

ON CHRISTOFFEL WORDS

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To The Memory Of My Father.

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Summary

Combinatorics on words has imposed itself as a powerful tool for the study of large number of discrete, linear, non-commutative objects. This report is intended to discuss words, in particular, Christoffel words.

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Introduction

The central notion of this report are words specially Christoffel words. A word is a finite or infinite sequence of symbols taken from a finite set. The natural environment of a finite word is a free monoid.

The first section of this report introduces the general notation on words. The remaining sections present the combinatorics of Christoffel words, named after the German mathematician and physicist Elwin B. Christoffel (1829-1900), and relationships between Christoffel words and topics in discrete mathematics.

The history of combinatorics on words goes back to the beginning of the last century, when A. Thue started to work on repetition-free words. He proved, among other things, the existence of an infinite square-free word over a ternary alphabet. Thue was the first to study systematically problems on words, and moreover as problems of their own.

Once the foundation of the theory was laid down it developed rapidly. In 1983, the active research on words culminated to the first book of the topic, namely Lothaire's book *Combinatorics on Words*. And this book had an enormous influence on the further development of the field.

Chapter 1

Preliminaries

In this chapter we introduce some notations on words which are useful in the latter chapters.

1.1 Semigroups

A semigroup is a set equipped with a binary associative operation.

A semigroup morphism from a semigroup S into a semigroup T is a mapping $f : S \rightarrow T$ such that $f(uv) = f(u)f(v)$ for all $u, v \in S$.

A subsemigroup is a subset closed under the operation.

A monoid M is a semigroup with a neutral element, i.e. an element e such that $me = em = m$ for all $m \in M$.

A submonoid of a monoid M is a sub set of M closed under the operation and containing the neutral element of M .

A monoid morphism $f : M \rightarrow N$ is a semigroup morphism such that $f(e_M) = e_N$.

Given a semigroup S and a set $X \subseteq S$, $X^+ = \{x_1, x_2 \dots x_n \mid n \geq 1, x_i \in X\}$, is the sub-semigroup generated by X . If S is a monoid, define $X^* = X^+ \cup e$ which is submonoid generated by X .

A semigroup (or monoid) S is called free if it has a subset B such that each element of S can be uniquely expressed as a product of elements of B . Such a B is referred to as a free generating set of S , or a base of S .

1.2 Words

A finite set of symbols A is called an alphabet. The elements of A are called letters.

A word w over an alphabet A is a sequence of symbols from A .

A word can be finite or infinite. The set of words over A is denoted by A^* and the empty word is denoted by e .

Concatenation or product of words is an operation defined by : $a_1 \dots a_n . b_1 \dots b_n = a_1 \dots a_n b_1 \dots b_n$; for $n \in \mathbb{N}$, and $a_i \in A$ for each i .

Concatenation of words is associative; i.e.

$$(a_1 \dots a_n . b_1 \dots b_n) . c_1 \dots c_n = (a_1 \dots a_n b_1 \dots b_n) . c_1 \dots c_n = a_1 \dots a_n b_1 \dots b_n c_1 \dots c_n;$$

$$\text{And } a_1 \dots a_n . (b_1 \dots b_n . c_1 \dots c_n) = a_1 \dots a_n . (b_1 \dots b_n c_1 \dots c_n) = a_1 \dots a_n b_1 \dots b_n c_1 \dots c_n.$$

The empty word e is the neutral element with respect to this operation.

Hence, $A^* = (A^*, \cdot)$ is a monoid and moreover, A^* is a free monoid generated by A .

Example 1.2.1. 1. *For any alphabet A the monoid A^* is free with the base A , and is called the free monoid generated by A .*

2. Let $X = \{a, ab, ba\}$. Then X^* is a monoid but not free, since $a.ba = aba = ab.a$.

Given a word $w \in A^*$, the square of w is the monoid product $w^2 = ww \in A^*$. Higher powers of w are defined analogously.

For a word w , $|w|$ denotes the length of w , and it is the total number of letters in w : $|w|_a$ denotes the number of a 's in w and $\text{alph } w$ denotes the set of letters having at least one occurrence in the word. We use the notation $A^+ = A^* - e$ (i.e the set of non empty words.)

A word u is a factor of a word w (resp. left factor or a prefix, a right factor or a suffix) if there exist words x and y such that $w = xuy$ (resp. $w = uy, w = xu$). A factor (resp. the prefix, the suffix) is proper if $xy \neq e$ (resp. $y \neq e, x \neq e$).

The reversal of a word $w = a_1a_2\dots a_n$, where a_1, \dots, a_n are letters, is the word $\tilde{w} = a_n a_{n-1} \dots a_1$.

A language is a subset of A^* .

A palindrome is a word w such that $w = \tilde{w}$. If $|w|$ is even, then w is a palindrome if and only if $w = x\tilde{x}$ for some word x . Otherwise w is a palindrome if and only if $w = xa\tilde{x}$ for some word x and some letter a .

An infinite word is a map from \mathbb{N} to A . We denote by $A^{\mathbb{N}}$ the set of infinite words. It is the set of sequences of symbols in A indexed by nonnegative integers, i.e. $w = w_0w_1\dots$. Also $A^\infty = A^* \cup A^{\mathbb{N}}$ denotes the set of finite or infinite words.

A factorization of a finite word w over A is a sequence (w_1, w_2, \dots, w_r) of words over A such that the relation $w = w_1w_2\dots w_r$ holds in the monoid A^* .

An integer $p \geq 1$ is a period of a word $w = a_1\dots a_n$ where $a_i \in A$ if $a_i = a_{i+p}$ for $i = 1, \dots, n - p$. The smallest period of w is called the period of w .

Example 1.2.2. *A word can have several periods. For example the word abababa has periods 2, 4, 6. Moreover, any number $\geq |w|$ is always a period of w .*

A word $w \in A^+$ is primitive if $w = u^n$ for $u \in A^+$ implies $n = 1$. In other words $w \in A^+$ is primitive if w is not the power of another word.

Two words x, y are conjugates if there exist words u, v such that $x = uv$ and $y = vu$.

Thus conjugate words are just cyclic shifts of one another.

Conjugacy is an equivalence relation. That is,

- (i). For any word $w, w = ew = we \Rightarrow w \sim w$;
- (ii). Let $w, w' \in A^*$ such that $w \sim w'$. Then there exist words u, v such that $w = uv$ and $w' = vu$. This implies that the conjugate of w' is uv which is the word w .

$\therefore w' \sim w$; and

- (iii). Let w, w' and $w'' \in A^*$ such that $w \sim w'$ and $w' \sim w''$. $w \sim w' \Rightarrow$ there exist words u, v such that $w = uv$ and $w' = vu$ and $w' \sim w'' \Rightarrow w'' = uv = w$. Thus, by (i) $w \sim w''$.

The conjugacy class of a word of length n and period p has p elements if p divides n and has n elements otherwise. In particular, a primitive word of length n has n distinct conjugates.

Given two alphabets A, B , a function $f : A^* \rightarrow B^*$ is a morphism if $f(xy) = f(x)f(y)$ for all $x, y \in A^*$. A morphism is uniquely determined by its values on the alphabet.

A morphism is literal if the image of a letter is a letter. A morphism is non erasing if the image of a letter is always a non empty word. $f(e_{A^*}) = e_{B^*}$, since the empty word e is the only element in a free monoid satisfying $w^2 = w$. The identity morphism on A^* is the morphism sending each $w \in A^*$ to itself. The trivial morphism from A^* to

B^* is the morphism sending each $w \in A^*$ to e_{B^*} .

Let \mathbb{N} denote the set of nonnegative integers. If a, b and n are integers, then the notation $a \equiv b \pmod{n}$ shall mean that $a - b$ is divisible by n . Equivalently, $a \equiv b \pmod{n}$ if and only if a and b have the same remainder upon division by n .

A binary relation \leq defined on a set A is a partial order on the set A if the following conditions hold identically in A :

P1: $a \leq a$ (reflexivity)

P2: $a \leq b$ and $b \leq a$ imply $a = b$ (antisymmetric)

P3: $a \leq b$ and $b \leq c$ imply $a \leq c$ (transitivity).

If, in addition, for every a, b in A

P4: $a \leq b$ or $b \leq a$

then we say \leq is a total order on A .

A poset (L, \leq) is called lattice if both $\inf\{a, b\}$ and $\sup\{a, b\}$ exist for all $a, b \in L$.

A path in $\mathbb{Z} \times \mathbb{Z}$ from (a, b) to (c, d) is a continuous map $\alpha : [0, 1] \rightarrow (\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z})$ such that $\alpha(0) = (a, b)$ and $\alpha(1) = (c, d)$. Since such paths are essentially determined by the points of $\mathbb{Z} \times \mathbb{Z}$ that lie on the path, we identify such a path with a sequence of points in $\mathbb{Z} \times \mathbb{Z}$ with consecutive points of the sequence differing by \vec{e}_1 or \vec{e}_2 , where \vec{e}_1 and \vec{e}_2 are the standard basis vectors of \mathbb{R}^2 .

Chapter 2

Christoffel words

Although the theory of Christoffel words began to take shape in the late 1800s, the term was not introduced until 1990 by Jean Brestel. By now there are numerous equivalent definitions and characterizations of Christoffel words.

A Christoffel word is a "discretization" of a line segment in the plane by a path in the integer lattice $\mathbb{Z} \times \mathbb{Z}$.

2.1 Geometric definition

Notation. If $a, b \in \mathbb{N}$, then a and b are said to be relatively prime if 1 is the only positive integer that divides both a and b . The notation $a \perp b$ shall mean "a and b are relatively prime".

Suppose $a, b \in \mathbb{N}$ and $a \perp b$. The lower Christoffel path of slope $\frac{b}{a}$ is the path from $(0,0)$ to (a,b) in the integer lattice $\mathbb{Z} \times \mathbb{Z}$ that satisfies the following two conditions.

- i. The path lies below the line segment that begins at the origin and ends at (a, b) .

- ii. The region in the plane enclosed by the path and the line segment contains no other points of $\mathbb{Z} \times \mathbb{Z}$ besides those of the path.

Upper Christoffel paths are defined analogously.

Suppose $a, b \in \mathbb{N}$ and $a \perp b$. The upper Christoffel path of slope $\frac{b}{a}$ is the path from $(0,0)$ to (a,b) in the integer lattice $\mathbb{Z} \times \mathbb{Z}$ that satisfies the following two conditions.

- i. The path lies above the line segment that begins at the origin and ends at (a,b) .
- ii. The region in the plane enclosed by the path and the line segment contains no other points of $\mathbb{Z} \times \mathbb{Z}$ besides those of the path.

The unmodified term Christoffel path will always mean lower Christoffel path.

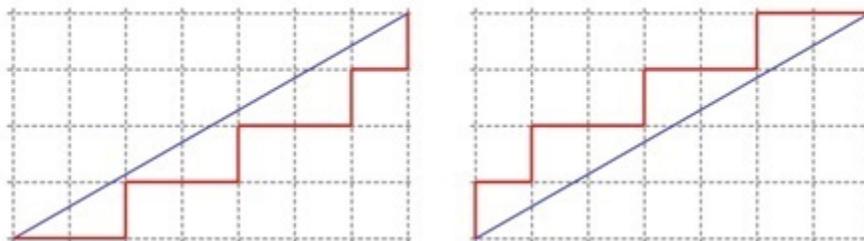


Figure 2.1: The lower and upper Christoffel words of slope $\frac{4}{7}$.

Every step in a Christoffel path moves from a point $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ to either the point $(i + 1, j)$ or the point $(i, j + 1)$. Thus, a Christoffel path of slope $\frac{b}{a}$ determines a word $C(a, b)$ in the alphabet $\{x, y\}$ by encoding steps of the first type by the letter x and steps of the second type by the letter y . See figure 2.2

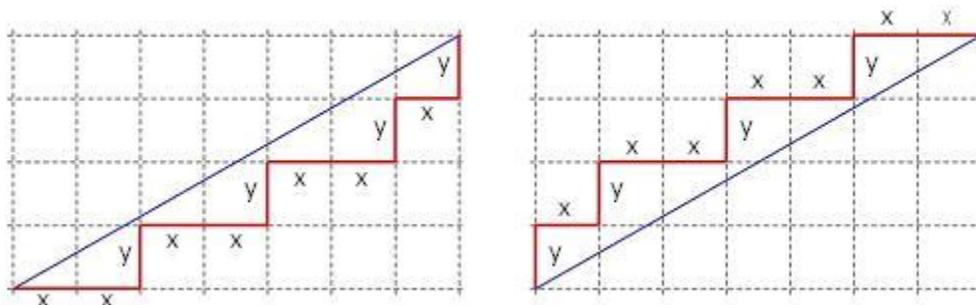


Figure 2.2: The lower and upper Christoffel words of slope $\frac{4}{7}$ are $xyxyxyxyxy$ and $yxyxyxyxyx$.

Definition 2.1.1. Let $a, b \in \mathbb{N}$. A word $w \in \{x, y\}^*$ is a lower Christoffel word of slope $\frac{b}{a}$ if $a \perp b$ and $w = C(a, b)$. A Christoffel word is trivial if its length is at most 1, and is nontrivial otherwise.

Upper Christoffel words are defined analogously.

Since every positive rational number can be expressed as $\frac{b}{a}$ where $a \perp b$ in only one way, there is a unique lower Christoffel word of slope r for all positive rational numbers r .

Example 2.1.1. *The following are examples of Christoffel words.*

1. *The Christoffel word of slope 0 is x , since $C(1, 0) = x$.*
2. *The Christoffel word of slope ∞ is y , since $C(0, 1) = y$.*
3. *The Christoffel word of slope 1 is xy , since $xy = C(1, 1)$.*
4. *The Christoffel word of slope $\frac{4}{7}$ is $xyxyxyxyxy$, (see figure 2.2).*

Remarks. 1. The empty word e is not a Christoffel word since $0 \perp 0$. Therefore, x and y are the only trivial Christoffel words.

2. The square or higher power of a Christoffel word is not a Christoffel word. Nor is a Christoffel word the power of a shorter word, that is, Christoffel words are primitive words.

2.2 Cayley Graph definition

Here we introduce an equivalent definition for the Christoffel word of slope $\frac{b}{a}$. In fact, it is the definition originally used by Christoffel, it amounts to reading edge-labellings of the cayley graph of $\mathbb{Z}/(a+b)\mathbb{Z}$.

Definition 2.2.1. Suppose $a \perp b$ and $(a, b) \neq (0, 1)$. The label of a point (i, j) on the (lower) Christoffel path of slope $\frac{b}{a}$ is the number $\frac{ib - ja}{a}$. That is, the label of (i, j) is the vertical distance from the point (i, j) to the line segment from $(0,0)$ to (a, b) .

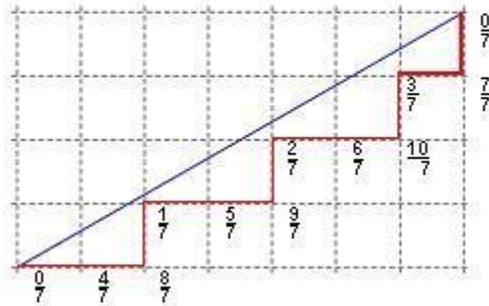


Figure 2.3: The labels of the points on the Christoffel path of slope $\frac{4}{7}$.

Now, suppose w is a lower Christoffel word of slope $\frac{b}{a}$ and suppose $\frac{s}{a}, \frac{t}{a}$ are two consecutive labels on the Christoffel path from $(0,0)$ to (a,b) . Either $(\frac{s}{a}, \frac{t}{a})$ represents a horizontal step (in which case $t = s + b$) or it represents a vertical step (in which case $t = s - a$). The following lemma summarizes these observations.

Lemma 2.2.1. *Suppose w is a lower Christoffel word of slope $\frac{b}{a}$ and $a \perp b$. If $\frac{s}{a}$ and $\frac{t}{a}$ two consecutive labels on the christoffel path from $(0,0)$ to (a,b) then $t \equiv s + b \pmod{a + b}$. Moreover, t takes as value each integer $0, 1, \dots, a + b - 1$ exactly once as $(\frac{s}{a}, \frac{t}{a})$ ranges over all consecutive pairs of labels.*

Definition 2.2.2. Suppose $a \perp b$. Consider the Cayley graph of $\mathbb{Z}/(a + b)\mathbb{Z}$ with generator b . It is a cycle, with vertices $0, b, 2b, \dots, 0 \pmod{a + b}$. Starting from zero and proceeding in the order listed above,

- (i). label those edges (s, t) satisfying $s < t$ by x ;
- (ii). label those edges (s, t) satisfying $s > t$ by y ;
- (iii). read edge-labels in the prescribed order, i.e., $0 \xrightarrow{x} b \xrightarrow{*} \dots \xrightarrow{*} a \xrightarrow{y} 0$.

The lower Christoffel word of slope $\frac{b}{a}$ is the word $x\dots y$ formed above.

Example 2.2.2. *Pick $a = 7$ and $b = 4$. Figure 2.4 shows the Cayley graph of $\mathbb{Z}/11\mathbb{Z}$ with generator 4 and edges $u \rightarrow v$ labeled x or y according to whether or not $u < v$. Reading the edges clockwise from 0 yields the word $xyxyxyxyxy$, which is the Christoffel word of slope $\frac{4}{7}$.*

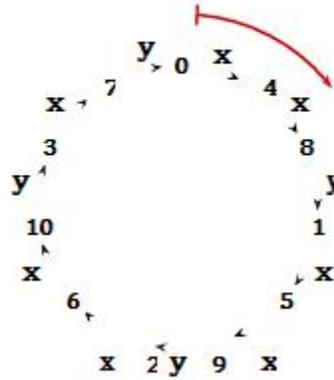


Figure 2.4: The Cayley graph of $\mathbb{Z}/(7+4)\mathbb{Z}$ with generator 4 and the associated Christoffel word.

Note. If we choose the generator a instead of b for $\mathbb{Z}/(a+b)\mathbb{Z}$ and swap the roles of x and y in definition 2.2.2 , the resulting word becomes the upper Christoffel word of slope $\frac{b}{a}$.

Lemma 2.2.3. *Suppose $a \perp b$. Let (i, j) be the point on the Christoffel path path from $(0,0)$ to (a, b) with label $\frac{t}{a}$. Then*

$$t \equiv (i + j)b \pmod{a + b} \text{ and } t \equiv ((a - i) + (b - j))a \pmod{a + b}.$$

Proof. By definition, the label of the point (i, j) on the (lower) Christoffel path of

$$\text{slope } \frac{b}{a} \text{ is the number } \frac{ib - ja}{a} = \frac{t}{a}.$$

$$\Rightarrow t = ib - ja.$$

$$\text{Now, } t - (i + j)b = (-j)(a + b) \Rightarrow t - (i + j)b \text{ is divisible by } (a+b).$$

$$\text{Hence, } t \equiv (i + j)b \pmod{a + b}.$$

$$\text{Similarly, } t \equiv ((a - i) + (b - j))a \pmod{a + b}.$$

□

Chapter 3

Christoffel Morphisms

In this chapter we introduce the monoid of Christoffel morphisms and exhibit a minimal set of generators for the monoid.

3.1 Christoffel Morphisms

Definition 3.1.1. A Christoffel morphism is an endomorphism of the free monoid $\{x, y\}^*$ that sends each Christoffel word on to a conjugate of a Christoffel word.

Example 3.1.1. 1. *The identity morphism $I : \{x, y\}^* \rightarrow \{x, y\}^*$ is a Christoffel morphism.*

2. *The morphism $f : \{x, y\}^* \rightarrow \{x, y\}^*$ given by $f(w) = e$, for each word $w \in \{x, y\}^*$, is not a Christoffel morphism.*

Lemma 3.1.2. *If f is an endomorphism of the free monoid A^* and w and w' conjugate words in A^* , then $f(w)$ and $f(w')$ are conjugate.*

Proof. since w and w' are conjugate, there exist words $u, v \in A^*$ such that $w = uv$ and $w' = vu$.

Now, $f(w) = f(uv) = f(u)f(v)$ and $f(w') = f(vu) = f(v)f(u)$.

\Rightarrow there exist words $f(u), f(v) \in A^*$ such that $f(w) = f(u)f(v)$ and $f(w') = f(v)f(u)$.

Hence, $f(w)$ and $f(w')$ are conjugate. □

Theorem 3.1.3. *The set of Christoffel morphisms is closed under composition.*

Proof. let G and H be Christoffel morphisms. We need to prove that $G \circ H : \{x, y\}^* \rightarrow \{x, y\}^*$ is also Christoffel morphism.

(i). $G \circ H$ is a morphism.

Let $w, w' \in \{x, y\}^*$. Then

$$\begin{aligned} G \circ H(ww') &= G(H(ww')) = G(H(w)H(w')) \dots \text{ because } H \text{ is a morphism} \\ &= G(H(w))G(H(w')) \dots \text{ because } G \text{ is a morphism} \\ &= G \circ H(w)G \circ H(w'). \end{aligned}$$

Hence, $G \circ H$ is an endomorphism of $\{x, y\}^*$.

(ii). By Lemma 3.1.2, any endomorphism of the free monoid $\{x, y\}^*$ maps conjugate words to conjugate words.

Therefore, $G \circ H$ is Christoffel morphism. □

If H is an endomorphism of $\{x, y\}^*$ and $w = a_0a_1 \dots a_r$ is a word in $\{x, y\}^*$ with $a_0, a_1, \dots, a_r \in \{x, y\}$, then $H(w) = H(a_0a_1 \dots a_r) = H(a_0)H(a_1) \dots H(a_r)$. Therefore, H is determined by the images of x and y , so we identify H with the order pair $(H(x), H(y))$.

Example 3.1.4. We use the above notation to define the following five important endomorphisms of $\{x, y\}^*$.

$$\begin{aligned} G &= (x, xy), & D &= (yx, y), \\ \tilde{G} &= (x, yx), & \tilde{D} &= (xy, y), \\ E &= (y, x). \end{aligned}$$

The remainder of this section is devoted to showing that they are also Christoffel morphisms.

Lemma 3.1.5. *The morphism \tilde{D} maps the Christoffel word of slope $\frac{b}{a}$ to the Christoffel word of slope $\frac{a+b}{a}$. The morphism G maps the Christoffel word of slope $\frac{b}{a}$ to the Christoffel word of slope $\frac{b}{a+b}$.*

Proof. We first prove the result for \tilde{D} . Suppose $a \perp b$. The Christoffel word of slope $\frac{b}{a}$, by definition, encodes the steps of the Christoffel path from $(0,0)$ to (a,b) : the letter x encodes the step \vec{e}_1 and the letter y encodes the step \vec{e}_2 , where \vec{e}_1 and \vec{e}_2 are the standard basis vectors of \mathbb{R}^2 . Since \tilde{D} maps x to xy and y to y , the word $\tilde{D}(w)$ corresponds to the path obtained from the Christoffel path from $(0,0)$ to (a,b) by replacing each step \vec{e}_1 by the two steps \vec{e}_1 and \vec{e}_2 . We argue that, this path is the Christoffel path from $(0,0)$ to $(a, a+b)$ implying that $\tilde{D}(w)$ is the Christoffel word of slope $\frac{a+b}{a}$.

Define a linear transformation $\tilde{D} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $\tilde{D}(c, d) = (c, c+d)$ for all $(c, d) \in \mathbb{R}^2$. Let W denote the Christoffel path from $(0,0)$ to (a,b) . Then $\tilde{D}(W)$ is the path in the integer lattice $\mathbb{Z} \times \mathbb{Z}$ consisting of steps $\tilde{D}(\vec{e}_1) = \vec{e}_1 + \vec{e}_2$ and $\tilde{D}(\vec{e}_2) = \vec{e}_2$. We argue that the path obtained from $\tilde{D}(W)$ by replacing the steps $\vec{e}_1 + \vec{e}_2$ with the pair of steps \vec{e}_1 and \vec{e}_2 is the Christoffel path from $(0,0)$ to $(a, a+b)$.

Let R denote the region between the Christoffel path W and the line segment from $(0,0)$ to (a,b) . Then there are no integer points in the interior of the region $\tilde{D}(R)$ because \tilde{D} is a linear transformation and the region R contains no integer points in its interior. Therefore, there are no integer points in the interior of the region obtained from $\tilde{D}(R)$ by adjoining the triangles with vertices \vec{v} , $\vec{v} + \vec{e}_1$ and $\vec{v} + (\vec{e}_1 + \vec{e}_2)$ whenever \vec{v} and $\vec{v} + (\vec{e}_1 + \vec{e}_2)$ are in $\tilde{D}(R)$. The boundary of this new region consists of the line segment from $(0,0)$ to $(a, a+b)$ and the path P obtained from the path $\tilde{D}(W)$ by replacing the steps $\vec{e}_1 + \vec{e}_2$ with the steps \vec{e}_1 and \vec{e}_2 .

Also, $(a+b) \perp a$, since $a \perp b$. Therefore, P is the Christoffel path from $(0,0)$ to $(a, a+b)$. Moreover, P is the path encoded by the word $\tilde{D}(W)$, since P is obtained from the Christoffel path from $(0,0)$ to (a,b) by replacing each step \vec{e}_1 with the steps \vec{e}_1 and \vec{e}_2 .

Hence, $\tilde{D}(w)$ is the Christoffel word of slope $\frac{a+b}{a}$.

The proof that $G(w)$ is a Christoffel word for any Christoffel word w is similar: define a linear transformation $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $G(c, d) = (c+d, d)$ for all $(c, d) \in \mathbb{R}^2$ and argue as above that G maps the Christoffel word of slope $\frac{b}{a}$ to the Christoffel word of slope $\frac{b}{a+b}$. \square

Tracing backwards through the proof we also have the following result.

Corollary 3.1.6. *If u is a Christoffel word of slope at least one, then the unique word w such that $\tilde{D}(w) = u$ is a Christoffel word. If u is a Christoffel word of slope at most one, then the unique word w such that $G(w) = u$ is a Christoffel word.*

The next Lemma relates the image of D with that of \tilde{D} , and the image of G with that of \tilde{G} .

Lemma 3.1.7. *For every word $w \in \{x, y\}^*$, there exists a word $v \in \{x, y\}^*$ such that $D(w) = yv$ and $\tilde{D}(w) = vy$, and a word $u \in \{x, y\}^*$ such that $G(w) = xu$ and $\tilde{G}(w) = ux$.*

Proof. we prove the result of D and \tilde{D} , the proof of G and \tilde{G} is the same.

Using induction on the length of w .

If $|w| = 1$, then w is either x or y . So $D(x) = yx$ and $\tilde{D}(x) = xy$ with $v = x$, or $D(y) = y$ and $\tilde{D}(y) = y$ with $v = e$. This establishes the base case of induction. Let w be a word of length $r \geq 1$ and suppose the claim holds for all words of length less than r . If $w = xw'$ for some $w' \in \{x, y\}^*$, then the induction hypothesis implies there exist a word $v' \in \{x, y\}^*$ such that $D(w') = yv'$ and $\tilde{D}(w') = v'y$. Therefore, $D(w) = D(xw') = yxyv'$ and $\tilde{D}(w) = \tilde{D}(xw') = xyv'y$. Taking $v = xyv'$, we are done. Suppose, instead, that $w = yw'$. Then the induction hypothesis implies there exist $v' \in \{x, y\}^*$ such that $D(w') = yv'$ and $\tilde{D}(w') = v'y$.

Hence, $D(w) = D(yw') = yyv'$ and $\tilde{D}(w) = \tilde{D}(yw') = yv'y$, so take $v = yv'$. \square

Corollary 3.1.8. *The morphisms G, D, \tilde{G} and \tilde{D} are Christoffel morphisms.*

Proof. By Lemma 3.1.5, G and \tilde{D} map Christoffel words to Christoffel words. Hence, they are Christoffel morphisms. We prove that D is a Christoffel morphism; the same argument proves that \tilde{G} is a Christoffel morphism. Let w be a Christoffel word. Lemma 3.1.7 implies there exists a word $v \in \{x, y\}^*$ such that $D(w) = yv$ and $\tilde{D}(w) = vy$. Therefore, $D(w)$ and $\tilde{D}(w)$ are conjugate words. Since $\tilde{D}(w)$ is a Christoffel word, $D(w)$ is a conjugate of a Christoffel word; that is, D is a Christoffel morphism. \square

We now turn to proving that E is a Christoffel morphism.

Lemma 3.1.9. *The morphism E maps lower Christoffel words of slope r onto upper Christoffel words of slope $\frac{1}{r}$.*

Proof. Suppose $a \perp b$. The Christoffel word of slope $\frac{b}{a}$, by definition, encodes the steps of the Christoffel path from $(0,0)$ to (a,b) : the letter x encodes the step \vec{e}_1 and the letter y encodes the step \vec{e}_2 , where \vec{e}_1 and \vec{e}_2 are the standard basis vectors of \mathbb{R}^2 . Since E maps x to y and y to x , the word $E(w)$ corresponds to the path obtained from the Christoffel path from $(0,0)$ to (a,b) by replacing each step \vec{e}_1 by \vec{e}_2 and vice versa.

Consider a reflection E over a line $x = y$, it is a transformation in which each point of the original figure (pre-image) has an image that is the same distance from the line of reflection as the original point but is on the opposite side of the line. Let W denote the Christoffel path from $(0,0)$ to (a,b) . Then $E(W)$ is the path in the integer lattice $\mathbb{Z} \times \mathbb{Z}$ consisting of steps $E(\vec{e}_1) = \vec{e}_2$ and $E(\vec{e}_2) = \vec{e}_1$.

Let R denote the region between the Christoffel path W and the line segment from $(0,0)$ to (a,b) . Then there are no integer points in the interior of the region $E(R)$ because E is a reflection and the region R contains no integer points in its interior. Therefore, $E(W)$ is the upper Christoffel path from $(0,0)$ to (b,a) and is also the path encoded by the word $E(w)$, since $E(W)$ is obtained from the Christoffel path from $(0,0)$ to (a,b) by replacing each step \vec{e}_1 by \vec{e}_2 and vice versa.

Therefore, $E(W)$ is the upper Christoffel word of slope $\frac{a}{b}$. □

Lemma 3.1.10. *Suppose $a \perp b$. The lower and upper Christoffel words of slope $\frac{b}{a}$ are conjugates.*

Proof. Suppose $a \perp b$ and let w be the Christoffel word of slope $\frac{b}{a}$. The word ww encodes a path in $\mathbb{Z} \times \mathbb{Z}$ from $(0,0)$ to $(2a, 2b)$. It consists of two copies of the Christoffel

path of slope $\frac{b}{a}$, the first starting at the origin and the second starting at (a, b) . See Figure 3.2.

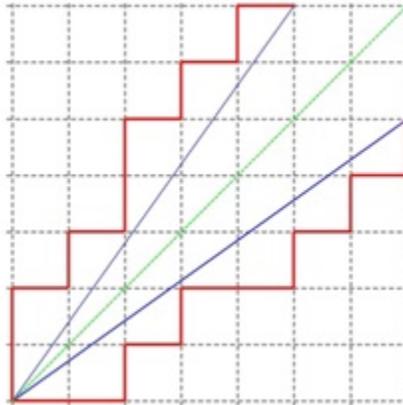


Figure 3.1: The geometric interpretation of the morphism E is reflection about the line $x = y$.

Let P denote the point on the first copy of the Christoffel path that is farthest (vertically) from the line segment defining the Christoffel path. By Lemma 2.2.1, this distance is $\frac{a+b-1}{a}$. Let P' denote the corresponding point on the translated copy of the Christoffel path. Then P and P' determine a word $w' \in \{x, y\}^*$ by encoding the part of the path from P to P' as a word in the letters x, y . Note that w' is a factor of ww of length equal to that of w . Since w' is a factor of ww and $|w'| = |w|$, the words w' and w are conjugate.

It remains to show that w' is the upper Christoffel word of slope $\frac{b}{a}$.

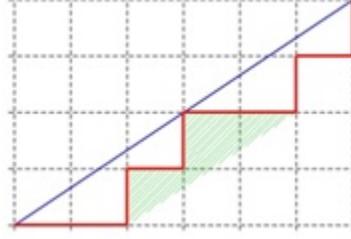


Figure 3.2: The path in $\mathbb{Z} \times \mathbb{Z}$ corresponding to the word $ww = xyxyxyxyxy$. The factor corresponding to the path from P to P' is $w' = yxyxx$. There are no integer points in the shaded region, so w' is an upper Christoffel word.

We will argue that there is no other integer point in the region (shaded in Figure 3.2) enclosed by the line segment PP' and the path from P to P' . Suppose $(i, j - 1)$ is an integer point directly below a point (i, j) on the path from P to P' , with $i > 0$. Then $(i - 1, j)$ is also a point on the path. Suppose the point $(i, j - 1)$ lies above or on the segment PP' . Then the vertical distance from $(i, j - 1)$ to the segment from $(0,0)$ to $(2a, 2b)$ is at most the vertical distance from the segment to p , by the choice of p . The former is $\frac{ib}{a} - (j - 1)$ and the latter is $\frac{a + b - 1}{a}$. That is,

$$\frac{ib}{a} - (j - 1) \leq \frac{a + b - 1}{a}.$$

Equivalently, $\frac{(i - 1)b - ja}{a} \leq \frac{-1}{a}$. But $\frac{(i - 1)b - ja}{a}$ is non negative because it is the distance from the point $(i - 1, j)$ to the segment from $(0,0)$ to $(2a, 2b)$. This is a contradiction, so there is no integer point within the region enclosed by the line segment pp' (of slope $\frac{b}{a}$) and the path from p to p' . That is, w' is the upper Christoffel word of slope $\frac{b}{a}$. \square

Theorem 3.1.11. *The morphisms $G, D, \tilde{D}, \tilde{D}, E$ are Christoffel morphisms.*

Proof. The first four morphisms are Christoffel morphisms by Corollary 3.1.8. It

remains to show that E is a Christoffel morphism. This follows from the previous two results: if w is a Christoffel word, then $E(w)$ is an upper Christoffel word, which is a conjugate of the corresponding lower Christoffel word. \square

3.2 Generators

The following theorem gives a manageable characterization of the monoid of Christoffel morphisms.

Theorem 3.2.1. *The monoid of Christoffel morphisms is generated by $G, D, \tilde{G}, \tilde{D}$ and E .*

In fact, $D = E \circ G \circ E$ and $\tilde{G} = E \circ \tilde{D} \circ E$, but it will simplify the proof to retain these superfluous generators. The proof makes frequent use of the following easy fact, so we separate it as a lemma.

Lemma 3.2.2. *If w is a Christoffel word or a conjugate of a Christoffel word, then xx and yy cannot both be factors of w .*

Proof. Indeed, let w be the Christoffel word of slope $\frac{b}{a}$. If $\frac{b}{a} < 1$, then w begins with xx and yy is not a factor of w . In the case $\frac{b}{a} > 1$, w ends with yy and xx is not a factor of w . The only nontrivial Christoffel word with neither xx nor yy is $C(1, 1) = xy$. Since w begins with an x and ends with a y , xx and yy are not both factors of the square w^2 . Finally, since every conjugate of w appears as a factor of w^2 , the property holds for conjugates of Christoffel words as well. \square

Proof. proof of Theorem 3.2.1.in five steps.

1. A Christoffel morphism f is non erasing, that is, the length of $f(w)$ is at least the length of w .

An erasing morphism f must necessarily send x or y to the empty word. In the first case, $f(xyy)$ is not primitive, in the second $f(xxy)$ is not primitive. On the other hand, all conjugates of a Christoffel word are primitive.

2. If f is a nonidentity Christoffel morphism, then $f(x)$ and $f(y)$ must begin or end by the same letter. Assume $f(x)$ begins by x (*studyE* $\circ f$ otherwise) and $f(y)$ begins by y . There are two possibilities:

(i): Suppose $f(x)$ ends by y . Either $f(y)$ ends by x or we are done. In the remaining case, $f(xy) = f(x)f(y)$ becomes $x...yy...x$, so xx and yy are both factors of every conjugate of $f(xy)$, save perhaps for $f(y)f(x) = yxxy$. On the other hand, $f(xy)$ is a conjugate of a Christoffel word u that is not equal to $f(x)f(y)$ or $f(y)f(x)$ since u begins by x and ends by y . Lemma 3.2.2. yields a contradiction.

(ii): Suppose instead $f(x)$ ends by x and $f(y)$ ends by y . Note that in this case, xx is a factor of $f(xxy) = f(x)f(x)f(y) = (xx)(xx)(yy)$. Hence, yy is not a factor of $f(xxy)$ by the lemma. In particular, yy is a factor of neither $f(x)$ nor $f(y)$. Similarly, by considering $f(xyy)$, we see that xx is a factor of neither $f(x)$ nor $f(y)$. This in turn forces $f(x) = (xy)^i x$, $f(y) = y(xy)^j$ and $f(xy) = (xy)^{i+j+1}$. Whence $i+j = 0$. Then f is the identity morphism, contrary to our hypothesis.

3. If f is a nonidentity Christoffel morphism, then there exists a morphism $g : \{x, y\}^* \rightarrow \{x, y\}^*$ and an $H \in \{G, D, \tilde{G}, \tilde{D}\}$ such that $f = H \circ g$.

A non empty word w on $\{x, y\}$ belongs to $\{x, y\}^*$ (i.e., is a word on the

"letters" x and xy) if and only if w begins by x and does not contain the factor yy . Similar descriptions hold for words in $\{y, xy\}^*$, $\{x, yx\}^*$ and $\{xy, y\}^*$. We argue that the image of f belongs to one of these monoids.

Since G, D, \tilde{G} and \tilde{D} are injective morphisms, with images in the respective monoids above, this will allow us to compose f with $G^{-1}, D^{-1}, \tilde{G}^{-1}, \tilde{D}^{-1}$, respectively to find g .

Since $f(xy)$ is a conjugate of a Christoffel word, xx and yy are not both factors of $f(xy)$. Assuming yy is not a factor of $f(xy)$, it follows that yy is a factor of neither $f(x)$ nor $f(y)$. By Step 2, $f(x)$ and $f(y)$ must then begin or end by the same letter, setting up several cases to check.

(i): If $f(x)$ and $f(y)$ both begin by x , then the image of f is a subset of $\{x, xy\}^*$. Therefore, $G^{-1} \circ f = g$ is also a morphism of $\{x, y\}^*$.

(ii): If $f(x)$ and $f(y)$ both begin by y , then neither may end by y (on account of the lemma and our assumption that yy is not a factor of $f(xy)$). Thus $f(x)$ and $f(y)$ both end by x and neither contain yy as a factor. That is, $f(x), f(y) \in \{x, yx\}^*$ and $G^{-1} \circ f$ is a morphism of $\{x, y\}^*$.

(iii): The case where $f(x)$ and $f(y)$ end by the same letter are handled analogous to the case above.

4. In the composition $f = H \circ g$ built above, g is a Christoffel morphism.

We now have that $f = H \circ g$, with $H \in \{G, D, \tilde{G}, \tilde{D}\}$, and that f sends Christoffel words onto conjugates of Christoffel words. We aim to show that g does as well. We analyze the case $H = G$, the rest being similar.

First, recall that if $G(w)$ is a Christoffel word, then w is a Christoffel word too (Corollary 3.1.6). We must show that if $G(w)$ is a conjugate of a Christoffel

word then w is as well. This is now easy, for if $G(w) = uv$ with vu a Christoffel word, then v begins by x and u ends by y . Moreover, by the definition of G , u must begin by x and yy is a factor of neither u, v nor uv . This implies that $u, v \in \{x, xy\}^*$, so $G^{-1}(u)$ and $G^{-1}(v)$ are defined, $w = G^{-1}(u)G^{-1}(v)$ and $G^{-1}(v)G^{-1}(u)$ is a Christoffel word.

5. There exist $H_i \in \{G, D, \tilde{G}, \tilde{D}\}$ such that $f = H_1 \circ \dots \circ H_s$.

From step 4, $f = H_1 \circ g$ for some Christoffel morphism g and some $H_1 \in \{G, D, \tilde{G}, \tilde{D}\}$. Moreover, $|f(x)| + |f(y)| > |g(x)| + |g(y)|$.

□

Corollary 3.2.3. *A morphism f on $\{x, y\}^*$ is a Christoffel morphism if and only if $f(xy), f(xxy)$ and $f(xyy)$ are conjugates of Christoffel words.*

Chapter 4

Standard Factorization

4.1 The Standard Factorization

This section proves that every Christoffel word can be expressed as the product of two Christoffel words in a unique way. This factorization is called the standard factorization.

Given that $a \perp b$, recall the method of labelling the Christoffel path from $(0,0)$ to (a,b) for nontrivial Christoffel words. By Lemma 2.2.1, if a and b are nonzero, then there is a unique point C on this path having label $\frac{1}{a}$. We call C the closest point for the path. It is the lattice point on the Christoffel path from $(0,0)$ to (a,b) with minimum nonzero distance to the line segment from $(0,0)$ to (a,b) .

Definition 4.1.1. Suppose $a \perp b$ with $a, b > 0$. The standard factorization of the Christoffel word w of slope $\frac{b}{a}$ is the factorization $w = (w_1, w_2)$, where w_1 encodes the portion of the Christoffel path from $(0,0)$ to the closest point C and w_2 encodes the portion from C to (a,b) .

Example 4.1.1. The standard factorization of the Christoffel word of slope $\frac{4}{7}$ is $(xy, xyxyxy)$. See Figure 4.1.

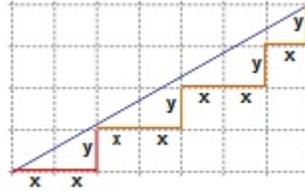


Figure 4.1: The closest point for the Christoffel path of slope $\frac{4}{7}$ occurs between the third and fourth steps, thus the standard factorization of $xyxxxyxxxyxy$ is

$$(xxy, xxyxxxyxy).$$

Proposition 4.1.2. *If (w_1, w_2) is the standard factorization of a nontrivial Christoffel word, then w_1 and w_2 are Christoffel words.*

Proof. Suppose w is a Christoffel word of slope $\frac{b}{a}$ and let (i, j) be the point on the Christoffel path from $(0, 0)$ to (a, b) labelled $\frac{1}{a}$. Then w_1 encodes the subpath P_1 from $(0, 0)$ to (i, j) and w_2 encodes the subpath P_2 from (i, j) to (a, b) .

Since (i, j) is the point on the Christoffel path that is closest to the line segment from $(0, 0)$ to (a, b) without being on the segment, no point of the Christoffel path besides $(0, 0)$, (a, b) and (i, j) lies in the triangle determined by these three points. Let S_1 be the line segment from $(0, 0)$ to (i, j) and S_2 be the line segment from (i, j) to (a, b) . Note that the region bounded by P_1 and S_1 contains no interior lattice points. Since, moreover, no integer points lie in the interior of the

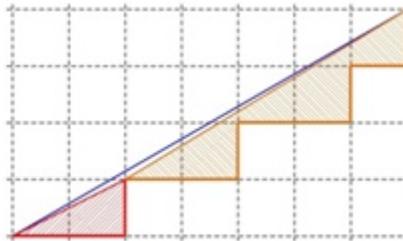


Figure 4.2: The standard factorization of a Christoffel word gives two Christoffel words.

line segment S_1 , it follows that $i \perp j$ and w_1 is the Christoffel word of slope $\frac{j}{i}$. Also the region bounded by P_2 and S_2 contains no interior lattice points. Since, no integer points lie in the interior of the line segment S_2 , it follows that $a - i \perp b - j$ and w_2 is the Christoffel word of slope $\frac{b - j}{a - i}$. \square

Theorem 4.1.3. (Pick's Theorem). *Let P be a simple polygon (that is, the boundary of P has no self-intersections) with vertices in $\mathbb{Z} \times \mathbb{Z}$. Then the area of P is $i + \frac{1}{2}b - 1$, where i is the number of integer points in the interior of P and b is the number of integer points of the boundary of P .*

Example 4.1.4. *Pick's Theorem provides a method to calculate the area of simple polygons whose vertices lie on lattice points—points with integer coordinates in the $x - y$ plane. The word "simple" in "simple polygon" only means that the polygon has no holes, and that its edges do not intersect.*

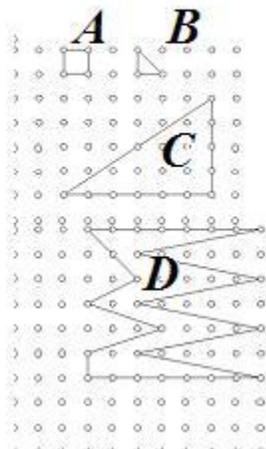


Figure 4.3: Pick's Theorem Examples.

For all the examples below, we'll let I be the number of interior vertices, and B be the number of boundary vertices. We will use the notation $A(P)$ to indicate the area of polygon P .

$$A). I = 0, B = 4, A(A) = 1, I + \frac{B}{2} = 2.$$

$$B). I = 0, B = 3, A(B) = \frac{1}{2}, I + \frac{B}{2} = \frac{3}{2}.$$

$$C). I = 7, B = 12, A(C) = 12, I + \frac{B}{2} = 13.$$

D). It is even uglier to calculate the area for this one, but after some addition and subtraction of areas, we find that: $I = 9, B = 26, A(D) = 21, I + \frac{B}{2} = 22$.

Theorem 4.1.5. *A nontrivial Christoffel word w has a unique factorization $w = (w_1, w_2)$ with w_1 and w_2 Christoffel words.*

Proof. Let (w_1, w_2) denote the standard factorization of w . Recall that this factorization is obtained from cutting the Christoffel path at its closest point C . Suppose there is another factorization $w = (u, v)$ with u and v Christoffel words. That is, $C' = (c, d)$ is another point on the path having no integer point on its corresponding regions and satisfying $c \perp d$. We reach a contradiction by comparing triangles ABC and ABC' . Since w_1, w_2 are Christoffel words, we know there are no integer lattice points in the interior of triangle ABC . Moreover, the only lattice points on its boundary occur at A, B and C . By Pick's Theorem, we have

$$\text{Area } ABC = i + \frac{1}{2}b - 1 = 0 + \frac{3}{2} - 1 = \frac{1}{2},$$

where i is the number of lattice points interior to ABC and b is the number of lattice points on its boundary. The same may be said for triangle ABC' : since u, v are Christoffel words, the line segments AC' and BC' do not cross the Christoffel path for w ; since w is a Christoffel word, this implies there are no interior lattice points in ABC' ; there are only 3 boundary lattice points by the same reasoning.

Now we have two triangles with the same base, the same area, but different heights. Contradiction. \square

Finally, we add some additional facts about the factorization (w_1, w_2) that will be useful in what follows. Recall that $SL_2(\mathbb{Z})$ is the group of invertible matrices with integer entries and determinant equal to 1.

Lemma 4.1.6. *Suppose (w_1, w_2) is the standard factorization of the Christoffel word w of slope $\frac{b}{a}$, where $a \perp b$. Then*

$$\begin{pmatrix} |w_1|_x & |w_2|_x \\ |w_1|_y & |w_2|_y \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Proof. The point (i, j) of the Christoffel path labelled $\frac{1}{a}$ is $(|w_1|_x, |w_1|_y)$. Also, $(a - i, b - j) = (|w_2|_x, |w_2|_y)$. Since exactly three integer points lie in the triangle with vertices $(0, 0), (i, j), (a, b)$, it follows from Pick's Theorem and the fact that determinant is twice the area of the triangle that

$$\det \begin{pmatrix} i & j \\ a - i & b - j \end{pmatrix} = 1.$$

Therefore, the matrix is an element of $SL_2(\mathbb{Z})$. \square

Lemma 4.1.7. *Let w denote the Christoffel word of slope $\frac{b}{a}$ and let (w_1, w_2) denote its standard factorization. Then $|w_1|b \equiv 1 \pmod{a+b}$ and $|w_2|a \equiv 1 \pmod{a+b}$. Moreover, $|w_1|$ and $|w_2|$ are relatively prime.*

Proof. By lemma 2.2.3, the point (i, j) on the Christoffel path from $(0,0)$ to (a, b) has label $\frac{b}{a}$, where t satisfies

$$t \equiv (i + j)b \pmod{a + b},$$

$$t \equiv (a - i) + (b - j)a \pmod{a + b}$$

(recall that $|w|_x = a$ and $|w|_y = b$). Since $(|w_1|_x, |w_1|_y)$ is the closest point of the Christoffel path to the line segment from $(0,0)$ to (a,b) , it has label $\frac{b}{a}$. Applying the above to $t = 1$ and the point $(i, j) = (|w_1|_x, |w_1|_y)$, we have $|w_1|b \equiv 1 \pmod{a + b}$ and $|w_2|a \equiv 1 \pmod{a + b}$.

It remains to show that $|w_1|$ and $|w_2|$ are relatively prime. By Corollary 4.1.6

$$\begin{pmatrix} |w_1|_x & |w_2|_x \\ |w_1|_y & |w_2|_y \end{pmatrix} \in SL_2(\mathbb{Z}).$$

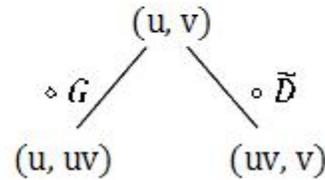
This implies that

$$\det \begin{pmatrix} |w_1| & |w_2| \\ |w_1|_y & |w_2|_y \end{pmatrix} = \det \begin{pmatrix} |w_1|_x & |w_2|_x \\ |w_1|_y & |w_2|_y \end{pmatrix} = 1.$$

That is, there exist integers k and l such that $|w_1|k + |w_2|l = 1$, which implies $|w_1| \perp |w_2|$. □

4.2 The Christoffel Tree

We close this chapter with a description of the Christoffel tree. This is the infinite, complete binary tree whose root is labelled (x, y) and whose vertices are labelled by pairs (u, v) of words in $\{x, y\}^*$ subject to the following branching rules.



View the vertex (u, v) above as a morphism $(x, y) \xrightarrow{f} (u, v)$. We have labelled the edges to indicate that $f = (u, v)$ has two branches, $f \circ G$ and $f \circ \tilde{D}$. The first few levels of the Christoffel tree are given below.

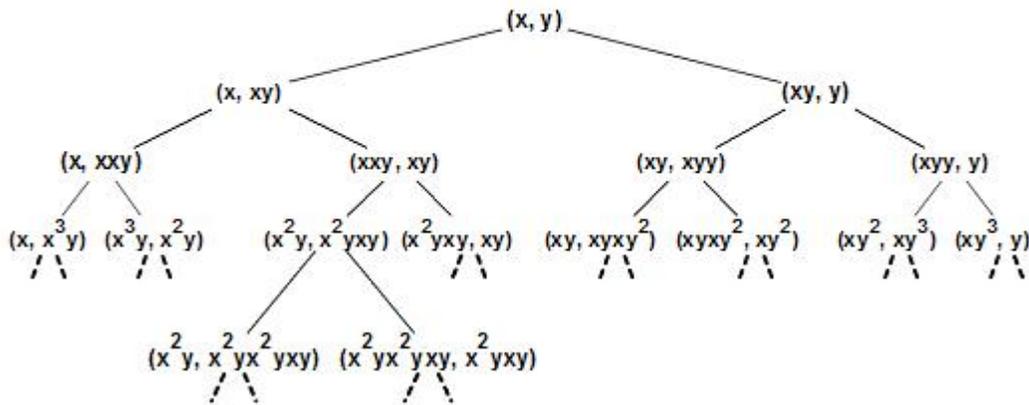


Figure 4.4: The Christoffel tree.

Theorem 4.2.1. *The Christoffel tree contains exactly once the standard factorization of each (lower) Christoffel word.*

Example 4.2.2. *Recall that $(xxy, xxyxxyxy)$ is the standard factorization of the*

Christoffel word of slope $\frac{4}{7}$. It appears in Figure 4.4 at the fifth level as $(G \circ \tilde{D} \circ G \circ G)(x, y)$.

Proof. Proof of Theorem 4.2.1. in three steps.

1. Each vertex (u, v) on the tree has the property that u, v and uv are Christoffel words.

We have seen that G and \tilde{D} send Christoffel words to Christoffel words. Since each $f = (u, v)$ on the tree corresponds to a composition of G s and \tilde{D} s, we get immediately that $u = f(x), v = f(y)$ and $uv = f(xy)$ are Christoffel words.

2. A vertex (u, v) on the Christoffel tree is the standard factorization of the Christoffel word uv .

By Step 1, u, v and uv are Christoffel words. By Theorem 4.1.5, the only way to factor uv as Christoffel words is the standard factorization (u, v) .

3. The standard factorization (w_1, w_2) of a Christoffel word w appears exactly once in the Christoffel tree.

We demonstrate how to write

$$(w_1, w_2) = (H_1 \circ H_2 \circ \dots \circ H_r)(x, y)$$

for some r and $H_i \in \{G, w_1, w_2\}$, thereby explicitly describing a path in the Christoffel tree from the root to the vertex (w_1, w_2) . The argument is illustrated in the example following this proof. We need only argue that standard factorizations may be lifted via G or \tilde{D} . Specifically, we apply G^{-1} if w_1w_2 has slope less than 1 and \tilde{D}^{-1} if w_1w_2 has slope greater than 1 (see Corollary 3.1.5).

Note that the closest point does not change under a change of basis. More precisely, we claim that if (u, v) is the standard factorization of the Christoffel

word uv of slope less than 1, then $(G^{-1}(u), G^{-1}(v))$ is the standard factorization of $G^{-1}(uv)$. First, since yy is not a factor of uv , yy is not a factor of u or v . Hence, u and v are in the image of G . Moreover, u and v are Christoffel words, so $G^{-1}(u)$ and $G^{-1}(v)$ are as well (Corollary 3.1.5). This is the standard factorization of $G^{-1}(uv)$ by Theorem 4.1.5. The same argument works for \tilde{D} .

Finally, the fact that (w_1, w_2) can occur at most once in the Christoffel tree comes from the following property of binary trees. Each vertex describes a unique path back to the root, a finite sequence of statements of the form, "I was a left branch" or "I was a right branch." Since being a left branch corresponds to precomposition by G and being a right branch corresponds to precomposition by \tilde{D} , if (w_1, w_2) appears at two distinct vertices of the graph, then we have two expressions of the form

$$(w_1, w_2) = (H_1 \circ H_2 \circ \dots \circ H_r)(x, y)$$

$$(w_1, w_2) = (H'_1 \circ H'_2 \circ \dots \circ H'_s)(x, y)$$

for some $r, s \in N$ and $H_i, H'_i \in \{G, \tilde{D}\}$. Since the only Christoffel word in the image of both G and D is xy (corresponding to the root of the tree), it follows that $H_1 = H'_1, H_2 = H'_2 \dots H_r = H'_s$. Therefore, both vertices of the graph describe the same (unique) path back to the root, contradicting the assumption that the two vertices are distinct.

□

Chapter 5

Palindromization

Recall that a word u is a palindrome if it is equal to its own reversal ($u = \tilde{u}$). This chapter begins with the observation that if w is a non trivial Christoffel word, then $w = xuy$ with u a (possibly empty) palindrome. It continues by investigating the set of palindromes u for which xuy is a Christoffel word.

5.1 Christoffel words and palindromes

We prove that every nontrivial (lower) Christoffel word can be expressed as xuy with u a palindrome, and that the corresponding upper Christoffel word is yux .

Lemma 5.1.1. *Suppose $a \perp b$. Translation by the vector $\vec{e}_2 - \vec{e}_1$ and rotation about the point $(\frac{a}{2}, \frac{b}{2})$ each map the interior points of the lower Christoffel path from $(0,0)$ to (a,b) onto the interior points of the upper Christoffel path from $(0, 0)$ to (a,b) .*

Proof. Translation: Let Q be a point different from $(0,0)$ and (a,b) on the lower Christoffel path from $(0,0)$ to (a,b) . Then the translated point $Q + (\vec{e}_2 - \vec{e}_1)$ is an integer point lying above the lower Christoffel path, and so it lies above the segment from $(0,0)$ to (a,b) . Since there is no path in the integer lattice consisting of steps \vec{e}_1

and \vec{e}_2 that avoids Q and $Q + (\vec{e}_2 - \vec{e}_1)$, and that has Q and $Q + (\vec{e}_2 - \vec{e}_1)$ on opposite sides of the path, it follows that $Q + (\vec{e}_2 - \vec{e}_1)$ lies on the upper Christoffel path from $(0,0)$ to (a,b) .

Rotation: Since there are no lattice points enclosed by the (upper or lower) Christoffel path and the segment from $(0,0)$ to (a,b) , a half-turn

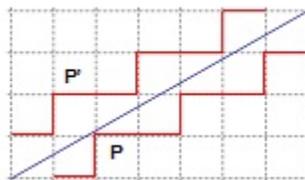


Figure 5.1: Translation by $\vec{e}_2 - \vec{e}_1$ maps P to P' ; rotation about $(\frac{7}{2}, 2)$ maps P onto the reverse P' .

about the midpoint of the line segment from $(0,0)$ to (a,b) maps the lower Christoffel path to the upper Christoffel path. \square

The following result is the consequence of the above geometric lemma.

Lemma 5.1.2. *Suppose $a \perp b$. If w is a nontrivial lower Christoffel word of slope $\frac{b}{a}$, then $w = xuy$ with u a palindrome. If w' is the upper Christoffel word of slope $\frac{b}{a}$, then $w' = yux$. In particular, $w' = \tilde{w}$.*

Proof. Let w and w' be the nontrivial lower and upper Christoffel words of slope $\frac{b}{a}$, respectively. By construction any lower Christoffel word begins by x and ends by y , so $w = xuy$ for some $u \in \{x,y\}^*$. Similarly, $w' = yu'x$ for some $u' \in \{x,y\}^*$. The words u and u' correspond to the subpaths P and P' obtained from the lower and upper Christoffel paths from $(0,0)$ to (a,b) , respectively, by removing the endpoints. By Lemma 5.1.1, P is a translate of P' , so $u = u'$. Also by Lemma 5.1.1, a half-turn

rotation maps P' onto P . Since rotation reverses the direction of P' , it follows that $u = \tilde{u}' = u$. So u is a palindrome. Finally, $w' = yu'x = yux = y\tilde{u}x = \tilde{w}$. \square

5.2 Palindromic closures

We next determine those palindromes u for which xuy is a Christoffel word, following the work of Aldo de Luca and others. A function Pal that maps words to palindromes is defined and it will be shown that $xPal(v)y$ is a Christoffel word for every $v \in \{x, y\}^*$. It will be useful to have the terminology palindromic prefix and palindromic suffix, that is, a prefix (respectively, suffix) u of a word w such that u is a palindrome.

Proposition 5.2.1. *Let w be a word. Write $w = uv$, where v is the longest suffix of w that is a palindrome. Then $w^+ = w\tilde{u}$ is the unique shortest palindrome having w as a prefix.*

Definition 5.2.1. Let w be a word. The word w^+ constructed in Proposition 5.2.1 is called the (right) palindromic closure of w .

Example 5.2.2. *Let $w = yxyxxy$. The longest palindromic suffix of w is $v = yxy$. Putting $u = yx$, we have $w^+ = w\tilde{u} = yxyxxyxy$ and w^+ is indeed a palindrome.*

Definition 5.2.2. Define a function $Pal : \{x, y\}^* \rightarrow \{x, y\}^*$ recursively as follows. For the empty word e , let $Pal(e) = e$. If $w = vz \in \{x, y\}^*$ for some $z \in \{x, y\}$, then let $Pal(w) = Pal(vz) = (pal(v)z)^+$.

The word $Pal(w)$ is called the iterated palindromic closure of w .

Example 5.2.3. *We compute $Pal(xyxx)$.*

$$Pal(x) = (Pal(e)x)^+ = x^+ = x$$

$$Pal(xy) = (Pal(x)y)^+ = (xy)^+ = xyx.$$

$$Pal(xyxx) = (Pal(xy)x)^+ = ((xyx)x)^+ = xyxxxyx.$$

$$Pal(xyxxx) = (Pal(xyxx)x)^+ = ((xyxxxyx)x)^+ = xyxxxxyxxx.$$

Note that the Christoffel word of slope $\frac{4}{7}$ is $xyxxxxyxxxxy = xPal(xyxxx)y$.

The map $w \mapsto Pal(w)$ is injective. Briefly, the inverse map is obtained by taking the first letter after each palindromic prefix of $Pal(w)$ (excluding $Pal(w)$, but including the empty prefix e). The fact that this procedure works follows from the observation that the only palindromic prefixes of $Pal(w)$ are those obtained during the iterated construction of $Pal(w)$.

Example 5.2.4. Suppose $Pal(w) = xyxxxxyxxx$. The palindromic prefixes of $Pal(w)$ excluding $Pal(w)$ are: $e; x; xyx;$ and $xyxxx$. The first letter after these prefixes are: $x; y; x; x$. Therefore, $w = xyxx$. This agrees with the computation of $Pal(xyxx)$ in the previous example. Moreover, from that computation we note that the words $Pal(e), Pal(x), Pal(xy), Pal(xyxx)$ and $Pal(xyxxx)$ are palindromic prefixes of $Pal(xyxxx)$, and that they are the only palindromic prefixes of $Pal(xyxxx)$.

The remainder of this section is devoted to proving result that implies, among other things, that xuy is a Christoffel word if u is in the image of Pal .

Theorem 5.2.5. Let $v \in \{x, y\}^*$. Then $w = xPal(v)y$ is a Christoffel word. If (w_1, w_2) is the standard factorization of w , then

$$\mu(v) = \begin{pmatrix} |w_1|_x & |w_2|_x \\ |w_1|_y & |w_2|_y \end{pmatrix} \in SL_2(\mathbb{Z}).$$

where $\mu : \{x, y\}^* \rightarrow SL_2(\mathbb{Z})$ is the multiplicative monoid morphism defined by

$$\mu(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \mu(y) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and $Pal(v)$ has relatively prime periods $|w_1|$ and $|w_2|$.

Example 5.2.6. Let $w = xyxxxyxyxy$ denote the Christoffel word of slope $\frac{4}{7}$. Note that $xyxxxyxyx$ has periods 3 and 8. In previous examples we saw that $w = xPal(xyxx)y$ and that the standard factorization of w is $(w_1, w_2) = (xy, xyxxxyxy)$.

Therefore,

$$\begin{aligned} \mu(xyxx) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} |w_1|_x & |w_2|_x \\ |w_1|_y & |w_2|_y \end{pmatrix}. \end{aligned}$$

The proof is divided into three propositions. We begin by proving that $xPal(v)y$ is a Christoffel word.

The following formulae of Jacques Justin give a very useful method for computing $Pal(v)$.

Lemma 5.2.7. For any word $w \in \{x, y\}^*$,

$$Pal(xw) = G(Pal(w)x) = G(Pal(w))x,$$

$$Pal(yw) = D(Pal(w)y) = D(Pal(w))y.$$

Example 5.2.8. Compute $Pal(xyxx)$.

$$\begin{aligned} Pal(xyxx) &= G(Pal(yxx))x \\ &= G(D(Pal(xx))y)x \\ &= G(D(xx)y)x \\ &= G(yxyxy)x \\ &= xyxxxyxyx. \end{aligned}$$

Proposition 5.2.9. Suppose $v \in \{x, y\}^*$. Then $w = xPal(v)y$ is a Christoffel word.

Proof. Proceed by induction on $|v|$. Suppose the length of v is zero. Then $Pal(v) = e$ and $w = xy$, which is a Christoffel word. Suppose that $xPal(v)y$ is a Christoffel word for all words v of length at most r and let $v' \in \{x, y\}^*$ be a word of length $r + 1$. If v' begins with x , then write $v' = xv$ for some $v \in \{x, y\}^*$. Then, by the formulae of Justin,

$xPal(v')y = xPal(xv)y = xG(Pal(v)x = G(xPal(v)y)$. This is a Christoffel word because $xPal(v)y$ is a Christoffel word (by the induction hypothesis) and because G maps Christoffel words to Christoffel words.

If $v' = yv$, then $xPal(v')y = xPal(yv)yxD(Pal(v)y)y$. Lemma 3.1.7 implies there exists a word u such that $\tilde{D}(Pal(v)y) = uy$ and $D(Pal(v)y) = yu$. The first equality together with $\tilde{D}(y) = y$ implies that $u = \tilde{D}(Pal(v))$. Therefore, $Pal(v)y = y\tilde{D}(Pal(v))$.

Hence, $xPal(v')y = xD(Pal(v)y)y = xy\tilde{D}(Pal(v))y = \tilde{D}(xPal(v)y)$.

This is a Christoffel word because $xPal(v)y$ is a Christoffel word (by the induction hypothesis) and because \tilde{D} maps Christoffel words to Christoffel words. \square

We next prove that the entries of the matrix $\mu(v)$ are given by the numbers of occurrences of the letters x and y in the words w_1 and w_2 , where (w_1, w_2) is the standard factorization of the Christoffel word $xPal(v)y$.

Proposition 5.2.10. *Suppose $v \in \{x, y\}^*$. If (w_1, w_2) is the standard factorization of the Christoffel word $xPal(v)y$, then*

$$\mu(v) = \begin{pmatrix} |w_1|_x & |w_2|_x \\ |w_1|_y & |w_2|_y \end{pmatrix}.$$

Proof. We proceed by induction on the length of v . If $|v| = 0$, then $v = e$. So the Christoffel word $xPal(e)y$ is xy and its standard factorization is (x, y) . Therefore,

$$\mu(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} |x|_x & |y|_x \\ |x|_y & |y|_y \end{pmatrix}.$$

This establishes the base case of the induction. Suppose the result holds for all words v of length at most $r - 1 > 0$ and let v' be a word of length r . If v' begins with x , then $v' = xv$ for some $v \in \{x, y\}^*$. By the induction hypothesis,

$$\mu(v') = \mu(x)\mu(v) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} |w_1|_x & |w_2|_x \\ |w_1|_y & |w_2|_y \end{pmatrix} = \begin{pmatrix} |w_1| & |w_2| \\ |w_1|_y & |w_2|_y \end{pmatrix},$$

where (w_1, w_2) is the standard factorization of $xPal(v)y$. Writing (w'_1, w'_2) for the standard factorization of the Christoffel word $xPal(v')y$, we would like to show that

$$\begin{pmatrix} |w_1| & |w_2| \\ |w_1|_y & |w_2|_y \end{pmatrix} = \begin{pmatrix} |w'_1|_x & |w'_2|_x \\ |w'_1|_y & |w'_2|_y \end{pmatrix}.$$

In view of Lemma 4.1.6, it suffices to show that $|w'_1|_x + |w'_2|_x = |w_1| + |w_2|$ and $|w'_1|_y + |w'_2|_y = |w_1|_y + |w_2|_y$. Equivalently, we need to show that $|xPal(v')y|_x = |xPal(v)y|_x$ and $|xPal(v')y|_y = |xPal(v)y|_y$. By the formula of Justin, $xPal(v')y = xPal(xv)y = xG(Pal(v))xy = G(xpal(v)y)$. Since $G = (x, xy)$ replaces each letter of a word $m \in \{x, y\}^*$ with a word having exactly one occurrence of x , the number of occurrences of the letter x in $G(m)$ is the length of m . Therefore,

$$|xPal(v')y|_x = |G(xPal(v)y)|_x = |xPal(v)y|_x.$$

Since $G = (x, xy)$ fixes the letter x and replaces y with a word having exactly one occurrence of y , we have $|G(m)|_y = |m|_y$ for any word $m \in \{x, y\}^*$. Therefore,

$$|xPal(v')y|_y = |G(xPal(v)y)|_y = |xPal(v)y|_y.$$

This completes the induction for words beginning with the letter x .

If, instead, v' begins with the letter y , then $v' = yv$ for some $v \in \{x, y\}^*$, and

$$\mu(v') = \mu(y)\mu(v) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} |w_1|_x & |w_2|_x \\ |w_1|_y & |w_2|_y \end{pmatrix} = \begin{pmatrix} |w_1|_x & |w_2|_x \\ |w_1| & |w_2| \end{pmatrix},$$

where (w_1, w_2) is the standard factorization of $xPal(v)y$. As above, we need only show that $|xPal(v')y|_y = |xPal(v)y|$ and $|xPal(v')y|_x = |xPal(v)y|_x$. By the formulae of Justin,

$$xPal(v')y = xPal(yv)y = xD(Pal(v))yy.$$

Since $D = (yx, y)$, it follows that $|D(m)|_y = |m|$ and $|D(m)|_x = |m|_x$ for any word $m \in \{x, y\}^*$, so

$$\begin{aligned} |xPal(v')y|_y &= |xD(Pal(v))yy|_y = |Pal(v)yy| = |xPal(v)y|, \\ |xPal(v')y|_x &= |xD(Pal(v))yy|_x = |xPal(v)y|_x. \end{aligned}$$

This completes the induction. \square

We now turn to the computation of periods of the word $Pal(v)$. The following result determines a period of palindromes having palindromic prefixes.

Lemma 5.2.11. *If a palindrome u has a palindromic prefix $p \neq u$, then u has a period $|u| - |p|$.*

Proof. Write $u = pv$ for some word v . Then $u = \tilde{v}p$ because u and p are palindromes. Since $u = pv$, we have $u_i = p_i$ for $0 \leq i < |p|$. And since $u = \tilde{v}p$, we have $u_{i+|v|} = p_i$ for $0 \leq i < |p|$. Therefore, $u_i = u_{i+|v|}$ for $0 \leq i < |p|$. That is, u has period $|v| = |u| - |p|$. \square

Lemma 5.2.12. *Suppose $v \in \{x, y\}^*$. The word $Pal(v)$ has periods $|w_1|$ and $|w_2|$, where (w_1, w_2) is the standard factorization of the Christoffel word $xPal(v)y$. Moreover, the periods $|w_1|$ and $|w_2|$ are relatively prime and $|Pal(v)| = |w_1| + |w_2| - 2$.*

Proof. Let (w_1, w_2) is the standard factorization of $xPal(v)y$. Then w_1 and w_2 are Christoffel words by Proposition 4.1.2. There are two cases to consider.

Case 1: w_1 or w_2 is a trivial Christoffel word. If $w_1 = x$, then

$$\mu(v) = \begin{pmatrix} 1 & |w_2|_x \\ 0 & |w_2|_y \end{pmatrix}.$$

Since $\det(\mu(v)) = 1$, it follows that $|w_2|_y = 1$. So $|w_2| = x^e y$ for some $e \in N$.

Hence, $Pal(v) = x^e$, which has periods $|w_1| = 1$ and $|w_2| = e + 1$, and length $|w_1| + |w_2| = 1 + (e + 1) - 2 = e$. The same argument holds if $w_2 = y$.

Case 2: w_1 and w_2 are nontrivial Christoffel words. By Proposition 5.1.2, there exist palindromes u_1 and u_2 such that $w_1 = xu_1y$ and $w_2 = xu_2y$. Therefore, $Pal(v) = u_1xyu_2$. The previous lemma implies that $Pal(v)$ has periods $|Pal(v)| - |u_1| = |u_2| + 2 = |w_2|$ and $|Pal(v)| - |u_2| = |u_1| + 2 = |w_1|$. The fact that $|w_1|$ and $|w_2|$ are relatively prime follows from Lemma 4.1.7. \square

Reference

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