ADDIS ABABA UNIVERSITY

COLLEGE OF NATURAL SCIENCE
DEPARTMENT OF MATHEMATICS

GRADUATE PROJECT REPORT ON TRANSPORTATION PROBLEM UNDER SHORTAGE OF SUPPLY AND PENALTY FOR DEFICIENCY

Submited in partial fulfilment of requirements for the Degree of Master of Science in Mathematics

Prepared by: Gashaw Merga
Advisor: Dr.Berhanu Guta

April, 2015
Addis Ababa, Ethiopia

Contents

Addis Ababa University
Department of Mathematics

The undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled **Transportation problem under shortage of supply and penalty for unmet demand** by **Gashaw Merga** in partial fulfillment of the requirements for the degree of master of Science.

Dated: April, 2015

Advisor: ______________________
Dr.Berhanu Guta

Examining committee: ______________________

____

Notation

c_{ij}=the cost of transporting a unit of product from source i to destination j.
$\overline{c}_{ij}$= the arc reduced costs.
x_{ij}=the quantity of commodity transported from source i to destination j.
$c_{ij}x_{ij}$=the cost involved in moving a unit quantity from source i to destination j.
m=the number of rows.
n=the number of columns.
$m + n - 1$=the number of basic variables in basic initial solution.
$\sum$=sum
d_j=the demand required at the destination j.
s_i=supply available at the source i.
(i, j)= an arc whose initial node is i and terminal node is j. λ_i=dual variable at node i.
T= the set of arcs in the spanning tree T.
L= the set of nontree arcs whose flow is restricted to value zero.
U=the set of nontree arcs whose flow is restricted in value to the arcs' flow capacities.

Acknowledgement

First,I would like to express my deepest gratitude to God for giving me patience; I am also grateful to my advisor Dr. Berhanu Guta for his helpful discussion, comments and providing the necessary materials in the preparation of this paper.
Secondly I would like to extend my thank to my families and all who encouraged me to complete my project and their heart full help while writing the paper. Lastly I would like thanks to Department of mathematics, Addis Ababa University for giving the necessary materials throughout the preparation of this paper.

Abstract

The objective of this project is to minimize total transportation cost plus penalty cost for shortage of supply.
The work focused on formulating transportation problem under shortage of supply and penalty for unmet demand mathematically and converting the problem in to minimum cost flow network .
The network simplex method is used to solve the problem.
The extended transportation problem which is the transshipment problem also considered in the work. The problem is supported by numerical illustration and Matlab implementation.

Introduction

The transportation problem which is one of network integer programming problems is a problem that deals with distributing any commodity from any group of sources to any group of sink in the most cost effective way with a given supply and demand constraints.
This work focused on transportation problem under shortage of supply and penalty for deficiency and to minimise total transportation cost plus penalty cost.
In order to achieve this objective the problem is formulated mathematically as a model and converted in to minimum cost flow network . The network simplex algorithm is used to solve the problem.
The problem is mathematically formulated as follows;
*Shortage of supply means when

$$\sum_{i=1}^{m} s_i < \sum_{j=1}^{n} d_j.$$

where s_i is supply available at the source and d_j is the demand required at the destination, $s_i \geq 0$ and $d_j \geq 0$. where $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$. and the cost involved in moving a unit quantity from source i to destination j is c_{ij}.
Then the model of the problem is;

$$min \quad \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{j=1}^{n} c_{(m+1)j} x_{(m+1)j} \tag{1}$$

Subject to,

$$\sum_{j=1}^{n} x_{ij} = s_i, \qquad i = 1, 2, ..., m+1. \tag{2}$$

$$\sum_{i=1}^{m} x_{ij} + x_{(m+1)j} = d_j, \quad j = 1, 2, ..., n. \tag{3}$$

$$x_{ij} \geq 0, \quad i = 1, 2, ..., m+1. \ \ and \ \ j = 1, 2, ..., n.$$

Where $c_{(m+1)j}$ is penalty cost per unit of unmet demand at the j^{th} destination,
and

$$s_{m+1} = \sum_{j=1}^{n} d_j - \sum_{i=1}^{m} s_i$$

The slack variable $x_{(m+1)j}$ represents the quantity of unsatisfied demand at the j^{th} destination.

⋆ The transshipment problem is an extension of the framework of the transportation problems.The extension is in allowing the presence of a set of transshipment points that can serve as intermediate stops for shipments, possibly with a net gain or loss in units.
Mathematical formulation of the transshipment problem is:

$$Minimise \quad \sum_{(i,j)\in A} c_{ij}x_{ij}$$

subject to

$$\sum_{(i,j)\in A} x_{ij} = s_i \quad , i = 1, 2, ...m. \quad supply \ \ constraints.$$

$$\sum_{(i,j)\in A} x_{ij} = \sum_{(j,k)\in A} x_{jk} \quad flow \ conservation \ at \ transit \ nodes.$$

$$\sum_{(i,j)\in A} x_{ij} = d_j \quad , j = 1, 2, ..., n. \quad demand \ constraints.$$

$$0 \leq x_{ij} \leq u_{ij} \ \ , for \ all \ \ (i, j) \in A.$$

Chapter 1

PRELIMINARIES

1.1 Linear Programming(L.P)

It refers to a planning process that allocates resources; labor,materials,machines,capital in best possible (optimal) way so that costs are minimized or profits are maximized.In L.P,these resources are known as decision variables.
A linear programming problem is said to be in canonical form when it is written as:

$$Maximize \quad \sum_{j=1}^{n} c_j x_j \tag{1.1}$$

subject to,

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \qquad i = 1, 2, ..., m. \tag{1.2}$$

$$x_j \geq 0, \qquad j = 1, 2, ..., n.$$

The problem has m variables and n constraints. It may be written using vector terminology as:

$$Maximize \quad c^T X \tag{1.3}$$

subject to,

$$AX \leq b, \qquad X \geq 0. \tag{1.4}$$

A linear programming problem is said to be in standard form when it is written as:

$$Maximize \quad \sum_{j=1}^{n} c_j x_j \tag{1.5}$$

subject to,

$$\sum_{j=1}^{n} a_{ij} x_j + s_i = b_i, \qquad i = 1, 2, ..., m. \tag{1.6}$$

$$x_j \geq 0, \qquad j = 1, 2, ..., n, s_i \geq 0 \ i = 1 : m.$$

The problem has $m + n$ variables and n constraints. It may be written using vector terminology as:

$$Maximize \quad c^T X \tag{1.7}$$

subject to,

$$AX + S = b, \qquad X \geq 0, S \geq 0. \tag{1.8}$$

Note that a problem where we would like to minimize the cost function instead of maximize it may be rewritten in standard form by negating the cost coefficients $c_j(c^T)$.

1.2 Transportation Problem

1.2.1 Representation of transportation problem as a L.P problem

If x_{ij} represent the quantity of commodity transported from source i to destination j, and the cost of transporting a unit of product from source i to destination j is c_{ij} , then the cost involved in moving a unit quantity from source i to destination j is given as; $Cost = c_{ij} x_{ij}$.
The cost of transporting products from source i to all destinations is given as

$$cost = \sum_{j=1}^{n} c_{ij} x_{ij}$$

The total cost of transporting products from all the sources to all the destinations is given by;

$$Min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}, \tag{1.9}$$

$$\text{where} \quad x_{ij} \geq 0, i = 1, 2, ..., m. \text{ and } \quad j = 1, ..., n.$$

Definition 1.2.1. A transportation problem is a special case of linear programming problem of the following form:
If s_i and d_j are supply and demand respectively then, the objective function is;

$$minimize \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \tag{1.10}$$

Subject to,

$$\sum_{j=1}^{n} x_{ij} \leq s_i, \quad i = 1, 2, ..., m. \tag{1.11}$$

$$\sum_{i=1}^{m} x_{ij} \geq d_j, \quad j = 1, 2, ..., n. \tag{1.12}$$

$$x_{ij} \geq \ 0 \ , \ for \ \ all \ \ i \ \ and \ \ j.$$

The first constraint or equation means the sum of all shipments from a source cannot exceed the available supply and the second constraint means the sum of all shipments to a destination should be at least as large as the demand. Equation four simply means shipment value should not be negative. From the two constraints we can say that the total supply is either greater or equal to the demand.

1.2.2 Balanced transportation problem as a L.P. problem

Definition 1.2.2. A balanced transportation problem (BTP) is a transportation problem in which total supply is equal to total demand.
Mathematically this can be written as;

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j \tag{1.13}$$

Where s_i is supply available at the source and d_j is the demand required at the destination, $s_i \geq 0$ and $d_j \geq 0$. where i=1,...,m and j=1,...,n.
The objective function of the balanced transportation problem is;

$$\min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \tag{1.14}$$

Subject to,

$$\sum_{i=1}^{m} x_{ij} = d_j, \quad j = 1, 2, ..., n. \tag{1.15}$$

$$\sum_{j=1}^{n} x_{ij} = s_i, \quad i = 1, 2, ..., m. \tag{1.16}$$

$$x_{ij} \geq 0, \ \ for \ \ all \ \ i = 1, ..., m. \ \ and \ \ j = 1, ..., n.$$

For a feasible solution to exist, it is necessary that total supply be equal to total demand and there should be m+n-1 basic independent variables out of m+n variables, where m is the number of sources and n is the number of destinations.

1.2.3 unbalanced transportation problem

A situation in the transportation problem where total supply is not equal to the total demand is known as unbalanced transportation problem. This situation of the transportation problem is mostly encountered in everyday life, where there is high demand of a particular commodity than a factory can supply or demand is less than what a factory can produce. There are two situations, one is total supply is greater than total demand, mathematically written as,

$$\sum_{i=1}^{m} s_i > \sum_{j=1}^{n} d_j$$

In this case, demand is made to be equal to the surplus by creating a dummy destination. The other situation is where total demand is greater than the total supply stated mathematically as follows;

$$\sum_{i=1}^{m} s_i < \sum_{j=1}^{n} d_j.$$

A dummy source is created with a supply equal to the excess of the demand.

1.2.4 Network flow model of the transportation problem

The figure below shows a network representation of the transportation problem with m sources and n destinations.The sources are from 1 to m and the destinations are also from 1 to n.Each source is represented by a node and each sink is also represented by a node.The routes from supply node to demand node are represented by the arrows.

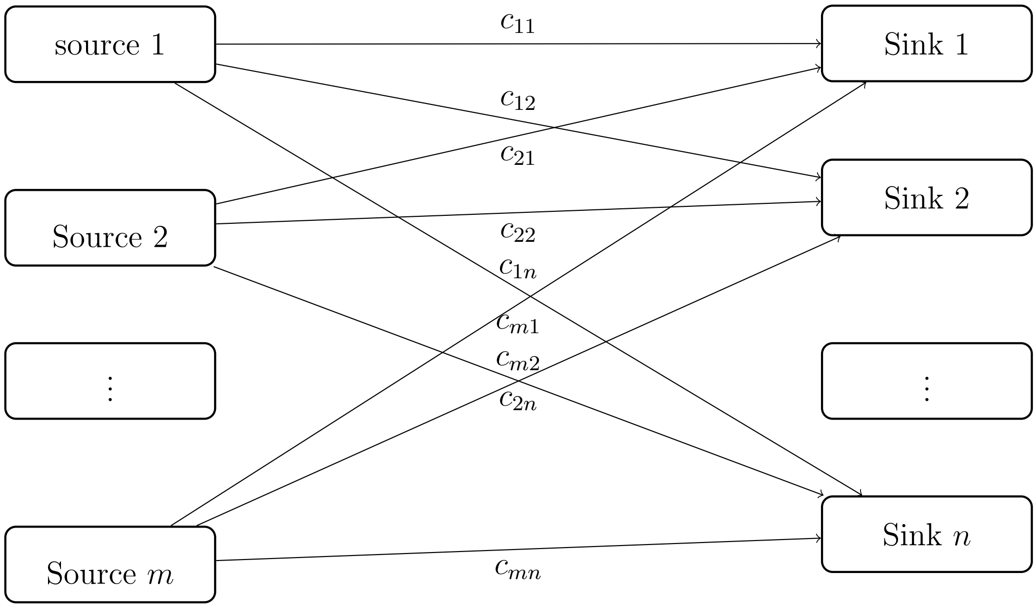


Figure 1.2.4

Here sources indicate the place from where transportation will begin, sink indicates the place where the product has to be arrived and c_{ij} indicates the transportation cost transporting from source to destination and sink denotes the destination.

1.2.5 General transportation model in table form

The transportation model can also be portrayed in a tabular form by means of a transportation table as shown below:

Origin(i)	Destination(j)				supply(s_i)
	1	2	...	n	
1	x_{11} c_{11}	x_{12} c_{12}	...	x_{1n} c_{1n}	s_1
2	x_{21} c_{21}	x_{22} c_{22}	...	x_{2n} c_{2n}	s_2
⋮	⋮	⋮	...	⋮	⋮
m	x_{m1} c_{m1}	x_{m2} c_{m2}	...	x_{mn} c_{mn}	s_m
Demand(j)	d_1	d_2	...	d_n	$\sum s_i = \sum d_j$

Figure 1.2.5

The number of constraints in transportation table is (m+n), where m denotes the number of rows and n denotes the number of columns.

1.2.6 The transportation algorithm

This algorithm minimizes the cost of transporting goods from m origins to n destinations.
The transportation algorithm consists four steps:

Step1. balance it, if the transportation problem is unbalanced.

Step2. find an initial basic feasible solution by north west corner method.

Step3. test the initial solution for optimality using modified distribution(MODI) method. If the solution is optimal stop, otherwise, determine a new improved solution.

Step4. Updating the solution. Repeat step 3 until the optimal solution arrived at.

1.2.7 The transshipment problem

In a transportation problem, shipments are allowed only between source-sink pairs.It is often the case that shipments may be allowed between sources and between sinks.Moreover,there may also exist points through which units of a product can be transshipped from a source to a sink.
Models with these additional features are called **Transshipment problems**.
Mathematical formulation of the transshipment problem is:

$$Minimise \quad \sum_{(i,j)\in A} c_{ij}x_{ij}$$

subject to

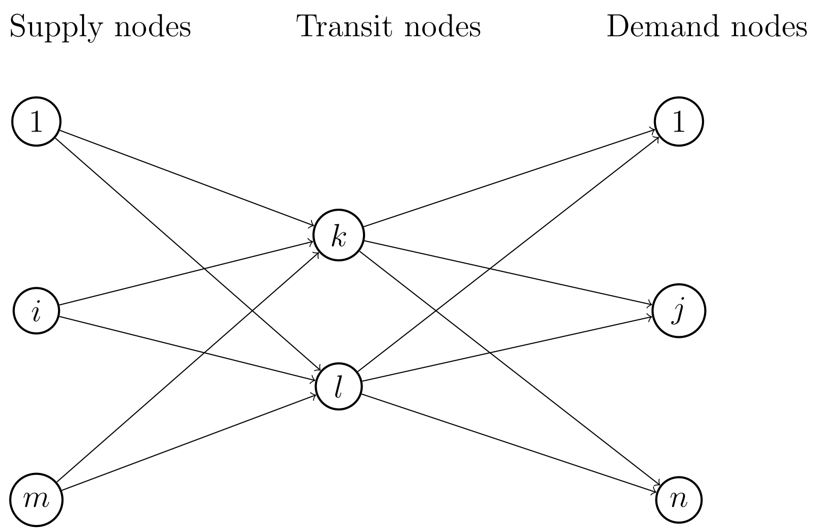
$$\sum_{(i,j)\in A} x_{ij} = s_i \quad supply \;\; constraints.$$

$$\sum_{(i,j)\in A} x_{ij} = \sum_{(j,k)\in A} x_{jk} \quad flow\ conservation\ at\ transit\ nodes.$$

$$\sum_{(i,j)\in A} x_{ij} = d_j \quad demand\ constraints.$$

$$0 \le x_{ij} \le u_{ij} \;\; , for\ all \;\; (i,j) \in A.$$

Network formulation of the transshipment problem is:



⋆ The transshipment problem is an extension of the framework of the transportation problems.The extension is in allowing the presence of a set of transshipment points that can serve as intermediate stops for shipments, possibly with a net gain or loss in units.
⋆To solve the transshipment problem we need to use the more general solution method which is the network simplex method.

1.2.8 Definitions of basic terms

The following terms are to be defined with reference to the transportation problem.

1. **Feasible solution(F.S)** A set of non negative allocation ,$x_{ij} \geq 0$, which satisfies the row and column restriction is known as feasible solution.

2. **Basic feasible solution(BFS)**
   A feasible solution to m-origins and n-destinations is said to be a basic feasible solution if the number of positive allocations are (m+n-1). If the number of allocations in basic feasible solution are less than (m+n-1), it is called degenerate basic feasible solution(DBFS),otherwise non degenerate.

3. **Optimal solution** A feasible solution is said to be optimal if minimizes the total transportation cost.

4. **Dummy origin or destination** A dummy origin or destination is an imaginary origin or destination with zero cost introduced to make an unbalanced transportation balanced.

5. **Destination or sink** It is the location to which shipments are transported to, these could be depots, warehouses or factories.

6. **Origin** This is also called **source**, it is a location from which commodities are transported from.

7. **Node potential** is the set of dual variables corresponding to the mass balance constraint of nodes.

Chapter 2

SOLUTION METHOD

The solution procedure for the transportation problem consists of two phases:
Phase 1.Finding the initial basic feasible solution.
Phase 2.Optimization of the initial basic feasible solution which is obtained in phase 1.

2.1 METHODS TO FIND AN INITIAL BASIC FEASIBLE SOLUTION

The most commonly used methods to find basic feasible solution of transportation problem are;

1. North west corner rule(NWCR) It is a procedure in transportation model where one starts at the upper left-hand cell of a table(the northwest corner)and systematically allocates units to shipping routes.

2. Least cost method(LCM) It is a cost based approach in an initial solution to transportation problem.This method makes initial allocations based on lowest cost.

3. Vogel's approximation method(VAM). It is the other important technique in addition to NWCR and LCM.

2.1.1 Algorithm for vogel's approximation method(VAM)

The vogel's approximation method(VAM) is an iterative procedure for computing a basic feasible solution of a transportation problem. This method is better than the other two methods. Because the basic feasible solution obtained by this method is nearer to the optimal solution. The algorithm of this method is given below:

Step1. Identify the boxes having minimum and next to minimum transportation cost in each row and write the difference(penalty) along the side of the table against the corresponding row.

Step2. Identify the boxes having minimum and next to minimum transportation cost in each column and write the difference(penalty) along the side of the table against the corresponding column. If minimum cost appear in two or more times in a row or column then select these same cost as a minimum and next to minimum cost and penalty will be zero.

Step3:

a. Identify the row and column with the largest penalty, breaking ties arbitrarily. Allocate as much as possible to the variable with the least cost in the selected row or column. Adjust the supply and demand and cross out the satisfied row or column. If row and column are satisfied simultaneously, only one of them is crossed out and remaining row or column is assigned a zero supply or demand.

b. If two or more penalty costs have same large magnitude, then select any one of them(or select the most top row or extreme left column).

Step4:

a. If exactly one row or one column with zero supply or demand remains uncrossed out, stop.

b. If only one row or column with positive supply or demand remains uncrossed out, determine the basic variables in the row or column by the least cost method.

c. If all uncrossed out rows or column have zero supply or demand, determine the zero basic variables by the least cost method. Stop.

d. Otherwise, go to step1.

Step5. Go to phase 2.

2.2 Optimizing the basic feasible solution applying U-V method

2.2.1 Algorithm of modified distribution method(MODI) (or U-V method)

The following are steps of MODI.

Step1. For an initial basic feasible solution with $(m+n-1)$ occupied(basic) cells, calculate u_i and$v_j$values for rows and columns, respectively using the relationship $c_{ij} = u_i + v_j$ for all allocated cells only.
To start with assume any one of the u_i or v_j to be zero.

Step2. For the unoccupied (non basic) cells calculate the cell evaluations or the net evaluations as $d_{ij} = c_{ij} - (u_i + v_j)$.

Step3:

a. If $d_{ij} \geq 0$, then the current solution is optimal.

b. If any $d_{ij} < 0$, then an improved solution can be obtained; by converting one of basic cells to non basic cells and one of non basic cells to a basic cell.

Step4. Select the cell corresponding to most negative cell evaluation. This cell is called the entering cell. Identify a closed path or a loop which starts and ends at the entering cell and connects some basic cells at every corner. It may be noted that right angle turns in this path are permitted.

Step5. put + sign in the entering cell and mark the remaining corners of the loop alternatively with − and + sign, with a plus sign at the cell being evaluated.

Step6. Determine the maximum number of units that should be shipped to this unoccupied cell. This quantity is added to all the cells on the path marked with plus sign and subtract from those cells mark with minus sign.

Step7. Repeat the whole procedure until an optimal solution is attained. i.e. $d_{ij} > 0$.

2.3 THE NETWORK SIMPLEX METHOD

The network simplex method is a specialized version of the well-known linear programming simplex method.It maintains a spanning tree solution and at each iteration transforms the current spanning tree solution in to an improved spanning tree solution until optimality is reached.
The implementation of network simplex method involves three main computational steps:

step1. Calculate(update) the dual price vector.

step2. Determine an entering arc.

step3. Find the leaving arc on the cycle created by the entering arc and update the flows.

In this project we shall see the implementation of **network simplex method** to solve balanced transportation problem.

Chapter 3

TRANSPORTATION PROBLEM UNDER SHORTAGE OF SUPPLY AND PENALTY FOR UNMET DEMAND

3.1 Balancing an unbalanced transportation problem

The unbalanced transportation problem is solved by first changing the un balanced problem in to a balanced transportation problem by creating a dummy column or row depending on where the storage is coming from, that is from supply or demand. The dummy row or column created in the unbalanced transportation problem is just a row or column with zero(0) as a unit cost of transportation.
However, a meaningful problem exists if there are penalty costs for unsatisfied demands and storage costs for surpluses at the origins.

Theorem 3.1.1. (Existence of feasible solution)
A necessary and sufficient condition for the existence of feasible solution of a $m \times n$ transportation problem is

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j.$$

Proof 3.1.1. The condition is necessary:
Let there exists a feasible solution to the transportation problem, then

$$\sum_{j=1}^{n} x_{ij} = s_i, \quad i = 1, 2, ..., m. \quad and$$

$$\sum_{i=1}^{m} x_{ij} = d_j, \quad j = 1, 2, ..., n.$$

Summing over all i and j respectively, we get

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} s_i \quad and \quad \sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} d_j$$

$$But \quad \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij}$$

$$\Rightarrow \qquad \sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j.$$

The condition is sufficient:

$$Let \quad \sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j = k$$

$$if \quad x_{ij} = \phi_i d_j$$

for all i and j, where $\phi_i \neq 0$ is any real number, then

$$\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \phi_i d_j = \phi_i \sum_{j=1}^{n} d_j = k\phi_i.$$

$$\Rightarrow \qquad \phi_i = \frac{1}{k}\sum_{j=1}^{n} x_{ij} = \frac{s_i}{k}.$$

Thus,

$$x_{ij} = \phi_i d_j = \frac{s_i d_j}{k}$$

, for all $i$and j.
as, $s_i \geq 0$, $d_j \geq 0$ so $x_{ij} \geq 0$, for all i and$j$.
Hence the theorem is proved.

3.1.1 Mathematical formulation of the problem

In general an unbalanced transportation problem can be balanced using storage cost and penalty cost as follows;
Introducing the slack variables $x_{i(n+1)}$ and $x_{(m+1)j}$ in supply and demand constraints respectively,

$$min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{i=1}^{m} c_{i(n+1)}x_{i(n+1)} + \sum_{j=1}^{n} c_{(m+1)j}x_{(m+1)j} \tag{3.1}$$

Subject to,

$$\sum_{j=1}^{n} x_{ij} + x_{i(n+1)} = s_i, \qquad i = 1, 2, ..., m. \tag{3.2}$$

$$\sum_{i=1}^{m} x_{ij} + x_{(m+1)j} = d_j, \qquad j = 1, 2, ..., n. \tag{3.3}$$

$$x_{ij} \geq 0 \quad , i = 1, 2, ..., m+1. \quad and \quad j = 1, 2, ..., n+1.$$

Where $c_{i(n+1)}$ is the storage cost per unit at the i^{th} origin, $c_{(m+1)j}$ is the penalty cost per unit of unsatisfied demand at the j^{th} destination
$x_{i(n+1)}$ is the unutilized item at the i^{th} origin and $x_{(m+1)j}$ is the unsatisfied demand at the j^{th} destination.

•In this project we consider the transportation problem under shortage of supply and penalty for deficiency.
Shortage of supply in transportation problem can be written mathematically as follows;

$$\sum_{i=1}^{m} s_i < \sum_{j=1}^{n} d_j.$$

where s_i is the supply at the i^{th} origin and d_j is the demand at the j^{th} destination.

•**The model of the problem is**;

If s_i is supply available at the source and d_j is the demand required at the destination, $s_i \geq 0$ and $d_j \geq 0$. where $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$. and the cost involved in moving a unit quantity from source i to destination j is $c_{ij}x_{ij}$.

$$min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} c_{(m+1)j}x_{(m+1)j} \tag{3.4}$$

Subject to,

$$\sum_{j=1}^{n} x_{ij} = s_i, \quad i = 1, 2, ..., m + 1. \tag{3.5}$$

$$\sum_{i=1}^{m} x_{ij} + x_{(m+1)j} = d_j, \quad j = 1, 2, ..., n. \tag{3.6}$$

$$x_{ij} \geq 0, \quad i = 1, 2, ..., m + 1. \ \ and \ \ j = 1, 2, ..., n.$$

Where $c_{(m+1)j}$ is penalty cost per unit of unmet demand at the j^{th} destination, and

$$s_{m+1} = \sum_{j=1}^{n} d_j - \sum_{i=1}^{m} s_i$$

The slack variable $x_{(m+1)j}$ represents the quantity of unsatisfied demand at the j^{th} destination.

3.1.2 The tabular form of the problem

The transportation problem under shortage of supply and penalty for deficiency can be shown in the table below.

Origin(i)	Destination(j)				Supply(s_i)
	1	2	...	n	
1	x_{11} c_{11}	x_{12} c_{12}	... ...	x_{1n} c_{1n}	s_1
2	x_{21} c_{21}	x_{22} c_{22}	... ...	x_{2n} c_{2n}	s_2
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$
m	x_{m1} c_{m1}	x_{m2} c_{m2}	... ...	x_{mn} c_{mn}	s_m
m+1	$x_{(m+1)1}$ $c_{(m+1)1}$	$x_{(m+1)2}$ $c_{(m+1)2}$	... ...	$x_{(m+1)n}$ $c_{(m+1)n}$	s_{m+1}
Demand(j)	d_1	d_2	...	d_n	$\sum_{i=1}^{m+1} s_i = \sum_{j=1}^{n} d_j$

Figure 3.2.2

Theorem 3.1.2. *Out of $(m+n)$ equations, there are only $(m+n-1)$ independent equations in a transportation problem, m and n being the number of origin and destinations and any one equation can be dropped as the redundant equation.*

Proof 3.1.2. Consider m row equations and n-1 column equations of the transportation problem as:

$$\sum_{j=1}^{n} x_{ij} = s_i, \qquad i = 1, 2, ..., m. \qquad (1)$$

$$\sum_{i=1}^{m} x_{ij} = d_j, \qquad j = 1, 2, ..., n-1 \qquad ..(2)$$

Now adding m origin constraints given in (1),we get

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} s_i. \qquad (3)$$

Also adding n-1 destination constraints given in (2), we get

$$\sum_{j=1}^{n-1} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{(n-1)} d_j. \qquad (4)$$

Subtracting (4) from (3), we get

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n-1} \sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} s_i - \sum_{j=1}^{n-1} d_j.$$

$$\Rightarrow \qquad \sum_{i=1}^{m} [\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n-1} x_{ij}] = \sum_{j=1}^{n} d_j - \sum_{j=1}^{n-1} d_j.$$

$$since, \sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j.$$

$$or \qquad \sum_{i=1}^{m} [\sum_{j=1}^{n-1} x_{ij} + x_{in} - \sum_{j=2}^{n-1} x_{ij}] = \sum_{j=1}^{n-1} d_j + d_n - \sum_{j=1}^{n-1} d_j.$$

$$\Rightarrow \qquad \sum_{i=1}^{m} x_{in} = d_n.$$

which is the n^{th} column equation(or destination constraint).
Thus, we have only $(m+n-1)$ linearly independent equations out of $(m+n)$ equations, one(any) is redundant.
Hence the theorem is proved.

Theorem 3.1.3. There always exists an optimal solution to balanced transportation problem.

Proof 3.1.3.

$$we \quad have \qquad \sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j.$$

It follows that a feasible solution exists of the problem.
i.e. $x_{ij} \geq 0$ for all i and j.
from the constraints of the problem each $x_{ij} \leq min\{s_i, d_j\}$.
Thus

$$0 \leq x_{ij} < min\{s_i, d_j\}.$$

i.e. the feasible region of the problem is non empty, closed and bounded.
Hence, there exists an optimal solution.

3.2 solving (TP) under shortage of supply and penalty for unmet demand

In case the total demand is more than the availability, we add a dummy origin to fill the balance requirement and the shipping costs are again set equal to zero.
However, in real life, the costs of unfilled demand is seldom zero, since it may involve lost sales, lesser profit, possibility of losing the customers or even business or the use of a more costly substitute.
•In this project we need to convert the balanced transportation problem in to minimum cost flow problem and solve the problem by network simplex algorithm.

3.2.1 Minimum cost flow problem

Let G=(N,A) be a directed network with a cost c_{ij} and a capacity u_{ij} associated with every arc $(i,j) \in A$.We associate with each node $i \in N$ a number b(i)which indicates its supply or demand depending on whether ‘$b(i) > 0 \ or \ b(i) < 0$. The minimum cost flow problem can be stated as follows:

$$\text{Minimize } \sum_{(i,j)\in A} c_{ij}x_{ij}$$

subject to

$$\sum_{(i,j)\in A} x_{ij} - \sum_{(j,i)\in A} x_{ji} = b(i), for\ all\ i \in N.$$

$$0 \leq x_{ij} \leq u_{ij}, \quad forall(i,j) \in A.$$

•A *tree* is a connected network which contains no cycle.
•A *spanning tree* in network G is a tree which connects every vertex of G.
• A cycle is a set of arcs forming a closed path.
In a minimum cost flow problem,a solution is defined by specifying the flow x_{ij} in each arc (i,j) of the network.
A solution is feasible if and only if:

i. for each transit node, total incoming flow equals total outgoing flow.

ii. for each sink, the total incoming flow equals node demand plus total outgoing flow.

iii. for each source, the total outgoing flow equals node supply plus total incoming flow.

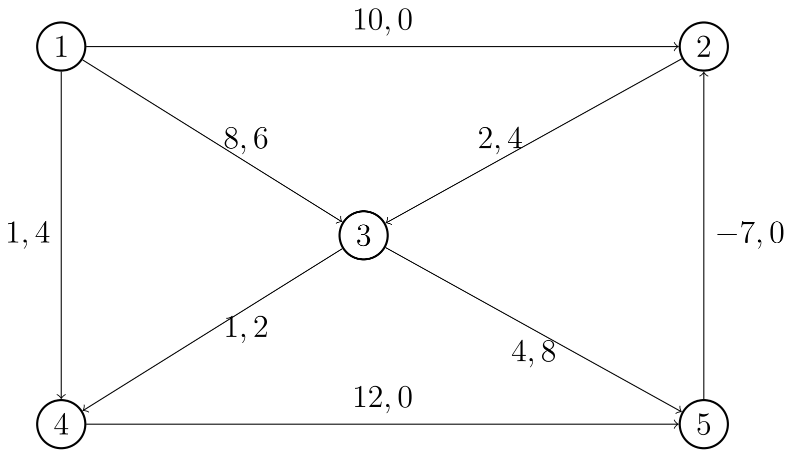
iv. All arc flows are non-negative.

✓ Conditions $(i) - (iii)$are encompassed by the constraints,for each node i:

$\sum_{(i,j)} x_{ij} - \sum_{(j,i)} x_{ji} = b_i$, where b_i is net supply at a node i.

For example,we can easily check for the transshipment problem below a feasible flow is:

$x_{13} = 6, x_{14} = 4, x_{23} = 4, x_{34} = 2, x_{35} = 8, x_{12} = 0, x_{25} = 0, x_{45} = 0$
with costs:
$c_{12} = 10, c_{13} = 8, c_{14} = 1, c_{23} = 2, c_{34} = 1, c_{35} = 4, c_{45} = 12, c_{52} = -7.$



Figure(1.) A feasible flow

✓The transportation problem can be rephrased into minimum cost flow problem.
Consider an arbitrary transportation problem (TP):

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \tag{3.7}$$

Subject to,

$$\sum_{i=1}^{m} x_{ij} = d_j, \quad j = 1, 2, ..., n. \tag{3.8}$$

$$\sum_{j=1}^{n} x_{ij} = s_i, \quad i = 1, 2, ..., m. \tag{3.9}$$

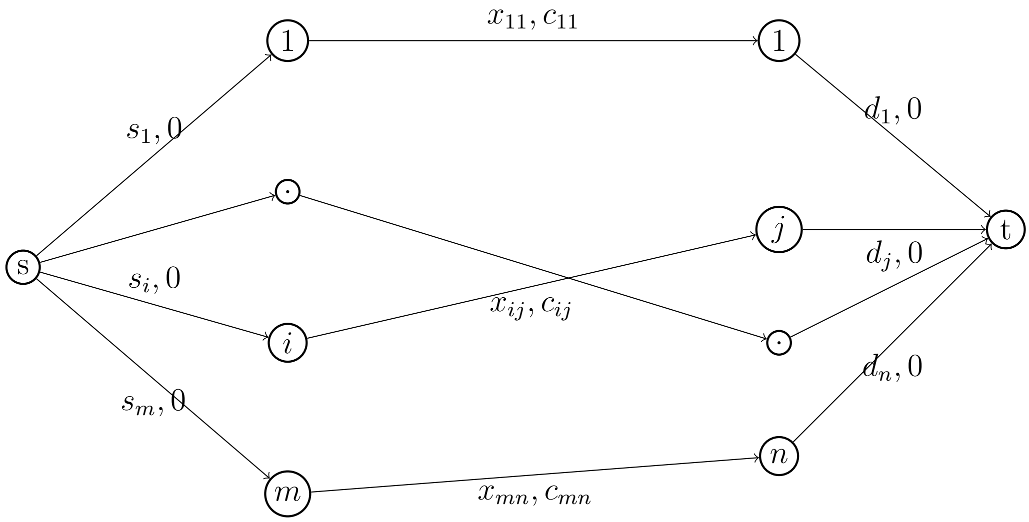
$$x_{ij} \geq 0, \text{ for all } i = 1, ..., m. \text{ and } j = 1, ..., n.$$

Design a minimum cost flow network graph G as follows:
It consists of a source s,m nodes$(1, 2, ..., m)$ corresponding to the depots, n nodes$(1, 2, 3, ..., n)$ corresponding to the customers, and a sink t.G_{Trp} has the following edges:

- • an edge from s to every depot i with capacity s_i and cost 0,

- • an edge from every depot i to every customer j with capacity ∞ and cost c_{ij},and

- •an edge from every customer j to the sink t with capacity d_j and cost 0.



Let $V_{TrP} = \sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$.
Consider the minimum cost flow problem in G_{TrP} for value V_{TrP}.
The required flow value forces the outgoing edges of s and the ingoing edges of t to be filled to their full capacity.
The constraints in the transportation problem are precisely equal to the flow conservation constraints in the nodes i $(1 \leq i \leq m)$ and j$(1 \leq j \leq n)$ in G_{TrP}.
Thus every transportation problem can be reduced to this corresponding minimum cost flow problem.
• The network simplex algorithm is one of the fastest algorithms to solve the minimum cost flow problem in practice.

✓The lagrangian of the minimum cost flow problem is

$$L_(x,\lambda) = \sum_{(i,j)\in E} c_{ij}x_{ij} - \sum_{i\in V} \lambda_i(\sum_{(i,j)\in E} x_{ij} - \sum_{(j,i)\in E} x_{ji} - b_i)$$

$$= \sum_{(i,j)\in E} (c_{ij} - \lambda_i + \lambda_j)x_{ij} + \sum_{i\in V} \lambda_i b_i$$

Let $\bar{c}_{ij} = c_{ij} - \lambda_i + \lambda_j$ be the reduced cost of edge $(i,j) \in E$. Minimizing $L(x,\lambda)$ subject to the reginal constraints $\underline{m}_{ij} \leq x_{ij} \leq \overline{m}_{ij}$ for all $(i,j) \in E$ then yields the following complementary slackness conditions:

$$\bar{c}_{ij} > 0 \Rightarrow x_{ij} = \underline{m}_{ij}$$

$$\bar{c}_{ij} < 0 \Rightarrow x_{ij} = \overline{m}_{ij}, \quad \text{and}$$

$$\underline{m}_{ij} < x_{ij} < \overline{m}_{ij} \Rightarrow \bar{c}_{ij} = 0.$$

Assume that x is a basic feasible solution associated with sets T,U and L.Then the system of equations

$$c_{ij} - \lambda_i + \lambda_j \quad \text{for all} \quad (i,j) \in T.$$

has a unique solution, which in turn allows us to compute $\bar{c}_{ij}$ for all edges $(i,j) \in E$.Note that by construction, $\bar{c}_{ij} = 0$ for all $(i,j) \in T$.

Theorem 3.2.1. *Minimum cost flow problems with arc costs* c_{ij} *or* $\overline{c}_{ij}$ *have the same optimal solutions.*

Proof. Let W be a directed cycle in network graph G.Then

$$\sum_{(i,j)\in W} \overline{c}_{ij} = \sum_{(i,j)\in W} (c_{ij} - \lambda_i + \lambda_j)$$
$$= \sum_{(i,j)\in W} c_{ij} - \sum_{(i,j)\in W} (\lambda_i - \lambda_j)$$
$$= \sum_{(i,j)\in W} c_{ij}.$$

the expretion $\sum_{(i,j)\in W}(\lambda_i - \lambda_j)$ sums to zero because for each node i in the cycle W , λ_i occures once with a possitive sign and once with a negative sign.

□

Theorem 3.2.2. *(Minimum cost flow optimality condition)*
A spanning tree structure (T, L, U) *is an optimal spanning tree structure of the minimum cost flow problem if it is feasible and for some choice of node potential* λ *, the arc reduced costs* $\overline{c}_{ij}$ *satisfy the following conditions:*

a. $\overline{c}_{ij} = 0$ *,for all* $(i, j) \in T$

b. $\overline{c}_{ij} \geq 0$ *,for all* $(i, j) \in L$

c. $\overline{c}_{ij} \leq 0$ *, for all* $(i, j) \in U$

Proof. Let $x^\star$ be the solution associated with the spanning tree structure (T, L, U) ,
• some set of node potentials λ together with the spanning tree structure (T, L, U) ,satisfies (a,b and c)

We need to show that $x^\star$ is an optimal solution of the minimum cost flow problem .

By theorem(3.2.1) minimizing $\sum_{(i,j)\in A} c_{ij}x_{ij}$ is equivalent to minimizing $\sum_{(i,j)\in A} \overline{c}_{ij}x_{ij}$. The conditions stated in (a,b and c) imply that for the given node potential λ minimizing $\sum_{(i,j)\in A} \overline{c}_{ij}x_{ij}$ is equivalent to the following expression:

minimizing $\sum_{(i,j)\in L} \overline{c}_{ij}x_{ij} - \sum_{(i,j)\in U} \mid \overline{c}_{ij} \mid x_{ij}$.

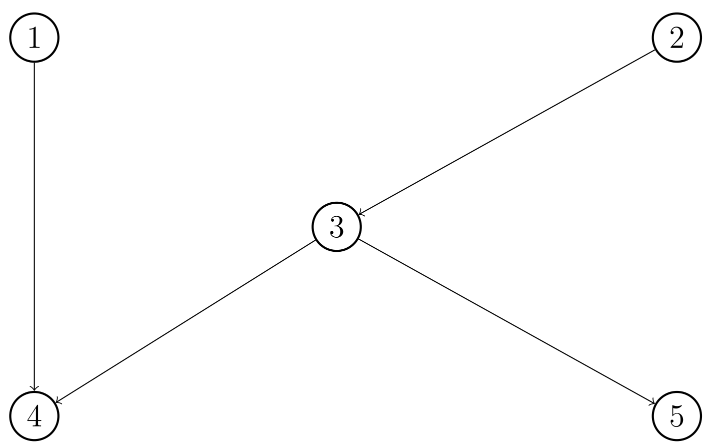
The definition of the solution $x^\star$ implies that for any arbitrary solution x,

- $x_{ij} \geq x^\star_{ij}$ for all $(i,j) \in L$.
- $x_{ij} \leq x^\star_{ij}$ for all $(i,j) \in U$.

The expression $\sum_{(i,j)\in L} \overline{c}_{ij}x_{ij} - \sum_{(i,j)\in U} \mid \overline{c}_{ij} \mid x_{ij}$ implies that the objective function value of the solution x will be greater than or equal to that of $x^\star$. □

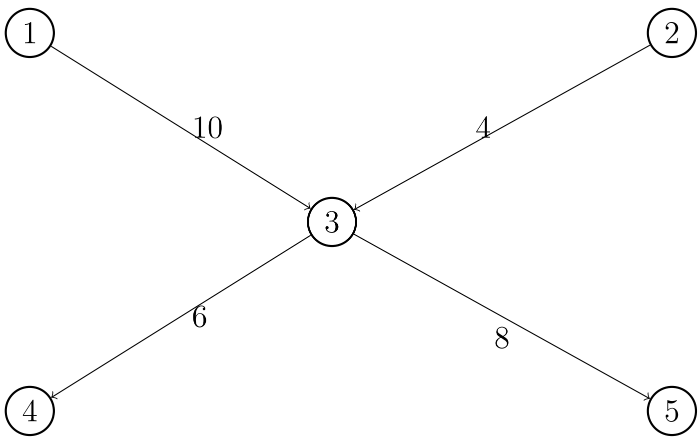
3.2.2 Basic feasible solutions and Spanning trees

•A spanning tree in network G is a tree which meets every vertex of G.
For any tree T,(the number of arcs in T)=(the number of vertices in T - 1)
An example of a spanning tree from figure 1.



Figure(2.) A spanning tree

Solving a min-cost flow problem with the network simplex algorithm, one has therefore that all the basic feasible solutions explored by the algorithm are spanning trees of the flow network.
For example,A basic feasible solution of the above problem is shown by a spanning tree below:



Figure(3.) A basic feasible solution

3.2.3 Optimality criterion

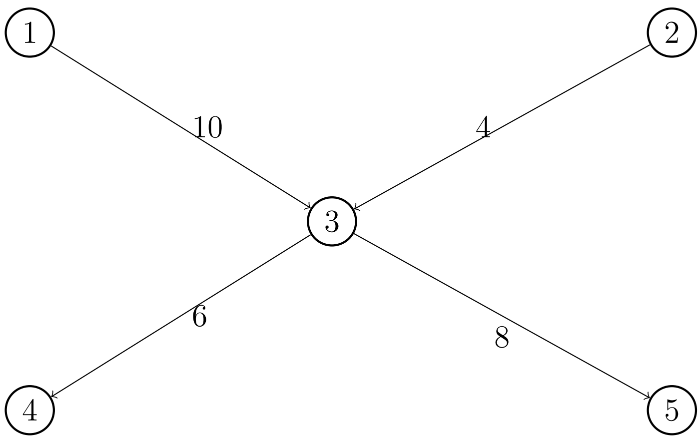
The dual variable λ are often called node potentials.Consider then a basic feasible solution, and partition A in to two sets B (the arcs of the spanning tree)and F (the other arcs of the network).If an arc is basic in the optimal solution,the corresponding dual constraint must be satisfied at equality,

$$i.e.\ \ \lambda_i - \lambda_j = c_{ij} \qquad \forall (i,j) \in B \tag{3.10}$$

Equation (3.10)are n-1, in n variables.one variable can be arbitrarily fixed to zero.we can easily determine a solution λ , and check dual feasibility,

$$i.e.\ \lambda_i - \lambda_j \leq c_{ij} \qquad \forall (i,j) \in F \tag{3.11}$$

Hence,if given a basic feasible solution x, the value λ obtained through (3.10)also respect the (3.11), x is optimal.
For instance,



Figure(4.) A basic feasible solution

For the feasible bases in Figure(3),and with the costs in Figure(1) , we have:

$\lambda_1 - \lambda_3 = c_{13} = 8$
$\lambda_2 - \lambda_3 = c_{23} = 2$
$\lambda_3 - \lambda_4 = c_{34} = 1$
$\lambda_3 - \lambda_5 = c_{35} = 4$

From this, arbitrarily letting
$\lambda_3 = 0, we have \ \ \lambda_1 = 8, \lambda_2 = -2, \lambda_4 = -1, \lambda_5 = -4.$
Putting these values in to (3.11) one gets:

$\lambda_1 - \lambda_2 = 10 \leq c_{12} = 10$
$\lambda_1 - \lambda_4 = 9 \nleq c_{14} = 1$
$\lambda_4 - \lambda_5 = 3 \leq c_{45} = 12$
$\lambda_5 - \lambda_2 = -2 \nleq c_{52} = -7$

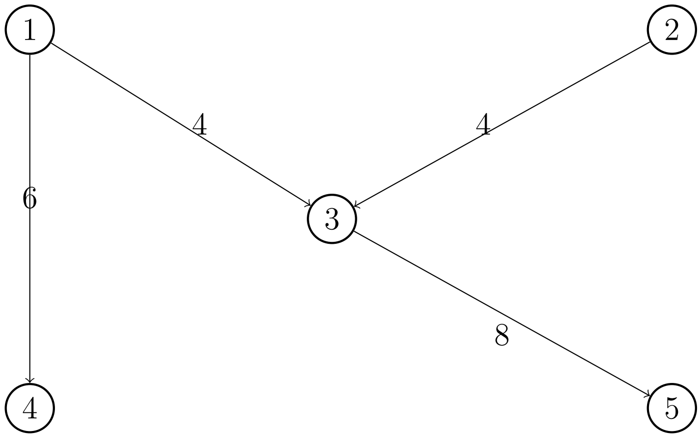
Hence, the first and the third constraints are satisfied, while the second and the fourth are violated.Therefore,the corresponding bases are not optimal.

3.2.4 Pivot operation

If a solution x does not verify the optimality conditions, by (3.11) there must exist an arc $(i,j) \in$ F(arcs not in the tree) such that $\lambda_i - \lambda_j > c_{ij}$ .in other words,the reduced cost of the variable x_{ij} is, $\overline{c_{ij}} = c_{ij} - \lambda_i + \lambda_j < 0$, and then it will be profitable to bring variable x_{ij} in to the base, i.e. activate the arc (i,j).The arc (i,j) forms a cycle with the arcs of B(the spanning tree). Let C be the set of arcs in the cycle.Since the new basis will still be a spanning tree, the arc that leave the basis must be an arc of C.
In fact, since $\overline{c_{ij}} < 0$, it is profitable to increase the flow on (i,j).This implies that,in order to maintain the feasibility of the current solution, one must necessarily alter the value of the flow on all the arcs of C, increasing the flow of the arcs that, in the cycle, have the same orientation as (i,j) and decreasing the flow of the arcs that have opposite orientation.
If there is at least one arc, say (h,k), having opposite orientation with respect to the cycle, and such that, as the flow on (i,j) is increased, its flow reaches zero before all the other arcs.Arc (h,k) leaves the basis. The maximum feasible value v of the flow in arc (i,j)is:
$v = min\{x_{um} : (u,m) \in C, (u,m) \textit{ is oppositely oriented to } (i,j)\}$.
The new basic feasible solution is therefore obtained simply increasing by v the flow of the arcs of C having the same orientation as (i,j), and decreasing by the same amount v the flow of the arcs of C having opposite orientation than (i,j).
Example,let us consider figure(4.),
A possible arc entering the basis $(i,j) = (1,4)$ which forms the cycle with the spanning tree $C = \{(1,4),(1,3),(3,4)\}$. Arc (1,3) and (3,4) have opposite orientation to (1,4). The flow in these arcs is, $x_{13} = 10 \textit{ and } x_{34} = 6$. Therefore v=6 , the arc leaving the basis is (3,4) and the new basic variables are :

$$\text{X} = \begin{pmatrix} x_{13} \\ x_{23} \\ x_{14} \\ x_{35} \end{pmatrix} = \begin{pmatrix} 10-6=4 \\ 4 \\ 0+6=6 \\ 8 \end{pmatrix}$$

The pivot operation is:



Figure(5.) Pivot operation

We can now proceed with the new iteration:
Computation of variables λ. From complementary slackness conditions one has:

$\lambda_1 - \lambda_3 = c_{13} = 8$
$\lambda_2 - \lambda_3 = c_{23} = 2$
$\lambda_1 - \lambda_4 = c_{14} = 1$
$\lambda_3 - \lambda_5 = c_{35} = 4$

Arbitrarily fixing $\lambda_1 = 0$ one has;
$\lambda_3 = -8, \lambda_2 = -6, \lambda_4 = -1, \lambda_5 = -12.$
Dual feasibility requires:

$\lambda_1 - \lambda_2 = 10 \leq c_{12} = 10$
$\lambda_3 - \lambda_4 = -7 \leq c_{34} = 1$
$\lambda_4 - \lambda_5 = 11 \leq c_{45} = 12$
$\lambda_5 - \lambda_2 = -6 \not\leq c_{52} = -7$

Note that the fourth condition is violated. Hence arc (5,2) enters the basis and creates the cycle $C = \{(5,2),(2,3),(3,5)\}$.
Since all the arcs of C have the same orientation as (5,2) , we conclude that the problem is unbounded, and there is no optimal solution.

3.2.5 Network simplex Algorithm

step1. Obtain an initial extended basic feasible solution (T),where T is a spanning tree of G.

step2. Use the fact that$c_{ij} = \lambda_i - \mu_j$ for all basic variables x_{ij} ,$(i,j) \in T$,and then one variable,say λ_1,can be fixed to 0,to find all dual variables for the current basic feasible solution.

step3. Compute the reduced costs $\overline{c_{ij}} = c_{ij} - \lambda_i + \mu_j$, for all (i,j) not in T .

step4. If $\overline{c_{ij}} \geq 0$ for all (i,j) not in T , STOP.We are at an optimal solution.Otherwise, choose an entering variable among those nonbasic variables violating these conditions.

step5. Using the unique cycle formed by adding the entering variable,identify the leaving variable and update the basic feasible solution(T).

step6. Using the new basic feasible solution ,return to step 2.

3.2.6 Numerical illustration

Example 1. *Consider the transportation problem in the table below, the total demand exceeds total supply. Suppose that the penalty costs per unit of unsatisfied demand are* 5,3 *and* 2 *for destinations A,B and C.*

Origins	Destinations			Supply
	A	B	C	
1	5	1	7	10
2	6	4	6	40
3	3	2	5	15
Demand	25	20	50	

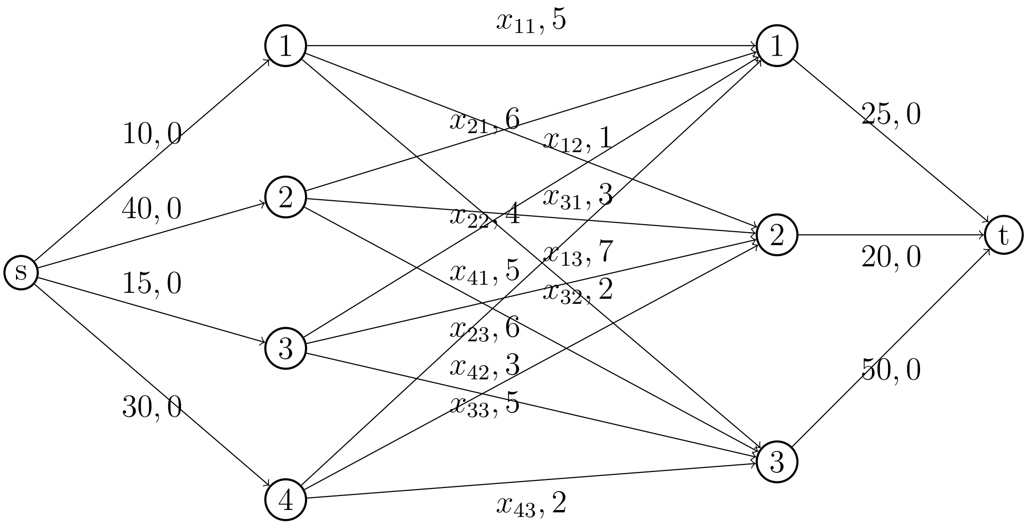
Find an optimal solution.

Solutions

The given problem is unbalanced,since total supply $<$ total demand.
which are total supply = 65 and total demand = 95
From this we have Dummy supply =95-65= 30
Then the balanced problem is as follows:

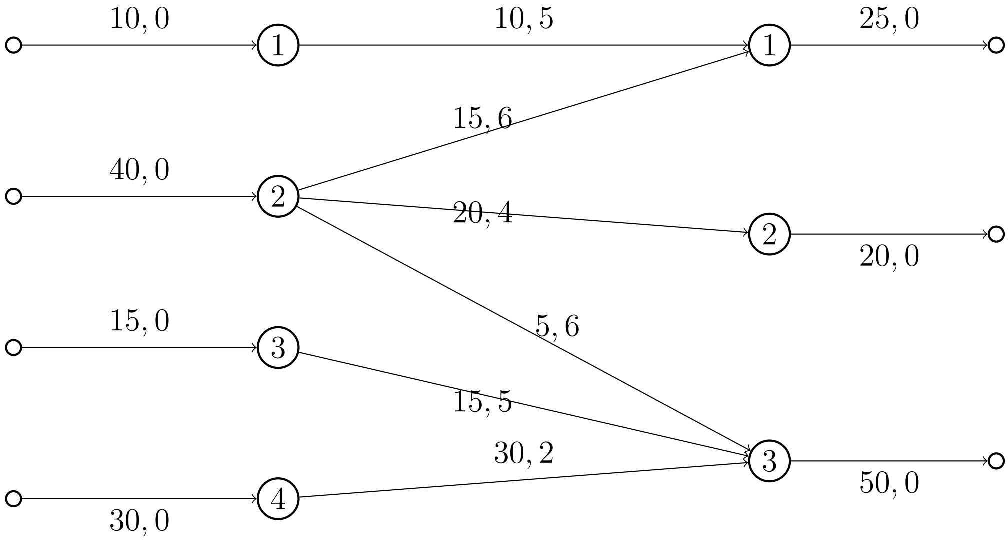
Origins	Destinations			Supply
	A	B	C	
1	5	1	7	10
2	6	4	6	40
3	3	2	5	15
Dummy origin	5	3	2	30
Demand	25	20	50	95

Now let us convert the above problem in to minimum cost flow problem.



•now let us solve the above problem by using NETWORK SIMPLEX ALGORITHM.

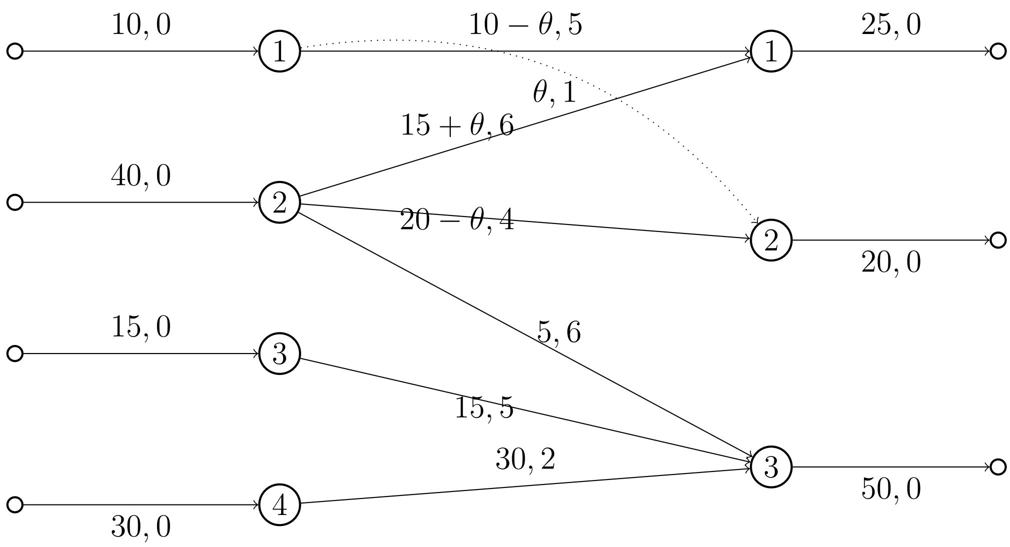
An initial BFS can be found by iteratively considering pairs $(i, j)$of supplier i and customer j,increasing x_{ij}until either the supply s_i or the demand d_j is satisfied, and moving to the next supplier as spanning tree below.The spanning tree below shows an initial basic feasible solution.



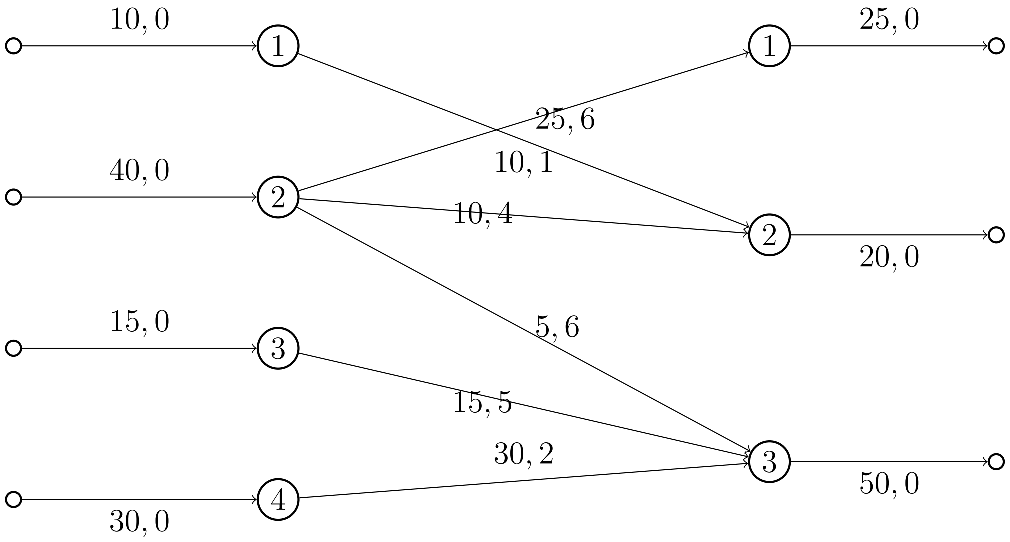
• To determine the values of the dual variables λ_i for $i = 1, 2, 3, 4$and μ_j,$j = 1, 2, 3$ by using $c_{ij} = \lambda_i - \mu_j$ for all $(i, j) \in T$.
we have$\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 0, \lambda_4 = -3$
$\mu_1 = -5, \mu_2 = -3, \mu_3 = -5$.

•For all (i, j) not in the tree(T) we have
$\overline{c_{12}} = -2, \overline{c_{13}} = 2, \overline{c_{31}} = -2, \overline{c_{32}} = -1$
$\overline{c_{41}} = 8, \overline{c_{42}} = 3$.

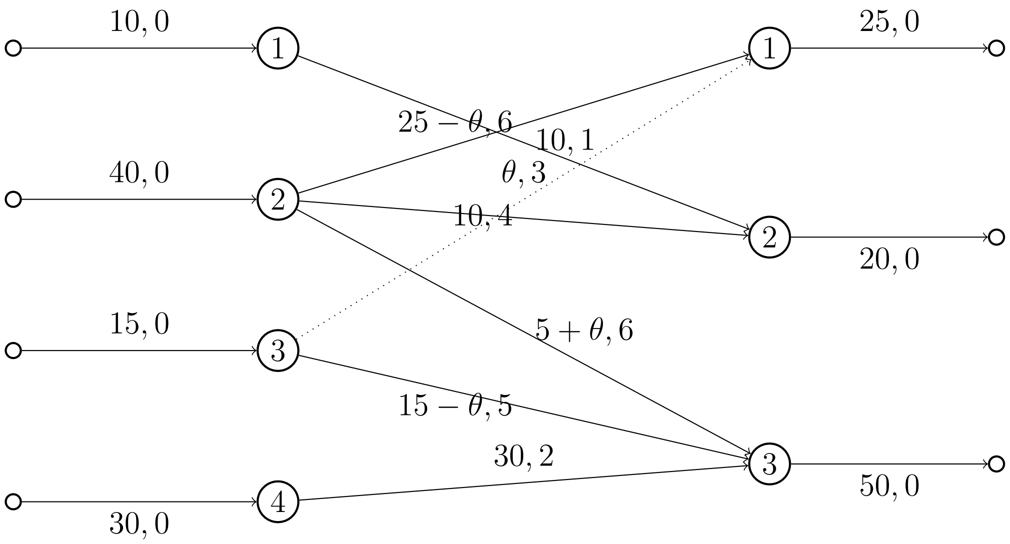
• But the optimality condition is violated by $\overline{c_{12}} = -2, \overline{c_{31}} = -2, \overline{c_{32}} = -1$. Then by choosing edge $(1, 2)$ forms a unique cycle with the spanning tree(T) as follows:



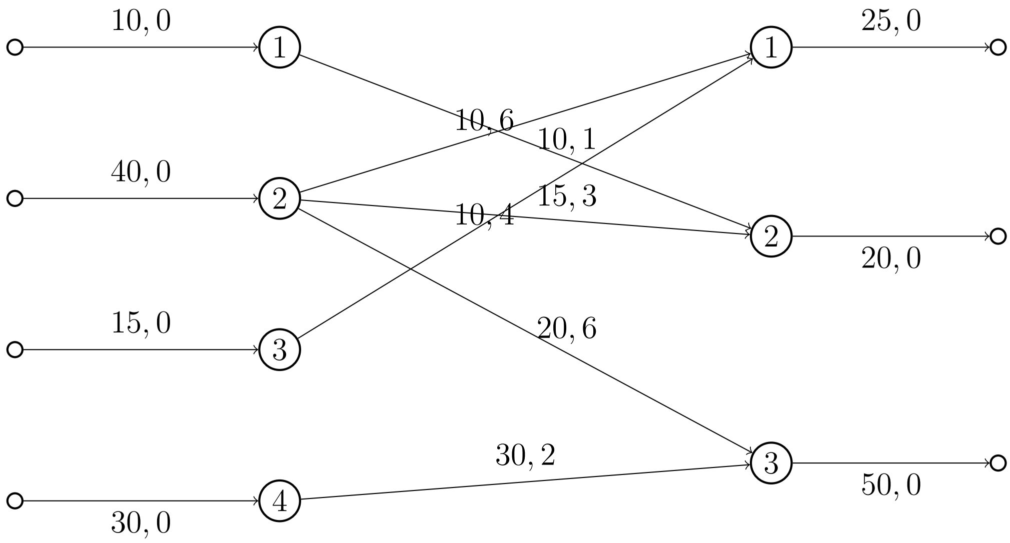
Due to the special structure of the network, we will alternately increase and decrease the flow for edges along the cycle. So increasing x_{12} , x_{21} and decreasing x_{11}, x_{22} by $\theta = 10$ amount we get the spanning tree below:



•Re-computing the values of the dual variables for all $(i, j) \in T$ we have
:
$\lambda_1 = 0, \lambda_2 = 3, \lambda_3 = 2, \lambda_4 = --1$
$\mu_1 = -3, \mu_2 = -1, \mu_3 = -3.$
and for all (i,j) not in T we get:
$\overline{c_{11}} = 2, \overline{c_{13}} = 4, \overline{c_{31}} = -2, \overline{c_{32}} = -1$
$\overline{c_{41}} = 3, \overline{c_{42}} = 3.$
•Again the optimality condition fails , since $\overline{c_{31}} = -2, \overline{c_{32}} = -1$. Then by choosing edge $(3, 1)$ forms a unique cycle with the spanning tree(T) as follows:



So increasing x_{31} , x_{23} and decreasing x_{21}, x_{33} by $\theta = 15$ amount we get the spanning tree below:



•Re-computing the values of the dual variables for all $(i, j) \in T$ we have :

$\lambda_1 = 0, \lambda_2 = 3, \lambda_3 = 0, \lambda_4 = -1$
$\mu_1 = -3, \mu_2 = -1, \mu_3 = -3.$
and for all (i,j) not in T we get:
$\overline{c_{11}} = 2, \overline{c_{13}} = 4, \overline{c_{32}} = 1, \overline{c_{33}} = 2$
$\overline{c_{41}} = 3, \overline{c_{42}} = 3.$
This shows optimality reached.

i.e. $x_{12} = 10,\ x_{21} = 10,\ x_{22} = 10,\ x_{23} = 20,\ x_{31} = 15,$ and $x_{43} = 30$(dummy) and the cost of penalty for unmet demand is equal to $2(c_{43} = 2)$, all others non basic variables are equal to zero.

$\therefore$ minimum cost=10(1) + 10(6) + 10(4) + 20(6) + 15(3) + 30(2) = 335.

Matlab implementation of the problem

```
>> f=[5 1 7 6 4 6 3 2 5 5 3 2];
Aeq=[1 1 1 0 0 0 0 0 0 0 0 0;
    0 0 0 1 1 1 0 0 0 0 0 0 ;
    0 0 0 0 0 0 1 1 1 0 0 0;
    0 0 0 0 0 0 0 0 0 1 1 1;
    -1 0 0 -1 0 0 -1 0 0 -1 0 0;
    0 -1 0 0 -1 0 0 -1 0 0 -1 0;
    0 0 -1 0 0 -1 0 0 -1 0 0 -1];
beq=[10;40;15;30;-25;-20;-50];
lb=zeros(1,12);
[x,fval,exitflag,output,lambda]=linprog(f,Aeq,beq,[],[],lb)
Optimization terminated.

x =

    0.0000
   10.0000
    0.0000
   10.0000
   10.0000
   20.0000
   15.0000
    0.0000
    0.0000
    0.0000
    0.0000
   30.0000


fval =

  335.0000


exitflag =

     1


output =

      iterations: 6
       algorithm: 'large-scale: interior point'
    cgiterations: 0
         message: 'Optimization terminated.'


lambda =

    ineqlin: [7x1 double]
      eqlin: [0x1 double]
      upper: [12x1 double]
      lower: [12x1 double]
```

Summary

In this project, we have discussed transportation problem with shortage of supply and penalties for deficiency.we have balanced unbalanced transportation problem and converted in to minimum cost flow network and solved by using the efficient algorithm which is the Network simplex method and got optimal solution for the transportation problem under shortage of supply and penalty for unmet demand.

Bibliography

[1] **S.M. Sinha** (*mathematical programming*)(*Theory and method*).

[2] **Roorkee M.E**(Elect.)1978 ; Delhi M.B.A 1983.

[3] **Hamdy A. Taha**, *Transportation model and its variants*,"*operational research* 8^{th} *edition*",**Prentice Hall** 2006.

[4] **R.Panneersel Vam** (*Operations Research*).

[5] **RAVINDRA K.AHUJA,THOMAS L.MAGNANTI** (*Theory,Algorithms and Applications*)