

LINEAR THREE-LEVEL PROGRAMMING PROBLEM WITH THE APPLICATION TO HIERARCHICAL ORGANIZATIONS



By
Esubalew Lakie

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Abstract

In many decision processes there is an hierarchy of decision-makers and decisions are taken at different levels in this hierarchy. The decentralized planning problem has long been recognized as an important decision-making problem. In many practical decision making activities, decision making structure has changed in the last few decades, from a single person (or decision maker) and single criterion (or constraint factor) to multi-person (or decision maker) and multi-criteria and even to hierarchical (or multi-level) situations. In any organization with hierarchical decision systems, the sequential and preemptive nature of the decision process makes the problem of making an optimal decision, and it is different from the usual operations research methods.

In hierarchical decision process decision-makers are often arranged within a hierarchical administrative structure, each with his/her objective (perhaps conflicting). A planner at one level of the hierarchy may have an objective and a set of feasible decision space determined, in part, by other levels. However, his control instruments may allow him/her to influence the policies at other levels and in this manner improve his/her own objective function. Therefore a multilevel programming problem approach is developed for modelling such type of decentralized planning problems.

A multilevel programming problem is a nested optimization problem over a single feasible region. This approach partitions control of the decision variables among several decision makers, in the hierarchy. Each decision-maker in the hierarchy acting in a sequence to maximize his/her own objective function. These decision-makers interact through a set of "corporate" constraints involving the decision variables of all divisions. The general structure of the multilevel programming problem method will be discussed in detail and we will focus on three level linear programming problems.

Dedication

This thesis is dedicated to
my mother *Wubit Aragaw* and
my uncle *Ashagrie Yideg*.

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Chapter 1

Introduction

Many resource planning problems require compromises among the objectives of several interacting individuals or agencies often, these groups are arranged within an administrative or hierarchical structure with independent and perhaps conflicting objectives. For example, the water resource policies set forth by the Federal government affect the objectives and options and hence the strategies of the state officials. This process continues within a hierarchy of decision-makers, including local governments, planning agencies, and basic economic units such as firms and households. Each unit of the hierarchy wishes to maximize its individual benefit function in view of the partial exogenous control exercised at other levels of the hierarchy. In this example, the actions at the state level also affect the benefits sought by the Federal government. The Federal government can control this effect by exercising preemptive partial control over the state through budget modification and regulations.

Planning such type of hierarchical organizations tends to be dominated by issues of central administrations and the coordination of lower level activities, and typically proceeds from the point of view of top management. A

mathematical programming the so called **multilevel programming** has often served as a basis for structuring the underlying goals and behavioral norm of such organizations.

An important feature of multilevel optimization problems is that a planner at one level of the hierarchy may have his/her objective function determined, in part, by variables controlled at other levels. However, his/her control instruments may allow him/her to influence the policies at other levels, and thereby improve his/her own objective function. Such policies may include the control of the allocation and use of resources at lower levels, and the control of the benefits conferred upon subordinate levels.

The following are some of the examples which can be modeled by multilevel optimization.

1. *Flood control:* Seeking to reduce flood risk, government can provide structural flood control measures, implement floodplain zoning program, and subsidize food insurance. These policies will influence floodplain development based on the individual objectives and benefits of land users. If not carefully conceived, the government programs can be rendered useless if they inadvertently encourage expanded floodplain development.
2. *International water system:* An international river basin agency desires to equitably distribute the benefits conferred from a multi-national, multi-reservoir, multi-purpose water system. To do so, the agency will control the operating policies of some, but perhaps not all, reservoirs in the system. In view of these controls, individual nations will determine their water use policies to best meet their individual requirements.

Note that the above problems have the following characteristics in common:

1. The systems have interacting decision-making units with in a predominantly hierarchical structure.
2. Each decision-making unit maximizes net benefits independently of other units, is affected by the action of other units as an external effect.
3. The external effect on decision-makers problem is reflected in both his/her objective function and his/her set of feasible decisions.

The design of large and complex systems involve decomposition of the system into a number of smaller subsystems, each with its own goals and constraints. One of the most common interconnections between such systems has a hierarchical structure. In hierarchical structures a decision maker at one level controls or coordinates the decision makers on the levels below it and in turn it is controlled by the decision maker on the level above it. Moreover, it is assumed that a decision maker at each level has a certain degree of autonomy, i.e., he/she has the freedom to choose the best options among the alternatives in his/her domain (or control area) and has a different objective (possibly not in harmony with objectives in the other levels). A system decomposed in this manner is referred to as *a decentralized multi-level system*.

In decentralized decision systems, one decision maker can influence the outcome and decisions of others, and thereby improve his/her own objective. Planning in such environment has long been recognized as an important decision making problem as described in Bialas and Karwan [9].

Multilevel decomposition methods lend themselves easily to an economic interpretation of the algorithmic process. The procedure is viewed as an adjustment phase with the supral planner sending tentative information to the lower level subunits, observing their reactions, and then updating the corporate information. This information can take the form of placing prices

on the scarce resources (i.e. price directive decomposition [13]), partitioning the resources among the subunits (i.e. resource directive decomposition, [17]) or integrating the two. That is, the multi-divisional organization is coordinated through pricing or partitioning of resources that are under the direct control of the headquarters or top management (called center). The center sends tentative information (on prices or quantities) to the divisions, each of which is asked to submit a separate optimal production plan based on this information and its individual objective. In successive stages of the algorithm, the tentative plans of the divisions are coordinated at the center in order to update and modify the centers information. An optimal plan for the center (and thus the firm) emerges when no further informational modification is required.

Dantzing and Wolfe type decomposition techniques merely partition the decision variables and the objective functions into small individual sets. The implicit assumption behind the algorithm is that all variables are under central control, and either that the center's objectives can be decomposed into objectives for divisions, or that it will be the same as the sum of the division objectives, i.e. the center decision-maker is willing to accept the aggregate objectives of the subunits as his/her own. Kornai and Liptak [19] recognized that some of the variables may not be under the direct control of the center. However, their algorithm also assumes that the objective at the center is the sum of divisional objectives. In most decentralized organizations, the objectives of the divisions, and especially the divisional decision-makers, could conflict with those at the center. Also, for a general hierarchical system, such an assumption is less restrictive than the ones implicit in the decomposition techniques. Thus a different formulation is needed for coordinating and optimizing hierarchical systems.

It is also known that multi-objective programming approaches seek to find a simultaneous compromise among the various goals of different divisions [35, 34]. Some techniques used in hierarchical systems include vector maximization [26]. However, such techniques also assume that all objectives are those of single decision-maker or a coherent group of decision-makers and can not fully account for the independent behavior of each division. In contrast, a game theoretic approach explicitly considers the individual decision-making units by assigning each a unique objective function and a control set. The Stackelberg game, an n-person non-cooperative game with leader follower strategy is conceptually extended to the multi-level programming problem, in which the players are required to move in turn and strategy sets are no longer assumed to be disjointed.

In order to overcome the short comings of the above approaches, multi-level programming (MLP) is developed to solve the decentralized planning problem. The general multi-level programming problem is a set of nested optimizations over a single feasible region. Control over the decision variables is partitioned among the levels, but a decision variable of one level may affect the objective functions of other levels. Thus, an important feature of multi-level programming problem is that a planner at one level of the hierarchy may have his/her objective function and decision space determined, in part, by variable controlled at other levels. However, his control instruments may allow him to influence the policies at other levels and thereby improve his own objective function.

There are many planning and/or decision making situations that can be properly represented by a multi-level programming model. All of them appear whenever a hierarchical structure is present and, in general, this model can be characterized as follows:

1. There exist interacting decision-making units within a predominantly hierarchical structure.
2. There are two or more decision-makers involved in the process with independent and some times conflicting goals, at the same or different levels of the hierarchy.
3. Each decision-maker can exercise direct control over only certain variables.
4. The execution of decision-making units is sequential from higher level to lower level. The lower level decision-maker executes its policies after and in view of, the decisions of the higher level. That is, for two adjacent levels in the decision tree, the decision-making process is carried out in two sequential stages; first, the higher level decision maker announces his/her plan of action; in the second: the lower level decision maker reacts rationally to the plan put forth by the higher level decision maker.
5. Each decision-making unit optimizes its own objective function independent of other units, but it is affected by the actions of other levels. In other words the plan announced by the higher level decision-maker is taken as exogenous data, and the lower decision maker independently optimizes his/her plan of action according to his/her goals and limitations, disregarding the goals of the higher level decision-maker.
6. We assume that there is no uncertainty involved in the decision making process. In particular, decision-maker at higher level knows the objective function and the constraints of the lower level problem and the lower level decision maker knows the decisions of the higher level

decision makers.

7. The external effect on the decision-maker's problem is reflected in both his/her objective function and his set of feasible decisions.

In general, multilevel programming consist of decision makers in the hierarchy who make decisions in a structured, "leader-follower" ordering. By convention, if the hierarchy has n decision makers, decision maker one (the decision maker at the top of the hierarchy), goes first. He/she makes his/her decisions based on his/her objective function and the rational reactions of the decision makers who make their decisions after him/her. After the first decision maker has made his/her decisions, decision maker two makes his/her decisions based on his/her objective function, the decisions made by the first decision maker and the rational reactions of the decision makers who make their decisions after him/her. This process continues down the hierarchy until the final decision maker(decision maker n) makes his/her decisions based on his/her objective function and decisions made by the $n - 1$ decision makers above him/her.

1.1 Mathematical Models of Hierarchical Decision Making Organizations

One of the prevalent decision-making structures throughout history has been the hierarchy. Hierarchies in managements have existed for as long as people have tried to organize their effort work. In today's society hierarchies exist in nearly every facet of life.They occur in the government, the business world, our church, and even in our family unit.

In the hierarchy decision makers' position and individual problems are well-defined. This is not to imply that the solution obtained from the hierarchy is optimal. In fact, the decisions made from organizations in the hierarchical form often seem to make no sense. Therefore, by understanding them, we can eliminate their inherent inefficiencies and better utilize the resources which they expend.

With few exceptions, existing mathematical models for optimizing hierarchical systems have relied heavily on the Danzig-Wolfe Decomposition principle [14]. In such formulations, the decision space is partitioned among subunits of the decomposable system. The subunits interact through a set of "corporate" constraints involving decision variables of all subunits. The remaining constraints can be to each division, with each constraint a function of the decision variables under the control of a single subunit.

These multilevel decomposition models lend themselves readily to an economic interpretation of the algorithmic process. The procedure is viewed as an adjustment phase with the master planner (higher level decision maker) sending tentative information to the lower level decision makers, observing their reactions, and updating his decision based on this information. Ultimately, the master planner establishes a policy which causes the system as a whole to operate feasibly and optimally with respect to his/her objective function.

1.2 Statement of the problem

We know that a government has several levels. In any level of the government one of the most crucial decisions which must be made concerns the allocation of resources to different sectors of the economy and several divisions

of the government in a given country. Consider the resource allocation to universities in Ethiopia by the federal government. The federal government of Ethiopia allocates a certain amount of budget to universities in the country, and decides on the policy to be accomplished by them. Each University after receiving the budget in turn allocates (or decides on the amount of) budget to its faculties (of course part of the budget could be used by the central administration of the University itself) and pass decisions on certain resources that the faculties should consider. The third level of budget and decision units, the faculties decide on what to do and how to effectively use the budget. It is obvious that every reaction of the lower level unit affects the value of the upper level and that of the middle level also affects both upper and lower levels. This thesis takes into account the fact that decisions about resource usage at the sub-levels cannot be controlled (although they may be predictable) once resources have been allocated.

The objective of this thesis is to show the federal government how to allocate its budget to the universities and in turn the university how to allocate its budget received from the federal government to the faculties efficiently and optimally. Here we take into account, the fact that decision about the usage of the budget at the sublevels can not be controlled (but it may be predictable) by the higher level, if the budget once allocated. Thus, one of the aims of our model is to provide rationality in decision making for the higher level decisions by the lower level decision maker. This is done by sub-models which predict how sublevels will react when they are given various amount of resources.

As we can see it is a hierarchical decision making process containing three levels and one (or a group of) decision maker at each level. Thus it is modeled by three-level linear programming problem. Mathematical programming models

to solve problems of the above type has been studied since 1960s, (Dantzig and Wolfe 1960 [14]).

1.3 Historical Development

Decentralized planning has been long recognized as an important decision making problem. Mathematical programming methods to solve such problems trace back early in the development of linear programming. The decomposition method of Dantzig and Wolfe [14] for the solution of certain large-scale linear programming problems has served well as the underpinning for much of this study. Such a formulation partitions the decision space among several planning division. The subproblem solved by a division maximizes that portion of overall objective function controlled by the division, subject to the divisional constraints. The Dantzig and Wolfe method can then be viewed as providing inducements to the division to encourage overall optimal behavior of the corporation.

These techniques were further discussed by the work of Charnes, et al [13], who recognized that when subdivision have alternative solution for their individual optimization problem, they must receive information from master planner in order to operate coherently. The decomposition approach has been successfully applied by Haimes and he associates to a wider range of multilevel planning problem (Haimes, Foley and Yu [16]). The decomposing approach includes coordinating mechanism of dual price preventing the various division or agencies from working against the goal of master planner. Cassidy, et al [12] proposed a model and solution procedure for a specific case where such a coordinating mechanism doesn't exist.

Many solution approaches have been developed for the case of linear bi-level

programming problems. Candler and Townsly [11] proposed an algorithm for bi-level optimization, known as the T-set algorithm, that focuses on generating and enumerating bases from the lower level activities. The solution method involves an implicit search of all feasible behavioral optimal bases, with out re-examining any previously explored basis. But the algorithm may not stop as soon as the goal optimal is attained; it is one of the limitation of this algorithm. Narula and Nwasu [22, 23] also proposed a procedure via regular simplex pivots with modification after taking the dual of the lower level problem for two-level hierarchical programming problems.

An algorithm proposed by Bialas and Karwan [10] for bi-level programming problems uses simplex method for bounded variables and finds extreme points in the set of rational reactions of the lower level problem; it then move among the extreme points of the lower level problem, never allowing the upper level objective function to decrease. However only the local optimal solution is obtained. The intercepting algorithm by Parragar [25] suggests adding a cut to the original feasible region after a local optimal solution has been found. The k^{th} -best algorithm has been proposed by Wen [30] and Bialas and Karwin [10]. First, it solves the upper level problem over the overall solution space in order to get the first best solution. If the solution is not in the set of rational reaction of the lower level problem, then the second best solution may be found among the extreme points which are adjacent to the first best. The algorithm moves sequentially through these ordered extreme points of the overall solution space until one, the k^{th} -best is found in the rational reaction set of the lower level problems, and then terminates with a global optimal solution. Computational experience with k^{th} -best algorithm has demonstrated that it finds a global solution for most linear bi-level programming problems, although long time may be needed before the solution

is found, it is one of the weakness of this algorithm. Even if there are some weakness, this approach to the bi-level linear programming can be intuitively extended to the general n-level linear resource control problem.

Bard [3] proposed an algorithm for solving the general bi-level programming problem. The algorithm is based on the grid search algorithm which exhibits the desirable property of monotonicity and the algorithm is based on two necessary optimality conditions on the paper for stationarity and local optimality. Visweswaran, Floudas, Ierapetritou and Pistikopoulos [28] offered a decomposition based optimization approach to bi-level linear and quadratic programming problems. By replacing the inner problem by its corresponding KKT optimality conditions, the problem is transformed to a single non-convex (due to the complementarity condition) mathematical programming problem. Based on the primal-dual global optimization approach of Floudas and Visweswaran [15], the problem is decomposed into a series of primal and relaxed-dual subproblems whose solutions provide lower and upper bounds to the global optimal solution.

There are a few varying approaches yet known in solving three level programming problem. One of the approaches is a hybrid method offered by Wen [29] which combines the “ k^{th} -best” vertex enumeration algorithm proposed by Bialas and Karwan [10], and a complementary pivot algorithm. Though this method works satisfactorily for most problems, its computational load grows geometrically with the number of constraints (i.e. the size of the coefficient matrices) and the hierarchies (i.e. the number of levels in hierarchy). The second approach is offered by Bard [4]. Bard extended the idea of the grid search algorithm that is designed to solve two level hierarchy to a model of three level hierarchy. The algorithm that Bard proposed for solving the three-level programming problem includes a cutting plane approach for solving a

bi-linear programming problem and a vertex search procedure for the third level, at each iteration. One of the main advantages of this algorithm is that, it can be extended beyond three level hierarchies and can be used for general multilevel linear programming problems. Its principal limitation seems to be the bookkeeping burden imposed by the prospect of multiple optimal solutions.

With this respect until now several algorithms have been developed that can find an optimal solution for the linear bi-level programming problem and some algorithms that solves non-linear programming problem. However the computational efficiency of these algorithm does not consistently perform well, owing to the complexity of the problem. And the algorithms proposed for solving three level programming problem have some limitations as we have seen above. It will be helpful to develop more efficient algorithm for solving the linear bi-level programming problem and extend it as well to the general n -level programming problem, In addition, there exist several areas involving multi-level programming, further study of which will be interesting.

Chapter 2

General Definition of Multi-level Programming Problem

Consider an organization composed of p levels, each characterized by individual objective functions f^i for $i = 1, 2, \dots, p$, defined over a jointly dependent strategy set S , which are to be maximized by the respective planners. Assume that decisions are made sequentially beginning with planner 1 who has control over a vector $x^1 \in \mathbf{X}^1$, followed by planner 2 who has control over a vector $x^2 \in \mathbf{X}^2$ down through planner p who has control over a vector $x^p \in \mathbf{X}^p$, where \mathbf{X}^i is nonempty subsets of \mathbb{R}^{n^i} , $i = 1, 2, \dots, p$, $n = n^1 + n^2 + \dots + n^p$ and $x = (x^1, x^2, \dots, x^p) \in \mathbf{X}^n$.

Further assume that f^i is defined from $\mathbf{X}^1 \times \mathbf{X}^2 \times \dots \times \mathbf{X}^p$ to \mathbb{R} , for all $i = 1, 2, \dots, p$ and S is a compact subset of \mathbb{R}^n with $x \in S$.

The following nested optimization problem, known as the multi-level programming problem (MLPP) captures this structure

$$\begin{aligned}
& \max_{x^1 \in \mathbf{X}^1} f^1(x), \text{ where } x^2 \text{ solves} \\
& \max_{x^2 \in \mathbf{X}^2} f^2(x), \text{ where } x^3 \text{ solves} \\
& \max_{x^3 \in \mathbf{X}^3} f^3(x), \text{ where } x^4 \text{ solves} \tag{2.0.1} \\
& \vdots \\
& \max_{x^p \in \mathbf{X}^p} f^p(x) \\
& \text{Subject to } x \in \mathbf{S}
\end{aligned}$$

Now if $p = 1$ and f^p is nonlinear, problem (2.0.1) reduces to standard non-linear program. If each f^i , $i = 1, 2, \dots, p$ is linear and \mathbf{S} has the form $S = \{x \in \mathbf{R}^n : Ax \leq b\}$ that is, \mathbf{S} is a polyhedral set where A is an $m \times n$ matrix and b is m dimensional constant column vector, then problem (2.0.1) is known as linear multi-level programming problem.

When $p=2$ then problem (2.0.1) is known as **bi-level programming problem**. Most of the work todate in multi-level programming has concentrated on algorithmic development and some optimality conditions for linear bi-level programming problems [20] although parallel efforts have produced a partial characterization of optimality conditions for a more general case. When $p=3$ problem (2.0.1) is known as **three level programming problem**.

In the next chapter we will describe basic properties of three-level linear programming problem and introduce some concepts on three-level linear programming problems.

2.1 The p -level Linear Resource Control Problem

Let the vector of decision variable $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ be partitioned among p decision makers and let $x^k = (x_1^k, x_2^k, \dots, x_{n_k}^k) \in \mathbb{R}^{n_k}$ for $k = 1, 2, \dots, p$ where $\sum_{k=1}^p n_k = n$. Let

$$\max \{f(x) : (x^k \mid x^1, x^2, \dots, x^{k-1})\}$$

denote the maximization of a function $f(x)$ over a compact region $S \subseteq \mathbb{R}^n$ by varying only $x^k \in \mathbb{R}^{n_k}$, given fixed $(x^1, x^2, \dots, x^{k-1}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \dots \times \mathbb{R}^{n_{k-1}}$. Note that x^{k+i} is a function of $x^1, x^2, \dots, x^{k+i-1}$ for $i = 1, 2, \dots, p-k$.

Definition 2.1.1 *The set $W_f(S)$ given by $W_f(S) = \left\{ \hat{x} \in S \mid f(\hat{x}) = \max \{f(x) : (x^k \mid x^1, x^2, \dots, x^{k-1})\} \right\}$ is known as the set of rational reactions of f over S .*

Let $f_1(x), f_2(x), \dots, f_p(x)$ be bounded functions defined over S and $S^p = S$ be a compact set.

The problem at the lower level of the hierarchy, level p , is defined as :

$$(P^p) : \max \{f_p(x) : (x^p \mid x^1, x^2, \dots, x^{p-1})\} \\ \text{s.t. } x \in S^p.$$

This represents the maximization of the level p objective function over S^p , parametrically given the decision variables x^1, x^2, \dots, x^{p-1} of decision makers at levels $1, 2, \dots, p-1$ respectively. The set, S^p , is defined to be the level- p feasible region.

The rational reaction set

$$S^{p-1} = W_{f_p}(S^p)$$

is defined to be the level $p - 1$ feasible region. Then the problem at level $p - 1$ is given by

$$(P^{p-1}) : \max\{f_{p-1}(x) : (x^{p-1} | x^1, x^2, \dots, x^{p-2})\}$$

$$s.t. x \in S^{p-1}.$$

In general, the level k feasible region is defined as

$$S^k = W_{f^{k+1}}(S^{k+1})$$

and the optimization problem at level k can be written as

$$(P^k) : \max\{f_k(x) : (x^k | x^1, x^2, \dots, x^{k-1})\}$$

$$s.t. x \in S^k.$$

Now the p -level linear resource control problem is the multilevel programming problem where

$$f^k = c^k x, \quad (k = 1, 2, \dots, p) \text{ and } S = \{x \in \mathbb{R}^n | Ax \leq b, x \geq 0\}.$$

in the above formulation, where A is an $n \times m$ matrix and b is m dimensional constant column vector.

This problem may be more clearly represented as

$$\begin{aligned}
 \max_{x^1} f^1(x) &= c^1 x \\
 &\text{where } x^2 \text{ solves} \\
 \max_{x^2} f^2(x) &= c^2 x \\
 &\text{where } x^3 \text{ solves} \\
 \max_{x^3} f^3(x) &= c^3 x \\
 &\text{where } x^4 \text{ solves} \\
 &\dots \\
 \max_{x^p} f^p(x) &= c^p x \\
 &\text{Subject to } x \in \mathbf{S}
 \end{aligned} \tag{2.1.1}$$

Note that the objective function at each level k , $f^k(x)$, is defined over the decision space of all levels. Thus the level- k decision maker may have his/her objective function determined, in part, by variables controlled at other levels. However, by controlling x^k , after decisions at levels 1 to $k-1$ have been made, level k may influence the decisions made at level $k+1$ and all lower levels to improve his own objective function.

Chapter 3

Linear Three Level Programming Problem

Three level linear programming problems are mathematical optimization problems where the set of all variables are partitioned among three vectors x^1, x^2 and x^3 ; and x^3 to be chosen as an optimal solution of the third (lower level) mathematical programming problem parameterized in x^1 and x^2 . After finding the optimal value x^3 of the lower level mathematical programming problem which is parameterized in x^1 and x^2 . Then x^2 is to be chosen as an optimal solution of the second mathematical programming problem parameterized in x^1 using an optimal solution x^3 of the lower level. Finally x^1 is chosen as an optimal solution of the first (higher level) mathematical programming problem using the optimal solutions x^2 and x^3 of the second and the third level problems as given values.

Let the lower level problem is introduced as follows:

$$\max_{x^3} \{f^3(x^1, x^2, x^3) : g^3(x^1, x^2, x^3) \leq 0, h^3(x^1, x^2, x^3) = 0\} \quad (3.0.1)$$

where

$$\begin{aligned} f^3 &: \mathbb{R}^{n^1} \times \mathbb{R}^{n^2} \times \mathbb{R}^{n^3} \longrightarrow \mathbb{R} \\ g^3 &: \mathbb{R}^{n^1} \times \mathbb{R}^{n^2} \times \mathbb{R}^{n^3} \longrightarrow \mathbb{R}^q \\ h^3 &: \mathbb{R}^{n^1} \times \mathbb{R}^{n^2} \times \mathbb{R}^{n^3} \longrightarrow \mathbb{R}^r, \end{aligned}$$

$$\begin{aligned} g^3(x^1, x^2, x^3) &= (g^{31}(x^1, x^2, x^3), g^{32}(x^1, x^2, x^3), \dots, g^{3q}(x^1, x^2, x^3)) \\ h^3(x^1, x^2, x^3) &= (h^{31}(x^1, x^2, x^3), h^{32}(x^1, x^2, x^3), \dots, h^{3r}(x^1, x^2, x^3)) \end{aligned}$$

Let $\Psi(x^1, x^2)$ denote the solution set of problem (3.0.1) for fixed $(x^1, x^2) \in \mathbb{R}^{n^1} \times \mathbb{R}^{n^2}$. Then Ψ is a point-to-set mapping from $(x^1, x^2) \in \mathbb{R}^{n^1} \times \mathbb{R}^{n^2}$ into a power set of \mathbb{R}^{n^3} denoted by $\Psi : \mathbb{R}^{n^1} \times \mathbb{R}^{n^2} \longrightarrow \mathbb{R}^{n^3}$.

Denote some elements of $\Psi(x^1, x^2)$ by $x^3(x^1, x^2)$ and assume for the moment that this choice is unique for all possible values of x^1 and x^2 .

Then the second level problem is formulated as follows:

$$\max_{x^2} \{f^2(x^1, x^2, x^3) : g^2(x^1, x^2, x^3) \leq 0, h^2(x^1, x^2, x^3) = 0, x^3 \in \Psi(x^1, x^2)\} \quad (3.0.2)$$

where

$$\begin{aligned} f^2 &: \mathbb{R}^{n^1} \times \mathbb{R}^{n^2} \times \mathbb{R}^{n^3} \longrightarrow \mathbb{R} \\ g^2 &: \mathbb{R}^{n^1} \times \mathbb{R}^{n^2} \times \mathbb{R}^{n^3} \longrightarrow \mathbb{R}^k \\ h^2 &: \mathbb{R}^{n^1} \times \mathbb{R}^{n^2} \times \mathbb{R}^{n^3} \longrightarrow \mathbb{R}^l, \end{aligned}$$

$$\begin{aligned} g^2(x^1, x^2, x^3) &= (g^{21}(x^1, x^2, x^3), g^{22}(x^1, x^2, x^3), \dots, g^{2k}(x^1, x^2, x^3)) \\ h^2(x^1, x^2, x^3) &= (h^{21}(x^1, x^2, x^3), h^{22}(x^1, x^2, x^3), \dots, h^{2l}(x^1, x^2, x^3)) \end{aligned}$$

Let $\varphi(x^1)$ denote the solution set of problem (3.0.2) for fixed $(x^1) \in \mathbb{R}^{n^1}$.

Then φ is also a point-to-set mapping from \mathbb{R}^{n^1} into a power set of \mathbb{R}^{n^3} .

Denote some elements of $\varphi(x^1)$ by $x^2(x^1)$ and now again assume that this choice is unique for all possible values of x^1 . Then the aim of the three level programming problem is to select the parameter vector x^1 such that this selection of x^1 is conducted so that certain equality and (or) inequality

constraints

$$\begin{aligned} g^1(x^1, x^2(x^1), x^3(x^1, x^2(x^1))) &\leq 0, \\ h^1(x^1, x^2(x^1), x^3(x^1, x^2(x^1))) &= 0 \end{aligned} \quad (3.0.3)$$

are satisfied and an objective function $f^1(x^1, x^2(x^1), x^3(x^1, x^2(x^1)))$ is minimized.

where

$$\begin{aligned} f^1 : \mathbb{R}^{n^1} \times \mathbb{R}^{n^2} \times \mathbb{R}^{n^3} &\longrightarrow \mathbb{R} \\ g^1 : \mathbb{R}^{n^1} \times \mathbb{R}^{n^2} \times \mathbb{R}^{n^3} &\longrightarrow \mathbb{R}^s \\ h^1 : \mathbb{R}^{n^1} \times \mathbb{R}^{n^2} \times \mathbb{R}^{n^3} &\longrightarrow \mathbb{R}^t \end{aligned}$$

The problem of determining a best solution \hat{x}^1 can thus be described as that of finding a best vector \hat{x}^1 of parameters for the parametric optimization problems (3.0.1) and (3.0.2) which together with the responses $x^2(\hat{x}^1) \in \varphi(\hat{x}^1)$ and $x^3(\hat{x}^1, x^2(\hat{x}^1)) \in \Psi(\hat{x}^1, x^2(\hat{x}^1))$. That is

$$\begin{aligned} \text{“max”}_{x^1} & f^1(x^1, x^2(x^1), x^3(x^1, x^2(x^1))) \\ \text{s.t.} & \quad g^1(x^1, x^2(x^1), x^3(x^1, x^2(x^1))) \leq 0 \\ & \quad h^1(x^1, x^2(x^1), x^3(x^1, x^2(x^1))) = 0 \\ & \quad x^2(\hat{x}^1) \in \varphi(\hat{x}^1) \\ & \quad x^3(\hat{x}^1, x^2(\hat{x}^1)) \in \Psi(\hat{x}^1, x^2(\hat{x}^1)) \end{aligned} \quad (3.0.4)$$

This problem is the three-level programming problem. The function f^1 is called the upper (higher) level objective function and the functions g^1 and h^1 are called the upper level constraint functions. Strongly speaking, this definition of three-level programming problem is valid only in the case when the lower level solutions are uniquely determined for each possible value of x^1 . The quotation marks have been used to express this uncertainty in the definition of the three-level programming problem. If the lower level problem has at most one (global) optimal solution for all values of the parameter, the quotation marks can be dropped and the familiar notation of three-level optimization problem arises.

Simply, the following nested optimization problem, known as the three-level programming problem which represents the above problem:

$$(p^1) \left\{ \begin{array}{l} \max_{x^1 \in \mathbf{X}^1} f^1(x^1, x^2, x^3), \\ \text{where } x^2 \text{ solves} \\ (p^2) \left\{ \begin{array}{l} \max_{x^2 \in \mathbf{X}^2} f^2(x^1, x^2, x^3), \\ \text{where } x^3 \text{ solves} \\ (p^3) \left\{ \begin{array}{l} \max_{x^3 \in \mathbf{X}^3} f^3(x^1, x^2, x^3) \\ \text{Subject to } x \in S \end{array} \right. \end{array} \right. \end{array} \right. \quad (3.0.5)$$

Where S is the over all constraint set, and $\mathbf{X}^1 \subseteq \mathbb{R}^{n^1}$, $\mathbf{X}^2 \subseteq \mathbb{R}^{n^2}$ and $\mathbf{X}^3 \subseteq \mathbb{R}^{n^3}$.

3.1 Existence of multiple optimal solution

For any three-level programming problem, care must be taken when the solution to (p^3) is not unique for x^1 fixed at \hat{x}^1 and x^2 fixed at \hat{x}^2 . Although not affecting the value of the level-three objective function, $f^3(x)$, these solutions can have a greatly varying impact on the objective function of the level two decision maker. Therefore, the control over the choice among multiple optima at level three may have to be delegated to level two or outside referee.

Bialas and Karwan [9] proposed an incentive scheme to overcome the problem of multiple optimal solution for the lower level problem to bi-level programming problem. We use this method to three level programming problem. This method perturbs the level-three objective function, replacing the the original objective function by $f^{3*}(x) = f^3(x) + \epsilon f^2(x)$ for x^1 fixed at \hat{x}^1 and

x^2 fixed at \hat{x}^2 , where the value of $\epsilon > 0$ is suitably small. This would require that level two “kick-back” a small portion of its earning to encourage level three to choose a desirable solution.

Similarly, if there are multiple optimal solution to the second level objective function, $f^2(x)$, (the second decision maker has multiple response) for a given x^1 fixed at \hat{x}^1 , then we use the above formulation to reduce the existence of multiple optimal solution. That is we replace the original objective function $f^2(x)$ by $f^{2*}(x) = f^2(x) + \epsilon f^1(x)$ for x^1 fixed at \hat{x}^1 where the value of $\epsilon > 0$ is suitably small.

Note that, in general, such a perturbation method may still not determine a unique solution since level one may have the same objective function value for a number of level two optimal solution and level two may have the same objective function value for a number of level three optimal solution.

3.2 Notation and definitions

The multi-level programming problem as viewed by planner k can be thought of as a standard mathematical programming whose feasible region has been augmented to include a series of implicitly defined constraints. We will now use the notation proposed by Bialas and Karwan [9]. This will ultimately lead to a reformulation of problem (3.0.5).

To begin, let (x^1, x^2) be an arbitrary partitioning of $x \in \mathbb{R}^n$ and denote the maximization of a bounded function $f(x^1, x^2)$ over a compact region $S \subseteq \mathbb{R}^n$ for a fixed value x^1 by

$$\max\{ f(x^1, x^2) : (x^2 \mid x^1) \} \tag{3.2.1}$$

That is (3.2.1) represents maximization of $f(x^1, x^2)$ for fixed value of x^1 with respect to x^2 such that $(x^1, x^2) \in S$.

In the case of bi-level programming if we think of x^1 as the vector under the control of planner 1 then planner 2 is faced with parameterized problem given in (3.2.1).

Definition 3.2.1 *The set $W_f(S)$ given by $W_f(S) = \left\{ (\hat{x}^1, \hat{x}^2) \in S : f(\hat{x}^1, \hat{x}^2) = \max\{f(\hat{x}^1, x^2) : (x^2 | \hat{x}^1), (x^1, x^2) \in S\} \right\}$ is known as the set of rational reactions of f over S .*

Note that the maximization in the above definition is only taken over x^2 for each fixed \hat{x}^1 . If planner 1 selects \hat{x}^1 and there exists an \hat{x}^2 which uniquely maximizes $f(\hat{x}^1, x^2)$ for all $(\hat{x}^1, x^2) \in S$, then there is an induced mapping ϕ which maps each \hat{x}^1 into \hat{x}^2 , where $(\hat{x}^1, \hat{x}^2) \in S$. Then the induced mapping $\hat{x}^2 = \phi(\hat{x}^1)$ provides the rational reaction of planner 2 for each \hat{x}^1 . This suggests an alternative representation of $W_f(s)$ which can be expressed as $W_f(s) = \{x = (x^1, x^2) \in S : x^2 = \phi(x^1)\}$. Here $W_f(S)$ is also called the higher level feasible region over which upper level objective function is maximized by varying x^1 .

Then the bi-level programming problem can be written as $\max_{x^1} \{f^1(x) : x \in W_{f^2}(S)\}$ where f^2 is lower level objective function, f^1 is upper level objective function and S is a constraint set.

Example 3.2.1 *Let x^1 and x^2 be single component vectors. Suppose $S = \{x \in \mathbb{R}^2 : A^1 x^1 + A^2 x^2 \leq b\}$ is a polyhedron shown in Figure 3.1.*

Let $f^2(x) = cx$. Then, for any fixed feasible choice of \bar{x}^1 , level two solves the

following linear programming problem

$$\begin{aligned} \max_{x^2} cx &= c^1 \bar{x}^1 + c^2 x^2 \\ \text{s.t. } A^2 x^2 &\leq b - A^1 \bar{x}^1 \end{aligned} \quad (3.2.2)$$

The solution \bar{x}^2 to (3.2.2) together with \bar{x}^1 , results a point (\bar{x}^1, \bar{x}^2) which is an element of level one feasible region $W_{f^2}(S)$, the hatched region in Figure 3.1.

If level one decision maker wishes to maximize his/her objective function, $f^1(x^1, x^2)$ by controlling the vector x^1 , he/she must solve the following mathematical programming problem

$$\begin{aligned} \max_x f^1(x^1, x^2) \\ \text{s.t. } (x^1, x^2) \in W_{f^2}(S). \end{aligned} \quad (3.2.3)$$

Or equivalently

$$\max_x f^1(x^1, \phi(x^1)) \quad \text{s.t. } (x^1, \phi(x^1)) \in W_{f^2}(S)$$

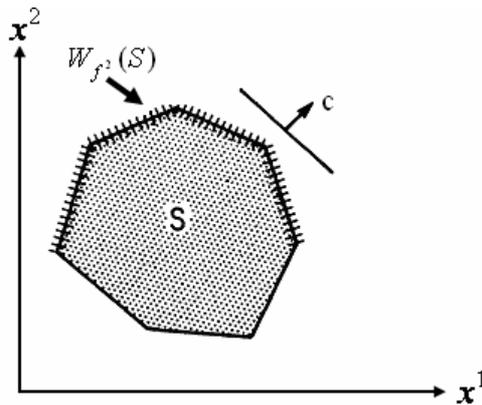


Figure 3.1: Example of a Rational Reaction Set for a Bilevel problem

The problem given by (3.2.3) is called a two level programming problem and

may be more explicitly written as

$$\begin{aligned} & \max_{x^1} f^1(x^1, x^2), \text{ where } x^2 \text{ solves} \\ & \quad \max_{x^2} f^2(x^1, x^2), \\ & \quad \text{s.t. } (x^1, x^2) \in \mathbf{S} \end{aligned}$$

Now we extend the above development to three level programming problem. Let the decision variable $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ be partitioned among 3 levels and let $x^k = (x_1^k, x_2^k, \dots, x_{n_k}^k)$ for $k = 1, 2, 3$ where $\sum_{k=1}^3 n_k = n$. Denote the maximization of a function $f(x)$ over a compact region $S \subseteq \mathbb{R}^n$ by varying only $x^k \in \mathbb{R}^{n_k}$, for fixed $(x^1, x^2, \dots, x^{k-1}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \dots \times \mathbb{R}^{n_{k-1}}$ by $\max \{f(x) : (x^k \mid x^1, x^2, \dots, x^{k-1})\}$.

Note that x^{k+i} is a function of $x^1, x^2, \dots, x^{k+i-1}$ for $i = 1, 2$.

Now we will depict this formulation to three level programming problem. Let the full set of constraints for all levels be denoted by S and $x \in S$ is partitioned among three levels, i.e. $x = (x^1, x^2, x^3) \in \mathbb{R}^n$.

Then the problem at the lowest level of hierarchy, level three, is given by

$$(P^3) \begin{cases} \max\{f^3(x) : (x^3 \mid x^1, x^2)\} \\ \text{s.t. } x \in S^3 = S \end{cases}$$

Which is a standard optimization problem parameterized by the decision vectors x^1 and x^2 of the respective higher level planners.

The feasible region $S = S^3$ is defined as the level-three feasible region. The solution to (P^3) in \mathbb{R}^{n^3} for each fixed x^1 and x^2 form a set $S^2 = W_{f^3}(S^3) = \left\{ \hat{x} \in S^3 : f^3(\hat{x}) = \max\{f^3(x) : (x^3 \mid x^1, x^2)\} \right\}$ called the level-two feasible region (also called rational reaction set of level three problem) over which $f^2(x)$ is maximized by varying x^2 for fixed x^1 . Thus the problem at the second level is given by

$$(p^2) \begin{cases} \max\{f^2(x) : (x^2 | x^1)\} \\ s.t. x \in S^2. \end{cases}$$

Where x^3 does not explicitly appear in the preparation of x but it is contained implicitly in the arguments of the objective function and (P^2) is parameterized by decision vector x^1 . The solutions to (P^2) in \mathbb{R}^{n_2} for each fixed x^1 form a set

$$S^1 = W_{f^2}(s^2) = \left\{ \hat{x} \in S^2 : f^2(\hat{x}) = \max\{f^2(x) : (x^2 | x^1)\} \right\}.$$

This is called the level one feasible region over which $f^1(x)$ is maximized.

Therefore, the problem at level one is given by

$$(p^1) \begin{cases} \max f^1(x) \\ s.t. x \in S^1 \end{cases}$$

Note that the objective function at level k , $f^k(x)$ ($k = 1, 2, 3$) is defined over the decision space of all levels. Thus, the level k planner may have an objective function determined, in part, by variables controlled at other levels. However, by controlling x^1 level one may influence the policies at level two and level three to improve his objective function. Similarly, by controlling x^1 , after decisions from level one have been made, level two may influence the policies at level three to improve his objective function. At the same time the choice of lower level decision maker affects level one's and level two's feasible regions and the choice of level two decision maker affects level one's feasible region. Hence the decisions of the lower level decision makers also influences the policies of the higher level decision maker. This establishes the interdependence of each planner's problem. The above discussion of three level programming problem can be written as a nested optimization as follows:

$$\begin{aligned}
& \left. \begin{array}{l} (p^1) \end{array} \right\} \left\{ \begin{array}{l} \max_{x^1} f^1(x) \\ \text{where } x^2 \text{ solves} \\ (p^2) \left\{ \begin{array}{l} \max_{x^2} f^2(x) \\ \text{where } x^3 \text{ solves} \\ (p^3) \left\{ \begin{array}{l} \max_{x^3} f^3(x) \\ s.t. \ x \in S \end{array} \right. \end{array} \right. \end{array} \right. \end{array} \quad (3.2.4)
\end{aligned}$$

When all functions f^1 , f^2 , and f^3 are linear and a set S is a polyhedral set then the above problem is known as a *three-level linear programming problem*. In the next sections basic properties of three-level linear programming problem.

The next example is a three level linear programming problem and its feasible region is given below it.

Example 3.2.2 Let x^1 , x^2 and x^3 be single component vectors.

$$\begin{aligned}
\max_{x^1} f^1 &= x^1 + x^2, \\
&\text{where } x^2 \text{ solves :} \\
\max_{x^2} f^2 &= x^2 + x^3, \\
&\text{where } x^3 \text{ solves :} \\
\max_{x^3} f^3 &= x^3
\end{aligned}$$

Subject to

$$\left. \begin{aligned}
 x^1 + x^2 + x^3 &\leq 3 \\
 x^1 + x^2 - x^3 &\geq 1 \\
 x^1 - x^2 + 2x^3 &\leq 1 \\
 -x^1 + x^2 + x^3 &\leq 1 \\
 x^3 &\geq 1/2 \\
 x^1, x^2, x^3 &\geq 0
 \end{aligned} \right\} \dots\dots\dots (S)$$

The rational reaction set of the lower level problem is given by the hatched region and the rational reaction set of the second level is given by the emboldened edge in the Figure 3.2.

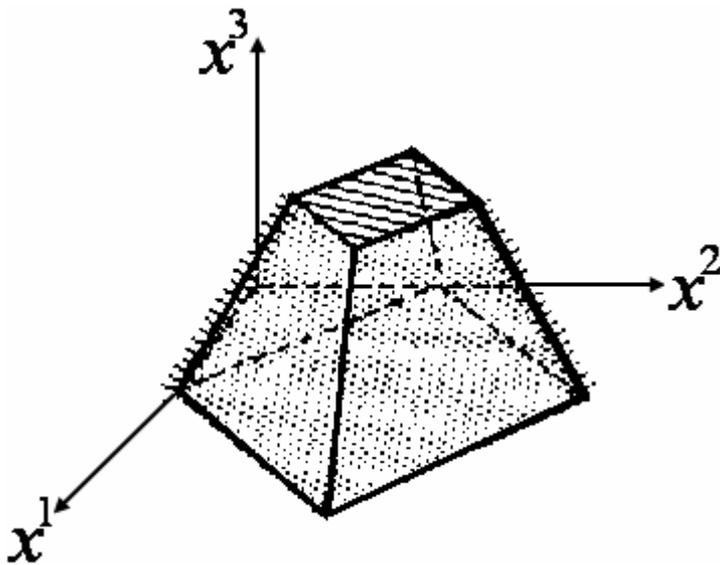


Figure 3.2: Feasible regions for three-level programming problem

3.3 Geometry of the Three-Level Linear Programming Problem

In this section we develop some of the geometric properties of the linear three-level programming problem.

The result can be viewed as an extension of those obtained by Bard [6] for the linear bi-level programming problem. Our goal is to show that; if all the functions in (3.2.4) are linear and S is a polyhedron, then the optimal solution occurs at a vertex of S . The linear three-level programming problem is given by:

$$\begin{array}{l}
 (p^1) \left\{ \begin{array}{l}
 \max_{x^1} f^1(x) = c^{11}x^1 + c^{12}x^2 + c^{13}x^3 \\
 \text{where } x^2 \text{ solves} \\
 (p^2) \left\{ \begin{array}{l}
 \max_{x^2} f^2(x) = c^{21}x^1 + c^{22}x^2 + c^{23}x^3 \\
 \text{where } x^3 \text{ solves} \\
 (p^3) \left\{ \begin{array}{l}
 \max_{x^3} f^2(x) = c^{31}x^1 + c^{32}x^2 + c^{33}x^3 \\
 \text{Suppose to} \\
 S = \{x \in \mathbb{R}^n : A^1x^1 + A^2x^2 + A^3x^3 \leq b\}
 \end{array} \right.
 \end{array} \right.
 \end{array} \right.
 \end{array} \tag{3.3.1}$$

Although the set of rational reactions for level-three and level-two may be non convex, it may possess some of the important properties of convex sets. This section will highlight some of the currently known results regarding the geometric properties of three-level linear programming problem. Some of these results are exploited by some of the algorithms in the next sections.

The following theorem and its corollaries help to characterize both S^2 and S^1 and the optimal solution for problems (P^2) and (P^1) .

Theorem 3.3.1 *Suppose $S^3 = \{x \in \mathbb{R}^n : Ax \leq b\}$ is bounded. Let S^2 and S^1 be the rational reactions of (P^2) and (P^1) respectively then the following hold.*

1. $S^1 \subseteq S^2 \subseteq S^3$.
2. Let y_1, y_2, \dots, y_r be any r points of S^3 and $\lambda_1, \lambda_2, \dots, \lambda_r \geq 0$ be scalars with $\sum_{i=1}^r \lambda_i = 1$ such that $\sum_{i=1}^r \lambda_i y_i \in S^2$. Then $\lambda_i > 0$ implies $y_i \in S^2$. Similarly, let y_1, y_2, \dots, y_r be any r points of S^2 and $\lambda_1, \lambda_2, \dots, \lambda_r \geq 0$ be scalars with $\sum_{i=1}^r \lambda_i = 1$ such that $\sum_{i=1}^r \lambda_i y_i \in S^1$. Then $\lambda_i > 0$ implies $y_i \in S^1$.

Proof: ►

1. By the definition of S^2 and S^3 , it is clear that $S^1 \subseteq S^2 \subseteq S^3$.
2. (Proof by contradiction) Let $y_1, y_2, \dots, y_r \in S^3$ with $x = (x^1, x^2, x^3) = \sum_{i=1}^r \lambda_i y_i \in S^2$, $\lambda_i \geq 0$ for $i = 2, \dots, r$, and $\lambda_1 > 0$, $\sum_{i=1}^r \lambda_i = 1$. Suppose $y_1 = (y_1^1, y_1^2, y_1^3) \notin S^2$. Then there exists \bar{y} such that $\bar{y} = (\bar{y}_1^1, \bar{y}_1^2, \bar{y}_1^3) \in S^2$ with $y_1^1 = \bar{y}_1^1$, $y_1^2 = \bar{y}_1^2$ and $c^{31} \bar{y}_1^3 > c^{31} y_1^3$. Now using number(1) we have $\bar{y} = (\bar{y}_1^1, \bar{y}_1^2, \bar{y}_1^3) \in S^3$. Therefore $\bar{x} = (\bar{x}^1, \bar{x}^2, \bar{x}^3) = \lambda_1 \bar{y} + \sum_{i=2}^r \lambda_i y_i \in S^2$. Nothing that $x^1 = \bar{x}^1$, $x^2 = \bar{x}^2$ and $\lambda_1 > 0$ with the following $c^{31} x^3 = \lambda_1 c^{31} y_1^3 + \sum_{i=2}^r \lambda_i c^{31} y_i^3 < \lambda_1 c^{31} \bar{y}_1^3 + \sum_{i=2}^r \lambda_i c^{31} y_i^3 = c^{31} \bar{x}^3$.

Therefore we have established \bar{x} with the following properties

- $\bar{x} = (\bar{x}^1, \bar{x}^2, \bar{x}^3) \in S^3$
- $x^1 = \bar{x}^1, x^2 = \bar{x}^2$ and
- $f^3(x) < f^3(\bar{x})$

This contradicts the definition of S^2 . Since $x \in S^2$ maximizes f^3 for the fixed values of x^1 and x^2 . Therefore $\lambda_1 > 0$ implies that $y_1 \in S^2$. Since the choice of y_i among the y 's was arbitrary, we have proved that $\lambda_i > 0$ implies that $y_i \in S^2$. Using the same procedure we can prove the second result. ◀

The set S^2 , posses a weak convex-like property with respect to the set S^3 . Hence any point in S^3 which is positively contributes in any convex combinations to form a point in S^2 must also be elements of S^2 . Since this is true for any point of S^3 it must be true for the extreme point of S^3 which results in the following corollary:

Corollary 3.3.2 *If x is an extreme point of S^2 , then x is extreme point of S^3 and similarly, if x is an extreme point of S^1 , then x is extreme point of S^2 so is to S^3 .*

Proof: ► (Proof by contradiction)

Let x be an extreme point of S^2 . Suppose x is not an extreme point of S^3 . Then there exists extreme points $y_1, y_2, \dots, y_r \in S^3$ and $\lambda_i > 0$ for $i = 1, 2, \dots, r$ with $\sum_{i=1}^r \lambda_i = 1$ such that $\sum_{i=1}^r \lambda_i y_i \in S^2$ (since S is convex). From theorem(3.3.1) this implies $y_1, y_2, \dots, y_r \in S^2$ and hence x can not be an extreme point of S^2 , it is a contradiction. Similarly we can show for extreme point of S^1 . ◀

Using the above results, one can conclude that S^2 is a very special portion of the boundary of S^3 and S^1 is also special portion of boundary of S^2 .

Definition 3.3.3 *A point-to-set mapping $\Gamma : \mathbb{R}^p \rightarrow 2^{\mathbb{R}^q}$ is called polyhedral if its graph*

graph $\Gamma := \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : y \in \Gamma(x)\}$ is equal to the union of a finite number of polyhedral sets.

Here a polyhedral set is the intersection of a finite number of half spaces; it is a closed and convex set.

Theorem 3.3.4 *The point-to-set mapping $\psi(\cdot)$ is polyhedral, where $\psi(x^1, x^2) = \text{Arg max}\{c^{31}x^1 + c^{32}x^2 + c^{33}x^3 : A^3x^3 \leq b - (A^1x^1 + A^2x^2), x^1, x^2 \geq 0\}$*

denotes the set of optimal solutions of the lower level problem

$$\begin{aligned} & \max_{x^3} c^{31}x^1 + c^{32}x^2 + c^{33}x^3 \\ & \text{s.t. } A^3x^3 \leq b - (A^1x^1 + A^2x^2), \quad x^1, x^2 \geq 0 \end{aligned}$$

Proof: ► By linear programming duality, $x \in \psi(x^1, x^2)$ if and only if there exists $\lambda \in R^{n_3}$ such that

$$\begin{aligned} A^1x^1 + A^2x^2 + A^3x^3 &\leq b, \quad x^3 \geq 0 \\ A^{1T}\lambda - c^3 &\geq 0, \quad \lambda \geq 0 \\ \lambda(b - A^1x^1 - A^2x^2 - A^3x^3) &= 0 \\ x^3(A^{1T}\lambda - c^3) &= 0 \end{aligned} \tag{3.3.2}$$

For any sets $I \subseteq \{1, \dots, n_3\}$, $J \subseteq \{1, \dots, m\}$ consider the solution set

$M(I, J)$ of the system of linear (in) equalities

$$\begin{aligned} (A^1x^1 + A^2x^2 + A^3x^3 - b)_j &= 0, \quad j \in J \\ (A^1x^1 + A^2x^2 + A^3x^3 - b)_j &\leq 0, \quad j \notin J \\ (A^{1T}\lambda - c^3)_i &= 0, \quad i \in I, \\ (A^{1T}\lambda - c^3)_i &\geq 0, \quad i \notin I, \\ \lambda_j &\geq 0, \quad j \in J, \quad \lambda_j = 0, \quad j \notin J \\ x_i^3 &\geq 0, \quad i \in I, \quad x_i^3 = 0, \quad i \notin I \end{aligned}$$

Then, conditions (3.3.2) are satisfied. The set $M(I, J)$ is polyhedral. Since the graph of $\psi(\cdot)$ is equal to the union of the sets $M(I, J)$ so the assertion follows. ◀

Corollary 3.3.5 *The level-one feasible region, S^1 for the linear bi-level programming problem is composed of the union of connected faces and edges of lower level constraint set S .*

Direct application of Theorem 3.3.4 is that a vertex of S^2 is also a vertex of S ; hence the optimal solution of the bi-level programming problem occurs at a vertex of S . The following theorem indicates a similar conclusion for the linear three-level programming problem.

Theorem 3.3.6 *If the solution to the three-level linear programming problem given by equation 3.3.1 is unique, then*

1. *The level-two feasible region, S^2 is formed from the union of connected faces and edges of S .*
2. *The level-one feasible region, S^1 is formed from the union of connected faces and edges of S^2 .*

The proof of Theorem(3.3.6) is similar to the Theorem 3.3.4. Now as a corollary of Theorem (3.3.6) we obtain the following:

Corollary 3.3.7 1. *If the optimal solution of the lower level problem is uniquely determined for each value of the parameter (x^1, x^2) , then there is an optimal solution of the level-two problem, which is a vertex of S^2 and hence at a vertex of S .*

2. *If the optimal solution of the level-two problem is uniquely determined for each value of the parameter x^1 , then there is an optimal solution of the higher level problem, which is a vertex of S .*

The following example illustrates the above properties of the linear three-level programming problem. The fact that a solution occurs at a vertex of S suggest that extreme point search procedures could be used to form the bases of an algorithm called enumeration algorithm to solve (3.0.4). Bard [2] and Wen [22] summarizes a number of such efforts primarily directed at the bi-level case.

Example 3.3.1

$$\begin{aligned} \max_{x^1} f^1 &= 2x^2 - x^1, \\ &\text{where } x^2 \text{ solves :} \\ \max_{x^2} f^2 &= 10x^1 + 6x^2 - x^3, \\ &\text{where } x^3 \text{ solves :} \\ \max_{x^3} f^3 &= -x^1 + 12x^2 - x^3 \end{aligned}$$

S.t.

$$\left. \begin{aligned} -x^1 + 8x^2 - x^3 &\leq 8 \\ 3x^1 + 3x^2 - x^3 &\leq 15 \\ x^2 + 2x^3 &\leq 12 \\ -2x^2 + 3x^3 &\geq 6 \\ -x^2 + x^3 &\geq 3 \\ x^1, x^2, x^3 &\geq 0 \end{aligned} \right\} \dots\dots\dots (S)$$

The three feasible regions for this example are depicted in Figure 3.3. The set of all triples (x^1, x^2, x^3) satisfying the above constraints must fall within the polyhedron S which is the level-three feasible region. From f^3 it can be seen that once x^1 and x^2 are chosen, planner 1 will react by picking the smallest possible value of x^3 such that $(x^1, x^2, x^3) \in S^1$. This produces the rational reaction set S^2 of f^3 (it is the level-two feasible region) denoted by the hatched faces in the Figure 3.3. Thus, for x^1 fixed at \bar{x}^1 , planner two's parametric problem is to maximize f^2 over

$$S^2(\bar{x}^1) = \{\bar{x} \in S : f^3(\bar{x}) = \max_{\hat{x}^3} \{f^3 = -\bar{x}^1 + 12x^2 - \hat{x}^3 : x^2\}\}$$

Planner one's problem is then to maximize $f^1 = 2x^2 - x^1$ over S^1 the reaction set of the second problem (the level-one feasible region) which is marked by

the emboldened edges. Specifically, we have

$$S^1 = \{x \in S : f^3(x) = \max(f^2 = 10x^1 + 6\hat{x}^2 - x^3 : (\hat{x}^2 | x^1), (x^1, x^2, x^3) \in S^2)\}.$$

The optimal solution to the over all problem is found by solving the highest level problem and is denoted by a point H in Figure 3.3. As expected it occurs at a vertex of S^1 as well as S.

Point F represents a local optimum, since movement in any feasible direction away from this point will produce a decrease of f^1 . Finally, an examination of S reveals a pareto optimum at (the infeasible) point G; that is a strict improvement in each objective function realized when we move from H to G. Table 3.1 summarizes the results for these three situations.

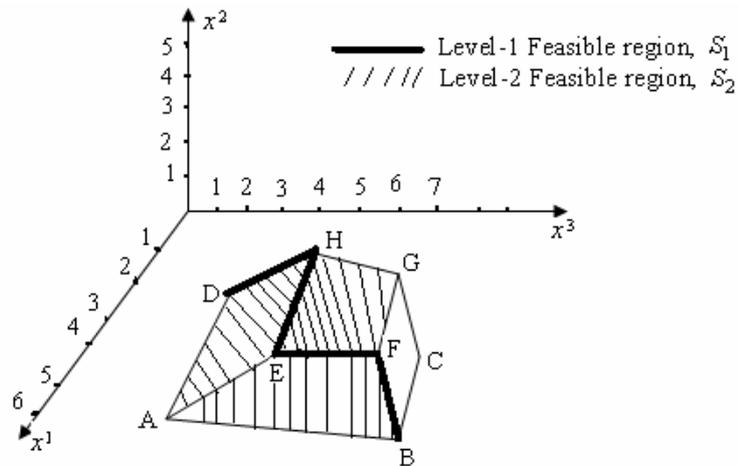


Figure 3.3: Feasible regions for Example 3.3.1

A basic assumption in a multilevel programming problem rules out cooperation among the planners. As demonstrated in Example 3.3, if this restriction is relaxed, the payoff realized at each level may increase. The possibility of non efficient solutions has wider implications for our work because it discour-

Vertex	x	f^1	f^2	f^3
Global optimum, H	(3.05, 1.75, 1.52)	1.62	21.57	14.27
Local optimum, F	(4.47, 2.27, 4.91)	-0.07	53.41	17.86
Patero optimum, G	(5.08, 1.84, 1.62)	2.07	22.06	15.39

Table 3.1: Objective function values for various optima

ages the adaptation of algorithms designed to solve the vector maximization problem [6].

3.4 Optimality condition for three-level linear programming problem

Optimality conditions in mathematical programming problem have two significant purposes:

1. They provide a means of determining whether or not a feasible point is a local (global) optimal solution for the given problem and
2. They may suggest a structure for development of an algorithm that solves the given problem.

For standard nonlinear programming problem the Kuhn-Tucker conditions under certain regularity assumptions provide some important optimality conditions. Now we will see some important optimality conditions for three level linear programming problem by using the Kuhn-Tucker conditions. Suppose

a three-level linear programming problem is given as follow:

$$\begin{aligned}
\max_{x^1} f^1(x) &= c^{11}x^1 + c^{12}x^2 + c^{13}x^3 \\
&\text{where } x^2 \text{ solves :} \\
\max_{x^2} f^2(x) &= c^{21}x^1 + c^{22}x^2 + c^{23}x^3 \\
&\text{where } x^3 \text{ solves :} \\
\max_{x^3} f^3(x) &= c^{31}x^1 + c^{32}x^2 + c^{33}x^3 \\
&\text{s.t.} \\
S &= \{x \in R^n : A^1x^1 + A^2x^2 + A^3x^3 \leq b, x \geq 0\}
\end{aligned}$$

Here, since x^1 is fixed when the second level problem is solved we can assume that $c^{21} = 0$ and similarly, since x^1 and x^2 are fixed when the lower level problem is solved we can put $c^{31} = c^{32} = 0$. Latter on we add $c^{21}x^1$, $c^{31}x^1$ and $c^{32}x^2$ on its appropriate position, whenever necessary. And hence we have the following three level linear programming problem.

$$\max_{x^1} f^1(x) = c^{11}x^1 + c^{12}x^2 + c^{13}x^3, \quad (12a)$$

$$\max_{x^2} f^2(x) = c^{22}x^2 + c^{23}x^3, \quad (12b)$$

$$\max_{x^3} f^3(x) = c^{33}x^3 \quad (12c) \quad (3.4.1)$$

$$\text{s.t.} \\ S = \{x \in R^n : A^1x^1 + A^2x^2 + A^3x^3 \leq b, x \geq 0\} \quad (12d)$$

Where $c^1 = (c^{11}, c^{12}, c^{13})$, $c^2 = (c^{22}, c^{23})$, and c^{33} are constant row vectors and b is a column vector with appropriate dimension $b \in R^m$, A^i is an $m \times n$ matrix for $i = 1, 2, 3$ and S , given by (12d) is assumed to be a compact polyhedral set.

Most of the algorithms developed for solving linear bi-level programming problem are based on the understanding that the reaction set (inducible region) of the lower level problem is union of connected faces and edges of the over all constraint set S . In [3] it is described that the lower level can be

replaced by a system of inequalities derived from its Kuhn-Tucker conditions. A global solution to the resultant new problem gives us the solution to the original bi-level programming problem. Similarly we will use this result to the three-level programming problem.

If the three level programming problem given by (1) is viewed as a three level optimization problem, then once x^1 and x^2 are specified then the inside problem (12c) with constraint set (12d) become a standard mathematical programming problem, which is given by

$$\begin{aligned} \max_{x^3} f^3(x) &= c^{33}x^1 & (12e) \\ \text{s.t. } x \in S &= \{x \in R^n : A^3x^3 \leq b - A^1x^1 - A^2x^2, x^3 \geq 0\} \end{aligned}$$

For the given choice x^1 and x^2 of level one and level two decision makers, x^3 solves the lower level problem if and only if x^3 satisfies the Kuhn-Tucker conditions of problem (12e). That is x^3 satisfies the following conditions

$$\begin{aligned} A^1x^1 + A^2x^1 + A^3x^1 &\leq b, \\ uA^3 &\geq c^{33} \\ x &\geq 0, u \geq 0 \\ u(A^1x^1 + A^2x^2 + A^3x^3 - b) &= 0 \end{aligned}$$

Where u is an m-dimensional row vector of dual variable and the existence of u is determined from the Kuhn-Tucker theorem. Now since the second decision maker optimizes his/her objective function f^2 over the rational reaction set of f^3 given by

$W_{f^3}(S) = \left\{ \hat{x} \in S : f^3(\hat{x}) = \max\{f(x) : (x^3 \mid \hat{x}^1, \hat{x}^2)\} \right\}$ and any point in $W_{f^3}(S)$ satisfies the above Kuhn-Tucker condition, we can replace $W_{f^3}(S)$ by the above Kuhn-Tucker condition of the lower level problem. Hence for a given x^1 the level two problem can be written as

$$\begin{aligned} \max_{x^2, x^3, u} f^2(x) &= c^{22}x^2 + c^{23}x^3 \\ \text{Subject to} & \end{aligned}$$

$$\begin{aligned}
A^1x^1 + A^2x^2 + A^3x^3 &\leq b, \\
uA^3 &\geq c^{33} \\
x &\geq 0, u \geq 0 \\
u(A^1x^1 + A^2x^2 + A^3x^3 - b) &= 0
\end{aligned}$$

Making use of this result an alternative form of problem 3.4.1 can be given as follow:

$$\max_{x^1} f^1(x) = c^{11}x^1 + c^{12}x^2 + c^{13}x^3, \quad (13a)$$

$$\begin{aligned} &\text{where } (x^2, x^3) \text{ solves :} \\ \max_{x^2, x^3, u} f^2(x) &= c^{22}x^2 + c^{23}x^3 \end{aligned} \quad (13b)$$

$$\text{Subject to} \quad (3.4.2)$$

$$A^1x^1 + A^2x^2 + A^3x^3 \leq b \quad (13c)$$

$$uA^3 \geq c^{33} \quad (13d)$$

$$x \geq 0, u \geq 0 \quad (13e)$$

$$u(A^1x^1 + A^2x^2 + A^3x^3 - b) = 0 \quad (13f)$$

Notice that player two now has control over x^2 and x^3 as well as u .

Due to the presence of complimentary condition (13f), the above problem is a nonlinear bi-level programming problem. If player one is detached from the above formulation by temporarily fixing x^1 the inside problem (13b)-(13f) is easily solved by currently known algorithms such as branch and bound [8], complimentary pivoting [29], or implicit enumeration [11]. If we repeat the transformation we have used to obtain (3.4.2), we get an explicit approximation to S^1 , but this approximation of S^1 gives us a highly non convex set and extremely difficult to solve the resulting problem globally. But even if such a solution is obtained, there would be no guarantee that it would be an element of the level two reaction set S^2 . This, of course, stems from the fact that the Kuhn-Tucker conditions associated with the inside problem (13b)-(13f) for x^1 fixed are only necessary condition. It is because of the existence of (13f) which prevents sufficiency condition. This difficulty can be avoided by using penalty function approach; that is by attaching an appropriately large weight to the complimentary term and then shifting it to the objective function

(13b). This gives us the following problem:

$$\max_{x^1} f^1(x) = c^{11}x^1 + c^{12}x^2 + c^{13}x^3, \quad (14a)$$

where (x^2, x^3) solves :

$$\max_{x^2, x^3, u} f^2(x) = c^{22}x^2 + c^{23}x^3 + ku(A^1x^1 + A^2x^2 + A^3x^3 - b) \quad (14b)$$

Subject to (3.4.3)

$$A^1x^1 + A^2x^2 + A^3x^3 \leq b \quad (14c)$$

$$uA^3 \geq c^{33} \quad (14d)$$

$$x \geq 0 \quad (14e)$$

$$u \geq 0 \quad (14f)$$

Where k is a sufficiently large finite constant number. When the higher level decision maker is detached by fixing x^1 , the inside problem (14b)-(14e) of the above problem is essentially a bilinear programming problem such that the solution of this problem exists at the vertex of the constraint set (14c)-(14f). (This is due to the work of T.H. Vaish and C.M. Shitty [27]. Now we make use of this result in the following theorem that establishes the equivalence of problem (3.4.2) and problem (3.4.3) and hence to problem (3.4.1).

Theorem 3.4.1 *For x^1 fixed at x^{1*} there exists a finite number k such that (x^{2*}, x^{3*}, u^*) solves problem (13b)-(13f) if and only if it solves problem (14b)-(14f).*

Proof: ► To find the appropriate k for the case where (x^{2*}, x^{3*}, u^*) solves (13b)-(13f) let $X = \{x : x^1 = x^{1*}\}$ be the set of distinct vertices of (14c) and (14e) and U be the set of distinct vertices of (14d) and (14f). From bi-level programming method we know that $x^{1*} \in X$, and from bilinear programming we know that a solution to (14b)-(14f) must be in $X \times U$. If (14b) is evaluated at each of $r \times p$ vertices of $X \times U$, the complementarity condition will be negative or zero. For the latter case (x^*, u^*) clearly provides the largest

payoff.

Alternatively when $u(A^1x^1 + A^2x^2 + A^3x^3 - b) < 0$ and if k is selected such that $k > \max_{x \in X, u \in U} (c^{22}x^{2*} + c^{23}x^{3*} - c^{22}x^2 - c^{23}x^3)/u(A^1x^1 + A^2x^2 + A^3x^3 - b)$ the optimality of (x^*, u^*) would still be assured. Because $c^{22}x^2 + c^{23}x^3$ is bounded above on the constraint set (14c), hence the first part of the theorem holds. For the case where (x^{2*}, x^{3*}, u^*) solves (14b)-(14e) we can pick the above k . So this point must necessarily solve (13b)-(13f). ◀

As we can see, the constraint set of Problem (3.4.3), is a convex polytope and hence it is easier to work with it than that of problem (3.4.2). Therefore it is possible to develop first order necessary optimality condition for the linear three-level programming problem by adding the Kuhn-Tucker conditions of the inside problem (3.4.3) to the higher level problem. In the following theorem we once more apply the Kuhn-Tucker conditions of the inside problem to obtain a necessary optimality condition to the linear three level programming problem and an over approximation of the level two inducible region.

Theorem 3.4.2 *A necessary condition that x^* solves three level linear programming problem (3.4.1) is that there exist vectors $u^*, \bar{u}^*, v^* \in R^m$ such that $(x^*, u^*, \bar{u}^*, v^*)$ is feasible to the following problem:*

$$\begin{aligned}
& \max_{x, u, \bar{u}, v} f^1(x) = c^{11}x^1 + c^{12}x^2 + c^{13}x^3, \\
& \text{Subject to} \\
& A^1x^1 + A^2x^2 + A^3x^3 \leq b \\
& uA^3 \geq c^{33} \\
& \bar{u}A^2 \geq c^{22} \\
& \bar{u}A^3 \geq c^{23} \\
& ku - v - \bar{u} = 0 \\
& u(A^1x^1 + A^2x^2 + A^3x^3 - b) = 0 \\
& v(A^1x^1 + A^2x^2 + A^3x^3 - b) = 0 \\
& x \geq 0, \quad u \geq 0 \quad v \geq 0
\end{aligned} \tag{3.4.4}$$

Proof: ► A straight forward application of the Kuhn-Tucker theorem to the inside problem of (3.4.3) with x^1 fixed gives us the desired result. The only complication arises from the stationary term associated with u that shown to be redundant. Let v^1 , v^2 , and v^3 be the m -dimensional Kuhn-Tucker multipliers associated with (14c)-(14f), respectively. First order stationary condition with respect to u gives us

$$k(A^1x^1 + A^2x^2 + A^3x^3 - b) - A^3v^1 - I_mv^2 = 0 \quad (3.4.5)$$

Where I_m is m -dimensional identity matrix, v^1 and v^2 are nonnegative. In addition, complementarity requires that $uv^2 = 0$. If we multiply both sides of (3.4.5) by u and introduce the initial complimentary term $u(A^1x^1 + A^2x^2 + A^3x^3 - b) = 0$ to the new formulation we can always set $v^1 = 0$ and hence we get $v^2 = k(A^1x^1 + A^2x^2 + A^3x^3 - b)$. Thus we have the following. $uv^2 = ku(A^1x^1 + A^2x^2 + A^3x^3 - b) = 0$. Therefore, the result holds. ◀

3.5 Solution Procedure

The possible existence of local optimal solution, even for linear multilevel programming problem aggravates the general task for algorithmic development. Many solution approaches to solve multilevel programming problem have been developed to date. Most of these algorithmic developments have been devoted to the linear two-level programming problem which can be defined as:

$$\begin{aligned} & \max_{x^1} c^{11}x^1 + c^{12}x^2, \\ & \qquad \qquad \qquad \text{where } x^2 \text{ solves :} \\ & \max_{x^2} c^{21}x^1 + c^{22}x^2 \qquad \qquad \qquad (3.5.1) \\ & \text{S.t. } A^1x^1 + A^2x^2 \leq b \\ & \qquad \qquad \qquad x^1, x^2 \geq 0 \end{aligned}$$

Here, since x^1 is fixed when the lower level problem is solved by the level two decision maker, we can assume $c^{21} = 0$ and latter on add the term $c^{21}x^1$, if necessary. Solution techniques for bi-level programming problems can be classified as:

1. the vertex enumeration approach
2. the complimentary pivoting approach
3. the Kuhn-Tucker approach and
4. the branch and bound approach.

These have been described in specific paper in the literature (see the paper by Bialas and Karwan [10]). Some mathematicians are working on the extension of these solution procedures to three level programming problems and the general n-level programming problems.

A hybrid algorithm has been developed by Wen and Bialas [31] to solve the three-level programming problem. It is based on the k^{th} best vertex enumeration technique to the bi-level programming problem developed by Bialas and Karwan [10]. At any iteration k the hybrid method maximizes the level one objective function over the overall constraint set S to obtain the leader's best solution (x_k^1, x^2, x^3) . This solution is checked for feasibility by solving the second level problem, maximizing $f^2(x)$ over its constraint set for x^1 fixed at x_k^1 to get a solution (x_k^2, x^3) . This also is checked for feasibility for the level two constraint set by solving the lower problem maximize $f^3(x)$ over its constraint set S for x^1 fixed at x_k^1 and x^2 fixed at x_k^2 to find a solution x_k^3 . If x_k^3 is an element of level one reaction set S^2 and (x_k^2, x_k^3) is an element of level two reaction set S^1 then the algorithm terminates with global optimum.

Otherwise, the adjacent extreme point of (x_k^1, x_k^2, x_k^3) is generated and the procedure is repeated until an optimal solution is obtained.

While the hybrid algorithm seems to work well in small problems (see an example in Wen and Bialas [31]). It is likely to pose computational problems when the number of constraints and number of variables increases. In this paper, we will see another procedure that solves three-level linear programming problem better than the hybrid algorithm. The procedure that we will describe here for globally solving three-level linear programming problem combines simplex like vertex enumeration for the approximate programming problem of the three-level linear programming problem developed in the previous section using the Kuhn-Tucker optimality conditions with the branch and bound algorithm. Because of which we call this algorithm **simplex-cutting plane algorithm**. Before going to the detail description of this algorithm let us see vertex enumeration algorithm for bi-level programming problem developed by Bialas and Karwan [10].

3.5.1 Vertex enumeration algorithm for bilevel programming problem

The idea of vertex enumeration algorithm is based on the characteristic that the extreme point of level one feasible region, S^1 is also an extreme point of the level two feasible region S . When the higher level decision variable x^1 is fixed to be \bar{x}^1 , then the lower level problem may be given by

$$\begin{aligned} \max_{x^2} \quad & c^{21}\bar{x}^1 + c^{22}x^2 \\ \text{s.t.} \quad & A^2x^2 \leq b - A^1\bar{x}^1 \\ & x^2 \geq 0. \end{aligned}$$

Let the optimal solution of this problem is \bar{x}^2 , then $(\bar{x}^1, \bar{x}^2) \in W_{f^2}(S)$, where

$$f^2(x) = c^{21}x^1 + c^{22}x^2.$$

Candeler and Townsley [11] have proposed an algorithm, known as T-set algorithm, that focuses on generating and enumerating basis for lower level programming problem. This solution method involves an implicit search of all potential optimal basis, with out reconsidering any previously explored basis. But the algorithm may not stop as soon as the global solution is obtained.

An algorithm proposed by Bialas and Karwan [10] uses the simplex method for bounded variables and finds an extreme point in the level one reaction set $W_{f^2}(S)$; it then moves among the extreme points of $W_{f^2}(S)$, never allowing $f^2(S)$ to decrease. However, by using this procedure we get only a local optimal solution. The interesting algorithm by Parraga [25] suggests adding a cut to the original feasible region after a local optimal solution has been found. Then the method due to Matheiss and Rubin [21] may be used to find the cut points and to check all of them so as to obtain a better point while maintaining feasibility. This algorithm guarantees the existence of global optimal solution. Parraga also suggests that the number as well as the size of subproblems required to solve global optimal solutions are rather small in comparison with other algorithms.

3.5.2 The k^{th} -best Algorithm

The k^{th} -best Algorithm has been proposed by Bialas and Karwan [10] and Wen [30]. First it solves $\max (c^{11}x^1 + c^{12}x^2)$ over the over all constraint set in order to get the first best solution and upper bound for f^1 . If the solution is not on the level one feasible region then the second best solution may be found among the extreme points which are adjacent to the first best. And

the algorithm moves sequentially through these ordered extreme points of S until one, the k^{th} -best solution is found in level one feasible set S^1 . When this point is found the algorithm terminates with global optimal solution.

Assume the over all constraint set (level two feasible region) S^2 is bounded and a unique solution exists to the lower level problem for any feasible x^1 . From corollary of theorem (3.3.6) the solution to the higher level problem must occur at the extreme point of S . Let $\bar{x}_1, \bar{x}_2, \bar{x}_3, \dots, \bar{x}_N$ denotes the first N ordered basic feasible solutions to the linear programming problem given by

$$\begin{aligned} \max \quad & c^1 x \\ \text{S.t.} \quad & Ax \leq b, \quad x \geq 0 \end{aligned} \tag{3.5.2}$$

such that $c^1 \bar{x}_i \geq c^1 \bar{x}_{i+1}$ for $i = 1, 2, \dots, N-1$. Then solving the higher level problem is equivalent to finding the index $k = \min \{i \in \{1, 2, \dots, N\} : \bar{x}_i \in S^1\}$, where S^1 is the higher level feasible region; results the global optimal solution. This requires finding the k^{th} - best extreme point solution to the problem given in (3.5.1). The k^{th} -best algorithm performs this search and the resulting solution is a global solution to the given bi-level programming problem.

The “ k^{th} -best” Algorithm:

Step 1 Put $i=1$. Solve problem (3.5.2) to get optimal solution \bar{x}_1 by simplex method. Let $W = \{\bar{x}_1\}$ and $T = \emptyset$. Go to step 2.

Step 2 Solve the following linear programming problem by simplex algorithm:

$$\begin{aligned} \max \quad & c^2 x \\ \text{S.t.} \quad & Ax \leq b, \quad x^1 = \bar{x}_i^1, \quad x^2 \geq 0 \end{aligned} \tag{3.5.3}$$

Let \hat{x} be the optimal solution to (3.5.3). If $\hat{x} = \bar{x}_i$, stop; with global optimal solution \bar{x}_i and $k=i$. Otherwise, go to step 3.

Step 3 Let W_i be the set of extreme points of S , which are adjacent to \bar{x}_i . Let $T = T \cup \{\bar{x}_i\}$ and $W = (W \cup W_i) \setminus T$. Go to step 4.

Step 4 Put $i=i+1$ and chose \bar{x}_i such that $c^1\bar{x}_i = \max\{c^1x : x \in W\}$. Go to step 2.

Step 2 repeatedly tests extreme points to determine whether they are elements of level one feasible region S^1 or not. Since each successive pair of extreme points is adjacent, this process can be performed by using dual simplex algorithm. Also for the same reasoning, a simple implementation of this algorithm can consist of storing the basis column indices for the first $(k-1)$ best points and then determining which basis adjacent to these points which is in the level one feasible region S^1 , this results a maximum value of level one objective function.

Computational experience with the k^{th} -best algorithm demonstrates that it finds a solution easily for most bi-level programming problems, but occasionally an unacceptably large time may be needed before a solution is found. This approach to the bi-level programming problem can be intuitively extended to the three-level programming problem. This can be done easily by

1. Replacing problem (3.5.2) by:

$$\max f^1(x) = c^{11}x^1 + c^{12}x^2 + c^{13}x^3,$$

$$S.t. \quad S = \{x : A^1x^1 + A^2x^2 + A^3x^3 \leq b \quad x \geq 0\}$$

2. Replacing problem (3.5.3) by:

For x^1 fixed at \bar{x}_i^1 solve the following level two optimization problem

$$\max f^2(x) = c^{21}x^1 + c^{22}x^2 + c^{23}x^3,$$

$$S.t. \quad x \in W_{f^3}(S)$$

where $W_{f^3}(S)$ is the reaction set of the lower level problem.

3.5.3 Simplex-Cutting Plane Algorithm

Now we can describe a simplex-cutting plane algorithm for three-level linear programming problem. It uses the approximate problem (3.4.3) of the three-level linear programming problem which is discussed in section (3). The following lemma summarizes some properties of problem (3.3.1) and problem (3.4.3) and serves to support the algorithmic structures.

- Lemma 3.5.1** 1. *The functional value of the objective function f^1 evaluated at a solution to (3.4.4) provides an upper bound on the optimal value of the linear three-level linear programming problem.*
2. *A solution to problem (3.4.4) is always in the lower level reaction set $W_{f^3}(S)$.*
3. *For the local solution x^* to the three-level linear programming problem (3.3.1) the hyperplane $f^1(x) = f^1(x^*)$ supports the convex hull of the level-one constraint set S^1 if and only if x^* is a global optimal solution of the three-level linear programming problem.*

Simplex-Cutting Plane Algorithm:

Step 1 Solve problem (3.4.4) to get a solution \hat{x} and put $f^1(\hat{x})$ is an upper bound of f^1 . Put S^* to be the constraint set of problem (3.4.4). Go to step 2.

Step 2 For x^1 fixed at \hat{x}^1 solve the following problem (level two-problem)

$$\begin{aligned} \max \quad & f^2(x) = c^{22}x^2 + c^{23}x^3, \\ \text{S.t.} \quad & x \in W_{f^3}(S) \end{aligned} \tag{3.5.4}$$

Let $(x)^1$ denote the optimal solution to (3.5.4). If $\hat{x} = (x)^1$ or $f^1(x)^1 = f^1(\hat{x})$, then stop with global optimal solution \hat{x} . Otherwise define $f^1 = f^1(x)^1$, which gives a lower bound for level one optimal objective functional value and go to step 3.

Step 3 Search a vertex adjacent to $(x)^k$ that lies in the level-one feasible region S^1 and in the direction of improvement of level one objective function. If there is no such a vertex found, go to step 4. Otherwise move to this vertex and repeat the search until a local optimum is achieved. Suppose x^* be the local optimal solution. Then update the lower bound by setting $f^1 = f^1(x^*)$.

Step 4 Add the cutting constraint $f^1(x) = f^1 + \epsilon$ to the constraint set of problem (3.3.1) and solve problem (3.3.1) using the new constraint set to get a solution \hat{x} . Where $\epsilon \geq 0$ is a sufficiently small constant number chosen so that no unexplored vertices of S are eliminated.

Step 5 Test whether the solution \hat{x} is in S^1 or not by solving problem (3.5.4) for x^1 fixed at \hat{x}^1 . If the test has positive response (that is \hat{x} is an element of S^1) put $(x)^{k+1} = \hat{x}$, remove the cutting constraint and go to step 3 with $k=k+1$; otherwise search a vertex which is adjacent to \hat{x} in the direction of the improvement of level one objective function to get a point $(x)^{k+1}$ in S^1 . Repeat this search if necessary starting from each of the other multiple solutions obtained at step 4 until such a point is found. If there exist a point $(x)^{k+1}$ in S^1 then go to step 3

with $k=k+1$; otherwise stop with global optimal solution x^* .

As we can see the algorithm is developed by using the advantage of the connectedness of level one inducible region (level one feasible region) S^1 of Theorem 3.3.6 and using the fact that the solution of the three-level linear programming problem must occur at a vertex of the over all constraint set S of the Corollary 3.3.7. At the initial stage, a convenient starting point is found by solving the proper approximate problem (3.4.4) of the three-level linear programming problem (3.3.1) to get a solution x^* , which gives the functional value $f^1 = f^1(\hat{x})$. This functional value gives upper bound for the optimal functional value of the objective function $f^1(x)$. At step 2 global optimality is tested by solving the second player's problem for x^1 fixed at \hat{x}^1 to get a solution $(x)^1$ and the termination of the algorithm occurs if either \hat{x} is an element of level one constraint S^1 or a solution $(x)^1$ which is an element of S^1 by construction gives an objective function value which is equal to the upper bound $f^1(\hat{x})$ obtained by solving problem (3.4.4). At step 3 a simplex-type search is conducted over the constraint set S^1 to find a local optimal solution. At this step a vertex adjacent to a point $(x)^k$ is obtained in the direction of improvement of the objective function f^1 over a set S^1 . This requires before a vertex adjacent to $(x)^k$ be included along the path of improvement of the objective function f^1 , problem (3.5.4) be solved to assure its membership in S^1 .

At step 4 a new constraint, using the current best lower bound is added to the constraint set of problem (3.4.4) to remove the explored solution from the constraint region. Parraga [25] used this cutting plane procedure in his approach to the bi-level programming problem. By resolving problem (3.4.4) with the new constraint we get another trial solution \hat{x} which gives the new best upper bound $f^1(\hat{x})$ for the optimal objective function value. In general

this trial solution will not be a vertex of the over all constraint set S and it may be an element of level one inducible region S^1 . However, from part (iii) of the lemma 3.5.1 we see that if the current local optimal solution x^* is not a global optimal solution the cutting plane $f^1(x) = f^1(x^*)$ intersects the level one feasible region S^1 . Step 5 is designed to find a point of intersection of the cutting plane with the original constraint set of problem (3.3.1). The search is facilitated by the parallel nature of the cutting plane, which permits the ready identification of potential elements of the level two reaction set. This can be achieved by pivoting through the multiple optimal solutions obtained when (3.3.1) is solved using the new constraint. The actual procedure implemented was based on the work of Mattheis [21].

These steps, (Step 3 to step 5) either result in a solution to the three-level programming problem, in which case the algorithm terminates, or if not, generate a point in S^1 which is a vertex of S , which gives a lower bound for the optimal functional value of f^1 . A cutting constraint, using the current best lower bound, is added and the algorithm returns to Step 3, replacing the constraint set of problem (3.4.4) by constraint set of problem 3.4.4) plus the new additional cutting plane constraint. The algorithm is iterated until a solution to problem (3.4.4) is found, in a finite number of iterations. The algorithm may be terminated when the difference between the current lower and upper bound is small enough, noting that lower bounds are non decreasing and the upper bounds are non increasing as a function of the iteration number. Also, at each iteration, the previous additional constraint is merely replaced by a new one, So that the constraint set is never supplemented by more than one constraint.

Example 3.5.1 Consider the following three-level programming problem.

$$\begin{aligned} \max_{x^1} f^1 &= -4x^1 + 2x^2 - 5x^3 \text{ where } x^2 \text{ solves} \\ \max_{x^2} f^2 &= -x^2 + 4x^3 \text{ where } x^2 \text{ solves} \\ \max_{x^3} f^2 &= 2x^3 \\ \text{subject to} \end{aligned}$$

$$\begin{aligned} 2x^2 - x^3 &\geq 2 \\ -3x^1 + x^2 - x^3 &\geq -12 \\ -3x^2 - x^3 &\geq -24 \\ x^1 &\geq 2 \\ -x^3 &\geq -6 \\ x^1 \geq 0, x^2 \geq 0, x^3 &\geq 0 \end{aligned}$$

Figure 3.4 displays S and the reaction sets S^2 and S^1 for level three and level two objectives respectively. S^2 is denoted by the hatched area and S^1 by the emboldened line.

When the above algorithm is applied to this example, Step 1 produces the point $\hat{x} = (2, 1, 0)$ that is seen in Figure 3.4, it is not an element of S^1 . By fixing x^1 at 2 and solving the inside problem, (level two problem) we get the point B, which is an element of S^1 and the value of the level one function at \hat{x} and at the point B. The search at step 3 leads to the conclusion that this point $(x)^1 = (2, 4, 6)$ is a local solution x^* because there no adjacent vertices that both lie in S^1 and in the direction of improvement of f^1 . At step 4, the cutting plane $-4x^1 + 2x^2 - 5x^3 \leq -30 + \epsilon$ is added to the constraint region and problem (3.4.4) for this problem and it gives a new point $\hat{x} = (4.57, 6.43, 4.71)$. Here ϵ was chosen as 1. The point \hat{x} lies along the edge \overline{FG} and it is not in S^1 : therefore, at step 5 a search of adjacent vertices must be conducted to find an element of the level-one

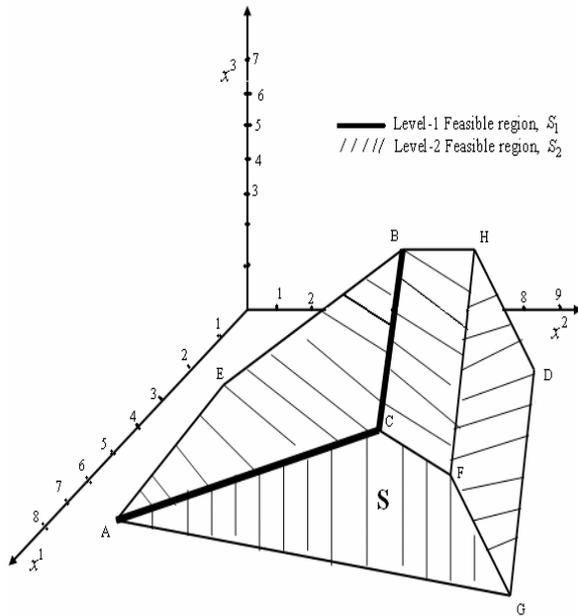


Figure 3.4: Feasible regions for Example 3.5.1

inducible region that intersects the cut, should exist. The search leads to the point $(x)^2 = (3.65, 3.05, 4.1)$ that lies along the edge \overline{AC} . Returning to step 3 brings us to a point A , which turns out to be the local optimum $x^* = (14/3, 3.1, 0)$. A second cutting plane must now be added but the algorithm terminates at step 5 when no new point in S^1 can be found, that produces a value of f^1 larger than $f^1(x^*) = -50/3$, it is the best lower bound. Hence x^* is the global solution of the problem.

Chapter 4

Application of three level programming problem

4.1 Problem definition

As it is mentioned in the introduction part of this paper that resource allocation for universities by the federal government can be considered as a three level decision making problem. The federal government has a certain budget to be distributed among n university. Then by taking a budget allocated to a university, the decision makers at the university level in turn distribute the budget to the faculties with in the university, and the faculty uses the budget to execute the activities planned at various departments.

The objective of this thesis is to show how the central government body can allocate resources in terms of the effectiveness the universities, and how the universities can utilize the allocated resources effectively to maximize their benefits at their faculty levels. Here we take into account, the fact that decision on the usage of the budget at the sublevels can not be controlled (but it could be predictable) by the higher level, if the budget once allocated.

Thus, one of the aims of our model is to provide rationality in decision making for the higher level decisions by the lower level decision maker. This is done by sub-models which predict how sublevels will react when they are given various amount of resources.

Traditionally the federal government allocates a budget to universities by considering the following main factors:

- i.* the financial capacity of the country
- ii.* the previous year budget approved to the universities
- iii.* the budget request of universities for the coming budget year
- iv.* the efficiency of the universities
- v.* current market price (standard rate) of the materials
- vi.* taking into consideration of the governmental policy.

Similarly the decision maker of the university distribute a budget to the faculties by taking mainly the following points into consideration

- a)* the budget allocated to the university from the federal government
- b)* the previous year budget approved to the faculties
- c)* the budget request of the faculties for the coming budget year
- d)* the efficiency of the faculties
- e)* current market price (standard rate) of the materials
- f)* governmental policy for higher learning and research
- g)* priority of the university.

4.1.1 Existing Procedure

The existing budget allocation process is performed in the following fashion. The federal government decides the amount of money to be allocated to the universities by forming a budget ceiling, which is the maximum amount of money that the university could request for the given budget year. This ceiling is decided by the decision maker at the federal government level mainly based on the financial capacity of the country, priority given to higher learning institution by the policy and the previous year approved budget to the universities.

After a budget ceiling to each university is decided, the office of the MOFED send a letter to each university that contains the budget ceiling which will be allocated to the given university. In the mean time, the federal government ask the universities to send their budget request which is based on their ceiling given to them. Then by receiving this ceiling from the central government, the decision maker at the university level decides its own budget ceiling which will be given to the faculties with in the university, it is based on the ceiling of the university received from the federal government and the previous year approved budget of the faculties. Then this ceiling will be given to the faculties and asks the faculties to send their budget request.

The decision maker of the faculty will perform their annual action plan of the budget year by considering the last year budget request without taking full consideration of the ceiling given to the faculty. So, they perform their budget request, it is almost the same as the previous year budget request with some percent increment from the last year budget request. Because of this, most of the time there is a big difference between this budget request and the ceiling given to the faculty. Then by receiving a budget request from the faculties,

the decision maker of the university perform his/her budget request which contains the budget request of the faculties with small modification. This modification is performed to reduce the budget request of the faculties which has too large difference from the ceiling of the faculty, but this modification do not satisfy the ceiling of the federal government.

Finally, by receiving the budget request from the universities the decision maker of the federal government approves the budget which will be allocated to the universities by taking into consideration of the budget request of the universities and the ceiling which is given to the universities. But, most of the time, this approved budget could be too small to satisfy the request of the universities. Similarly, by receiving the approved budget, the decision maker at the university level decides on the amount of the budget which will actually be allocated to the faculties. This budget allocated to the faculties could not satisfy the request of the faculties.

The following are some of the drawback of this procedure

1. The central government (office of the MOFED) determine the ceiling of a budget that will be allocated to the universities in Ethiopia using only the last year approved budget without considering the action plan of the universities. Similarly the universities determine the ceilings to their faculties without considering the action plan of the faculties, simply they use the last year approved budget of the faculties.
2. Mostly the faculty as well as the university will prepare their action plan with out considering the ceiling given from the higher level decision makers. Instead they prepare their budget request of the previous year with some percent increment and it results the amount of money which is far more than the ceiling.

3. The decision makers found in the university and central government know, why the levels below it will request that amount of money which is higher than the ceiling. Traditionally these decision makers assume that this budget request is exaggerated, so they offer a budget which is less than the budget request with out considering the action plan of the faculties and the universities.
4. The decision makers at each level decides randomly without considering the decisions of the decision maker above it and without forecasting the decisions of the decision maker below it.
5. Some times there is a budget transfer from one faculty to the other and there is also a budget return at the end of the budget year in some faculties and in some universities. This shows us there is no appropriate budget allocation procedure.
6. The decision maker at the central government enforces the decision maker at university level to satisfy the central government objective without satisfying the need of the universities. Similarly the decision maker at university level enforces the decision maker at university level.

These drawbacks have different impacts on the improvements of the universities and the faculties, and it is difficult to know the efficiency of the universities and faculties, since this procedure prevents them to have new and well designed action plan at each budget year. Budget approved above the need of each level has got a bad consequence on the efficiency of the budget usage.

Now to remove such type of problems we will model this decision process using multi-level linear programming method. We assume that a federal

government has a total budget of T units to be distributed among n different universities in the country. Each university i has m_i faculties and in each faculty j of the university there exists p_{ij} various activities (teaching and research) which can be undertaken in the given budget year. Each activity has a cost and a value (output) to the faculties and the universities. The value may be expressed in monetary terms or in terms of some numerical scale (may be in terms of a ratio against the satisfaction of the government policy or any other terms) which serves to reflect the desirability of the activities. The aim of the decision maker of the university is to maximize the value of all the activities carried out in its particular faculties.

The above problem is a three level problem and resource allocations can usually be modeled using linear functional relationships. Similar problem was studied by Cassidy, *et. al.* [12]. However, at the lower level of the hierarchy, it is assumed that projects (activities) are chosen to be funded or not. This will be a kind of 0-1 problem at the lower level. At a department of a university, however, activities could be undertaken possibly with a lower scale but may not be omitted.

Since a certain budget is allocated to the faculties, we assume that some percentage of an activity must be carried out in each faculty.

The objective and constraints to the faculty serve to generate submodels, one for each university, which describe the above decision process. The decision makers at the university level, viewing such decisions from above, must arrive at a means of assigning a budget (inputs) to each of the university submodels; we suppose that the federal government's division of its resource to universities is to be accomplished in such a way that each university is as satisfied as possible with the budget it is given. Similarly, the decision maker of the university allocation of resources to the faculties is to be performed in

such a way that each faculty is as satisfied as possible with the budget given to it.

There is an objective function available which describe minimizing the “dissatisfaction” of the universities with their budgets. Using this function Kirby and Raike [18] hypothesize an objective function, which they term a “relative regret” function, measuring the relative disappointment of each user in a computer center as a result of the “job turnaround” times he faces. (The analysis in [18] is directed at equalizing the satisfaction of computation center users). We adopt a modified version of that “relative regret” function to our model.

4.2 Mathematical Formulation

We suppose that the federal government has a total budget of T units to be distributed among (or allocated to) n universities in the country. University i has m_i ($i = 1, 2, \dots, n$) faculties and in Faculty j there are p_{ij} ($j = 1, 2, \dots, m_i$) activities performed in the given budget year. Let P_{ijk} be the teaching and research activity to be performed in Faculty j of University i . Now we introduce variables x_{ijk} , $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m_i$, $k = 1, 2, \dots, p_{ij}$ where x_{ijk} represents the ratio of an activity (planned job) P_{ijk} to be completed in the budget year for a given budget in the Faculty j of University i . Associated to x_{ijk} there is a cost c_{ijk} , for undertaking an activity P_{ijk} and a value w_{ijk} , the benefit or output from the activity P_{ijk} .

Now we can express the objective of Faculty j of University i (we shall call it a lower level) by maximizing the total value of all activities performed in the faculty, mathematically it is given by

$$\max_{x_{ijk}} \sum_{k=1}^{p_{ij}} w_{ijk} x_{ijk} .$$

The variables x_{ijk} are required to satisfy the condition that at least a certain ratio of an activity be performed in each faculty, and at most p_{ijk} activity can be done in faculty j , so that the constraint $\sum_{k=1}^{p_{ij}} x_{ijk} \leq p_{ij}$ for each i will be satisfied and x_{ijk} must satisfy the following condition

$$x_{ijk} \geq 0, \quad (j = 1, 2, \dots, m_i, i = 1, 2, \dots, n).$$

Further, if we put F_{ij} be the budget allocated to faculty j by university i , then we have the following constraint

$$\sum_{k=1}^{p_{ij}} c_{ijk} x_{ijk} \leq F_{ij} \quad (j = 1, 2, \dots, m_i, i = 1, 2, \dots, n).$$

Thus the decision maker of Faculty j in University i has the following problem:

$$\begin{aligned} \max_{x_{ijk}} \quad & \sum_{k=1}^{p_{ij}} w_{ijk} x_{ijk} \\ \text{s.t.} \quad & \sum_{k=1}^{p_{ij}} c_{ijk} x_{ijk} \leq F_{ij} \\ & \sum_{k=1}^{p_{ij}} x_{ijk} \leq p_{ij} \\ & x_{ijk} \geq 0 \quad \text{for all } k. \end{aligned} \tag{4.2.1}$$

We will use problem (4.2.1) not only to provide a rational decision making model for the faculty, but also for predictive purpose for the decision maker of the university and the federal government. That is, the decision maker of the university will use it as a tool for predicting the response of the faculty to the possible budget allocation (i.e. for potential choices F_{ij} by the university, where F_{ij} is a budget allocated to a faculty j by the university i).

The significant part of the development of this problem is the formulation of

the universities' and the federal government's decisions. The decision maker at the university level have three objectives

- minimize each faculties' dissatisfaction with their budget
- distribute the budget effectively to the faculties in the university
- improve the capacity and the quality of the university.

To formulate each faculties' dissatisfaction with their budget mathematically we adopt the relative regret function from [18]. This function measures the difference between the value that a faculty can actually attain as a result of a given budget and the value it would attain if all activities are done (i.e. if the faculty were allowed to undertake all its planned activities in full capacity). Mathematically the difference of these values is given by

$$\sum_{k=1}^{p_{ij}} w_{ijk} - \sum_{k=1}^{p_{ij}} w_{ijk} x_{ijk}. \quad (4.2.2)$$

Here x_{ijk} is determined from the lower level problem (4.2.1) and it is expressed in terms of F_{ij} .

In order to normalize the regret of each faculty, we let the relative regret of Faculty j of University i be r_{ij} where r_{ij} is given by

$$\begin{aligned} r_{ij} &= \frac{\sum_{k=1}^{p_{ij}} w_{ijk} - \sum_{k=1}^{p_{ij}} w_{ijk} x_{ijk}}{\sum_{k=1}^{p_{ij}} w_{ijk}} \\ &= 1 - \frac{\sum_{k=1}^{p_{ij}} w_{ijk} x_{ijk}}{\sum_{k=1}^{p_{ij}} w_{ijk}} \end{aligned}$$

Then one of University i 's objective is to minimize the over all dissatisfaction of the faculties. In terms of relative regret, it is equivalent to minimizing the sum of the values of r_{ij} , $i = 1, 2, \dots, n$. In the mean time, it is expected from the universities that, the activities performed by the universities must satisfy the government policy, and the budget allocated to the faculties is determined by the coherence efficiency of the faculties to the government policy and the action plan of the faculties. That is, each university gives a weight to their faculties according to the government policy and the efficiency of the faculties. Here the efficiency of a Faculty j of University i is determined by considering the action plan of Faculty j relative to that of University i and the government policy. And then the university tries to allocate a budget to the faculties based on their priority order, it is due to their weight order. Therefore, each university gives a weight to the faculties which measures the priority of the faculties. Now let $a_{ij} \geq 0$ be a weight given to a Faculty j by University i .

Not only this, there are several activities performed in the central administration of each university to improve the capacity and the quality of the university. This shows that some part of the budget U_i is used by the central administration of University i to carry out such activities. Let University i has m_{il} activities in the given budget year. Let P_{ik} ($k = 1, 2, \dots, m_{il}$) be the activities to be carried out in the given budget year in University i and let x_{ik} represent the ratio of an activity P_{ik} to be completed in the budget year for the given budget U_i in University i . Now put c_{ik} be the cost of an activity P_{ik} and w_{ik} be the benefit (out put) obtained from activity P_{ik} . Then University i tries to maximize $\sum_{j=1}^{m_i} a_{ij}F_{ij} + \sum_{k=1}^{m_{il}} w_{ik}x_{ik}$. It is equivalent to minimizing the negative of it. That is, University i tries to minimize

$$\left(- \left(\sum_{j=1}^{m_i} a_{ij} F_{ij} + \sum_{k=1}^{p_{ij}} w_{ik} x_{ik} \right) \right).$$

Then the over all objective of the second level decision maker is formulated mathematically as follows:

$$\min_{F_{ij}, x_{ik}} \left(\sum_{j=1}^{m_i} \left(1 - \frac{\sum_{k=1}^{p_{ij}} w_{ijk} x_{ijk}}{\sum_{k=1}^{p_{ij}} w_{ijk}} \right) - \left(\sum_{j=1}^{m_i} a_{ij} F_{ij} + \sum_{k=1}^{m_{il}} w_{ik} x_{ik} \right) \right) \quad (4.2.3)$$

where x_{ijk} solves the lower level problem given by (4.2.1) and the value of x_{ijk} in (4.2.2) is the optimal values of these variables, which in turn depends on F_{ij} , that is each x_{ij} is implicitly a function of F_{ij} .

The decision to be made by the University i decision maker consist of a choice of values for F_{ij} , x_{ik} , $j = 1, 2, \dots, m_i$ and $k = 1, 2, \dots, m_{il}$. The allowable range of such choices is restricted by the amount of a budget U_i , where U_i is a budget allocated to University i by the Federal government. Thus the budget constraint for the University i is given by

$$\sum_{j=1}^{m_i} F_{ij} + \sum_{k=1}^{p_{ij}} c_{ik} x_{ik} \leq U_i, \text{ and } F_{ij}, x_{ik} \geq 0 \text{ for all } j \text{ and } k.$$

Similar to decision maker at the university level, the decision maker at the central government have three objectives

- minimize each universities' dissatisfaction with their budget
- distribute the budget effectively to the universities in the country
- improve the capacity and the quality of the higher education in the country

To formulate each universities' dissatisfaction with their budget mathematically, now again we use the relative regret function which measures the dissatisfaction of the universities. That is, this function measures the difference between the value that a university can attain as a result of a given budget and the value it would attain if given an unlimited budget to the university from the federal government. Mathematically this difference is given by:

$$\sum_{j=1}^{m_i} \sum_{k=1}^{p_{ij}} w_{ijk} + \sum_{k=1}^{m_{il}} w_{ik} - \left(\sum_{j=1}^{m_i} \sum_{k=1}^{p_{ij}} w_{ijk} x_{ijk} + \sum_{k=1}^{m_{il}} w_{ik} x_{ik} \right). \quad (4.2.4)$$

Here x_{ijk} is determined from the lower level problems which is expressed in terms of U_i .

To normalize the regret experienced by each university, we let the relative regret of University i be R_i where R_i is given by

$$\begin{aligned} R_i &= \frac{\sum_{j=1}^{m_i} \sum_{k=1}^{p_{ij}} w_{ijk} + \sum_{k=1}^{m_{il}} w_{ik} - \left(\sum_{j=1}^{m_i} \sum_{k=1}^{p_{ij}} w_{ijk} x_{ijk} + \sum_{k=1}^{m_{il}} w_{ik} x_{ik} \right)}{\sum_{j=1}^{m_i} \sum_{k=1}^{p_{ij}} w_{ijk} + \sum_{k=1}^{m_{il}} w_{ik}} \\ &= 1 - \frac{\left(\sum_{j=1}^{m_i} \sum_{k=1}^{p_{ij}} w_{ijk} x_{ijk} + \sum_{k=1}^{m_{il}} w_{ik} x_{ik} \right)}{\sum_{j=1}^{m_i} \sum_{k=1}^{p_{ij}} w_{ijk} + \sum_{k=1}^{m_{il}} w_{ik}}. \end{aligned}$$

In terms of this relative regret R_i , one of the federal government's objective is to minimize the dissatisfaction of universities, which is equivalent to minimizing the sum of the values of R_i , $i = 1, 2, \dots, n$. Mathematically it is formulated as follows:

$$\min \sum_{i=1}^n R_i = \sum_{i=1}^n \left(1 - \frac{\left(\sum_{j=1}^{m_i} \sum_{k=1}^{p_{ij}} w_{ijk} x_{ijk} + \sum_{k=1}^{m_{il}} w_{ik} x_{ik} \right)}{\sum_{j=1}^{m_i} \sum_{k=1}^{p_{ij}} w_{ijk} + \sum_{k=1}^{m_{il}} w_{ik}} \right).$$

But the Federal government allocates a resource U_i to University i based on its policy and the efficiency of the university. That is, the resources flow to the universities according to the policy of the government and the action plan proposed by universities. (Since we measure the efficiency of a university by taking into consideration of the action plan of the university.) Now again the government gives a weight $b_i \geq 0$ to University i , which measures the priority of University i given by the government due to its policy and its efficiency. Thus the Federal government wishes to allocate a resource to the universities such that the goal of its policy is attained. That is the Federal government allocate a resource to the universities in the country according to their priority order (their weight).

Similar to the universities, there are several activities performed by federal government to improve the capacity and the quality of the higher education in the country. This shows that some part of the budget T is used by federal government to carry out such activities. Let the Federal government has r activities in the given budget year. Let P_k ($k = 1, 2, \dots, r$) be the activities to be carried out in the given budget year by the federal government and let x_k represent the ratio of an activity P_k to be completed in the budget year for the given budget T . Now put c_k to be the cost of an activity P_k and w_k be the benefit (out put) obtained from activity P_k . Then the federal government tries to maximize $\sum_{j=1}^{m_i} b_i U_i + \sum_{k=1}^r w_k x_k$. It is equivalent to minimizing the

negative of it. That is, the federal government tries to minimize $(-\sum_{i=1}^n b_i U_i + \sum_{k=1}^r w_k x_k)$.

Thus the objective function of the federal government can be formulated mathematically as follows:

$$\min_{U_i, x_k} \left(\sum_{i=1}^n \left(1 - \frac{\sum_{j=1}^{m_i} \sum_{k=1}^{p_{ij}} w_{ijk} x_{ijk} + \sum_{k=1}^{m_{il}} w_{ik} x_{ik}}{\sum_{j=1}^{m_i} \sum_{k=1}^{p_{ij}} w_{ijk} + \sum_{k=1}^{m_{il}} w_{ik}} \right) - \left(\sum_{j=1}^{m_i} b_i U_i + \sum_{k=1}^r w_k x_k \right) \right)$$

where x_{ijk} is determined from the lower level problem and F_{ij} and x_{ik} are determined from the second level problem.

However, the amount of a budget U_i allocated to University i is restricted to the amount of a budget T in Federal government hand and the choice of x_k . That is, we have the following budget constraint for the federal government, which is given by:

$$\sum_{i=1}^n U_i + \sum_{k=1}^r c_k x_k \leq T, \quad U_i \geq 0, \quad \text{and } x_k \geq 0 \text{ for all } i \text{ and } k.$$

The above problem formulation can be written as a nested optimization (a three level programming problem) as follows:

$$\min_{U_i, x_k} \left(\sum_{i=1}^n \left(1 - \frac{\sum_{j=1}^{m_i} \sum_{k=1}^{p_{ij}} w_{ijk} x_{ijk} + \sum_{k=1}^{m_{il}} w_{ik} x_{ik}}{\sum_{j=1}^{m_i} \sum_{k=1}^{p_{ij}} w_{ijk} + \sum_{k=1}^{m_{il}} w_{ik}} \right) - \left(\sum_{j=1}^{m_i} b_i U_i + \sum_{k=1}^r w_k x_k \right) \right)$$

where F_{ij} and x_{ik} solve

$$\min_{F_{ij}, x_{ik}} \left(\sum_{j=1}^{m_i} \left(1 - \frac{\sum_{k=1}^{p_{ij}} w_{ijk} x_{ijk}}{\sum_{k=1}^{p_{ij}} w_{ijk}} \right) - \left(\sum_{j=1}^{m_i} a_{ij} F_{ij} + \sum_{k=1}^{m_{il}} w_{ik} x_{ik} \right) \right)$$

where x_{ijk} solves

$$\begin{aligned} \max_{x_{ijk}} & \sum_{k=1}^{p_{ij}} w_{ijk} x_{ijk} \\ \text{s.t.} & \sum_{k=1}^{p_{ij}} c_{ijk} x_{ijk} \leq F_{ij} \\ & \sum_{j=1}^{m_i} F_{ij} + \sum_{k=1}^{p_{ij}} c_{ik} x_{ik} \leq U_i \\ & \sum_{i=1}^n U_i + \sum_{k=1}^r c_k x_k \leq T \\ & x_k \geq 0, \quad k = 1, 2, \dots, r \\ & x_{ik} \geq 0, \quad i = 1, 2, \dots, n, \quad k = 1, 2, \dots, m_{il} \\ & U_i \geq 0, \quad i = 1, 2, \dots, n \\ & F_{ij} \geq 0, \quad i = 1, 2, \dots, n, \quad j = 1, 2, \dots, m_i \\ & x_{ijk} \geq 0, \quad i = 1, 2, \dots, n, \quad j = 1, 2, \dots, m_i, \\ & \quad \quad \quad k = 1, 2, \dots, p_{ij}. \end{aligned} \tag{4.2.5}$$

As we can see problem (4.2.5) is a linear three level programming problem. And it is possible to find the optimal solution of problem (4.2.5) by using the Simplex-cutting plane algorithm.

Chapter 5

Summery and Conclusion

Multilevel mathematical programming problems, if carefully defined, can serve as useful tools in modelling decision processes in hierarchical organizations. Such models can predict the inefficiencies of optimal decisions and identify the seats of true control within hierarchical organizations.

This thesis has presented an approach to model and solve certain multilevel decision making problems. The proposed formulation partitions control over decision variables among ordered levels with in the hierarchical planning structure. The planner at one level attempts to maximize his/her individual objective function which may depend, in part, on variables controlled at other levels of the hierarchy. His/her control instrument (variable) may allow him/her to influence, rather than dictate the decision strategy (policy) of other decision makers at other levels in the hierarchy, and thereby improve his/her own objective function. The decisive execution of the decisions is sequential, from highest level to lowest level.

A model for determining how a central government can most efficiently allocate resources among other levels of the government is presented in this thesis. As a particular example, we focussed on the resource allocation to

the universities from the federal government, and its implementation at the faculties level, which makes a three-level decision making process. The model explicitly includes the fact that lower levels of the government can make their independent decisions once they have been given resources from the federal government. A key feature of this model is the mathematical formulation of the central government's objective of distributing resources effectively, while at the same time being as fair as possible to all lower levels, those receiving this resource allocations.

The discussion in this thesis has mainly focussed on the three-level linear programming problem. Although this problem is non-convex programming problem, the feasible region of this problem posses properties which make the solution to this problem acceptable. For this, Theorem 3.3.6 illustrates a key property of the feasible regions to the higher levels and Corollary 3.3.7 shows that the global solution to a three-level linear programming problem exists on the extreme points of the feasible regions.

In this thesis first order necessary optimality condition is established. Theorem 3.4.2 gives us this necessary conditions for the solutions of three-level linear programming problem. This optimality condition is constructed by using the Kuhn-Tucker conditions of the lower level decision makers problem as a constraints for the higher level, and using the idea of penalty function approach to remove the complementarity conditions. Using this optimality condition a simplex cutting plane algorithm is also constructed. This algorithm mainly relies on the fact that the solutions to a three-level linear programming problem exists on the vertices of the feasible region.

We have reviewed the development of the two level and three level programming problems and their characteristics, solution approaches and applications. Several algorithms have been developed that can find an optimal solu-

tion for the linear two-level and some algorithms such as hybrid and simplex cutting plane algorithms are developed for three-level linear programming problem. However, the computational efficiency of these algorithms does not consistently perform well, owing to the the complexity of the problem. There is no claim that any algorithm procedure will be the best for all two-level and three-level linear programming problems. It will be helpful in the future to develop a more efficient algorithm for solving both two-level and three-level linear programming problems and extend it as well to the general n -level programming problem, for which the geometry of the problem is more complex than that suggested by the results of section 3.3.

Another interesting feature of multilevel programming problem also requires further study: is that their solution may not be pareto-optimal. That is, there may exist feasible decisions where some levels can increase their objective function with out decreasing the objective function of any level. Therefore, it is worth while to investigate the conditions of existence pareto-optimal like solutions and to analyze the relationship between multi-level and multi-criteria programming problem.

Most two-level and three-level programming problems are formulated in such a way that there is only a single decision maker at each level. In fact, the model involving multi-units in the lower level is very common to the real world. Bard [5] initiates this subject of the multi-divisional organization through two levels of management and Anandalingam [2] also introduces the Stackelberg equilibrium solution concept to two levels hierarchical system. It can be further divided into two cases depending on whether the cooperation among units in the same level is allowed or not.

Finally and probably the most important area for further research would involve real applications with real decision makers. The particular structure

of multilevel programming problems facilitates the formulation of a number of practical problems that involve a hierarchical decision making process. Among the several application areas which need multilevel programming problems, the following are remarkable examples:

- **Transportation**- Network design problem and trip demand estimation problem.
- **Management**- Coordination of multi-divisional firms, network facility location delivered price competition and credit allocation.
- **Planning**- Application of agricultural policies and electric utility planning.
- **Engineering Design**- Optimal design problems, and soon.

We have seen in section 4.1.1 that the current procedure that the decision makers use to decide their decision has different drawbacks. To overcome these problems, to improve the effectiveness of their decision and to improve the efficiency of their work it is recommended that the decision makers have to use the following:

- ★ The decision makers have to decide independently. That is, once the budget is assigned to the decision makers they have to use their budget without any influence of the decision makers at the other levels.
- ★ The decision makers have to decide sequentially.
- ★ The decision makers of the central government has to make their decision by taking into full consideration of the action plane of the universities and the rational reaction of the universities for the given budget

by forecasting it from starting from the action plan and the last year reaction of the universities.

- ★ The decision makers at the university level have to decide their decision by using the decisions of the central government and the action plane of their faculties. They have to use the rational reaction of the faculties.
- ★ The decision makers at the lower levels have to react rationally for any decision of higher level decision makers above their level.

Therefore the decision makers have to use a scientific method to make their decision and to forecast the rational reaction of the decision makers at the levels below them. That is, they have to use three level programming model.

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Declaration

This thesis is my original work and has not been presented for a degree in any other university and that all sources of information used for this thesis have been fully acknowledged.

Esubalew Lakie

Signature

This thesis is submitted for Examination with my approval as a university advisor.

Advisor

Signature

Dr. Semu Mitiku
