ 2 - t \end{pmatrix}$

So :$h_1(t) = -3 - t^2 + 4t, and, h_2(t) = 2 - t$

$Min_{u\in[-1,1]}w(u,\lambda,t) = (h_1, h_2)\begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix}$

$Min_{u\in[-1,1]}W(u,\lambda,t) = h_1u_1^* + h_2u_2^*$

$Min_{u\in[-1,1]}W(u,\lambda,t) = (-3 - t^2 + 4t)u_1^* + (2 - t)u_2^*$

a)For $W_1(u) = (-3 - t^2 + 4t)u_1^*$

i)Let, $h_1 \geq 0$

i.e $-3 - t^2 + 4t \geq 0$

$\Rightarrow t^2 - 4t + 3 \leq 0$

$\Rightarrow 1 \leq t \leq 3$

hence, $u_1^* = -1$ is the minimizer of W_1,for,$1 \leq t \leq 3$

ii)$Let, h_1 \leq 0$

i.e $-3 - t^2 + 4t \leq 0$

$\Rightarrow t^2 - 4t + 3 \geq 0$

$\Rightarrow t \leq 1, or, t \geq 3$
hence, $u_1^* = 1$, is the minimizer of $W_1, for, t \leq 1, or, t \geq 3$
b)For $W_2(u) = (2-t)u_2^*$
i)$Let, h_2 \geq 0$
i.e $2 - t \geq 0 \Rightarrow t \leq 2$
So, $u_2^* =$ -1 is the minimizer of $W_2, for, t \leq 2$
ii)$Let, h_2 \leq 0$
i.e $2 - t \leq 0 \Rightarrow t \geq 2$
So,$u_2^* = 1$ is the minimizer of $W_2, for, t \geq 2$
Therefore we have
$u_1^*(t) = -1, for, 1 \leq t \leq 3$
$u_1^*(t) = 1, for, t \leq 1, or, t \geq 3$
And also we have
$u_2^*(t) = -1, for, t \leq 2$
$u_2^*(t) = 1, for, t \geq 2$
So we can generalize $u_1^*(t), for, 0 \leq t \leq 4$

$$u_1^*(t) = \begin{cases} 1 & t \in [0,1) \\ -1 & t \in [1,3) \\ 1 & t \in [3,4] \end{cases}$$

And we can generalize $u_2^*(t), for, 0 \leq t \leq 4$

$$u_2^*(t) = \begin{cases} -1 & t \in [0,2) \\ 1 & t \in [2,4] \end{cases}$$

Finally the solution u^*(t) can be expressed as:

$$u^{T*}(t) = (u_1^*(t), u_2^*(t)) = \begin{cases} (1,-1) & t \in [0,1) \\ (-1,-1) & t \in [1,2) \\ (-1,1) & t \in [2,3) \\ (1,1) & t \in [3,4] \end{cases}$$

3)**Validity of ODE**: The solution u*(t),gives us the possibility to calculate x by the system of differential equations:
i.e :$\dot{x}(t) = Ax(t) + B(t)u^*(t)$

$$\Rightarrow \begin{pmatrix} \dot{x}_1(t) \\ \dot{x_2}(t) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

case(i): $For, t \in [0,1), we, have \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} t \\ -t \end{pmatrix}$$

Now let's solve the homogenous system :

i.e $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\Rightarrow |A - \lambda I_2| = 0$

$\Rightarrow Det \begin{pmatrix} -1-\lambda & 0 \\ -1 & -1-\lambda \end{pmatrix} = 0$

$\Rightarrow (-1-\lambda)^2 = 0$

$\Rightarrow \lambda_1 = -1, and, \lambda_2 = -1$

$From \begin{pmatrix} -1-\lambda & 0 \\ -1 & -1-\lambda \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Case(i):For $\lambda_1 = -1$

$\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow k_1 = 0, k_2 = 1$

so,$s_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$

Case(ii):For $\lambda_2 = -1$

$\Rightarrow \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow k_1 = -1, k_2 = 0$

so :$s_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{-t}$

So,the general solution of the homogeneous system is

x = $c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} + c_2[\begin{pmatrix} 0 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{-t}]$

$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -c_2e^{-t} \\ c_1e^{-t} + c_2e - t \end{pmatrix}$

To find the particular solution

Let $x = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ be the assumed solution.

This implies x' = Ax + F(t)

$\Rightarrow (\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix})' = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} [\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}] + \begin{pmatrix} t \\ -t \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} a_1t + b_1 \\ a_2t + b_2 \end{pmatrix} + \begin{pmatrix} t \\ -t \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -a_1+1 \\ -a_1-a_2-1 \end{pmatrix} t + \begin{pmatrix} -b_1 \\ -b_1-b_2 \end{pmatrix}$$

$\Rightarrow a_1 = 1, a_2 = 2, b_1 = -1, b_2 = 3$

So,the particular solution is

$x = \begin{pmatrix} 1 \\ -2 \end{pmatrix} t + \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t-1 \\ -2t+3 \end{pmatrix}$

Hence the general solution of the system is

$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} -1+t-c_2e^{-t} \\ 3-2t+c_1e^{-t}+c_2te^{-t} \end{pmatrix}$

Using the boundary conditon $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$,

For $t \in [0,1)$ we have the optimal solution;

$\begin{pmatrix} x_1^*(t) \\ x_2^*(t) \end{pmatrix} = \begin{pmatrix} -1+t \\ 3-2t \end{pmatrix}, for, \begin{pmatrix} u_1^*(t) \\ u_2^*(t) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Case(ii) $t \in [1,2), for \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$,we have

$$\Rightarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

Solving this system of ODE,we get;

$x_1(t) = 1-t-c_2e^{-t}, and, x_2(t) = -1+c_1e^{-t}+c_2te^{-t}$

Since x is continuous at x=1 from case(i),we have $\begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,(is the boundary condition for case(ii)

For t $\in [1,2)$ we have the optimal solution of

$\begin{pmatrix} x_1^*(t) \\ x_2^*(t) \end{pmatrix} = \begin{pmatrix} 1-t \\ -1+2e^{1-t} \end{pmatrix}, for, \begin{pmatrix} u_2^*(t) \\ u_2^*(t) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

case(iii) $t \in [2,3), for, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$,we,have;

$$\Rightarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Solving this system of ODE,weget,

$x_1(t) = 1-t-c_2e^{-t}, and, x_2(t) = -3+2t+c_1e^{-t}+c_2te^{-t}$

Since x is continuous at x=2 from case(ii),we have $\begin{pmatrix} x_1(2) \\ x_2(2) \end{pmatrix} = \begin{pmatrix} -1 \\ -1+2e^{-t} \end{pmatrix}$(is the boundary condition for case(iii)

For $t \in [2,3)$ we have an optimal solution of

$\begin{pmatrix} x_1^*(t) \\ x_2^*(t) \end{pmatrix} = \begin{pmatrix} 1-t \\ -3+2t+2e(1-e)e^{-t} \end{pmatrix}, for, \begin{pmatrix} u_1^*(t) \\ u_2^*(t) \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Case(iv) $t \in [3,4], for \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,we have

$$\Rightarrow \begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

solving this system of ODE,we get

$x_1(t) = -1 + t - c_2e^{-t}, and, x_2(t) = 1 + c_1e^{-t} + c_2te^{-t}$

Since x is continuous at x=3 from case(iii) we have $\begin{pmatrix} x_1(3) \\ x_2(3) \end{pmatrix} = \begin{pmatrix} -2 \\ 3 + 2e^{-2}(1-e) \end{pmatrix}$(is the bounday condition for case(iv))

For $t \in [3,4]$ we have an optimal solution of

$$\begin{pmatrix} x_1^*(t) \\ x_2^*(t) \end{pmatrix} = \begin{pmatrix} -1 + t - 4e^{3-t} \\ 1 - 2e^{1-t}(-1 + e + 5e^2) + 4te^{3-t} \end{pmatrix}, for, \begin{pmatrix} u_2^*(t) \\ u_2^*(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

3.5 Quadratic optimal control problems

3.5.1 Quadratic optimal control problem with free right end point

Let $\Gamma \in \mathbb{R}^{n\times n}$ be a positive semi definite symmetrical matrix.

$A \in RS[a,b]^{n\times n}$

$B \in RS[a,b]^{n\times m}$

$C \in RS[a,b]^{n\times n}$ be a positive semi definite and symmetrical matrix $\forall t \in [a,b]$

$D \in RS[a,b]^{m\times m}$ be a positive definite and symmetrical matrix $\forall t \in [a,b]$

$\gamma \in \mathbb{R}^n$

$c \in RS[a,b]^n$

$d \in RS[a,b]^m$

In this section we consider optimal control problems ,where the functions H and g are quadratic with respect to x and u,and the differential equation is linear .

That is problem of the following type:

$f(x,u) = \frac{1}{2}x^T(b)\Gamma x(b) + \gamma^T x(b) + \int_a^b (\frac{1}{2}x^T Cx + c^T x + \frac{1}{2}u^T Du + d^T u)dt \to Min, \quad , (x,u) \in R$

R $= \{(x,u) \in K \times Q : \dot{x} = Ax + Bu, t \in [a,b]\}$

K $= \{x \in RCS^{(1)}[a,b]^n : x(a) = \alpha\}$

Q $= \{u \in RS[a,b]^m\}$

This type of problem is called Quadratic optimal control problem

. Then we have the following Lagrange function

$L(x,\dot{x},u,\lambda,t) = \frac{1}{2}x^T Cx + c^T x + \frac{1}{2}u^T Du + d^T u)dt + \lambda^T(\dot{x} - Ax - Bu)$

$L(x,\dot{x},u,\lambda,t) = \frac{1}{2}x^T Cx + c^T x + \frac{1}{2}u^T Du + d^T u)dt + \lambda^T\dot{x} - \lambda^T Ax - \lambda^T Bu$

$$L(x,\dot{x},u,\lambda,t) = \underbrace{\frac{1}{2}x^T Cx + c^T x + \lambda^T\dot{x} - \lambda^T Ax}_{} + \underbrace{\frac{1}{2}u^T Du + d^T u - \lambda^T Bu}_{}$$

And then we have :

$G(x, \dot{x}, \lambda, t) = \frac{1}{2}x^T Cx + c^T x + \lambda^T \dot{x} - \lambda^T Ax$

$W(u, \lambda, t) = \frac{1}{2}u^T Du + d^T u$ - $\lambda^T Bu$

Obviously G is convex with regard to x and $\dot{x}$, W is convex with regard to u.

So we have a sufficient condition for(x*,u*) to be a solution of our

optimal control problem.

a) From :$W(u, \lambda, t) = \frac{1}{2}u^T Du + d^T u$ - $\lambda^T Bu$

And by the condition of Min of Theorem 3.1.1 we have.

$Min_{u \in Q} W(u, \lambda, t) = Min_{u \in Q}\{\frac{1}{2}u^T Du + d^T u - \lambda^T Bu\}$

So by convexity and differentiability of W,we have

$\frac{\partial W}{\partial u} = 0$

$\Rightarrow \frac{\partial}{\partial u} (\frac{1}{2}u^T Du + d^T u - \lambda^T Bu) = 0$

$\Rightarrow u^{*T} D + d^T - \lambda^T B = 0$

$\Rightarrow (u^{*T} D + d^T - \lambda^T B)^T = (0)^T$

$\Rightarrow (u^{*T} D)^T + (d^T)^T - (\lambda^T B)^T = (0)^T$

$\Rightarrow (D^T u^*) + d - B^T \lambda = 0$

Since D is symmetric; we have

$\Rightarrow Du^* + d - B^T \lambda = 0$

Since D is positive definite then the determinant of D $\neq 0$; then we have

$Du^* + d - B^T \lambda = 0$

$\Rightarrow Du^* = B^T \lambda - d$

$\Rightarrow u^* = D^{-1}B^T \lambda - D^{-1}d$

b)When we substitute this solution(u*) in the differential equation $\dot{x} = Ax + Bu$

we have $\dot{x} = Ax + Bu^*$

$\Rightarrow \dot{x} = Ax + B(D^{-1}B^T \lambda - D^{-1}d)$

$\Rightarrow \dot{x} = Ax + BD^{-1}B^T \lambda - BD^{-1}d$,such that $x^*(a) = \alpha$ (IC)

c) From:$G(x, \dot{x}, \lambda, t) = \frac{1}{2}x^T Cx + c^T x + \lambda^T \dot{x} - \lambda^T Ax$

we have $G_x = Cx + c^T - \lambda^T A$ and $G_{\dot{x}} = \lambda^T$

By theorem 3.1.3,if x is a minimizer of fλ(x,u) on K,it satisfies ELDE and TR condition,then,

$\frac{d}{dt} G_{\dot{x}} = G_x$

$\Rightarrow \frac{d}{dt} \lambda^T = Cx + c^T - \lambda^T A$

$\Rightarrow \dot{\lambda}^T = Cx + c^T - \lambda^T A$

$\Rightarrow (\dot{\lambda}^T)^T = (Cx + c^T + \lambda^T A)^T$

$\Rightarrow \dot{\lambda} = Cx + c - A^T \lambda$

$\Rightarrow \dot{\lambda} = -A^T \lambda + Cx + c$

d) From Transversality condition(TR)

$G_{\dot{x}}(x(b), \dot{x}(b), \lambda(b), b) = -H'(x(b))$

$G_{\dot{x}}(x(b), \dot{x}(b), \lambda(b), b) = -[\frac{1}{2}x^T(b)\Gamma x(b) + \gamma^T x(b)]$

$\Rightarrow \lambda^T(b) = -[x(b)\Gamma + \gamma^T]$

$\Rightarrow \lambda^T(b) = -\Gamma x(b) - \gamma^T$

$\Rightarrow \lambda(b) = -\Gamma x(b) - \gamma$

So From equations (b) and (c) we have

$$\begin{cases} \dot{x} = Ax + BD^{-1}B^T\lambda - BD^{-1}d \\ \dot{\lambda} = Cx - A^T\lambda + c \end{cases}$$

Where $x(a) = \alpha$(IC) and $\lambda(b) = -\Gamma x(b) - \gamma$(TR)
Hence there will be systems of differential equations

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & BD^{-1}B^T \\ C & -A^T \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} + \begin{pmatrix} -BD^{-1} \\ c \end{pmatrix}$$

If we have the solution(x^*, λ^*) of this system of differential equations,then we can calculate the control function u* by substitute λ^* in the equation $u^* = D^{-1}B^T\lambda^* - D^{-1}d$

Normal boundary value problem:-
If $\Gamma = 0$,we have optimal control problem;
$f(x, u) = \gamma x(b) + \int_a^b (\frac{1}{2}x^T Cx + c^T x + \frac{1}{2}u^T Du + d^T u)dt \rightarrow Min, \quad ,(x, u) \in \mathbb{R}$
R $= \{(x, u) \in K \times Q : \dot{x} = Ax + Bu, t \in [a, b]\}$
K $= \{x \in RCS^{(1)}[a, b]^n : x(a) = \alpha\}$
Q $= \{u \in RS[a, b]^m\}$
Then we get the solutions;x* and λ^* as

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & BD^{-1}B^T \\ C & -A^T \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} + \begin{pmatrix} -BD^{-1} \\ c \end{pmatrix}, \quad x(a) = \alpha, \quad x(b) = -\gamma$$

If we have the solution(x^*, λ^*) of this system of differential equations,then we can calculate the control function u* by substitute λ^* in the equation $u^* = D^{-1}B^T\lambda^* - D^{-1}d$

Example:Find the optimal control u*(t) and the corresponding optimal trajectory x*(t) for the following optimal control problem
f(x,u) $= \frac{-14}{9}x(1) + \int_0^1 [\frac{5}{2}x^2(t) + \frac{1}{2}u^2(t) + u(t)]dt \dashrightarrow Min, (x, u) \in R$
R $= \{(x, u) \in K \times Q : \dot{x}(t) = 2x(t) + u(t); t \in [0, 1]\}$
K $= \{x \in RCS^{(1)}[0, 1] : x(0) = 2/9\}$;
Q $= \{u \in RS[0, 1]\}$ Where γ = -14/9,Γ = 0,C = 5,D = 1,A = 2,B = 1,c = 0,d = 1.
From the Lagrange function
$L(x, \dot{x}, u, \lambda, t) = \frac{5}{2}x^2 + \frac{1}{2}u^2 + u + \lambda[\dot{x} - 2x - u]$
$L(x, \dot{x}, u, \lambda, t) = \underbrace{\frac{5}{2}x^2 + \lambda\dot{x} - 2\lambda x} + \underbrace{\frac{1}{2}u^2 + u - \lambda u}$
Where,$G(x, \dot{x}, \lambda, t) = \frac{5}{2}x^2 + \lambda\dot{x} - 2\lambda x$
And,$W(u, \lambda, t) = \frac{1}{2}u^2 + u - \lambda u$
Obviously,G is convex with respect to x and $\dot{x}$. And also W is convex with respect to u,then we have sufficient condition for (x^*, u^*) to be an optimal solution of the given problem
. 1) **Minimum Condition**
If u is a minimizer of f_λ(x,u) on Q then u satisfies:

$Min_{u\in Q}W(u,\lambda,t) = Min_{u\in Q}\{\frac{1}{2}u^2 + u - \lambda u\}$

So,by convexity and differentiability of W,we have :

$\frac{\partial W}{\partial u} = 0$

$\Rightarrow W_u(\frac{1}{2}u^2 + u - \lambda u) = 0$

$\Rightarrow u(t) = \lambda(t) - 1$

2)**Validity of ODE**:

since the solution,u = u* satisfies ODE,

$\Rightarrow \dot{x} = 2x + u^*$

So,we,have $\dot{x} = 2x + \lambda - 1, with, IC, x(0) = 2/9$

3)**Validity of ELDE and TR**:

If x is a minimizer of $f_\lambda(x,u)$ on K then x satisfies ELDE and TR.

$G(x,\dot{x},\lambda,t) = \frac{5}{2}x^2 + \lambda\dot{x} - 2\lambda x$

$G_x = 5x - 2\lambda$ and $G_{\dot{x}} = \lambda$

i)Let x be an extremal of the variational problem,then it satisfies ELDE,i.e

$\frac{d}{dt}G_{\dot{x}} = G_x$

$\Rightarrow \frac{d}{dt}\lambda = 5x - 2\lambda$

$\Rightarrow \dot{\lambda} = 5x - 2\lambda$

ii)From TR Condition

$G_{\dot{x}}(x(b),\dot{x}(b),\lambda(b),b) = -H'(x(b))$

$G_{\dot{x}}(x(1),\dot{x}(1),\lambda(1),1) = -H'(x(1))$

$\lambda(1) = -(\frac{-14}{9}x(1))$

$\lambda(1) = \frac{14}{9}$

So we have,$\dot{\lambda} = 5x - 2\lambda, with, TR, \lambda(1) = \frac{14}{9}$

Now taking system of differential equations;

$\dot{x} = 2x + \lambda - 1, if, x(0) = 2/9$

$\dot{\lambda} = 5x - 2\lambda if, \lambda(1) = 14/9$

This can be written as: $\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

First,Let's solve the homogeneous part:

$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix}$

Let $x(t) = k_1e^{\alpha t}$ and $\lambda(t) = k_2e^{\alpha t}$ are assumed soltions

$\Rightarrow \dot{x} = \alpha k_1e^{\alpha t}$ and $\dot{\lambda} = \alpha k_2e^{\alpha t}$

$$\begin{pmatrix} \alpha k_1e^{\alpha t} \\ \alpha k_2e^{\alpha t} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} k_1e^{\alpha t} \\ k_2e^{\alpha t} \end{pmatrix}$$

This implies

$\alpha k_1 = 2k_1 + k_2$

$\alpha k_2 = 5k_1 - 2k_2$

This implies

$(2 - \alpha)k_1 + k_2 = 0$

$5k_1 + (-2 - \alpha)k_2 = 0$

If the determinant of the coefficient matrix of homogeneous equation is zero,then the system has infinite solution

$$\Rightarrow Determinantof \begin{pmatrix} 2-\alpha & 1 \\ 5 & -2-\alpha \end{pmatrix} = 0$$

$\Rightarrow (2-\alpha)(-2-\alpha) - 5 = 0$

$\Rightarrow -4 - \alpha^2 - 5 = 0$

$\Rightarrow \alpha^2 - 9 = 0$

$\Rightarrow \alpha_1 = -3, \alpha_2 = 3$

From the given homogeneouse equation,

$$\begin{pmatrix} 2-\alpha & 1 \\ 5 & -2-\alpha \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i) for,$\alpha = -3$,we have

$$\begin{pmatrix} 5 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow k_1 = -1, k_2 = 5$

$$\Rightarrow S_1 = \begin{pmatrix} -1 \\ 5 \end{pmatrix} e^{-3t}$$

ii)And for $\alpha = 3$,we have

$$\begin{pmatrix} -1 & 1 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow k_1 = 1, k_2 = 1$

$$\Rightarrow S_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

So the general solution of he homogenous system is

$$\begin{pmatrix} x \\ \lambda \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 5 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

To find he particular solution,Let x = $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

$\Rightarrow x' = Ax + F(t)$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}' = \begin{pmatrix} 2 & 1 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

This imlies

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2a_1 + a_2 \\ 5a_1 - 2a_2 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

This implies

$2a_1 + a_2 = -1$

$5a_2 - 2a_2 = 0$

$\Rightarrow; a_1 = 2/9, a_2 = 5/9$

Hence the general solution of the system is

$$\begin{pmatrix} x \\ \lambda \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 5 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + \begin{pmatrix} 2/9 \\ 5/9 \end{pmatrix}$$

Therefore

$x(t) = -c_1 e^{-3t} + c_2 e^{3t} + 2/9, x(0) = 2/9$

$\lambda(t) = 5c_1 e^{-3t} + c_2 e^{3t} + 5/9, \lambda(1) = 14/9$

Hence

$x^*(t) = \frac{-1}{5e^{-3}+e^{-3}} e^{-3t} + \frac{1}{5e^{-3}+e^{-3}} e^{3t} + 2/9$

$\lambda^*(t) = \frac{5}{5e^{-3}+e^{-3}} e^{-3t} + \frac{1}{5e^{-3}+e^{-3}} e^{3t} + 5/9$

And we have $u(t) = \lambda(t) - 1$

$u^*(t) = \frac{5}{5e^{-3}+e^{-3}} e^{-3t} + \frac{1}{5e^{-3}+e^{-3}} e^{3t} + 4/9$

So the optimal control u^*(t) and the optimal trajectory x*(t) of the problem are

$u^*(t) = \frac{5}{5e^{-3}+e^{-3}} e^{-3t} + \frac{1}{5e^{-3}+e^{-3}} e^{3t} + 4/9$

$x^*(t) = \frac{-1}{5e^{-3}+e^{-3}} e^{-3t} + \frac{1}{5e^{-3}+e^{-3}} e^{3t} + 2/9$

3.5.2 Linear-Quadratic(LQ)optimal control problem

From the quadratic control problem,if γ=0,c=0,d=0;we get

$f(x,u) = \frac{1}{2}x^T(b)\Gamma x(b) + \int_a^b (\frac{1}{2}x^T Cx + \frac{1}{2}u^T Du)$dt

R = $\{(x,u) \in K \times Q : \dot{x} = Ax + Bu, t \in [a,b]\}$

K = $\{x \in RCS^{(1)}[a,b]^n : x(a) = \alpha\}$

Q = $\{u \in RS[a,b]^m\}$

which is called Linear-Quadratic optimal control problem.

To solve the Linear-Quadratic optimal control problem : Let : $\lambda(t) = -P(t)x(t)$, P$\in$ $C^{(1)}[a,b]^{n\times n}$

When we substitute -Px in differential equation $\dot{\lambda} = -A^T\lambda + Cx$ we get :

$\dot{\lambda} = -A^T(-Px) + Cx$

$\dot{\lambda} = A^T(Px) + Cx$

$\dot{\lambda} = (A^TP + C)x$

But by using product rule on ;$\lambda = -Px$, we have

$\dot{\lambda} = -\dot{P}x - P\dot{x}$

$\dot{\lambda} + \dot{P}x + P\dot{x}$=0

Substituting;$\dot{\lambda} = (A^TP + C)x$ in $\dot{\lambda} + \dot{P}x + P\dot{x}$=0,we get

$(A^TP + C)x + \dot{P}x + P\dot{x} = 0, \rightarrow (*)$

Substituting;$\dot{x} = Ax + BD^{-1}B^T\lambda$,in (*),we get

$(A^TP + C)x + \dot{P}x + P(Ax + BD^{-1}B^T\lambda) = 0$

Finally Substituting -Px in place of λ gives

$(A^TP + C)x + \dot{P}x + PAx + PBD^{-1}B^T(-Px) = 0$

$(A^TP + C)x + \dot{P}x + PAx - PBD^{-1}B^TPx = 0$
$(A^TP + C + \dot{P} + PA - PBD^{-1}B^TP)x = 0$, we have x=0 as a trivial solution
$\dot{P}x = (P(BD^{-1}B^T)P - A^TP - PA - C)x$
$\dot{P} = P(BD^{-1}B^T)P - A^TP - PA - C.$
$\dot{P} = (BD^{-1}B^T)P^2 - (A^T + A)P - C,$ P(b) = Γ,(Ricatti differential equation)
By solving the Ricatti ODE,we get a unique solution P^*,where P^* is a symmetrical matrix
If we have P^*,then we can calculate x^* as a solution of the differential equation:
$\dot{x} = [A - (BD^{-1}B^TP^*]x, \quad x(a) = \alpha$
After that we can calculate λ^* by;
$\lambda^* = -P^*x^*, \qquad \lambda^*(b) = -\Gamma x^*(b)$
Then we can calculate u^* by;
$u^* = -D^{-1}B^TP^*x^*$(is called optimal controller)
Then (x^*, u^*) is a solution of our optimal control problem.
Hence,the optimal control input u is described in the form of a feed back by a linear function of the state variable x.
The matrix R $= -D^{-1}B^TP^*$,is called feed back matrix

.

ExampleFind the optimal state function and the optimal control function of the following quadratic optimal control problem. :
$f(x, u) = \gamma^Tx(1) + \int_0^1(c^Tx + \frac{1}{2}x^TCx + \frac{1}{2}u^TDu)dt \dashrightarrow Min, (x, u) \in R$
$R = \{(x, u) \in K \times Q : \dot{x}(t) = Ax(t) + Bu(t), t \in [0, 1]\}$
$k = \{x \in RCS^{(1)}[0, 1]^2 : x(0) = (-1/3, -1/3)^T\}$
$Q = \{u \in RS[0, 1]^2\}$

where A=$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$,B = $\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$,D = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,C = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,c=$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$,$\gamma$ = $\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$

From the lagrange function we have:
$L(x, \dot{x}, u, \lambda, t) = c^Tx + \frac{1}{2}x^TCx + \frac{1}{2}u^TDu + \lambda^T(\dot{x} - Ax - Bu)$
$$L(x, \dot{x}, u, \lambda, t) = \underbrace{c^Tx + \frac{1}{2}x^TCx + \lambda^T\dot{x} - \lambda^TAx}_{} + \underbrace{\frac{1}{2}u^TDu - \lambda^TBu}_{}$$

Where
$G(x, \dot{x}, \lambda, t) = c^Tx + \frac{1}{2}x^TCx + \lambda^T\dot{x} - \lambda^TAx$
$W(u, \lambda, t) = \frac{1}{2}u^TDu - \lambda^TBu$
i)Clearly G is convex with respect (x,$\dot{x}$), and W is convex with respect to u,then we have a sufficient condition for (x^*, u^*) to be an optimal soluion of the given OCP.
ii)**Minimum condition**
If u is a minimizer of f(x,u) on Q then u satisfies ;
$Min_{u \in Q}W(u, \lambda, t) = Min_{u \in Q}\{\frac{1}{2}u^TDu - \lambda^TBu\}$
By convexity and differentiability of W,we have:
$\Rightarrow \frac{\partial w}{\partial u} = 0$

$\Rightarrow W_u(\frac{1}{2}u^TDu - \lambda^TBu) = 0$
$\Rightarrow u^TD - \lambda^TB = 0$
$\Rightarrow Du^* - B^T\lambda = 0$
$\Rightarrow u^* = D^{-1}B^T\lambda$
iii)**Validity of ODE**:
since u = u* satisfy ODE,
$\Rightarrow \dot{x} = Ax + Bu^*$
$\Rightarrow \dot{x} = Ax + BD^{-1}B^T\lambda$
iv)**Validity of ElDE and TR** :
If x is a minimizer of f_λ(x,u) on K then x satisfies ELDE and TR.
From;$G(x, \dot{x}, \lambda, t) = c^Tx + \frac{1}{2}x^TCx + \lambda^T\dot{x} - \lambda^TAx$
$\Rightarrow G_x = c^T + x^TC - \lambda^TA$ and $G_{\dot{x}} = \lambda^T$
i)Let x be an extremal of the variational problem,then it satisfies ELDE;
$\Rightarrow \frac{d}{dt}G_{\dot{x}} = G_x$
$\Rightarrow \frac{d}{dt}\lambda^T = c^T + x^TC - \lambda^TA$
$\Rightarrow \dot{\lambda}^T = c^T + x^TC - \lambda^TA$
$\Rightarrow \dot{\lambda}(t) = Cx - A^T\lambda + c$
ii)From TR Condition
$G_{\dot{x}}(x(b), \dot{x}(b), \lambda_0(b), b) = -H'(x(b))$
$G_{\dot{x}}(x(1), \dot{x}(1), \lambda_0(b), 1) = -H'(x(1))$
$\lambda(t)|_{t=1} = -\gamma$

$$\lambda(1) = \begin{pmatrix} -2/3 \\ -1/3 \end{pmatrix}$$

From (iii) and (iv),we have

$$\dot{x}(t) = Ax + BD^{-1}B^T\lambda, x(0) = \begin{pmatrix} -1/3 \\ -1/3 \end{pmatrix}$$

$$\dot{\lambda}(t) = Cx - A^T\lambda + c, \lambda(1) = \begin{pmatrix} -2/3 \\ -1/3 \end{pmatrix}$$

So,we let's solve the following system of differential equations

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & BD^{-1}B^T \\ C & -A^T \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} + \begin{pmatrix} 0 \\ c \end{pmatrix}$$

But,$BD^{-1}B^T = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

And, $-A^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

So from the system of equations

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & BD^{-1}B^T \\ C & -A^T \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} + \begin{pmatrix} 0 \\ c \end{pmatrix}$$

This implies

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

where $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} -1/3 \\ -1/3 \end{pmatrix}, and, \begin{pmatrix} \lambda_1(1) \\ \lambda_2(1) \end{pmatrix} = \begin{pmatrix} -2/3 \\ -1/3 \end{pmatrix}$

First Let's solve the homogeneous system: i.e

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1 \\ \lambda_1 \\ \lambda_2 \end{pmatrix}$$

To find the eigen values ; let $|A - tI_4| = 0$,i.e

$$Det \begin{pmatrix} -1-t & 0 & 1 & -1 \\ 0 & 1-t & -1 & 1 \\ 1 & 0 & 1-t & 0 \\ 0 & 1 & 0 & -1-t \end{pmatrix} = 0$$

$\Rightarrow t^4 - 4t^2 + 3 = 0$
$\Rightarrow (t-1)(t^3 + t^2 - 3t - 3) = 0$
$\Rightarrow t_1 = -1, t_2 = -\sqrt{3}, t_3 = 1, t_4 = \sqrt{3}$

So,from $\begin{pmatrix} -1-t & 0 & 1 & -1 \\ 0 & 1-t & -1 & 1 \\ 1 & 0 & 1-t & 0 \\ 0 & 1 & 0 & -1-t \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

1) For t = -1,we have

$$\begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 2 & -1 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow S_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix} e^{-t}$$

For,t $= -\sqrt{3}$ we have

$$\begin{pmatrix} -1+\sqrt{3} & 0 & 1 & -1 \\ 0 & 1+\sqrt{3} & -1 & 1 \\ 1 & 0 & 1+\sqrt{3} & 0 \\ 0 & 1 & 0 & -1+\sqrt{3} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow S_2 = \begin{pmatrix} 1+\sqrt{3} \\ 1-\sqrt{3} \\ -1 \\ 1 \end{pmatrix} e^{-\sqrt{3}t}$$

For,t = 1,we have;

$$\begin{pmatrix} -2 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow S_3 = \begin{pmatrix} -0 \\ 2 \\ 1 \\ 1 \end{pmatrix} e^{-t}$$

For,t $= \sqrt{3} we, have$

$$\begin{pmatrix} -1-\sqrt{3} & 0 & 1 & -1 \\ 0 & 1-\sqrt{3} & -1 & 1 \\ 1 & 0 & 1-\sqrt{3} & 0 \\ 0 & 1 & 0 & -1-\sqrt{3} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow S_4 = \begin{pmatrix} 1-\sqrt{3} \\ 1+\sqrt{3} \\ -1 \\ 1 \end{pmatrix} e^{-\sqrt{3}t}$$

So the general solution of the homogenous system is;

$$c_1 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1+\sqrt{3} \\ 1-\sqrt{3} \\ -1 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix} e^{-\sqrt{3}t} + c_4 \begin{pmatrix} 1-\sqrt{3} \\ 1+\sqrt{3} \\ -1 \\ 1 \end{pmatrix} e^{\sqrt{3}t}$$

To find he particular solution

Let $x = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$

So,we have:x' = Ax + F(t)

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}' = \begin{pmatrix} -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -a_1+a_3-a_4 \\ a_2-a_3+a_4 \\ a_1+a_3 \\ a_2-a_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$-a_1 + a_3 - a_4 = 0$

$a_2 - a_3 + a_4 = 0$

$a_1 + a_3 + 1 = 0$

$a_2 - a_4 = 0$

Hence : $a_1 = 0, a_2 = 0, a_3 = -1, a_4 = 0, \quad \Rightarrow x = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$

The general solutions for the system of differential equation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ is given by } \begin{pmatrix} x_1(t) \\ x_2(t) \\ \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} =$$

$$c_1 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1+\sqrt{3} \\ 1-\sqrt{3} \\ -1 \\ 1 \end{pmatrix} e^{-\sqrt{3}t} + c_3 \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix} e^{t} + c_4 \begin{pmatrix} 1-\sqrt{3} \\ 1+\sqrt{3} \\ -1 \\ 1 \end{pmatrix} e^{\sqrt{3}t} + \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

This can be written as
$x_1(t) = -2c_1e^{-t} + c_2e^{-\sqrt{3}t} + \sqrt{3}c_2e^{-\sqrt{3}t} + c_4e^{\sqrt{3}t} - \sqrt{3}c_4e^{\sqrt{3}t}$
$x_2(t) = c_2e^{-\sqrt{3}t} - \sqrt{3}c_2e^{-\sqrt{3}t} + 2c_3e^{-t} + c_4e^{\sqrt{3}t} + \sqrt{3}c_4e^{\sqrt{3}t}$
$\lambda_1(t) = c_1e^{-t} - c_2e^{-\sqrt{3}t} + c_3e^{t} - c_4e^{-\sqrt{3}t} - 1$
$\lambda_2(t) = c_1e^{-t} + c_2e^{-\sqrt{3}t} + c_3e^{t} + c_4e^{\sqrt{3}t}$

And we have boundary and transversality conditions,i.e : $\begin{pmatrix} x_1(0) \\ x_2(0) \\ \lambda_1(1) \\ \lambda_2(1) \end{pmatrix} = \begin{pmatrix} -1/3 \\ -1/3 \\ -2/3 \\ -1/3 \end{pmatrix}$

Thus the solution of the system of differential equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{is given by}$$

$x_1(t) = \frac{-1}{3} - 2c_1e^{-t} + c_2e^{-\sqrt{3}t} + \sqrt{3}c_2e^{-\sqrt{3}t} + c_4e^{\sqrt{3}t} - \sqrt{3}c_4e^{\sqrt{3}t}$
$x_2(t) = \frac{-1}{3} + c_2e^{-\sqrt{3}t} - \sqrt{3}c_2e^{-\sqrt{3}t} + 2c_3e^{-t} + c_4e^{\sqrt{3}t} + \sqrt{3}c_4e^{\sqrt{3}t}$
$\lambda_1(t) = \frac{-2}{3} + c_1e^{-t} - c_2e^{-\sqrt{3}t} + c_3e^{t} - c_4e^{-\sqrt{3}t} - 1$
$\lambda_2(t) = \frac{-1}{3} + c_1e^{-t} + c_2e^{-\sqrt{3}t} + c_3e^{t} + c_4e^{\sqrt{3}t}$

When we solve the following system of linear equations:
$x_1(0) = -2c_1 + (1+\sqrt{3})c_2 + (1-\sqrt{3})c_4 = -1/3$
$x_2(0) = (1-\sqrt{3})c_2 + 2c_3 + (1+\sqrt{3})c_4 = -1/3$
$\lambda(1) = -e^{-1}c_1 + e^{-\sqrt{3}}c_2 - ec_3 + e^{\sqrt{3}}c_4 = -1/3$
$\lambda(1) = e^{-1}c_1 + e^{-\sqrt{3}}c_2 + ec_3 + e^{\sqrt{3}}c_4 = -1/3, then,$
$c_1 \approx$ 1.4
$c_2 \approx$ 12.3
$c_3 \approx$ -0.2
$c_4 \approx$ -0.4

But from Minimum condition we have: $u^* = D^{-1}B^T\lambda^*$

$$\Rightarrow u^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \lambda$$
$$\Rightarrow \begin{pmatrix} u_2 \\ u_1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \begin{pmatrix} \lambda_1 - \lambda_2 \\ 0 \end{pmatrix}$$
So $u_1(t) = -2c_2e^{-\sqrt{3}t} - 2c_4e^{\sqrt{3}t} - 1$
$and \quad u_2(t) = 0$

Hence The optimal control function and the optimal state function of the problem are:
$x_1^*(t) = -2.8e^{-t} + 12.3e^{-\sqrt{3}t} + 12.3\sqrt{3}e^{-\sqrt{3}t} - 0.4e^{\sqrt{3}t} + 0.4\sqrt{3}e^{\sqrt{3}t}$
$x_2^*(t) = 12.3e^{-\sqrt{3}t} - 12.3\sqrt{3}e^{-\sqrt{3}t} - 0.4e^{-t} - 0.4e^{\sqrt{3}t} - 0.4\sqrt{3}e^{\sqrt{3}t}$
$u_1^*(t) = -24.6e^{-\sqrt{3}t} + 0.8e^{\sqrt{3}t} - 1$
$u_2^*(t) = 0$

3.5.3 Quadratic Control problems with fixed end points

In this part we consider the following quadratic optimal control problems with fixed end points
$f(x,u) = \frac{1}{2}x^T(b)\Gamma x(b) + \gamma^T x(b) + \int_a^b(\frac{1}{2}x^T Cx + c^T x + \frac{1}{2}u^T Du + d^T u)dt \rightarrow Min, (x,u) \in \mathbb{R}$
$\mathrm{R} = \{(x,u) \in K \times Q : \dot{x} = Ax + Bu, t \in [a,b]\}$
$\mathrm{K} = \{x \in RCS^{(1)}[a,b]^n : x(a) = \alpha, x(b) = \beta\}$
$\mathrm{Q} = \{u \in RS[a,b]^m\}$
Now it is possible to take constant x(b) - $\beta = 0$ (or better only the term x(b) in the objective function as additional term of the function H(x(b)).Then we have an optimal control problem with free right end point.Among the solutions we have to choose those for which x(b)-β=0 holds.

Then by adding this term we get the following quadratic optimal control problem with free right end point
Min;$\hat{f}(x,u) = \frac{1}{2}x^T(b)\Gamma x(b) + \gamma^T x(b) + \mu^T x(b) + \int_a^b(\frac{1}{2}x^T Cx + c^T x + \frac{1}{2}u^T Du + d^T u)dt, (x,u) \in R$
$\hat{R} = \{(x,u) \in K \times Q : \dot{x} = Ax + Bu, t \in [a,b]\}$
$\hat{K} = \{x \in RCS^{(1)}[a,b]^n : x(a) = \alpha\}$
$\mathrm{Q} = \{u \in RS[a,b]^m\}$

conclusion

This paper tries to give optimal solutions of the variational problems and optimal control problems by using optimality conditions.The optimal solutions are summarized as follows;
a)consider variational problem:

$$(P) \quad f(y) = \int_a^b L(y(t), \dot{y}(t), t)dt \rightarrow \quad min \quad y \in s$$

ii)Let L is continuous and continuously partially differentiable with respect to the first two arguments and let y_0 is a solution of the variational problem.
For y_0 to be a solution of the variational problem then ELDE and TR condition(for the problem of free right end point) hold and if L is convex with regard to the first two components for all t in [a,b] then y_0 is the optimal solution of of the problem.

From the given practical examples of minimizing length of a curve and minimizing surface area of solid revolution; variational calculus gives us economical advantage interims of time and material resources.

b)Consider an optimal control problem:
(P) f(x,u) = $\int_a^b g(x(t), u(t), t)dt \dashrightarrow$ Min, (x,u) $\in$ R

i)The necessary optimality conditions for $x^*(t)$ and $u^*(t)$ to be extremal functions of the problem are the validity of
i)Pontryagin minimum principle
ii)Euler Lagrange Differential Equation(ELDE) and transversaliy condition(TR)(for the problem of free right end point).
iii) the ordinary differential equation(ODE)
And a sufficient optimality condition for the extremals $x^*(t)$ and $u^*(t)$ to be the optimal solutions of the problem is the convexity of the separated problems.i.e the convexity of G with regard to x and $\dot{x}$,and the convexity of W with regard to u.
If these necessary and sufficient conditions are satisfied then the extremals $x^*(t)$ and $u^*(t)$ are the optimal solution of he problem.

In the case of Quadratic optimal control problems; the separated problems G and W are convex with respect to (x,$\dot{x}$) and u respectively(i.e a sufficient condition holds).
If the three necessary conditions are satisfied then the extremals $x^*(t)$ and $u^*(t)$ are the optimal solutions of the problem.

From production-Inventory problem; minimizing the cost of holding x(t) items and minimizing the cost of producing u(t) items helps the manufacturing firm to maximize its profit and to meet known demand over a planning period of time.And also in Rocket launching problem;minimizing the cost(fuel/energy) gives economical advantage.

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