THREE-PERSON COOPERATIVE GAME AND ITS APPLICATION IN DECISION MAKING PROCESS OF HIERARCHICAL ORGANIZATIONS

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Abstract

The solution of an n-level linear problem, when the levels make decisions sequentially and independently, is not necessarily pareto-optimal, i.e.there exist feasible points which offer increased payoffs to some levels without diminishing the payoffs to other levels. These increased payoffs may be obtained if the levels coalesce.

In particular if the number of decision makers on each level form coalition to work cooperatively they can get a better payoff. A game theoretic methodology for predicting coalition formation in the decision makers on each level is presented. The problem is modeled as an abstract game. If a core exists for the characteristic function game, then there exists a set of enforceable points which offer the increased payoffs available to the system, but a core may not exist. When the core exist, for games with non-empty cores, it would be an advantageous property for a power index to assign values to the players that comprise a solution in the core. We use Shapley value, as a solution concept, which has got a drawback that it might not always been an element of the core. So we take the Willick's power index as a best solution concept, which has got a better relation with the core than Shapley.

The n-level Stackleberg problem, which represent a class of n-level linear problem, as a special case the 3-level Stackleberg problem, is defined. In the university budget allocation system coalition among decision makers in each level, science faculty, is used to demonstrate the suggested methodology.

Addis Ababa University

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Dedication

To my family and my best friend Abraham Kassaye.

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I would like to express my deepest gratitude and thanks to my thesis advisor Dr.Semu Mitiku for his unreserved material support and invaluable advise. Without his insight in the problem and patience with me, I would, no doubt, still be on square one. I also recall and appreciate his invaluable financial and moral support to start my study earlier and to complete it on time. His encouragement and appreciation throughout my study at the university has been very indispensable. I will remember his generosity, consideration, and friendship long after I have forgotten the contents of this thesis.

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Chapter 1

Introduction

The behaviors of organizational structures have been extensively studied and debated over the last half century. The group decision-making process is a driving force in the world today. It has been known for sometime that the successful design of large, complex systems invariably involves decomposition of the system into a number of smaller subsystems, each with its own goal and constraints. The resulting interconnection of subsystems may take on many forms. Such as committees, hierarchies, and matrix organizations to name a few. But one of the most prevalent decision making structure through out history has been the hierarchy. Hierarchies exist in nearly every facet of life. They occurs in the government, business world, and even in the family unit.

A question that could arise is why hierarchy is so prevalent, even in nature. Some suggest that the hierarchy exhibits features that lend itself to obtaining solutions more easily than other structures. Hierarchical form in which a given level unit (or decision maker) controls or coordinates the units (or decision makers) on the level below it, and in turn is controlled by the units on the level above it. A system decomposed in this way is termed as decentralized multi-level system.

In a decentralized system, for example, top management is most often concerned with the development of corporate strategy and issues relating to overall profitability, market share and long-term liquidity and not with the coordination of day-to-day activities. As such, it rarely exercises direct control over its decision but may influence their operation by setting transfer prices, establishing product lines and channeling capital investment. At the divisional level, if management's objectives are not in line with those of head quarters, a situation may arise where individual units either overproduce or under produce,as they attempt to maximize net benefits. Finally at the lowest level of the hierarchy, labor employs uses a different set of response mechanisms to further its objective with in the collective bargaining frame work.

In game theoretic approaches explicitly consider the individual decision-making units or players by assessing each a unique objective function and control set. In the hierarchy, player's position and individual problem is well defined. This does not imply the solution obtained from a hierarchy is optimal. In fact, the decision made from organizations in the hierarchical form often seems to make no-sense. Therefore, by understanding them, we can eliminate their inherent efficiencies and better utilize the source which they expand.

Since the advent of linear programming and game theory in the 1940's and 1950's, a substantial effort has been directed toward analyzing the behaviors of interacting decision makers, each attempting to optimize individual objectives in view of decisions made by others. Many scientific disciplines have contributed towards analyzing these problems, including operations research, control theory, economics, psychology, human factors, organizational behavior, sociology and political science.

Game theory is the formal study of conflict and cooperation. Game theoretic concepts apply whenever the actions of several agents are interdependent. These agents may be individuals, groups, firms or any combination of these. The concept of game theory provides a language to formulate, analyze, structure, and understand strategic scenarios.

The internal consistency and mathematical foundation of game theory make it a prime tool for modeling and designing automated decision making process in interactive environment. As a mathematical tool for decision maker the strength of game theory is the methodology it provides for structuring and analyzing problems of strategic choice. The process of formally modeling a situation as a game requires the decision maker to enumerate explicitly the players and their strategic options and to consider their preference and reaction.

The aim of game theory is to investigate the manner in which rational people should interact when they have conflicting interests. Most two person games are usually zero-sum type where everything that one player wins must be lost by other which is strictly competitive. But, as often happens in most real life situations, it is not generally true that the interests of two players are always exactly opposed. Very often, both can benefit through cooperation. Cooperation behavior often emerges at a group, rather than social level; in many instances we observe the formation of independent groups, teams, clubs cooperatives each of them persecuting the same goal (in turn provision of commodities, maximization of profits, raising of public funds,standards of behavior etc). It is then natural to focus on what players ought, in some sense, to agree. Two natural restrictions on such agreements are, first, that each player gets at least as large payoffs as he can guarantee himself and the second, the coalition consisting of all players in the game receive a payoff vector which is Pareto optimal (the payoff cannot be improved further to all players' advantage). When n-persons work to cooperate to be benefited together we call the game n-persons cooperative game. As a special case of n-person cooperative game, when we have three persons cooperating to work together we call that 3-persons cooperative game.

In any level of government one of the most crucial decisions which must be made concerns the allocation of resources to different sectors of the economy and different divisions of government. The allocation of resources by a state or national government is often a multi level allocation process where one decision-maker of government distributes resources to several lower level decision makers. These decision makers in turn may either allocate the money to be distributed among even lower level players of the government hierarchy, or they may budget the resources for their own purposes.

For example, the federal government has a budget to be distributed among different offices of ministers and sectors, consider higher educations, in particular Addis Ababa University and the university has got faculties in it that a budget has to be distributed to them. So the intent of this paper is to model this allocation process in a way which takes into account both the initial allocation by the higher level of government and the usage of resources which is determined by the lower level. We do this in such a way that we take into account the fact that decisions about resource usage at the sublevels cannot be controlled once resources have been allocated. Thus one of the aims of our model is to provide rationality in decision making for the initial decisions by the state or national government. And the objective of this paper is to show the federal government how to allocate its resources in terms of the effect of the use of these resources by the sectors and also is the objective of this paper to show how this decision makers at each level make a better payoff or use of resource by working cooperatively.

Bialas and Chew [2] presented a model of cooperation among decision makers in hierarchical organization. The mathematical foundation of this work were heavily based on the theory of stacklberg games, since Stackleberg games offer us a good tool to model hierarchies, due to their independent and sequential nature. This work was based, largely on a class of mathematical programming problems called "multi level programming " problems.

In Stackleberg games, the system has interacting players with in a hierarchical structure where the leader will begin the game by announcing his decision, and the process continues for each player down through the hierarchy. Each subordinate player executes his policies after, and with full knowledge of, the decisions of the superior players. And the decision of a player can impact any other player's objective function, and a subsequent player's set of feasible choices.

In the hierarchy of decision makers each unit of the hierarchy wishes to maximize its individual benefit function in view of partial exogenous control exercise at other level.

N-person cooperative game theory is applicable for today's society since many activities take place in coalition. However, because of informational problems the allocation of agents in coalitional structures may take social dilemmas. So in order to make a good outcome by working together cooperatively and having a good information flow in the coalition, we need to use the concept of an n-person cooperative game theory and its methods and theories like the theory of Stackleberg games but the solution of an n-player Stackleberg game, when the players make decision sequentially and independently, is not necessarily pareto-optimal. There exists feasible points which offer increased

payoffs to some players without diminishing the payoffs to other players in each level. The players can improve their payoffs by forming coalitions.
By a coalition we mean a subset of players that has the right to make binding agreement among themselves. Coalition formation is an important issue. Gamson [14] suggests
"in every historical description of a revolution,in every political biographer's description of the ascent of his subject, there is a more or less explicit account of the coalition and alliances which furthered the outcomes. "
Thus, the payoff possibilities available to each coalition can be described by means of characteristic function, denoted *v(S)*, that associate with each coalition S the total utility that members of that coalition can achieve when they act in concert, whatever the remaining players may do, and an imputation is a payoff vector that gives each player at least as much as he can guarantee himself and gives all players together *v(N)* [utility of the total (grand) coalition]. A key concept in the study of n-person cooperative game is domination, which refers to the power a coalition can exert through its ability to go it alone.
The questions that will immediately be raised in forming coalition are

- Which coalitions will tend to form,
- Are the coalitions enforceable (if no new coalition can form such that everyone in this new coalition will have a greater payoff than they were ordered in their previous coalition), and
- What will be the final distribution of wealth to each of the players?

So, in order to answer these questions in this Stackleberg setting we use abstract game and accompanying solution concepts like core, the set of all undominated imputation for a game, which is the first cooperative game solution concept that predated formal game theory. It is an appealing solution; it does suffer from two faults: some games have no outcomes in the core and others have vastly many outcomes in the core. However, for games with non-empty core, We use other power indices like Shaple value and Willick's power index [33] to determine the power of individuals in each level.

1.1 Historical development

Many scientific disciplines have contributed toward analyzing problems associated with hierarchical organizations, including operation research, control theory, economics, psychology, human factors, organizational behavior, sociology and political science. Much work has been done in the field of organizational theory. In the sixties, mathematical applications by Bonini, Robert, and Wagner [5] and Cooper,Leavit, and Shelly [10] led to greater understanding of organizations and why they work the way they do.

In the seventies, the work by Marschak and Radner [21], economic theory of teams, formally modeled the multi-person organizations under asymmetric information with jointly shared goals. Groves [15] generalized their work to include members with goals by considering incentives. Considerable effort has been focused on agency theory which also considers to individuals with differing goals and asymmetric information. Williamson [31] considers the efficient combination of markets and organizations in Markets and Hierarchies.

In eighties, Burton and Obel [6] incorporated Williamson's premises into mathematical representation of organizational structure to investigate efficiency hypothesis. Drenik [12] developed mathematical models of the consistent organization based on Simons' [28] bounded rationality in mathematical Organization theory. In 1994, Burton and Obel [7] provide an overview of organizational models.

Parallel to the work on organizational theory, many strides were made in other fields. Since the advent of linear programming and game theory in the 1940's and 1950's [11, 30], a substantial effort has been directed toward analyzing the behavior of interacting decision makers , each attempting to optimize individual objectives in view of decision made by others. In 1944, Von Neumann and Morgenstern [30] developed the theory of n-person games in their classic, Theory of Games and Economic Behavior. Their work led to the formulation of a real valued characteristic function defined on the set of all subsets of a set of players. In 1951, Shapley [24] axiomatically derived a value for each player in an n-person game. This value has been the basis for much research over the last 40 years in game theory.

In 1957, Luce and Raiffa [20] provided a detailed survey of game theory. In the sixties, Lucas and Thrall [19] formulated a theory of cooperation with games in partition function form. As opposed to characteristic games, their formulation assigns a real numbered outcome to each coalition in each partition of the set of players. Maschler [22] studied the power of coalition and Lucas [18] provided an overview of the mathematical theory of games.

In the eighties, Bialas and Chew [2] presented a model of cooperation among decision makers in the hierarchical organizations. The mathematical foundation of this work were based heavily on the theory of stackleberg games [29], which offer a good tool to model hierarchies, due to their independent and sequential nature. A limitation of the early work by Bialas and Chew was its restrictions to linear objective functions for each of the player and a requirement that all feasible decisions had to reside with in a convex polytope.

Bialas[1] then extended these results to problems with continuous objective functions and compact feasible regions. Necessary and sufficient conditions for the existence of the set of non-dominated solutions, known as the core, were developed for these games by Willick [32].

In recent years, researchers have shown relation between a variation of the shapely value and the core. In 1995 Willick[32] provides the need for considering non-super additive games via hierarchical organizations, he axiomatically derived a new power index which has got a strong relationship with the core of the game (set of stable solutions)and compared it to it's predecessors, including the shapely value. In 1997 Chien-Hsin Yang [34] shows a relationship between a family of values and shapely value, the latter is always a member of the previous, using a relaxation of efficiency of the shapely value. Using Stackleberg games [29] and their resulting non-convex programming problems, which can be used to model the behavior of independent decision-makers acting within a hierarchy, Willick [33] examines the formation of coalitions within such organizations of optimizers for a large class of hierarchical problems; in his work he developed a necessary and sufficient condition for the existence of a core. In 2001 Ulrich Bindseil [4] analyzes the "one country-one vote" rule for monetary policy decision making of the governing council of the European Central Bank in a framework of cooperative game theory;the shapely value is used as a solution concept.

1.2 Overview

In each level there are a number of decision makers who make decision heterogeneously and finally decides to have a single decision maker who takes the decision of all. The main focus of this thesis is to show the importance of coalition within individual decision makers in each level. In this thesis we will discuss how to determine the value of individual decision maker in each level.

In chapter 2, the Stackleberg game is formally introduced. For a decision maker to make a rational decision, he needs to determine the rational reaction of the decision maker who are lower in the hierarchy. This rational set is mathematically derived. Inherent inefficient behavior of the hierarchy are demonstrated.

In chapter 3, coalition formation is examined as a method to avoid inefficient solutions, and the power of an individual in a game is expressed by using indices of power.

In chapter 4, we define a problem on the budget allocation of the federal government to Addis Ababa University and the university to Science faculty and mathematical formulation of the problem as a three-player Stackleberg game, which represent a class of 3-level linear programming problem is possible. And we finally show the need of coalition and determine the value of individual decision maker in the coalition.

Chapter 2

The Stackleberg Model

This section will introduce the Stackleberg model presented by Bialas and Chew [2] and by Willick [33]. That work was based, largely, on a class of mathematical programming problems often called "multilevel programming" problems.

In this model, a decision-maker at one level of the hierarchy may have his objective function and decision space determined by decisions made at other levels. Each player's control instruments may allow him to influence the decisions at other levels, and thereby improve his own objective function. In some cases, these instruments may include the allocation and use of resources at lower levels, or, perhaps, the benefits conferred upon other levels.

In other words, Stackleberg games consist of players who make decisions in structured, "leader-follower" ordering. By convention, if the game has n players, player n goes first (the player at the top of the hierarchy). He makes his decisions based on his objective function and the rational reactions of the players who make their decisions after him. After player n has made his decisions, player n − 1 makes his decisions based on his objective function, the decisions made by player n and the rational reaction of the players who make their decisions after him. This process continues down the hierarchy until finally player 1 makes his decisions based on his objective function and the decisions made by the n $-$ 1 players above him.

Definition 2.0.1 *:- Let the vector $x \in I\!\!R^M$ be partitioned as $x = (x^a, x^b)$. Let $S \subset R^M$ be compact, and let $f : I\!\!R^M \longrightarrow I\!\!R$ be continuous on S. Then*

$$\psi_f(S) = \{\hat{x} \in S | f(\hat{x}) = \max\{f(x) : (x^a|\hat{x}^b)\}\} \tag{2.0.1}$$

is the set of rational reaction of f over S.

In the above definition the maximization is taken over x^a which uniquely maximizes $f(x^a, \hat{x}^b)$ over all $(x^a, \hat{x}^b) \in S$, then there is induced mapping $\hat{x}^a = \varphi(x^b)$ which provides the rational reaction for each $\hat{x}^b$, and we can then express (1) as

$$\psi_f(S) = S \bigcap \{(x^a, x^b)/x^a = \varphi(x^b)\} \tag{2.0.2}$$

.

Example 2.0.1 *Let $x = (x^1; x^2)$ where x^1 and x^2 are single component vectors. Suppose $S = \{x \in I\!\!R^2/A_1x_1 + A_2x_2 \leq b\}$ is the polyhedron shown in figure 2.1. Then for any fixed, feasible choice of x^2_*, level one solves the linear programming problem*

$$\begin{aligned} &\max Cx = C^1x^1 + C^2x^2 \\ &s.t : A_1x^1 \leq b - A_2x^2_* \end{aligned} \tag{2.0.3}$$

The solution x^1_*, together with x^2_*, results a point (x^1_*, x^2_*) which is an element of $\psi_{Cx}(S)$, the shaded region in figure 2.1.

If level two wishes to maximizes its objective function, $f_2(x^1, x^2)$, by controlling only the vector x^2, it must solve the mathematical programming problem

$$\max f_2(x^1, x^2)$$
$$s.t : (x^1, x^2) \in \psi_{f_1}(S) \tag{2.0.4}$$

Or equivalently

$$\max f_2(\varphi(x^2), x^2)$$
$$s.t : (\varphi(x^2), x^2) \in S$$

We call 2.0.4 as a two-level programming problem

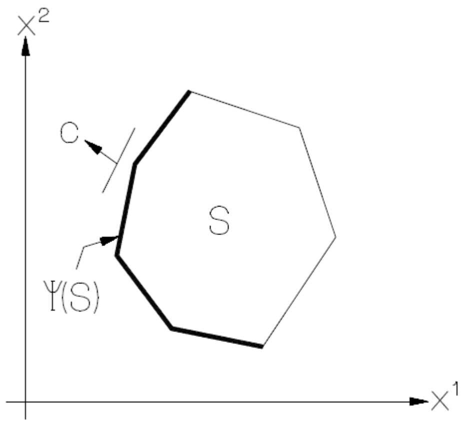

Figure 2.1: Example of a Rational Reaction Set for a two-level problem

2.1 The n-player Stackelberg Game

To formally define the n-player Stackleberg game, let the vector of decision variables for all players, denoted by $x \in \mathbb{R}^M$, be partitioned among n players with

$$x^k = (x_1^k, x_2^k, \ldots, x_{N_k}^k) \qquad (k = 1, 2, \ldots, n)$$

Where $\sum_{k=1}^{n} N_k = M$. We will require that the n players choose values of $x \in S^1 \subset \mathbb{R}^M$, where S^1 is compact. The shape of S^1 will determine the ability of one player to affect the set of feasible choices of the other players. Let $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ be a set of continuous functions with $f_i(x) : S^1 \longrightarrow \mathbb{R}$ for all $i = 1, \ldots, n$

Definition 2.1.1 *Let the vector $x \in \mathbb{R}^M$ be partitioned as $x = (x^a, x^b)$ with $x^a = (x^1, \ldots, x^{k-1})$ and $x^b = (x^k, \ldots, x^n)$. The level-k feasible region, S^k, is recursively defined as*

$$S^k = \psi_{f_{k-1}}(S^{k-1}) \text{ for } k=2,3,\ldots,n.$$

The set S^k represents the feasible outcomes resulting from the rational reaction of players at levels $1, 2, \ldots, k-1$. Hence S^k contains all of the information necessary for player k to asses the behaviors of these players.

Given the preemptive decisions $(\hat{x}^{k+1}, \ldots, \hat{x}^n)$ of the first n-k leading players, the optimization problem which must be solved by the player at level k is then

$$(L^k :) \max \{f_k(x) : x^k | (x^{k+1}, \ldots, x^n)\}$$

$$s.t. \quad \mathrm{x} \in S^k$$

This establishes a collection of nested mathematical programming problems $(L^1, \ldots, L^n)$ jointly representing the decision problems of n-players in a hierarchical organization. This collection of nested mathematical programming problems define the bounded Stackleberg game (Bialas and Karwan [3])

When the number of players reduced from n to 3 then we will have a 3-player Stackleberg game. Since the intension of this paper is to deal with 3-person cooperative games, let's define the 3-player Stackleberg game.

2.2 The 3-player Stackelberg Game

Let the vector of decision variables for all players, denoted by $x \in \mathbb{R}^M$, be partitioned among three players with

$$x^k = (x_1^k, x_2^k, \ldots, x_{N_k}^k) \qquad (k = 1, 2, 3)$$

where $\sum_{k=1}^{3} N_k = M$. We will require that the 3 players choose values of $x \in S^1 \subset R^M$, where S^1 is compact. Let$\{f_1(x), f_2(x), f_3(x)\}$ be a set of continuous functions with

$$f_i(x) : S^1 \longrightarrow R \text{ for all } i = 1, 2, 3$$

Now the 3-player Stackleberg game can be expressed as a 3-level programming problem as follows

$$\begin{aligned}
&\max\{f_1(x) : (x^1|x^2, x^3)\} \\
&s.t : x \in S^1 \\
&\max\{f_2(x) : (x^2|x^3)\} \\
&s.t : x \in \psi_{f_1}(S^1) = S^2 \\
&\max\{f_3(x) : x^3\} \\
&s.t : x \in \psi_{f_2}(S^2)
\end{aligned} \tag{2.2.1}$$

In general, the solution of stackelberg game, when the players make decision sequentially and independently, is not necessarily pareto-optimal. That is there exist feasible points which offer increased payoffs to some players without diminishing the payoff to other players. During this hierarchy, the decision makers in each level also could improve their payoff. The players can improve their payoffs by forming coalition. In order to show this ,let's consider the following two level linear problem.(see chew[9])

$$\max \{C^2x : (x^2)\}$$
$$\text{s.t: } \max \{C^1x : (x^1|x^2)\}$$
$$\text{s.t: } Ax \leq b$$
$$x \geq 0$$

The objective functions on each level has got a control of only one variable vector x^1 for level one and x^2 for level two, where level one's choice is restricted by decision made at level two. Level two, top level, makes his decision first, taking in consideration that level one will maximize its objective function with respect to level two's decision. For example (on figure 2.2), if level two were to choose the x^2 coordinate of point a, level one would choose the x^1 coordinate of point b. Therefore, it is wise for level two to consider level one's set of "rational reaction". This set is denoted by S^2. Having, then identified S^2, level two will choose the point in S^2 which maximizes his objective function. Level one being a rational decision maker, is now forced to choose the same point since it maximizes his payoff. The Stackleberg game of the above two level linear problem is the game which satisfies all important things to be considered by each players and we can re-write it as a Stackleberg game as follows.

$$\max\{C^1x : (x^1|x^2)\}$$
$$\text{s.t}$$
$$x \in S^1 = \{x \in \mathbb{R}^2 | Ax \le b, x \ge 0\}$$
$$max\{C^2x\}$$
$$\text{s.t: } x \in \psi_{f_1}(S^1)$$

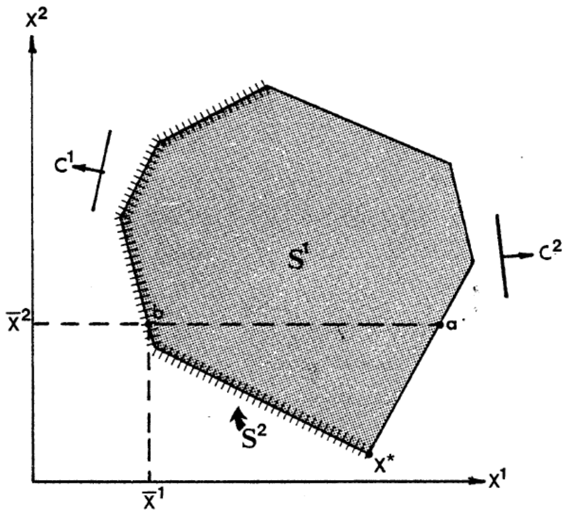

Figure 2.2: A two-level programming problem

Thus, for example, x^* is the solution, but x^* is not pareto optimal since we can find a point with a better payoff for each level. For example $\bar{x}$ (see Figure 2.3). But $\bar{x}$ to be the solution it is needed to have some agreement between players at each level. To clarify this, let $\bar{x}$ have coordinates ($\bar{x}^1$,$\bar{x}^2$). Suppose player two chooses the payoff at point $\bar{x}$ and choose $\bar{x}^2$. player one being rational chooses x^1 coordinate of a instead of $\bar{x}^1$, to make $\bar{x}$ the solution, which maximizes his payoff. This makes player two to have less payoff than the payoff at x^*. So here is why we need the agreement. Otherwise, points with increased payoffs which is the section M in Figure 2.4, than x^* would be untenable if each players act independently. Therefore, the players(decision-makers) must realize that if these increased payoffs are to be attained, it is necessary for them to form coalition. In coalition formation, the most important question to be answered is, what will be the final distribution of payoff to each players, so we will use indices of power as a solution concept.

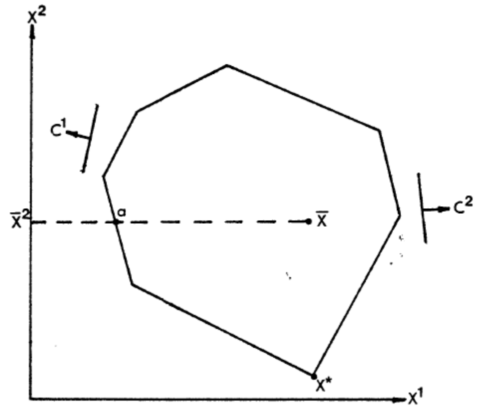

Figure 2.3: Instability of improved solution

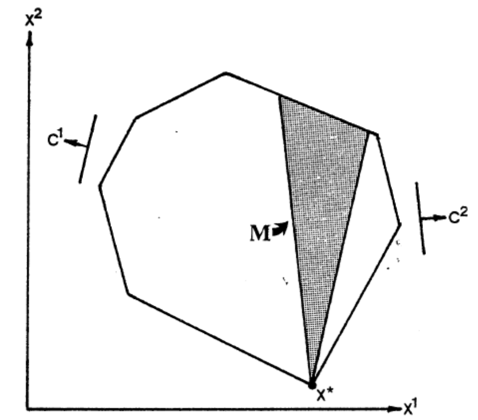

Figure 2.4: Strong agreement region

Chapter 3

Coalition

As suggested in the previous chapter, the formation of coalition among subsets of players could provide a means to achieve pareto optimality. In this chapter, we will try to answer the questions what a coalition is, which coalitions to form and how to distribute the payoff(outcome) in the coalition among the players at each level. The abstract game and the solution concept proposed by Shenoy[27] provides the foundation for answering these questions. In particular we are interested in games that are characteristic function form.

3.1 Coalitional Form

Characteristic Functions.

Let $n \geq 2$ denote the number of players in the game, numbered from 1 to n, and let N denote the set of players, $N = \{1, 2, \ldots, n\}$. A coalition, S, is defined to be a subset of N, $S \subset N$, which is a subset of players that has the right to make binding agreement by themselves, and the set of all coalitions is denoted by 2^N. By convention, we also speak of the empty set, ϕ, as a coalition, the **empty coalition**. The set N is also a coalition, called the **grand coalition**. If there are just two players, n = 2, then there are four coalition, $\{\phi, \{1\}, \{2\}, N\}$.

Let $G = \{G_1, G_2, \ldots, G_n\}$ denote a coalition structure or a partition of N into nonempty coalition where $G_1 \bigcap G_2 = \phi \quad for \quad all \quad i \neq j$ and $\bigcup_{i=1}^{n} G_i = N$. Since there is a coalition formation, each player in the coalition works for the group. Hence, the objective function of each player in coalition G_i becomes,

$$f'_{G_i}(x) = \sum_{i \in G_j} f_i(x)$$

Although the sequence of the players' decisions have not been changed, their objective functions have. Let G(i) denote the unique coalition $G_j \in G$ such that player $i \in G_j$. Now player i will be maximizing $f'_{G_j}(x)$. Let the solution to the n-level optimization problem which results from these objective functions be denoted by $\bar{x}(G)$.

Definition 3.1.1 *suppose that S^1 is compact and $\bar{x}(G)$ is unique. The value of (or payoffs to) coalition $G_j \in G$, denoted by $v(G_j, G)$, is given by*

$$v(G_j, G) \equiv \sum_{i \in G_j} f_i(\bar{x}(G)) = f'_{G_i}(\bar{x}(G))$$

Definition 3.1.2 *By an n-person game in characteristic function form, we mean is the pair (N, v), where $N = \{1, 2, \ldots, n\}$ is the set of players and v is a real-valued function, called the characteristic function of the game, defined on the set, 2^N, of all coalitions (subsets of N), and satisfying*

1. *$v(\phi) = 0$ i.e. empty set has no value, and*

2. *(superadditivity) if S and T are disjoint coalitions ($S \bigcap T = \phi$), then $v(S) + v(T) \leq v(S \bigcup T)$ i.e. value of two disjoint coalitions is at least as great as when they work together than when they work apart.*

Here we can consider v as the value, or worth, or power, of coalition S when its members act together as a unit.

Definition 3.1.3 *An abstract game is a pair $(\Theta; dom)$ where Θ is a set whose members are called outcomes and **dom** is a binary relation on Θ called **domination**.*

An example of a 3-person game in characteristic function form is:

Example 3.1.1

$$
\begin{aligned}
v(\{1\}) &= 4 \\
v(\{2\}) &= 4 \\
v(\{3\}) &= 6 \\
v(\{1,2\}) &= 16 \\
v(\{1,3\}) &= 18 \\
v(\{2,3\}) &= 14 \\
v(\{1,2,3\}) &= 24
\end{aligned}
$$

A value is assigned to each subset, referred to as a coalition, of N ={1,2,3}. As we can see from the definition, this game is superadditive. In this game, player 1 alone obtains a value of 4. If he forms a coalition with player 2, they together obtain a value of 16. If all three players form a grand coalition, the group obtains a value of 24. One question that arises is which coalitions the players will form. In order to extract as much value as possible from the game, the players in this game should form a grand coalition. The question of much debate is how the value of 24, obtained from the grand coalition, should be distributed among the three players.

There are an infinite number of ways that the value of 24 can be distributed among the three players. It can be argued, though, that many of these distributions are unfair to at least one of the players. For instance, the distribution which gives the entire allotment to player 2 does not seem reasonable to the other two players. Player 1 would argue that he deserves a distribution of at least 4 because that is the amount that he can obtain by himself, without any help from the other players.

Furthermore, the distribution which gives the three players (14,4,6), respectively, also does not seem equitable. Player 2 and player 3 should threaten to break up the grand coalition and form a coalition between them selves if they do not jointly receive a value of 14, the value of their coalition. This is because they have the power to obtain a value of 14 with no help from player1.

The preceding argument is the basis for the game theoretical concept of the core of a game in characteristic function form. In the next section we formally define the core of a game in this form. We will also demonstrate its weakness for providing a technique to distribute the allotment among the players.

3.2 Indices of Power

In this section we will review some of the previous works done in the area of the power of an individual in a game, which are useful for the distribution of payoffs among players.

3.2.1 Imputation and the Core

Definition 3.2.1 *An **imputation** x is a payoff vector that is group rational and individually rational.*
The set of imputations may be written as

$$\{x = (x_1, x_2, \ldots, x_n) : \textstyle\sum_{i \in N} x_i = v(N), \textit{and} \quad x_i \geq v(\{i\}) \textit{ for all } i \in N\}$$

In the previous example ,
$v(\{1\}) = 4, v(\{2\}) = 4,$
$v(\{3\}) = 6, v(\{1, 2\}) = 16,$
$v(\{1, 3\}) = 18, v(\{2, 3\}) = 14 \quad \textit{and} \quad v(N) = 24.$
The set of imputations is
$\{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 24, x_1 \geq 4, x_2 \geq 4, x_3 \geq 6\}.$

Definition 3.2.2 *A solution configuration (X,G), is a feasible solution configuration if and only if*
$\sum_{i \in R} x_i \leq v(R, G) \quad for \quad all \quad R \in G$

Definition 3.2.3 *let $(X, G_x), (Y, G_y)$ be a solution configuration. Then (X, G_x) dominates (Y, G_y), denoted by, (X, G_x) dom (Y, G_y) if and only if there exists a nonempty $R \in G_x$ such that*

1. *$x_i \geq y_i$ for all $i \in R$, i.e. each decision maker in R prefers coalition structure G_x to G_y.*

2. *$\sum_{i \in R} x_i \leq v(R, G_x)$ i.e. R is a feasible coalition in G_x, which means R must not demand more from the imputation x than its value $v(R, G_x)$.*

Definition 3.2.4 *The set, C, of stable imputations is called the* ***core****,*
$C = \{x = (x_1, x_2, \ldots, x_n) : \sum_{i \in N} x_i = v(N),\ \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subset N\}$

A solution to the core represents a fair distribution of the wealth among the players of the game. And any solution in the core is stable.

Definition 3.2.5 *A solution of a game is said to be* ***stable*** *if no subset of players are able to improve upon their joint distribution allotted to them by the solution.*

Thus, when looking for the value of players in a game, it is reasonable to search for solutions in the core. Referring back example 3.1.1, the core of the game consists of the distribution $(x_1; x_2; x_3) = (10;\ 6\ ;\ 8)$. This is the only distribution that seems to be equitable to all players considered.
But here, first, the core in some games is empty. Does this suggest that the players have no value in a game with an empty core? Probably not. Second, even when the core of a game is non-empty, often there are multiple solutions in the core. In such a situation, which solution do we use to describe the relative value of the players in the game? These are just some of the questions that arise when trying to implement the core concept for determining the values of the players in a game.
When the core is nonempty, each of its elements represents an enforceable solution configuration within the hierarchy. i.e. there always exists a solution configuration involving the grand coalition among the solution configurations in the core.

3.2.2 Shapley Value

Value Function

A value function, ϕ, is a function that assigns to each possible characteristic function of an n-person game, v, an n-tuple, $\phi(v) = (\phi_1(v), \phi_2(v), \ldots, \phi_n(v))$ of real numbers. Here $\phi_i(v)$ represents the worth or value of player i in the game with characteristic function v.
The axioms of fairness are placed on the function ϕ,
The Shapley Axioms for $\phi(v)$

1. **Efficency**: $\sum_{i \in N} \phi_i(v) = v(N)$

2. **Symmetry**: If i and j are players such that $v(S \bigcup \{i\}) = v(S \bigcup \{j\})$ for every coalition S not containing i and j, then $\phi_i(v) = \phi_j(v)$

3. **Dummy Axiom**: If i is player such that $v(S) = v(S \bigcup \{i\})$ for every coalition S not containing i, then $\phi_i(v) = 0$.

4. **Additivity**: If u and v are characteristic functions, then

$$\phi(u + v) = \phi(u) + \phi(v).$$

Axiom 1 is group rationality, that the total value of the players is the value of the grand coalition.
The second axiom says that if the characteristic function is symmetric in players i and j, then the values assigned to i and j should be equal.
The third axiom says that if player i is a dummy in the sense that he neither helps nor harms any coalition he may join, then his value should be zero.
The strongest axiom is number 4. It reflects the feeling that the arbitrated value of two games played at the same time should be the sum of the arbitrated values of the games if they are played at different times. It should be

noted that if u and v are characteristic functions, then so is $u + v$.

The Shapley value is given by $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ where for $i = 1, \ldots, n$

$$\phi_i(v) = \sum_{\substack{S \subset N \\ i \in S}} \frac{(|S| - 1)!(n - |S|)!}{n!}[v(S) - v(S - \{i\})]$$

The summation in this formula is the summation over all coalitions S that contain i. The quantity, $v(S) - v(S - \{i\})$, is the amount by which the value of coalition $S - \{i\}$ increases when player i joins it. Thus to find $\phi(v)$, merely list all coalitions containing i, compute the value of player is contribution to that coalition, multiply this by $(|S| - 1)!(n - |S|)!/n!$, and take the sum.

The interpretation of this formula is as follows. Suppose we choose a random order of the players with all $n!$ orders (permutations) of the players equally likely. Then we enter the players according to this order. If, when player i enters, he forms coalition S (that is, if he finds $S - i$ there already), he receives the amount $[v(S) - v(S - \{i\})]$.

The probability that when i enters he will find coalition $S - i$ there already is $(|S| - 1)!(n - |S|)!/n!$. The denominator is the total number of permutations of the n players. The numerator is number of these permutations in which the $|S| - 1$ members of $S - i$ come first ($((|S| - 1)!$ ways), then player i, and then the remaining $n - |S|$ players ($(n - |S|)!$ ways). So this formula shows that $\phi(v)$ is just the average amount player i contributes to the grand coalition if the players sequentially form this coalition in a random order.

Returning to the previous example, the shapely value of the players of the game are;

$$(\phi_1, \phi_2, \phi_3) = (\tfrac{26}{3}, 10, \tfrac{26}{3})$$

Notice that even though the core of this game is non-empty, the values assigned to the players by Shapley comprise an unstable solution. It should be pointed out that the Shapley value was never necessarily intended to be an element of the core.

One result that the Shapley value satisfies, when game v is superadditive, is the concept of *individual rationality.* Which is the concept that each player should receive from a game at least that amount that he can obtain by himself i.e. at least $v(\{i\})$. Any solution not satisfying individual rationality has no chance of being accepted by the players and should be regarded as a poor indicator of the power of the individuals. Since it enforces such a play to be out of the coalition to improve his solution.

When a game is not superadditive, individual rationality is not necessarily satisfied using the Shapley value.

This can be demonstrated by the following example.

$$\begin{aligned} v(\{1\}) &= 8 \\ v(\{2\}) &= 1 \\ v(\{3\}) &= 6 \\ v(\{1,2\}) &= 7 \\ v(\{1,3\}) &= 12 \\ v(\{2,3\}) &= 8 \\ v(\{1,2,3\}) &= 16 \end{aligned}$$

This game is not super additive because $v(\{1\}) + v(\{2\}) \geq v(\{1,2\})$. The Shapley value of this game is$(\frac{22}{3}, \frac{11}{6}, \frac{41}{6})$. Notice that player 1 should be able to demand at least 8, but is assigned a value of only $\frac{22}{3}$.

Some researchers argue that only superadditive games need to be considered; that in reality, non-super additive games do not occur. Willick[33] shows that this is not the case and that we need to be careful when applying traditional techniques to structures that promote non-superadditive games.

Relation of Shapley Value with the Core

We have seen that when dealing with games which are not superadditive, the Shapley value do not satisfy individual rationality. We argue that individual rationality is a minimal criterion that a power index should satisfy. This is because individual rationality represents a quantity that a player can guarantee himself without any cooperation with the other players.
Even if we limit our scope to games which are superadditive, it would be preferable that the solutions obtained by these power indices have some relationship with the core. In particular, when the core of a game is non-empty, it would be ideal if a power index resulted in a distribution that was an element of the core. This would be preferred due to the stability of the solutions in the core.
There is no known general relationship between Shapley value with the core, even when dealing with superadditive games. An example of this can be seen by referring to Example 3.1.1. This game was superadditive with a non-empty core, yet the Shapley value solution was not an element of the core.
It should be pointed out that there are classes of games for which the Shapley value is an element of the core. Iñarra and Usategui [16] introduce two such games: the average convex games and the partially average convex games. Since the average convex games are a subset of the partially convex games, we will present only the latter of the two classes of games. We first need to introduce one definition.

Definition 3.2.6 *Define the function* g *as follows:*
$g(A, B) = v(A \bigcup B) - v(A) - v(B)$ *for all* $A, B \subseteq N, A \bigcap B = \phi$

Definition 3.2.7 *A game v is **partially average convex***

$$\begin{pmatrix} |B| \\ |R| \end{pmatrix}^{-1} \sum_{R \subseteq B} g(A,R) \leq \begin{pmatrix} |B| \\ |R| \end{pmatrix}^{-1} \sum_{C \subseteq B} g(A, B/C)$$

$$\text{for all } A, B \subseteq N, B \subseteq N/A \text{ and for any } R \text{ and } C \text{ such that}$$
$$|R| > \tfrac{|A||B|}{n-|B|} > |B| - |C| \quad if \quad A \bigcup B \subset N, and$$
$$|R| = |B| - |C| \qquad if A \bigcup B = N.$$

For a game to be an element of this class of games, it needs to satisfy stringent conditions. Therefore, most games do not have the property of the Shapley value being an element of the core.

3.2.3 Willick's Power Index

This power index always has a unique solution that is also related to the concept of the core. Furthermore, it satisfies individual rationality even for games that are not superadditive.

Games will not be required to be superadditive; however, we will limit our focus to characteristic function games that yield the most wealth when the grand coalition is formed. That is, we will assume that games are *superadditive with respect to the grand coalition.*

Definition 3.2.8 *Let v be a game and α be a scalar. Then the game $v \boxplus \alpha$ is defined as the following:*

$$[v \boxplus \alpha] = \left\{ \begin{array}{ll} v(N) + \alpha & \textit{If P=N} \\ v(P) & \textit{otherwise} \end{array} \right\}$$

Definition 3.2.9 *Let N_1 and N_2 represent disjoint player sets with game v being composed of players from N_1 and game u being composed of players* *from N_2. Then, by $v \oplus u$, we mean the game w such that for any $P \subseteq N_1 \bigcup N_2$, $w(P) = v(P_1) + u(P_2)$, where $P_1 = P \bigcap N_1$ and $P_2 = P \bigcap N_2$*

Definition 3.2.10 *The game v is said to be the* ***superadditive cover*** *of the game u if for all $P \subseteq N$*

$$v(P) = \max_{p_P} \sum_{R \in p_P} u(R)$$

Where p_P is a coalition structure of the set of players in P.

With this definition, and five axioms for the basis of Willick's power index. Let $\rho(v)$ denote an n-dimensional vector satisfying the following axioms

Axiom W_1 $\quad \rho_i[v \boxplus \alpha] = \rho_i[v] + \alpha$

Axiom W_2 $\quad \rho_i[\gamma v] = \gamma \rho_i[v]$

Axiom W_3 Let u and v be games with disjoint player sets N_1 and N_2, respectively. Then,

$$\rho_i[u \oplus v] = \left\{ \begin{array}{ll} \rho_i[u] & \text{if i} \in N_1 \\ \rho_i[v] & \text{if i} \in N_2 \end{array} \right\}$$

Axiom W_4 Let u and v be superadditive games. Then,

$$\rho_i[u + v] = \rho_i[u] + \rho_i[v]$$

Axiom W_5 Let v be any game (possibly non-superadditive) and let u be the superadditive cover of v. Then,

$$\rho_i[u] = \rho_i[v]$$

Axiom 1 states that when game u is generated by adding a constant to the value of the grand coalition in game v, then the power index of a player in u is equal to the power index of that player in v plus the constant. In other words, adding a constant to the value of the grand coalition increases every players' power index by the constant

Axiom 2 states that the power index of a player in a multiple of a game is the product of the multiple and the power index of the player in the original game.

Axiom 3 states that the power index of a player in a sum of two games that have disjoint player sets is the power index of that player in the game that has that player in its player set.

Axiom 4 is the same as Shapley's Axiom 3.

Axiom 5 states that the power index of a player in a game is the same as the power index of that player in the superadditive cover of the game.

Definition 3.2.11 *The maximum value that the sum of all players except for player i can achieve is denoted by $u^*(N - \{i\})$ and is defined as*

$$u^*(N - \{i\}) = \max_{p_{N-\{i\}}} \sum_{P \in p_{N-\{i\}}} u(P)$$

Further, define $p^*_{N-\{i\}}$ as that coalition structure for which $u^*(N - \{i\})$ is obtained.

Notice that if the game u is superadditive, $p^*_{N-\{i\}} = \{N - \{i\}\}$.

Willick established that there is a unique function satisfying individual rationality defined on all games satisfying the axioms, which is given below:-

$$\rho_i[u] = u(N) - \max_{p_{N-\{i\}}} \sum_{p \in p_{N-\{i\}}} u(p)$$

Relation of Willick's Power Index and the Core

Lemma 3.2.12 *If* $(x_1, x_2, \ldots, x_n) \in C \neq \phi$,*then* $v^*(N-i) \leq \sum_{j\neq i} x_j$ *for all* $i \in N$

Let v be an n-person game with $C \neq \phi$ and let $(x_1, x_2, \ldots, x_n) \in C$. Then, from the definition of the core, it follows that

$$\sum_{j\in P} x_j \geq v(P)$$

For all $P \in N$. Therefore, it must be true that for each $P \in P^*_{N-\{i\}}$

$$\sum_{j\in P} x_j \geq v(P)$$

By summing both sides of the above inequality over the elements of $P^*_{N-\{i\}}$, we get

$$\sum_{P\in p^*_{N-\{i\}}} \sum_{j\in P} x_j \geq \sum_{P\in p^*_{N-\{i\}}} v(P) = v^*(N - \{i\})$$

Thus we obtain the desired result,

$$\sum_{j\neq i} x_j \geq v^*(N - \{i\})$$

Theorem 3.2.13 *Let* v *be an n-person game.*
If $\sum_{i\in N} \rho_i[v] < v(N)$ *then* $C = \phi$

Assume v is an n-person game with $(x_1, x_2, \ldots, x_n) \in C$. From lemma 3.2.11, We have $v^*(N-i) \leq \sum_{j\neq i} x_j$ for for all $i \in N$. It therefore follows that

$$\sum_{i\in N} v^*(N - \{i\}) \leq \sum_{i\in N} \sum_{j\neq i} x_j$$

Now we can derive the desired result.

$$\sum_{i\in N}\rho_i[v] = nv(N) - v^*(N-\{i\})$$
$$\sum_{i\in N}\rho_i[v] \geq nv(N) - \sum_{i\in N}\sum_{j\neq i} x_j$$
$$\sum_{i\in N}\rho_i[v] \geq nv(N) - (n-1)\sum_{j\neq i} x_j$$
$$\sum_{i\in N}\rho_i[v] \geq nv(N) - (n-1)v(N)$$

It therefore follows that

$$\sum_{i\in N}\rho_i[v] \geq v(N)$$

Next, we show that when the sum of the $\rho_i[v]$ is equal to the value of the grand coalition, the core is either empty or is the unique solution$\{(\rho_1[v], \rho_2[v], \ldots, \rho_n[v])\}$
For 3-person games the converse of theorem 3.15 is also true i.e. let v be a 3-person game. If $\sum_{i\in N}\rho_i[v] \geq v(N)$, then $C \neq \phi$ (see Willick [33]).

Example:-

$$v(\{1\}) = 0$$
$$v(\{2\}) = 0$$
$$v(\{3\}) = 0$$
$$v(\{1,2\}) = 5$$
$$v(\{1,3\}) = 5$$
$$v(\{2,3\}) = 5$$
$$v(\{1,2,3\}) = 8$$

Here, $(\rho_1[v], \rho_2[v], \rho_3[v]) = (3, 3, 3)$ but $C \neq \phi$

Theorem 3.2.14 *Let v be an n-person game with* $\sum\limits_{i \in N} \rho_i[v] = v(N)$ *then* $C \in \{\phi, (\rho_1[v], \rho_2[v], \ldots, \rho_n[v])\}$

assume v is a game. Furthermore, it is reasonable to assume that $C \in \phi$ otherwise, the above result holds trivially.

Let $x = (x_1, x_2, \ldots, x_n) \in C$. From our assumption, we have

$$v(N) = \sum_{i \in N} \rho_i[v]$$
$$= \sum_{i \in N} v^*(N - \{i\})$$

Rewriting the last equality, we obtain the following equation

$$\textstyle\sum_{i \in N} v^*(N - \{i\}) = (n-1)v(N)$$
$$= (n-1) \textstyle\sum_{i \in N} x_j$$
$$= \sum_{i \neq j} \sum_{i \in N} x_j$$

Since $x = (x_1, x_2, \ldots, x_n) \in C$, from the above lemma, the following must hold for each player

$$v^*(N - \{i\}) \leq \sum_{j \neq 1} x_j.$$

By adding slack variables, S_i, to the inequalities, we get

$$v^*(N - \{i\}) + S_i = \sum_{j \neq 1} x_j.$$

Where $S_i \geq 0$. Summing over all players, we obtain

$$\sum_{i\in N} v^*(N-\{i\}) + \sum_{i\in N} S_i = \sum_{i\in N}\sum_{j\neq i} x_j.$$

But , since $v^*(N-\{i\}) = \sum_{j\neq 1} x_j$, it follows that $\sum_{i\in N} S_i = 0$. Furthermore, since $S_i \geq 0$ for each i, it follows that $S_i = 0$. Thus, the sum of the slack variables equal zero, implying

$$v^*(N-\{i\}) = \sum_{j\neq i} x_j, \text{ for all i.}$$

Hence,

$$\sum_{j\neq i} x_j - v^*(N-\{i\}) = 0.$$

Finally, we compute,

$$\begin{aligned}\rho_i[v] &= v(N) - v^*(N-\{i\}) \\ &= \sum_{i\in N} x_j - v^*(N-\{i\}). \\ &= x_i + \sum_{i\neq j} x_j - v^*(N-\{i\}) \\ &= x_i\end{aligned}$$

This is the desired result.

Hence, if $\sum_{i\in N} \rho_i[v] = v(N)$,

then either $C \in \{(\rho_1[v], \rho_2[v], \ldots, \rho_n[v])\}$ or $C = \phi$.

Example:-

$$\begin{aligned} v(\{1\}) &= 4 \\ v(\{2\}) &= 4 \\ v(\{3\}) &= 6 \\ v(\{1,2\}) &= 16 \\ v(\{1,3\}) &= 18 \\ v(\{2,3\}) &= 14 \\ v(\{1,2,3\}) &= 24 \end{aligned}$$

$\rho_1[v] = 24 - 14 = 10$

$\rho_2[v] = 24 - 18 = 6$

$\rho_3[v] = 24 - 16 = 8$

In this example. $\sum_{i \in N} \rho_i[v] = v(N)$ and the core C=[10,6,8]

The last case to consider is when $\sum_{i \in N} \rho_i[v] > v(N)$. Here the core is a subset of the $\rho_i[v]$.

Theorem 3.2.15 *If* $\sum_{i \in N} \rho_i[v] > v(N)$ *then*
$C \subseteq \{(\rho_1[v] - \alpha_1, \rho_2[v] - \alpha_2, \ldots, \rho_n[v] - \alpha_n)/\sum_{i \in N} \alpha_i = \sum_{i \in N} \rho_i[v] - v(N), \alpha_i \geq 0, \forall i\}$

Assume v is a game with $\sum_{i \in N} \rho_i[v] > v(N)$. Again, it is reasonable to assume that $C \neq \phi$. Otherwise, the above result holds trivially.

Let$(x_1, x_2, \ldots, x_n) \in C$. Since $(x_1, x_2, \ldots, x_n) \in C$, from the above lemma, the following holds for each player i,

$$\sum_{j \neq i} x_j \geq v^*(N - \{i\}).$$

This implies that,

$$\sum_{j \neq i} x_j - v^*(N - \{i\}) \geq 0.$$

Now consider

$$\begin{aligned} \rho_i[v] &= v(N) - v^*(N - \{i\}). \\ &= \sum_{i \in N} x_i - v^*(N - \{i\}) \\ &= x_i \sum_{i \neq j} x_j - v^*(N - \{i\}) \\ &\geq x_i. \end{aligned}$$

Therefore $\rho_i[v] \geq x_i$ for all players i. This combined with the fact that $\sum_{i \in N} x_i = v(N)$, insures that there exists $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ in which $\sum_{i \in N} \alpha_i = \sum_{i \in N} \rho_i[v] - v(N)$ such that $(x_1, x_2, \ldots, x_n) = (\rho_1[v] - \alpha_1, \rho_2[v] - \alpha_2, \ldots, \rho_n[v] - \alpha_n)$.

Example:-

$$\begin{aligned} v(\{1\}) &= 8 \\ v(\{2\}) &= 1 \\ v(\{3\}) &= 6 \\ v(\{1,2\}) &= 7 \\ v(\{1,3\}) &= 12 \end{aligned}$$

$$v(\{2,3\}) = 8$$
$$v(\{1,2,3\}) = 16$$

For this game, the core consists of a set of elements which are the points $(8, 2-\alpha, 7+\alpha)$ for $0 \leq \alpha \leq 1$,
The Shapley values of this game are $\rho[v] = (\frac{22}{3}, \frac{11}{6}, \frac{41}{6})$.
Our power index result is:-
$\rho_1[v] = 16 - 8 = 8$
$\rho_2[v] = 16 - 14 = 2 = V\{1,2,3\} - \max\{v\{1,3\} = 12, v\{1\} + v3 = 14\}$
$\rho_3[v] = 16 - 9 = 7$

	Shapley	Willick
Individual rationality superadditive games	√	√
Individual rationality non-superadditive games	-	√
General relation with core	-	√
Uniqueness	√	√

Table 3.1: Comparison between power indices

So as we have tried to show the relation of willick's power index with the core is in a better situation than shapley value, on table 3.1, it is recommended on this thesis to use willick's power index to know the power of individual in the coalition, which helps for distributing the wealth among players during coalition.

Chapter 4

Application of 3-person Cooperative Game

4.1 Problem Definition

One's country Budget allocation system is in the hierarchical fashion where the federal government is the one at the top of the hierarchy and the minister offices will be on the next level and the offices in the minister offices will be on the third level and so on. In this study we will only consider the three level hierarchical budget allocation and in particular, the federal government will be considered as the top in hierarchy, and Addis Ababa University will be consider at the next level and finally the faculty of science will be consider at the third and last level.

The budget allocation is done in the scheme to be expressed as follows. The central government will decide the amount of money to be allocated to the university depending on the budget on hand; by giving a ceiling (floor), which is the maximum amount of money that the university might request for the budget year. Depending on the ceiling given the university will be called for preparing its action plan. The floor will be decided by the experts in the central government using the universities previous year approved budget and total amount of money on the hand of the government.

The university will inquire the faculty to do its action plan depending on the ceiling that will be given from the university which in turn depends on the ceiling given by the government. Decision of the ceiling on the university will be performed by the experts in it, according to the ceiling given from the government and the approved budget of the previous year for the faculty.

The experts in the faculty will perform annual plan of the budget year and submit it to the university. The experts in the university will make some improvement on the action plan and submit it to the central government and there also be an improvement to be done by the experts in it.

The experts in the university makes improvement on the action plan of the faculties by considering the following points.

a) the budget allocated to the university from the federal government

b) the previous year budget approved to the faculties

c) the budget request of the faculties for the coming budget year

d) the efficiency of the faculties

e) current market price (standard rate) of the materials

f) taking into consideration of the governmental policy.

g) priority of the university.

Similarly, the experts in the federal government makes improvement on the action plan of the university by considering the following main factors:

i. the financial capacity of the country

ii. the previous year budget approved to the universities

iii. the budget request of universities for the coming budget year

iv. the efficiency of the universities

v. current market price (standard rate) of the materials

vi. taking into consideration of the governmental policy.

Our objective is to show how the central government body can allocate its resources in terms of the effectiveness of the use of these resources by the universities, and how the universities utilize can the allocated resources effectively to maximize their benefits at their faculty levels. Here we take into account, the fact that decision about the usage of the budget at the sublevels can not be controlled (but it may be predictable) by the higher level, if the budget once allocated. Thus, one of the aims of our model is to provide rationality in decision making for the higher level decisions by the lower level decision maker. This is done by sub-models which predict how sublevels will react when they are given various amount of resources.

4.2 Mathematical Formulation

It is known that the mathematical formulation of the budget allocation problem is a three player Stackleberg game, class of three level linear programming problem. where the player at the top level is the federal government, then the university at the second level and finally the faculty at the third level, where individual decision maker in each level has got its own objective function and decision variables determined by decisions made at other levels.

The above discussion of three level programming problem can be written as a nested optimization as follows:

$$(p^1)\begin{cases} \max\limits_{x_3} f_3(x) \\ \quad \text{where } x_2 \text{ solves} \\ \quad (p^2)\begin{cases} \max\limits_{x_2} f_2(x) \\ \quad \text{where } x_1 \text{ solves} \\ \quad (p^3)\begin{cases} \max\limits_{x_1} f_1(x) \\ \quad s.t. \;\; x \in S \end{cases} \end{cases} \end{cases} \tag{4.2.1}$$

When all functions $f_1,\ f_2,$ and f_3 are linear and a set S is a polyhedral set then the above problem is known as a *three-level linear programming problem.* Where,

1. $f_1(x)$ express the objective function of the faculty
2. $f_2(x)$ express the objective function of the university and
3. $f_3(x)$ express the objective function of the federal government.

This three level linear programming problem, resulting the decision problems of three players in a hierarchical organization, will be solved by using different algorithms like Simplex cutting method and a solution will be found. (see Esubalew Lakie,[13]).

Assume x^* is the solution found from the three level programming problem. However, as expressed on chapter three, the solution x^* will not necessarily be pareto optimal and also the payoffs in each level are not pareto optimal since it is possible to find a solution with a better payoff. This points, which has got a better payoff is found in the conical section of the constraint set of the problem S, with its edges are drawn by drawing a hyperplane on each points of x^* and bounded by the compact set S.

However, the rational behavior of the players compels them not to move unilaterally, since each player chooses a point on its coordinate maximizing its payoffs. This is to mean, let $\tilde{x}$ have coordinates $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$. Suppose player 2 and 3 chooses $\tilde{x}_2$, $\tilde{x}_3$ of $\tilde{x}$. For $\tilde{x}$ to be a solution the first player need to choose $\tilde{x}_1$ but the first player will choose a point that is on its coordinate since that point will maximize his payoff. So the need of players to get maximum payoff and not to lose any payoff for the sake of others makes these points to be untenable, but this problem will be solved by having a binding agreement between decision makers in each level, which is called coalition among decision makers. In a similar situation the decision makers in each level will be better benefited together in forming coalition.

4.3 Coalition

A coalition is an agreement among the group of decision makers to work co-operatively and benefit together. On this section the questions that will be raised when forming a coalition between decision makers in each level will be addressed. These question are, which coalitions to form, are the coalitions enforceable and how to distribute the additional payoff to each individuals in the coalition. So we need to find a stable solution to the core, represents fair distribution of allotments among players.

On the budget allocation problem, there are different decision makers which can perform cooperation in coalition with one another in each level. In general, the total number of coalition is 2^N where N is the number of players. If we assume there are three decision makers on the level k, suppose faculty level and the decision makers of Biology, Mathematics and Physics departments, the total number of coalition between these decision makers is $2^3 = 8$. Which are:-

$\{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$, where

- {i} indicates the decision maker on level i not making any coaliton.
- {i,j} decision makers on level i and j forming a calition, and finally
- {1,2,3} is the grand coalition

Now we are going to address the above most important questions by using the concept of indices of power, chapter three, Shapley and Willick. Since we are going to find a solution which is stable element of the core during coalition, the enforceability of the coalition is must. Power indices measure a player's worth to the coalition.

4.3.1 Shapley Value

The Shapley value at each level is given by $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ where for $i = 1, \ldots, n$

$$\phi_i(v) = \sum_{\substack{S \subset N \\ i \in S}} \frac{(|S|-1)!(n-|S|)!}{n!}[v(S) - v(S - \{i\})]$$

So by using this formula we get the players worth to the coalition as indicated below

$\phi_1(v) = \frac{1}{6}[2v(\{1\}) + 2v(\{1,2,3\}) + v(\{1,2\}) - (2v(\{2,3\}) + v(\{2\}))]$

$\phi_2(v) = \frac{1}{6}[2v(\{2\}) + 2v(\{1,2,3\}) + v(\{1,2\}) + v(\{2,3\}) - (2v(\{1,3\})v(\{1\}) + v(\{3\}))]$

$\phi_3(v) = \frac{1}{6}[2v(\{3\}) + 2v(\{1,2,3\}) + v(\{2,3\}) - (2v(\{1,2\}) + v(\{2\}))]$

Now we will take the ratio of each decision makers worth to a coalition by the total worth to address the question how to distribute the payoff to the decision makers during coalition.
i.e. Let $\alpha_i = \frac{\phi_i}{\phi_1+\phi_2+\phi_3}$, on each level, for $i = 1, 2, 3$
We use α_i as a ratio to the additional payoff that a decision maker i on each level, for i=1,2,3, gets in the coalition since we distribute the payoff by using the power of decision maker in the sense that the one with higher power needs to have a better payoff than the others which are less powerful.

Next to address the question that which coalitions to form in each level we need to find a coalition with the best maximum payoff than the other six possible coalitions we have. This can be done by using an optimization problem modeled as follows.

$$\begin{aligned}
&\max \sum_{i=1}^{3} \alpha_i w_i \\
&s.t : w_i \geq 0 \\
&\sum_{i=1}^{3} w_i \leq v(N) - \sum_{i=1}^{3} f_i(x^*) \\
&x \in S
\end{aligned} \tag{4.3.1}$$

This is a linear programming problem.

Here, the Shapley value has got a drawback of having a general relation with the core, that makes the coalition enforceable, and also violates individual rationality for non-super additive games, games where the coalition has got less value than working individually. But the Willick's power index fulfill the drawback of Shapley and can be considered as a general solution concept for most games.

4.3.2 Willick's Power Index

Using similar procedure as Shapley we can address the above questions on each level by finding the power index of Willick. Willick's power index is given by

$$\rho_i[u] = u(N) - \max_{p_{N-\{i\}}} \sum_{p \in p_{N-\{i\}}} u(p)$$

By applying the above formula we will find the power of each individual as follows,

$\rho_1[u] = u(N) - \max\{u(\{2,3\}), (u(\{2\}) + u(\{3\}))\}$

$\rho_2[u] = u(N) - (u(\{1\}) + u(\{3\}))$

$\rho_3[u] = u(N) - \max\{u(\{1,2\}), (u(\{1\}) + u(\{2\}))\}$

and to know how to distribute payoff among decision makers we let

$\gamma_i = \frac{\rho_i}{\rho_1+\rho_2+\rho_3}$ for i=1,2,3

as an amount of payoff to be distributed to the decision makers in the coalition on the power they have.

Next to address the question which coalition will be formed, we model the linear optimization problem as follows,

$$\begin{aligned} &\max \sum_{i=1}^{3} \gamma_i w_i \\ &s.t : w_i \geq 0 \\ &\sum_{i=1}^{3} w_i \leq v(N) - \sum_{i=1}^{3} f_i(x^*) \\ &x \in S \end{aligned} \tag{4.3.2}$$

In solving the linear programming problem either(4.3.1) or (4.3.2) our solution will be a coalition on each level which has got a better additional payoff(w_i) than the other six possible coalition we have, and that additional payoff will be distributed to the decision makers by using their indices of power. Then the additional amount of payoff (A_{p_i}), on the payoff of x^*, each player can get is indices of power of it divided by total power index of players and multiplied by the additional payoff. So the total amount(T_{p_i}) of payoff each player can get in the best coalition, having maximum payoff, will be the additional amount of payoff plus the amount of payoff the player get on x^*.

Mathematically it can be represented as :-

$$\begin{aligned} A_{p_i} &= \alpha_i \times w_{ij} \qquad and \\ T_{p_i} &= A_{p_i} + x_i^*, \qquad for \qquad i = 1, 2, 3 \end{aligned} \tag{4.3.3}$$

, or

$$\begin{aligned} A_{p_i} &= \gamma_i \times w_i \qquad and \\ T_{p_i} &= A_{p_i} + x_i^*, \qquad for \qquad i = 1, 2, 3 \end{aligned} \tag{4.3.4}$$

Equation (4.3.3) for the Shapley value and Equation(4.3.4) for the Willick's power index.

4.4 General Steps of the Application

It is important to follow the following steps when dealing with the application of three person cooperative games in the hierarchical organizations. After defining the problem, clearly and deeply by describing the context of the problem and then stating the problem with in this context from problems drawn from real world.

Then a mathematical formulation of the problem will be given as a three player Stackleberg game, a class of three level linear programming problem and and the solution will be found by different algorithms and assume that solution is x^*.

But, x^* will not necessarily be a pareto-optimal solution and also payoff of individuals on each level is not necessarily pareto- optimal. since it is possible to find a point with a better payoff. This points will be untenable because of rational behavior of decision makers, so to make this points attainable we need an agreement what we call it cooperation(coalition)and there needs to be a discussion or hearing among decision makers in forming coalition. Taking formation of coalition to improve x^* as an open problem we will only consider how individual decision makers on a particular level can coalesce to get a better payoff.

In general, for most games and for m decision makers on each level the following steps will help us to answer the questions that will be raised in the formation of coalition on each level, which coalition to form and how to distribute the payoff.

for i=1,...,m

Step 1 : we will find ρ_i's using

$$\rho_i[u] = u(N) - \max_{p_{N-\{i\}}} \sum_{p \in p_{N-\{i\}}} u(p)$$

This values gives us the power of the individuals in the coalition. The values of $u(\{.\})$ can be described during the discussion time, for instance it can be hearing in our case.

Step 2 : we will calculate $\gamma_i' s$ by taking the ratio of the individual power index by the total power index.

$$\gamma_i = \frac{\rho_i}{\rho_1 + ... + \rho_m} = \frac{\rho_i}{\sum_{i=1}^{n} \rho_i}$$

This helps us to know how to distribute the additional payoff among players.

Step 3 : Now we need to know which coalition has the largest amount of additional payoff, to answer this we will model a linear optimization problem of type (4.4.1):

$$\begin{aligned} &\max \sum_{i=1}^{m} \gamma_i w_i \\ &s.t : w_i \geq 0 \\ &\sum_{i=1}^{m} w_i \leq v(N) - \sum_{i=1}^{m} f_i(x^*) \\ &x \in S \end{aligned} \tag{4.4.1}$$

where the first and the second constraint helps to bound points of coalition on the feasible set.

Step 4 : Finally for the coalition with highest additional payoff, found in step 3 we will distribute this additional payoff to individuals by using (4.4.2).

$$\begin{aligned} A_{p_i} &= \gamma_i \times w_i \qquad \textit{and} \\ T_{p_i} &= A_{p_i} + x_i^*, \end{aligned} \tag{4.4.2}$$

where, A_{p_i} is the additional amount of payoff on the payoff of x^*, and T_{p_i} the total payoff each player can get in the best coalition, having maximum payoff.

With all this four steps above we can help the decision makers on each level to coalesce and get rid of problems happening during coalition.

Chapter 5

Conclusion and Recommendations

5.1 Conclusion

This paper has taken the university budget allocation system as a three-person Stackleberg games and indicates the need of coalition among levels. In particular, this paper has presented a method for evaluating the effect of coalition formation among decision makers in each level. The solution to such problems might suggest ways in which a hierarchical system could be restricted to remove overall system inefficiencies.

The methods developed here have characterized the 3-level budget allocation problem as an abstract game. If a non-empty core exists for such a game, then enforceable coalitions exists which could gain additional benefits from the system. And on each level the decision makers can be coalesced for the additional payoff. A drawback to the core concept is that some games could have empty cores. However, for those games with non-empty cores, it would be an advantageous property for power indices, Shapley value and Willick's power index, to assign values to the players that comprise a solution in the core.

5.2 Recommendations

In the existing procedure of the budget allocation system it is necessary to have an agreement on the decision makers of each level that helps to get an additional payoff. For example, if the decision makers, in department, of the faculty wants to form a coalition and work together to have a single decision maker that has got the heterogenous decision of the the decision makers in that level, the departments in this new coalition will get additional payoff which in turn can maximize the payoff of the university.
There are, however, several issues that remain untouched by the results of this thesis. For example, the payoff found from three level programming problem is not pareto optimal so the questions how to get benefit for all levels, if the core is empty, is there efficient algorithm for constructing it and so on. As of my knowledge this parts are not yet addressed so it will be an open problem to be solved.

Bibliography

[1] W. F. Bialas, *"Cooperation n-person a Stackelberg games,"* proceedings of the 28th IEEE Conference on Decision andControl, December 1989, 2439-2444.

[2] W. F Bialas and M. H. Chew, *"Coalition Formation in n-Person Stackelberg games,* " Proceedings of the 21st IEEE Conference on Decision and Control, January 1980, 761-765.

[3] W. F Bialas and M. H Karwan, *"On Two -level Optimization,"* IEEE Trans.Autom.Control, 27, 1982, 211-214.

[4] U.Binseil : *" A Coalition-form analysis of the core One Country- One Vote Rule in the Governing Council of the European Central Bank."* Vol. 15, Number 1, spring 2001.

[5] C. P. Bonini, K. J. Robert and H. M. Wagner, *"Management controls: New Directions in Basic Research* ", Mcgraw-Hill,inc., New York,1964.

[6] R. M. Burton and B. Obel , *"A Computer Simulation Test of the M-form Hypotheses,"* administrative science quarterly, 25,1980,457-566.

[7] R. M. Burton and B. Obel, *"Mathematical models of organization design,"* Mathematical Models of Organizational Design Workshop, Eiasm, Brussels, March 14-15,1994.

[8] R. Cassidy, M. J. Kirby and W. M. Raike (1971) *"Efficient Distribution of Resources through Three Levels of Governments"*, Management Science Vol. 17, No. 8 pp 462-473

[9] M. N. Chew, *"A Game Theoretic Approach To Coalition Formation in Multilevel Decision Making Organizations"*, M.S. Thesis , Operations Reaserch Program,Dept. of Industrial Eng., SUNY at Buffalo,Buffalo , New York ,1981.

[10] W. W. Cooper ,H. J. Leavitt, and M. W. ShellyII , *"New Perspectives in Organization Reasearch,"* John Wiley and Sons , Inc., New York,1964.

[11] G. B. Dantizg, *"Linear Programming and Extensions"*, Princeton University Press, Princton,New Jersey ,1963.

[12] R. F. Drenick, *"A Mathematical Organizational Theory",* North-Holland, New York, 1986.

[13] L. Esubalew , *"Three level linear programming problem with its application in decision making process of hierachical organization"*, M. S. Thesis, Dept. of math., Addis Ababa University, 2007

[14] W. A. Gamson, *"A Theory of coalition formation,"* American Sociological Review, 21,1961

[15] T. Groves , *"Incentives in Teams,"* Econometrica , 41, July , 1973, 617-631.

[16] E. Inarra and J. M. Usategui, *" The Shapley Value and Average Convex Games,"* International Journal of Game Theory, 22, 1993, 13-29.

[17] M. J. L. Kirby and W.M. Raike (1968) *"Priority Planning in a University Computation Center"*, CBA Working Paper PP 69-114, University of Texas, Austin, Texas.

[18] W. F. Lucas, *" An Overview of the Mathematical Theory of Games,"* Management Science, 18, January 1972.

[19] W. F. Lucas,and R. M. Thrall, *" n-person Games in Partition Form,"* Naval Research logistics Quarterly, 10, 1963, 281-298.

[20] R.D.Luce and H.Raiffa, *"Games and Decisions"*: Introduction and Critical Survey, John Willey and Sons, Inc., New York, 1957.

[21] J. Marschak and R. Radner, *"Economic Theory of Teams"*, *Yale University press*, New Haven, 1972.

[22] M. Maschler, *"The power of a Coalition,"*Management science, 10, October, 1963.

[23] D. Monderer, D. Samet , and L. S. Shapley, *"Weighted Values and the Core,"* International journal of Game Theory, 21, 1992, 27-39.

[24] L. S. Shapley, *" A value for n-person Games,"* Annals of Mathematics Studies, 28, 1951.

[25] L. S. Shapley, *" On Balanced Sets and Cores,* Naval Research Logistics Quarterly, 14, December, 1967.

[26] L. S. Shapley, *" On Market Games,"* Journal of Economic Theory, 1, June, 1964.

[27] P. Shenoy, *"On Coalition Formation: A Game Theoretic Approach,"* Intl,Jour.Of Game Theory,May 8,1978.

[28] H. A. Simon, Models of Man: *"Social and Rational Mathematical Essays on Rational Behaviour in a Social Setting"*, John Wiley and Sons, Inc., New York, 1957.

[29] H. von Stackelberg, *"The Theory of the Market Economy"*, Oxford University Press,Oxford, 1952.

[30] J. von Neumann and Morgenstern, *"Theory of Games and Economic Behaviour"*,2nd edn., Princeton university Press, Princeton, New Jersey, 1947.

[31] O. E. Williamson , *"Markets and Hierarchies : Analysis and Antitrust Implications"*, The Free Press,New York, 1975.

[32] W. J. Willick, *" Solution Techniques for Cooperative n-person Stackelberg Games"* ,M.S. Thesis , Operations Research Program, Dept. of Industrial Eng., SUNY at Buffalo, Buffalo, New York ,1990.

[33] W. J. Willick: *" A Power Index for Cooperative Games with Applications to Hierarchial Organizations,"* Ph.D. Thesis , Operations Research Program, Dept. of Industrial Eng., SUNY at Buffalo, Buffalo, New York, January, 1995.

[34] C. H. Yang: *"A family of Values for n-person Cooperative Transferable Utility Games"*, M. S. Thesis, University at Buffalo, 1997.

Declaration

This thesis is my original work and has not been presented for a degree in any other university and that all sources of information used for this thesis have been fully acknowledged.

Abiy Dejenee

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Signature

This thesis is submitted for Examination with my approval as a university advisor.

Advisor | Signature

Dr. Semu Mitiku | - - - - - - -