Addis Ababa University

Office of Graduate Program

Faculty of Computer and Mathematical Science

Department of Mathematics

Integral Representation

For

Solutions of Elliptic Differential Operators

By: Berhe Kalayu

Advisor: Dr.Tsegaye Gedif

A Project submitted to the Office of Graduate Programs
of Addis Ababa University in Partial fulfillment of the requirements for the Degree of
Master of Science in Applied Mathematics

January, 2011

Addis Ababa Ethiopia

Declaration

“I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any degree, diploma, associate ship, fellowship or any other similar title to me.

$\overline{Author's\ signature}$ “

Permission

“This is to certify that this project is compiled by Berhe Kalayu in the Department of mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

$\overline{Advisior's\ signature}$ “

Acknowledgement

I would like to express my sincere gratitude to my advisor, Dr Tsegaye Gedif, for his support, patience, and encouragement throughout the whole task from the initial to the final level and enabled me to develop an understanding of the subject.

My thanks also go to Hailu Bikila one of the members of mathematics teachers on differential equation for his technical and editorial advice which was essential to the completion of my project.

I also like to express my eternal gratitude to my lovely parents and brothers for their everlasting love and support.

Lastly, I offer my regards and blessings to all of those who supported me in any respect during the completion of the project.

Abstract

This paper is devoted to integral representation for solutions of elliptic differential operators. Given Ω an open bounded domain in R^n, $n \geq 2$, with a smooth boundary $\partial\Omega$, and L an elliptic differential operator, u is a function such that $u \in C^k(\Omega)$ and f is continuously differentiable function ,we show that the solution u for the given equation

$$Lu = f$$

has an integral representation that can be derived from fundamental solution and Green's function.

Table of content

Contents page

Introduction

One of the important tools in the theory of PDEs is the integral representation of solutions. An integral representation is a formula of the solutions of a problem in terms of an integral depending on the Greens function and fundamental solutions for the operator given in the problem.

Integral representation plays a central role in various fields of pure and applied mathematics. In this paper we define elliptic operators, which are a special sort of partial differential operators. Many of the differential operators that crop up in problems in geometry, applied mathematics and physics are elliptic.

This paper consists of two chapters; the first one has a notations and preliminary results with the goals of developing and studying basic definitions and facts. Here I have tried to emphasize, whenever possible, familiar notations and basic results of denotation of partial derivative with few examples, in order to provide intuition and feeling for the project. For this part, a few important facts from result of advanced calculus like the Green's identities, integrals in polar coordinates and change of variable formulas are introduced. Chapter two mainly treats the integral representation for solutions of elliptic differential operators, chiefly for Laplace operator and other general elliptic differential operators. Here definitions, theorems with their proofs and examples with their explanations are provided and also a concept of fundamental and distributional solutions, Green's functions and its properties are discussed.

In the next sections i will define the general elliptic operators, and give a few examples and basic facts. Suppose L is an elliptic differential operator of order k. Moreover let u be a function defined and k times continuously differentiable in the closure of a domain Ω of R^n. Provided the adjoint elliptic differential operator possesses a fundamental solution, we shall see that u can be recovered from Lu and the boundary values of u. Strictly speaking, we shall get an integral representation of u.

CHAPTER ONE

Notations and preliminary results

We fix some familiar notations used throughout this project.

$x = (x_1, x_2, \dots x_n)$ is a variable point in the domain real n-dimensional Euclidean space R^n.

$$|x| = \left(\sum_{i=1}^{n} {x_i}^2\right)^{1/2}$$

Let $u(x) = u(x_1 x_2, \dots, x_n)$ be a differential function. We use any of the following notations to denote the partial derivative of u with respect to the i^{th} variable x_i.

$$u_{x_i} \quad , \quad \partial_i u \quad , \quad D_i u \quad , \quad \frac{\partial u}{\partial x_i}$$

The second partial derivatives of a twice continuously differentiable function u with respect to x_i and x_j will be denoting by any of the following notations.

$$u_{x_i x_j} \ , \quad \partial^2{}_{ij} u, \quad D_{ij} u \ , \quad \frac{\partial^2}{\partial x_i \partial x_j} u$$

We set $D^0 = I$, and for any positive integer k. We let $D^k u$ to be the set of all partial derivatives of u of order k.

Du denotes the gradient of u , that is

$$Du = (u_1, u_2, \dots, u_{n)}$$

Sometimes a much more convenient notation, the so called multi-index notation is used.

An n multi-index is an n-tuple of non negative integers

$$\alpha \coloneqq (\alpha_1, \alpha_2, \dots, \alpha_n)$$

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n = \sum_{i=1}^{n} \alpha_i$$

So,

$$D^{\alpha} = \frac{\partial^{\alpha_1}}{\partial {x_1}^{\alpha_1}} \cdot \frac{\partial^{\alpha_2}}{\partial {x_2}^{\alpha_2}} \cdots \cdot \frac{\partial^{\alpha_n}}{\partial {x_n}^{\alpha_n}}$$

Example 1 for $n = 2$,$\alpha = (0, 0)$ corresponds to $D^{\alpha} u = u$

$|\alpha| = 1$ is associated with $\alpha = (1, 0)$ or $\alpha = (0, 1)$.

$$\alpha = (1, 0): D^{\alpha} u = \frac{\partial u}{\partial x_1}, or\ \alpha = (0, 1): D^{\alpha} u = \frac{\partial u}{\partial x_2}$$

$|\alpha| = 2$ is associated with $\alpha = (2, 0), \alpha = (1, 1)$ or $\alpha = (0, 2)$

$$\alpha = (2, 0): D^{\alpha} u = \frac{\partial^2 u}{\partial {x_1}^2}$$

$$\alpha = (1, 1): D^{\alpha} u = \frac{\partial^2 u}{\partial x_1 \partial x_2}$$

$$\alpha = (0, 2): D^{\alpha} u = \frac{\partial^2 u}{\partial {x_2}^2}$$

For $n = 3$ just at $\alpha = (1, 1, 2)$ with $u = u(x_1, x_2, \dots, x_n)$

$$D^{\alpha} u = \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2} \frac{\partial^2 u}{\partial {x_3}^2} = \frac{\partial^4 u}{\partial x_1 \partial x_2 \partial {x_3}^2}$$

In general for,

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), u(x) = u(x_1, x_2, \dots, x_n), \qquad D^{\alpha} u = \frac{\partial^{|\alpha|} u}{\partial {x_1}^{\alpha_1} . \partial {x_2}^{\alpha_2} \dots \partial {x_n}^{\alpha_n}}$$

Example 2 The second order linear differential equation

$$\sum_{|\alpha|\leq 2}^{n} a_\alpha(x) D^\alpha u(x) = f(x)$$

Can be written as

$$a_{(2,0)}u_{xx} + a_{(1,1)}(x)u_{xy} + a_{(0,1)}(x)u_{yy} \qquad |\alpha| = 2$$

$$+ \quad a_{(1,0)}(x)u_x + a_{(0,1)}(x)u_y \qquad |\alpha| = 1$$

$$+ \qquad a_{(0,0)}(x)u \qquad |\alpha| = 0$$

$$= f(x)$$

We will see few important facts from calculus below

<u>1.1 Green's identities</u>

a) Let Ω be a bounded, open subset of R^n such that $\partial\Omega$ is C^1 boundary of Ω . And Let
$n(x) = (n_1(x), \dots, n_n(x))$ be the unit outward normal derivative to $\partial\Omega$ at $x \in \partial\Omega$.
Ω to be a C^1 boundary and $u \in C^1(\overline{\Omega})$,then

$$\int_\Omega u_{xi}\, dx = \int_{\partial\Omega} u\, n_i d\sigma \quad , (i = 1, \dots, n)$$

In general we have,

$$\int_\Omega divw dx = \int_{\partial\Omega} w.n d\sigma$$

Where w is a C^1 vector field on $\overline{\Omega}$ and the dot " • " denotes the Euclidean product of vectors in $\mathbb{R}^n$, and dσ is the volume element of $\partial\Omega$.

b) Let $u, w \in C^1\bar{\Omega}$ then for $(i = 1, \dots, n)$

$$\int_{\Omega} u_{x_i} w dx = \int_{\partial\Omega} u w n_i d\sigma - \int_{\Omega} u w_{x_i} dx$$

Proof

$$\int_{\Omega} (uw)_{x_i} = \int_{\partial\Omega} (uw) n_i d\sigma \qquad \textit{from } (a)$$

But,

$$\int_{\Omega} (uw)_{x_i} = \int_{\Omega} (u_{x_i} w + u w_{x_i}) dx = \int_{\partial\Omega} (uw).n_i d\sigma$$

Therefore,

$$\int_{\Omega} u_{x_i} w \, dx = \int_{\partial\Omega} uw.n_i d\sigma - \int_{\Omega} u w_{x_i} \, dx$$

c) Let $u, w \in C^2(\bar{\Omega})$.Then we have

(i) $\int_{\Omega} \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial n} d\sigma$

Proof:

$\Delta u = div(Du)$

$$\int_{\Omega} \Delta u = \int_{\Omega} div(Du) dx = \int_{\Omega} (Du).n d\sigma = \int_{\partial\Omega} \frac{\partial u}{\partial n} d\sigma$$

.

(ii) $\int_{\Omega} u \Delta w dx = -\int_{\Omega} Du.Dw dx + \int_{\partial\Omega} u \frac{\partial w}{\partial n} d\sigma$

(Green's First Identity)

Proof

$$div(uDw) = div((uw_{x_1}, \dots, uw_{x_n})), \qquad where \; Dw = (w_{x_1}, \dots, w_{x_n})$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(u\frac{\partial w}{\partial x_i})$$

$$= \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}.\frac{\partial w}{\partial x_i} + \sum_{i=1}^{n} u\frac{\partial^2 w}{\partial {x_i}^2} = Du.Dw + u\Delta w.$$

Therefore,

$u\Delta w = div(uDw) - Du.Dw$

Integrating with respect to dx on Ω we get;

$$\int_\Omega u\ \Delta w dx = \int_\Omega div(uDw)\,dx - \int_\Omega Du\ Dwdx$$

$$= \int_{\partial\Omega} uDw.n\,d\sigma - \int_\Omega Du\ Dwdx \quad \ldots.by \ \text{(a)}$$

$$= \int_{\partial\Omega} u\tfrac{\partial w}{\partial n}\,d\sigma - \int_\Omega Du\ Dwdx$$

Therefore,

$$\int_\Omega u\Delta wdx = -\int_\Omega Du.Dwdx + \int_{\partial\Omega} u\tfrac{\partial w}{\partial n}\,d\sigma$$

.

(iii) $$\int_\Omega (u\Delta w - w\Delta \mathrm{u})dx = \int_{\partial\Omega} (u\tfrac{\partial w}{\partial n} - w\frac{\partial u}{\partial n})d\sigma$$

(Green's Second Identity)

Proof We consider the first Green's identity;

$$\int_\Omega u\Delta wdx = -\int_\Omega Du.Dwdx + \int_{\partial\Omega} u\tfrac{\partial w}{\partial n}\,d\sigma$$

and

$$\int_{\Omega} w\Delta u dx = -\int_{\Omega} Du.Dw dx + \int_{\partial\Omega} w\frac{\partial u}{\partial n} d\,\sigma \; for \; u, w \in C^2(\bar{\Omega})$$

Then subtract the two equations yields the Green's second identity;

$$\int_{\Omega} (u\Delta w - w\Delta u) dx = \int_{\partial\Omega} (u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}) d\sigma$$

Note that $Du.Dw = Dw.Du$ implies

$$-\int_{\Omega} Du.Dw dx + \int_{\Omega} Dw.Du dx = 0$$

1.2 Integrals in polar coordinates

If $x \in \mathbb{R}^n$, $x = |x|\frac{x}{|x|} = rw$, w here $r \in (0, \infty)$ and $w = \frac{x}{|x|} \in s^{n-1}\{x \in \mathbb{R}^n =: \; |x| = 1\}$ which is a unit sphere.

If f is lebesgue measurable function in $\mathbb{R}^n$ such that either $f \geq 0$ in $\mathbb{R}^n$ or $\in L^1(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_0^{\infty} \int_{s^{n-1}} f(rw)\, r^{n-1} d\sigma(w) dr$$

Also,

$$\int_{B(x^0, r)} f(x) dx \; = \int_0^{r} \int_{s^{n-1}} f(x^0 + \rho w)\, \rho^{n-1} d\sigma(w) d\rho$$

$$= \int_0^{r} \int_{\partial B(x^0, t)} f(w)\, r^{n-1} d\sigma(w) dt$$

The above formulas allow us to convert n-dimensional lebesgue integrals into integrals over spheres.

1.3 Change of Variable Formula

Let Ω be an open set in $\mathbb{R}^n$, and $\Psi: \Omega \to \mathbb{R}^n$ be a one-to-one C^1 function such that

$\Psi^{-1}: \Psi(\Omega) \to \Omega$ is al so C^1.suppose t hat f is l ebesgue integrable on $\Psi(\Omega)$.Then $f \text{o} \Psi$ is lebesgue integrable on Ω, and

$$\int_{\Psi(\Omega)} f(x)dx = \int_{\Omega} f(\Psi(x))\,|detJ\Psi(x)|dx$$

where $J\Psi(x)$ is the jacobian matrix of Ψ at $x \in \Omega$.

We also need to fix some notations

ω_n denotes the volume of the unit ball B (0, 1) in $\mathbb{R}^n$ (i.e. $\omega_n := |B(0,1)|$)

It follows that

$$|\partial B(0,1)| := n\omega_n.$$

where

$$\partial B(0,1) = \{x \in R^n : |x| = 1\}$$

Let

$$\Psi(\text{x}) = y + rx. then\, J\Psi(x) = rI.$$

Note that

$$|B(x,r)| = \int_{B(x,r)} dx = \int_{\Psi(B(0,1))} dx = \int_{B(0,1)} |\det(J\Psi(x))|dx$$

Since

$$\Psi(x) = (y_1 + rx_1, y_2 + rx_2, \dots, y_n + rx_n)$$

$$J\Psi(x) = \begin{bmatrix} r & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r \end{bmatrix}$$

$$|detJ\Psi(x)| = r^n$$

Thus

$$|B(x,r)| = \int_{B(0,1)} r^n \, dx = r^n \int_{B(0,1)} dx = r^n |B(0,1)| = r^n \omega_n$$

Therefore,

$$|B(x,r)| = r^n \omega_n$$

Similarly,

$$|\partial B(x,r)| = r^{n-1} n\omega_n$$

Theorem let $u \epsilon C^2(\Omega)$ satisfy $\Delta u = 0$ in Ω. Then for any ball $B = B_R(y) \subset\subset \Omega$, we have

(i) $$u(y) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B} u d\sigma$$

(ii) $$u(y) = \frac{1}{\omega_n R^n} \int_B u dx$$

For harmonic functions, the above theorem asserts that the function value at the center of the ball B is equal to the integral mean values over both the surface ∂B and B itself. These results known as the mean value theorems.

Proof let $\rho \epsilon (0, R)$ and apply

$$\int_\Omega \Delta u = \int_{\partial \Omega} \frac{\partial u}{\partial n} d\sigma$$

to the ball $B_\rho = B_\rho(y)$. We obtain

$$\int_{\partial B_\rho} \frac{\partial u}{\partial n} d\sigma = \int_{B_\rho} \Delta u = 0$$

Introducing a radial and angular coordinates $= |x - y|, \omega = \frac{x-y}{r}$, and writing

$u(x) = u(y + r\omega)$, we have

$$\int_{\partial B_\rho} \frac{\partial u}{\partial n} d\sigma = \int_{\partial B_\rho} \frac{\partial u}{\partial r}(y + \rho\omega) d\sigma = \rho^{n-1} \int_{|\omega|=1} \frac{\partial u}{\partial r}(y + \rho\omega) d\omega$$

$$= \rho^{n-1} \frac{\partial}{\partial \rho} \int_{|\omega|=1} u(y + \rho\omega) d\omega = \rho^{n-1} \frac{\partial}{\partial \rho}\left[\rho^{1-n} \int_{\partial B_\rho} u d\sigma\right] = 0$$

Consequently for any $\rho \epsilon (0, R)$,

$$\rho^{1-n} \int_{\partial B_\rho} u d\sigma = R^{1-n} \int_{\partial B_R} u d\sigma$$

And since

$$\lim_{\rho \to 0} \rho^{1-n} \int_{\partial B_\rho} u d\sigma = n\omega_n u(y)$$

relation (i) follow. To get relation (ii) we write (i) in the from

$$n\omega_n \rho^{n-1} u(y) = \int_{\partial B_\rho} u d\sigma \, , \rho \leq R$$

and integrate with respect to ρ from 0 to R. The relation (ii) follows immediately.

CHAPTER TWO

Integral Representation for Solutions of Elliptic Differential Operators

2.1 Elliptic differential operators

We begin this section with some preliminary definitions.

Definition 1 A domain Ω in RR^n is an open, bounded subset of R^n.

Definition 2 The bounda ry of a s et Ω is t he i ntersection o f t he cl osure o f Ω and t he closure of the complement Ω, which is denoted by $\partial\Omega$. that is $\partial\Omega = \bar{\Omega} \cap \overline{RR^n \backslash \Omega}$.

Let L be a lin ear p artial d ifferential o perator o f o rder m defined i n a n ope n s ubset Ω of RR^n. Assume that L can be represented by means of the standard coordinate system x in the following way:

$$Lu \coloneqq L(x,D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha u$$

Or simply,

$$L \coloneqq L(x,D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha \qquad , D^\alpha = \frac{\partial^{\alpha_1}}{\partial {x_1}^{\alpha_1}} \cdots \frac{\partial^{\alpha_m}}{\partial x^{\alpha_m}} \qquad (2.1.1)$$

Where t he co efficients $a_\alpha(x)$, $(|\alpha| \le m)$ are r eal valued f unctions de fined i n Ω.The principal p art (or l eading p art) o f L is th e o perator o btained b y d eleting a ll lo wer o rder terms:

$$L_\alpha(x,D) = \sum_{|\alpha| = m} a_\alpha(x) D^\alpha$$

The function $L_\alpha(x,\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha$, which is a homogeneous polynomial of degree m with respect to ξ, is said to be the principal of L.

Definition 3 an operator L is said to be elliptic of order m at x if and only if $L_\alpha(x,\xi) \neq 0$ for $\xi \in R$R^n and $\xi \neq 0$.

L is cal led el liptic i n Ω if L is e lliptic a t e very p oint o f Ω. A nd L is c alled uni formly elliptic in Ω if there exist positive numbers c_1 and c_2 (independent of(x,ξ)) such that the following inequality holds:

$$c_1|\xi|^m \leq |L_1(x,\xi)| \leq c_2|\xi|^m if\ x \in \bar{\Omega}\ and\ \xi \epsilon \mathrm{R}\ R^n \qquad 2.1.2$$

Remark1 ellipticity is a condition only on the leading part of L, no restriction is imposed on the coefficients of lower order terms.

Note the s ame d efinition o f e llipticity (Definition 3) a pplies to o rdinary d ifferential operators.

Suppose t hat a n n^{th} order or dinary di fferential e quation $Lu = f$, w ith s mooth coefficients $a_i(x)$ and a smooth f, is given by

$$a_n(x)\frac{d^n u(x)}{dx^n} + \cdots + a_1(x)\frac{du(x)}{dx} + a_0(x)u(x) = f(x)$$

$$\Rightarrow \quad L = a_n(x)\frac{d^n}{dx^n} + \cdots + a_1(x)\frac{d}{dx} + a_0(x)$$

Then the principal of L is just

$$L_n(x,\xi) = a_n(x)\xi^n.$$

So that L is elliptic on R^n provided $a_n(x)$ is non-vanishing on R.we use immediately that examples of ODEs are easy to obtain. All ODEs with constant coefficients are elliptic.

Consider a Bessel's equation

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + (x^2 - \lambda^2)u = 0 \ \ ,\lambda \in R$$

$$L = x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} + (x^2 - \lambda^2)$$

The principal of L is

$$L_2(x, \xi) = x^2 \xi^2$$

Which vanishes at $x = 0$, hence it shows L is non elliptic operator.

Definition 4 an $n \times n$ real symmetric matrix A is said to be a positive definite if

$$x^T A x > 0$$

For all non zero vector x with real entries ($x \in R^n$), where x^T denotes the transpose of x.

Example 2.1

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is positive definite for a vector with the entries $x = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$ the quadratic form is

$$x^T A x = [x_0, x_1] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = {x_0}^2 + {x_1}^2 > 0$$

Where the entries x_0, x_1 are real and at least one of them non zero, this shows A is positive definite.

Definition 5 an operator of second order with real coefficients of the form

$$L = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c(x). \qquad 2.1.3$$

With $a_{ij}(x) = a_{ji}(x)$ is elliptic at $x = (x_1, x_2, \dots x_n)$ if and only if the matrix $\left(a_{ij}\right)_{i,j=1}^{n}$ is positive definite.

Definition 6 L is also elliptic in(2.1.3), if there exists a positive function $\mu(x)$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \mu(x)\sum_{i=1}^{n} {\xi_i}^2 \; for\ all\ (\xi_1, \xi_2 \dots, \xi_n) \in \mathrm{R}\, R^n \qquad 2.1.4$$

Example 2.2 the Laplace operator Δ is elliptic

Check

$$Lu = \Delta u \quad \Rightarrow \sum_{i,j=1}^{n} \delta_{ij} D_{ij}\, u \quad \Rightarrow a_{ij} = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$\Rightarrow \quad a_{ij}\,\xi_i\xi_j = \delta_{ij}\,\xi_i\xi_j$$

$$\Rightarrow \qquad \sum_{i,j=1}^{n} a_{ij}\,\xi_i\xi_j = \sum_{i,j=1}^{n} \delta_{ij}\,\xi_i\xi_j$$

$$\geq \sum_{i,j=1}^{n} \delta_{ij}\,(\xi_i)^2 = \sum_{i,j=1}^{n} (\xi_i)^2 = \|\xi\|^2$$

$$\Rightarrow \qquad \sum_{i,j=1}^{n} a_{ij}\,\xi_i\xi_j \geq \|\xi\|^2$$

which is the ellipticity condition with $\mu(x) = 1$.

Consequently, the Laplace operator Δ is elliptic.

Example 2.3 given the minimum surface equation

$$Lu = div\left(\frac{Du}{\sqrt{1 + \|Du\|^2}}\right) = 0$$

The operator L is elliptic.

Check

$$div\left(\frac{Du}{\sqrt{1+\|Du\|^2}}\right)=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{D_i u}{\sqrt{1+\|Du\|^2}}\right) \qquad (*)$$

But,

$$\frac{\partial}{\partial x_i}\left(\frac{D_i u}{\sqrt{1+\|Du\|^2}}\right)=\frac{\sqrt{1+\|Du\|^2}\ \frac{\partial}{\partial x_i}(D_i u)-D_i u\frac{\partial}{\partial x_i}\sqrt{1+\|Du\|^2}}{1+\|Du\|^2}$$

$$=\frac{\sqrt{1+\|Du\|^2}\ D_{ii}u-D_i u\frac{\partial}{\partial x_i}\sqrt{\left(1+\sum_{j=1}^{n}(D_j u)^2\right)}}{1+\|Du\|^2}$$

$$=\frac{(1+\|Du\|^2)D_{ii}u-D_i u\sum_{j=1}^{n}D_j uD_{ij}u}{(1+\|Du\|^2)^{3/2}} \qquad (**)$$

Now, applying (**) into (*), we have

$$0=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{D_i u}{\sqrt{1+\|Du\|^2}}\right)=\sum_{i=1}^{n}\frac{(1+\|Du\|^2)D_{ii}u-D_i u\sum_{j=1}^{n}D_j uD_{ij}u}{(1+\|Du\|^2)^{3/2}}$$

$$\Rightarrow\ \sum_{i=1}^{n}\left((1+\|Du\|^2)D_{ii}u-D_i u\sum_{j=1}^{n}D_j uD_{ij}u\right)=0$$

$$\Rightarrow \quad \sum_{i=1}^{n}(1+\|Du\|^2)\,D_{ii}u-\sum_{i=1}^{n}\sum_{j=1}^{n}D_iuD_juD_{ij}u=0$$

$$\Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n}(1+\|Du\|^2)\delta_{ij}\,D_{ij}u-\sum_{i=1}^{n}\sum_{j=1}^{n}D_iuD_juD_{ij}u=0 \; where\; \delta_{ij}=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}$$

$$\Rightarrow \quad \sum_{i,j=1}^{n}\big([1+\|Du\|^2]\delta_{ij}-D_iuD_ju\big)D_{ij}u=0$$

$$\Rightarrow \quad a_{ij}=[1+\|Du\|^2]\delta_{ij}-D_iuD_ju$$

$$\Rightarrow \quad a_{ij}\,\xi_i\xi_j=[1+\|Du\|^2]\delta_{ij}\,\xi_i\xi_j-D_iuD_ju\xi_i\xi_j$$

$$\Rightarrow \sum_{i,j=1}^{n}a_{ij}\,\xi_i\xi_j=[1+\|Du\|^2]\sum_{i,j=1}^{n}\delta_{ij}\,\xi_i\xi_j-\sum_{i,j=1}^{n}D_iu\xi_iD_ju\xi_j$$

$$=[1+\|Du\|^2]\sum_{i=1}^{n}(\xi_i)^2-\sum_{i=1}^{n}(D_iu\xi_i)^2$$

$$=[1+\|Du\|^2]\|\xi\|^2-\|Du\xi\|^2 \;\geq\; [1+\|Du\|^2]\|\xi\|^2-\|Du\|^2\|\xi\|^2 \;=\; \|\xi\|^2$$

$$\Rightarrow \sum_{i,j=1}^{n} a_{ij}\,\xi_i\xi_j \;\geq \|\xi\|^2$$

which is the ellipticity condition with $\mu(x) = 1$

Consequently the operator L is elliptic

Definition 7 the operator

$$(L+h) = \sum_{i,j=1}^{n} a_{ij}\,(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + h \;,$$

where h is an operator of order less than the order of L.

is said to be elliptic at x if

$$L = \sum_{i,j=1}^{n} a_{ij}\,\frac{\partial^2}{\partial x_i \partial x_j}$$

is elliptic there. (See Remark 1)

Example 2.4 We consider a general secondary linear partial differential equation in$\mathbb{R}^2$.

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u + h(x,y) = 0$$

From this,

$$[a_{ij}] = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

The corresponding eigen values of a_{ij}, $\lambda_1 = \lambda_1(x,y)$ and $\lambda_2 = \lambda_2(x,y)$ are real and $\lambda_1\lambda_2 = ac - b^2$.So in this case the operator is elliptic at (x,y) if and only if $ac - b^2 > 0$.

2.1.1 The Laplace operator

The simplest partial differential equation of elliptic type is the familiar Laplace equation.

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial xi^2}, u \in C^2(\Omega)$$

Where

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial {x_i}^2}$$

is the Laplace operator.

2.1.1.1 The fundamental solution of Laplace operator

Let

$$L = L(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$$

be an elliptic differential operator of order m in an open subset Ω of $R\mathrm{R}^n$, where Ω may be a small neighborhood of a point or the whole space R^n.

Definition a f unction $E(x, y)$ is c alled f undamental s olution of L if fo r every $f(x) \in C_0{}^\infty(\Omega)$,

$$u(x) = \int_\Omega E(x, y) f(y) dy$$

Solves the equation

$$L(x, D)u(x) = f(x) \quad in\ \Omega.$$

which can be interpreted in the sense of the theory of generalized functions as

$$L(x, D)E(x, y) = \delta(x - y)$$

where δ is the Dirac delta-function defined as

$$\int f(x)\delta(x)dx = f(0).$$

Where $C_0^{\infty}(\Omega)$ in t he a bove de notes a s et of i nfinitely di fferentiable f unction w ith compact support.

A f undamental s olution $E(x, y)$ of t he Laplace o perator Δ is t he s olution of t he inhomogeneous equation

$$\Delta E(x, y) = \delta(x - y)$$

<u>Definition</u>:-A radial f unction is a f unction of t he f orm $u(x) = \gamma(|x|)$ for s ome functions $\gamma: [0, \infty) \to \mathbb{R}$.

Let

$$r(x) = |x| = \sqrt{x_1{}^2 +, \dots, +x_n{}^2}$$

For $i = 1, \dots, n$

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}(x_1{}^2 +, \dots, +x_n{}^2)^{\frac{-1}{2}}.2x_i = \frac{x_i}{r} \quad , (x \neq 0)$$

Now for $u(x) = \gamma(|x|)$ for some $\gamma \in C^2(0, \infty)$

$$\frac{\partial u}{\partial x_i} = \gamma_r(r)\frac{\partial r}{\partial x_i} = \gamma_r(r)\frac{x_i}{r}$$

$$\frac{\partial^2 u}{\partial x_i{}^2} = \frac{\partial}{x_i}(\gamma_r(r)\frac{x_i}{r})$$

$$= \gamma_{rr}\frac{x_i{}^2}{r^2} + \gamma_r(\frac{1}{r} + x_i \partial x_i \frac{1}{r})$$

Where

$$\partial x_i \frac{1}{r} = \partial x_i (x_1{}^2 +, \dots, +x_n{}^2)^{\frac{-1}{2}} = -\frac{1}{2}(x_1{}^2 +, \dots, +x_n{}^2)^{\frac{-3}{2}} 2x_i = -\frac{x_i}{r^3}$$

Then,

$$\frac{\partial^2 u}{\partial x_i{}^2} = \gamma_{rr}\frac{x_i{}^2}{r^2} + \gamma_r(\frac{1}{r} - \frac{x_i{}^2}{r^3})$$

Thus,

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial {x_i}^2}$$

$$= \sum_{i=1}^{n} \gamma_{rr} \frac{{x_i}^2}{r^2} + \gamma_r (\frac{1}{r} - \frac{{x_i}^2}{r^3})$$

$$= \gamma_{rr} + \gamma_r \left(\frac{n}{r} - \frac{1}{r}\right), x \neq 0$$

Therefore to find u such that $\Delta u = 0$ in $R\mathrm{R}^n \backslash \{0\}$, we solve the differential equation

$$\gamma_{rr} + \gamma_r \left(\frac{n-1}{r}\right) = 0$$

Assuming

$$\gamma(r) \neq 0 \quad , \qquad \frac{\gamma_{rr}}{\gamma_r} = \frac{1-n}{r}$$

Integrate in r

$$\Rightarrow \ln\gamma_r = (1-\mathrm{n})\ln \mathrm{r} + \mathrm{k}$$

$$\Rightarrow \gamma_r = c_1 r^{1-n}$$

Where, k *and* c_1 are constants.

Integrate again

$$for\ n = 2: \quad \gamma_r \frac{c_1}{r} \Rightarrow \gamma = c_1 ln r + c_2$$

$$n > 2: \quad \gamma_r \frac{c_1}{r^{n-1}} \Rightarrow \gamma = -\frac{1}{n-2} \frac{c_1}{r^{n-2}} + c_2$$

Consequently if $r > 0$, we have

$$\gamma(r) = \begin{cases} \frac{1}{2-n}\frac{c_1}{r^{n-2}} + c_2 & if\ n > 2 \\ c_1 \ln r + c_2 & if\ n = 2 \end{cases}$$

Where $c_1 and\ c_2$ are co nstants. T hat sh ows t hat $u(x): x \longmapsto \gamma(|x|)$ is a s olution o f $\Delta u = 0$ in$\mathbb{R}^n \backslash \{0\}$.

Therefore we set that by the translation invariance of the Laplace operator the function

$$u: x \longmapsto \gamma(|x-y|)$$

is a solution of

$$\Delta u = 0 \ \ on\ \mathbb{R}^n \backslash \{y\}$$

.

Note that there are infinitely many solutions of $\Delta u = 0 \ \ on\ \mathbb{R}^n \backslash \{y\}$. Since the constant c_2 is harmonic even at 0, it contributes nothing and can be omitted. We take $c_2 = 0$ but would like to choose the constant c_1 such that

$$\Delta_x \gamma(|x-y|) = \delta_y(x)\,, \ \ (\, x \in \ \mathbb{R}^n \backslash \{y\})$$

In the sense that

$$\int_{\mathbb{R}^n} \gamma(|x-y|)\, \Delta\varphi(y) dy = \varphi(x)$$

for all $\varphi \in C_c^{\ \infty}(\mathbb{R}^n)$

For this, suppose $\varphi \in c^{\infty}(\mathbb{R}^n)$ say φ is supported in the open set Ω and let $\epsilon > 0$ small enough

Such that

$$B_\epsilon(x) \subseteq \Omega\,, let\ \Omega_\in = \Omega \backslash \overline{B_\epsilon}(x)$$

We will assume for the case $n \geq 3$ the case $n = 2$ being similar.

Let $\upsilon(y)$ denote the outer normal to $\partial\Omega_\in$ at y. Note that at $y \in \partial B_\epsilon(x)$

$$v(y) = \frac{x-y}{|x-y|}$$

By Green's second identity we see that

$$\int_{\Omega_\epsilon} (\gamma(|x-y|)\Delta\varphi(y) - \Delta\gamma(|x-y|)\varphi(y))dy$$

$$= \int_{\partial\Omega_\in} \left(\gamma(|x-y|)\frac{\partial\varphi}{\partial v} - \varphi(y)\frac{\partial\gamma(|x-y|)}{\partial v}\right) d\sigma(y)$$

Since

$$\Delta\gamma(|x-y|) = 0 \; on \; \Omega_\in \; ,$$

and recalling that $\varphi = 0$ near $\partial\Omega$ we see that

$$\int_{\Omega_\epsilon} \gamma(|x-y|)\Delta\varphi(y)dy = \int_{\partial\Omega_\in} \left(\gamma(|x-y|)\frac{\partial\varphi}{\partial v} - \varphi(y)\frac{\partial\gamma(|x-y|)}{\partial v}\right) d\sigma(y)$$

$$= \int_{|x-y|=\in} \left(\gamma(|x-y|)\frac{\partial\varphi}{\partial v} - \varphi(y)\frac{\partial\gamma(|x-y|)}{\partial v}\right) d\sigma(y)$$

$$= \frac{c_1}{(2-n)\in^{n-2}} \int_{|x-y|=\in} \frac{\partial\varphi}{\partial v} d\sigma(y) - \int_{|x-y|=\in} \varphi(y)\frac{\partial\gamma(|x-y|)}{\partial v} d\sigma(y)$$

Here v is the outer normal vector field, note that since

$$\gamma(|x-y|)|\Delta\varphi(y)|x_{\Omega_\epsilon}(y) \leq \gamma(|x-y|)|\Delta\varphi(y)|$$

For a ll $\mathcal{E} > 0$, and $\gamma(|x-y|)\Delta\varphi(y)$ is in tegrable o n $\mathbb{R}^n$,by t he l ebesgue dom inating convergence theorem we see that

$$\lim_{\in\to 0} \int_{\Omega_\epsilon} \gamma(|x-y|)\Delta\varphi(y)dy = \int_{\Omega} \gamma(|x-y|)\Delta\varphi(y)dy$$

Since $\varphi \in c^\infty(\Omega)$, and hence $D\varphi(y)$ is bounded it is clear that

$$\lim_{\in\to 0} \frac{c_1}{(2-n)\in^{n-2}} \int_{|x-y|=\in} \frac{\partial\varphi}{\partial v} \, d\sigma(y) = 0$$

Note that this limit is still true provided that $D\varphi$ is locally bounded on Ω.

Also, on $\partial B_\in(x)$

$$\frac{\partial \gamma(|x-y|)}{\partial \upsilon} = D_y\ \gamma(|x-y|)\ .\ \upsilon$$

$$= D_y\ \gamma(|x-y|)\,\frac{x-y}{|x-y|}$$

$$= \frac{c_1}{(2-n)}(n-2)\frac{|y-x|}{|x-y|}|x-y|^{1-n}.\frac{x-y}{|x-y|}$$

$$= c_1|x-y|^{1-n}$$

Therefore,

$$\int_{|x-y|=\epsilon} \varphi(y)\frac{\partial \gamma(|x-y|)}{\partial \upsilon}\,d\sigma(y) = \frac{c_1}{\epsilon^{n-1}}\int_{|x-y|=\epsilon} \varphi(y)d\sigma(y)$$

$$= c_1 n\omega_n ave \int_{|x-y|=\epsilon} \varphi(y)d\sigma(y)$$

$$\rightarrow\ =\ c_1 n\omega_n \varphi(x)$$

as $\epsilon \rightarrow 0$

Therefore we see that

$$\int_{\Omega} \gamma(|x-y|)\Delta\varphi(y)dy = c_1 n\omega_n \varphi(x) = \varphi(x)$$

Hence we choose $c_1 = \frac{1}{n\omega_n}$

Similarly, For case n=2, we have,

$\int_{\Omega} \gamma(|x-y|)\Delta\varphi(y)dy = c_1 2\pi\varphi(x)$,$since\ \partial B(0,1) = 2\pi, when\ n = 2$

Hence we choose $c_1 = \frac{1}{2\pi}$

Therefore for $x \neq 0$, we define

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi} ln|x| & if\ n = 2 \\ \frac{1}{n(2-n)\omega_n |x|^{n-2}} & if\ n > 2 \end{cases}$$

Note $\Delta\Gamma = 0$ for $x \neq 0$ but not defined if $x = 0$. $\Delta\Gamma$ can be defined everywhere in the sense of distributions,

$$\Delta\Gamma = \delta \qquad , \quad \delta_0(x) = \begin{cases} 1 & , x = 0 \\ 0 & , x \neq 0 \end{cases}$$

$$\Delta\Gamma(x-y) = \begin{cases} 0 & , x \neq y \\ 1 & , x = y \end{cases}$$

Formally, this suggests a solution formula for a poison equation

$\Delta u = f$ in $R\mathrm{R}^n$

Check

$$\Delta u = \int_{R^n} \Delta_x \Gamma(x-y) f(y) dy \ = \int_{R^n} \delta_x(y) f(y) dy = f(x)$$

Definition:-The function

$$\Gamma(|x-y|) = \begin{cases} \frac{1}{2\pi} ln|x-y| & if\ n = 2 \\ \frac{1}{n(2-n)\omega_n |x-y|^{n-2}} & if\ n > 2 \end{cases}$$

is the fundamental solution of Laplace operator.

By simple computation we have

$$D_i\Gamma(x-y) = \frac{1}{n\omega_n}(x_i - y_i)|x-y|^{-n}$$

$$D_{ij}\Gamma(x-y) = \frac{1}{n\omega_n}\{|x-y|^2\delta_{ij} - n(x_i-y_i)(x_j-y_j)\}|x-y|^{-n-2}$$

Clearly, Γ is harmonic for $x \neq y$.

<u>Lemma 2.1.1.1</u> we have

a) $$|D\Gamma(x)| \le \frac{c}{|x|^{n-1}} \quad , x \neq 0$$

b) $$|D^2\Gamma(x)| \le \frac{c}{|x|^{n}} \quad x \neq 0$$

c) $$\frac{\partial \Gamma(x)}{\partial v} = \frac{1}{n\omega_n r^{n-1}} \forall\, x \in \partial B(0,r)$$

<u>Note</u>

$$\frac{\partial \Gamma}{\partial v}(x) = D\Gamma(x).\frac{x}{|x|} \qquad where\ v(x) = \frac{x}{|x|}$$

Proof

a) $$for\ n = 2: \qquad \frac{\partial ln|x|}{\partial x_i} = \frac{1}{|x|}\frac{\partial |x|}{\partial x_i} = \frac{1}{|x|}\frac{x_i}{|x|}$$

$$D\Gamma(x) = \frac{1}{2\pi}\frac{x}{|x|^2} \quad \Rightarrow |D\Gamma(x)| \le \frac{c}{|x|}$$

$$for\ n > 2: \qquad \frac{\partial}{\partial x_i}\frac{1}{|x|^{n-2}} = \frac{\partial}{\partial x_i}\left[\sum {x_k}^2\right]^{\frac{-(n-2)}{2}}$$

$$= -\frac{(n-2)}{2}\left[\sum {x_k}^2\right]^{-(\frac{n-2}{2}+1)} 2x_i = -(n-2)\frac{x_i}{|x|^n}$$

$$D\Gamma(x) = \frac{-1}{n(2-n)\omega_n}.\left[\frac{-(n-2)x}{|x|^n}\right]$$

$$D\Gamma(x) = \frac{1}{n\omega_n}\frac{x}{|x|^n}$$

$$|D\Gamma(x)| \leq \frac{1}{n\omega_n}\frac{|x|}{|x|^n} = \frac{1}{n\omega_n}\frac{1}{|x|^{n-1}}$$

$$\Rightarrow \quad |D\Gamma(x)| \leq \frac{c}{|x|^{n-1}}$$

$$b) \qquad for\ \ n = 2\ ;\ \ \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} ln|x| = \frac{\partial}{\partial x_j}\left[\frac{x_i}{|x|^2}\right] = \frac{\delta_{ij}}{|x|^2} + x_i \frac{\partial}{\partial x_j}\frac{1}{|x|^2}$$

But

$$\frac{\partial}{\partial x_j}\frac{1}{|x|^2} = \frac{\partial}{\partial x_j}\left[\sum {x_k}^2\right]^{-1} = -\left[\sum {x_k}^2\right]^{-2} 2x_j = -2\frac{x_j}{|x|^4}$$

$$D^2\Gamma(x) = \frac{1}{2\pi}\left[\frac{\delta_{ij}}{|x|^2} - \frac{x_i x_j}{|x|^4}\right]$$

$$|D^2\Gamma(x)| \leq \frac{1}{2\pi}\left[\frac{1}{|x|^2} - \frac{2|x_i||x_j|}{|x|^4}\right] \leq \frac{1}{2\pi}\frac{3}{|x|^2}$$

$$\Rightarrow \qquad D^2\Gamma(x) \leq \frac{c}{|x|^2}$$

$$for\ \ \ n > 2\ \ ;\quad \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\left[\frac{1}{|x|^{n-2}}\right] = \frac{\partial}{x_j}\left(-(n-2)\frac{x_i}{|x|^n}\right)$$

$$= -(n-2)\left[\frac{\partial}{\partial x_j} x_i \frac{1}{|x|^n} + x_i \frac{\partial}{\partial x_j}\frac{1}{|x|^n}\right]$$

$$= -(n-2)\left[\frac{\delta_{ij}}{|x|^n} + x_i \frac{\partial}{\partial x_j}\frac{1}{|x|^n}\right]$$

But

$$\frac{\partial}{\partial x_j}\frac{1}{|x|^n} = \frac{\partial}{\partial x_j}\left[\sum {x_k}^2\right]^{\frac{-n}{2}} = \frac{-n}{2}\left[\sum {x_k}^2\right]^{\frac{-(n+1)}{2}} . 2x_i = \frac{-nx_j}{|x|^{n+1}}$$

$$D^2\Gamma(x) = \frac{1}{n(2-n)\omega_n}(-1)(n-2))\left[\frac{\delta_{ij}}{|x|^n} - nx_i x_j \frac{1}{|x|^{n+1}}\right]$$

$$= \frac{1}{n\omega_n}\left[\frac{\delta_{ij}}{|x|^n} - nx_i x_j \frac{1}{|x|^{n+1}}\right]$$

$$|D^2\Gamma(x)| \leq \frac{1}{n\omega_n}\left[\frac{1}{|x|^n} - n\frac{|x_i||x_j|}{|x|^{n+1}}\right] \leq \frac{1}{n\omega_n}\left[\frac{1}{|x|^n} - \frac{n}{|x|^n}\right] \leq \frac{1-n}{n\omega_n}\frac{1}{|x|^n}$$

$$\Rightarrow \quad |D^2\Gamma(x)| \leq \frac{c}{|x|^n}$$

.

$$c) \quad for \quad n = 2 \quad ; \; D\Gamma(x) = \frac{1}{2\pi} Dln|x| = \frac{1}{2\pi}\frac{1}{|x|}\frac{x}{|x|}$$

$$\Rightarrow \quad \frac{\partial\Gamma}{\partial v}(x) = \frac{1}{2\pi}\frac{x}{|x|^2}\frac{x}{|x|} = \frac{1}{2\pi}\frac{|x|^2}{|x|^2|x|} = \frac{1}{2\pi}\frac{1}{|x|}$$

$$for \;\; n > 2 \;\; ; \; \frac{\partial\Gamma}{\partial v}(x) = \; D\Gamma(x).\frac{x}{|x|} = \frac{1}{n\omega_n}\frac{x}{|x|^n}\frac{x}{|x|} = \frac{1}{n\omega_n}\frac{|x|^2}{|x|^{n+1}} = \frac{1}{n\omega_n |x|^{n-1}}$$

$$= \frac{1}{n\omega_n r^{n-1}}$$

Therefore,

$$\frac{\partial\Gamma}{\partial v}(x) = \frac{1}{n\omega_n r^{n-1}} \;\; \forall \, x \in \partial B(0,r)$$

Theorem (Integral Representation)

Let Ω be a domain with c^1 boundary. If $u \in c^2(\overline{\Omega})$ .Then for any $x \in \Omega$.

$$u(x) = \int_{\Omega} \Gamma(x-y)\Delta\, u(y)dy$$
$$- \int_{\partial\Omega} \left(\Gamma(x-y)\frac{\partial u(y)}{\partial v} - \mathrm{u}(y)\frac{\partial\Gamma(x-y)}{\partial v} \right) d\sigma(y) \qquad (1)$$

<u>Proof</u> Γ is defined on $\mathbb{R}^n \backslash \{y\}$ as follows

$$\Gamma(x-y) = \begin{cases} \dfrac{1}{2\pi} ln|x-y| & if\ n=2 \\ \dfrac{1}{n(2-n)\omega_n} \dfrac{1}{|x-y|^{n-2}} & if\ n>2 \end{cases}$$

$\Gamma(x-y)$ Satisfies $\Delta\Gamma(x-y) = 0$ in $\mathbb{R}^n \backslash \{y\}$ for $n \geq 2$. let $x \in \Omega$.

Then $B(y,\epsilon) \subset\subset \Omega$ for some $\epsilon > 0$. Let $\Omega_\epsilon = \Omega \backslash \overline{B}(y,\epsilon)$

Now $u(y)$ *and* $\Gamma(x-y)$ are in $c^2(\overline{\Omega_\epsilon})$ .We apply Green's second identity

$$\int_{\Omega_\epsilon} (\Gamma(x-y)\Delta\, u(y) - u(y)\Delta\Gamma(x-y))dy$$
$$= \int_{\partial\Omega_\epsilon} \left(\Gamma(x-y)\frac{\partial u}{\partial v}(y) - \mathrm{u}(y)\frac{\partial\Gamma(x-y)}{\partial v} \right) d\sigma(y)$$

Since $\Gamma(x-y)$ is harmonic in Ω_ϵ ,

$$\int_{\Omega_\epsilon} \Gamma(x-y)\Delta\, u(y)dy = \int_{\partial\Omega_\epsilon} \left(\Gamma(x-y)\frac{\partial u}{\partial v}(y) - \mathrm{u}(y)\frac{\partial\Gamma(x-y)}{\partial v} \right) d\sigma(y)$$
$$= \int_{\partial\Omega} \left(\Gamma(x-y)\frac{\partial u}{\partial v}(y) - \mathrm{u}(y)\frac{\partial\Gamma(x-y)}{\partial v} \right) d\sigma(y)$$
$$+ \int_{\partial B(y,\epsilon)} \left(\Gamma(x-y)\frac{\partial u}{\partial v}(y) - \mathrm{u}(y)\frac{\partial\Gamma(x-y)}{\partial v} \right) d\sigma(y) \qquad (2)$$

$$\underbrace{\int_{\Omega_\epsilon} \Gamma(x-y)\Delta\, u(y)dy}_{=I_\epsilon}$$

$$= \int_{\partial\Omega} \left(\Gamma(x-y) \frac{\partial u}{\partial \upsilon}(y) - \mathrm{u}(y) \frac{\partial \Gamma(x-y)}{\partial \upsilon} \right) d\sigma(y)$$

$$+ \underbrace{\int_{\partial B(y,\epsilon)} \left(\Gamma(x-y) \frac{\partial u}{\partial \upsilon}(y) \right) d\sigma(y)}_{=J_\epsilon} - \underbrace{\int_{\partial B(y,\epsilon)} \left(\mathrm{u}(y) \frac{\partial \Gamma(x-y)}{\partial \upsilon} \right) d\sigma(y)}_{K_\epsilon}$$

But $\int_{\Omega_\epsilon} u(y)\Delta\Gamma(x-y) = 0, since\ \Gamma(x-y)$ satisfies Laplace equation.

$$I_\epsilon = \int_{\Omega_\epsilon} \Gamma(x-y)\Delta\, u(y)\,\mathrm{dy}$$

$$= \int_{\Omega} \Gamma(x-y)\Delta\, u(y) x_{\Omega_\epsilon}(y)\,\mathrm{dy}$$

Note $x_{\Omega_\epsilon}(y) \to x_\Omega\,(y)\ as\ \epsilon \to 0$

$$\left|\Gamma(x-y)\Delta\, u(y) x_{\Omega_\epsilon}(y)\right| \leq |\Gamma(x-y)||\Delta\, u(y)|$$

$$I_\epsilon \to \int_{\Omega} \Gamma(x-y)\Delta\, u(y) x_\Omega\,(y)\,\mathrm{dy} = \int_{\Omega} \Gamma(x-y)\Delta\, u(y)\,\mathrm{dy}$$

as $\epsilon \to 0$

$$|J_\epsilon| = \left| \int_{\partial \mathrm{B}(y,\epsilon)} \Gamma(x-y) \frac{\partial u}{\partial \upsilon}(y) d\sigma(y) \right|$$

$$= c_n \max_{\Omega} |Du| \int_{\partial \mathrm{B}(y,\epsilon)} \frac{1}{|\mathrm{x}-\mathrm{y}|^{\mathrm{n}-2}} d\sigma(y)$$

$$= c_n \max_{\Omega} |Du| \int_0^\epsilon \int_{S^{n-1}} \frac{1}{\mathrm{r}^{\mathrm{n}-2}} \mathrm{r}^{\mathrm{n}-1} d\sigma(w) dr$$

$$= n\omega_n c_n \max_{\Omega} |Du| \int_0^\epsilon r\, dr$$

$$\Longrightarrow |J_\epsilon| \le c_n \max_{\Omega}|Du|\,\epsilon^2 \to 0 \; as\; \epsilon \to 0$$

$$\therefore \quad \lim_{\epsilon\to 0} J_\epsilon = 0$$

$$K_\epsilon = \int_{\partial B(y,\epsilon)} \mathrm{u}(y)\frac{\partial \Gamma(x-y)}{\partial \upsilon}\,d\sigma(y), y\epsilon\,\partial B(y,\epsilon)$$

$$\frac{\partial \Gamma(x-y)}{\partial \upsilon} = D\Gamma(x-y).\upsilon(y)$$

$$= D\Gamma(x-y).\frac{x-y}{|x-y|}\;, \upsilon(y) = \frac{x-y}{|x-y|}\; for\; x \in \partial B(y,\epsilon)$$

$$\Gamma(x-y) = \frac{1}{n(2-n)\omega_n}|\mathrm{x}-\mathrm{y}|^{2-\mathrm{n}}$$

$$D\Gamma(x-y) = \frac{1}{n\omega_n}|\mathrm{x}-\mathrm{y}|^{1-\mathrm{n}}\frac{\mathrm{y}-\mathrm{x}}{|\mathrm{y}-\mathrm{x}|}$$

$$\therefore \qquad \frac{\partial \Gamma(x-y)}{\partial \upsilon} = \frac{1}{n\omega_n}|\mathrm{y}-\mathrm{x}|^{-\mathrm{n}}(\mathrm{y}-\mathrm{x})\frac{\mathrm{x}-\mathrm{y}}{|\mathrm{x}-\mathrm{y}|}$$

$$= -\frac{1}{n\omega_n}|\mathrm{y}-\mathrm{x}|^{-\mathrm{n}}\frac{|\mathrm{y}-\mathrm{x}|^2}{|\mathrm{y}-\mathrm{x}|}$$

$$= -\frac{1}{n\omega_n}|\mathrm{y}-\mathrm{x}|^{1-\mathrm{n}}$$

$$K_\epsilon = \int_{\partial B(y,\epsilon)} \mathrm{u}(y)\frac{\partial \Gamma(x-y)}{\partial \upsilon}\,d\sigma(y)$$

$$= -\frac{1}{n\omega_n}\int_{\partial B(y,\epsilon)} \frac{\mathrm{u(y)}}{|\mathrm{x}-\mathrm{y}|^{\mathrm{n}-1}}\,d\sigma(y)$$

$$= -\frac{1}{n\omega_n\epsilon^{n-1}}\int_{\partial B(y,\epsilon)} \mathrm{u(y)}\,d\sigma(y)$$

$$= \quad -\frac{1}{|\partial B(y,\epsilon)|}\int_{\partial B(y,\epsilon)} \mathrm{u(y)}\,d\sigma(y)$$

But

$$\frac{1}{|\partial \mathrm{B}(y,\epsilon)|}\int_{\partial \mathrm{B}(y,\epsilon)} \mathrm{u}(\mathrm{y})d\sigma(y) = \frac{1}{|\partial \mathrm{B}(y,\epsilon)|}\int_{\partial \mathrm{B}(y,\epsilon)} \big(\mathrm{u}(\mathrm{y}) - \mathrm{u}(\mathrm{x})\big)d\sigma(y) + u(x)$$

Since

$$\frac{1}{|\partial \mathrm{B}(y,\epsilon)|}\int_{\partial \mathrm{B}(y,\epsilon)} \mathrm{u}(\mathrm{x})d\sigma(y) = u(x)$$

After taking limit as $\epsilon \to 0$ (2) becomes;

$$\int_{\Omega} \Gamma(x-y)\Delta\, u(y)dy = \int_{\partial\Omega} \left(\Gamma(x-y)\frac{\partial u}{\partial v}(y) - \mathrm{u}(y)\frac{\partial \Gamma(x-y)}{\partial v}\right)d\sigma(y) + u(x)$$

Hence

$$u(x) = \int_{\Omega} \Gamma(x-y)\Delta\, u(y)dy - \int_{\partial\Omega} \left(\Gamma(x-y)\frac{\partial u}{\partial v}(y) - \mathrm{u}(y)\frac{\partial \Gamma(x-y)}{\partial v}\right)d\sigma(y)$$

<u>Lemma 2.1.1.2</u>: If $u \in c^2(\overline{\Omega})$ satisfies $\Delta u = 0 \quad in\ \Omega$ then

$$u(x) = -\int_{\partial\Omega} \left(\Gamma(x-y)\frac{\partial u}{\partial v}(y) - \mathrm{u}(y)\frac{\partial \Gamma(x-y)}{\partial v}\right)d\sigma(y)$$

<u>Proof</u>

The proof follows from the above theorem

$$u(x) = \underbrace{\int_{\Omega} \Gamma(x-y)\Delta\, u(y)dy}_{=0} - \int_{\partial\Omega} \left(\Gamma(x-y)\frac{\partial u}{\partial v}(y) - \mathrm{u}(y)\frac{\partial \Gamma(x-y)}{\partial v}\right)d\sigma(y)$$

Since

$$\Delta u = 0$$

Therefore

$$u(x) = -\int_{\partial\Omega} \left(\Gamma(x-y)\frac{\partial u}{\partial v}(y) - \mathrm{u}(y)\frac{\partial \Gamma(x-y)}{\partial v}\right)d\sigma(y)$$

Note that the normal derivative $\frac{\partial u}{\partial v}$ on $\partial\Omega$ is not known, so we would like to modify the above formula to get a formula which does not involve $\frac{\partial u}{\partial v}$.

Suppose for each $x \in \Omega$, there is a function ϕ^x such that

$$\begin{cases} \Delta\phi^x(y) = 0 & in\ \Omega \\ \phi^x(y) = -\Gamma(x-y) & on\ \partial\Omega \end{cases}$$

Using Green's second identity on Ω with $u(y)$ and $\phi^x(y)$ we get

$$\int_\Omega \phi^x(y)\Delta u(y)dy = \int_{\partial\Omega}\left(\phi^x(y)\frac{\partial u}{\partial v}(y) - \mathrm{u}(y)\frac{\partial \phi^x(y)}{\partial v}\right)d\sigma(y) \qquad (3)$$

Adding (3) to (1) we get,

$$u(x) = \int_\Omega \big(\Gamma(x-y) + \phi^x(y)\big)\Delta u(y)dy$$

$$- \int_{\partial\Omega}\left((\Gamma(x-y) + \phi^x(y))\frac{\partial u}{\partial v}(y) - \mathrm{u}(y)\frac{\partial(\Gamma(x-y) + \phi^x(y))}{\partial v}\right)d\sigma(y)$$

$$u(x) = \int_\Omega \big(\Gamma(x-y) + \phi^x(y)\big)\Delta u(y)dy + \int_{\partial\Omega} \mathrm{u}(y)\frac{\partial(\Gamma(x-y) + \phi^x(y))}{\partial v}d\sigma(y)$$

$$- \int_{\partial\Omega}(\Gamma(x-y) + \phi^x(y))\frac{\partial u}{\partial v}(y)d\sigma(y) \qquad (4)$$

2.1.1.2 Green's function of Laplace operator

Definition a function $G(x,y)$ defined for $x, y \in \bar{\Omega}\ x \neq y$ is called the Green's function for the Laplace operator on Ω. If

(1) $G(x,y) = 0\ \ for\ x \in \partial\Omega$

(2) $G(x,y) = \Gamma(x-y) + \phi^x(y)$ is harmonic in $x \in \Omega$

Now (4) becomes

$$u(x) = \int_\Omega \mathrm{G(x,y)}\Delta u(y)dy + \int_{\partial\Omega} \mathrm{u}(y)\frac{\partial G(x,y)}{\partial v}d\sigma(y) - \int_{\partial\Omega} \underbrace{\mathrm{G(x,y))}}_{=0}\frac{\partial u}{\partial v}(y)d\sigma(y)$$

$$u(x) = \int_\Omega \mathrm{G(x,y)}\Delta u(y)dy + \int_{\partial\Omega} \mathrm{u}(y)\frac{\partial G(x,y)}{\partial v}d\sigma(y)$$

Using this formula we can obtain a solution of the Dirichlet problem for the poison equation

$$\begin{cases}\Delta u = f\ in\ \Omega \\ u = \varphi\ on\ \partial\Omega\end{cases}$$

By the representation formula

$$u(x) = \int_{\Omega} \mathrm{G(x,y)} f(y) dy + \int_{\partial\Omega} \varphi(\mathrm{y}) \frac{\partial G(x,y)}{\partial \upsilon} d\sigma(y)$$

Similar to the Dirichlet problem if

$$\begin{cases} \Delta u = 0 \ in\ \Omega \\ u = \varphi \ on\ \partial\Omega \end{cases} \qquad (5)$$

The solution is given by

$$u(x) = \int_{\partial\Omega} \varphi(\mathrm{y}) \frac{\partial G(x,y)}{\partial \upsilon} d\sigma(y)$$

Here $u(x)$ is the integral representation for the solution of the equation (5)

Theorem: The G reen's f unction $G(x,y)$ is s ymmetric th at i s $(x,y) = G(y,x)$, fo r $x, y \in \Omega\,, x \neq y$.

Proof Take $x, y \in \Omega\, with\, x \neq y$. choose $r > 0$ small enough s uch t hat $B(x,r) \cap B(y,r) = \emptyset$. A nd c onsider $B(x,r) \subset \Omega$ and $B(y,r)$. Se t $G_x(z) = G(x,z)$ a nd $G_y(z) = G(y,z)$ .By applying Green's formula in $\Omega \backslash [(B(x,r) \cap B(y,r)]$ we get,

$$\int_{\Omega \backslash B(x,r) \cap B(y,r)} \left(G_x \Delta\, G_y - G_y \Delta\, G_x \right) dz = \int_{\partial\Omega} (G_x \frac{\partial G_y}{\partial \upsilon} - G_y \frac{\partial G_2}{\partial \upsilon}) d\sigma$$

$$- \int_{\partial B(x,r)} (G_x \frac{\partial G_y}{\partial \upsilon} - G_y \frac{\partial G_x}{\partial \upsilon}) d\sigma - \int_{\partial B(y,r)} (G_x \frac{\partial G_y}{\partial \upsilon} - G_y \frac{\partial G_x}{\partial \upsilon}) d\sigma$$

Since G_x and G_y are harmonic for $x \neq z$ and $y \neq z$ respectively, and vanishes on $\partial\Omega$ we have

$$\int_{\partial B(x,r)} (G_x \frac{\partial G_y}{\partial \upsilon} - G_y \frac{\partial G_x}{\partial \upsilon}) d\sigma + \int_{\partial B(x_y,r)} (G_x \frac{\partial G_y}{\partial \upsilon} - G_y \frac{\partial G_x}{\partial \upsilon}) d\sigma = 0$$

Note that the left side has the same limit as the left side in the following as $r \to 0$

$$\int_{\partial B(x,r)} (\Gamma \frac{\partial G_x}{\partial \upsilon} - G_y \frac{\partial \Gamma}{\partial \upsilon}) d\sigma + \int_{\partial B(y,r)} (G_x \frac{\partial \Gamma}{\partial \upsilon} - \Gamma \frac{\partial G_x}{\partial \upsilon}) d\sigma = 0$$

While we have

$$\int_{\partial B(x,r)} \Gamma \frac{\partial G_y}{\partial v} d\sigma \to 0, \int_{\partial B(y,r)} \Gamma \frac{\partial G_x}{\partial v} d\sigma \to 0 \; as \;\; r \to 0$$

And

$$\int_{\partial B(x,r)} G_y \frac{\partial \Gamma}{\partial v} d\sigma \to G_y(x), \int_{\partial B(y,r)} G_x \frac{\partial \Gamma}{\partial v} d\sigma \to G_x(y) \; as \; r \to 0$$

This implies

$$G_y(x) - G_x(y), or \; G(y,x) = G(x,y)$$

2.2 Integral Representations for Solutions of an operator

Let Ω be a n ope n t hree dimension r egion in R^3 and t he bounda ry $\partial\Omega$ be a si mply connected, infinitely smooth surface.

Let $a \in (R^3), a(x) > 0$

Consider the following scalar differential operators;

$$L_a = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left[a(x)\frac{\partial}{\partial x_i}\right]$$

$$E_a(u,v) = \sum_{i=1}^{3} a(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_i} = a(x)\nabla u(x)\nabla v(x)$$

$$T_a(x, n(x), \partial_x) = \sum_{i=1}^{3} a(x) n_i(x) \frac{\partial}{\partial x_i} = a(x)\frac{\partial}{\partial n(x)}$$

We want to show that L_a is elliptic operator as follows;

L_a can be written as:

$$L_a = \sum_{i=1}^{3} a(x)\frac{\partial^2}{\partial {x_i}^2} + \sum_{i=1}^{3} b_i \frac{\partial}{\partial x_i} \qquad , \quad b_i = \frac{\partial}{\partial x_i} a(x) \qquad\qquad (2.2.1)$$

Comparing (2.2.1) to Eq.(2.1.3), L_a is an operator of second order to show its ellipticity, we use Defintion6.

Here in (2.2.1) $a_{ij} = a(x) and\ i = j$

$$a_{ij}\,\xi_i\xi_j = a(x)\xi_i\xi_j$$

$$\sum_{i,j=1}^{3} a_{ij}\,\xi_i\xi_j = \sum_{i,j=1}^{3} a(x)\xi_i\xi_j = a(x)\sum_{i,j=1}^{3} \xi_i\xi_j = a(x)\sum_{i=1}^{3} {\xi_i}^2 = a(x)\|\xi\|^2$$

This implies that L_a is elliptic with $\mu(x) = a(x) > 0$.

We can get the first and second Green's identity for an operator L_a ;

$$\int_\Omega [vL_a u + E_a(u,v)]dx = \int_{\partial\Omega} vT_a u d\sigma$$

(First Green's identity for an operator L_a)

$$\int_\Omega (vL_a u - uL_a v)dx = \int_{\partial\Omega} (vT_a u - uT_a v)d\sigma$$

(Second Green's identity for an operator L_a)

To get the first identity for an operator L_a , we begin by

$$L_a u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[a(x)\frac{\partial u}{\partial x_i}\right] = div(aDu)$$

Recall

$$div(aDu) = Da.Du + a\Delta u$$

Integrate both sides over Ω yields

$$\int_{\Omega}(Da.Du + a\Delta u)dx = \int_{\Omega} div(aDu)dx = \int_{\partial\Omega} aDu.nd\sigma = \int_{\partial\Omega} a\frac{\partial u}{\partial n}d\sigma = \int_{\partial\Omega} T_a u d\sigma$$

$$vL_a u = vdiv(aDu) = vDa.Du + av\Delta u$$

$$\int_{\Omega} vL_a udx = \int_{\Omega}(vDa.Du + av\Delta u)dx \qquad (2.2.2)$$

But,

$$\int_{\Omega} av\Delta udx = -\int_{\Omega} D(av).Dudx + \int_{\partial\Omega} av\frac{\partial u}{\partial n}d\sigma$$

$$= -\int_{\Omega} vDa.Dudx - \int_{\Omega} aDv.Dudx + \int_{\partial\Omega} av\frac{\partial u}{\partial n}d\sigma \qquad (2.2.3)$$

Inserting $(**)$ in to $(*)$ yields

$$\int_{\Omega} vL_a udx = -\int_{\Omega} aDu.Dvdx + \int_{\partial\Omega} av\frac{\partial u}{\partial n}d\sigma$$

$$\int_{\Omega}[vL_a u + E_a(u,v)]dx = \int_{\partial\Omega} vT_a ud\sigma \qquad (2.2.4)$$

The equation (2.2.4) is the first Green's identity for operator L_a .

To get the second Green's identity for an operator L_a we use the (2.2.4)

$$\int_{\Omega}[vL_a u + E_a(u,v)]dx = \int_{\partial\Omega} vT_a ud\sigma \qquad (2.2.5)$$

And similarly,

$$\int_{\Omega}[uL_a v + E_a(v,u)]dx = \int_{\partial\Omega} uT_a vd\sigma \qquad (2.2.6)$$

Subtracting (2.2.6) from (2.2.5) we get the second Green's identity for operator L_a .

$$\int_{\Omega}(vL_a u - uL_a v)dx = \int_{\partial\Omega}(vT_a u - uT_a v)d\sigma$$

2.2.1 Parametrix

A function $P_a(x,y)$ of two variables $x,y \in R^3$ is called a parametrix for an operator L_a if

$$L_a P_a(x,y) = \delta(x-y) + R_a(x,y) \qquad (2.2.7)$$

Example the function

$$P_a(x,y) = \frac{-1}{4\pi a(y)|x-y|} \qquad x,y \in R^3$$

is a Parametrix with the corresponding weakly singular remainder

$$R_a(x,y) = \sum_{i=1}^{3} \frac{x_i - y_i}{4\pi a(y)|x-y|^3} \frac{\partial a(x)}{\partial x_i} \qquad x,y \in R^3$$

Check

$$L_a P_a(x,y) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left[a(x)\frac{\partial P_a(x,y)}{\partial x_i}\right] = Da.\,DP_a(x,y) + a\Delta P_a(x,y)$$

(i) $$a\Delta P_a(x,y) = a\Delta\left(\frac{-1}{4\pi a(y)|x-y|}\right) = \frac{a(x)}{a(y)}\left(\frac{-1}{4\pi|x-y|}\right) = \frac{a(x)}{a(y)}\delta(x-y)$$

Recall from d istribution if $a \in C^\infty$, t hen $a\delta = a(0)\delta$ it means t ake $\varphi\epsilon\, D(R^n)$ and consider

$$\langle a\delta, \varphi\rangle = \langle \delta, a\varphi\rangle = \langle \delta, \psi\rangle, \;\; \psi = a\varphi\epsilon D(R^n)$$

$$\langle \delta, \psi\rangle = \psi(0) = a(0)\varphi(0) = a(0)\langle \delta, \varphi\rangle = \langle a(0)\delta, \varphi\rangle$$

$$\Rightarrow \;\; \langle a\delta, \varphi\rangle = \langle a(0)\delta, \varphi\rangle$$

$$\frac{a(x)}{a(y)}\delta(x-y) = a(x,y)\delta(x-y) = \underbrace{a(x,y)}_{=1}\Big|_{x=y} \delta(x-y) = \delta(x-y)$$

Therefore,

$$a\Delta P_a(x,y) = \delta(x-y)$$

Next I will show that

$$(ii) \qquad\qquad Da.DP_a(x,y) = R_a(x,y)$$

$$DP_a(x,y) = D\left(\frac{-1}{4\pi a(y)|x-y|}\right) = \frac{1}{4\pi a(y)}\frac{\partial}{\partial x_i}\left(\frac{-1}{|x-y|}\right) = \frac{1}{4\pi a(y)}\frac{(-(1)' + |x-y|')}{|x-y|^2}$$
$$= \frac{1}{4\pi a(y)}\sum_{i=1}^{3}\frac{x_i - y_i}{|x-y|^3}$$

$$Da.DP_a(x,y) = \sum_{i=1}^{3}\frac{\partial a(x)}{\partial x_i}\frac{1}{4\pi a(y)}\sum_{i=1}^{3}\frac{x_i - y_i}{|x-y|^3} = \sum_{i=1}^{3}\frac{1}{4\pi a(y)}\frac{x_i - y_i}{|x-y|^3}\frac{\partial a(x)}{\partial x_i}$$
$$= R_a(x,y)$$

From this we get

$$L_a P_a(x,y) = \delta(x-y) + R_a(x,y).$$

Recall Green's second identity for an operator L_a

$$\int_{\Omega}(vL_a u - uL_a v)dx = \int_{\partial\Omega}(vT_a u - uT_a v)d\sigma \qquad (2.2.8)$$

Let u be a solution of

$$\begin{cases} L_a u = f \ in \ \Omega \\ u = \varphi \ on \ \partial\Omega \end{cases} \qquad (2.2.9)$$

And substitute v by $P_a(x,y)$ in (2.2.7) we have

$$\int_{\Omega}\left(P_a(x,y)f(x) - u(x)\underbrace{L_a P_a(x,y)}_{=\delta(x-y)+R_a(x,y)}\right)dx$$
$$= \int_{\partial\Omega}(P_a(x,y)T_a u(x) - u(x)T_a P_a(x,y))d\sigma$$

$$\int_{\Omega}(P_a(x,y)f(x) - u(x)(\delta(x-y) + R_a(x,y)))dx$$
$$= \int_{\partial\Omega}(P_a(x,y)T_a u(x) - u(x)T_a P_a(x,y))d\sigma$$

$$\int_{\Omega} P_a(x,y) f(x) dx - \int_{\Omega} u(x)\, \delta(x-y) dx - \int_{\Omega} u(x)\, R_a(x,y) dx$$

$$= \int_{\partial\Omega} P_a(x,y) T_a u(x) d\sigma - \int_{\partial\Omega} u(x) T_a P_a(x,y)$$

$$u(y) = \int_{\Omega} P_a(x,y) f(x) dx - \int_{\Omega} u(x)\, R_a(x,y) dx$$

$$- \int_{\partial\Omega} P_a(x,y) T_a u(x) d\sigma + \int_{\partial\Omega} \varphi T_a P_a(x,y) d\sigma \qquad (2.2.10)$$

Note that $u(y) = u * \delta = \int u(x)\delta(x-y)dx$

From Eq. (2.2.10), $u(y)$ is the integral representation for the solution of Eq.(2.2.9).

If

$$a = 1\,, \qquad R_a(x,y) = \sum_{i=1}^{3} \frac{x_i - y_i}{4\pi |x-y|^3} \frac{\partial(1)}{\partial x_i} = 0$$

This implies from Eq. (2.2.7) $L_a P_a(x,y) = \delta(x-y)$

$\Rightarrow$ $P_a(x,y)$ is a fundamental solution for an operator L_a for $a = 1$.

Basically, for $a = 1$,

$$L_a u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[a(x)\frac{\partial u}{\partial x_i}\right] = div(aDu) = Da.Du + a\Delta u = \Delta u$$

$$\Rightarrow \quad L_a = \Delta$$

Substitute a fundamental solution $P_a(x,y)$ by $E(x,y)$, and

$$T_a(x, n(x), \partial_x) = a(x)\frac{\partial}{\partial n(x)} = \frac{\partial}{\partial n(x)} \; for\ a = 1$$

in 2.2.10) becomes

$$u(y) = \int_\Omega E(x,y)f(x)dx - \int_\Omega u(x)\,.\,0dx$$

$$-\int_{\partial\Omega} E(x,y)\frac{\partial u(x)}{\partial n(x)}d\sigma + \int_{\partial\Omega} \varphi\frac{\partial E(x,y)}{\partial n(x)}d\sigma$$

$$u(y) = \int_\Omega E(x,y)f(x)dx + \int_{\partial\Omega}\left(\varphi\frac{\partial E(x,y)}{\partial n} - E(x,y)\frac{\partial u}{\partial n}\right)d\sigma \qquad (2.2.11)$$

Eq.(2.2.11) shows us $u(y)$ is the integral representation for $\begin{cases}\Delta u = f & in\ \Omega \\ u = \varphi & on\ \partial\Omega\end{cases}$

$\Rightarrow$ $Eq.(2.2.10)$ reduces to $Eq.(2.2.11)$ for $a = 1$.

2.3 Fundamental solution for Ordinary differential operators

Definition let L be a differential operator of the form

$$L = \sum_{|\alpha|\le k} a_\alpha(x)D^k \qquad , \qquad D = \frac{d}{dx}$$

A function $E(x,y)$ is called a fundamental solution of L, if for every function $f(x)$,which is sufficiently regular and vanishes outside a bounded set

$$L\left[\int E(x,y)f(y)dy\right] = f(x).$$

Symbolically this amount to $E(x,y)$ being a solution of the inhomogeneous differential equation

$$LE(x,y) = \delta(x-y)$$

Where δ denotes the so called Dirac delta function defined as;

$$\delta(x) = \begin{cases}\infty, & x = 0 \\ 0, & x \neq 0\end{cases}$$

and which is also constrained to satisfy the identity

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

Example 1 consider the scalar problem

$$Lu \coloneqq \frac{d^2u}{dx^2} = 0$$

We would like to find a fundamental solution $E(x,y)$ satisfying

$$LE(x,y) = \delta(x-y)$$

$$\Rightarrow \frac{d^2}{dx^2}E(x,y) = \delta(x-y)$$

Since the general solution of the ODE for $x \neq y$ can be written as $A + Bx$. we may take as ansatz for $E(x,y)$

$$E(x,y) = \begin{cases} A_1 + B_1x & if\ x < y \\ A_2 + B_2x & if\ x > y \end{cases}$$

Using integration, we obtain for sufficiently small ε that

$$\frac{dE}{dx}(y+\varepsilon; y) - \frac{dE}{dx}(y-\varepsilon; y) = \int_{y-\varepsilon}^{y+\varepsilon} \delta(x-y)dx = 1.$$

Applying these conditions to the solution above, we find

$$B_2 - B_1 = 1.$$

Since E is apparently continuous at $x = y$, we also find

$$A_1 + B_1y = A_2 + B_2y.$$

Hence we obtain the fundamental solution

$$E(x;y) = \begin{cases} A_1 + B_1 x & if\ x \le y \\ A_1 - y + (B_1 + 1)x & if\ x > y. \end{cases}$$

Where A_1 and B_1 are arbitrary constants.

Recall The Heaviside function $H(x)$ is defined to be equal to zero for every negative value of x and to unify for every positive value of , that is

$$H(x) = \begin{cases} 0, x < o \\ 1\ , x > 0 \end{cases}$$

It has jump discontinuity at $x = 0$

$$H^{'} = \delta$$

.

This is sometimes written as

$$H(x) = \int_{-\infty}^{x} \delta(t)dt$$

Example2 find the fundamental solution for ordinary differential equation $u^{''} = f$.

To find the fundamental solution $E(x)$, we want to solve $E^{''}(x) = \delta(x)$. Recall that the Heaviside function $H(x)$ satisfies $(H(x) + c)^{'} = \delta(x)$ for any constant c. Let us take $c = \frac{-1}{2}$ and try to solve

$$E^{'}(x) = \begin{cases} \frac{-1}{2} & if\ x > 0 \\ \frac{1}{2} & if\ x < 0. \end{cases} \qquad (*)$$

We may integrate (*) to obtain a particular solution

$$E(x) = \frac{1}{2}|x| \qquad (**)$$

as our fundamental solution. (The choice $c = -{}^{1}\!/_{2}$ was made to simplify the resultant form (**) of the fundamental solution.)

References

[1] **Ahmmed Mohammed** Lectures On Second Order Linear Elliptic PDEs, 2010.

[2] **Charles Nash**, Differential Topology and quantum field theory, Academic Press, 1991.

[3] David Gilbarg, Neil.Trudinger, Elliptic partial differential equations of second order, Springer-Verlag Berlin Heidelberg Newyork, 2001.

[4] **Gregory T. Von Nessi**, Elementary Theoretical Methods in Partial Differential Equation, 2005.

[5] **Lawrence C. Evans**, Partial differential equations, American Mathematical Society, 2010.

[6] **Wolfgang Hackbusch**, Elliptic Differential equations: Theory and numerical Treatment, Springer-verlag Berlin Heidelberg, 1992.