SCHOOL OF GRADUATE STUDIES
DEPARTMENT OF MATHEMATICS

GRADUATE PROJECT REPORT ON

EBERLEIN-ŠMULIAN THEOREM

SUBMITED IN PARTIAL FULFILMENT OF THE REQUIREMENT FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

Prepared by:

Eshetu Demessie
Advisor: Mengistu Goa(Ph.D)

Feb 2013
Addis Ababa, Ethiopia

Contents

Permission

This is to certify that this project is compiled by Eshetu Demessie in the Department of Mathematics Addis Ababa University under my supervision. I here by also confirm that the project can be submitted for evaluation by examiner and eventual defence

Addis Ababa University
Feb 2013

Dr. Megistu Goa

Acknowledgement

First, I would like to express my deepest gratitude to **God** for giving me patience; I am also grateful to my advisor Dr. Mengistu Goa for his helpful discussion, comments and providing the necessary materials in the preparation of this paper.
Secondly I would like to extend my thank to my families and all who encouraged me to complete my project and their heart full help while writing the paper. Lastly I would like thanks to Department of mathematics, Addis Ababa University for giving the necessary materials throughout the preparation of this paper.

Abstract

A subset A of a Banach space X is called weakly sequentially compact if every sequence in A has a weak cluster point in X. The difficulty implication of the Eberlein-Šmulian theorem states that such a set is already relatively weakly compact. This implication was proved by W. Eberlein .

Introduction

A subset of a Banach space is relatively, weakly compact if and only if it is relatively, weakly, sequentially compact.
In the first chapter we deal with introducing a Banach spaces and compactness in general normed linear spaces.
We start by basic notations, definitions and F. Riesz lemma is noted. After this a theorem which states that all n-dimensional normed linear spaces are isomorphic. From this we conclude that, in order for each bounded sequence in the normed linear space X to have a norm convergent subsequence, it is necessary and sufficient that X be finite dimensional. Finally, we shown that any norm-compact subset K of a normed linear space is contained in the closed convex hull of some null sequence.

In the second chapter we deal with the two weaker-than-norm topologies of greatest importance in Banach space theory are the weak topology and the weak-star (or weak*) topology. The weak topology is present in every normed linear space and the weak* topology is present only in dual spaces.
We start by defining the weak topology a normed linear space X and the weak convergence of a net in X. From this we shown that, a linear map $T : X \to Y$ between the normed linear spaces X and Y is norm-to-norm continuous if and only if T is weak-to-weak continuous.
After this we define the weak* topology of a normed linear space and the weak* convergence of a sequence in X. From this we shown that, for any normed linear space X, B_{X^*} is weak* compact (Alaoglu's Theorem)

Finally we shown that, B_X is weakly compact if and only if X is **reflexive space**.

In the last chapter we deal with how does a subset K of a Banach space X to be compact. After this, some definitions and lemmas are noted. Finally we state and proof the Eberlein-$\check{S}$mulian theorem.

Chapter 1

INTRODUCTION TO BANACH SPACES

In this chapter we deal with compactness in general normed linear spaces. The aim is to convey the notion that in normed linear spaces, norm-compact sets are small -both algebraically and topologically.

1.1 Basic Definitions

Definition 1.1.1. *Let X be a vector space over the field* $\mathbb{K}$ *.Then the mapping* $\|.\|$ *:*$X \to [0, \infty)$ *is said to be a* ***norm*** *on X if and only if every x,y* $\in X$ *and* $\lambda \in \mathbb{K}$ *satisfy the following conditions*
(1) $\|x\| \geq 0$ *and* $\|x\| = 0 \Leftrightarrow x = 0$
(2) $\|\lambda x\| = |\lambda| \|x\|$
(3) $\|x + y\| \leq \|x\| + \|y\|$*(triangular inequality).*
The pair $(X, \|.\|)$ *is called the normed space .*

Each normed space X can be considered as a **metric** space by definition of the canonical metric in the following way .

$$d(x, y) = \|x - y\| \; x, y \in X$$

But a metric space may have no algebraic (*vector*) structure; i.e. it may not be a vector space.

Example 1.1.1. *Let* $X = \{a, b\}$ *or any other finite set*
Define $d : X \times X \to \Re_+$ *as follows .*

$$d(x,y) = \begin{cases} 0 & \textit{if } x = y \\ 1 & \textit{otherwise} \end{cases}$$

Then d *is a metric on* X*. This is an example of a metric space that is not a normed vector space because there is no way to define vector addition or scalar multiplication for a finite set.*

Definition 1.1.2. *Let* X *be a normed space. A vector* $f \in X$ *is a limit point of a set* $S \subseteq X$ *if there exist vectors* $g_n \in S$ *that converges to* f*.*
A subset S *of a normed space* X *is* ***closed*** *if it contains all of it's limit points.*

Theorem 1.1.1. *(Minkowski-inequality for sequences)*
Let $1 < p < \infty$ *and let x,y* $\in \ell^p$*. Then*

$$\|x + y\|_p = \|x\|_p + \|y\|_p$$

Example 1.1.2. *Let* $X = C[a, b]$ *be the space consisting of all continuous real valued functions defined on the bounded closed interval [a,b] with:*

$$\|x\|_2 = (\int_a^b |x(t)|^2 dt)^{\frac{1}{2}}$$

is a normed space.

Solution: (1) $\|x\|_2 = 0$

$$\Rightarrow (\int_a^b |x(t)|^2 dt)^{\frac{1}{2}} = 0$$
$$\Rightarrow x(t) = 0, \text{for all} \quad t \in [a, b]$$
$$\Rightarrow x = 0$$

(2) Let x,y$\in \ell^2$

$\|x+y\|_2 = (\int_a^b |x(t)+y(t)|^2 dt)^{\frac{1}{2}} \leq \|x\|_2 + \|y\|_2$ (by Minkowski inequality)

$$
\begin{aligned}
(3) \qquad \|\alpha x\|_2 &= (\int_a^b |\alpha x(t)|^2 dt)^{\frac{1}{2}} \\
&= (\int_a^b |\alpha|^2 |x(t)|^2 dt)^{\frac{1}{2}} \\
&= |\alpha| (\int_a^b |x(t)|^2 dt)^{\frac{1}{2}} \\
&= |\alpha| \|x\|_2
\end{aligned}
$$

Notations

B_X is the closed unit ball of X defined by

$$B_X = \{x \in X : \|x\| \leq 1\}$$

S_X is the closed unit sphere of X defined by

$$S_X = \{x \in X : \|x\| = 1\}$$

For a fixed X, the continuous dual is denoted by X^* and a typical member of X^* might be called x^*.

c_o = all sequences of real numbers converging to zero with the norm defined by

$$\|x\| = \sup_n |\zeta_n|$$

where $x = \{\zeta_n\} \in c_o$

ℓ_1 = all absolutely summable sequences, i.e satisfying $\sum_n |a_n| < \infty$ with the norm given by

$$\|x\| = \sum_{n=1}^{\infty} |a_n|$$

where $x = \{a_n\} \in \ell_1$.

Definition 1.1.3. *Let X and Y be linear spaces over the same field $\mathbb{K}$.*
*A mapping $T : X \to Y$ is called a **linear operator** if and only if*
$T(x+y) = Tx + Ty, \forall x, y \in X$(Additive) and
$T(\alpha x) = \alpha Tx$ for all $\alpha \in \mathbb{K}$ and for all $x \in X$ (Homogeneous)

Definition 1.1.4. *Let $A : X \to Y$ be a linear operator. Then A is said to be **bounded** if and only if there is a $\lambda \geq 0$ such that*
$\|Ax\| \leq \lambda \|x\|$ for all $x \in X$.

Definition 1.1.5. *Let X be a normed space and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of elements of X.*
(a) $\{f_{n\}_{n\in\mathbb{N}}}$ converges to $f \in X$ if for all $\epsilon > 0, \exists N > 0$,all $n \geq N$

$$\|f - f_n\| < \epsilon$$

In this case we write $f_n \to f$ or

$$\lim_{n\to\infty} f_n = f$$

*(b) $\{f_n\}_{n\in\mathbb{N}}$ is a **Cauchy sequence** if for all $\epsilon > 0$,$\exists N > 0$*

$$\|f_n - f_m\| < \epsilon$$

for all $m, n \geq N$

Definition 1.1.6. *A normed linear space X which does have the property that all cauchy sequences are convergent is said to be **complete***
*Acomplete normed linear space is called a **Banach space**.*

Example 1.1.3. $C[0,1]$ *. This is the space consisting of all continuous real valued functions on the closed unit interval $[0,1]$ with*

$$\|x\|_\infty = \sup_{0\leq t\leq 1} |x(t)| = \max_{0\leq t\leq 1} |x(t)|$$

is a Banach space.

Definition 1.1.7. *A space X can be embedded in to the double dual* X^{**} *by* $x \to T_x$*, where* $T_x(\phi) = \phi(x)$*.*
Thus $T : X \to X^{**}$ *is an injective linear mapping, though not necessarily surjective (spaces for which this canonical embedding is surjective are called* ***reflexive space****).*

Theorem 1.1.2. *(Rank-Nullity Theorem)*
Let X and Y be vector spaces over a field $\mathbb{K}$*; and*
let $T : X \to Y$ *be a linear transformation. Assuming the dimension of X is finite. Then*

$$dim(X) = dim(Ker(T)) + dim(Im(T)),$$

where dim(X) is the dimension of X; Ker is the Kernel, and Im the image. Note that dim(Ker(T)) is called the nullity of T and dim(Im(T)) is called the rank of T.

Example 1.1.4. *Finite dimensional normed spaces are reflexive spaces, because in this case the space, its dual and bidual all have the same linear dimension; hence the linear injection j from the definition is bijective; by the Rank-Nullity theorem. But*
The Banach space c_o *of scalar sequences tending to zero at infinity, equipped with the supremum norm, is not a reflexive space. It follows from the general properties bellow that* ℓ^1 *and* ℓ^∞ *are not reflexive spaces; because* ℓ^1 *is isomorphic to the dual of* c_o*, and* ℓ^∞ *is isomorphic to the dual of* ℓ^1*.*

1.2 Riesz Lemma

Lemma 1.2.1. *Let Y be a proper closed linear subspace of the normed linear space X and* $0 < \theta < 1$*. Then there is an* $x_\theta \in S_X$*, for which* $\|x_\theta - y\| > \theta$ *,for every* $y \in Y$

Proof. Pick any $x \in X \setminus Y$. Since Y is closed the distance from x to Y is positive i.e. $0 < d = inf\{\|x - z\| : z \in Y\} < \frac{d}{\theta}$

Therefore there is a $z \in Y$ such that $\|x - z\| < \frac{d}{\theta}$
Let $x_\theta = \frac{x-z}{\|x-z\|}$. Furthermore, if $y \in Y$, then

$$\begin{aligned}
\|x_\theta - y\| &= \|\frac{x-z}{\|x-z\|} - y\| \\
&= \|\frac{x}{\|x-z\|} - \frac{z}{\|x-z\|} - \frac{\|x-z\|y}{\|x-z\|}\| \\
&= \frac{1}{\|x-z\|}\|x - \underbrace{(z + \|x-z\|y)\|} \\
&= \frac{1}{\|x-z\|}\|x - p\| > \frac{\theta}{d}d = \theta,
\end{aligned}$$

where p $\in Y$and p $= z + \|x - z\|y.$

Therefore $\quad \|x_\theta - y\| > \theta \quad$ for all y $\in Y$

□

Definition 1.2.1. *A collection C of subsets of a space X is said to* ***cover*** *X if the union of elements of C is equal to X. It is an open covering of X if its elements are open subsets of X.*

Definition 1.2.2. *A subset A of a topological space X is* ***compact*** *if every open cover of A contains a finite subcover. Precisely, if $A \subseteq \cup_\alpha u_\alpha$ for some collection of open sets u_α, then*

$$A \subseteq \bigcup_{k=1}^{n} u_{\alpha_k}$$

for some sub collection u_{α_k}

Example 1.2.1. *Let $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. Then X is compact.*
Solution.
Let e be an open covering of X.
Since 0 is a limit point of $\{\frac{1}{n} : n \in \mathbb{N}\}$ every neighborhood U of 0 contains almost all elements of $\{\frac{1}{n}, n \in \mathbb{N}\}$ since

$$0 \in X \subseteq \bigcup_{U \in e} U$$

,$\exists v \in e$ such that $0 \in v$

Let $N \in \mathbb{N}$ such that $\frac{1}{N} \in v$

Let $U_1 \in e$ such that $1 \in U_1$

Let $U_2 \in e$ such that $\frac{1}{2} \in U_2$

.

.

.

Let $U_{N-1} \in e$ such that $\frac{1}{N-1} \in U_{N-1}$

Consider $\{U_1, U_2, ..., U_{N-1}, v\} \subseteq e$. Then

$$X \subseteq (\bigcup_{i=1}^{N-1} U_i) \bigcup v$$

Therefore X is compact

Example 1.2.2. $(0, 1)$ *is not compact*

Solution.

Let $e = \{(\frac{1}{n}, 1) : n \in \mathbb{N}\}$ then e is an open covering of $(0, 1)$

Let $\{(\frac{1}{n_1}, 1), (\frac{1}{n_2}, 1), \ldots, (\frac{1}{n_k}, 1)\}$ be a finite sub collection.

Let $N = max(n_1, n_2, \ldots, n_k)$

$\Rightarrow (\frac{1}{N}, 1) \subseteq (0, 1)$ *and* $(\frac{1}{N}, 1) \neq (0, 1)$. *Then e is an open covering of $(0, 1)$ which admits no finite sub cover*

Lemma 1.2.2. *Let X be a normed space and let $x_1, x_2, \ldots, x_n$ be linearly independent vectors of X. Then there is a $\mu > 0$ such that*
$|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n| \leq \mu \|\alpha_1 x_1 + \cdots + \alpha_n x_n\|$ *for all* $\alpha_1, \alpha_2, \ldots, \alpha_n \in K$

Proof. **Case 1**.

If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, then the lemma is rue

Case 2.

Let $|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n| = 1$

Define the linear mapping

$T : \ell^1(n) \to X$ by $T(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$

The mapping T is continuous .

Let $M = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) : |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n| = 1\}$
Then M is compact subset of $\ell^1(n)$.
Then let $T(M)$ is also compact .
Consider the function $f(x) = \|x\|, x \in X$. f is continuous and therefore f has a minimum $r \geq 0$ on $T(M)$, i.e. $r \leq \|\alpha_1 x_1 + \cdots + \alpha_n x_n\|$ for all $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in M$. Since $\alpha_i = 1$ for all $i = 1, 2, \ldots, n$ and $x_i^{'}s$ are a basis for X . $r \neq 0$
Let $\mu = \frac{1}{r}$.
Then we have, $1 = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n| \leq \mu\|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n\|$

Case 3.
Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be scalars not all zero.
Define:

$$\beta_i = \frac{\alpha_i}{\sum_{i=1}^{n} |\alpha_i|}, i = 1, 2, \ldots, n. \sum_{i=1}^{n} |\beta_i| = \sum_{i=1}^{n} \frac{|\alpha_i|}{\sum_{i=1}^{n} |\alpha_i|} = \sum_{i=1}^{n} |\frac{\alpha_i}{\alpha_i}| = 1$$

Therefore

$$\sum_{i=1}^{n} |\beta_i| = 1.$$

Then by case 2 it follows that
$|\beta_1| + |\beta_2| + \cdots + |\beta_n| \leq \mu\|\beta_1 x_1 + \cdots + \beta_n x_n\|$
Hence $|\alpha_1| + \cdots + |\alpha_n| \leq \mu\|\alpha_1 x_1 + \cdots + \alpha_n x_n\|$ □

Definition 1.2.3. *A vector space X is said to be* ***finite dimensional*** *if there is a positive integer n such that X contains a linearly independent set of n vectors, where as any set of $n+1$ or more vectors of X is linearly dependent. n is called the dimension of X, written $n = dimX$. If $X = \{0\}$, we define $dimX = 0$. If X is does't finite dimensional it is said to be infinite dimensional.*

Example 1.2.3. *$C[a, b]$ is infinite dimensional where as $\Re^n$ and C^n are n dimensional.*

Definition 1.2.4. *A subset V of a linear vector space X is called a* ***Hamel basis*** *of X if for every vector $x \in X$ can be uniquely expressed as a finite linear combination of some elements of V*

Lemma 1.2.3. *In a finite dimensional normed space X. The norm convergence is equivalent to the componentwise convergence.*

Proof. Let $x_1, x_2, \ldots, x_n$ be a basis of X .
Assume that $\alpha_i^k \to \alpha_i$ for all $i \in \{1, 2, \ldots, n\}$.
By the continuity of vector addition and scalar multiplication we get

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} \sum_{i=1}^{n} \alpha_i^k x_i = \sum_{i=1}^{n} (\lim_{k\to\infty} \alpha_i^{(k)}) x_i = \sum_{i=1}^{n} \alpha_i x_i = y$$

Now let $y_k \to y$ (in the sense of norm) we get

$$\sum_{i=1}^{n} |\alpha_i^k - \alpha_i| \leq \mu \|y_k - y\|$$

Where

$$y_K = \sum_{i=1}^{n} \alpha_i^{(k)} x_i$$

and

$$y = \sum_{i=1}^{n} \alpha_i x_i$$

. Since

$$\lim_{k\to\infty} \|y_k - y\| = 0$$

$\alpha_i^k \to \alpha_i \quad (k \to \infty)$ for all $i \in \{1, 2, \ldots, n\}$. But this is the component wise convergence □

Definition 1.2.5. *Let X and Y be vector spaces over the same field $\mathbb{K}$.*
A bijective linear operator $T : X \to Y$ is called an ***isomorphism*** *of X onto Y. We also say that X and Y are isomorphic*

Theorem 1.2.1. *If X and Y are finite dimensional normed linear spaces of the same dimension, then they are isomorphic.*

Proof. We show that if X has dimension n, then X is isomorphic to ℓ_1^n.
Recall that the norm of an n-tuple

$$(a_1, a_2, \dots, a_n)$$

in ℓ_1^n is given by

$$\|(a_1, a_2, \dots, a_n)\| = |a_1| + |a_2| + \cdots + |a_n|$$

Let $x_1, x_2, \dots, x_n$ be a Hamel basis for X. Define the linear map $I : \ell_1^n \to X$ by $I((a_1, a_2, \dots, a_n)) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$
I is a linear space isomorphism of ℓ_1^n onto X.
Moreover for each $a_1, a_2, \dots, a_n$ in ℓ_1^n

$$\|a_1x_1 + a_2x_2 + \cdots + a_nx_n\| \leq (\max_{1\leq i\leq n} \|x_i\|)(|a_1| + |a_2| + \cdots + |a_n|)$$

(by triangular inequality). Therefore I is a bounded linear operator. Since T is a bounded linear operator and X is a Banach space.
Then the open mapping theorem would come immediately. Letting us conclude that I is an open map and therefore an isomorphism. We did not know this though ; so we continue. To prove i^{-1} is continuous ,we need only show that I is bounded bellow by some $m > 0$ on the closed unit sphere $S\ell_1^n$ of ℓ_1^n. We define the function $f : S_{\ell_1(n)} \to \Re$ by $f((a_1, a_2, \dots, a_n)) = \|a_1x_1 + a_2x_2 + \cdots + a_nx_n\|$. The axioms of norm quickly show that f is on the compact subset $S_{\ell_1(n)}$ of $\Re^n$. Therefore f attains a maximum value $m \geq o$ at some $(a_1^0, a_2^0, \dots, a_n^0)$ in $S_{\ell_1(n)}$. Let us assume that $m = 0$.
The $\|a_1^0x_1 + a_2^0x_2 + \cdots + a_n^0x_n\| = 0$, so that $a_1^0x_1 + a_2^0x_2 + \cdots + a_n^0x_n = 0$. Since $x_1, x_2, \dots, x_n$ constitute a Hamel basis for X, the only way this can happen is for $a_1^0 = a_2^0 = \cdots = a_n^0 = 0$,a hard task for any $a_1^0, a_2^0, \dots, a_n^0 \in S_{\ell_1(n)}$ □

Definition 1.2.6. *We know that two norms $\|.\|_1, \|.\|_2$ are said to be **equivalent** if and only if there are two positives r_1, r_2*
such that $r_1 \leq \frac{\|x\|_1}{\|x\|_2} \leq r_2$ for all $x \in X, x \neq 0$, ... (1)

Proposition 1.2.1. *All norms on a finite dimensional space X are equivalent*

Proof. Let $\|.\|_1, \|.\|_2$ be two norms on X. Then by theorem 1.2.1 we have that

$$\|x_n\|_1 \to \|x\|_1 \Leftrightarrow \|x_n\|_2 \to \|x\|_2 \text{.............................}(2)$$

Now let the two normed spaces $X_1 := (X, \|.\|_1), X_2 := (X, \|.\|_2)$ be given. Then we can consider the mapping I_x as a mapping from x_1 to x_2 or from x_2 to x_1 from (2) we get $\|I_x x_n\| \to \|I_x x\|$, that means I_x is in both cases a continuous function. Then there are two constants μ_1 and μ_2 such that
$\|x\|_1 = \|I_x x\|_1 \leq \mu_1 \|x\|_2$ and $\|x\|_2 = \|I_x x\|_2 \leq \mu_2 \|x\|_1$ .... (3)
Therefore

$$\frac{1}{\mu_1}\|x\|_2 \leq \|x\|_1 \leq \mu_2 \|x\|_2$$

From this (1) follows □

Corollary 1.2.1. *Finite dimensional linear spaces are complete*

Proof. Let (y_k) be a Cauchy sequence in X, where

$$y_k = \sum_{i=1}^{n} \alpha_i^k x_i$$

then by lemma 1.2.2, we have

$$\sum_{1}^{n} |\alpha_i^k - \alpha_i^j| \leq \mu \|y_k - y_j\| \to 0 \quad (k, j \to \infty)$$

for all $i = 1, 2, \ldots, n$ that means $|\alpha_i^{(k)} - \alpha_i^j| \to 0$ for all $i \in \{1, 2, \ldots, n\}$ then the sequence $(\alpha_i^{(k)})$ is Cauchy sequences in $\mathbb{K}$, for each $i \in \{1, 2, \ldots, n\}$. Since $\mathbb{K}$ is complete, we have $\alpha_i^{(k)} \to \alpha_i$. Then by lemma 1.2.3 we have

$$y_k \to y = \sum_{1}^{n} \alpha_i x_i \in X$$

So we have that X is complete. □

Corollary 1.2.2. . *If Y is a finite dimensional linear subspace of the normed linear space X, then Y is a closed subspace of X.*

Proof. Suppose Y is a finite dimensional subspace of a normed space X. Then Y is complete. By corollary 1.2.1, therefore Y is closed. □

Definition 1.2.7. *Let A be a set and $A \neq \emptyset$, we say that A is* ***finite*** *if $A \sim \mathbb{N}_k$ for some k, where $\mathbb{N}_k = \{1, 2, 3, \ldots, k\}$. And A is* ***countable*** *if A is finite or $A \sim \mathbb{N}$.*

Theorem 1.2.2. *In order for each closed bounded subset of the normed linear space X to be compact, it is necessary and sufficient that X be finite dimensional*

Proof. Should the dimension of X be n, then X is isomorphic to ℓ_2^n
(by theorem 1.2.1). Therefore the compactness of closed bounded subset of X follow from the classical Heine-Borel theorem.
Should X be infinite dimensional, then S_X is not compact, though it is closed and bounded. In fact we show that there is a sequence x_n in S_X such that for any distinct m and n, $\|x_n - x_m\| \geq \frac{1}{2}$. To start pick $x_1 \in S_X$. Then the linear span of x_1 is proper closed linear subspace of X (proper because it is 1 dimensional, closed because of corollary 1.2.2). So by lemma 1.2.1 there is an x_2 in S_X such that $\|x_2 - \alpha x_1\| \geq \frac{3}{4}$ for all $\alpha \in \Re$. Then the linear span of x_1 and x_2 is proper closed linear subspace of X (by corollary 1.2.2 and lemma 1.2.1). Then there is an x_3 in S_X such that $\|x_3 - (\beta x_2 + \alpha x_1)\| \geq \frac{3}{4}$ for all $\alpha, \beta \in \Re$.
Continue the sequence so generated does all that is expected of it. Which implies that a sequence $\{x_n\}$ has no a convergent subsequence.
Hence S_X is not compact. □

Definition 1.2.8. *A subset K of a normed space X is said to be* ***convex*** *provided that given two points $u, v \in K$, then the set*

$$(1 - \lambda)u + \lambda v \in K$$

for $\lambda \in [0, 1]$

Example 1.2.4. *$[a, b]$ is a convex set*

Solution. $a \leq (1-\lambda)c + \lambda c = c < (1-\lambda)c + \lambda d < (1-\lambda)d + \lambda d = d \leq b$ where $c, d \in [a, b]$ and $\lambda \in [0, 1]$

Definition 1.2.9. *The* ***convex hull*** *of a set C is the intersection of all convex set which contain the set C*

Example 1.2.5. *Suppose that $[a, b]$ and $[c, d]$ are two intervals on the real line with $b < c$, so that the intervals are disjoint. Then the convex hull of the set $[a, b] \cup [c, d]$ is just the interval $[a, d]$*

Definition 1.2.10. *Let S be a non-empty set. A binary relation denoted by $\leq$, on S is said to be a* ***partially ordering*** *if and only if it satisfies the following conditions.*
(1) $a \leq a$ for every $a \in S$; (Reflexive).
(2) [$a \leq b$ and $b \leq a$] $\Rightarrow a = b$, for each a,b $\in S$ (Antisymmetry)
(3) [$a \leq b$ and $b \leq c$]$\Rightarrow a \leq c$, for each a,b,c $\in S$ (Transitivity).

Definition 1.2.11. *A* ***totally ordered*** *set is a partially ordered set such that every two elements of the set are comparable.*

Definition 1.2.12. *We say that $(D, \leq)$ is a* ***directed set****, if $\leq$ is a relation on D such that*
(i) $x \leq y$ and $y \leq z \Rightarrow x \leq z$ for each x,y,z $\in D$.
(ii) $x \leq x$, for each $x \in D$.
(iii) for each x,y $\in D$, there exists $z \in D$ with $x \leq z$ and $y \leq z$

Definition 1.2.13. *A subset A of a directed set D is called* ***residual (or eventual)*** *if there is some $d_o \in D$ such that $d \geq d_o$ implies $d \in A$*

Definition 1.2.14. *A* ***net*** *in a topological space (or in a set X) is a map from any non-empty directed set D to X. It is denoted by $(x_d)_{d \in D}$.*
If (x_α) is a net in the topological space X, and x is an element of X we say that the net converges towards x or has limit x and write $\lim x_\alpha = x$ if and only if for every neighborhood u of x, (x_α) is eventually in u.

Example 1.2.6. *Every non-empty totally ordered set is directed set. Therefore every function on such a set is a net.*
In particular the natural numbers with usual order form such a set and a sequence is a function on the natural numbers, so every sequence is a net.

Theorem 1.2.3. *If K is a compact subset of the normed linear space X, then there is a sequence $\{x_n\}$ in X such that*

$$\lim_n \|x_n\| = 0$$

and K is contained in the closed convex hull of $\{x_n\}$

Proof. K is compact; thus $2K$ is compact. Pick a finite $\frac{1}{4}$ net for $2K$, i.e. pick $x_1, x_2, ..., x_{n(1)}$ in $2K$ such that each point of $2K$ is with in $\frac{1}{4}$ of an x_i $1 \leq i \leq n(1)$. Denote by $B(x, \epsilon)$ the set $\{y : \|x - y\| \leq \epsilon\}$
Look at the compact chunks of $2K$: $[2K \cap B(x_1, \frac{1}{4})], \ldots, [2K \cap B(x_{n(1)}, \frac{1}{4})]$.
Move them to the origin: $[2K \cap B(x_1, \frac{1}{4})] - x_1, \ldots, [2K \cap B(x_{n(1)}, \frac{1}{4})] - x_{n(1)}$.
Translation is continuous; so chunks move to compact sets. Let K_2 be the union of the resultant chunks
i.e. $K_2 = \{[2K \cap B(x_1, \frac{1}{4})] - x_1\} \cup \cdots \cup \{[2K \cap B(x_{n(1)}, \frac{1}{4})] - x_{n(1)}\}$.
K_2 is compact, thus $2K_2$ is compact . Pick a finite $\frac{1}{16}$ net for $2K_2$, i.e.
Pick $x_{n(1)+1}, \ldots, x_{n(2)}$ in $2K_2$ such that each point of $2K_2$ is within $\frac{1}{16}$ of an $x_i, n(1) + 1 \leq i \leq n(2)$
Look at the compact chunks of $2K_2$: $[2K_2 \cap B(x_{n(1)+1}, \frac{1}{16})], \ldots, [2K_2 \cap B(x_{n(2)}, \frac{1}{16})]$. Move them to the origin : $[2K_2 \cap B(x_{n(1)+1}, \frac{1}{16})] - x_{n(1)+1}, \ldots, [2K_2 \cap B(x_{n(2)}, \frac{1}{16})] - x_{n(2)}$.
Translation is still continuous; so the chunks ,once moved, are still compact.
Let k_3 be the union of the replaced chunks:
$K_3 = \{[2K_2 \cap B(x_{N(1)+1}, \frac{1}{16})] - x_{n(1)+1}\} \cup \cdots \cup \{[2K_2 \cap B(x_2, \frac{1}{16})] - x_{n(2)}\}$.
K_3 is compact, and we continue in a similar manner. Observe that if

$$x \in K$$

$$2x \in 2K$$

$$2x - x_{i(1)} \in K_2, \text{ for some } 1 \leq i(1) \leq n(1); \text{ so}$$

$$4x - 2x_{i(1)} \in 2K_2,$$

$$4x - 2x_{i(1)} - x_{i(2)} \in K_3, \text{ for some } n(1)+1 \leq i(2) \leq n(2); \text{ so}$$

$$8x - 4x_{i(1)} - 2x_{i(2)} \leq 2K_3,$$

$$8x - 4x_{i(2)} - 2x_{i(2)} - x_{i(3)} \in K_4, \text{ for some } n(2)+1 \leq i(3) \leq n(3); \text{ so}$$

etc.Alternatively,

$$x - \tfrac{x_{i(1)}}{2} \in K,$$

$$x - \tfrac{x_{i(1)}}{2} - \tfrac{x_{i(2)}}{4} \in \tfrac{1}{4}K_3,$$

$$x - \tfrac{x_{i(1)}}{2} - \tfrac{x_{i(2)}}{4} - \tfrac{x_{i(3)}}{8} \in \tfrac{1}{8}K_4, \ldots$$

It follows that

$$x = \lim_n \sum_{k=1}^{n} \frac{x_{i(k)}}{2^k}$$

and $x \in \overline{co}(0, x_{i(1)}, x_{i(2)}, \ldots,) \subseteq \overline{co}(0, x_1, x_2, \ldots,)$ □

Chapter 2

The Weak and Weak* Topologies

As we saw in our study compactness in normed linear spaces, the norm topology is too strong to allow any widely applicable subsequential extraction principles. Indeed in order that each bounded sequence in X have a norm convergent subsequence, it is necessary and sufficient that X be finite dimensional. This fact leads us to consider other, weaker topologies on normed linear spaces which are related to the linear structure of the spaces and to search for subsequential extraction principles therein.
The two weaker than norm topologies of greatest importance in Banach space theory are the weak topology and the weak star (weak*) topology. The weak topology is present in every normed linear spaces. The weak* topology is present only in dual spaces.

2.1 Weak convergence

Definition 2.1.1. *A sequence $\{x_n\}$ in a normed space X converges weakly to a vector $x \in X$ if $f(x_n) \to f(x)$, for every $f \in X^*$*

Theorem 2.1.1. *(Bessel's inequality) let $\{e_k\}$ be an orthonormal sequence* *in an inner product space X. Then for every $x \in X$*

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leq \|x\|^2$$

Proposition 2.1.1. *Norm convergence implies weakly convergence*

Proof. Suppose $x_n \to x$ (strongly),
Now $|f(x_n) - f(x)| = |f(x_n - x)| \leq \lambda \|x_n - x\| \to 0 \Rightarrow x_n \to x$ weakly □

The converse of the proposition is not always true

Example 2.1.1. *An orthonormal system $\{x_n\}$ in a Hilbert space X converges weakly to zero, and it is does not converge strongly. By Reisz's Representation theorem the weak convergence to zero is equivalent to $\langle x_n, x\rangle \to 0$ for every $x \in X$. By Bessel$'$s inequality*
$\sum |\langle x_n, x\rangle|^2 \leq \|x\|^2$
Therefore $\{x_n\}$ is not strong covergent

Definition 2.1.2. *Let X be a normed space. A sub set C of X is said to be weakly closed if there exists a sequence $\{f_n\}$ in C such that $\{f_n\}$ converges to $x \in X$. The closure of C is weakly open in X.*

Definition 2.1.3. *The weak topology on a normed space X is defined as the weakest topology in which all maps $f \in X^*$ ($i.e. f : X \to \Re$) are continuous.*

Definition 2.1.4. *In a Banach space B a weakly compact set is a set that is compact for the weak topology.*

Properties:-
The weak topology is
(1) linear (addition and scalar multiplications are continuous)
(2) Hausdorff (the weak limits are unique)
Alternatively we can describe a basis for the weak topology. Since the weak topology is patently linear we need only specify the neighborhoods of 0, translation will carry these neighborhoods throughout X. A typical basic

neighborhoods of 0 is generated by an $\epsilon > 0$ and finitely many members $x_1^*, x_2^*, ..., x_n^*$ of X^*.
It's form is :
$W(0; x_1^*, x_2^*, \ldots, x_n^*, \epsilon) = \max\{x \in X : |x_1^* x|, |x_2^* x|, \ldots, |x_n^* x|\} < \epsilon$
In fact each basic neighborhood $W(0; x_1^*, x_2^*, \ldots, x_n^*, \epsilon)$ of 0 contains the intersection $\cap \ker x_i^*$ of the null space $\ker x_i^*$ of the x_i^* a linear sub space of finite codimension.

Lemma 2.1.1. *Let E be a linear space and $f, f_1, f_2, \ldots, f_n$ be linear functionals on E such that $\ker f \supseteq \cap_{i=1}^n \ker f_i$. Then f is a linear combination of the $f_i's$, $i \in (1, 2, 3, \ldots, n)$*

Proof. *W.L.O.G.* Assume that $\{f_1, f_2, \ldots, f_n\}$ is a linear independent. We proceed by induction on n.
For $n = 1$. Assume that $\ker f_1 \subseteq \ker f$. Since $f_1 \neq 0$, there is an $x_1 \in X$ such that $f_1(x_1) \neq 0$
$x \in X$, $u_x = x - \frac{f_1(x)}{f_1(x_1)} x$.
$f_1(u_x) = f_1(x) - \frac{f_1(x)}{f_1(x_1)} f_1(x_1) = 0$.
$\Rightarrow u_x \in \ker f_1 \subseteq \ker f$.
$\Rightarrow u_x \in \ker f$
Let $x = z + \alpha x_1$.
$f(x) = f(z) + \alpha f(x_1) = \frac{f_1(x)}{f_1 x_1} f(x_1) = \frac{f(x_1)}{f_1(x_1)} f_1(x) = \alpha f_1(x)$, where $\alpha = \frac{f(x_1)}{f_1(x_1)}$ and $f(z) = 0$
Therefore $f = \alpha f_1$
Inductively assume that the lemma is true for $n - 1$ suppose $\bigcap_{i=1}^{n-1} \ker f_i \subseteq \ker f$
It follows from the induction assumption that $\bigcap_{i=1}^{n} \ker f_i \not\subseteq \ker f_j$ for all $j = \{1, 2, \ldots, n\}$ and $i \neq j$.
Then there exist $x_1, x_2, \ldots, x_n \in X$ such that $f_i(x_j) = \delta_{ij}$ for $i, j = \{1, 2, \ldots, n\}$.
Let $x \in X$, put $y_x = x - \sum_{i=1}^n f_i x_i$.
Then for each $j = \{1, 2, \ldots, n\}$.
$f_j(y_x) = f_j(x) - \sum_{i=1}^n f_i(x) f_j(x_i) = f_j(x) - f_j(x) = 0 \Rightarrow y_x \in \ker f_i \subseteq \ker f$.
Therefore $x = y_x + \sum_{i=1}^n f_i(x) x_i$

Hance $f(x) = f(y_x) + \sum_{i=1}^{n} f_i(x)f(x_i) = \sum_{i=1}^{n} \alpha_i f_i(x)$.
Therefore $f = \sum_{i=1}^{n} \alpha_i f_i$, where $\alpha_i = f(x_i)$ □

Note. Though the weak topology is smaller than the norm topology, it produces the same continuous linear functional.
To see this. Suppose f be a weakly continuous linear functional on the normed linear space X, then $U = \{x : |f(x)| < 1\}$ a weak neighborhood of 0. As such U contains $W(0; x_1^*, \dots, x_n^*, \epsilon)$.
Since f is linear and $W(0; x_1^*, \dots, x_n^*, \epsilon)$ contains the linear space $\bigcap_{i=1}^{n} \ker x_i^*$, it follows that $\ker f$ contains $\cap_{i=1}^{n} \ker x_i^*$
which implies f must be a linear combination of $x_1^*, \dots, x_n^*$ and so $f \in X^*$.

Definition 2.1.5. *If X is a topological space. X is said to be* ***metrizable*** *if there exists a metric d on the set X that induces the topology of X*

Definition 2.1.6. *A space X is said to have a* ***countable basis*** *at each $x \in X$, if there is a countable collection β of neighborhood of x such that each neighborhood of x contains at least one of the elements of β. A space that has a countable basis at each of its points is said to satisfy the* ***first countability axiom*** *(or to be first countable)*

Note: Every metrizable space satisfies the first countability axiom.
The weak topology is really of quite different character than is the norm topology (at least in the case of infinite dimensional normed spaces).

Definition 2.1.7. *Let S be a subspace of the normed space. Then the closure in X of $span(S), \overline{span(S)}$ is called the closed linear manifold (sub space) generated by S*

Theorem 2.1.2. *If the weak topology of a normed linear space X is metrizable, then X is finite dimensional*

Proof. Suppose the weak topology of X is metrizable, there exists a sequence (x_n^*) in X^* such that given any weak neighborhood u of 0

we can find a rational $\epsilon > 0$ and $n(u)$ such that u contains $W(0; x_1^*, x_2^*, ..., x_n^*, \epsilon)$. Each $x^* \in X^*$ generates the weak neighborhood $(0; x^*, \epsilon)$ of 0,which inturn contains one of the sets $W(0; x_1^*, \ldots, x^*_{n(W(0;x^*,\epsilon))}, \epsilon)$. However we have seen that this entails x^* being a linear combination of $x_1^*, \ldots, x^*_{n(W)}$.
If we let F_m be the linear span of $x_1^*, \ldots, x_m^*$, then each F_m is a finite dimensional linear sub space of X^* which is a fortiori closed; Moreover we have just seen that $X^* = \bigcup_m F_m$. The Baire Category theorem now alerts us to the fact that one of the F_m has $non-empty$ interior, a fact which tells us that the F_m has to be all of X^*. $X^*(and \quad hence X)$ must be finite dimensional □

Proposition 2.1.2. *If X is infinite dimensional, then the weak topology of X is not complete*

Proof. Let $\{x_n\}$ be a cauchy sequence in X. Since $\{x_n\}$ does not converge (by theorem 2)
Therefore X is not complete □

Definition 2.1.8. *Let A be a subspace of a normed space X. The closure of A is the smallest closed sub set $\overline{A}$ of X that contains A. If $\overline{A} = X$, then A is said to be* ***dense*** *in X.*

Theorem 2.1.3. *If K is a convex subset of the normed linear space X, then the closure of K in the norm topology concides with the weak closure of K*

Proof. There are no more open sets in the weak topology than there are in the norm topology.
If K is a convex set and if there is a point $x_0 \in \overline{K}^{weak} \backslash \overline{K}^{\|\cdot\|}$, then there would be an $x_0^* \in X^*$ such that $\sup x_0^* \overline{K}^{\|\cdot\|} \leq \alpha < \beta \leq x_0^*(x_0)$, for some α, β
This follows from the Separation theorem and the convexity of $\overline{K}^{\|\cdot\|}$. However $x_0 \in \overline{K}^{weak}$ implies there is a net (x_d) in K such that $x_o =$ weak $\lim_d x_d$
It follows that $x_0^* x_o = \lim_d x_0^* x_d$
it is contradiction to the fact that $x_o^*(x_o)$ is separated from all the $x_o^*(x_d)$ by the gulf between α and β □

Corollary 2.1.1. *If $\{x_n\}$ is a sequence in the normed linear space X for which weak* $\lim_n x_n = 0$*, then there is a sequence (δ_n) of convex combinations of the x_n such that* $\lim_n x_n = 0$

Proof. suppose $x \notin K := \overline{conv}(x_n)$. By using a Separation theorem, we can separate the closed convex set K from $\{x\} = \{0\}$; Namely there exists a functional $f \in X^*$ such that $\sup_{y\in K} f(y) < f(x)$.
Since $x_n \in K$, this implies that $\sup_n f(x_n) < f(x) = f(0) = 0$ which is contradicts weak convergence.
Suppose $x_n \to x(weakly)$ and $x = 0$
Let $f \in X^*$, a supporting functional of X, i.e. $\|f\| = 1, f(x) = \|x\|$

$$f(x_n) \to f(x) = \|x\|$$
$$\Rightarrow |f(x_n)| \to |f(x)| = \|f\|\|x\| = \|x\| = 0$$
$$\Rightarrow \|x_n\| \to \|x\| = 0$$
$$\Rightarrow \lim_n \|x_n\| = 0$$

□

Corollary 2.1.2. *If Y is a linear subspace of the normed linear space X, then* $\overline{Y}^{weak} = \overline{Y}^{\|\cdot\|}$

Corollary 2.1.3. *If K is a convex set in the normed linear space X, then K is norm closed if and only if K is weakly closed*

Proof. Assume K is closed and convex. By corollary of intersection of half-spaces to Hahn-Banach theorem K is the intersection of the closed half spaces that contain K: Each closed half space has the form
$A_{f,a} = \{x \in X : f(x) \leq a\}$ for some $f \in X^*$ and $a \in \Re$. Hence $A_{f,a}$ is weakly closed. The intersection K of the closed half spaces is therefore weakly closed. □

Theorem 2.1.4. . *A linear map $T : X \to Y$ between the normed linear spaces X and Y is norm-to-norm continuous if and only if T is weak-to-weak continuous*

Proof. Suppose T is weak-to-weak continuous if and only if for each $y^* \in Y^*$, y^*T is a weakly continuous linear functional on X. This inturn occurs if and only if y^*T is a norm continuous linear functional on X for each $y^* \in Y^*$.
On the other direction suppose T is not norm-to-norm continuous, then TB_x is not a bounded set of Y. Therefore by Banach-Steinhause theorem a $y^* \in Y^*$ such that y^*TB_x is not bounded , y^*T is not bounded linear functional.Which implies that T is not weak-to-weak continuous □

2.2 The Weak* Topology

On X^* there are two natural weaker topologies. The weak topology that we already considered makes all functionals on X^{**} continuous functionals on X^*. The other topology called weak* topology, is only concerned with continuity of functionals that come from $X \subseteq X^{**}$. Let X be a normed linear space. We describe the weak* topology of X* by indicating how a net (x_d^*) in X^* converges weak* to a member x_0^* of X*.
We say that (x_d^*) converges weak* to $x_0^* \in X^*$ if for each $x \in X$,

$$(x_0^*)x = \lim_d (x_d^*)x.$$

As with the weak topology, we can give a description of a typical basic weak* neighborhood of 0 in X^*; this time such a neighborhood is generated by an $\epsilon > 0$ and a finite collection of elements in X, say $x_1, ..., x_n$.
Then the form is

$$W^*(0; x_1, \ldots, x_n, \epsilon) = \max\{x^* \in X^* : |x^*x_1|, \ldots, |x^*x_n|\} < \epsilon$$

The weak* topology is a linear topology; so it is enough to describe the neighborhoods of 0, and neighborhoods of other points in X* can be obtained by translation.
Notice that weak* basic neighborhoods of 0 are also weak neighborhoods of 0; in fact, they are just the basic neighborhoods generated by those members of

X^{**} that are actually in X. For, any x^{**} that are left over in X^{**} after taking away X give weak neighborhoods of 0 in X* that are not weak* neighborhoods. A conclusion to be drawn is this: the weak * topology is no bigger than the weak topology. Like the weak topology, excepting finite dimensional spaces, duals are never weak* metrizable or weak* complete. Also proceeding as we did with the weak topology, we can show that the weak* dual of X* is X. An important consequence of this is the following theorem.

Theorem 2.2.1. *(Goldstine's Theorem). For any normed linear space X, B_X is weak* dense in $B_{X^{**}}$, and so X is weak* dense in X^{**}.*

Proof. The second assertion follows from the first; so we concentrate our attentions on proving B_X is always weak* dense in $B_{X^{**}}$.
Let $x^{**} \in X^{**}$ be any point not in $\overline{B_X}^{weak*}$. Since $\overline{B_X}^{weak*}$ is a weak* closed convex set and $x^{**} \notin \overline{B_X}^{weak*}$, there is an $x^* \in X^{**}$ weak* dual X^* such that

$$\sup\{x^* y^{**} : y^{**} \in \overline{B_X}^{weak*}\} < x^{**} x^*$$

Of course we can assume $\|x^*\| = 1$; but now the quantity on the left is at least $\|x^*\| = 1$, and so $\|x^{**}\| > 1$. It follows that every member of $B_{X^{**}}$ falls inside $\overline{B_X}^{weak*}$. □

As important and useful fact as **Goldstine's theorem** is, the most important feature of the *weak** topology is contained in the following compactness result.

Theorem 2.2.2. *(Alaoglu's Theorem). For any normed linear space X, B_{X^*} is weak* compact.*
Consequently, weak closed bounded subsets of X^* are weak* compact.*

Proof. If $x^* \in B_{X^*}$, then for each $x \in B_X, |x^* x| \leq 1$. Consequently, each $x^* \in B_{X^*}$ maps B_X in to the set D of scalars of modulus ≤ 1. We can there fore identify each member of B_{X^*} with a point in the product space D^{B_X}.
Tychonoff's theorem tells us this latter space is compact. On the other hand, the weak* topology is defined to be that of point wise convergence on B_X, and

so this identification of B_x^* with a sub set of D^{B_X} leaves the weak* topology unscathed; it need only be established that B_{X^*} is closed in D^{B_X} to complete the proof.
Let (x_d^*) be a net in B_{X^*} converging point wise on B_X to $f \in D^{B_X}$. Then it is easy to see that f is "linear" on B_X : in fact, if $x_1, x_2 \in B_X$ and a_1, a_2 are scalars such that $a_1x_1 + a_2x_2 \in B_X$, then

$$\begin{aligned} f(a_1x_1 + a_2x_2) &= \lim_d x_d^*(a_1x_1 + a_2x_2) \\ &= \lim_d(a_1x_d^*(x_1) + a_2x_d^*(x_2)) \\ &= \lim_d a_1x_d^*(x_1) + \lim_d a_2x_d^*(x_2) \\ &= a_1f(x_1) + a_2f(x_2) \end{aligned}$$

It follows that f is indeed the restriction to B_X of a linear functional x' on X; moreover, since f(x) has modulus ≤ 1 for $x \in B_X$, this x' is even in B_{X^*}. This completes the proof. □

Chapter 3

The Eberlein-$\check{S}$mulian Theorem

We saw in the previous chapter that regardless of the normed linear space X, weak* closed, bounded sets in X^* are weak* compact. How does a subset K of a Banach space X get to be weakly compact?

Lemma 3.0.1. *Weakly compact sets are norm closed and norm bounded.*

Proof. Let K be weakly compact set in the normed linear space X. If $x^* \in X^*$, then x^* is weakly continuous
Therefore x^*K is a compact set of scalars.
$\Rightarrow x^*K$ is bounded, for each $x^* \in X^*$
$\Rightarrow$ K is bounded (by the uniform boundedness theorem)
Further K is weakly compact, hence weakly closed, and so norm closed. $\square$

The converse of the lemma is not true.

Example 3.0.1. *B_{co} is closed bounded set but not weakly compact*

Proof. Suppose B_{co} is weakly compact.
Since each sequence in B_{co} would have a weak cluster point in B_{co}. consider the sequence (δ_n) defined by $\delta_n = e_1 + ... + e_n$, where e_k is the k^{th} unit vector in c_o. The supremum norm of c_o is given by

$$\|x\| = \sup_n |\zeta_n|$$

where $x = \{\zeta_n\} \in c_o$
So that $\|\delta_n\| = 1$ for all n. Take $\lambda \in B_{co}$ that is a weak cluster point of (δ_n), for each $x^* \in c_o^*$, $(x^*\delta_n)$ has $x^*\lambda$ for a cluster point.
Now evaluation of a sequence in c_o at its k^{th} coordinate is a continuous linear functional; call it e_k^*. Note that $e_k^*(\delta_n) = 1$ for all $n \geq k : e_k^*\lambda = 1$ for all k.
Hence $\lambda = (1, 1, \ldots, 1 \ldots) \notin c_o$.
Therefore B_{co} is not weakly compact. □

Lemma 3.0.2. *B_X is weakly compact if and only if X is* ***reflexive space****.*

Proof. suppose $B_X = B_{X^{**}}$, naturally this occurs when and only when $X = X^{**}$; such X are called reflexive. Then the natural embeding of X in to X^{**} is a weak-to-weak* homeomorphism of X on to X^{**} that carries B_X exactly on to $B_{X^{**}}$. By Alaoglue theorem. B_X is weakly compact.
On the other direction, suppose B_x is weakly compact, then any $x^{**} \in X^{**}$ not in B_x can be separated from the weak* compact convex set B_X by an element of the weak* dual of X^{**}. i.e there is an $x^* \in B_{X^*}$ such that

$$\sup_{\|x\| \leq 1} x^*x (= \|x^*\| = 1) < x^{**}x^* \Rightarrow \|x^{**} \geq 1 \Rightarrow x^{**} \notin B_{X^{**}}$$

Therefore $B_X = B_{X^{**}}$
Hence X is reflexive. □

Remark. A bounded set A in a Banach space X is relatively weakly compact.
To see this:. Since $\overline{A}^{weak*}$ is weak* compact. Should each element in $\overline{A}^{weak*}$ actually be in X; then $\overline{A}^{weak*}$ is just $\overline{A}^{weak}$
Therefore $\overline{A}^{weak}$ is weakly compact.

Note. A linear space has many norms. In general the space may be complete with relative to one norm not complete with relative to another norm. Unless the two norms are equivalent.

Definition 3.0.1. *A subset K of a topological space X is said to be sequentially compact if for any sequence $\{x_n\} \subset K$, there exists a convergent subsequence with limit in K.*

Definition 3.0.2. *A space X is said to be **limit point compact** if every infinite subset of X has a limit point.*

Definition 3.0.3. *A subset K of a Banach space X is weakly, sequentially compact if for any sequence $\{x_n\}$ in K there is a weakly convergent subsequence with limit in K.*

Definition 3.0.4. *Let X and Y be topological spaces.*
Let $f : X \to Y$ be a bijective map. Then f is said to be a homeomorphism if both f and f^{-1} are continuous.

Lemma 3.0.3. *Let $f : X \to Y$ be a bijective continuous map. If X is compact and Y is Hausdorff, then f^{-1} is continuous.*

Proof. Let C be closed in X

$$\begin{aligned} &\Rightarrow C \text{ is compact} \\ &\Rightarrow f(C) \text{ is compact in Y} \\ &\Rightarrow f(C) \text{ is closed in Y} \end{aligned}$$

Therefore f^{-1} is continuous.
Hence f is homeomorphism. □

Definition 3.0.5. *A space having a countable dense subset is often said to be **separable**.*

Lemma 3.0.4. *Let X be metrizable. The following are equivalent.*
(a) compact
(b) Limit point compact
(c) sequentially compact

Proof. $(a \Rightarrow b)$
Let X be compact.
Let $A \subseteq X$ be infinite. Suppose A has no limit point. which implies A is closed. If $a \in A$ and a is not a limit point, then there is a neighborhood u_a of a such that
$u_a \cap A \backslash \{a\} = \emptyset$. Then $\{u_a : a \in A\}$ is an open covering of A. Since A is compact there exists $u_{a_1}, \ldots, u_{a_n}$ such that

$$A \subseteq \bigcup_{i=1}^{n} u_{a_i}$$

Therefore $A = \{a_1, a_2, \ldots, a_n\}$ which is a contradiction to the fact that A is infinite,which implies that A has a limit point.
Hence X is limit point compact.
$(b \Rightarrow c)$
Let X be limit point compact.
Let $\{x_n\}$ be a sequence of points.
Let $A = \{x_n | n \in \mathbb{N}\}$. Then A is either finite or infinite.
Case 1.
A is finite.
Let $f : \mathbb{N} \to A$, $f(n) = x_n$. Then $f(\mathbb{N})$ is finite.
Now

$$f^{-1}(A) = \bigcup_{i=1}^{n} f^{-1}(\{x_i\})$$

where $A = \{y_1, y_2, ..., y_n\}$. Thus for some i $f^{-1}(\{x_i\})$ is infinite.
Put $f^{-1}(\{y_i\}) = \{n_k | k = 1, 2, \ldots\}$.
Then $\{x_{n_k}\}_{k=1}^{\infty}$ is constant sequence as $x_{n_k} = y_i$ for all k
Therefore $\{x_n\}$ has a convergent sub sequence in X.
Case 2.
A is infinite. Since X is limit point compact, then A has a limit point say x.
Then any ball centered at x infinitely many points of A

$$\text{For } n = 1, \text{ let } x_{n_1} \in B(x, 1) \bigcap A$$

$$\text{For } n = 2, \text{ let } x_{n_2} \in B(x, 2) \bigcap A$$
$$\text{For } n = 3, \text{ let } x_{n_3} \in B(x, 3) \bigcap A$$
Continue in this way.

Consider $\{x_{nk}\}_{k=1}^{\infty}$, then this a sub sequence of $\{x_n\}$ and as $k \to \infty, x_{n_k} \to x$
Hence X is sequentially compact.
$(c \Rightarrow a)$. Suppose X is sequentially compact. We need to show that X is compact.
Suppose X is not compact. Then for some $\epsilon > 0$, no finite $\epsilon - balls$ cover X. Let $x_1 \in X$ and consider $B(x_1, \epsilon)$. By assumption $X \backslash B(x_1, \epsilon) \neq \emptyset$. So pick $x_2 \backslash B(x_2, \epsilon)$. Then $X \backslash (B(x_1, \epsilon) \cup B(x_2 \epsilon)) \neq \emptyset$. Continue in this manner.
Suppose you choose x_n such that $x_n \notin \cup_{i=1}^{n-1}(B(x_i, \epsilon))$.
Now $\cup B(x_i, \epsilon) \neq X$. There exists $x_{n+1} \notin \bigcup_{i=1}^{n} B(x_i, \epsilon))$.
Consider $\{x_n\}$. If a sequence of points in X. Note that the condition

$$d(x_{n+1}, x_i) \geq \epsilon$$

for all $i = 1, 2, \ldots$
Which implies $\{x_n\}$ is not Cauchy sequence.
In fact for any $x \in X$, $B(x, \frac{\epsilon}{2})$ contains at most one value of $\{x_n\}$.
If $x_n, x_m \in B(x, \frac{\epsilon}{2})$
W.L.O.G. $(n > m)$

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon$$

Which is a contradiction.
Therefore any x can not be a limit point.
Hence no $x \in X$ can be a limit point of $\{x_n\}$. Which implies $\{x_n\}$ has no a sub sequence which is convergent. Which is a contradiction to the fact that X is sequentially compact.
Therefore X is compact. □

Theorem 3.0.3. *(The Eberlein-Šmulian Theorem). A subset of a Banach space is relatively weakly compact if and only if it is relatively weakly sequentially compact.*

In particular, a subset of a Banach space is weakly compact if and only if it is weakly sequentially compact.

Proof. To start, we will show that a relatively weakly compact subset of a a Banach space is relatively weakly sequentially compact. This will accomplished in two easy steps.
*Step*1. If K is a (relatively)‘weakly compact set in a Banach space X and X^* contains a countable total set, then $\overline{K}^{weak}$ is metrizable . Recall that a set $F \subseteq X^*$ is called **total** if $f(x) = 0$ for each $f \in F$ implies $x = 0$.
Suppose that K is weakly compact and $\{x_n^*\}$ is a countable total subset of nonzero members of X^*. The function $d : X \times X \to \Re$ defined by

$$d(x, x') = \sum_n |x_n^*(x - x')| \|x_n^*\|^{-1} 2^{-1}$$

is a metric on X. The formal identity map is weakly-to-d continuous on the bounded set K. Since a continuous one-to-one map from a compact space to a Hausdorff space is a Homeomorphism, we conclude that d restricted to $K \times X$ is a metric that generates the weak topology of K.
*Step*2. Suppose A is a relatively weakly compact subset of the Banach space X and $\{a_n\}$ be a sequence of members of A. Look at the closed linear span$[a_n]$ of the $\{a_n\}$; $[a_n]$ is weakly closed in X. Therefore, $A \bigcap [a_n]$ is relatively weakly compact in the separable Banach space $[a_n]$. Now the dual of a separable Banach space contains a countable total set: if $\{d_n\}$ is a countable dense set in the unit sphere of the separable space and $\{d_n^*\}$ is chosen in the dual to satisfy $d_n d_n^* = 1$, and $\{d_n^*\}$ is total.
From our first step we know that $\overline{A \bigcap [a_n]}^{weak}$ is metrizable in the weak topology of $[a_n]$. Since compactness and sequential compactness are equivalent in metric space, $\overline{A \bigcap [a_n]}^{weak}$ is a weakly sequentially compact subset of $[a_n]$. In particular, if a is any weak limit point of (a_n), then there is a subsequence $\{a_n'\}$ of $\{a_n\}$ that converges weakly to a in $[a_n]$. It is plain that $\{a_n'\}$ also converges weakly to a in X.

We now turn to the the other direction. We start with an observation: if E is a finite-dimensional subspace of X^{**}, then there is a finite set E' of S_{x*} such that for any x^{**} in E

$$\frac{\|x^{**}\|}{2} \leq \max\{|x^{**}x^{*}| : x^{*} \in E'\}$$

In fact, S_E is norm compact.
Therefore, there is a finite set $\frac{1}{4}$ net $F = \{x_1^{**}, ..., x_n^{**}\}$ for S_E.
Pick $x_1^{*}, ..., x_n^{*} \in S_{x*}$ such that

$$x_k^{**}x_k^{*} \geq \frac{3}{4}$$

Then whenever $x^{**} \in S_{E'}$, we have

$$x^{**}x_k^{*} = x_k^{**}x_k^{*} + (x^{**}x_k^{*} - x_k^{**}x_k^{*}) \geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

for a suitable choice of K. This observation is the basis of our proof.
Let A be a relatively weakly sequentially compact subset of X; each infinite subset of A has a weak cluster point in X. Since A is also relatively weakly countably compact. Consider $\overline{A}^{weak*}$. $\overline{A}^{weak*}$ is weak* compact since A, and therefore $\overline{A}^{weak*}$, is bounded due to the relative weak sequential compactness of A. We use the strategy espoused at the start of this section to show A is relatively weakly compact; that is, we show $\overline{A}^{weak*}$ actually lies in X.

Take $x^{**} \in \overline{A}^{weak*}$, and let $x_1^{*} \in S_{x*}$. Since $x^{**} \in \overline{A}^{weak*}$ each weak* neighborhood of x^{**} contains a member of A. In particular, the weak* neighborhood generated by $\epsilon = 1$ and x_1^{*}, $\{y^{**} \in X^{**} : |(y^{**} - x^{**})(x_1^{*})| < 1\}$, contains a member a_1 of A. From this we get

$$|(x^{**} - a_1)(x_1^{*})| < 1$$

Consider the linear span $[x^{**}, x^{**} - a_1]$ of x^{**} and $x^{**} - a_1$; this a finite-dimensional subspace of X^{**}. Our observation deals us $x_2^{*}, ..., x_{n(2)}^{*} \in S_{x*}$ such that for any y^{**} in $[x^{**}, x^{**} - a_1]$,

$$\frac{\|y^{**}\|}{2} \leq \max\{|y^{**}(x_k)| : 1 \leq k \leq n(2)\}$$

x^{**} is not going anywhere, i.e, it is still in $\overline{A}^{weak*}$; so each weak* neighborhood of x^{**} intersects A. In particular, the weak* neighborhood about x^{**} generated by $\frac{1}{2}$ and $x_1^*, ..., x_{n(2)}^*$ intersects A to give us an a_2 in A such that

$$|(x^{**} - a_2)(x_1^*)|, |(x^{**} - a_2)(x_2^*)|, ..., |(x^{**} - a_2)(x_{n(2)}^*)| < \frac{1}{2}$$

Now look at the linear span $[x^{**}, x^{**} - a_1, x^{**} - a_2]$ of $x^{**}, x^{**} - a_1$ and $x^{**} - a_2$. As a finite-dimensional subspace, $[x^{**}, x^{**} - a_1, x^{**} - a_2]$ provides us with $x_{n(2)+1}^*, ..., x_{n(3)}^*$ in S_{x*} such that

$$\frac{\|y^{**}\|}{2} \leq \max\{|y^{**}(x_k^*)| : 1 \leq k \leq n(3)\}$$

for any $y^{**} \in [x^{**}, x^{**} - a_1, x^{**} - a_2]$.
Once more quickly choose a_3 in A such that $x^{**} - a_3$ charges againest $x_1^*, ..., x_{n(3)}^*$ for no more than $\frac{1}{3}$ value. Observe that the finite-dimensional linear space $[x^{**}, x^{**} - a_1, x^{**} - a_2, x^{**} - a_3]$ provides us with a finite subset $x_{n(3)+1}^*, ..., x_{n(4)}^*$ in S_{x*} such that

$$\frac{\|y^{**}\|}{2} \leq \max\{|y^{**}(x_k^*)| : 1 \leq k \leq n(4)\}$$

for any $y^{**} \in [x^{**}, x^{**} - a_1, x^{**} - a_2, x^{**} - a_3]$.
Where does all this lead us? Our hypothesis on A (being relatively weakly sequentially compact) allows us to find an $x \in X$ that is a weak cluster point of the constructed sequence $\{a_n\} \subseteq A$. Since the closed linear span $[a_n]$ of the a_n is weakly closed, $x \in [a_n]$. It follows that $x^{**} - x$ is in the weak* closed linear span of $\{x^{**}, x^{**} - a_1, x^{**} - a_2, \dots\}$. Our construction of the x_i^* and the a_i assures that

$$\frac{\|y^{**}\|}{2} \leq \sup_m |y^{**} x_m^*| \text{..} (1)$$

holds for any y^{**} in the linear span of $x^{**}, x^{**} - a_1, x^{**} - a_2, \dots$. An easy continuity argument shows that (1) applies as well to any y^{**} in the weak* closed linear span of $x^{**}, x^{**} - a_1, x^{**} - a_2, \dots$. In particular, we can apply (1) to $x^{**} - x$. However,

$$|(x^{**} - x)(x_m^*)| \leq |(x^{**} - a_k)(x_m^*)| + |x_m^*(a_k) - x_m^*(x)| \leq \frac{1}{p}$$

as little as you pleas

if $m \leq n(p), p \leq k$ and you take advantage of the fact that x is a weak cluster point of (a_n). So $x^{**} - x = 0$, and this ensures that $x^{**} = x$ in X. $\square$

Bibliography

(1) Joseph Diestel, Sequences and Series in Banach Spaces, 1984 by Springer-Verlag New York.
(2) Richard R. Goldberg, Methods of Real Analysis, 1964 by Blaisdell publishing campany.
(3) James R. Menkres, Topology, a first course, 1975 by Prentice-Hall.
(4) R. Deumlich, Functional Analysis I, II, Addis Ababa University(Unpublished), 2000.