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ON

IDEALS OF LATTICE ORDERED MONOID

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The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a project entitled ″ IDEALS OF LATTICE ORDERED MONOIDS″ by Alemu Baye in partial fulfillment of the requirements for the degree of Master of Science in Mathematics (Algebra).

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Abstract

In this project the notion of an ideal of a lattice ordered monoid A is introduced. The notion of congruence relations on a lattice ordered monoid (l-monoid) A is also introduced and its relation with ideals of A is investigated. In addition, we will introduce the notion of normal ideals of lattice ordered monoid and dually residuated lattice ordered monoid (DRl-monoid) in order to study their connection with congruence relations. The ideal induced by congruence relations on an l-monoid need not be normal. By imposing additional conditions on the congruence relations on a DRl-monoid, we will prove that the induced ideal will be normal.

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Chapter 1

Introduction

A Partial order on a non-empty set P is a relation "≤"on P which is reflexive, anti-symmetric and transitive. Lattice can be defined as an Algebraic lattice on a non-empty set with binary operations join and meet, which are commutative, associative and satisfy the absorption identities and equivalently an order lattice is a partially ordered set in which every doubleton {x, y} has greatest lower bound or inf{x, y} and least upper bound or sup{x, y}.

A lattice ordered monoid is a monoid which has also a lattice structure. Ideals of lattice ordered groups were investigated in detail by Birkhoff. Dually residuated lattice ordered semi-groups were studied by Swamy as a common abstraction of Boolean rings and abelian lattice ordered groups. Ideals of dually residuated lattice ordered semi-groups were investigated by Kovar, Hansen and Rachunek. Kuhr studied ideals of dually-residuated lattice ordered monoid and extended results of Birkhoff to DRl-monoids.

The aim of this project is to extend the concept of an ideal to any lattice ordered monoid and study relations between ideals of an l-monoid A and congruence relations on A. This project work contains five chapters. The second chapter deals with the theory of lattices. In the third chapter the ideal of lattice order groups will be discussed. In chapter four notions of ideals and normal ideals of lattice ordered monoid and dually residuated lattice ordered monoid are introduced. Moreover, connection of ideals and congruence relations are studied. In the last chapter concluding remarks are given.

Chapter 2

Lattice Theory

In this chapter we will give definitions and some results on lattice theory and lattice ordered groups. Some of these are partially ordered set, lattice, and lattice, distributive lattice, etc.

2.1 Partial Ordered Set (Poset)

Definition 2.1.1: A non empty set P together with a relation " $\leq$ " on P is said to be a partially ordered set (poset) if the following are satisfied: for all x ,y ,z $\in$ P

P_1. $x \leq x$ (reflexivity),

P_2.if $x \leq y$ and $y \leq x$, then $x = y$ (anti-symmetry),

P_3.if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).

Note: A partially ordered set which satisfies either $x \leq y$ or $y \leq x$ for all x , y, z $\in$ P is called chain.

Examples 2.1.1:

1. Let X be a non empty set. Then the power set of X is a poset ordered by set inclusion.
2. Let N be the set of natural numbers

Define on N by $n \leq m \Leftrightarrow$ n divides m then $(N, \leq)$ is a poset.

3. The set of real numbers, with the usual less or equal to, is a chain.

Definition 2.1.2: Let $(P, \leq)$ be a partially ordered set and S be a non empty subset of P. Then an element:

1. u of P is said to be an upper bound of S if $x \leq u$ for all $x \in S$. An upper bound u of S is said to be a least upper bound or supremum (sup) or join of S if $u \leq y$ for each upper bound y of S.
2. ℓ of P is said to be a lower bound of S if $\ell \leq s$ for each $s \in S$. ℓ is said to be greatest lower bound or infmum (inf) or meet of S if $z \leq \ell$ for each lower bound z of S.

Remark 2.1.1: In any chain, for each non-empty finite subset the least upper bound and the greatest lower bound exist.

Definition 2.1.3: Let P and Q be poset. Then a function f: P→ Q is called order preserving map or isotone, if x ≤ y, then f(x) ≤ f(y), $\forall x, y \in P$.

Definition 2.1.4: An isomorphism between two posets P and Q is a bijection which satisfies f(a) ≤ f(b) if and only if a ≤ b.

Definition 2.1.5: A function f: P → Q is called antitone (order reversing) if and only if a ≤ b implies f(b) ≤ f(a) for all a, b ∈ P.

Definition 2.1.6: A dual isomorphism between two posets P and Q is a bijection which Satisfies a ≤ b if and only if f(b) ≤ f(a) for all a, b ∈ P.

Note: A dual isomorphism from poset P to itself is a dual automorphism.

2.2 Lattice

A Lattice can be defined in two different but equivalent ways as an algebraic lattice and order lattice.

Definition 2.2.1: An upper semi lattice is a poset (P, ≤) in which every doubleton $\{x, y\}$ has a least upper bound in P (sup$\{x, y\}$) denoted by x V y and called **join** of x and y.

Definition 2.2.2: A lower semi-lattice is a poset (P, ≤) in which every doubleton $\{x, y\}$has a greatest lower bound in P (inf$\{x, y\}$) denoted by x Λ y and called **meet** of x and y.

Definition 2.2.3: An ordered Lattice is a poset that is simultaneously an upper semi-lattice and a lower semi-lattice.

Definition 2.2.4: Algebraic lattice is a non empty set L together with two binary operations V and Λ on L which satisfies, for all a, b, c ∈ L

(L_1). a V b = b V a and a Λ b = b Λ a (commutativity),

(L_2). a V (b V c) = (a V b) V c and a Λ (b Λ c) = (a Λ b) Λc (associativity),

(L_3). a V (a Λ b) = a and a Λ (a V b) = a (absorption law).

Theorem 2.2.1: In any algebraic lattice (L, V, Λ) both operations are idempotent. That is, a Λ a = a and a V a = a.

Proof: For any a ∈ L, take b ∈ L and let c = a V b. Then

a V a =a V [a Λ (a V b)] (by absorption law).

=a V (a Λ c) (by supposition)

=a (again absorption law)

Similarly, a Λ a = a. ■

Definition 2.2.5: Given an algebraic lattice (L, V, Λ), define "≤ " on L by a ≤ b iff a Λ b = a for all a, b ∈ L.

Theorem 2.2.2: In an algebraic lattice (L, V, Λ), for all a, b ∈ L; a Λ b = a iff a V b = b.

Proof: (⇐) Suppose a V b = b

a Λ b = a Λ (a V b) = a (by absorption law)

(⇒) Suppose a Λ b = a

a V b = (a Λ b) V b = b (by absorption law) ■

Remark 2.2.1: Thus in an algebraic lattice (L, V, Λ), for all a, b∈L, a ≤ b if and only if a Λ b = a and a V b = b.

Example 2.2.1: (1). The power set of a finite set form a lattice, where join and meet are union and intersection, respectively.

(2). Positive integers form a lattice, where a $\leq$ b if and only if b is a multiple of a. The join and meet are the least common multiple and greatest common divisor, respectively.

Theorem 2.2.3: An algebraic lattice and an ordered lattice are equivalent.

Proof: ($\Rightarrow$) Suppose (L, V, Λ) is an algebraic lattice. We want to show that (L,$\leq$) is order Lattice. We first show that (L, $\leq$) is a poset.

(i). For a $\in$ L, let a = a Λ a. (by idempotent Law)

Then, a $\leq$ a. Hence, "$\leq$" is reflexive.

(ii). For a, b$\in$L, if a $\leq$ b and b $\leq$ a, then a = a Λ b and b = b Λ a. Then a = a Λ b = b Λ a = b. Thus "$\leq$" is anti-symmetry.

(iii). If a $\leq$ b and b $\leq$ c, then a Λ b = a and b Λ c = b. This implies that a = a Λ b = a Λ (b Λ c) = (a Λ b) Λ c = a Λ c. Thus, a $\leq$ c. Hence "$\leq$" is transitive.

Therefore, (L, $\leq$) is a poset.

We then show that (L, $\leq$) is an ordered lattice that is (L, $\leq$) is upper and lower semi-lattices. Let a, b $\in$ L. Now, from absorption law, we have a Λ (a V b) = a. Thus, a $\leq$ a V b. Similarly, b Λ (b V a) = b. Hence, b $\leq$ b V a. Therefore, a V b is the upper bound of $\{a, b\}$.

Let u be any upper bound of $\{a, b\}$.Then a $\leq$ u and b $\leq$ u. This implies that a V u = u & b V u = u.

Therefore (a V b) Vu = a V (b V u) (by associativity)

=a V u

=u

Thus a V b $\leq$ u. Therefore a V b = lub{a, b}.

Similarly we can show that inf$\{a, b\}$ exists in L and inf$\{a, b\}$ = a Λ b. Hence (L, ≤) is an order lattice.

(⇐) Let (L, ≤) bean order Lattice. Then L is both upper and lower semi lattices.
i.e. for elements a, b in L, glb$\{a, b\}$and lub$\{a, b\}$both exist in L.
Define "Λ" and "V" on L by

$$a \Lambda b = \inf\{a, b\} = glb\{a, b\} \text{ and } a V b = \sup\{a, b\} = lub\{a, b\}.$$

We first show that Λ and V are a binary operations on L.
(a). L is closed under Λ and V (by definition 2.2.1)
(b). Let a = a' and b = b'. Then, a Λ b = inf$\{a, b\}$ = inf$\{a', b'\}$ = a' Λ b'. Similarly, a V b = a' V b'.
Hence Λ and V are binary operations on L.

Next we show that (L, Λ, V) is algebraic lattice.
(i). Let a, b ∈ L. Then a Λ b = inf$\{a, b\}$ = inf$\{b, a\}$ = b Λ a & a V b = sup{a, b} = {b, a} = a V b.
Hence Λ and V are commutative.

(ii) To show the associative property of Λ and V first, we shall prove that

(a). glb$\{a, b, c\}$ = glb$\{a, glb\{b, c\}\}$, ∀a, b, c∈L.

(b). lub$\{a, b, c\}$ = lub$\{a, lub\{b, c\}\}$, ∀a, b, c∈L.

Let d = glb$\{a, b, c\}$ and e = glb$\{a, glb\{b, c\}\}$. Since d ≤ b; d ≤ c, we have d is a lower bound of $\{b, c\}$. Hence d ≤ glb$\{b, c\}$. Therefore, d is a lower bound of $\{a, glb\{b, c\}\}$. Thus d ≤ e. (*)
Now, e ≤ a and e ≤ glb$\{b, c\}$. Then e is a lower bound of $\{a, b, c\}$.Thus e ≤ glb$\{a, b, c\}$.
Thus, e ≤ d. (**)
Hence, from (*) & (**), d = e and hence glb$\{a, b, c\}$ = glb$\{a, glb\{b, c\}\}$.
Similarly, we can show that lub$\{a, b, c\}$ = lub$\{a, lub\{b, c\}\}$.
Let a, b, c∈L; then a Λ (b Λ c) = a Λ inf$\{b, c\}$.

$= \inf\{a, \inf\{b, c\}\}$

$= \inf\{a, b, c\}$

$= \inf\{\inf\{a, b\}, c\}$

$= \inf\{a \Lambda b, c\}$

$= (a \Lambda b) \Lambda c$

Therefore Λ is associative. By the same argument, V is associative.

(iii). Let a, b $\in$L. Then, a Λ (a V b) = a Λ $\sup\{a, b\}$

Let $\sup\{a, b\}$ = x. Then, a $\leq$ x and b $\leq$ x. Then, a Λ (a V b) = $\inf\{a, \sup\{a, b\}\} = \inf\{a, x\}$ = a.

Thus, a Λ (a V b) = a

a V (a Λ b) = a V $\inf\{a, b\}$

Let $\inf\{a, b\}$ = x. Then x $\leq$ a and x $\leq$ b. Now a V (a Λ b) = $\sup\{a, \inf\{a, b\}\} = \sup\{a, x\}$ = a.

Thus, a V (a Λ b) = a. Hence we have the absorption law.

Therefore, (L, Λ, V) is an algebraic lattice. ■

Definition 2.2.6: Two lattices (L_1, Λ, V) and (L_2, V, Λ) are said to be isomorphic if there is bijection f : $L_1 \rightarrow L_2$ such that for every a, b $\in L_1$ the following are true.

(i). f(a V b) = f(a) V f(b).

(ii). f(a Λ b) = f(a) Λ f(b).

Theorem 2.2.4: Two Lattices (L_1, V, Λ) and (L_2, V, Λ) are isomorphic if and only if there is a bijection f: $L_1 \rightarrow L_2$ such that both f and f^{-1} are order preserving.

Proof: ($\Rightarrow$) suppose f: $L_1 \rightarrow L_2$ is an isomorphism. Let a, b $\in L_1$ and a $\leq$ b.

In any lattice a $\leq$ b if and only if a Λ b = a. Then f(a) = f(a Λ b) = f(a) Λ f(b) (by definition 1.2.6(ii)). Thus f(a) = f(a) Λ f(b). Then f(a) $\leq$ f(b). Hence, f is order preserving.

Clearly, f^{-1} is bijective since inverse of a bijective map is bijective.

Let a, b $\in$ L_2. Then a $\leq$ b if and only if a Λ b = a. Then $f^{-1}(a) = f^{-1}(a \Lambda b) = f^{-1}(a) \Lambda f^{-1}(b)$

Thus, $f^{-1}(a) \leq f^{-1}(b)$. Hence f^{-1} is order preserving.

(⇐) Let f: $L_1 \rightarrow L_2$ be a bijective map, such that f and f^{-1} are order preserving

For a, b ∈ L_1, we have a ≤ a V b and b ≤ a V b. Then f(a) ≤ f(a V b) and f(b) ≤ f(a V b) because f is order preserving. Hence f(a) V f(b) ≤ f(a V b) since f(a) V f(b) = f(a) or f(a) V f(b) = f(b). (*)

Conversely f(a) ≤ f(a) V f(b) and f(b) ≤ f(a) V f(b). Then a ≤ $f^{-1}(f(a) \vee f(b))$ and b ≤ $f^{-1}(f(a) \vee f(b))$ because f^{-1} is order preserving and f is one to one, then $f^{-1}(f(a)) = a$.

Thus a V b ≤ $f^{-1}(f(a) \vee f(b))$. Then f(a V b) ≤ f(a) V f(b) since f is order preserving.

Hence, f(a V b) ≤ f(a) V f(b) . (**)

From (*) & (**), f(a V b) = f(a) V f(b). Similarly, one can shows that f(a Λ b) = f(a) Λ f(b).

Therefore f is an isomorphism. ■

Theorem 2.2.5: In any lattice L the two distributive laws are equivalent. That is, for all a, b, c ∈ L, the following statements are equivalent:

(1). a Λ (b V c) = (a Λ b) V (a Λ c)

(2). a V (b Λ c) = (a V b) Λ (a V c)

Proof: (1 ⇒ 2) Suppose 1 holds.

a V (b Λ c)) = (a V (a Λ c)) V (b Λ c) (by absorption law)

= a V ((a Λ c) V (b Λ c)) (by association law)

= a V ((c Λ a) V (c Λ b)) (by commutative law)

= a V (c Λ (a V b)) (by 1)

= a V ((a V b) Λ c) (by commutative law)

= (a Λ (a V b)) V ((a V b) Λ c) (by absorption law)

= ((a V b) Λ a) V ((a V b) Λ c) (by commutative law)

= (a V b) Λ (a V c) (by 1)

Thus (2) holds.

(2⇒1) Suppose 2 holds. Then

(a Λ b) V (a Λ c) = [(a Λ b) V a] Λ [(a Λ b) V c] (by 2)

= a Λ [(a V c) Λ (b V c)] (by 2 and absorption law)

$= [a \wedge (a \vee c)] \wedge (b \vee c)$ (by association law)

$= a \wedge (b \vee c)$ (by absorption law) ■

Definition 2.2.7: A distributive lattice is a lattice which satisfies either (and hence both) of the distributive laws given in the above theorem.

Definition 2.2.8: A lattice is said to satisfy the cancellation law, if whenever, $a \vee b = c \vee b$ and $a \wedge b = c \wedge b$, then we have $a = c$.

Proposition 2.2.1: A lattice is distributive if and only if it satisfies the cancellation law.

Proof: ($\Rightarrow$) Suppose a lattice $(L, \vee, \wedge)$ is distributive. Suppose for $a, b, c \in L$, $a \vee b = c \vee b$ and $a \wedge b = c \wedge b$. Then, $c = c \vee (c \wedge b)$

$= c \vee (a \wedge b)$

$= (c \vee a) \wedge (c \vee b)$ distributive law

$= (c \vee a) \wedge (a \vee b)$

$= a \vee (c \wedge b)$ distributive law

$= a \vee (a \wedge b)$

$= a$

($\Leftarrow$) Suppose a lattice $(L, \vee, \wedge)$ satisfies the cancelation law.

Let $a, b, c \in L$ such that $a \vee b = c \vee b$ and $a \wedge b = c \wedge b$. Then $a = c$.

$a = a \vee (a \wedge b)$ (by absorption law)

$= a \vee (c \wedge b)$ since $a \wedge b = c \wedge b$.

$= a \vee (b \wedge c)$ (by commutative law) (*)

and $(a \vee b) \wedge (a \vee c) = (a \vee b) \wedge (a \vee a)$ since $a = c$.

$= (a \vee b) \wedge a$ (by idempotent law)

$= a \wedge (a \vee b)$ (by commutative law)

$= a$ (by absorption law) (**)

From (*) and (**), $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Therefore, the lattice $(L, \vee, \wedge)$ is distributive. ■

Definition 2.2.9: Let L be a lattice and S be a non-empty subset of L such that for every pair of elements a and b in S, both a ∨ b and a ∧ b are in S, where ∨ and ∧ are the operations of L, then we say that S with the same operations (restricted to S) is a sub-lattice of L.

Example 2.2.2:

1. L is a sub lattice of itself.
2. P = {a, c, d, e} is a poset it is a lattice but P is not a sub-lattice of a lattice {a, b, c, d, e} as shown in the diagram below.

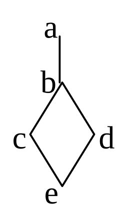

Figure 1

2.3 Group

Definition 2.3.1: A non-empty set G with a binary operation "+" is said to be a group if it satisfies the following:

1. Closurety, i.e. for all a, b $\in$ G $\Rightarrow$ (a + b) $\in$ G;
2. Associativity, i.e. for all a, b, c $\in$ G, (a + b) + c = a + (b + c);
3. Existence of identity, i.e. $\exists$e $\in$ G such that a + e = a = e + a for all a in G;
4. Existence of inverse, i.e. for every a $\in$ G, there exists b $\in$ G such that a + b = e = b + a.

Example 2.3.1:

a) The set of integers with addition is a group.
b) The set of real numbers with addition is a group.
c) (IN, +), where IN is the set of natural numbers, is not a group.

Chapter 3

Lattice Ordered Groups

3.1 Homogeneity

Definition 3.1.1: A relation "$\leq$" on a Group G is called left homogeneous if $x \leq y$, then $a + x \leq a + y$ and right homogeneous if $x \leq y$, then $x + b \leq y + b$ for all, a, $b \in G$. A relation which is both left and right homogeneous is called homogeneous.

Example 3.1.1: (Z, +) is a group and ″$\leq$″ is the usual less than or equal to. Then ″$\leq$″ is homogeneous on Z (the set of integers).

Proposition 3.1.1: Suppose the relation "$\leq$" on G is homogenous. If $x \leq y$, then $-y \leq -x$.

Proof: Given: $x \leq y$. We want to show $-y \leq -x$. Set $a = -x$ and $b = -y$, we get $a + x + b \leq a + y + b$. Then. $-x + x - y \leq -x + y - y$. Thus, $-y \leq -x$. ■

Theorem 3.1.1: On an additive group (G, +, $\leq$), lattice homogeneity is equivalent to the assertion that every group translation $x \rightarrow a + x + b$ is a lattice automorphism.

Proof: ($\Rightarrow$) Suppose $\leq$ is homogeneous on G.

Define $f: G \rightarrow G$ such that $f(x) = a + x + b$ for a, $b \in G$.

(i). Let $x \leq y$. Then $a + x + b \leq a + y + b$. Then $f(x) \leq f(y)$.Thus, f preserves order (isotone).

(ii). Let $f(x) = f(y)$. Then $a + x + b = a + y + b$. Then $x + b = y + b$. (by left cancellation)

Thus $x = y$ (by right cancellation).

Therefore f is 1–1.

(iii). Let $y \in G$; for a, $b \in G$, $x = -a + y - b \in G$. Then, by definition of f, $f(x) = f(-a + y - b) =$ $a + (-a + y - b) + b = y$. Thus f is onto because the pri-image of y is $-a + y - b$.

Hence from (ii) & (iii), f is bijective. Thus, f^{-1} exists.

Define, $f^{-1}: G \to G$ such that $f^{-1}(x) = -a + x - b$.

If $x \leq y$, then $-a + x + (-b) \leq -a + y + (-b)$ (by homogeneity)

Thus, $f^{-1}(x) \leq f^{-1}(y)$.Then, f^{-1} is order preserving.

Therefore, by theorem 2.2.4, f and f^{-1} are order preserving. Hence f is automorphism.

($\Leftarrow$) Suppose $f: G \to G$ defined by $f(x) = a + x + b$ is lattice automorphism.

And suppose that $x \leq y$. Then $f(x) \leq f(y)$ because f is order preserving.

This implies that $a + x + b \leq a + y + b$. Therefore "$\leq$" is homogenous. ■

Theorem 3.1.2: On a group G which is a lattice, homogeneity is equivalent to the assertion that every group translation of the form $x \to a - x + b$ is a dual automorphism.

Proof: ($\Rightarrow$) suppose "$\leq$" is homogenous.

(i). If $x \leq y$, then by proposition 2.2.1, $-y \leq -x$ and by homogeneity we have,

$a + (-y) + b \leq a + (-x) + b, \forall a, b \in G$.

Then, $a - y + b \leq a - x + b$. Therefore, $f(y) \leq f(x)$.Thus, f is order reversing (antitone)

(ii). Let $f(x) = f(y)$. Then $a - x + b = a - y + b$. By cancellation law, we have $-x = -y$.

Then, $x = y$. Therefore, f is one to one.

(iii). Let $y \in G$. Then for a, $b \in G$, we have $f(x) = a - (a - y + b) + b = a - a + y - b + b = y$

Hence, f is on to. Thus, f is bijective.

It remains to show that $f^{-1}$is order- reversing. We have $f: G \to G$ defined by $f(x) = a - x + b$.

Let x, y ∈ G. Then, there exist s, t ∈ G such that f(s) = x and f(t) = y.

Suppose $x \leq y$. Then, $f(s) \leq f(t)$. This implies that $a - s + b \leq a - t + b, \forall a, b \in G$. By cancelation law, we have $-s \leq -t$.

Then, $-f^{-1}(x) \leq -f^{-1}(y)$. Thus $f^{-1}(y) \leq f^{-1}(x)$. Hence f^{-1} is order reversing.

Therefore f is a dual automorphism.

(⇐) Suppose f is a dual automorphism.

Let $x \leq y$. Then $f(y) \leq f(x)$ since f is order reversing. This implies that $a - y + b \leq a - x + b$. Then, $-(a - x + b) \leq -(a - y + b)$. Thus $-a + x - b \leq -a + y - b$. Therefore, the relation "≤" is homogenous. ■

3.2 Lattice Ordered Group (l-group)

Definition 3.2.1: A Lattice ordered group (l-group) is a system (G, +, V,Λ) such that

(i). (G, +) is a group;

(ii).The relation"≤ "is homogenous, i.e. if $x \leq y$, then $a + x + b \leq a + y + b, \forall a, b \in G$ and

(iii). (G, ≤) is Lattice.

Note: Condition III asserts that the relation "≤" satisfies the usual conditions, ∀x, y, z ∈P.

(p_1). For all x, $x \leq x$, (reflexive)

(P_2). If $x \leq y$ and $y \leq x$, then $x = y$, (anti-symmetry)

(P_3). If $x \leq y$ and $y \leq z$, then $x \leq z$, (transitive)

(L'). For any two elements x and y of G the set {x, y}has a lub $x \vee y$ and a glb $x \wedge y$.

Example 3.2.1: Let (IR, +) be the group of real numbers with the usual addition" + " and let "≤" be the usual less than or equal to relation. Then (R, +, ≤) is an l- Group.

Definition 3.2.2: An element a of an l-group G is called positive if $a \geq 0$ and a is called negative if $a \leq 0$.The set of all positive elements of G will be denoted by $G^+ = \{a \in G : a \geq 0\}$ which is called the positive cone of G.

Theorem 3.2.1: In an l-group G, left homogeneity is equivalent to either of the left distributive laws.

(i). $a + (x \vee y) = (a + x) \vee (a + y)$

(ii). $a + (x \wedge y) = (a + x) \wedge (a + y)$

Proof: (⇒) Suppose Left homogeneity holds (i.e. $x \leq y$ implies $a + x \leq a + y$ for all $a \in G$).

Then, $(a + x) \vee (a + y) = a + y$. (*)

Moreover, $x \leq y$ implies $x \vee y = y$, and hence $a + (x \vee y) = a + y$. (**)

Therefore, by (*) and (**), $a + (x \vee y) = (a + x) \vee (a + y)$. Hence (i) holds.

(⇐) Suppose (i) holds.

Let $x \leq y$. Then, $x \vee y = y$. By right homogeneity, $a + (x \vee y) = a + y$. Then, $(a + x) \vee (a + y) = a + y$. Hence, $a + x \leq a + y$. Therefore, left homogeneity holds. ■

Remark 3.2.1: Right homogeneity is equivalent to either of the right distributive laws.

(i). $(x \vee y) + a = (x + a) \vee (y + a)$,

(ii). $(x \wedge y) + a = (x + a) \wedge (y + a)$

for every a in G.

Theorem 3.2.2: Homogeneity is equivalent to monotonicity Laws. That is, if $x \leq x'$ and $y \leq y'$, then $x + y \leq x' + y'$.

Proof: (⇒) Suppose Homogeneity holds.

Suppose $x \leq x'$ and $y \leq y'$. Then, by right homogeneity, $x + y' \leq x' + y'$ and by left homogeneity, $x + y \leq x + y'$. Thus, $x + y \leq x + y' \leq x' + y'$. Therefore, by transitivity, $x + y \leq x' + y'$. Hence, monotonicity holds.

($\Leftarrow$) Suppose monotonicity holds.

Suppose $x \leq x'$. Since $y \leq y$, by monotonicity, $x + y \leq x' + y$. Therefore right homogeneity holds. Moreover, $x \leq x$ and $y \leq y'$ imply $x + y \leq x + y'$, by monotonicity. Thus, left homogeneity holds. Therefore, homogeneity holds. ■

Theorem 3.2.3: Homogeneity is equivalent to the following "dualization laws"

(i). $a - (x \wedge y) + b = (a - x + b) \vee (a - y + b)$

(ii). $a - (x \vee y) + b = (a - x + b) \wedge (a - y + b)$

(iii). $x \wedge y = -(-x \vee -y)$

Proof: ($\Rightarrow$) Suppose homogeneity holds.

Let $x \leq y$. Then $a + x + b \leq a + y + b$ and $x \wedge y = x$. By dual automorphism, $a - y + b \leq a - x + b$. Thus $a - (x \wedge y) + b = a - x + b$, since $x \wedge y = x$ and $(a - x + b) \vee (a - y + b) = a - x + b$, since $a - y + b \leq a - x + b$. Then, $a - (x \wedge y) + b = (a - x + b) \vee (a - y + b)$ and also $(a - x + b) \wedge (a - y + b) = a - y + b$ and $a - (x \vee y) + b = a - y + b$ since $x \vee y = y$. Therefore, $a - (x \vee y) + b = (a - x + b) \wedge (a - y + b)$.

($\Leftarrow$) Suppose $a - (x \wedge y) + b = (a - x + b) \vee (a - y + b)$.

Let $x \leq y$. Then $x \wedge y = x$ and $(a - x + b) \vee (a - y + b) = a - (x \wedge y) + b = a - x + b$. Then $a - y + b \leq a - x + b$. By dual automorphism, $a + x + b \leq a + y + b$. Hence, it is homogeneous.
Now let $a = b = 0$. From dual automorphism, $x \leq y$ implies $0 - y + 0 \leq 0 - x + 0$. Thus, $-y \leq -x$.

Then, $-x \vee -y = -x$ implies $-(-x \vee -y) = -(-x)$. Thus $-(-x \vee -y) = x$ and $x \wedge y = x$ since $x \leq y$. Hence $x \wedge y = -(-x \vee -y)$. ■

Definition 3.2.3: Let a be an element of an l-group. The positive part of a is $a^+ = a \vee 0$ and the negative part of a is $a^- = a \wedge 0$.

Proposition 3.2.1: Using right homogeneity, we get

(i). $a \vee b = (b - a)^+ + a = (a - b)^+ + b$

(ii). $a \wedge b = -(-a + (a - b)^+) = -(a - b)^+ + a$

Proof: $(b - a)^+ + a = ((b - a) \vee 0) + a$

$= (b - a + a) \vee (0 + a)$ (by right distribute law)

$= b \vee a$ and

$(a - b)^+ + b = ((a - b) \vee 0) + b)$

$= (a - b + b) \vee (0 + b) = a \vee b$

Thus, $a \vee b = b \vee a = (a - b)^+ + b = (b - a)^+ + a$. ■

3.3 Some Properties of l-group

Theorem 3.3.1: In any l-group G, we have $\forall$ a, b $\in$ G,

$$a - (a \wedge b) + b = b \vee a.$$

Proof: From theorem 3.2.3, homogeneity is equivalent to

$a - (x \wedge y) + b = (a - x + b) \vee (a - y + b)$

Then, $a - (a \wedge b) + b = (a - a + b) \vee (a - b + b) = b \vee a$. ■

Corollary3.2.1: (Dedekind) In any commutative l-group G, we have $\forall$ a, b $\in$ G,

$$a + b = (a \vee b) + (a \wedge b)$$

Proof: From the above theorem,

$a - (a \wedge b) + b = a + b - (a \wedge b) = a \vee b$. Then $a + b = (a \vee b) + (a \wedge b)$. ■

Proposition 3.3.1: For every element a of an l-group we have, $a = a^+ + a^-$.

Proof: Set $b = 0$ in $a + b = (a \vee b) + (a \wedge b)$. Then $a + 0 = (a \vee 0) + (a \wedge 0)$.

Therefore, $a = a^+ + a^-$. ■

Theorem 3.3.2: Any l-group G is a distributive lattice.

Proof: Suppose $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$, $\forall a, x, y \in G$. Then,

$$a \wedge y = a - (a \vee y) + y \text{ by theorem 2.3.1}$$
$$= a - (a \vee x) + x - x + y$$
$$= (a \wedge x) - x + y$$

Thus, $0 = -x + y$. Therefore $x = y$. Hence, cancelation law holds. Therefore G is a distributive lattice. ■

Theorem 3.3.3: In any l-group G, We have

(i). If $a \wedge b = 0$ and $a \wedge c = 0$, then $a \wedge (b + c) = 0$.

(ii). If $a \vee b = 0$ and $a \vee c = 0$, then $a \vee (b + c) = 0$.

Proof: (i) Let G be an l-group and suppose $a \wedge b = 0 = a \wedge c$.

We want to show that $a \wedge (b + c) = 0$.Sincea $\wedge$ b = 0, we have

$$c = (a \wedge b) + c$$
$$= (a + c) \wedge (b + c) \qquad \text{(by theorem 3.3.2)}$$

Thus, $a \wedge c = a \wedge (a + c) \wedge (b + c)$

$= a \wedge (b + c)$ since $a \leq a + c$; a and c are positive.

Therefore, $a \wedge (b + c) = 0$ since $a \wedge c = 0$.

The proof of (ii) is immediate. ■

Definition 3.3.1: Two positive elements a and b of an l-group G are said to be **disjoint,** in symbols, a ⊥ b if a Λ b = 0.

Lemma 3.3.1: Disjoint (positive) elements are permutable.

(i). If a Λ b = 0, then a + b = b + a.

(ii). If a Λ b = 0, then $(a - b)^+$ = a & $(a - b)^-$ = –b.

Proof: (i). Suppose a Λ b = 0. From theorem 3.3.1, a – 0 + b = b V a. Then a + b = a V b and b + a = b V a. Hence a + b = b + a since V is commutative.

(ii) Suppose a Λ b = 0. Then, $(a - b)^+$ = (a – b) V 0 = (a – b) V 0 + b –b.

$(a - b)^+$ = ((a – b) V 0 + b) – b

= (a – b + b) V (0 + b) – b (by1)

= (a V b) – b

From theorem 2.3.1, a V b = a – (a Λ b) + b. Then (a V b) – b = a – (a Λ b), but (a Λ b) = 0. Hence $(a - b)^+$ = a. ■

Lemma 3.3.2: In any l-group G, if 0 ≤ na, then 0 ≤ a.

Proof: From left distributive law, n(a Λ 0) = na Λ (n–1)a Λ (n–2)a Λ ...Λ a Λ0. Since na Λ 0 = 0,

n(a Λ 0) = (n – 1)a Λ (n – 2)a Λ…Λ a Λ 0

= (n – 1)(a Λ 0)

Thus, n(a Λ 0) = (n – 1)(a Λ 0). Then , n(a Λ 0) = n(a Λ 0) – (a Λ 0). By cancelation law, (a Λ 0) = 0. Hence 0 ≤ a. ■

Lemma 3.3.3: The positive part and the inverse of negative part of any elements are disjoint. That is, for any a, (a V 0) Λ (–a V 0) = a^+Λ ($-a^-$) = 0.

Proof: Clearly – (a Λ 0) = –a V 0 and by distributive law, (a V 0) Λ (–a V 0) = (a Λ–a) V 0 = 0.

Then $0 \geq a \wedge -a$. (*)

This implies that $0 \leq -(a \wedge -a) = -a \vee a$ and we know that $a \wedge -a \leq a \vee -a$. By right homogeneity, $0 \leq (a \vee -a) - (a \wedge -a)$. Then $0 \leq (a \vee -a) - (-(a \vee -a))$.

$\Rightarrow 0 \leq (a \vee -a) + (a \vee -a)$

$\Rightarrow 0 \leq 2(a \vee -a)$

Now use lemma 3,3.2 with n = 2; $0 \leq a \vee -a$

$\Rightarrow -(a \vee -a) = a \wedge -a \leq 0$ (**)

From (*) and (**), $0 = a \wedge -a$. Hence $0 = (a \wedge -a) \vee 0 = (a \vee 0) \wedge (-a \vee 0) = a^+ \wedge (-a^-)$. ■

Definition 3.3.2: The absolute value of an element a of an l-group G is $a \vee (-a)$, that is, $|a| = a \vee (-a)$.

Theorem 3.3.5: In any l-group G, we have the following properties of absolute value:

(i). $|a| \geq 0$; $|a| = 0$ if and only if $a = 0$;

(ii). $|na| = |n||a|$ for any integer n;

(iii). $|a| = a^+ + (-a^-)$;

(iv). $|a - b| = (a \vee b) - (a \wedge b)$.

Proof: (i) We have, $-|a| = -[(-a) \vee a] = a \wedge (-a) \leq a \vee (-a) = |a|$. Hence, $0 = -|a| + |a| \leq |a| + |a|$. This implies, by right homogeneity, that $0 \leq 2|a|$ and thus $0 \leq |a|$ by lemma 3.3.2.

By definition $|0| = 0 \vee (-0) = 0 \vee 0 = 0$.

(ii). a^+ & $-a^- = (-a)^+$ are disjoint, i.e. $a^+ \wedge (-a)^+ = 0$. From lemma 3.3.1, a^+ and $(-a)^+$ are permutable and $na = n(a^+) + n(a^-)$. Thus, $n(a^+) \wedge n(-a^-) = 0$ and $(na)^+ = n(a^+)$ and $(na)^- = n(a^-)$. Therefore $|na| = |n||a|$ for positive integer n since $|-x| = |x|$ it is also true for negative n.

(iii). We have $a^+ - a^- = (a \vee 0) - (a \wedge 0)$

$= (a \vee 0) + (-a \vee 0)$

$= [(a \vee 0) + -a] \vee [(a \vee 0) + 0]$ (by distributive law)

$= [0 \vee -a] \vee [(a \vee 0)]$

$= [0 \vee 0] \vee [-a \vee a]$

$= -a \vee a = |a|$

(iv). Setting a – b in place of a in (iii), we will obtain,

$|a - b| = (a - b)^+ - (a - b)^-$

$= [(a - b) \vee 0] + b - b - [(a - b) \wedge 0]$

$= [(a - b) \vee 0 + b] - [b + (a - b) \wedge 0]$

Hence, $|a - b| = (a \vee b) - (a \wedge b)$. ■

Chapter 4

Ideals of Lattice Ordered Monoid

4.1 Lattice ordered monoid

Definition 4.1.1: A non-empty set A with binary operation "+" is said to be a monoid if it satisfies the following conditions:

1. Closurety, i.e. $a, b \in A \Rightarrow a + b \in A$;
2. Associativity, i.e. $a, b, c \in A$, $(a + b) + c = a + (b + c)$;
3. Existence of identity, i.e. there exists $e \in A$ such that $a + e = a = e + a$. for all $a \in A$.

Notation: We denote the identity element in a monoid by 0.

Definition 4.1.2: A lattice ordered monoid (l-monoid) is a system $(A, +, 0, \vee, \wedge)$ such that

1. $(A, +, 0)$ is a monoid;
2. $(A, \vee, \wedge)$ is a lattice; and
3. $a + (b \vee c) = (a + b) \vee (a + c)$, $(b \vee c) + a = (b + a) \vee (c + a)$,

$a + (b \wedge c) = (a + b) \wedge (a + c)$, $(b \wedge c) + a = (b + a) \wedge (c + a)$, for each $a, b, c \in A$.

Remark 4.1.1: The partial order induced by lattice operations $\vee$ and $\wedge$ is denoted by $\leq$. Clearly, if $a \leq b$, then $c + a \leq c + b$ & $a + d \leq b + d$, $\forall a, b, c, d \in A$.

Example 4.1.1: $A = (N_0, +)$ is a monoid of non-negative integers with "+" the usual addition and let the relation "$\leq$" be the usual less or equal to. Then $(N_0, +, \leq)$ is an l-monoid.

Definition 4.1.3: A subset S of a monoid $(A, +, 0)$ is called a submonoid of A if $0 \in S$ and $a + b \in S$, for each $a, b \in S$. A submonoid S of an l-monoid A is called an l-submonoid of A if S is also a sub-lattice of A.

Example 4.1.2: $(N_0, +, \leq)$ is an l-submonoid of $(Z, +, \leq)$. That is, the set of non-negative integers with the usual addition "+" and the usual less or equal to relation "≤" is an l-submonoid of the set of integers with the usual addition "+" and less or equal to relation "≤".

Note: **-** If for elements a and b of an l-monoid A there exist a least $x \in A$ such that $b + x \geq a$ and a least $y \in A$ such that $y + b \geq a$, then the element x is denoted by $a \leftharpoondown b$ and the element y by $a \rightharpoonup b$.

Definition 4.1.4: A system $(B, +, 0, \vee, \wedge, \leftharpoondown, \rightharpoonup)$ is called a dually residuated lattice ordered monoid (notation DRl-monoid) if

1. $(B, +, 0, \vee, \wedge)$ is an l-monoid;
2. for each a, b in B there exist elements $a \leftharpoondown b$ and $a \rightharpoonup b$;
3. $b + ((a \leftharpoondown b) \vee 0) \leq a \vee b$, $((a \rightharpoonup b) \vee 0) + b \leq a \vee b$, for each $a, b \in B$;
4. $a \leftharpoondown a \geq 0$, $a \rightharpoonup a \geq 0$ for each $a \in B$.

Note that the condition in(2) is equivalent to the following system of identities:

$(x \rightharpoonup y) + y \geq x$, $y + (x \leftharpoondown y) \geq x$,

$x \rightharpoonup y \leq (x \vee z) \rightharpoonup y$, $x \leftharpoondown y \leq (x \vee z) \leftharpoondown y$,

$(x + y) \rightharpoonup y \leq x$, $(y + x) \leftharpoondown y \leq x$.

Examples 4.1.3:

(1). Let $G = (G, +, 0, -, \vee, \wedge)$ be an l-group. Set $x \rightharpoonup y = x - y$ and $x \leftharpoondown y = -y + x$, $\forall x, y \in G$. Then $(G, +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ is a DRl-monoid.

(2). Let G be an l-group and G^+ be its positive cone, i. e. $G^+ = \{x \in G: 0 \leq x\}$. Set $x \rightharpoonup y = (x - y) \vee 0$ and $x \leftharpoondown y = (-y + x) \vee 0$ for any $x, y \in G^+$. Then $(G^+, +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ is a DRl-monoid.

Lemma 4.1.1: For elements x, y of any DRl-monoid, it holds

$$(x \rightharpoonup y) \vee (y \rightharpoonup x) = (x \vee y) \rightharpoonup (x \wedge y),$$

$$(x \leftharpoondown y) \vee (y \leftharpoondown x) = (x \vee y) \leftharpoondown (x \wedge y).$$

Proof: $(x \vee y) \rightharpoonup (x \wedge y) = [x \rightharpoonup (x \wedge y)] \vee [y \rightharpoonup (x \wedge y)]$ (by distributive law)

$= [(x \rightharpoonup x) \vee (x \rightharpoonup y)] \vee [(y \rightharpoonup x) \vee (y \rightharpoonup y)]$(by theorem 2.2.3)

$= [0 \vee (x \rightharpoonup y)] \vee [(y \rightharpoonup x) \vee 0]$

$= (x \rightharpoonup y) \vee (y \rightharpoonup x) \vee 0.$

However, $(x \rightharpoonup y) \vee (y \rightharpoonup x) \geq (x \rightharpoonup (x \vee y)) \vee (y \rightharpoonup (x \vee y)) = (x \vee y) \rightharpoonup (x \vee y) = 0.$

Therefore $(x \vee y) \rightharpoonup (x \wedge y) = (x \rightharpoonup y) \vee (y \rightharpoonup x).$

The proof of the second statement is analogous. ■

Lemma 4.1.2: If $x \geq y \geq z$, then $(x \rightharpoonup y) + (y \rightharpoonup z) = x \rightharpoonup z$ and $(y \leftharpoondown z) + (x \leftharpoondown y) = x \leftharpoondown z$.

Proof: If $y \geq z$ then $y \rightharpoonup z \geq 0$ and $y = y \vee z = (y \rightharpoonup z)^{+} + z = (y \rightharpoonup z) + z$.

Hence $x \rightharpoonup y = x \rightharpoonup ((y \rightharpoonup z) + z) = (x \rightharpoonup z) \rightharpoonup (y \rightharpoonup z)$.

Similarly, $x \geq y$ entails $x \rightharpoonup z \geq y \rightharpoonup z$ which yields $x \rightharpoonup z = ((x \rightharpoonup z) \rightharpoonup (y \rightharpoonup z)) + (y \rightharpoonup z)$.

Summarizing, $x \rightharpoonup z = (x \rightharpoonup y) + (y \rightharpoonup z)$. ■

Lemma 4.1.3: The following holds in any DRl-monoid:

1.$0 \rightharpoonup x = 0 \leftharpoondown x$,

2. $(x \rightharpoonup y) + (y \rightharpoonup z) \geq x \rightharpoonup z$,

3. $(y \leftharpoondown z) + (x \leftharpoondown y) \geq x \leftharpoondown z$.

Proof: (1). From $(x + (0 \rightharpoonup x)) + x = x + ((0 \rightharpoonup x) + x) \geq x + 0 = x$ it follows that $x + (0 \rightharpoonup x) \geq x \rightharpoonup x = 0$, thus $0 \rightharpoonup x \geq 0 \leftharpoondown x$. Similarly, $0 \leftharpoondown x \geq 0 \rightharpoonup x$. Therefore $0 \rightharpoonup x = 0 \leftharpoondown x$.

(2). $(x \rightharpoonup y) + (y \rightharpoonup z) + z \geq (x \rightharpoonup y) + y \geq x$. Thus, $(x \rightharpoonup y) + (y \rightharpoonup z) \geq x \rightharpoonup z$.

(3). Similar to (2). ■

Applying (2) and (3), we will obtain the following result:

Lemma 4.1.4: In every DRl-monoid we have

1. $y \rightharpoonup x \geq (z \rightharpoonup x) \leftharpoondown (z \rightharpoonup y)$,

2. $y \leftharpoondown x \geq (z \leftharpoondown x) \rightharpoonup (z \leftharpoondown y)$,

3. $y \rightharpoonup x \geq (y \rightharpoonup z) \rightharpoonup (x \rightharpoonup z)$,

4. $y \leftharpoondown x \geq (y \leftharpoondown z) \leftharpoondown (x \leftharpoondown z)$.

4.2. Distance Functions

Definition 4.2.1: Let A be a DRl-monoid. We define the distance function by

$$d_1(x, y) = (x \rightharpoonup y) \vee (y \rightharpoonup x),$$

$$d_2(x, y) = (x \leftharpoondown y) \vee (y \leftharpoondown x),$$

for any $x, y \in A$.

For an element a in an l-monoid A, we define the **positive part** of a by $a^+ = a \vee 0$ and the **negative part** of a by $a^- = a \wedge 0$. Moreover, a is called positive (negative) if $a \geq 0$ $(a \leq 0)$. The set of all positive (negative) elements of A will be denoted by A^+ (A^-). If a is an **invertible element** of A, then the inverse of a is denoted by –a. The set of all invertible elements of A is denoted by In(A).

From this time on we shall use the following assertions without any reference:

(A_1). For each element a of a l-monoid, $a = a^+ + a^- = a^- + a^+$.

(A_2). Each negative element of a DRl-monoid is invertible.

Proposition4.2.1: The distance functions have the following properties:

1. $d_1(x, y) = d_1(y, x)$,
2. $d_2(x, y) = d_2(y, x)$,
3. $d_1(x, 0) = d_2(x, 0)$,
4. $d_1(x, y) = (x \rightharpoonup y)^+ + (y \rightharpoonup x)^+$,
5. $d_2(x, y) = (y \leftharpoondown x)^+ + (x \leftharpoondown y)^+$,
6. $d_1(x, y) \geq 0$ with $d_1(x, y) = 0$ if and only if $x = y$,
7. $d_2(x, y) \geq 0$ with $d_2(x, y) = 0$ if and only if $x = y$,
8. $d_1(x, y) \leq d_1(x, z) + d_1(z, y) + d_1(x, z)$,
9. $d_1(x, y) \leq d_1(z, y) + d_1(x, z) + d_1(z, y)$,
10. $d_2(x, y) \leq d_2(x, z) + d_2(z, y) + d_2(x, z)$,

11. $d_2(x, y) \le d_2(z, y) + d_2(x, z) + d_2(z, y)$,
12. $d_1(x, y) \vee d_1(s, t) \ge d_1(x \vee s, y \vee t), d_1(x \wedge s, y \wedge t)$,
13. $d_2(x, y) \vee d_2(s, t) \ge d_2(x \vee s, y \vee t), d_2(x \wedge s, y \wedge t)$,
14. $d_2(z \rightharpoonup x, z \rightharpoonup y) \le d_1(x, y)$,
15. $d_1(z \leftharpoonup x, z \leftharpoonup y) \le d_2(x, y)$,
16. $d_1(x \rightharpoonup z, y \rightharpoonup z) \le d_1(x, y)$,
17. $d_2(x \leftharpoonup z, y \leftharpoonup z) \le d_2(x, y)$.

Proof: Obviously, (1) and (2) hold; (3) follows by Lemma 4.1.3 (1). To check (4), and similarly (5), we compute

$$\begin{aligned}
d_1(x, y) &= (x \rightharpoonup y) \vee (y \rightharpoonup x) = (x \vee y) \rightharpoonup (x \wedge y) \text{ by Lemma 4.1.1} \\
&= [(x \vee y) \rightharpoonup y] + [y \rightharpoonup (x \wedge y)] \text{ by Lemma 4.1.2} \\
&= [(x \rightharpoonup y) \vee (y \rightharpoonup y)] + [(y \rightharpoonup x) \vee (y \rightharpoonup y)] \\
&= [(x \rightharpoonup y) \vee 0] + [(y \rightharpoonup x) \vee 0] \\
&= (x \rightharpoonup y)^{+} + (y \rightharpoonup x)^{+}.
\end{aligned}$$

Further, (6) follows from (4) and (7) from (5), respectively, since

$d_1(x, y) = (x \rightharpoonup y)^{+} + (y \rightharpoonup x)^{+} \ge 0$.

It is clear that $x = y$ entails $d_1(x, y) = 0$. Conversely, if $d_1(x, y) = (x \rightharpoonup y)^{+} + (y \rightharpoonup x)^{+} = 0$, then $(x \rightharpoonup y)^{+} = (y \rightharpoonup x)^{+} = 0$. Hence $x \rightharpoonup y \le 0$ and $y \rightharpoonup x \le 0$, and so $x \le y$ and $y \le x$, thus $x = y$.

Now, we will prove (8) (similarly (9), (10) and (11)):

$$\begin{aligned}
d_1(x, z) + d_1(z, y) + d_1(x, z) &= [(x \rightharpoonup z) \vee (z \rightharpoonup x)] + [(z \rightharpoonup y) \vee (y \rightharpoonup z)] + [(x \rightharpoonup z) \vee (z \rightharpoonup x)] \\
&= [(x \rightharpoonup z) + (z \rightharpoonup y) + (x \rightharpoonup z)] \vee [(x \rightharpoonup z) + (z \rightharpoonup y) + (z \rightharpoonup x)] \\
&\quad \vee [(x \rightharpoonup z) + (y \rightharpoonup z) + (x \rightharpoonup z)] \vee [(x \rightharpoonup z) + (y \rightharpoonup z) + (z \rightharpoonup x)] \\
&\quad \vee [(z \rightharpoonup x) + (z \rightharpoonup y) + (x \rightharpoonup z)] \vee [(z \rightharpoonup x) + (z \rightharpoonup y) + (z \rightharpoonup x)] \\
&\quad \vee [(z \rightharpoonup x) + (y \rightharpoonup z) + (x \rightharpoonup z)] \vee [(z \rightharpoonup x) + (y \rightharpoonup z) + (z \rightharpoonup x)] \\
&\ge [(x \rightharpoonup z) + (z \rightharpoonup y) + (x \rightharpoonup z)] \vee [(x \rightharpoonup z) + (z \rightharpoonup y) + (z \rightharpoonup x)] \\
&\quad \vee [(x \rightharpoonup z) + (y \rightharpoonup z) + (z \rightharpoonup x)] \vee [(z \rightharpoonup x) + (y \rightharpoonup z) + (z \rightharpoonup x)]
\end{aligned}$$

$$\begin{aligned}
d1(x, z) + d_1(z ,y) + d_1(x, z) &\ge [((x \rightharpoonup z) + (z \rightharpoonup y)) + ((x \rightharpoonup z) \vee (z \rightharpoonup x))] \\
&\quad \vee [((x \rightharpoonup z) \vee (z \rightharpoonup x)) + ((y \rightharpoonup z) + (z \rightharpoonup x))]
\end{aligned}$$

$$(\text{using } (x \rightharpoonup z) \vee (z \rightharpoonup x) \geq 0, \text{ by } (4))$$

$$\geq [(x \rightharpoonup z) + (z \rightharpoonup y)] \vee [(y \rightharpoonup z) + (z \rightharpoonup x)]$$

$$\geq (x \rightharpoonup y) \vee (y \rightharpoonup x) = d_1(x, y).$$

Therefore $d_1(x, z) + d_1(z, y) + d_1(x, z) \geq d_1(x, y)$.

Let us verify (12):

$$d_1(x, y) \vee d_1(s, t) = (x \rightharpoonup y) \vee (y \rightharpoonup x) \vee (s \rightharpoonup t) \vee (t \rightharpoonup s)$$

$$= (x \rightharpoonup y) \vee (s \rightharpoonup t) \vee (y \rightharpoonup x) \vee (t \rightharpoonup s)$$

$$\geq [x \rightharpoonup (y \vee t)] \vee [s \rightharpoonup (y \vee t)] \vee [y \rightharpoonup (x \vee s)] \vee [t \rightharpoonup (x \vee s)]$$

$$= [(x \vee s) \rightharpoonup (y \vee t)] \vee [(y \vee t) \rightharpoonup (x \vee s)] = d_1(x \vee s, y \vee t).$$

Therefore, $d_1(x, y) \vee d_1(x, y) \geq d_1(x \vee s, y \vee t)$.

The rest of (12) and (13) is analogous. Finally, (14)–(17) are consequences of Lemma 4.1.4. ■

Proposition 4.2.2: Each negative element of a DRl-monoid is invertible.

Proof: Let x be a negative element of a DRl-monoid B. Then $x \leq 0$. By definition 4.1.4 (2), we have $0 \rightharpoonup x$, $0 \leftharpoondown x \in B$ such that $0 \rightharpoonup x + x = 0$ and $x + 0 \leftharpoondown x = 0$.

$\Rightarrow 0 \rightharpoonup x + x = 0 = x + 0 \leftharpoondown x$ since by lemma 4.1.3(1) $0 \rightharpoonup x = 0 \leftharpoondown x = -x$.

Hence every negative element of a DRl-monoid is invertible. ■

Theorem 4.2.1: The set of all invertible elements of an l-monid A (In(A)) is an l-group and a sub lattice of A.

Proof: Clearly, In(A) is a group. Let $a, b \in In(A)$.Then we have

$$[a \vee b] + [(-a) \wedge (-b)] = [(a \vee b) + (-a)] \wedge [(a \vee b) + (-b)]$$

$$= [0 \vee (b-a)] \wedge [(a-b) \vee 0] \leq 0, \text{ and}$$

$$[a \vee b] + [(-a) \wedge (-b)] = a + [(-a) \wedge (-b)] \vee b + [(-a) \wedge (-b)]$$

$$= [0 \wedge (a-b)] \vee [(b-a) \wedge 0] \geq 0$$

Hence $[a \vee b] + [(-a) \wedge (-b)] = 0$.

By similar argument, we can obtain $[(-a) \wedge (-b)] + [a \vee b] = 0$.

Hence $-(a \vee b) = (-a) \wedge (-b)$. Similarly, $-(a \wedge b) = (-a) \vee (-b)$. Therefore $a \wedge b, a \vee b \in In(A)$.

Hence In(A) is an l-group and a sub lattice of A. ■

Definition 4.2.2: The absolute value of an element x of a DRI-monoid is defined by $(x \rightharpoonup 0) \vee (0 \rightharpoonup x)$.That is, $|x| = (x \rightharpoonup 0) \vee (0 \rightharpoonup x) = d(x, 0)$.

Theorem 4.2.2: Let A be an l-monoid. Let In(A) be the set of all invertible elements of A, and let $P = \{ y \in A: y \wedge |z| = 0$ for each $z \in In(A) \}$, where $|z|$ is the absolute value of z in In(A). Then, (i). $P \subseteq A^+$, 0 is the least element of P, $In(A) \cap P = \{0\}$,

(ii). P is a convex subset of A,

(iii). P is a sub lattice of A and an l-submonoid of A.

Proof:

i.$0 \wedge |0| = 0 \Rightarrow 0 \in P$. And for $a \in P$ $a \wedge |z| = 0$, $\forall z \in In(A)$. Then $0 \leq a$. Thus, $P \subseteq A^+$. Hence, 0 is the least element of P. Now if $a \in In(A) \cap P$, then $a = a \wedge |a| = 0$.Thus, $In(A) \cap P = \{0\}$.

ii. Let $a, b \in P$, $x \in A$, and $a \geq x \geq b$. Then, $0 = a \wedge |z| \geq x \wedge |z| \geq b \wedge |z| = 0$ for each $z \in In(A)$. Thus $x \wedge |z| = 0$ for each $z \in In(A)$and hence $x \in P$. Therefore, P is a convex subset of A.

iii. Let $a, b \in P$. By theorem 3.3.3, $a + b \in P$ and by (i) $0 \in P$. Hence P is a sub monoid of A. Since $a \geq 0$, $b \geq 0$, we have $a + b \geq b$, $a + b \geq a$. Thus, $a + b \geq a \vee b \geq a \wedge b \geq 0$. From the convexity of P, it follows that $a \vee b$, $a \wedge b \in P$. Therefore P is a sub lattice of A. ■

Lemma 4.2.1: Let B be a DRl-monoid, $x \in B$. Then $|x| = x^+ + (-x^-) = (-x^-) + x^+$.

Proof: $|x| = x^+ \rightharpoonup x^-$and $x^+ \geq x^-$. This implies that $x^+ \rightharpoonup x^- \geq x^- \rightharpoonup x^-$.
Then $x^+ \rightharpoonup x^- \geq 0$ and $x^+ = x^+ \vee x^- = (x^+ \rightharpoonup x^-)^+ + x^- = (x^+ \rightharpoonup x^-) + x^-$.
Therefore $(-x^-) + x^+ = x^+ + (-x^-) = x^+ \rightharpoonup x^- = |x|$. ■

4.3 Ideals of Lattice ordered monoid

Definition4.3.1: Let B be a DRl-monoid. Then a non-empty subset I of B is said to be an ideal of

B if it satisfies the following conditions:

(I_1). $0 \in I$

(I_2). If $x, y \in I$, then $x + y \in I$.

(I_3) If $x \in B$, $y \in I$ and $|x| \leq |y|$, then $x \in I$.

Theorem 4.3.1: Let B be a DRl-monoid, $I \subseteq B$. Let $0 \in I$ and $u + v \in I$, for each $u, v \in I$. Then the following propositions are equivalent:

(i). If $x \in B$, $y \in I$ and $|x| \leq |y|$, then $x \in I$.

(ii). If $x \in B$, $a, b \in I$ and $x^+ + a^- \leq x^- + b^+$, then $x \in I$.

(iii). If $x \in B$, $a, b \in I$ and $a^- + x^+ \leq b^+ + x^-$, then $x \in I$.

Proof:

(i)⇒(ii): Let $x^+ + a^- \leq x^- + b^+$, for some $x \in B$, $a, b \in I$. Since $|b^+| \leq |b|$, $|-a^-| \leq |a|$, we have $b^+, -a^- \in I$. Hence, $b^+ + (-a^-) \in I$. Then, by lemma 4,2.1, $|x| = -x^- + x^+ \leq b^+ + (-a^-)$.This implies that $|x| \leq |b^+ + (-a^-)|$. Therefore, $x \in I$.

(ii) ⇒(i): Let $|x| \leq |y|$ for some $x \in B$, $y \in I$. Then $(-x^-) + x^+ \leq y^+ + (-y^-)$. Hence, $x^+ + y^- \leq x^- + y^+$. Therefore, $x \in I$.

Analogously we can prove that (i) ⇔ (iii). ■

Next we will give the definition of an ideal of an l-monoid.

Definition 4.3.2: Let B be an l-monoid. A Subset I of B is called a left (right) ideal of B, if the conditions I_1, I_2 and the following conditions I_3^l, (I_3^r respectively) are fulfilled:

(I_3^l). If $x \in B$, $a, b \in I$ and $x^+ + a^- \leq x^- + b^+$, then $x \in I$.

(I_3^r). If $x \in B$, $a, b \in I$ and $a^- + x^+ \leq b^+ + x^-$, then $x \in I$.

A subset I of an l-monoid B is an ideal of B if I is both a left and right ideals of B.

Example 4.3.1: $\{0\}$ and B are ideals of an l-monoid B.

Remark 4.3.1: If I is a submonoid of an l-monoid A with the least element u and the greatest element v, then I is a left (right) ideal of A iff for each $x \in A \setminus I$ from
$x^+ + u^- \leq x^- + v^+$ ($u^- + x^+ \leq v^+ + x^-$, resp.) it follows that $x \in I$.

Theorem 4.3.2: Let A be an l-monoid, I an ideal of A.

(i). If $x \in A$, $a, b \in I$ and $x^+ + a^- \leq b^+ + x^-$, then $x \in I$.

(ii). If $x \in A$, $a, b \in I$ and $a^- + x^+ \leq x^- + b^+$, then $x \in I$.

Proof: (i). Let $x \in A$, $a, b \in I$, $x^+ + a^- \leq b^+ + x^-$. Then, $(x^+)^+ + a^- = x^+ + a^- \leq b^+ + x^- \leq b^+ = (x^+)^- + b^+$. Thus, $(x^+)^+ + a^- \leq (x^+)^- + b^+$. Then, $x^+ \in I$. Moreover, $a^- + (x^-)^+ = a^- \leq x^+ + a^- \leq b^+ + x^- = b^+ + (x^-)^-$. This implies that $a^- + (x^-)^+ \leq b^+ + (x^-)^-$. Then $x^- \in I$. Therefore, $x = x^+ + x^- \in I$.

The argument for (ii) is similar. ■

Theorem 4.3.3: Let A be an l-monoid, I a left ideal of A, $x \in A$. Then

(i). $x \in I$ if and only if $x^+ \in I$ and $x^- \in I$,

(ii). I is a convex subset of A,

(iii). I is an l-submonoid of A.

Proof: (i). ($\Rightarrow$) Let $x \in I$. Then $(x^+)^+ + x^- \leq x^+ = (x^+)^- + x^+$. Thus, $(x^+)^+ + x^- \leq (x^+)^- + x^+$ and hence $x^+ \in I$. Since $(x^-)^+ + x^- = x^- \leq (x^-)^- + x^+$, we have $(x^-)^+ + x^- \leq (x^-)^- + x^+$ Then $x^- \in I$. Therefore, $x^+, x^- \in I$.

($\Leftarrow$) Let $x^+, x^- \in I$. Then $x = x^+ + x^- \in I$.

(ii). Let $x \leq z \leq y$; $x, y \in I$ and $z \in A$. Then $z^+ \leq y^+, x^- \leq z^-$. By (i), $y^+, x^- \in I$. Since $(z^+)^+ + (y^+)^- = z^+ \leq y^+ = (z^+)^- + (y^+)^+$, we have $z^+ \in I$. Again since $(z^-)^+ + (x^-)^- = x^- \leq z^- \leq (z^-)^- + (x^-)^+$, we have $z^- \in I$. Hence, $z = z^+ + z^- \in I$. Therefore I is a convex subset of A.

(iii) We need only to show that I is a sub lattice of A. Let $x, y \in I$. By (i), $x^+ + y^+, x^- + y^- \in I$. Then from $x^- + y^- \leq x^- \Lambda y^- \leq x \Lambda y \leq x \vee y \leq x^+ \vee y^+ \leq x^+ + y^+$ and by the convexity of I we get $x \wedge y$, $x \vee y \in I$. Hence I is a sub-lattice of A. ■

Remark 4.3.2: An analogous theorem is valid for a right ideal of an l-monoid A.

Corollary 4.3.1: If x is an element of a left ideal I of an l-monoid A, then the interval
$[x-, x+] \subseteq I$.

Proof: Let $x \in I$. Then, by theorem 4.3.3, $x^+ \in I$ and $x^- \in I$. Since I is convex and $x^- \leq x^+$, we have $[x-, x+] \subseteq I$. ■

Lemma 4.3.1: Let A be an l-monoid, I a left ideal of A.

(i). If $x \in A^-$, $y \in I^+$ and $0 \leq x + y$, then $x \in I$.

(ii). If $x \in A^+$, $y \in I^-$ and $x + y \leq 0$, then $x \in I$.

Proof: (i). Let $x \in A^-$, $y \in I^+$ and $0 \leq x + y$. Then $x^+ + y^- = 0 \leq x + y = x^- + y^+$, which intern implies that $x^+ + y^- \leq x^- + y^+$.Hence, by definition of left ideal, $x \in I$.
The proof of (ii) can be obtained dually. ■

Lemma 4.3.2: Let A be an l-monoid, I a right ideal of A.

(i). If $x \in A^-$, $y \in I^+$ and $0 \leq y + x$, then $x \in I$.

(ii). If $x \in A^+$, $y \in I^-$ and $y + x \leq 0$, then $x \in I$.

Proof: The proof is similar to the proof of Lemma 4.3.1. ■

Recall that the set In(A) of all invertible elements of an l-monoid A is an l-group and a sub-lattice of A and $-(x \wedge y) = (-x) \vee (-y)$, $-(x \vee y) = (-x) \wedge (-y)$ for each $x, y \in In(A)$. Hence, if x is an invertible element of an l-monoid A, then $x^+$and $x^-$are invertibleelements of A and $-(x^-) = (-x)^+$; $-(x^+) = (-x)^-$.

Lemma 4.3.3: Let A be an l-monoid, I a left ideal of A, $x \in I$. If x is invertible, then $-x \in I$.

Proof: Let x be an invertible element of a left ideal I. Then,$(-x)^+ + x^- = -(x^-) + x^- = 0$ and $-(x^+) + x^+ = (-x)^- + x^+ = 0$. Thus, $(-x)^+ + x^- = (-x)^- + x^+$. Therefore, $-x \in I$.■

4.4 Congruent Relations

If we are given an equivalence relation ϱ on a set S, then we will write a ϱ b instead of (a, b) $\in \varrho$. In this section we show that the equivalence class containing 0 determined by congruence relations on an l-monoid constitute an ideal.

Definition 4.4.1: An equivalence relation ϱ on an l-monoid A is called a congruence relation on A if for each a, b, c, d $\in$A the following condition is satisfied:

(C_1) If a ϱ b and c ϱ d, then $(a \wedge c)\ \varrho\ (b \wedge d)$, $(a \vee c)\ \varrho\ (b \vee d)$ and $(a + c)\ \varrho\ (b + d)$.

For a $\in$ A we denote the equivalence class containing a by $[a]_\varrho = \{x \in A;\ x\ \varrho\ a\}$.

Theorem 4.4.1: Let ϱ be a congruence relation on an l-monoid A. Then $[0]_\varrho$ is an ideal of A.

Proof: Clearly, $0 \in [0]_\varrho$ and thus $[0]_\varrho$ is non-empty. We first show that is a convex sublattice of A. To see this, let $z \in A$, a, $b \in [0]_\varrho$ such that $a \leq z \leq b$. Then $a + 0 \leq z + 0 \leq b + 0$, $a \wedge 0 \leq z \wedge 0 \leq b + 0$, $a \vee 0 \leq z \vee 0 \leq b \vee 0$ and $|a| \leq |z| \leq |a|$ or $|a| \geq |z| \geq |b|$

$\Leftrightarrow |d(a, 0)| \leq |d(z, 0)| \leq |d(b, 0)|$ or $|d(a, 0)| \geq |d(z, 0)| \geq |d(b, 0)|$.

$\Rightarrow z\ \varrho\ 0$.

Hence, $z \in [0]_\varrho$ and thus $[0]_\varrho$ is a convex subset of A.

Let a, b $\in [0]_\varrho$. Then a ϱ 0 and b ϱ 0. Since ϱ is a congruence relation we have

$(a + b)\ \varrho\ (0 + 0)$, $(a \wedge b)\ \varrho\ (0 \wedge 0)$ and $(a \vee b)\ \varrho\ (0 \vee 0)$.

$\Rightarrow (a + b)\ \varrho\ 0$, $(a \wedge b)\ \varrho\ 0$ and $(a \vee b)\ \varrho\ 0$.

$\Rightarrow a + b,\ a \wedge b,\ a \vee b \in [0]_\varrho$.

Therefore, $[0]_\varrho$ is a convex sub-lattice of A. From our observation above we have, $[0]_\varrho$ is submonoid of A.

Now let $z \in A$, a, $b \in [0]_\varrho$, such that $z^+ + a^- \leq z^- + b^+$. Then from $a^- \leq z^+ + a^- \leq z^- + b^+ \leq b^+$ and the convexity of $[0]_\varrho$ it follows that $(z^+ + a^-)\ \varrho\ 0$ and $(z^- + b^+)\ \varrho\ 0$. From $(a^-)\ \varrho\ 0$ and $(b^+)\ \varrho\ 0$, we get $(z^+ + a^-)\ \varrho\ (z^+)$ and $(z^- + b^+)\ \varrho\ (z^-)$. Hence $(z^+)\ \varrho\ 0$, $(z^-)\ \varrho\ 0$.

Thus $(z^{+} + z^{-})\ \varrho\ 0$ which implies that $z \in [0]_{\varrho}$ since $z = z^{+} + z^{-}$.

Therefore, $[0]_{\varrho}$ is a left ideal of A. Using a similar argument we can show that $[0]_{\varrho}$ is also a right ideal of A.

Therefore $[0]_{\varrho}$ is an ideal of A. ■

4.5 Normal Ideals of Lattice Ordered Monoid

In this section we will consider normal ideals of an l-monoid and their characterizations.

Definition 4.5.1: An ideal I of an l-monoid A is normal iff x + I = I + x for each $x \in$ A.

Lemma 4.5.1: An ideal I of an l-monoid A is normal iff $x + I^{+} = I^{+} + x$ and $x + I^{-} = I^{-} + x$, $\forall x \in A$.

Proof: ($\Rightarrow$) Let I be a normal ideal of A. Let $x \in A$, $a \in I^{+}$. Then x + a = b + x for some $b \in I$ because $x + a \in x + I = I + x$. Then $x + a = (x + a) \vee x = (b + x) \vee x = (b \vee 0) + x$ $= b^{+} + x$. Since $b \in I$, by theorem 4.3.3, $b^{+} \in I$ and hence $x + a \in I^{+} + x$.

Hence, $x + I^{+} \subseteq I^{+} + x$. Analogously we can show that $I^{+} + x \subseteq x + I^{+}$. Therefore $x + I^{+} = I^{+} + x$. Dually we can show that $x + I^{-} = I^{-} + x$.

($\Leftarrow$) Let $x + I^{+} = I^{+} + x$ and $x + I^{-} = I^{-} + x$ for each $x \in A$. Then for $z \in A$ and $d \in I$ we get

$$
\begin{aligned}
z + d &= z + d^{+} + d^{-} \\
&= (z + d^{+}) + d^{-} \in (z + d^{+}) + I^{-} = I^{-} + (z + d^{+}) \\
&= h + (z + d^{+}), \text{ for some } h \in I^{-} \\
&= h + (g + z) \text{ for some } g \in I^{+} \text{ because } z + d^{+} \in z + I^{+} = I^{+} + z \\
&= (h + g) + z \in I + z.
\end{aligned}
$$

Hence, $z + I \subseteq I + z$.

Similarly we can show that $I + z \subseteq z + I$. Thus, z + I = I + z for all $z \in$ I. Therefore, I is normal ideal of A. ■

Definition 4.5.2: An ideal I of a DRl-monoid B is said to be a normal ideal of B satisfying the following condition for each x, y$\in$B:

$(x \rightharpoonup y)^+ \in I \Leftrightarrow (x \leftharpoondown y)^+ \in I$.

Next we show that in the case of DRl-monoids our definition of a normal ideal of an l-monoid coincides with the above definition.

Proposition 4.5.1: For any ideal I of a DRl-monoid A, the following conditions are equivalent:

i. I is a normal ideal of A;

ii. $(x \rightharpoonup y)^+ \in I \Leftrightarrow (x \leftharpoondown y)^+ \in I$ for any $x \in A$.

Proof: (i) $\Rightarrow$ (ii). Let $x \in A$, $h \in I^+$ and set $y = h + x \in I^+ + x$. It is clear that $y \geq x$ and, consequently, $y = x \vee y = (y \rightharpoonup x)^+ + x = x + (y \leftharpoondown x)^+$. From $h + x \geq y$ it follows that $h \geq y \rightharpoonup x \geq 0$, since $y \geq x$. Hence $(y \rightharpoonup x)^+ = y \rightharpoonup x \in I^+$. But I is normal in A, so that $(y \leftharpoondown x)^+ \in I^+$. Thus $y \in x + I^+$. Hence $I^+ + x \subseteq x + I^+$. Similarly, $x + I^+ \subseteq I^+ + x$.
Therefore, $(x \rightharpoonup y)^+ \in I \Leftrightarrow (x \leftharpoondown y)^+ \in I$.

(ii) $\Rightarrow$ (i) If $(y \rightharpoonup x)^+ \in I$ then $x \vee y = (y \rightharpoonup x)^+ + x = x + h$ for some $h \in I^+$.
Therefore $y \leq x + h$, which yields $(y \leftharpoondown x)^+ \leq ((x + h) \leftharpoondown x)^+ \leq h \vee 0 = h \in I^+$.
Thus, $(y \leftharpoondown x)^+ \in I$. The converse is analogous. Hence $(x \rightharpoonup y)^+ \in I \Leftrightarrow (y \leftharpoondown x)^+ \in I$.
Therefore, I is a normal ideal of A. ■

Note:-We have already proved, in theorem 4.4.1, that the equivalence class $[0]_\varrho$ determined by a congruence relation gives us an ideal of an l-monoid. The following example shows that $[0]_\varrho$ need not be a normal ideal of for any congruence relation ϱ.

Example 4.5.1: Let A = {0, a, b, c, d}. The binary operation + on A is defined by Table 1. The partial order $\leq$ on A is defined by the diagram in Figure 2. Then (A, +, $\leq$) is a non-commutative l-monoid and the relation

ϱ = {(0, 0), (a, a), (b, b), (c, c), (d, d), (c, 0), (d, b), (a, b), (a, d), (0, c), (b, d), (b, a), (d, a)} is a congruence relation on the l-monoid A. The ideal $[0]_\varrho = \{0, c\}$ is not a normal ideal, since $d + [0]_\varrho = \{d, a\} \neq \{d\} = [0]_\varrho + d$.

+	0	a	b	c	d
0	0	a	b	c	d
a	a	a	a	a	a
b	b	a	b	a	a
c	c	a	d	c	d
d	d	a	d	a	a

Table 1

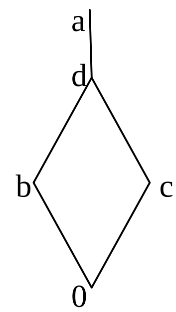

Figure 2

Note: If we imposed, in addition to (C_1), the condition

(C_2) if $a \varrho b$ and $c \varrho d$, then $(a \rightharpoonup c) \varrho (b \rightharpoonup d)$, $(a \leftharpoondown c) \varrho (b \leftharpoondown d)$

on an equivalence relation ϱ on a DRl-monoid B to be a congruence relation on B, then $[0]_\varrho$ becomes a normal ideal of B for each congruence relation ϱ on B.

Definition 4.5.3: Let A be an l-monoid, I a left ideal of A and J a right ideal of A. We define two binary relations ϱ_I^1 and ϱ_J^2 on A by:

$x\varrho_I^1 y$ if and only if there exist elements $g_1, h_1 \in I$ such that $x \leq g_1 + y$ and $y \leq h_1 + x$,

$x\varrho_J^2 y$ if and only if there exist elements $g_2, h_2 \in J$ such that $x \leq y + g_2$ and $y \leq x + h_2$

for each $x, y \in A$.

Definition 4.5.4: Let B be a DRl-monoid and I be an ideal of B. We define two binary relations $\theta_1(I)$ and $\theta_2(I)$ on B by:

$x\theta_1(I)y$ if and only if $(x \rightharpoonup y) \vee (y \rightharpoonup x) \in I$,

$x\theta_2(I)y$ if and only if $(x \leftharpoondown y) \vee (y \leftharpoondown x) \in I$, for each $x, y \in B$.

Lemma 3.5.2: For any ideal I, $\theta_1(I)$ and $\theta_2(I)$ are equivalence relations.

Proof: It is obvious that $\theta_1(I)$ is reflexive and symmetric. The transitivity follows from Proposition 4.2.1. Indeed, if $(x, y), (y, z) \in \theta_1(I)$ then $d_1(x, z) \le d_1(x, y) + d_1(y, z) + d_1(x, y) \in I$, hence $d_1(x, z) \in I$. $(x, z) \in \theta_1(I)$.Similarly for $\theta_2(I)$. Therefore $\theta_1(I)$ and $\theta_2(I)$ are equivalent relations. ■

Theorem 4.5.1: For any ideal I of a DRl-monoid B, the relations $\theta_1(I)$ and $\theta_2(I)$ are congruence relations on the lattice (B, V,Λ). Moreover, if I is normal, then $[0]_{\theta_1(I)} = [0]_{\theta_2(I)} = I$.

Proof: Let $(x, y), (s, t) \in \theta_1(I)$, i.e., $d_1(x, y), d_1(s, t) \in I$. Then, by Proposition 4.2.1,

$d_1(x \vee s, y \vee t) \le d_1(x, y) \vee d_1(s, t) \le d_1(x, y) + d_1(s, t) \in I$, and

$d_1(x \wedge s, y \wedge t) \le d_1(x, y) \vee d_1(s, t) \le d_1(x, y) + d_1(s, t) \in I$.

Hence,$(x \vee s, y \vee t), (x \wedge s, y \wedge t) \in \theta_1(I)$.Similarly, for $\theta_2(I)$.Therefore $\theta_1(I)$ and $\theta_2(I)$ are congruent relations on the lattice $(A, \vee, \wedge)$.

For each $x \in A$, $x \in [0]_{\theta_1}$ iff $(x, 0) \in \theta_1(I)$ iff $d_1(x, 0) = |x| \in I$ iff $x \in I$. Therefore $[0]_{\theta_1(I)} = [0]_{\theta_2(I)} = I$. ■

Proposition 4.5.2: Let A and B be DRl-monoids and $\varphi: A \rightarrow B$ is a homomorphism. Then $\varphi^{-1}(0) = \{x \in A; \varphi(x) = 0\}$ is a normal ideal of A.

Proof: Since A and B are a DRl-monoids. Then, $0 \in A \Rightarrow \varphi(0) = 0 \in B$.
$\Rightarrow 0 \in \varphi^{-1}(0)$. Thus, $\varphi^{-1}(0)$ is non-empty.

Let $x, y \in \varphi^{-1}(0)$. Then, $\varphi(x) = 0$ and $\varphi(y) = 0$. Thus, $\varphi(x + y) = \varphi(x) + \varphi(y) = 0 + 0 = 0$. Hence, $x + y \in \varphi^{-1}(0)$.

Suppose $\varphi(x) = 0$ and $|y| \le |x|$. Then $\varphi(|x|) = \varphi(x \vee (0 \rightharpoonup x)) = \varphi(x) \vee (0 \rightharpoonup \varphi(x)) = 0$ and, consequently, $\varphi(|y|) = 0$. Hence $\varphi(y \vee (0 \rightharpoonup y)) = \varphi(y) \vee (0 \rightharpoonup \varphi(y)) = 0$, which gives $\varphi(y) = 0$.

Thus $y \in \varphi(0)$.Therefore $\varphi^{-1}(0)$ is an ideal in A.

Finally, $(x \rightharpoonup y)^{+} \in \varphi^{-1}(0)$ if and only if $\varphi((x \rightharpoonup y) \vee 0) = (\varphi(x) \rightharpoonup \varphi(y)) \vee 0 = 0$.

Hence, $0 \geq \varphi(x) \rightharpoonup \varphi(y)$ iff $\varphi(y) \geq \varphi(x)$ iff $0 \geq \varphi(x) \leftharpoondown \varphi(y)$.Thus, $0 = (\varphi(x) \leftharpoondown \varphi(y)) \vee 0 = \varphi((x \leftharpoondown y) \vee 0)$, thus $(x \leftharpoondown y)^{+} \in \varphi^{-1}(0)$.Therefore $\varphi^{-1}(0)$ is a normal ideal of A. ■

Proposition 4.5.3: If I is a normal ideal of a DRl-monoid A, then, for all $x, y \in A$, $d_1(x, y) \in I$ if and only if $d_2(x, y) \in I$.

Proof: If $d_1(x, y) = (x \rightharpoonup y)^{+} + (y \rightharpoonup x)^{+} \in I$ then $(x \rightharpoonup y)^{+}, (y \rightharpoonup x)^{+} \in I$.

Since I is a normal ideal, this implies $(x \leftharpoondown y)^{+}, (y \leftharpoondown x)^{+} \in I$.

Hence, $d_2(x, y) = (x \leftharpoondown y)^{+} + (y \leftharpoondown x)^{+} \in I$. ■

Corollary 4.5.1: If I is a normal ideal then $\theta_1(I) = \theta_2(I)$; it will be denoted by $\theta(I)$.

Proof: Let $(x, y) \in \theta_1(x, y)$. Then $x\theta_1(I)y$ iff $(x \rightharpoonup y) \vee (y \rightharpoonup x)$.

$\Rightarrow x \rightharpoonup y, y \rightharpoonup x \in I$.

$\Rightarrow (x \rightharpoonup y)^{+}, (y \rightharpoonup x)^{+} \in I$.

$\Rightarrow (x \leftharpoondown y)^{+}, (y \leftharpoondown x)^{+} \in I$ since I is a normal ideal.

$\Rightarrow (x \leftharpoondown y)^{+} + y \leftharpoondown x)^{+} \in I$.

$\Rightarrow x\theta_2(I)y$, i.e. $(x, y) \in \theta_2(y)$.

$\Rightarrow \theta_1(I) \subseteq \theta_2(I)$.

By the same argument, if $(x, y) \in \theta_2(I)$, then $(x, y) \in \theta_1(I)$.

Hence, $\theta_2(I) \subseteq \theta_1(I)$.

Therefore, $\theta_1(I) = \theta_2(I) = \theta(I)$. ■

Lemma 4.5.3: Let I is a normal ideal of a DRl-monoid A. If $(x, y) \in \theta(I)$ then, for each $z \in A$,

$$(x \rightharpoonup z, y \rightharpoonup z) \in \theta(I), \qquad (x \leftharpoondown z, y \leftharpoondown z) \in \theta(I),$$

$$(z \rightharpoonup x, z \rightharpoonup y) \in \theta(I), \qquad (z \leftharpoondown x, z \leftharpoondown y) \in \theta(I).$$

Proof: This follows from Proposition 4.2.1 (14)–(17). ■

Theorem 4.5.2: If I is a normal ideal of a DRl-monoid A then $\theta(I)$ is a congruence relation on A. In addition, $[0]_{\theta(I)} = I$.

Proof: Let $(x, y)\in\theta(I)$ and $(s, t)\in\theta(I)$. Then $(x \rightharpoonup y)^+, (s \rightharpoonup t)^+ \in I$.

Obviously, $x \le x \vee y = (x \rightharpoonup y)^+ + y$ and $s \le s \vee t = (s \rightharpoonup t)^+ + t$. Hence, it follows that

$$x + s \le (x \rightharpoonup y)^+ + y + (s \rightharpoonup t)^+ + t = (x \rightharpoonup y)^+ + (y + (s \rightharpoonup t)^+) + t$$
$$= (x \rightharpoonup y)^+ + (h + y) + t$$
$$= ((x \rightharpoonup y)^+ + h) + (y + t)$$

for some $h \in I^+$ such that $y + (s \rightharpoonup t)^+ = h + y$.

However, $((x \rightharpoonup y)^+ + h) + (y + t) \ge x + s$ iff $(x \rightharpoonup y)^+ + h \ge (x + s) \rightharpoonup (y + t)$.

Therefore, $((x + s) \rightharpoonup (y + t))^+ \le ((x \rightharpoonup y)^+ + h)^+ = (x \rightharpoonup y)^+ + h \in I$. So, $((x + s) \rightharpoonup (y + t))^+ \in I$.

We can similarly show that $((y + t) \rightharpoonup (x + s))^+ \in I$.

Hence, we conclude that $d_1(x + s, y + t) = ((x + s) \rightharpoonup (y + t))^+ + ((y + t) \rightharpoonup (x + s))^+ \in I$ and $(x + s, y + t)\in\theta(I)$.

By Lemma 4.5.3, $(x \rightharpoonup s, y \rightharpoonup s)\in\theta(I)$ and $(y \rightharpoonup s, y \rightharpoonup t) \in\theta(I)$.This yields $(x\rightharpoonup s, y \rightharpoonup t)\in\theta(I)$. Similarly, $(x \leftharpoondown s, y \leftharpoondown t)\in\theta(I)$.

The rest follows by Theorem 4.5.1. ■

Theorem 4.5.3: If θ is a congruence on a DRl-monoid A, then $[0]_\theta = \{x \in A; (x, 0)\in\theta\}$ is a normal ideal in A. Moreover, $\theta = \theta([0]_\theta)$.

Proof: The first part follows by Proposition 3.5.2. Further, we claim that

(C) $\quad (x, y)\in\theta \Leftrightarrow (d_1(x, y), 0)\in\theta,$

or equivalently,

$$(x, y)\in\theta \Leftrightarrow (d_2(x, y), 0)\in\theta.$$

Indeed, if $(x, y)\in\theta$ then $(x \rightharpoonup y, 0)\in\theta$ and $(y \rightharpoonup x, 0)\in\theta$, whence

$(d_1(x, y), 0) = ((x \rightharpoonup y) \vee (y \rightharpoonup x), 0) \in \theta$. Conversely, $(d_1(x, y), 0) \in \theta$ if and only if $d_1(x, y) \in [0]_\theta$ which implies $(x \rightharpoonup y)^+, (y \rightharpoonup x)^+ \in [0]_\theta$. This gives $x \vee y = (x \rightharpoonup y)^+ + y \equiv 0 + y = y\ (\theta)$, and $x \vee y = (y \rightharpoonup x)^+ + x \equiv 0 + x = x\ (\theta)$.

Thus, by the transitivity, $(x, y) \in \theta$. Now, $\theta = \theta([0]_\theta)$ is an immediate consequence of (C). ■

We now show in the case that I is an ideal of a DRl-monoid B, ϱ_I^1 coincides with $\theta_1(I)$ and ϱ_I^2 coincides with $\theta_2(I)$.

Lemma 4.5.4: Let B be a DRl-monoid.

(i). If $x, y \in B$, then $0 \rightharpoonup (x \rightharpoonup y) \leq y \rightharpoonup x$, $0 \leftharpoondown (x \leftharpoondown y) \leq y \leftharpoondown x$.

(ii). If I is an ideal of B, then $\varrho_I^1 = \theta_1(I)$ and $\varrho_J^2 = \theta_2(I)$.

Proof: (i). Let $x, y \in B$. By Lemma 4.1.3, $(y \rightharpoonup x) + (x \rightharpoonup y) \geq y \rightharpoonup y = 0$ and $(x \leftharpoondown y) + (y \leftharpoondown x) \geq y \leftharpoondown y = 0$. This yields $0 \rightharpoonup (x \rightharpoonup y) \leq y \rightharpoonup x$, $0 \leftharpoondown (x \leftharpoondown y) \leq y \leftharpoondown x$.

(ii) Let I be an ideal of B, $x, y \in B$, $x\varrho_I^1 y$. Hence, $x \leq g + y$, $y \leq h + x$, for some $g, h \in I$. Thus, $x \rightharpoonup y \leq g$, $y \rightharpoonup x \leq h$. In view of Proposition 4.2.1(6) we have $0 \leq (x \rightharpoonup y) \vee (y \rightharpoonup x) \leq g \vee h$. From the convexity of I we obtain $(x \rightharpoonup y) \vee (y \rightharpoonup x) \in I$. Therefore $x\theta_1(I)y$.

Let $z, t \in B$, $z\theta_1(I)t$. Thus $(z \rightharpoonup t) \vee (t \rightharpoonup z) \in I$. In view of (i) we have $|z \rightharpoonup t| = (z \rightharpoonup t) \vee (0 \rightharpoonup (z \rightharpoonup t)) \leq (z \rightharpoonup t) \vee (t \rightharpoonup z) = |(z \rightharpoonup t) \vee (t \rightharpoonup z)|$. This yields $z \rightharpoonup t \in I$. Analogously, $t \rightharpoonup z \in I$. Since $z \leq (z \rightharpoonup t) + t$, $t \leq (t \rightharpoonup z) + z$, we have $z\varrho_I^1 t$.

Hence $\varrho_I^1 = \theta_1(I)$.

Similarly we can show that $\varrho_J^2 = \theta_2(I)$. ■

Theorem 4.5.4: Let A be an l-monoid, I a left ideal of A and J a right ideal of A. Then ϱ_I^1 and ϱ_J^2 are congruence relations on the lattice $(A, \vee, \wedge)$.

Proof: It is clear relation that the ϱ_I^1 is reflexive and symmetric. Let a, b, c ∈ A, a ϱ_I^1 b and bϱ_I^1 c. Hence a ≤ g + b, b ≤ h + a, b ≤ u + c, c ≤ v + b, for some g, h, u, v ∈ I. Then a ≤ g + u + c, c ≤ v + h + a. Since g + u, v + h ∈ I, we have a ϱ_I^1 c.

Let x, y, s, z ∈ A, x ϱ_I^1 y and s ϱ_I^1 z. Then x ≤ g′ + y, y ≤ h′ + x, s ≤ u′ + z, z ≤ v′ + s, for some g′, h′, u′, v′ ∈ I.

$\Rightarrow (g' \vee u') + (y \vee z) = (g' + y) \vee (g' + z) \vee (u' + y) \vee (u' + z) \geq (g' + y) \vee (u' + z) \geq x \vee s.$

$\Rightarrow (g' \vee u') + (y \vee z) \geq x \vee s.$

Analogously, $(h' \vee v') + (x \vee s) \geq y \vee z$. Further, we have

$(g' \vee u') + (y \wedge z) = [(g' + y) \vee (u' + y)] \wedge [(g' + z) \vee (u' + z)] \geq (g' + y) \wedge (u' + z) \geq x \wedge s.$

Then $(g' \vee u') + (y \wedge z) \geq x \wedge s$. Similarly $(h' \vee v') + (x \wedge s) \geq y \wedge z$. Since $g' \vee u', h' \vee v' \in I$.

Hence $(x \vee s)\ \varrho_I^1\ (y \vee z)$ and $(x \wedge s)\ \varrho_I^1\ (y \wedge z)$.

Therefore, ϱ_I^1 is a congruence relation on the lattice (A, ∨, ∧).

Similarly we can show that ϱ_J^2 is a congruence relation on the lattice (A, ∨, ∧). ■

Let A be an l-monoid and let I be a left ideal of A. The $(A, \vee, \wedge)/\varrho_I^1$ is a lattice and for the partial order relation ≤ of this factor lattice the following is valid.

Theorem 4.5.5: Let A be an l-monoid, I a left ideal of A, x, y ∈ A. Then the following conditions are equivalent:

(i). $[x]_{\varrho_I^1} \leq [y]_{\varrho_I^1}$,

(ii). $x \leq g + (x \wedge y)$, for some g ∈ I,

(iii). $x \vee y \leq h + y$, for some h ∈ I.

Proof: (i) ⇒(ii). Let x, y ∈ A, $[x]_{\varrho_I^1} \leq [y]_{\varrho_I^1}$. Then $[x \wedge y]_{\varrho_I^1} = [x]_{\varrho_I^1}$.

$$\Rightarrow (x \wedge y)\ \varrho_I^1\ x.$$

Therefore, $x \leq g + (x \wedge y)$ for some g ∈ I.

(ii) ⇒(iii). Let x, y ∈ A and $x \leq g + (x \wedge y)$, for some g ∈ I. Let $h = g^+$. Then h ∈I.

Since $x \leq h + (x \wedge y) \leq (h + x) \wedge (h + y)$, we have $x \leq h + y$. Clearly, $y \leq h + y$.
Therefore, $x \vee y \leq h + y$.
(iii) ⇒(i). Let $x, y \in A$, $x \vee y \leq h + y$, for some $h \in I$. Since $y \leq 0 + (x \vee y)$, we conclude that $y \varrho_I^1 (x \vee y)$.

$$\Rightarrow [y]_{\varrho_I^1} = [x]_{\varrho_I^1} \vee [y]_{\varrho_I^1}.$$

Therefore, $[x]_{\varrho_I^1} \leq [y]_{\varrho_I^1}$. ■

Note: Analogous theorem is valid for right ideal of an l-monoid.

Lemma 4.5.5: If ϱ is a congruence relation on an l-monoid A, then $\varrho^1_{[0]_\varrho} \subseteq \varrho$, $\varrho^2_{[0]_\varrho} \subseteq \varrho$.

Proof: Let $x, y \in A$, $x \varrho^1_{[0]_\varrho} y$. Then $x \leq g + y$ and $y \leq h + x$, for some $g, h \in [0]_\varrho$.
$\Rightarrow g \varrho 0$, $h \varrho 0$. Then $y \varrho (g + y)$, $x \varrho (h + x)$ and hence $(x \wedge y) \varrho (x \wedge (g + y))$,
$(y \wedge x) \varrho (y \wedge (h + x))$. Thus, $(x \wedge y) \varrho x$, $(x \wedge y) \varrho y$. This yields $x \varrho y$.
Analogously we can show that $\varrho^2_{[0]_\varrho} \subseteq \varrho$. ■

Unlike in a DRl-monoid, in an l-monoid A the relations $\varrho^1_{[0]_\varrho} = \varrho$ and $\varrho^2_{[0]_\varrho} = \varrho$ need not be valid for a congruence relation ϱ on A. For the congruence ϱ from Example 4.5.1 we have $\varrho^1_{[0]_\varrho} = \{(0, 0), (b, b), (c, c), (d, d), (a, a), (0, c), (c, 0), (b, d), (d, b)\} \neq \varrho$.

Lemma 4.5.6: If I is an ideal of A, then $[0]_{\varrho_I^1} \subseteq I$, $[0]_{\varrho_J^2} \subseteq I$.

Proof: Let I be an ideal of A, $p \in [0]_{\varrho_I^1}$. By Theorems 3.4.1 and 3.5.4, $[0]_{\varrho_I^1}$ is an idealof A and hence $(p^+) \varrho_I^1 0$, $(p^-) \varrho_I^1 0$. Then $0 \leq p^+ \leq q$, $0 \leq r + p^- \leq r$ for some $q, r \in I$. From the convexity of I, it follows that $p^+ \in I$. In view of Lemma 3.3.2 (i) from $0 \leq r + p^-$ we obtain $p^- \in I$. Then $p = p^+ + p^- \in I$. Thus, $[0]_{\varrho_I^1} \subseteq I$. Similarly, we can show that $[0]_{\varrho_J^2} \subseteq I$. ■

For a normal ideal I of an l-monoid the relations $[0]_{\varrho_I^1} = I$, $[0]_{\varrho_I^2} = I$ need not be valid. To see this let us consider the example below.

Example4.5.2: The set A = {0, a, b, c, d, e, f} with the binary operation ″+″ on A defined by Table 2 and the partial order ″≤″ on A defined by the diagram in Figure 3 is a commutative l-monoid. The ideal I = {0, a, c, e} of the l-monoid A is normal, $\varrho_I^1 = \varrho_J^2$ = {(0, 0), (a, a), (b, b), (c, c), (d, d), (e, e), (f, f), (0, e), (e, 0), (0, a), (a, 0), (e, a), (a, e), (d, f), (f, d)},
$[0]_{\varrho_I^1} = [0]_{\varrho_J^2}$ = {0, a, e} ≠ I.

+	0	a	b	c	d	e	f
0	0	a	b	c	d	e	f
d	a	a	b	c	f	a	f
b	b	b	b	b	b	b	b
d	c	c	b	c	b	c	b
d	d	f	b	b	d	d	f
e	e	a	b	c	d	e	f
f	f	f	b	b	f	f	f

Table 2

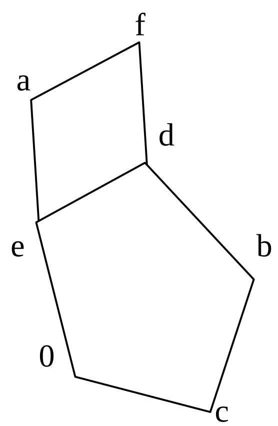

Figure 3

Definition 4.5.4:An ideal I of an l-monoid A is called tall if for each x ∈ I there exist b, c ∈ I such that 0 ≤ x + b, 0 ≤ c + x.

Theorem 4.5.6: Let I be a tall ideal of an l-monoid A. Then $[0]_{\varrho_I^1} = [0]_{\varrho_J^2}$ = I.

Proof: Let I be a tall ideal of A, x ∈ I. Then 0 ≤ b + x, 0 ≤ x + c, for some b, c ∈ I. Since $x \leq x^+ + 0$, $x \leq 0 + x^+$, we have x ϱ_I^1 0 and x ϱ_J^2 0.
Hence I $\subseteq [0]_{\varrho_I^1}$ and I $\subseteq [0]_{\varrho_J^2}$.Then Lemma 4.5.6 completes the proof. ■

Chapter 5

Conclusion

In this project, we have studied lattices and properties of lattices and lattice ordered group and its properties for the study of lattice ordered monoid (l-monoid) and ideals of lattice ordered monoid. We have seen generalizations of results from lattice ordered groups to ideals of l-monoid. In addition, we have studied properties of ideals of dually residuated lattice ordered monoid (DRl-monoid). The connection between normal ideals of DRl-monoid (l-monoid in general) and congruence relations were also studied.

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