PARAMETRIC OSCILLATION WITH TWO-MODE SQUEEZED VACUUM AND COHERENT LIGHT

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Degree of Master of Science in Physics

By

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The undersigned hereby certify that they have read and recommended to the Faculty of Science School of Graduate Studies for acceptance a thesis entitled "**Parametric Oscillation with Two-Mode Squeezed Vacuum and Coherent Light**" by **Tewodros Yirgashewa** in partial fulfillment of the requirements for the degree of **Master of Science in Physics**.

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To my parents.

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Abstract

We obtain stochastic differential equations for the nondegenerate parametric oscillator with the cavity modes driven by coherent light and coupled to a two-mode squeezed vacuum reservoir. With the aid of these equations, we calculate the quadrature variance and the squeezing spectrum. It is found that the two-mode squeezed vacuum increases the degree of squeezing. It is also found that for $\omega = 0$ the noise in the plus quadrature is completely suppressed. However, the two-mode driving light has no effect on the squeezing properties of the cavity modes.

On the other hand, using the solutions of the stochastic differential equations, we calculate the mean and the variance of the photon number. It turns out that the two-mode driving light and the two-mode squeezed vacuum increase the mean and the variance of the photon number. Furthermore, employing the same solutions, we also obtain the antinormally ordered characteristic function defined in the Heisenberg picture. With the help of the resulting characteristic function, we determine the Q function which is then used to calculate the photon number distribution. Moreover, we calculate the mean, the variance, the photon number distribution for the number of photon pairs, in the absence of the two-mode driving light.

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1. INTRODUCTION

Light has played a special role in our attempts to understand nature both classically and quantum mechanically. Squeezing is one of the interesting nonclassical features of light that has been attracting attention and studied by many authors [1-11]. In squeezed light the noise in one quadrature is below the vacuum or coherent level at the expense of enhanced fluctuations in the other quadrature, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation. Squeezed light has potential applications in low-noise communications and precision measurements [13,14]. Squeezed light can be generated by quantum optical processes such as parametric oscillation [1-10], second harmonic generation [1,9,10], and four-wave mixing [9,10].

A parametric oscillator has been considered as an important source of squeezed light. It is one of the most interesting and well characterized optical device in quantum optics. In this device a pump photon interacts with a nonlinear crystal inside a cavity and is down-converted into two highly correlated photons. If these photons have the same frequency the device is called a degenerate parametric oscillator, otherwise it is called a nondegenerate parametric oscillator.

The quantum fluctuations and photon statistics of the signal mode produced by a degenerate parametric oscillator coupled to a squeezed vacuum reservoir have been analyzed employing the pertinent Fokker-Planck equation [4] or the quantum Langevin equations [3]. Recently a degenerate parametric oscillator with the cavity mode driven by coherent light and coupled to a squeezed vacuum reservoir has been studied applying stochastic differential equations [6].

The quantum dynamics of a nondegenerate parametric oscillator coupled to two uncorrelated squeezed vacuum reservoirs has been analyzed employing the Q function obtained by solving the Fokker-Planck equation using the propagator method [7,15]. The variance of the quadrature operators and the photon number distribution for the signal-idler modes produced by a nondegenerate parametric oscillator coupled to a two-mode squeezed vacuum reservoir have also been studied applying the pertinent Langevin equations [3].

On the other hand, obtaining stochastic differential equations, associated with the normally ordering, for the cavity mode variables appears to involve a relatively less mathematical task. In view of this, the main objective of this thesis is to study, employing stochastic differential equations, the squeezing and statistical properties of the light produced by a nondegenerate parametric oscillator with the cavity modes driven by coherent light and coupled to a two-mode squeezed vacuum reservoir via a single port-mirror.

We first obtain stochastic differential equations for the cavity mode variables by applying the pertinent Master equation. With the aid of the resulting equations, we calculate the quadrature variance for the two-mode cavity radiation and the squeezing spectrum for the two-mode output radiation. In addition, we determine the mean photon number, the variance of the photon number, and the photon number distribution. We also calculate the mean, the variance, and the probability distribution for the number of photon pairs, in the absence of the driving coherent light modes.

2. STOCHASTIC DIFFERENTIAL EQUATIONS

We seek to study the statistical and squeezing properties of the two-mode light produced by a nondegenerate parametric oscillator, with the cavity modes driven by coherent light modes and coupled to a two-mode squeezed vacuum reservoir via a single-port mirror.

Fig. 2.1: A nondegenerate parametric oscillator.

In a nondegenerate parametric oscillator a pump photon of frequency ω is down converted into a highly correlated signal and idler photons of frequencies ω_a and ω_b such that $\omega = \omega_a + \omega_b$. The process of the parametric interaction can be described by the Hamiltonian

$$H' = ig(\hat{a}\hat{b}\hat{c}^{\dagger} - \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{c}), \tag{2.0.1}$$

where $\hat{a}(\hat{b})$ is the annihilation operator for the signal (idler) mode, $\hat{c}$ is the annihilation operator for the pump mode, and g is the coupling constant.

With the pump mode treated classically, the Hamiltonian can be written as

$$H' = i\lambda(\hat{a}\hat{b} - \hat{a}^\dagger\hat{b}^\dagger), \qquad (2.0.2)$$

where λ, considered to be real and constant, is proportional to the amplitude of the pump mode. We consider the case in which the cavity modes are driven by two coherent light modes of frequencies ω_a and ω_b. With the driving modes treated classically, the interaction of the cavity modes with the driving modes can be described by the Hamiltonian

$$H'' = i\varepsilon(\hat{a}^\dagger - \hat{a} + \hat{b}^\dagger - \hat{b}), \qquad (2.0.3)$$

where ε, considered to be real and constant, is proportional to the amplitude of mode a and mode b. Therefore the Hamiltonian describing the parametric interaction and the interaction of the cavity modes with the driving modes has the form

$$H = i\varepsilon(\hat{a}^\dagger - \hat{a} + \hat{b}^\dagger - \hat{b}) + i\lambda(\hat{a}\hat{b} - \hat{a}^\dagger\hat{b}^\dagger). \qquad (2.0.4)$$

Using (2.4) and taking into account the interaction of the cavity modes with a two-mode squeezed vacuum reservoir, we find the equation of evolution of the density operator for the two-mode cavity radiation to be

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= \varepsilon(\hat{a}^\dagger\hat{\rho} + \hat{b}^\dagger\hat{\rho} - \hat{a}\hat{\rho} - \hat{b}\hat{\rho} - \hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{b}^\dagger + \hat{\rho}\hat{a} + \hat{\rho}\hat{b}) + \lambda(\hat{a}\hat{b}\hat{\rho} - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{b} + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger)\\ &+ \frac{\kappa(N+1)}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{\kappa N}{2}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger)\\ &+ \frac{\kappa(N+1)}{2}(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}) + \frac{\kappa N}{2}(2\hat{b}^\dagger\hat{\rho}\hat{b} - \hat{b}\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^\dagger)\\ &+ \kappa M(\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger + \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger + \hat{a}\hat{\rho}\hat{b} + \hat{b}\hat{\rho}\hat{a} - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{a}\hat{b}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger - \hat{\rho}\hat{a}\hat{b}),\end{aligned} \qquad (2.0.5)$$

where κ is the cavity damping constant, N and M are the reservoir parameters given by

$$N = \sinh^2 r \tag{2.0.6}$$

and

$$M = \sinh r \cosh r, \tag{2.0.7}$$

with r being the squeeze parameter. We use this master equation to determine the equations of evolution for the expectation values of normally ordered cavity modes operators. The time evolution of the expectation value of an operator $\hat{A}$ in the Schrödinger picture can be expressed as [1]

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{A}). \tag{2.0.8}$$

Now taking into account (2.5) along with (2.8), one can write

$$\begin{aligned}
\frac{d}{dt}\langle\hat{a}(t)\rangle &= \varepsilon Tr(\hat{a}^\dagger\hat{\rho}\hat{a} + \hat{b}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{\rho}\hat{a} - \hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{a} + \hat{\rho}\hat{a}\hat{a} + \hat{\rho}\hat{b}\hat{a}) \\
&+ \lambda Tr(\hat{a}\hat{b}\hat{\rho}\hat{a} - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b}\hat{a} + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}) \\
&+ \frac{\kappa(N+1)}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}) + \frac{\kappa N}{2}Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}) \\
&+ \frac{\kappa(N+1)}{2}Tr(2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a}) + \frac{\kappa N}{2}Tr(2\hat{b}^\dagger\hat{\rho}\hat{b}\hat{a} - \hat{b}\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{b}\hat{b}^\dagger\hat{a}) \\
&+ \kappa M Tr(\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger\hat{a} + \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} + \hat{a}\hat{\rho}\hat{b}\hat{a} + \hat{b}\hat{\rho}\hat{a}\hat{a} - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a} - \hat{\rho}\hat{a}\hat{b}\hat{a}).
\end{aligned} \tag{2.0.9}$$

Applying the cyclic property of the trace operation together with the commutation relations

$$[\hat{a}, \hat{a}^\dagger] = 1, \tag{2.0.10}$$

$$[\hat{a}, \hat{b}] = [\hat{a}^\dagger, \hat{b}^\dagger] = [\hat{a}, \hat{b}^\dagger] = 0, \tag{2.0.11}$$

and

$$[\hat{a}^2, \hat{a}^\dagger] = 2\hat{a}, \tag{2.0.12}$$

we readily find

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -\frac{\kappa}{2}\langle\hat{a}(t)\rangle - \lambda\langle\hat{b}^\dagger(t)\rangle + \varepsilon. \tag{2.0.13}$$

It can also be shown in a similar manner that

$$\frac{d}{dt}\langle\hat{b}(t)\rangle = -\frac{\kappa}{2}\langle\hat{b}(t)\rangle - \lambda\langle\hat{a}^\dagger(t)\rangle + \varepsilon, \tag{2.0.14}$$

$$\frac{d}{dt}\langle\hat{a}(t)\hat{b}(t)\rangle = -\kappa\langle\hat{a}(t)\hat{b}(t)\rangle - \lambda\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle - \lambda\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle$$
$$+\varepsilon\langle\hat{a}(t)\rangle + \varepsilon\langle\hat{b}(t)\rangle - (\lambda + \kappa M), \tag{2.0.15}$$

$$\frac{d}{dt}\langle\hat{a}^2(t)\rangle = -\kappa\langle\hat{a}^2(t)\rangle - 2\lambda\langle\hat{a}(t)\hat{b}^\dagger(t)\rangle + 2\varepsilon\langle\hat{a}(t)\rangle, \tag{2.0.16}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle = -\kappa\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle + \varepsilon(\langle\hat{a}^\dagger(t)\rangle + \langle\hat{a}(t)\rangle)$$
$$- \lambda(\langle\hat{a}(t)\hat{b}(t)\rangle + \langle\hat{a}^\dagger(t)\hat{b}^\dagger(t)\rangle) + \kappa N. \tag{2.0.17}$$

The c-number equations corresponding to Eqs. (2.13), (2.14), (2.15), (2.16), and (2.17) are

$$\frac{d}{dt}\langle\alpha(t)\rangle = -\frac{\kappa}{2}\langle\alpha(t)\rangle - \lambda\langle\beta^*(t)\rangle + \varepsilon, \tag{2.0.18}$$

$$\frac{d}{dt}\langle\beta(t)\rangle = -\frac{\kappa}{2}\langle\beta(t)\rangle - \lambda\langle\alpha^*(t)\rangle + \varepsilon, \tag{2.0.19}$$

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = -\kappa\langle\alpha(t)\beta(t)\rangle - \lambda\langle\alpha^*(t)\alpha(t)\rangle - \lambda\langle\beta^*(t)\beta(t)\rangle$$
$$+\varepsilon\langle\alpha(t)\rangle + \varepsilon\langle\beta(t)\rangle - (\lambda + \kappa M), \tag{2.0.20}$$

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -\kappa\langle\alpha^2(t)\rangle - 2\lambda\langle\alpha(t)\beta^*(t)\rangle + 2\varepsilon\langle\alpha(t)\rangle, \tag{2.0.21}$$

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = -\kappa\langle\alpha^*(t)\alpha(t)\rangle + \varepsilon(\langle\alpha^*(t)\rangle + \langle\alpha(t)\rangle)$$
$$- \lambda(\langle\alpha(t)\beta(t)\rangle + \langle\alpha^*(t)\beta^*(t)\rangle) + \kappa N. \tag{2.0.22}$$

On the basis of Eqs. (2.18) and (2.19), one can write

$$\frac{d}{dt}\alpha(t) = -\frac{\kappa}{2}\alpha(t) - \lambda\beta^*(t) + \varepsilon + f_\alpha(t), \tag{2.0.23}$$

$$\frac{d}{dt}\beta(t) = -\frac{\kappa}{2}\beta(t) - \lambda\alpha^*(t) + \varepsilon + f_\beta(t), \tag{2.0.24}$$

where $f_\alpha(t)$ and $f_\beta(t)$ are noise forces. We next seek to determine the properties of the noise forces $f_\alpha(t)$ and $f_\beta(t)$. We note that Eq. (2.18) and the expectation value of Eq. (2.23) as well as Eq. (2.19) and the expectation value of Eq. (2.24) will have the same form if

$$\langle f_\alpha(t)\rangle = \langle f_\beta(t)\rangle = 0. \tag{2.0.25}$$

Using Eqs. (2.23) and (2.24) together with the relation

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = \langle\alpha(t)\dot{\beta}(t)\rangle + \langle\dot{\alpha}(t)\beta(t)\rangle, \tag{2.0.26}$$

we find

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = -\kappa\langle\alpha(t)\beta(t)\rangle - \lambda\langle\alpha^*(t)\alpha(t)\rangle - \lambda\langle\beta^*(t)\beta(t)\rangle + \varepsilon\langle\alpha(t)\rangle + \varepsilon\langle\beta(t)\rangle$$
$$+ \langle\alpha(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha(t)\rangle. \tag{2.0.27}$$

Comparison of Eqs. (2.20) and (2.27) indicates that

$$\langle\alpha(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha(t)\rangle = -(\lambda + \kappa M). \tag{2.0.28}$$

The formal solutions of Eqs. (2.23) and (2.24) can be written as

$$\alpha(t) = \alpha(0)e^{-(\kappa t)/2} + \int_0^t e^{-\kappa(t-t')/2}[f_\alpha(t') + \varepsilon - \lambda\beta^*(t')]dt', \tag{2.0.29}$$

$$\beta(t) = \beta(0)e^{-(\kappa t)/2} + \int_0^t e^{-\kappa(t-t')/2}[f_\beta(t') + \varepsilon - \lambda\alpha^*(t')]dt'. \tag{2.0.30}$$

Assuming the noise force $f_\beta(t)$ at time t cannot affect the system variables at the earlier times, we see that

$$\langle\alpha(0)f_\beta(t)\rangle = \langle\alpha(0)\rangle\langle f_\beta(t)\rangle = 0, \tag{2.0.31}$$

$$\langle\beta(t')^* f_\beta(t)\rangle = \langle\beta(t')^*\rangle\langle f_\beta(t)\rangle = 0. \tag{2.0.32}$$

Thus on account of Eqs. (2.25), (2.29), (2.31), and (2.32), we obtain

$$\langle\alpha(t)f_\beta(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f_\beta(t)f_\alpha(t')\rangle dt'. \tag{2.0.33}$$

It can also be verified in a similar fashion that

$$\langle\beta(t)f_\alpha(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t)f_\beta(t')\rangle dt'. \tag{2.0.34}$$

Therefore, in view of Eqs. (2.28), (2.33), and (2.34) and the assumption that

$$\langle f_\alpha(t)f_\beta(t')\rangle = \langle f_\beta(t)f_\alpha(t')\rangle, \tag{2.0.35}$$

we arrive at

$$\int_0^t e^{-\kappa(t-t')/2}\langle f_\beta(t)f_\alpha(t')\rangle dt' = \int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t)f_\beta(t')\rangle dt' = -\frac{1}{2}(\lambda+\kappa M). \tag{2.0.36}$$

Now on the basis of the relation [1]

$$\int_0^t e^{-a(t-t')/2}\langle f(t)g(t')\rangle dt' = D, \tag{2.0.37}$$

we assert that

$$\langle f(t)g(t')\rangle = 2D\delta(t-t'), \tag{2.0.38}$$

where a is a constant and D is a constant or some function of the time t. We then see that

$$\langle f_\beta(t)f_\alpha(t')\rangle = \langle f_\alpha(t)f_\beta(t')\rangle = -(\lambda+\kappa M)\delta(t-t'). \tag{2.0.39}$$

Furthermore, using Eq. (2.23) along with the relation

$$\frac{d}{dt}\langle\alpha(t)\alpha(t)\rangle = 2\langle\dot{\alpha}(t)\alpha(t)\rangle. \tag{2.0.40}$$

we obtain

$$\frac{d}{dt}\langle\alpha(t)\alpha(t)\rangle = -\kappa\langle\alpha^2(t)\rangle - 2\lambda\langle\alpha(t)\beta^*(t)\rangle + 2\epsilon\langle\alpha(t)\rangle + 2\langle\alpha(t)f_\alpha(t)\rangle. \tag{2.0.41}$$

Comparison of Eqs. (2.21) and (2.41) shows that

$$\langle\alpha(t)f_\alpha(t)\rangle = 0. \tag{2.0.42}$$

Applying Eq. (2.29) together with (2.42), we find

$$\int_0^t e^{-a(t-t')/2}\langle f_\alpha(t)f_\alpha(t')\rangle dt' = 0, \tag{2.0.43}$$

from which follows

$$\langle f_\alpha(t)f_\alpha(t')\rangle = 0. \tag{2.0.44}$$

It can also be established in a similar manner that

$$\langle f_\beta(t)f_\beta(t')\rangle = \langle f_\alpha^*(t)f_\beta(t')\rangle = 0. \tag{2.0.45}$$

Moreover, employing Eq. (2.23) and its complex conjugate along with the relation

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = \langle\alpha^*(t)\dot{\alpha}(t)\rangle + \langle\dot{\alpha^*}(t)\alpha(t)\rangle, \tag{2.0.46}$$

we find

$$\begin{aligned}\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = &-\kappa\langle\alpha^*(t)\alpha(t)\rangle + \varepsilon(\langle\alpha^*(t)\rangle + \langle\alpha(t)\rangle) - \lambda(\langle\alpha(t)\beta(t)\rangle + \langle\alpha^*(t)\beta^*(t)\rangle)\\ &+ \langle\alpha(t)f_\alpha^*(t)\rangle + \langle\alpha^*(t)f_\alpha(t)\rangle.\end{aligned} \tag{2.0.47}$$

Comparison of Eqs. (2.22) and (2.47) indicates that

$$\langle\alpha(t)f_\alpha^*(t)\rangle + \langle\alpha^*(t)f_\alpha(t)\rangle = \kappa N. \tag{2.0.48}$$

With the aid of Eq. (2.29) and its complex conjugate, we obtain

$$\langle\alpha(t)f_\alpha^*(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha^*(t)f_\alpha(t')\rangle dt', \tag{2.0.49}$$

$$\langle\alpha^*(t)f_\alpha(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t)f_\alpha^*(t')\rangle dt'. \tag{2.0.50}$$

Now in view of Eqs. (2.48), (2.49), (2.50) and the assumption that

$$\langle f_\alpha^*(t)f_\alpha(t')\rangle = \langle f_\alpha(t)f_\alpha^*(t')\rangle, \tag{2.0.51}$$

we get

$$\int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha^*(t)f_\alpha(t')\rangle dt' = \int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t)f_\alpha^*(t')\rangle dt' = \frac{\kappa N}{2}. \tag{2.0.52}$$

Hence on account of (2.37) and (2.38), we note that

$$\langle f_\alpha^*(t)f_\alpha(t')\rangle = \langle f_\alpha(t)f_\alpha^*(t')\rangle = \kappa N\delta(t-t'). \tag{2.0.53}$$

It can also be established in a similar procedure that

$$\langle f_\beta(t)f_\beta^*(t')\rangle = \langle f_\beta^*(t)f_\beta(t')\rangle = \kappa N\delta(t-t'). \tag{2.0.54}$$

Eqs. (2.39), (2.44), (2.45), (2.53), and (2.54) describe the correlation properties of the noise forces $f_\alpha(t)$ and $f_\beta(t)$ associated with the normal ordering.

In order to obtain the solutions of Eqs. (2.23) and (2.24), we introduce a new variable defined by

$$z_\pm = \alpha(t) \pm \beta^*(t). \tag{2.0.55}$$

Applying Eq. (2.23), along with the complex conjugate of Eq. (2.24), we obtain

$$\frac{d}{dt}z_\pm = -\frac{1}{2}\lambda_\pm z_\pm + \varepsilon \pm \varepsilon + f_\alpha \pm f_\beta^*, \tag{2.0.56}$$

where

$$\lambda_\pm = \kappa \pm 2\lambda. \tag{2.0.57}$$

According to Eq. (2.56) together with (2.57), the equation of evolution for $z_{-}(t)$ has no solution for $\kappa < 2\lambda$. We then identify $\kappa = 2\lambda$ as the threshold condition. Thus for $2\lambda < \kappa$ the solution of Eq. (2.56) can be written as

$$z_{\pm}(t) = z_{\pm}(0)e^{(-\lambda_{\pm}t)/2} + \int_0^t e^{-\lambda_{\pm}(t-t')/2}[\varepsilon \pm \varepsilon + f_{\alpha}(t') \pm f_{\beta}^{*}(t')]dt'. \quad (2.0.58)$$

It then follows that

$$\alpha(t) = E_{+}(t)\alpha(0) + E_{-}(t)\beta^{*}(0) + F_{+}(t) + F_{-}(t), \quad (2.0.59)$$

$$\beta(t) = E_{+}(t)\beta(0) + E_{-}(t)\alpha^{*}(0) + F_{+}^{*}(t) - F_{-}^{*}(t), \quad (2.0.60)$$

in which

$$E_{\pm} = \frac{1}{2}(e^{-\lambda_{+}t/2} \pm e^{-\lambda_{-}t/2}) \quad (2.0.61)$$

and

$$F_{\pm}(t) = \frac{1}{2}\int_0^t e^{-\lambda_{\pm}(t-t')/2}[\varepsilon \pm \varepsilon + f_{\alpha}(t') \pm f_{\beta}^{*}(t')]dt'. \quad (2.0.62)$$

3. QUADRATURE FLUCTUATIONS

We next proceed to determine the quadrature variance and squeezing spectrum for the cavity radiation produced by a nondegenerate parametric oscillator, with the cavity modes driven by coherent light modes and coupled to a two-mode squeezed vacuum reservoir.

3.1 Quadrature variance

We seek to describe the superposition of two light modes represented by $\hat{a}$ and $\hat{b}$ by the operator

$$\hat{c} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{b}). \tag{3.1.1}$$

Using the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1, \tag{3.1.2}$$

it can easily be shown that

$$[\hat{c}, \hat{c}^\dagger] = 1. \tag{3.1.3}$$

The squeezing properties of a two-mode light are described by the quadrature operators

$$\hat{c}_+ = \hat{c} + \hat{c}^\dagger \tag{3.1.4}$$

and

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}). \tag{3.1.5}$$

With the aid of Eqs. (3.3), (3.4), and (3.5), one can readily verify that

$$[\hat{c}_+, \hat{c}_-] = 2i. \tag{3.1.6}$$

A two-mode light is said to be in a squeezed state if either $\Delta c_+ < 1$ and $\Delta c_- > 1$ or $\Delta c_+ > 1$ and $\Delta c_- < 1$ such that $\Delta c_+ \Delta c_- \geq 1$. The variance of the quadrature operators are defined by

$$\Delta c_\pm^2(t) = \langle \hat{c}_\pm^2(t) \rangle - \langle \hat{c}_\pm(t) \rangle^2. \tag{3.1.7}$$

Applying Eqs. (3.4) and (3.5), one can write (3.7) in the normal order as

$$\Delta c_\pm^2(t) = 1 \pm \langle \hat{c}^{\dagger 2}(t) + \hat{c}^2(t) \pm 2\hat{c}^\dagger(t)\hat{c}(t) \rangle \mp \langle \hat{c}^\dagger(t) \pm \hat{c}(t) \rangle^2. \tag{3.1.8}$$

This can be expressed in terms of c-number variables associated with the normal ordering as

$$\Delta c_\pm^2(t) = 1 \pm \langle \gamma^{*2}(t) + \gamma^2(t) \pm 2\gamma^*(t)\gamma(t) \rangle \mp \langle \gamma^*(t) \pm \gamma(t) \rangle^2, \tag{3.1.9}$$

where $\gamma(t)$ is the c-number corresponding to the operator $\hat{c}(t)$. Introducing a new variable defined by

$$\gamma_\pm(t) = \gamma^*(t) \pm \gamma(t). \tag{3.1.10}$$

Eq. (3.9) can be rewritten as

$$\Delta c_\pm^2(t) = 1 \pm (\langle \gamma_\pm^2(t) \rangle - \langle \gamma_\pm(t) \rangle^2) \tag{3.1.11}$$

or

$$\Delta c_\pm^2(t) = 1 \pm \langle \gamma_\pm(t), \gamma_\pm(t) \rangle. \tag{3.1.12}$$

The c-number equation corresponding to Eq. (3.1) is expressible as

$$\gamma(t) = \frac{1}{\sqrt{2}}(\alpha(t) + \beta(t)). \tag{3.1.13}$$

We then note that

$$\frac{d\gamma(t)}{dt} = \frac{1}{\sqrt{2}}\Big(\frac{d\alpha(t)}{dt} + \frac{d\beta(t)}{dt}\Big). \tag{3.1.14}$$

On account of Eqs. (2.23) and (2.24), we obtain

$$\frac{d\gamma(t)}{dt} = -\frac{\kappa}{2}\gamma(t) - \lambda\gamma^*(t) + \sqrt{2}\varepsilon + \frac{1}{\sqrt{2}}(f_\alpha(t) + f_\beta(t)). \tag{3.1.15}$$

With the help of Eq. (3.10), one easily gets

$$\frac{d\gamma_\pm(t)}{dt} = \frac{d\gamma^*(t)}{dt} \pm \frac{d\gamma(t)}{dt}. \tag{3.1.16}$$

Employing Eq. (3.15) and its complex conjugate along with (3.16), we find

$$\frac{d\gamma_\pm(t)}{dt} = -\frac{1}{2}\lambda_\pm\gamma_\pm(t) + \sqrt{2}(\varepsilon \pm \varepsilon) + \frac{1}{\sqrt{2}}\big[(f^*_\alpha(t) \pm f_\alpha(t)) + (f^*_\beta(t) \pm f_\beta(t))\big], \tag{3.1.17}$$

where

$$\lambda_\pm = \kappa \pm 2\lambda. \tag{3.1.18}$$

The equation of evolution for $\gamma_-(t)$ has no solution for $\kappa < 2\lambda$. Then $\kappa = 2\lambda$ is identified as the threshold condition. We thus realize that our analysis will be confined to a nondegenerate parametric oscillator operating below threshold. The solution of Eq.(3.17) for $2\lambda < \kappa$ can be written as

$$\gamma_\pm(t) = \gamma_\pm(0)e^{-(\lambda_\pm t)/2} + \int_0^t e^{-\lambda_\pm(t-t')/2}\big[\sqrt{2}(\varepsilon\pm\varepsilon) + \frac{1}{\sqrt{2}}\big((f^*_\alpha(t')\pm f_\alpha(t')) + (f^*_\beta(t')\pm f_\beta(t'))\big)\big]dt'. \tag{3.1.19}$$

Now in view of Eq. (2.25), we obtain

$$\langle\gamma_\pm(t)\rangle = \langle\gamma_\pm(0)\rangle e^{-\lambda_\pm t/2} + \frac{2\sqrt{2}}{\lambda_\pm}(\varepsilon \pm \varepsilon)[1 - e^{-(\lambda_\pm t)/2}]. \tag{3.1.20}$$

Furthermore, applying the relation

$$\frac{d}{dt}(\gamma_\pm^2(t)) = 2\gamma_\pm(t)\frac{d\gamma_\pm(t)}{dt}, \tag{3.1.21}$$

together with Eq. (3.17), we find

$$\begin{aligned}\frac{d}{dt}\langle\gamma_\pm^2(t)\rangle &= -\lambda_\pm\langle\gamma_\pm^2(t)\rangle + (2\sqrt{2}(\varepsilon \pm \varepsilon))\langle\gamma_\pm(t)\rangle \\ &+ \sqrt{2}\langle\gamma_\pm(t)f_\alpha^*(t)\rangle \pm \sqrt{2}\langle\gamma_\pm(t)f_\alpha(t)\rangle \\ &+ \sqrt{2}\langle\gamma_\pm(t)f_\beta^*(t)\rangle \pm \sqrt{2}\langle\gamma_\pm(t)f_\beta(t)\rangle,\end{aligned} \tag{3.1.22}$$

so that multiplying Eq. (3.19) by f_α^*, we get

$$\begin{aligned}\langle\gamma_\pm(t)f_\alpha^*(t)\rangle &= \langle\gamma_\pm(0)f_\alpha^*(t)\rangle e^{(-\lambda_\pm t)/2} + \sqrt{2}(\varepsilon \pm \varepsilon)\int_0^t e^{-\lambda_\pm(t-t')/2}\langle f_\alpha^*\rangle dt' \\ &+ \frac{1}{\sqrt{2}}\int_0^t e^{-\lambda_\pm(t-t')/2}\big(\langle f_\alpha^*(t')f_\alpha^*(t)\rangle \pm \langle f_\alpha(t')f_\alpha^*(t)\rangle\big)dt' \\ &+ \frac{1}{\sqrt{2}}\int_0^t e^{-\lambda_\pm(t-t')/2}\big(\langle f_\beta^*(t')f_\alpha^*(t)\rangle \pm \langle f_\beta(t')f_\alpha^*(t)\rangle\big)dt'.\end{aligned} \tag{3.1.23}$$

Taking into account Eqs. (2.25), (2.39), (2.44), (2.53) and the fact that a noise force at time t can not affect the system variables at the earlier times, we readily obtain

$$\begin{aligned}\langle\gamma_\pm(t)f_\alpha^*(t)\rangle &= \pm\frac{\kappa N}{\sqrt{2}}\int_0^t e^{-\lambda_\pm(t-t')/2}\delta(t-t')dt' \\ &- \frac{1}{\sqrt{2}}(\lambda + \kappa M)\int_0^t e^{-\lambda_\pm(t-t')/2}\delta(t-t')dt'.\end{aligned} \tag{3.1.24}$$

Now on the basis of the relation

$$D\int_0^t e^{-a(t-t')/2}\delta(t-t')dt' = D/2, \tag{3.1.25}$$

we arrive at

$$\langle\gamma_\pm(t)f_\alpha^*(t)\rangle = -\frac{1}{2\sqrt{2}}(\lambda + \kappa M) \pm \frac{\kappa N}{2\sqrt{2}}. \tag{3.1.26}$$

It can also be verified in a similar way that

$$\langle \gamma_{\pm}(t) f_{\alpha}(t) \rangle = \frac{\kappa N}{2\sqrt{2}} \mp \frac{1}{2\sqrt{2}}(\lambda + \kappa M), \tag{3.1.27}$$

$$\langle \gamma_{\pm}(t) f^*_{\beta}(t) \rangle = -\frac{1}{2\sqrt{2}}(\lambda + \kappa M) \pm \frac{\kappa N}{2\sqrt{2}}, \tag{3.1.28}$$

and

$$\langle \gamma_{\pm}(t) f_{\beta}(t) \rangle = \frac{\kappa N}{2\sqrt{2}} \mp \frac{1}{2\sqrt{2}}(\lambda + \kappa M). \tag{3.1.29}$$

On account of Eqs. (3.26), (3.27), (3.28), and (3.29), one can write Eq. (3.22) as

$$\begin{aligned}\frac{d}{dt}\langle \gamma^2_{\pm}(t) \rangle &= -\lambda_{\pm}\langle \gamma^2_{\pm}(t) \rangle + 2\sqrt{2}(\varepsilon \pm \varepsilon)\langle \gamma_{\pm}(t) \rangle \\ &\quad - 2(\lambda + \kappa M) \pm 2\kappa N. \end{aligned} \tag{3.1.30}$$

Upon substituting Eq. (3.20) into (3.30), we find

$$\begin{aligned}\frac{d}{dt}\langle \gamma^2_{\pm}(t) \rangle &= -\lambda_{\pm}\langle \gamma^2_{\pm}(t) \rangle + 2\sqrt{2}(\varepsilon \pm \varepsilon)\langle \gamma_{\pm}(0) \rangle e^{-(\lambda_{\pm} t)/2} \\ &\quad + \frac{8}{\lambda_{\pm}}(\varepsilon \pm \varepsilon)^2(1 - e^{-(\lambda_{\pm} t)/2}) - 2(\lambda + \kappa M) \pm 2\kappa N. \end{aligned} \tag{3.1.31}$$

The solution of Eq. (3.31) can be written as

$$\begin{aligned}\langle \gamma^2_{\pm}(t) \rangle &= \langle \gamma^2_{\pm}(0) \rangle e^{-\lambda_{\pm} t} + 2\sqrt{2}(\varepsilon \pm \varepsilon)\langle \gamma_{\pm}(0) \rangle \int_0^t e^{-\lambda_{\pm}(t - t'/2)} dt' \\ &\quad + \frac{8}{\lambda_{\pm}}(\varepsilon \pm \varepsilon)^2 \int_0^t e^{-\lambda_{\pm}(t-t')} dt' - \frac{8}{\lambda_{\pm}}(\varepsilon \pm \varepsilon)^2 \int_0^t e^{-\lambda_{\pm}(t-t'/2)} dt' \\ &\quad + (-2(\lambda + \kappa M) \pm 2\kappa N) \int_0^t e^{-\lambda_{\pm}(t-t')} dt'. \end{aligned} \tag{3.1.32}$$

Carrying out the integration, we obtain

$$\begin{aligned}\langle \gamma^2_{\pm}(t) \rangle &= \langle \gamma^2_{\pm}(0) \rangle e^{-\lambda_{\pm} t} + \frac{4\sqrt{2}}{\lambda_{\pm}}(\varepsilon \pm \varepsilon)\langle \gamma_{\pm}(0) \rangle \big(e^{-\lambda_{\pm} t/2} - e^{-\lambda_{\pm} t}\big) \\ &\quad + \frac{8}{\lambda^2_{\pm}}(\varepsilon \pm \varepsilon)^2 \big(1 - 2e^{-\lambda_{\pm} t/2} + e^{-\lambda_{\pm} t}\big) \\ &\quad + \big(\frac{-2\lambda - 2\kappa M}{\lambda_{\pm}} \pm \frac{2\kappa N}{\lambda_{\pm}}\big)\big(1 - e^{-\lambda_{\pm} t}\big). \end{aligned} \tag{3.1.33}$$

On the other hand, using Eq. (3.20), one can write

$$\langle\gamma_\pm(t)\rangle^2 = \langle\gamma_\pm(0)\rangle^2 e^{-\lambda_\pm t} + \frac{4\sqrt{2}}{\lambda_\pm}(\varepsilon \pm \varepsilon)\langle\gamma_\pm(0)\rangle\big(e^{-\lambda_\pm t/2} - e^{-\lambda_\pm t}\big)$$
$$+ \frac{8}{\lambda_\pm^2}(\varepsilon \pm \varepsilon)^2\big(1 - 2e^{-\lambda_\pm t/2} + e^{-\lambda_\pm t}\big). \tag{3.1.34}$$

Hence substitution of Eqs. (3.33) and (3.41) into (3.11) results in

$$\Delta c_+^2(t) = (\Delta c_+^2(0))e^{-\lambda_+ t} + \frac{\kappa}{\lambda_+}(2N + 1 - 2M)(1 - e^{-\lambda_+ t}), \tag{3.1.35}$$

$$\Delta c_-^2(t) = (\Delta c_+^2(0))e^{-\lambda_- t} + \frac{\kappa}{\lambda_-}(2N + 2M + 1)(1 - e^{-\lambda_- t}), \tag{3.1.36}$$

so that in view of Eqs. (2.6) and (2.7), we see that

$$\Delta c_+^2(t) = (\Delta c_+^2(0))e^{-\lambda_+ t} + \frac{\kappa}{\lambda_+}e^{-2r}(1 - e^{-\lambda_+ t}), \tag{3.1.37}$$

$$\Delta c_-^2(t) = (\Delta c_+^2(0))e^{-\lambda_- t} + \frac{\kappa}{\lambda_-}e^{2r}(1 - e^{-\lambda_- t}), \tag{3.1.38}$$

where

$$\Delta c_\pm^2(0) = 1 \pm \big(\langle\gamma_\pm^2(0)\rangle - \langle\gamma_\pm(0)\rangle^2\big). \tag{3.1.39}$$

Assuming that the cavity radiation is initially in a two-mode vacuum state, we find [3]

$$\Delta c_+^2(t) = e^{-\lambda_+ t} + \frac{\kappa}{\lambda_+}e^{-2r}(1 - e^{-\lambda_+ t}), \tag{3.1.40}$$

$$\Delta c_-^2(t) = e^{-\lambda_- t} + \frac{\kappa}{\lambda_-}e^{2r}(1 - e^{-\lambda_- t}). \tag{3.1.41}$$

At steady state we have

$$(\Delta c_+^2)_{ss} = \frac{\kappa}{\lambda_+}e^{-2r}, \tag{3.1.42}$$

$$(\Delta c_-^2)_{ss} = \frac{\kappa}{\lambda_-}e^{2r}, \tag{3.1.43}$$

where ss stands for steady state. We easily notice that the squeezing occurs in the plus quadrature. It is also clear that the best squeezing in the nondegenerate parametric oscillator is achieved at threshold ($\kappa = 2\lambda$) giving [3]

$$(\Delta c_+^2)_{ss} = \frac{1}{2}e^{-2r}, \tag{3.1.44}$$

$$(\Delta c_-^2)_{ss} \longrightarrow \infty. \tag{3.1.45}$$

This shows that the variance of the plus quadrature is simply the product of the variance when the nondegenerate parametric oscillator is coupled to ordinary vacuum and the variance pertaining to the squeezed vacuum. We then observe that one effect of the squeezed vacuum is to enhance the degree of squeezing of the signal-idler modes. We also realize that the driving modes have no effect on the quadrature variance of the two-mode cavity radiation.

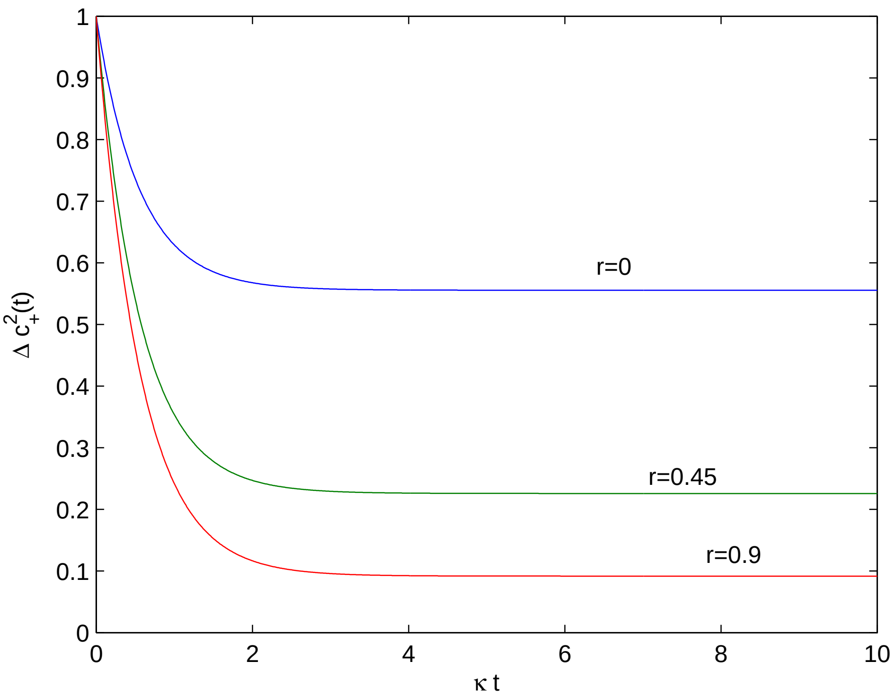

Fig. 3.1: Plots of $\Delta c_+^2(t)$ versus κt for $\lambda/\kappa = 0.4$ and for different values of r.

It is clear from Eq. (3.40) as well as the plots in Fig 3.1 that the variance of the quadrature operator $\hat{c}_+(t)$ decreases with time as well as the squeeze parameter.

3.2 Squeezing spectrum

The squeezing spectrum of a two-mode light is expressible as [1,11]

$$S_{\pm}^{out}(\omega) = 2Re\int_0^\infty \langle \hat{c}_{\pm}^{out}(t), \hat{c}_{\pm}^{out}(t+\tau)\rangle_{ss} e^{i\omega\tau} d\tau. \tag{3.2.1}$$

Taking into account Eq. (3.4) along with the commutation relation

$$[\hat{c}(t), \hat{c}^\dagger(t+\tau)] = \delta(\tau), \tag{3.2.2}$$

we readily find

$$\langle \hat{c}_+^{out}(t)\hat{c}_+^{out}(t+\tau)\rangle = \delta(\tau) + \langle \hat{c}^{\dagger out}(t)\hat{c}^{\dagger out}(t+\tau)\rangle + \langle \hat{c}^{\dagger out}(t)\hat{c}^{out}(t+\tau)\rangle + \langle \hat{c}^{\dagger out}(t)\hat{c}^{out}(t+\tau)\rangle + \langle \hat{c}^{out}(t)\hat{c}^{out}(t+\tau)\rangle. \quad (3.2.3)$$

This can be expressed in terms of c-number variables associated with the normal ordering as

$$\langle \hat{c}_+^{out}(t)\hat{c}_+^{out}(t+\tau)\rangle = \delta(\tau) + \langle \gamma_+^{out}(t)\gamma_+^{out}(t+\tau)\rangle. \quad (3.2.4)$$

in which $\gamma_+(t)$ is defined by Eq. (3.10). It can also be shown in a similar manner that

$$\langle \hat{c}_-^{out}(t)\hat{c}_-^{out}(t+\tau)\rangle = \delta(\tau) - \langle \gamma_-^{out}(t)\gamma_-^{out}(t+\tau)\rangle. \quad (3.2.5)$$

Hence combining Eqs. (3.49) and (3.50), we have

$$\langle \hat{c}_\pm^{out}(t)\hat{c}_\pm^{out}(t+\tau)\rangle = \delta(\tau) \pm \langle \gamma_\pm^{out}(t)\gamma_\pm^{out}(t+\tau)\rangle, \quad (3.2.6)$$

so that with the aid of this result Eq. (3.46) can be written as

$$S_\pm^{out}(\omega) = 2Re\left[\int_0^\infty \delta(\tau)e^{i\omega\tau}d\tau \pm \int_0^\infty \left[\langle \gamma_\pm^{out}(t)\gamma_\pm^{out}(t+\tau)\rangle_{ss} - \langle \gamma_\pm^{out}(t)\rangle_{ss}\langle \gamma_\pm^{out}(t+\tau)\rangle_{ss}\right]e^{i\omega \quad (3.2.7)$$

Now on the basis of the relation

$$\int_0^\infty \delta(x)f(x)dx = \frac{1}{2}f(0), \quad (3.2.8)$$

we arrive at

$$S_\pm^{out}(\omega) = 1 \pm 2Re\int_0^\infty \langle \gamma_\pm^{out}(t), \gamma_\pm^{out}(t+\tau)\rangle_{ss}e^{i\omega\tau}d\tau. \quad (3.2.9)$$

We recall that the c-number variables corresponding to the output and input modes are related by

$$\alpha^{out}(t) = \sqrt{\kappa}\alpha(t) - \alpha^{in}(t), \quad (3.2.10)$$

$$\beta^{out}(t) = \sqrt{\kappa}\beta(t) - \beta^{in}(t). \tag{3.2.11}$$

Applying Eqs. (3.10), (3.13), (3.56), and (3.57), we get

$$\gamma_{\pm}^{out}(t) = \sqrt{\kappa}\gamma_{\pm}(t) - \gamma_{\pm}^{in}(t). \tag{3.2.12}$$

The c-number variables corresponding to the input modes are defined as

$$\alpha^{in}(t) = \frac{1}{\sqrt{\kappa}} f_{\alpha R}(t), \tag{3.2.13}$$

$$\beta^{in}(t) = \frac{1}{\sqrt{\kappa}} f_{\beta R}(t), \tag{3.2.14}$$

with $f_{\alpha R}$ and $f_{\beta R}$ being the noise forces associated with the squeezed vacuum reservoir. Moreover, using Eqs. (3.10), (3.13), (3.58), and (3.59), we obtain

$$\gamma_{\pm}^{in}(t) = \frac{1}{\sqrt{2\kappa}} \left[\left(f_{\alpha R}^{*} \pm f_{\alpha R}\right) + \left(f_{\beta R}^{*} \pm f_{\beta R}\right)\right]. \tag{3.2.15}$$

Now in view of Eq. (3.57), the squeezing spectrum of the out put radiation can be put in the form

$$\begin{aligned} S_{\pm}^{out} = 1 &\pm 2\kappa Re \int_0^\infty \langle\gamma_{\pm}(t)\gamma_{\pm}(t+\tau)\rangle_{ss} e^{i\omega\tau} d\tau \mp 2\kappa Re \int_0^\infty \langle\gamma_{\pm}(t)\gamma_{\pm}^{in}(t+\tau)\rangle_{ss} e^{i\omega\tau} d\tau \\ &\mp 2\kappa Re \int_0^\infty \langle\gamma_{\pm}^{in}(t)\gamma_{\pm}(t+\tau)\rangle_{ss} e^{i\omega\tau} d\tau \pm 2Re \int_0^\infty \langle\gamma_{\pm}^{in}(t)\gamma_{\pm}^{in}(t+\tau)\rangle_{ss} e^{i\omega\tau} d\tau \\ &\mp 2\kappa Re \int_0^\infty \langle\gamma_{\pm}(t)\rangle_{ss}\langle\gamma_{\pm}(t+\tau)\rangle_{ss} e^{i\omega\tau} d\tau \pm 2\sqrt{\kappa} Re \int_0^\infty \langle\gamma_{\pm}(t)\rangle_{ss}\langle\gamma_{\pm}^{in}(t+\tau)\rangle_{ss} e^{i\omega\tau} d\tau \\ &\pm 2\sqrt{\kappa} Re \int_0^\infty \langle\gamma_{\pm}^{in}(t)\rangle_{ss}\langle\gamma_{\pm}(t+\tau)\rangle_{ss} e^{i\omega\tau} d\tau \mp 2Re \int_0^\infty \langle\gamma_{\pm}^{in}(t)\rangle_{ss}\langle\gamma_{\pm}^{in}(t+\tau)\rangle_{ss} e^{i\omega\tau} d\tau. \end{aligned} \tag{3.2.16}$$

Furthermore, the solution of Eq. (3.17) can be written as

$$\begin{aligned} \gamma_{\pm}(t+\tau) = \gamma_{\pm}(t) e^{-\lambda_{\pm}\tau/2} + \int_0^\infty e^{-\lambda_{\pm}(\tau-\tau')/2} \Bigg[&\sqrt{2}(\varepsilon \pm \varepsilon) + \frac{1}{\sqrt{2}}\left(f_{\alpha}^{*}(t+\tau') \pm f_{\alpha}(t+\tau')\right) \\ &+ \frac{1}{\sqrt{2}}\left(f_{\beta}^{*}(t+\tau') \pm f_{\beta}(t+\tau')\right)\Bigg] d\tau'. \end{aligned} \tag{3.2.17}$$

On account of the fact that a noise force at time $t+\tau'$ can not affect the system variables at the earlier time t, we find

$$\langle\gamma_{\pm}(t)\gamma_{\pm}(t{+}\tau)\rangle = \langle\gamma_{\pm}^{2}(t)\rangle_{ss}\sqrt{2}(\varepsilon{\pm}\varepsilon)\int_{0}^{\tau}\langle\gamma_{\pm}(t)\rangle_{ss}e^{-\lambda_{\pm}(\tau-\tau')/2}d\tau'. \quad (3.2.18)$$

We note that Eqs. (3.20) and (3.33) have at steady state the form

$$\langle\gamma_{\pm}(t)\rangle_{ss} = \frac{2\sqrt{2}}{\lambda_{\pm}}(\varepsilon\pm\varepsilon) \quad (3.2.19)$$

and

$$\langle\gamma_{\pm}^{2}(t)\rangle_{ss} = \frac{8}{\lambda_{\pm}^{2}}(\varepsilon^{2}\pm\varepsilon^{2})\big(\frac{-2\lambda-2\kappa M\pm2\kappa N}{\lambda_{\pm}}\big), \quad (3.2.20)$$

so that on substituting these into Eq. (3.63), we get

$$\langle\gamma_{\pm}(t)\gamma_{\pm}(t+\tau)\rangle = \frac{8}{\lambda_{\pm}^{2}}(\varepsilon^{2}\pm\varepsilon^{2})e^{-\lambda_{\pm}\tau/2}+\big(\frac{-2\lambda-2\kappa M\pm\kappa N}{\lambda_{\pm}}\big)e^{-\lambda_{\pm}\tau/2}$$
$$+\frac{8}{\lambda_{\pm}^{2}}(\varepsilon^{2}\pm\varepsilon^{2})\big(1-e^{-\lambda_{\pm}\tau/2}\big). \quad (3.2.21)$$

We also note that

$$\langle\gamma_{\pm}(t+\tau)\rangle_{ss} = \frac{2\sqrt{2}}{\lambda_{\pm}}(\varepsilon\pm\varepsilon)e^{-\lambda_{\pm}\tau/2}+\frac{2\sqrt{2}}{\lambda_{\pm}}(\varepsilon\pm\varepsilon)\big(1-e^{-\lambda_{\pm}\tau/2}\big). \quad (3.2.22)$$

In addition, with the aid of Eq. (3.62), one can write

$$\langle\gamma_{\pm}^{in}(t)\gamma_{\pm}(t+\tau)\rangle_{ss} = I_{1}e^{-\lambda_{\pm}\tau/2}+I_{2}\int_{0}^{\tau}e^{-\lambda_{\pm}(\tau-\tau')/2}d\tau'$$
$$+\int_{0}^{\tau}I_{3}e^{-\lambda_{\pm}(\tau-\tau')/2}d\tau'+\int_{0}^{\tau}I_{4}e^{-\lambda_{\pm}(\tau-\tau')/2}d\tau', \quad (3.2.23)$$

where

$$I_{1} = \langle\gamma_{\pm}^{in}(t)\gamma_{\pm}(t)\rangle, \quad (3.2.24)$$

$$I_{2} = \sqrt{2}(\varepsilon\pm\varepsilon)\langle\gamma_{\pm}^{in}(t)\rangle, \quad (3.2.25)$$

$$I_{3} = \frac{1}{\sqrt{2}}\big[\langle f_{\alpha}^{*}(t+\tau')\gamma_{\pm}^{in}(t)\rangle\pm\langle f_{\alpha}(t+\tau')\gamma_{\pm}^{in}(t)\rangle\big], \quad (3.2.26)$$

and

$$I_4 = \frac{1}{\sqrt{2}}\big[\langle f_\beta^*(t+\tau')\gamma_\pm^{in}(t)\rangle \pm \langle f_\beta(t+\tau')\gamma_\pm^{in}(t)\rangle\big]. \tag{3.2.27}$$

Now applying Eqs. (3.19), (3.60), and (2.25), along with the fact that the noise force at time t can not affect the system variables at the earlier times, we readily find

$$\begin{aligned}I_1 = &\frac{1}{2\sqrt{\kappa}}\int_0^t e^{-\lambda_\pm(t-t')/2}\big[\langle f_{\alpha R}^*(t)f_\alpha^*(t')\rangle \pm \langle f_{\alpha R}^*(t)f_\alpha(t')\rangle \pm \langle f_{\alpha R}(t)f_\alpha^*(t')\rangle + \langle f_{\alpha R}(t)f_\alpha(t')\rangle\\ &+ \langle f_{\alpha R}^*(t)f_\beta^*(t')\rangle \pm \langle f_{\alpha R}^*(t)f_\beta(t')\rangle \pm \langle f_{\alpha R}(t)f_\beta^*(t')\rangle + \langle f_{\alpha R}(t)f_\beta(t')\rangle\big]dt'\\ &+ \frac{1}{2\sqrt{\kappa}}\int_0^t e^{-\lambda_\pm(t-t')/2}\big[\langle f_{\beta R}^*(t)f_\alpha^*(t')\rangle \pm \langle f_{\beta R}^*(t)f_\alpha(t')\rangle \pm \langle f_{\beta R}(t)f_\alpha^*(t')\rangle + \langle f_{\beta R}(t)f_\alpha(t')\rangle\\ &+ \langle f_{\beta R}^*(t)f_\beta^*(t')\rangle \pm \langle f_{\beta R}^*(t)f_\beta(t')\rangle \pm \langle f_{\beta R}(t)f_\beta^*(t')\rangle + \langle f_{\beta R}(t)f_\beta(t')\rangle\big]dt'.\end{aligned} \tag{3.2.28}$$

We note that the noise force $f_\alpha(t)$ or $f_\beta(t)$ is the sum of the system and reservoir noise forces

$$f_\alpha(t) = f_{\alpha R}(t) + f_{\alpha S}(t), \tag{3.2.29}$$

or

$$f_\beta(t) = f_{\beta R}(t) + f_{\beta S}(t), \tag{3.2.30}$$

with $f_{\alpha S}(t)$ and $f_{\beta S}(t)$ being the system noise forces. Applying (3.74) and (3.75), Eq. (2.39) can be rewritten as

$$\begin{aligned}&\langle f_{\alpha R}(t)f_{\beta R}(t')\rangle + \langle f_{\alpha R}(t)f_{\beta S}(t')\rangle + \langle f_{\alpha S}(t)f_{\beta R}(t')\rangle\\ &+ \langle f_{\alpha S}(t)f_{\beta S}(t')\rangle = -(\lambda+\kappa M)\delta(t-t'),\end{aligned} \tag{3.2.31}$$

from which follows

$$\langle f_{\alpha R}(t)f_{\beta R}(t')\rangle = -\kappa M\delta(t-t'), \tag{3.2.32}$$

$$\langle f_{\alpha S}(t)f_{\beta S}(t')\rangle = -\lambda\delta(t-t'), \tag{3.2.33}$$

$$\langle f_{\alpha R}(t)f_{\beta S}(t')\rangle = \langle f_{\alpha S}(t)f_{\beta R}(t')\rangle = 0, \tag{3.2.34}$$

so that application of these results leads to

$$\langle f_{\alpha R}(t)f_{\beta}(t')\rangle = -\kappa M\delta(t-t'). \tag{3.2.35}$$

Similarly with the aid of Eqs. (2.39), (2.44), (2.45), (2.52), (2.53) along with (3.74) and (3.75), we obtain

$$\langle f_{\beta R}(t)f_{\alpha}(t)\rangle = -\kappa M\delta(t-t'), \tag{3.2.36}$$

$$\langle f_{\alpha R}(t)f_{\alpha}(t')\rangle = \langle f^{*}_{\alpha R}(t)f_{\beta}(t')\rangle = \langle f_{\beta R}(t)f_{\beta}(t')\rangle = 0, \tag{3.2.37}$$

and

$$\langle f^{*}_{\alpha R}(t)f_{\alpha}(t')\rangle = \langle f^{*}_{\beta R}(t)f_{\beta}(t')\rangle = \kappa N\delta(t-t'). \tag{3.2.38}$$

Substituting these results together with their complex conjugate into (3.73) yields

$$I_1 = \sqrt{\kappa}(-M\pm N)\int_0^t e^{-\lambda_{\pm}(t-t')/2}\delta(t-t')dt' + \sqrt{\kappa}(-M\pm N)\int_0^t e^{-\lambda_{\pm}(t-t')/2}\delta(t-t')dt'. \tag{3.2.39}$$

Hence on carrying out the integration, we get

$$I_1 = \sqrt{\kappa}(-M\pm N). \tag{3.2.40}$$

Moreover in view of Eqs. (3.60) and (2.25), we see that

$$\langle \gamma^{in}_{\pm}(t)\rangle_{ss} = \langle \gamma^{in}_{\pm}(t+\tau)\rangle_{ss} = 0. \tag{3.2.41}$$

With the aid of this result, one can readily verified that

$$I_2 = 0. \tag{3.2.42}$$

In addition , on account of (3.60), one can write

$$I_3 = \frac{1}{2\sqrt{\kappa}} \big[\big(\langle f^*_\alpha(t+\tau')f^*_{\alpha R}(t)\rangle \pm \langle f^*_\alpha(t+\tau')f_{\alpha R}(t)\rangle$$
$$+ \langle f^*_\alpha(t+\tau')f^*_{\beta R}(t)\rangle \pm \langle f^*_\alpha(t+\tau')f_{\beta R}(t)\rangle \big) \pm \big(\langle f_\alpha(t+\tau')f^*_{\alpha R}(t)\rangle$$
$$\pm \langle f_\alpha(t+\tau')f_{\alpha R}(t)\rangle + \langle f_\alpha(t+\tau')f^*_{\beta R}(t)\rangle \pm \langle f_\alpha(t+\tau')f_{\beta R}(t)\rangle \big) \big] . \tag{3.2.43}$$

Employing Eqs. (3.80), (3.81), (3.82), and (3.83), we obtain

$$I_3 = \sqrt{\kappa}(-M \pm N)\delta(\tau'). \tag{3.2.44}$$

Proceeding in a similar manner, one can also establish that

$$I_4 = \sqrt{\kappa}(-M \pm N)\delta(\tau'). \tag{3.2.45}$$

Hence combination of (3.85), (3.87), (3.89), (3.90), and (3.68) leads to

$$\langle \gamma^{in}_\pm(t)\gamma_\pm(t+\tau)\rangle_{ss} = \sqrt{\kappa}(-M \pm N)e^{-\lambda_\pm \tau/2} + \sqrt{\kappa}(-M \pm N)\int_0^\tau e^{-\lambda_\pm(\tau-\tau')/2}\delta(\tau')d\tau'$$
$$+ \sqrt{\kappa}(-M \pm N)\int_0^\tau e^{-\lambda_\pm(\tau-\tau')/2}\delta(\tau')d\tau'. \tag{3.2.46}$$

It then follows that

$$\langle \gamma^{in}_\pm(t)\gamma_\pm(t+\tau)\rangle_{ss} = 2\sqrt{\kappa}(-M \pm N)e^{-\lambda_\pm \tau/2}. \tag{3.2.47}$$

Moreover, with the help of Eq. (3.60), we find

$$\gamma^{in}_\pm(t+\tau) = \frac{1}{\sqrt{2\kappa}} \big[\big(f^*_{\alpha R}(t+\tau) \pm f_{\alpha R}(t+\tau) \big) + \big(f^*_{\beta R}(t+\tau) \pm f_{\beta R}(t+\tau) \big) \big] . \tag{3.2.48}$$

Using Eqs. (3.80), (3.81), (3.82), and (3.83) along with (3.60) and (3.93), we get

$$\langle \gamma^{in}_\pm(t)\gamma^{in}_\pm(t+\tau)\rangle_{ss} = 2(-M \pm N)\delta(\tau). \tag{3.2.49}$$

On the other hand in view of the fact that the noise force at time $t + \tau$ can not affect the system variables at the earlier time t, we see that

$$\langle \gamma_{\pm}(t)\gamma_{\pm}^{in}(t+\tau)\rangle_{ss} = 0. \tag{3.2.50}$$

Therefore, on account of (3.66), (3.67), (3.86), (3.92), (3.94), and (3.95) the squeezing spectrum of the out put radiation takes the form

$$S_{\pm}^{out}(\omega) = 1 \pm \frac{4\kappa}{\lambda_{\pm}}(-\lambda - \kappa M \pm \kappa N) Re \int_0^{\infty} e^{-(\lambda_{\pm}/2 - i\omega)\tau} d\tau$$
$$\mp 4\kappa(-M \pm N) Re \int_0^{\infty} e^{-(\lambda_{\pm}/2 - i\omega)\tau} d\tau \pm 4(-M \pm N) Re \int_0^{\infty} \delta(\tau) e^{i\omega\tau} d\tau. \tag{3.2.51}$$

On carrying out the integration, we obtain

$$S_{\pm}^{out}(\omega) = 1 \pm \frac{2\kappa}{\omega^2 + (\lambda_{\pm}/2)^2}\big[-\lambda - M \pm N\big]$$
$$\mp \frac{2\kappa\lambda_{\pm}}{\omega^2 + (\lambda_{\pm}/2)^2}\big[-M \pm N\big] \pm 2[M \pm N]. \tag{3.2.52}$$

It then follows that

$$S_{+}^{out}(\omega) = 1 - \frac{2\kappa\lambda}{\omega^2 + (\lambda_{+}/2)^2}\big[2(N-M)+1\big] + 2[N-M] \tag{3.2.53}$$

and

$$S_{-}^{out}(\omega) = 1 + \frac{2\kappa\lambda}{\omega^2 + (\lambda_{-}/2)^2}\big[2(N+M)+1\big] + 2[N+M], \tag{3.2.54}$$

so that in view of Eqs. (2.6) and (2.7), we have

$$S_{+}^{out}(\omega) = \big[1 - \frac{2\kappa\lambda}{(\kappa/2+\lambda)^2 + \omega^2}\big] e^{-2r} \tag{3.2.55}$$

and

$$S_{-}^{out}(\omega) = \big[1 + \frac{2\kappa\lambda}{(\kappa/2-\lambda)^2 + \omega^2}\big] e^{2r}. \tag{3.2.56}$$

At threshold, we see that

$$S_{+}^{out}(\omega) = \frac{\omega^2}{\kappa^2 + \omega^2} e^{-2r} \tag{3.2.57}$$

and

$$S_{-}^{out}(\omega) = \frac{\kappa^2 + \omega^2}{\omega^2} e^{2r}. \tag{3.2.58}$$

We observe that the squeezed vacuum reservoir has the effect of increasing the degree of squeezing, while the driving light modes have no effect on the squeezing spectra. We also realize that for $\omega = 0$, the noise in the plus quadrature is completely suppressed, with the noise in the minus quadrature going to infinity. Finally, for $r = 0$ Eqs. (3.102) and (3.103) reduce to

$$S_{+}^{out}(\omega) = \frac{\omega^2}{\kappa^2 + \omega^2} \tag{3.2.59}$$

and

$$S_{-}^{out}(\omega) = \frac{\kappa^2 + \omega^2}{\omega^2}. \tag{3.2.60}$$

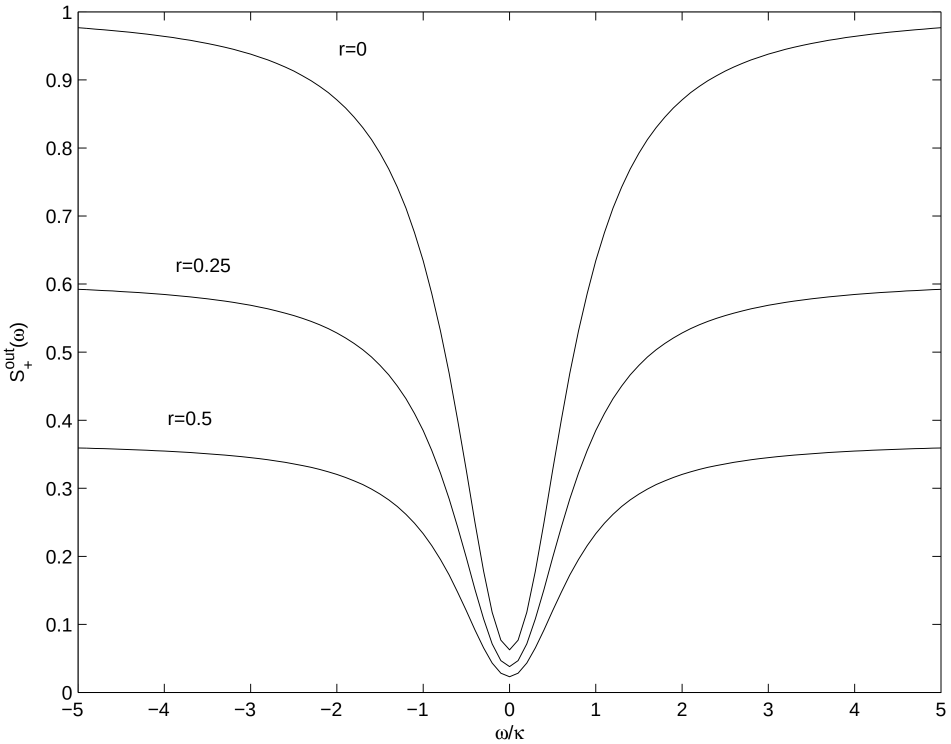

Fig. 3.2: Plots of $S_+^{out}(\omega)$ versus $\frac{\omega}{\kappa}$ for $\frac{\lambda}{\kappa} = 0.3$ and for different values of r.

We notice that the squeezing spectrum is an inverted Lorentizian. It is also clear that the maximum squeezing occurs for any values of r at $\omega = 0$.

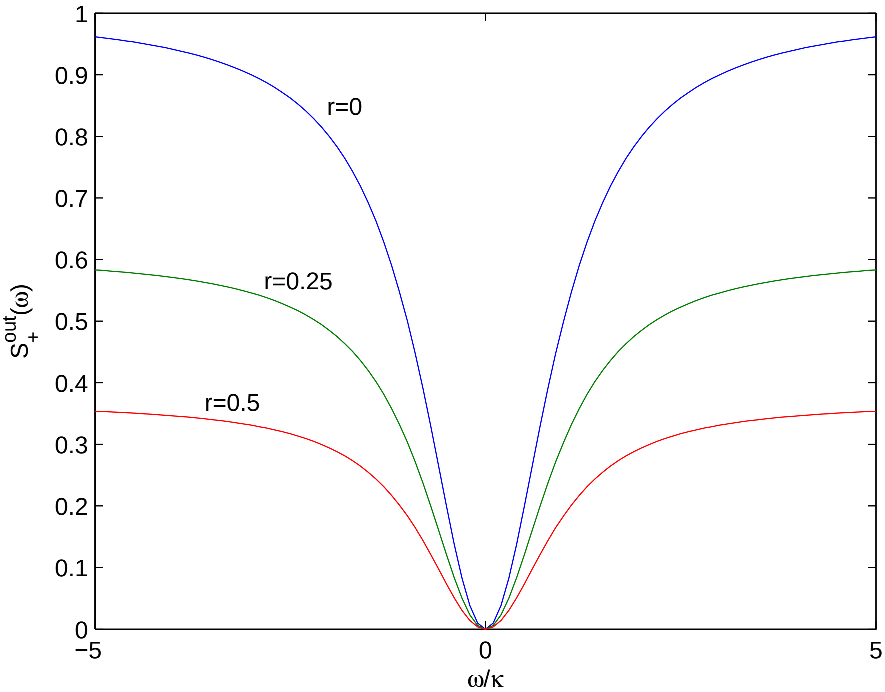

Fig. 3.3: Plots of $S_{+}^{out}(\omega)$ versus $\frac{\omega}{\kappa}$ for $\frac{\lambda}{\kappa} = 0.5$ at threshold and for different values of r.

4. PHOTON STATISTICS

4.1 The mean and variance of the photon number

For the system under consideration the number of photons in the cavity can be represented by the number operator

$$\hat{n} = \hat{n}_a + \hat{n}_b, \tag{4.1.1}$$

where $\hat{n}_a = \hat{a}^\dagger \hat{a}$ and $\hat{n}_b = \hat{b}^\dagger \hat{b}$ are the photon number operators for mode a and mode b. The mean photon number for the cavity mode can also be expressed in terms of c-number variables associated with the normal ordering as

$$\langle \hat{n} \rangle = \langle \alpha^*(t)\alpha(t) \rangle + \langle \beta^*(t)\beta(t) \rangle. \tag{4.1.2}$$

Now applying Eqs. (2.59) and (2.60) along with the assumption that the cavity radiation is initially in a two-mode vacuum state, we obtain

$$\langle \alpha^*(t)\alpha(t) \rangle = \langle F_+^*(t)F_+(t) \rangle + \langle F_-^*(t)F_-(t) \rangle + \langle F_+^*(t)F_-(t) \rangle + \langle F_-^*(t)F_+(t) \rangle. \tag{4.1.3}$$

Employing Eq. (2.62), we have

$$\langle F_-^*(t)F_-(t) \rangle = \frac{1}{4}\int_0^t e^{-\lambda_-(2t-t'-t'')/2}[\langle f_\alpha^*(t')f_\alpha(t'') \rangle + \langle f_\beta(t')f_\beta^*(t'') \rangle - \langle f_\alpha^*(t')f_\beta^*(t'') \rangle - \langle f_\beta(t')f_\alpha(t'') \rangle]dt'dt'', \tag{4.1.4}$$

so that on account of Eqs.(2.39), (2.53), and (2.54), we find

$$\langle F_-^*(t)F_-(t) \rangle = \frac{\kappa N + \lambda + \kappa M}{2}\int_0^t e^{-\lambda_-(2t-2t'-2t'')/2}\delta(t'-t'')dt'dt''. \tag{4.1.5}$$

Upon carrying out the integration, we obtain

$$\langle F_-^*(t)F_-(t)\rangle = \frac{\kappa N + \lambda + \kappa M}{2\lambda_-}(1 - e^{-\lambda_- t}). \tag{4.1.6}$$

Furthermore, it can also be established in a similar manner that

$$\langle F_+^*(t)F_+(t)\rangle = \frac{4\varepsilon^2}{\lambda_+^2}(1 - 2e^{-\lambda_+ t/2} + e^{-\lambda_+ t}) + \frac{\kappa N - \lambda - \kappa M}{2\lambda_+}(1 - e^{-\lambda_+ t}). \tag{4.1.7}$$

Moreover, using Eq. (2.62), we can write

$$\langle F_+^*(t)F_-(t)\rangle = \frac{1}{4}\int_0^t e^{-[(\lambda_+ + \lambda_-)t - \lambda_+ t' - \lambda_- t'']/2}[2\varepsilon(\langle f_\alpha(t)\rangle) - \langle f_\beta^*(t)\rangle + \langle f_\alpha(t')f_\alpha^*(t'')\rangle$$
$$- \langle f_\beta^*(t')f_\alpha^*(t'')\rangle + \langle f_\alpha(t')f_\beta(t'')\rangle - \langle f_\beta^*(t')f_\beta(t'')\rangle]dt'dt''. \tag{4.1.8}$$

With the aid of Eqs. (2.25), (2.39), (2.53), and (2.54), we get

$$\langle F_+^*(t)F_-(t)\rangle = 0. \tag{4.1.9}$$

We also note that

$$\langle F_-^*(t)F_+(t)\rangle = 0. \tag{4.1.10}$$

Hence substitution of (4.6), (4.7), (4.9), and (4.10) into (4.3) results in

$$\langle \alpha^*(t)\alpha(t)\rangle = \frac{4\varepsilon^2}{\lambda_+^2}(1 - 2e^{-\lambda_+ t/2} + e^{-\lambda_+ t}) + \frac{\kappa N - \lambda - \kappa M}{2\lambda_+}(1 - e^{-\lambda_+ t})$$
$$+ \frac{\kappa N + \lambda + \kappa M}{2\lambda_-}(1 - e^{-\lambda_- t}). \tag{4.1.11}$$

In addition, it can be verified following a similar procedure that

$$\langle \beta^*(t)\beta(t)\rangle = \frac{4\varepsilon^2}{\lambda_+^2}(1 - 2e^{-\lambda_+ t/2} + e^{-\lambda_+ t}) + \frac{\kappa N - \lambda - \kappa M}{2\lambda_+}(1 - e^{-\lambda_+ t})$$
$$+ \frac{\kappa N + \lambda + \kappa M}{2\lambda_-}(1 - e^{-\lambda_- t}). \tag{4.1.12}$$

Therefore, on combining Eqs. (4.11) and (4.12), we obtain

$$\langle\hat{n}\rangle = \frac{8\varepsilon^2}{\lambda_+^2}(1 - 2e^{-\lambda_+ t/2} + e^{-\lambda_+ t}) + \frac{\kappa N - \lambda - \kappa M}{\lambda_+}(1 - e^{-\lambda_+ t}) + \frac{\kappa N + \lambda + \kappa M}{\lambda_-}(1 - e^{-\lambda_- t}). \quad (4.1.13)$$

At steady state, we have

$$\langle\hat{n}\rangle_{ss} = \frac{8\varepsilon^2}{\lambda_+^2} + \frac{\kappa N - \lambda - \kappa M}{\lambda_+} + \frac{\kappa N + \lambda + \kappa M}{\lambda_-}. \quad (4.1.14)$$

In the absence of the driving coherent light modes ($\varepsilon = 0$), Eq. (4.14) reduces to

$$\langle\hat{n}\rangle_{ss} = \frac{\kappa N - \lambda - \kappa M}{\lambda_+} + \frac{\kappa N + \lambda + \kappa M}{\lambda_-}. \quad (4.1.15)$$

Furthermore, in the absence of the driving coherent light modes ($\varepsilon = 0$) and when the cavity is coupled to ordinary vacuum ($N = 0$ and $M = 0$), Eq. (4.14) becomes

$$\langle\hat{n}\rangle_{ss} = \frac{4\lambda^2}{\lambda_+\lambda_-}. \quad (4.1.16)$$

This represent the mean number of the signal-idler photons produced by the parametric oscillator. Moreover, when the driving coherent light modes and the parametric interaction are absent ($\varepsilon = 0$ and $\lambda = 0$), Eq. (4.14) takes the form

$$\langle\hat{n}\rangle_{ss} = 2N. \quad (4.1.17)$$

This is the mean photon number due to the two-mode squeezed vacuum reservoir. Finally, in the absence of the parametric interaction ($\lambda = 0$) and when the cavity is coupled to ordinary vacuum, we get

$$\langle\hat{n}\rangle_{ss} = \frac{8\varepsilon^2}{\kappa^2}. \quad (4.1.18)$$

This represents the mean photon number due to the driving coherent light modes. Hence the above results show that the driving coherent light modes and the two-mode squeezed vacuum reservoir increase the mean photon number.

The variance of the photon number can be expressed as

$$\Delta n^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2. \tag{4.1.19}$$

Applying Eq. (4.1) the variance can be put in the form

$$\Delta n^2 = \Delta n_a^2 + \Delta n_b^2 + 2(\langle \hat{n}_a \hat{n}_b \rangle - \langle \hat{n}_a \rangle \langle \hat{n}_b \rangle). \tag{4.1.20}$$

On the other hand, the photon number variance for mode a can be written in the normal order as

$$\Delta n_a^2 = \langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle + \langle \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2. \tag{4.1.21}$$

This can also be expressed in terms of c-number variables associated with the normal ordering as

$$\Delta n_a^2 = \langle \alpha^{*2} \alpha^2 \rangle + \langle \alpha^* \alpha \rangle - \langle \alpha^* \alpha \rangle^2. \tag{4.1.22}$$

Following a similar procedure the photon number variance for mode b can also be written as

$$\Delta n_b^2 = \langle \beta^{*2} \beta^2 \rangle + \langle \beta^* \beta \rangle - \langle \beta^* \beta \rangle^2. \tag{4.1.23}$$

In addition, one can also write

$$\langle \hat{n}_a \hat{n}_b \rangle = \langle \alpha^* \alpha \beta^* \beta \rangle. \tag{4.1.24}$$

Hence substitution of (4.22), (4.23), and (4.24) into (4.20) yields

$$\Delta n^2 = \langle \alpha^{*2} \alpha^2 \rangle + \langle \beta^{*2} \beta^2 \rangle + \langle \alpha^* \alpha \rangle + \langle \beta^* \beta \rangle - (\alpha^* \alpha \rangle + \langle \beta^* \beta \rangle)^2 + 2\langle \alpha^* \alpha \beta^* \beta \rangle \tag{4.1.25}$$

In view of Eq. (4.2), we see that

$$\Delta n^2 = \langle \hat{n} \rangle - \langle \hat{n} \rangle^2 + \langle \alpha^{*2}\alpha^2 \rangle + \langle \beta^{*2}\beta^2 \rangle + 2\langle \alpha^*\alpha\beta^*\beta \rangle. \tag{4.1.26}$$

Furthermore, Eqs. (2.59) and (2.60) can be rewritten as

$$\alpha(t) = E_+(t)\alpha(0) + E_-(t)\beta^*(0) + \sqrt{P} + \eta_\alpha(t), \tag{4.1.27}$$

$$\beta(t) = E_+(t)\beta(0) + E_-(t)\alpha^*(0) + \sqrt{P} + \eta_\beta(t), \tag{4.1.28}$$

where $E_\pm(t)$ is defined by Eq. (2.61),

$$P = \frac{4\varepsilon^2}{\lambda_+^2}(1 - 2e^{-\lambda_+ t/2} + e^{-\lambda_+ t}), \tag{4.1.29}$$

$$\eta_\alpha(t) = \frac{1}{2}[\int_0^t e^{-\lambda_+(t-t')/2}\big(f_\alpha(t') + f_\beta^*(t')\big)dt' + \int_0^t e^{-\lambda_-(t-t')/2}\big(f_\alpha(t') - f_\beta^*(t')\big)dt'], \tag{4.1.30}$$

and

$$\eta_\beta(t) = \frac{1}{2}[\int_0^t e^{-\lambda_+(t-t')/2}\big(f_\alpha^*(t') + f_\beta(t')\big)dt' - \int_0^t e^{-\lambda_-(t-t')/2}\big(f_\alpha^*(t') - f_\beta(t')\big)dt']. \tag{4.1.31}$$

Employing Eq. (4.27) and its complex conjugate along with the assumption that the cavity radiation is initially in a two- mode vacuum state, we get

$$\langle \alpha^{*2}\alpha^2 \rangle = P^2 + 2\sqrt{P}^3(\langle \eta_\alpha \rangle + \langle \eta_\alpha^* \rangle) + P(\langle \eta_\alpha^2 \rangle + \langle \eta_\alpha^{*2} \rangle) + \\ 4P\langle \eta_\alpha \eta_\alpha^* \rangle + 2\sqrt{P}(\langle \eta_\alpha \eta_\alpha^{*2} \rangle + \langle \eta_\alpha^* \eta_\alpha^2 \rangle) + \langle \eta_\alpha^2 \eta_\alpha^{*2} \rangle. \tag{4.1.32}$$

We recall that Gaussian variables with vanishing means satisfy the relation [1]

$$\langle ABCD \rangle = \langle AB \rangle\langle CD \rangle + \langle AC \rangle\langle BD \rangle + \langle AD \rangle\langle BC \rangle. \tag{4.1.33}$$

Next we need to show that $\eta_\alpha(t)$ is Gaussian variable. One can rewrite Eq. (4.30) as

$$\eta_\alpha = \eta_+ + \eta_-, \tag{4.1.34}$$

in which

$$\eta_+ = \frac{1}{2}\int_0^t e^{-\lambda_+(t-t')/2}\big(f_\alpha(t') + f_\beta^*(t')\big)dt' \tag{4.1.35}$$

and

$$\eta_- = \frac{1}{2}\int_0^t e^{-\lambda_-(t-t')/2}\big(f_\alpha(t') - f_\beta^*(t')\big)dt'. \tag{4.1.36}$$

Employing Eqs. (4.35) and (4.36), the time evolution for the expectation values of η_+ and η_- can be expressed as

$$\frac{d}{dt}\langle\eta_+\rangle = -\frac{\lambda_+}{4}\langle\eta_+\rangle \tag{4.1.37}$$

and

$$\frac{d}{dt}\langle\eta_-\rangle = -\frac{\lambda_-}{4}\langle\eta_-\rangle. \tag{4.1.38}$$

Inspection of (4.37) and (4.38) indicates that η_+ and η_- are Gaussian variables. Thus in view of Eq. (4.34), we see that η_α is also Gaussian variable. In addition, using (4.30), one easily gets

$$\langle\eta_\alpha(t)\rangle = 0. \tag{4.1.39}$$

Hence on the basis of the relation (4.33), we find

$$\langle\eta_\alpha\rangle = \langle\eta_\alpha^*\rangle = \langle\eta_\alpha\eta_\alpha^{*2}\rangle = \langle\eta_\alpha^*\eta_\alpha^2\rangle = 0, \tag{4.1.40}$$

$$\langle\eta_\alpha^2\eta_\alpha^{*2}\rangle = \langle\eta_\alpha^2\rangle\langle\eta_\alpha^{*2}\rangle + 2\langle\eta_\alpha\eta_\alpha^*\rangle^2, \tag{4.1.41}$$

so that application of these results in (4.32) leads to

$$\langle\alpha^{*2}\alpha^2\rangle = P^2 + P(\langle\eta_\alpha^2\rangle + \langle\eta_\alpha^{*2}\rangle) + 4P\langle\eta_\alpha\eta_\alpha^*\rangle + \langle\eta_\alpha^2\rangle\langle\eta_\alpha^{*2}\rangle + 2\langle\eta_\alpha\eta_\alpha^*\rangle^2. \tag{4.1.42}$$

Now with the help of Eq. (4.30), we can write

$$\langle\eta_\alpha^2\rangle = \frac{1}{4}[\int_0^t e^{-\lambda_+(2t-t'-t'')/2}\langle\big(f_\alpha(t')+f_\beta^*(t')\big)\big(f_\alpha(t'')+f_\beta^*(t'')\big)dt'dt''\rangle$$
$$+\int_0^t e^{-\lambda_-(2t-t'-t'')/2}\langle\big(f_\alpha(t')-f_\beta^*(t')\big)\big(f_\alpha(t'')-f_\beta^*(t'')\big)dt'dt''\rangle$$
$$+\int_0^t e^{[-(\lambda_++\lambda_-)t-\lambda_+t'-\lambda_-t'']/2}\langle\big(f_\alpha(t')+f_\beta^*(t')\big)\big(f_\alpha(t'')-f_\beta^*(t'')\big)dt'dt''\rangle$$
$$+\int_0^t e^{[-(\lambda_++\lambda_-)t-\lambda_-t'-\lambda_+t'']/2}\langle\big(f_\alpha(t')-f_\beta^*(t')\big)\big(f_\alpha(t'')-f_\beta^*(t'')\big)dt'dt''\rangle]. \tag{4.1.43}$$

Using Eqs.(2.44) and (2.45), we get

$$\langle\eta_\alpha^2\rangle = 0. \tag{4.1.44}$$

We also note that

$$\langle\eta_\alpha^{*2}\rangle = 0. \tag{4.1.45}$$

Furthermore, applying Eq. (4.30) and its complex conjugate, we find

$$\langle\eta_\alpha\eta_\alpha^*\rangle = \frac{1}{4}\big[\langle\tau_1\rangle+\langle\tau_2\rangle+\langle\tau_3\rangle+\langle\tau_4\rangle\big], \tag{4.1.46}$$

in which

$$\tau_1 = \int_0^t e^{-\lambda_+(2t-t'-t'')/2}\big(f_\alpha(t')+f_\beta^*(t')\big)\big(f_\alpha^*(t'')+f_\beta(t'')\big)dt'dt'', \tag{4.1.47}$$

$$\tau_2 = \int_0^t e^{-\lambda_-(2t-t'-t'')/2}\big(f_\alpha(t')-f_\beta^*(t')\big)\big(f_\alpha^*(t'')-f_\beta(t'')\big)dt'dt'', \tag{4.1.48}$$

$$\tau_3 = \int_0^t e^{[-(\lambda_++\lambda_-)t-\lambda_+t'-\lambda_-t'']/2}\big(f_\alpha(t')+f_\beta^*(t')\big)\big(f_\alpha^*(t'')+f_\beta^*(t'')\big)dt'dt'', \tag{4.1.49}$$

and

$$\tau_4 = \int_0^t e^{[-(\lambda_++\lambda_-)t-\lambda_-t'-\lambda_+t'']/2}\big(f_\alpha(t')-f_\beta^*(t')\big)\big(f_\alpha^*(t'')+f_\beta^*(t'')\big)dt'dt''. \tag{4.1.50}$$

With the aid of Eqs. (2.39), (2.53), and (2.54), we obtain

$$\langle\tau_1\rangle = 2\kappa N - 2(\lambda + \kappa M)\int_0^t e^{-\lambda_+(2t-t'-t'')/2}\delta(t-t'')dt'dt'', \quad (4.1.51)$$

$$\langle\tau_2\rangle = 2\kappa N + 2(\lambda + \kappa M)\int_0^t e^{-\lambda_-(2t-t'-t'')/2}\delta(t-t'')dt'dt'', \quad (4.1.52)$$

and

$$\langle\tau_3\rangle = \langle\tau_4\rangle = 0. \quad (4.1.53)$$

Upon carrying out the integration , Eqs. (4.51) and (4.52) turn over into

$$\langle\tau_1\rangle = \frac{2\kappa N - 2\lambda - 2\kappa M}{\lambda_+}(1 - e^{-\lambda_+ t}), \quad (4.1.54)$$

$$\langle\tau_2\rangle = \frac{2\kappa N + 2\lambda + 2\kappa M}{\lambda_-}(1 - e^{-\lambda_- t}). \quad (4.1.55)$$

Hence substitution of (4.54) and (4.55) into Eq. (4.46) results in

$$\langle\eta_\alpha\eta_\alpha^*\rangle = \frac{1}{2}(Q + R), \quad (4.1.56)$$

in which

$$Q = \frac{\kappa N - \lambda - \kappa M}{\lambda_+}(1 - e^{-\lambda_+ t}) \quad (4.1.57)$$

and

$$R = \frac{\kappa N + \lambda + \kappa M}{\lambda_-}(1 - e^{-\lambda_- t}). \quad (4.1.58)$$

Therefore, on account of (4.44), (4.45), and (4.56), Eq. (4.42) takes the form

$$\langle\alpha^{*2}\alpha^2\rangle = P^2 + 2P(Q + R) + \frac{1}{2}(Q + R)^2. \quad (4.1.59)$$

One can also establish in a similar manner that

$$\langle\beta^{*2}\beta^2\rangle = P^2 + 2P(Q + R) + \frac{1}{2}(Q + R)^2. \quad (4.1.60)$$

On the other hand, employing (4.27) and (4.28) along with the assumption that the cavity radiation is initially in a two-mode vacuum state, we can write

$$\begin{aligned}\langle\alpha^*\alpha\beta^*\beta\rangle &= P^2 + 2(\sqrt{P})^3(\langle\eta_\alpha\rangle + \langle\eta_\beta\rangle + \langle\eta_\alpha^*\rangle + \langle\eta_\beta^*\rangle)\\ &+ P(\langle\eta_\beta\eta_\beta^*\rangle + \langle\eta_\alpha\eta_\beta^*\rangle + \langle\eta_\alpha^*\eta_\beta^*\rangle + \langle\eta_\alpha\eta_\beta\rangle + \langle\eta_\alpha^*\eta_\beta\rangle + \langle\eta_\alpha\eta_\alpha^*\rangle)\\ &+ \sqrt{P}(\langle\eta_\alpha\eta_\beta\eta_\beta^*\rangle + \langle\eta_\alpha^*\eta_\beta\eta_\beta^*\rangle + \langle\eta_\alpha\eta_\alpha^*\eta_\beta^*\rangle + \langle\eta_\alpha\eta_\alpha^*\eta_\beta\rangle) + \langle\eta_\alpha\eta_\alpha^*\eta_\beta\eta_\beta^*\rangle.\end{aligned} \tag{4.1.61}$$

We recall that $\eta_\alpha(t)$ and $\eta_\beta(t)$ are Gaussian variables with vanishing means. Thus on the basis of the relation (4.33), we get

$$\langle\eta_\alpha\eta_\beta\eta_\beta^*\rangle = \langle\eta_\alpha^*\eta_\beta\eta_\beta^*\rangle = \langle\eta_\alpha\eta_\alpha^*\eta_\beta^*\rangle = \langle\eta_\alpha\eta_\alpha^*\eta_\beta\rangle) = 0, \tag{4.1.62}$$

$$\langle\eta_\alpha\eta_\alpha^*\eta_\beta\eta_\beta^*\rangle = \langle\eta_\alpha\eta_\alpha^*\rangle\langle\eta_\beta\eta_\beta^*\rangle + \langle\eta_\alpha\eta_\beta\rangle\langle\eta_\alpha^*\eta_\beta^*\rangle + \langle\eta_\alpha\eta_\beta^*\rangle\langle\eta_\alpha^*\eta_\beta\rangle, \tag{4.1.63}$$

so that applying these results together with the fact that $\eta_\alpha(t)$ and $\eta_\beta(t)$ have zero means, Eq. (4.61) reduces to

$$\begin{aligned}\langle\alpha^*\alpha\beta^*\beta\rangle &= P^2 + P(\langle\eta_\beta\eta_\beta^*\rangle + \langle\eta_\alpha\eta_\beta^*\rangle + \langle\eta_\alpha^*\eta_\beta^*\rangle + \langle\eta_\alpha\eta_\beta\rangle + \langle\eta_\alpha^*\eta_\beta\rangle + \langle\eta_\alpha\eta_\alpha^*\rangle)\\ &+ \langle\eta_\alpha\eta_\alpha^*\rangle\langle\eta_\beta\eta_\beta^*\rangle + \langle\eta_\alpha\eta_\beta\rangle\langle\eta_\alpha^*\eta_\beta^*\rangle + \langle\eta_\alpha\eta_\beta^*\rangle\langle\eta_\alpha^*\eta_\beta\rangle.\end{aligned} \tag{4.1.64}$$

Now with the help of (4.30) and (4.31), one can write

$$\begin{aligned}\langle\eta_\alpha\eta_\beta^*\rangle = \frac{1}{4}\Big[&\int_0^t e^{-\lambda_+(2t-t'-t'')/2}\langle\big(f_\alpha(t') + f_\beta^*(t')\big)\big(f_\alpha(t'') + f_\beta^*(t'')\big)\rangle dt'dt''\\ &- \int_0^t e^{-\lambda_-(2t-t'-t'')/2}\langle\big(f_\alpha(t') - f_\beta^*(t')\big)\big(f_\alpha(t'') - f_\beta^*(t'')\big)\rangle dt'dt''\\ &- \int_0^t e^{-[(\lambda_++\lambda_-)-\lambda_+t'-\lambda_-t'']/2}\langle\big(f_\alpha(t') + f_\beta^*(t')\big)\big(f_\alpha(t'') - f_\beta^*(t'')\big)\rangle dt'dt''\\ &- \int_0^t e^{-[(\lambda_++\lambda_-)-\lambda_-t'-\lambda_+t'']/2}\langle\big(f_\alpha(t') - f_\beta^*(t')\big)\big(f_\alpha(t'') + f_\beta^*(t'')\big)\rangle dt'dt''.\end{aligned} \tag{4.1.65}$$

On account of Eqs. (2.44) and (2.45), we obtain

$$\langle \eta_\alpha \eta^*_\beta \rangle = 0. \tag{4.1.66}$$

We also note that

$$\langle \eta^*_\alpha \eta_\beta \rangle = 0. \tag{4.1.67}$$

Furthermore, using (4.30) and (4.31), we can write

$$\begin{aligned}\langle \eta_\alpha \eta_\beta \rangle = \frac{1}{4}\Big[&\int_0^t e^{-\lambda_+(2t-t'-t'')/2} \langle \big(f_\alpha(t') + f^*_\beta(t')\big)\big(f_\alpha(t'')^* + f_\beta(t'')\big)\rangle dt'dt'' \\ &- \int_0^t e^{-\lambda_-(2t-t'-t'')/2} \langle \big(f_\alpha(t') - f^*_\beta(t')\big)\big(f_\alpha(t'')^* - f_\beta(t'')\big)\rangle dt'dt'' \\ &- \int_0^t e^{-[(\lambda_+ + \lambda_-) - \lambda_+ t' - \lambda_- t'']/2} \langle \big(f_\alpha(t') + f^*_\beta(t')\big)\big(f_\alpha(t'')^* - f_\beta(t'')\big)\rangle dt'dt'' \\ &+ \int_0^t e^{-[(\lambda_+ + \lambda_-) - \lambda_- t' - \lambda_+ t'']/2} \langle \big(f_\alpha(t') - f^*_\beta(t')\big)\big(f_\alpha(t'')^* + f_\beta(t'')\big)\rangle dt'dt''.\end{aligned} \tag{4.1.68}$$

Applying Eqs. (2.39), (2.53), (2.54), we get

$$\langle \eta_\alpha \eta_\beta \rangle = \frac{\kappa N - \lambda - \kappa M}{2\lambda_+}(1 - e^{-\lambda_+ t}) - (\frac{\kappa N + \lambda + \kappa M}{2\lambda_-})(1 - e^{-\lambda_- t}). \tag{4.1.69}$$

In view of (4.15) and (4.16), Eq. (4.71) can be rewritten as

$$\langle \eta_\alpha \eta_\beta \rangle = \frac{1}{2}(Q - R), \tag{4.1.70}$$

where Q and R are defined by (4.57) and (4.58). We also see that

$$\langle \eta^*_\alpha \eta^*_\beta \rangle = \frac{1}{2}(Q - R). \tag{4.1.71}$$

Hence combination of (4.56), (4.66), (4.67), (4.70), and (4.71) with (4.64) results in

$$\langle \alpha^* \alpha \beta^* \beta \rangle = P^2 + 2QP + \frac{1}{2}(Q^2 + R^2). \tag{4.1.72}$$

Moreover, with the aid of Eqs. (4.13), (4.29), (4.57), and (4.58), we find

$$\langle \hat{n} \rangle^2 = 4P^2 + Q^2 + R^2 + 4PQ + 4PR + 2QR. \tag{4.1.73}$$

Finally, on account of Eqs. (4.59), (4.60), (4.72), and (4.73), Eq. (4.26) can be put in the form

$$\Delta n^2 = \langle \hat{n} \rangle + Q^2 + R^2 + 4QR. \tag{4.1.74}$$

With the help of Eqs. (4.29), (4.57), and (4.58) the variance takes at steady state the form

$$\Delta n_{ss}^2 = \langle \hat{n} \rangle_{ss} + \Big(\frac{\kappa N - \lambda - \kappa M}{\lambda_+}\Big)^2 + \Big(\frac{\kappa N + \lambda + \kappa M}{\lambda_-}\Big)^2 + \frac{16\varepsilon^2}{\lambda_+^2}\Big(\frac{\kappa N - \lambda - \kappa M}{\lambda_+}\Big). \tag{4.1.75}$$

We note that $\Delta n_{ss}^2 > \langle \hat{n} \rangle_{ss}$ and hence the cavity radiation has super-Poissonian statistics.

In the absence of the coherent light modes ($\varepsilon = 0$), Eq. (4.75) reduces to

$$\Delta n_{ss}^2 = \Big(\frac{\kappa N - \lambda - \kappa M}{\lambda_+}\Big) + \Big(\frac{\kappa N + \lambda + \kappa M}{\lambda_-}\Big) + \Big(\frac{\kappa N - \lambda - \kappa M}{\lambda_+}\Big)^2 + \Big(\frac{\kappa N + \lambda + \kappa M}{\lambda_-}\Big)^2. \tag{4.1.76}$$

Furthermore, in the absence of the driving coherent light modes ($\varepsilon = 0$) and when the cavity modes are coupled to ordinary vacuum ($N = 0$ and $M = 0$), Eq. (4.75) becomes

$$\Delta n_{ss}^2 = \frac{4\lambda^2}{\lambda_+ \lambda_-} + \frac{\lambda^2}{\lambda_+^2} + \frac{\lambda^2}{\lambda_-^2}. \tag{4.1.77}$$

This represents the variance of the photon number of the signal-idler modes produced by the parametric oscillator. Moreover, when the driving coherent light modes and the parametric interaction are absent ($\varepsilon = 0$ and $\lambda = 0$), the variance of the photon number takes the form

$$\Delta n_{ss}^2 = 2M^2 + 2N(N+1), \tag{4.1.78}$$

on the basis of the relation

$$M = \sqrt{N(N+1)}, \tag{4.1.79}$$

we see that

$$\Delta n_{ss}^2 = 4N(N+1). \tag{4.1.80}$$

This represents the variance due to the two-mode squeezed vacuum. In addition, in the absence of the parametric interaction ($\lambda = 0$) and when the cavity mode is coupled to ordinary vacuum ($N = 0$ and $M = 0$), Eq. (4.75) reduces to

$$\Delta n_{ss}^2 = \frac{8\varepsilon^2}{\kappa^2}. \tag{4.1.81}$$

This represents the variance of the cavity radiation due to the two-mode driving coherent light.

4.2 The variance of the photon number difference

The photon number difference inside the cavity can be represented by an operator

$$\hat{n}_d = \hat{n}_a - \hat{n}_b. \tag{4.2.1}$$

Applying (4.82) the mean of the photon number difference can be expressed as

$$\langle \hat{n}_d \rangle = \langle \hat{n}_a \rangle - \langle \hat{n}_b \rangle. \tag{4.2.2}$$

One can also rewrite (4.83) in terms of the c-number variables as

$$\langle \hat{n}_d \rangle = \langle \alpha^* \alpha \rangle - \langle \beta^* \beta \rangle. \tag{4.2.3}$$

Hence on account of Eqs. (4.11) and (4.12), the mean of the photon number difference turns out to be

$$\langle \hat{n}_d \rangle = 0. \tag{4.2.4}$$

The variance of the photon number difference can be expressed as

$$\Delta n_d^2 = \langle (\hat{n}_a - \hat{n}_b)^2 \rangle - \langle \hat{n}_a - \hat{n}_b \rangle^2. \tag{4.2.5}$$

This can be rewritten as [12]

$$\Delta n_d^2 = \Delta n_a^2 + \Delta n_b^2 + 2(\langle \hat{n}_a \rangle \langle \hat{n}_b \rangle - \langle \hat{n}_a \hat{n}_b \rangle). \tag{4.2.6}$$

Now with the aid of (4.22), (4.23), (4.24), Eq. (4.87) can be put in the form

$$\Delta n_d^2 = \langle \hat{n} \rangle + \langle \alpha^{*2} \alpha^2 \rangle + \langle \beta^{*2} \beta^2 \rangle - 2\langle \alpha^* \alpha \beta^* \beta \rangle. \tag{4.2.7}$$

Therefore, substitution of (4.59), (4.60), and (4.72) into Eq. (4.88) yields

$$\Delta n_d^2 = \langle \hat{n} \rangle + 4PR + 2QR. \tag{4.2.8}$$

On account of (4.29), (4.57), and (4.58), the variance of the photon number difference takes at steady state the form

$$(\Delta n_d^2)_{ss} = \langle \hat{n} \rangle_{ss} + 2\Big(\frac{\kappa N - \lambda - \kappa M}{\lambda_+}\Big)\Big(\frac{\kappa N + \lambda + \kappa M}{\lambda_-}\Big) + \frac{16\varepsilon^2}{\lambda_+^2}\Big(\frac{\kappa N + \lambda + \kappa M}{\lambda_-}\Big). \tag{4.2.9}$$

Now in the absence of the driving coherent light modes ($\varepsilon = 0$), Eq. (4.90) reduces to

$$(\Delta n_d^2)_{ss} = \frac{\kappa N - \lambda - \kappa M}{\lambda_+} + \frac{\kappa N + \lambda + \kappa M}{\lambda_-} + 2\Big(\frac{\kappa N - \lambda - \kappa M}{\lambda_+}\Big)\Big(\frac{\kappa N + \lambda + \kappa M}{\lambda_-}\Big). \tag{4.2.10}$$

In the absence of the driving coherent light modes ($\varepsilon = 0$) and when the cavity mode is coupled to ordinary vacuum ($N = 0$ and $M = 0$), Eq. (4.90) becomes

$$(\Delta n_d^2)_{ss} = \frac{2\lambda^2}{\lambda_+ \lambda_-}. \tag{4.2.11}$$

This is the variance of the photon number difference due to the parametric oscillator. Furthermore, when the driving coherent light modes and the parametric interaction are absent ($\varepsilon = 0$ and $\lambda = 0$), Eq. (4.90) takes the form

$$(\Delta n_d^2)_{ss} = -2M^2 + 2N(N + 1). \tag{4.2.12}$$

In view of the relation (4.79), we see that

$$(\Delta n_d^2)_{ss} = 0. \tag{4.2.13}$$

We see that the variance of the photon number difference due to the two-mode squeezed vacuum is zero [12]. Moreover, in the absence of the parametric interaction ($\lambda = 0$) and when the cavity mode is coupled to ordinary vacuum ($N = 0$ and $M = 0$), Eq. (4.90) reduces to

$$(\Delta n_d^2)_{ss} = \frac{8\varepsilon^2}{\kappa^2}. \tag{4.2.14}$$

This represents the variance of the photon number difference due to the two-mode driving coherent light.

4.3 The photon number distribution

4.3.1 The Q function

We now proceed to obtain the Q function for the system under consideration. The Q function for a two-mode light is expressible as

$$Q(\alpha,\beta,t) = \frac{1}{\pi^4}\int d^2z d^2\chi \phi(z,\chi,t) exp(z^*\alpha + \chi^*\beta - z\alpha^* - \chi\beta^*), \tag{4.3.1}$$

with the characteristic function $\phi(z,\chi,t)$ defined in Heisenberg picture by

$$\phi(z,\chi,t) = Tr\{\hat{\rho(0)} e^{-z^*\hat{a}(t)} e^{-\chi^*\hat{b}(t)} e^{z\hat{a}^\dagger(t)} e^{\chi\hat{b}^\dagger(t)}\}. \tag{4.3.2}$$

Employing the identity

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{B}}e^{\hat{A}}e^{[\hat{A},\hat{B}]}, \tag{4.3.3}$$

the characteristic function can be expressed in terms of c-number variables associated with the normal ordering in the form

$$\phi(z,\chi,t) = exp\big[(z^*z + \chi^*\chi)\big]\langle exp(z\alpha^* - z^*\alpha + \chi\beta^* - \chi^*\beta)\rangle. \tag{4.3.4}$$

Now using Eqs. (4.27) and (4.28), we have

$$\begin{aligned}\phi(z,\chi,t) = exp\big[-(z^*z+\chi^*\chi)\big]\langle\exp\big[&zE_+\alpha^*(0)+zE_-\beta(0)+z\sqrt{P}+z\eta^*_\alpha\\ &-z^*E_+\alpha(0)-z^*E_-\beta^*(0)-z^*\sqrt{P}-z^*\eta_\alpha\\ &+\chi E_+\beta^*(0)+\chi E_-\alpha(0)+\chi\sqrt{P}+\chi\eta^*_\beta\\ &-\chi^*E_+\beta(0)-\chi^*E_-\alpha^*(0)-\chi^*\sqrt{P}-\chi^*\eta_\beta\big]\rangle. \end{aligned} \tag{4.3.5}$$

It then follows that

$$\begin{aligned}\phi(z,\chi,t) = exp\big[-(z^*z+\chi^*\chi)\big]\langle\exp\big[&(\chi E_- - z^*E_+)\alpha(0)\\ &+(zE_+-\chi^*E_-)\alpha^*(0)+(zE_--\chi^*E_+)\beta(0)+(\chi E_+-\chi^*E_-)\beta(0)\big]\rangle\\ &\times\exp\big[(z-z^*+\chi-\chi^*)\sqrt{P}\big]\langle\exp\big[z\eta^*_\alpha-z^*\eta_\alpha+\chi\eta^*_\beta-\chi^*\eta_\beta\big]\rangle.\end{aligned} \tag{4.3.6}$$

Considering the cavity radiation to be initially in a two-mode vacuum state, we find

$$\begin{aligned}&\langle\exp\big[(\chi E_- - z^*E_+)\alpha(0)+(zE_+-\chi^*E_-)\alpha^*(0)\\ &+(zE_--\chi^*E_+)\beta(0)+(\chi E_+-z^*E_-)\beta^*(0)\big]\rangle = 1.\end{aligned} \tag{4.3.7}$$

In view of (4.102), Eq. (4.101) takes the form

$$\begin{aligned}\phi(z,\chi,t) &= exp\big[-(z^*z+\chi^*\chi)+(z-z^*+\chi-\chi^*)\sqrt{P}\big]\\ &\times\langle\exp\big[z\eta^*_\alpha-z^*\eta_\alpha+\chi\eta^*_\beta-\chi^*\eta_\beta\big]\rangle.\end{aligned} \tag{4.3.8}$$

We recall that $\eta_\alpha(t)$ and $\eta_\beta(t)$ are Gaussian variables with zero means. Thus employing the relation [1,3,16]

$$\langle\exp[\eta_\alpha+\eta_\beta]\rangle = \exp\frac{1}{2}\langle(\eta_\alpha+\eta_\beta)^2\rangle, \tag{4.3.9}$$

the characteristic function can be put in the form

$$\begin{aligned}\phi(z,\chi,t) &= exp\big[-(z^*z+\chi^*\chi)+(z-z^*+\chi-\chi^*)\sqrt{P}\big]\\ &\times\exp\frac{1}{2}\langle\big[z\eta^*_\alpha-z^*\eta_\alpha+\chi\eta^*_\beta-\chi^*\eta_\beta\big]^2\rangle\end{aligned} \tag{4.3.10}$$

or

$$\phi(z,\chi,t) = \exp\big[-(z^*z+\chi^*\chi)+(z-z^*+\chi-\chi^*)\sqrt{P}\big] \times \exp\big[\frac{z^2}{2}\langle\eta_\alpha^{*2}\rangle+\frac{z^{*2}}{2}\langle\eta_\alpha^2\rangle+\frac{\chi^2}{2}\langle\eta_\beta^{*2}\rangle+\frac{\chi^{*2}}{2}\langle\eta_\beta^2\rangle - zz^*\langle\eta_\alpha\eta_\alpha^*\rangle+z\chi\langle\eta_\alpha^*\eta_\beta^*\rangle-z^*\chi\langle\eta_\alpha\eta_\beta^*\rangle-z\chi^*\langle\eta_\alpha^*\eta_\beta\rangle + z^*\chi^*\langle\eta_\alpha\eta_\beta\rangle-\chi\chi^*\langle\eta_\beta\eta_\beta^*\rangle\big]. \quad (4.3.11)$$

On account of Eqs. (4.44), (4.45), (4.56), (4.66), (4.67), (4.70), and (4.71), we obtain

$$\phi(z,\chi,t) = \exp\big[-(1+\frac{Q+R}{2})(z^*z+\chi^*\chi)+\frac{Q-R}{2}(\chi z+\chi^*z^*) + \sqrt{P}(z-z^*+\chi-\chi^*)\big], \quad (4.3.12)$$

where P, Q, and R are defined by (4.29), (4.57), (4.58). Finally, we can also put the characteristic function in the form

$$\phi(z,\chi,t) = \exp\big[-a(z^*z+\chi^*\chi)+b(\chi z+\chi^*z^*)+c(z-z^*+\chi-\chi^*)\big], \quad (4.3.13)$$

in which

$$a = 1+\frac{Q+R}{2} = 1+\frac{\kappa N-\lambda-\kappa M}{2\lambda_+}(1-e^{-\lambda_+t})+\frac{\kappa N+\lambda+\kappa M}{2\lambda_-}(1-e^{-\lambda_{+-}t}), \quad (4.3.14)$$

$$b = \frac{Q-R}{2} = \frac{\kappa N-\lambda-\kappa M}{2\lambda_+}(1-e^{-\lambda_+t})-(\frac{\kappa N-\lambda-\kappa M}{2\lambda_-})(1-e^{-\lambda_-t}), \quad (4.3.15)$$

$$c = \sqrt{P} = \frac{2\varepsilon}{\lambda_+}(1-e^{-\lambda_+t}). \quad (4.3.16)$$

Now upon introducing (4.108) into (4.96) and carrying out the integration, the Q function for our system turns out to be

$$Q(\alpha,\beta,t) = \frac{1}{\pi^2}[u^2-v^2]\exp(-2cw)\exp\big[-u(\alpha^*\alpha+\beta^*\beta)+v(\alpha\beta+\alpha^*\beta^*) + w(\alpha^*+\alpha+\beta^*+\beta)\big], \quad (4.3.17)$$

in which

$$u = \frac{a}{a^2 - b^2}, \tag{4.3.18}$$

$$v = \frac{b}{a^2 - b^2}, \tag{4.3.19}$$

$$w = c(u - v). \tag{4.3.20}$$

4.3.2 The photon number distribution

The joint probability to find n photons of mode a and m photons of mode b can be written in terms of the Q function as [1,3]

$$P(n,m,t) = \frac{\pi^2}{n!m!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}\frac{\partial^{2m}}{\partial\beta^{*m}\partial\beta^m}\big[Q(\alpha,\beta,t)\exp(\alpha^*\alpha+\beta^*\beta)\big]_{\alpha^*=\alpha=\beta^*=\beta=0}, \tag{4.3.21}$$

so that on account of Eq.(4.112), we have

$$P(n,m,t) = \frac{[u^2-v^2]}{n!m!}e^{-2cw}\sum_{ijklpqrs}\frac{(1-u)^{i+j}v^{k+l}w^{p+q+r+s}}{i!j!k!l!p!q!r!s!} \times \frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}\frac{\partial^{2m}}{\partial\beta^{*m}\partial\beta^m}\big[\alpha^{i+k+P}{\alpha^*}^{i+l+q}\beta^{j+k+r}{\beta^*}^{j+l+s}\big]_{\alpha^*=\alpha=\beta^*=\beta=0}. \tag{4.3.22}$$

Carrying out the differentiation and applying the condition $\alpha^* = \alpha = \beta^* = \beta = 0$, we obtain

$$P(n,m,t) = \frac{[u^2-v^2]}{n!m!}e^{-2cw}\sum_{ijklpqrs}\frac{(1-u)^{i+j}v^{k+l}w^{p+q+r+s+}}{i!j!k!l!p!q!r!s!} \times \frac{(i+k+p)!}{(i+k+p-n)!}\frac{(i+l+q)!}{(i+l+q-n)!}\frac{(j+k+r)!}{(j+k+r-m)!}\frac{(j+l+s)!}{(j+l+s-m)!} \times \delta_{i+k+p,n}\delta_{i+l+q,n}\delta_{j+k+r,m}\delta_{j+l+r,m}. \tag{4.3.23}$$

For $m = n$ the photon number distribution takes the form

$$P(n,n,t) = [u^2-v^2]e^{-2cw}\sum_{ijkl}n!^2\frac{(1-u)^{i+j}v^{k+l}w^{4n-2i-2j-2k-2l}}{i!j!k!l!} \times \frac{1}{(n-i-k)!}\frac{1}{(n-i-l)!}\frac{1}{(n-j-k)!}\frac{1}{(n-j-l)!}\,. \tag{4.3.24}$$

Now in the absence of the coherent driving light modes ($\varepsilon = 0$), we have

$$w = c = 0. \tag{4.3.25}$$

We also note that $k = l = n - i = n - j$, therefore, the photon number distribution reduces to [3]

$$P(n, n, t) = [u^2 - v^2]e^{-2cw} \sum_j n!^2 \frac{(1-u)^{2j} v^{2(n-j)}}{j!^2 \big[(n-j)!\big]^2}. \tag{4.3.26}$$

It is also interesting to consider the photon number distribution in the absence the driving light modes ($\varepsilon = 0$) and the parametric interaction ($\lambda = 0$) at steady state. Thus on taking into account Eqs. (4.109), (4.110), (4.113), and (4.114), we find for this case

$$a^2 - b^2 = \cosh^2 r, \tag{4.3.27}$$

$$1 - u = 0, \tag{4.3.28}$$

$$v = \tanh r. \tag{4.3.29}$$

In view of these results, the photon number distribution (Eq. (4.121)) reduces to [3]

$$P(n, n) = \frac{\tanh^{2n}(r)}{\cosh^2(r)}. \tag{4.3.30}$$

This represents at steady state the photon number distribution of the cavity modes due to a two-mode squeezed vacuum. Furthermore, in the absence of the parametric interaction ($\lambda = 0$) and when the cavity is coupled to ordinary vacuum ($N = 0$ and $M = 0$), we get

$$1 - u = 0, \tag{4.3.31}$$

$$v = 0, \tag{4.3.32}$$

$$w = c. \tag{4.3.33}$$

One can also readily verify that

$$i = j = k = l = 0, \tag{4.3.34}$$

$$p = q = r = s = n. \tag{4.3.35}$$

Hence with the aid of these results, the photon number distribution (Eq. (4.119)) takes the form

$$P(n, n, t) = e^{-2c^2}\frac{c^{4n}}{n!^2}. \tag{4.3.36}$$

In view of Eq. (4.29) and (4.81), we see that the mean number of photons for the coherent light modes can be expressed as

$$\overline{n} = 2c^2, \tag{4.3.37}$$

so that application of (4.132) in Eq. (4.131) leads to

$$P(n, n, t) = \frac{1}{n!^2}e^{-\overline{n}}\left(\frac{\overline{n}}{2}\right)^{2n}. \tag{4.3.38}$$

This represents the photon number distribution for the two-mode coherent light.

4.4 The mean and variance of the number of photon pairs

Employing Eq. (3.1), one can write

$$\langle\hat{c}^\dagger\hat{c}\rangle = \frac{1}{2}\langle\hat{a}^\dagger\hat{a}\rangle + \frac{1}{2}\langle\hat{b}^\dagger\hat{b}\rangle + \frac{1}{2}\langle\hat{a}^\dagger\hat{b}\rangle + \frac{1}{2}\langle\hat{a}\hat{b}^\dagger\rangle. \tag{4.4.1}$$

Suppose we have a two-mode light for which

$$\langle\hat{a}^\dagger\hat{b}\rangle = \langle\hat{a}\hat{b}^\dagger\rangle = 0. \tag{4.4.2}$$

For this case, Eq. (4.134) takes the form

$$\langle\hat{c}^\dagger\hat{c}\rangle = \frac{1}{2}\langle\hat{a}^\dagger\hat{a}\rangle + \frac{1}{2}\langle\hat{b}^\dagger\hat{b}\rangle. \tag{4.4.3}$$

This can be interpreted as mean number of photon pairs.

In order to determine $\langle\hat{c}^\dagger\hat{c}\rangle$ represents the mean number of photon pairs for the system under consideration or not, we need to check whether Eq. (4.135) holds or not. Thus using the c-number variables associated with the normal ordering, one can write

$$\langle\hat{a}^\dagger\hat{b}\rangle = \langle\alpha^*(t)\beta(t)\rangle. \tag{4.4.4}$$

Applying Eqs. (2.59) and (2.60) along with the assumption that the cavity radiation is initially in a two-mode vacuum state, we obtain

$$\langle\hat{a}^\dagger(t)\hat{b}(t)\rangle = \langle F_+^* F_+^*\rangle + \langle F_-^* F_+^*\rangle - \langle F_+^* F_-^*\rangle - \langle F_-^* F_-^*\rangle, \tag{4.4.5}$$

where $F_\pm$ is defined by Eq. (2.62). Employing the complex conjugate of (2.62), we have

$$\begin{aligned}\langle F_+^* F_+^*\rangle = \frac{1}{4}\int_0^t e^{-\lambda_+(t-t'-t'')/2}\big[4\varepsilon^2 + 2\varepsilon(\langle f_\alpha^*(t')\rangle + \langle f_\beta(t')\rangle + \langle f_\alpha^*(t'')\rangle + \langle f_\beta(t'')\rangle +)\\ + \langle f_\alpha^*(t') f_\alpha^*(t'')\rangle + \langle f_\alpha^*(t') f_\beta(t'')\rangle + \langle f_\beta(t') f_\beta(t'')\rangle\\ + \langle f_\beta(t') f_\alpha^*(t'')\rangle\big]dt'dt''. \end{aligned}\tag{4.4.6}$$

On account of Eqs. (2.25), (2.44), and (2.45), we find

$$\langle F_+^* F_+^*\rangle = \varepsilon^2\int_0^t e^{-\lambda_+(t-t-t)/2}dt'dt''. \tag{4.4.7}$$

Upon performing the integration, we get

$$\langle F_+^* F_+^*\rangle = \frac{4\varepsilon^2}{\lambda_+^2}\big[1 - 2e^{-\lambda_+ t/2} + e^{\lambda_+ t}\big]. \tag{4.4.8}$$

Furthermore, using the complex conjugate of (2.62), one can write

$$\begin{aligned}\langle F_-^* F_-^*\rangle = \frac{1}{4}\int_0^t e^{-\lambda_-(t-t'-t'')/2}\big[\langle f_\alpha^*(t') f_\alpha^*(t'')\rangle + \langle f_\beta(t') f_\beta(t'')\rangle - \langle f_\beta(t') f_\alpha^*(t'')\rangle\\ - \langle f_\alpha^*(t') f_\beta(t'')\rangle\big]dt'dt''. \end{aligned}\tag{4.4.9}$$

With the aid of (2.44) and (2.45), we obtain

$$\langle F_-^* F_-^* \rangle = 0. \tag{4.4.10}$$

Moreover, it can also be established in a similar fashion that

$$\langle F_+^* F_-^* \rangle = \langle F_-^* F_+^* \rangle = 0. \tag{4.4.11}$$

Finally, substitution of (4.141), (4.143), (4.144) into (4.138) results in

$$\langle \hat{a}^\dagger \hat{b} \rangle = \frac{4\varepsilon^2}{\lambda_+^2}\big[1 - 2e^{-\lambda_+ t/2} + e^{-\lambda_+ t}\big]. \tag{4.4.12}$$

We also note that

$$\langle \hat{a}\hat{b}^\dagger \rangle = \frac{4\varepsilon^2}{\lambda_+^2}\big[1 - 2e^{-\lambda_+ t/2} + e^{-\lambda_+ t}\big]. \tag{4.4.13}$$

These results indicate that $\langle \hat{c}^\dagger \hat{c} \rangle$ does not represent the mean number of photon pairs for the system under consideration. Thus we note that $\langle \hat{c}^\dagger \hat{c} \rangle$ can represent the mean number of photon pairs for the system under consideration provided that $\varepsilon = 0$.

Next we proceed to calculate the mean number of photon pairs for the system under consideration in the absence of the coherent driving light modes. To this end, the mean photon number can also be expressed in terms of the c-number variables associated with the normal ordering as

$$\overline{n}_\gamma = \langle \gamma^*(t)\gamma(t) \rangle. \tag{4.4.14}$$

Using Eq. (3.10), we have

$$\gamma^*(t) = \frac{1}{2}\big[\gamma_+(t) + \gamma_-(t)\big], \tag{4.4.15}$$

$$\gamma^*(t) = \frac{1}{2}\big[\gamma_+(t) - \gamma_-(t)\big]. \tag{4.4.16}$$

Now with the aid of (4.148), and (4.149), Eq. (4.147) can be put in the form

$$\overline{n}_\gamma = \frac{1}{4}\big[\langle\gamma_+^2(t)\rangle - \langle\gamma_-^2(t)\rangle\big]. \tag{4.4.17}$$

For this case, employing Eq. (3.33) along with the assumption that the cavity radiation is initially in a two-mode vacuum state, we obtain

$$\langle\gamma_+^2(t)\rangle = \frac{2\kappa N - 2\lambda - \kappa M}{\lambda_+}(1 - e^{-\lambda_+ t}), \tag{4.4.18}$$

$$\langle\gamma_-^2(t)\rangle = \frac{2\kappa N - +2\lambda + \kappa M}{\lambda_-}(1 - e^{-\lambda_- t}). \tag{4.4.19}$$

Therefore, the mean number of photon pairs turns out to be

$$\overline{n}_\gamma = \frac{\kappa N - \lambda - \kappa M}{2\lambda_+}(1 - e^{-\lambda_+ t}) + \frac{\kappa N + \lambda + \kappa M}{2\lambda_-}(1 - e^{\lambda_- t}). \tag{4.4.20}$$

At steady state, we see that

$$(\overline{n}_\gamma)_{ss} = \frac{\kappa N - \lambda - \kappa M}{2\lambda_+} + \frac{\kappa N + \lambda + \kappa M}{2\lambda_-}. \tag{4.4.21}$$

When the cavity is coupled to ordinary vacuum ($N = 0$ and $M = 0$), Eq. (4.154) reduces to

$$(\overline{n}_\gamma)_{ss} = \frac{2\lambda^2}{\lambda_+\lambda_-}. \tag{4.4.22}$$

This represents the mean number of photon pairs produced by the parametric oscillator. Moreover, in the absence of the parametric interaction ($\lambda = 0$), Eq. (4.154) takes the form

$$(\overline{n}_\gamma)_{ss} = N. \tag{4.4.23}$$

This result describes the mean number of photon pairs due to the two-mode squeezed vacuum.

In the absence of the driving coherent light modes, the variance of the number of photon pairs is expressible as

$$\Delta n_\gamma^2 = \langle(\hat{c}^\dagger\hat{c})^2\rangle - \langle\hat{c}^\dagger\hat{c}\rangle^2. \tag{4.4.24}$$

Employing the commutation relation (2.10), Eq. (4.157) can be written in the normal order as

$$\Delta n_\gamma^2 = \langle \hat{c}^{\dagger 2}\hat{c}^2\rangle + \langle \hat{c}^\dagger \hat{c}\rangle - \langle \hat{c}^\dagger \hat{c}\rangle^2. \tag{4.4.25}$$

The variance can also be expressed in terms of c-number variables associated with normal ordering as

$$\Delta n_\gamma^2 = \langle \gamma^{*2}\gamma^2\rangle + \langle \gamma^*\gamma\rangle - \langle \gamma^*\gamma\rangle^2. \tag{4.4.26}$$

Now applying Eqs. (4.149) and (4.150), the variance can be put in the form

$$\Delta n_\gamma^2 = \frac{1}{16}\big[\langle \gamma_+^4(t)\rangle + \langle \gamma_-^4(t)\rangle - 2\langle \gamma_+^2(t)\gamma_-^2(t)\rangle\big] + \overline{n}_\gamma - \overline{n}_\gamma^2. \tag{4.4.27}$$

For the case in which the driving coherent light modes are absent, Eq. (3.19) can be rewritten as

$$\gamma_+ = \gamma_+(0)e^{-\lambda_+ t/2} + \frac{1}{\sqrt{2}}\int_0^t e^{-\lambda_+(t-t')/2}\big[f_\alpha^*(t') + f_\alpha(t') + f_\beta^*(t') + f_\beta(t')\big]dt', \tag{4.4.28}$$

$$\gamma_- = \gamma_-(0)e^{-\lambda_- t/2} + \frac{1}{\sqrt{2}}\int_0^t e^{-\lambda_-(t-t')/2}\big[f_\alpha^*(t') - f_\alpha(t') + f_\beta^*(t') - f_\beta(t')\big]dt'. \tag{4.4.29}$$

With the help of (4.118) and (4.119), one easily finds

$$\langle \gamma_+(t)\gamma_-(t)\rangle = 0. \tag{4.4.30}$$

In addition, Eqs. (4.161) and (4.162) together with (4.120) indicate that $\gamma_+(t)$ as well as $\gamma_-(t)$ is a Gaussian variable with zero mean. Thus on the basis of the relation (4.37), we obtain

$$\langle \gamma_\pm^4(t)\rangle = 3\langle \gamma_\pm^2(t)\rangle^2. \tag{4.4.31}$$

Furthermore, with the aid of Eqs. (4.151) and (4.152) along with the assumption that the cavity radiation is initially in a two-mode vacuum state, we get

$$\langle\gamma_{\pm}^{4}(t)\rangle = 3\Big[\frac{2\kappa N - 2\lambda - 2\kappa M}{\lambda_{+}}(1 - e^{-\lambda_{+}t})\Big]^{2}, \tag{4.4.32}$$

$$\langle\gamma_{-}^{4}(t)\rangle = 3\Big[-(\frac{2\kappa N + 2\lambda + \kappa M}{\lambda_{-}})(1 - e^{-\lambda_{-}t})\Big]^{2}. \tag{4.4.33}$$

Moreover, we recall that $\gamma_{+}(t)$ as well as $\gamma_{-}(t)$ is a Gaussian variable with vanishing mean. Thus on the basis of the relation (4.37), we obtain

$$\langle\gamma_{+}^{2}(t)\gamma_{-}^{2}(t)\rangle = \langle\gamma_{+}^{2}(t)\rangle\langle\gamma_{-}^{2}(t)\rangle + 2\langle\gamma_{+}(t)\gamma_{-}(t)\rangle^{2}. \tag{4.4.34}$$

On the other hand, applying Eqs. (4.161) and (4.162), we can write

$$\begin{aligned}
\langle\gamma_{+}(t)\gamma_{-}(t)\rangle = {} & \langle\gamma_{+}(0)\rangle e^{-(\lambda_{+}+\lambda_{-})t/2} \\
& + \frac{1}{\sqrt{2}}e^{-\lambda_{+}t/2}\int_{0}^{t} e^{-\lambda_{-}(t-t'')}\big[\langle\gamma_{+}(0)f_{\alpha}^{*}(t'')\rangle - \langle\gamma_{+}(0)f_{\alpha}(t'')\rangle \\
& + \langle\gamma_{+}(0)f_{\beta}^{*}(t'')\rangle - \langle\gamma_{+}(0)f_{\beta}(t'')\rangle\big]dt'' \\
& + \frac{1}{\sqrt{2}}e^{-\lambda_{-}t/2}\int_{0}^{t} e^{-\lambda_{+}(t-t'')}\big[\langle\gamma_{-}(0)f_{\alpha}^{*}(t')\rangle + \langle\gamma_{-}(0)f_{\alpha}(t')\rangle \\
& + \langle\gamma_{-}(0)f_{\beta}^{*}(t')\rangle + \langle\gamma_{-}(0)f_{\beta}(t')\rangle\big]dt'' \\
& + \frac{1}{2}\int_{0}^{t} e^{-[(\lambda_{+}+\lambda_{-})t-\lambda_{+}t'-\lambda_{-}t'']/2}\big[\langle f_{\alpha}^{*}(t')f_{\alpha}^{*}(t'')\rangle - \langle f_{\alpha}(t')f_{\alpha}(t'')\rangle \\
& + \langle f_{\beta}^{*}(t')f_{\beta}^{*}(t'')\rangle - \langle f_{\beta}(t')f_{\beta}(t'')\rangle \\
& + \langle f_{\alpha}^{*}(t')f_{\beta}(t'')\rangle - \langle f_{\alpha}(t')f_{\beta}^{*}(t'')\rangle + \langle f_{\beta}^{*}(t')f_{\alpha}(t'')\rangle - \langle f_{\beta}(t')f_{\alpha}^{*}(t'')\rangle \\
& + \langle f_{\alpha}^{*}(t')f_{\alpha}(t'')\rangle - \langle f_{\alpha}(t')f_{\alpha}^{*}(t'')\rangle + \langle f_{\beta}^{*}(t')f_{\beta}(t'')\rangle - \langle f_{\beta}(t')f_{\beta}^{*}(t'')\rangle \\
& + \langle f_{\beta}^{*}(t')f_{\alpha}^{*}(t'')\rangle - \langle f_{\beta}^{*}(t')f_{\alpha}(t'')\rangle - \langle f_{\alpha}^{*}(t')f_{\beta}^{*}(t'')\rangle - \langle f_{\alpha}(t')f_{\beta}^{*}(t'')\rangle\big]dt'dt''.
\end{aligned} \tag{4.4.35}$$

On account of Eqs. (2.39), (2.44), (2.45), (2.53), (2.54), and the assumption that the cavity radiation is initially in a two-mode vacuum state along with the fact that the noise force at times t' and t'' can not affect the system variables at the earlier time $t = 0$, we readily find

$$\langle \gamma_+(t)\gamma_-(t)\rangle = 0. \qquad (4.4.36)$$

Thus with the aid of Eqs. (4.151) , (4.152), and (4.169) along with the assumption that the cavity radiation is initially in a two-mode vacuum state, expression (4.170), can be put in the form

$$\langle \gamma_+^2(t)\gamma_-^2(t)\rangle = -4QR. \qquad (4.4.37)$$

Finally, substitution of (4.151), (4.152), (4.154), and (4.170) into (4.160) yields

$$\Delta n_\gamma^2 = \overline{n}_\gamma + \frac{1}{2}(Q^2 + R^2). \qquad (4.4.38)$$

On account of (4.57) and (4.58), the variance of the number of photon pairs at steady state takes the form

$$(\Delta n_\gamma^2)_{ss} = (\overline{n}_\gamma)_{ss} + \frac{1}{2}\big[(\frac{\kappa N - \lambda - \kappa M}{\lambda_+})^2 + (\frac{\kappa N + \lambda + \kappa M}{\lambda_-})^2\big]. \qquad (4.4.39)$$

For the case in which the cavity is coupled to ordinary vacuum ($N = 0$ and $M = 0$), Eq. (4.172) reduces to

$$(\Delta n_\gamma^2)_{ss} = \frac{2\lambda^2}{\lambda_+\lambda_-} + \frac{1}{2}\big[\frac{\lambda^2}{\lambda_+^2} + \frac{\lambda^2}{\lambda_-^2}\big]. \qquad (4.4.40)$$

This is the variance of the number of photon pairs due to the parametric oscillator. Furthermore, in the absence of the parametric interaction ($\lambda = 0$), Eq. (4.172) becomes

$$(\Delta n_\gamma^2)_{ss} = M^2 + N(N+1), \qquad (4.4.41)$$

so that on the basis of the relation (4.79), we find

$$(\Delta n_\gamma^2)_{ss} = 2N(N+1). \qquad (4.4.42)$$

This represents the variance of the number of photon pairs due to the two-mode squeezed vacuum.

4.5 The photon number distribution for photon pairs

4.5.1 The Q function

We now proceed to obtain the Q function for the number of photon pairs produced by the system under consideration in the absence of the driving coherent light modes. To this end , we note that the Q function is expressible in the form

$$Q(\gamma,t)=\frac{1}{\pi^2}\int d^2z\phi(z,t)\exp{(z^*\gamma-z\gamma^*)}, \qquad (4.5.1)$$

with the characteristic function $\phi(z,t)$ defined in the Heisenberg picture by

$$\phi(z,t)=Tr(\hat{\rho}(0)e^{-z^*\hat{c}(t)}e^{z\hat{c}^\dagger(t)}). \qquad (4.5.2)$$

Employing the identity (4.98), the characteristic function can be expressed in terms of c-number variables associated with the normal ordering as

$$\phi(z,t)=e^{-zz^*}\langle\exp z\gamma^*-z^*\gamma\rangle. \qquad (4.5.3)$$

Now using Eqs. (4.148) and (4.149), we have

$$\phi(z,t)=e^{-zz^*}\langle\exp\frac{1}{2}\big[(z-z^*)\gamma_+(t)+(z+z^*)\gamma_-(t)\big]\rangle. \qquad (4.5.4)$$

Furthermore, we recall that $\gamma_+(t)$ and $\gamma_-(t)$ are Gaussian variables with zero mean. Thus on the basis of the relation (4.104), Eq. (4.179) can be expressed as

$$\phi(z,t)=e^{-zz^*}\exp\frac{1}{4}\langle\big[(z-z^*)\gamma_+(t)+(z+z^*)\gamma_-(t)\big]^2\rangle. \qquad (4.5.5)$$

It then follows that

$$\phi(z,t)=exp\big[-(1+\frac{1}{2}\langle\gamma_+^2(t)\rangle-\frac{1}{2}\langle\gamma_-^2(t)\rangle)zz^* \\ +\frac{1}{4}(\langle\gamma_+^2(t)\rangle+\langle\gamma_-^2(t)\rangle)(z^2+z^{*2})+\frac{1}{2}(z-z^*)(z+z^*)\langle\gamma_+(t)\gamma_-(t)\rangle\big]. \qquad (4.5.6)$$

Hence on account of Eqs. (4.1531), (4.152), and (4.169), we obtain

$$\phi(z,t) = \exp\,[-Azz^* - \frac{B}{2}(z^2 + z^{*2})], \tag{4.5.7}$$

in which

$$A = 1 + \frac{\kappa N - \lambda - \kappa M}{\lambda_+}(1 - e^{-\lambda_+ t}) + \frac{\kappa N + \lambda + \kappa M}{\lambda_-}(1 - e^{-\lambda_- t}), \tag{4.5.8}$$

$$B = \frac{-\kappa N + \lambda + \kappa M}{\lambda_+}(1 - e^{-\lambda_+ t}) + \frac{\kappa N + \lambda + \kappa M}{\lambda_-}(1 - e^{-\lambda_- t}). \tag{4.5.9}$$

Finally, introducing (4.182) into (4.176) and performing the integration, the Q function for the number of photon pairs turns out to be

$$Q(\gamma,t) = [U^2 - V^2]^{\frac{1}{2}} \exp\,[-U\gamma^*\gamma - \frac{V}{2}(\gamma^2 + \gamma^{*2})], \tag{4.5.10}$$

where

$$U = \frac{A}{A^2 - B^2}, \tag{4.5.11}$$

$$V = \frac{B}{A^2 - B^2}. \tag{4.5.12}$$

4.5.2 The photon number distribution

The photon number distribution for the number of photon pairs is expressible in terms of the Q function as

$$P(n,t) = \frac{\pi}{n!}\frac{\partial^{2n}}{\partial\gamma^{*n}\partial\gamma^n}\big[Q(\gamma,t)e^{-\gamma^*\gamma}\big]_{\gamma^*=\gamma=0}. \tag{4.5.13}$$

Now substitution of (4.185) into (4.188) results in

$$P(n,t) = \frac{1}{n!}[U^2 - V^2]^{\frac{1}{2}}\frac{\partial^{2n}}{\partial\gamma^{*n}\partial\gamma^n}\exp\big[(1-U)\gamma^*\gamma - \frac{V}{2}(\gamma^2 + \gamma^{*2})\big]_{\gamma^*=\gamma=0}. \tag{4.5.14}$$

Upon expanding the exponential functions in power series, we have

$$P(n,t) = \frac{1}{n!}[U^2 - V^2]^{\frac{1}{2}}\sum_{prs}\frac{(-1)^{r+s}(1-U)^p(V)^{r+s}}{2^{r+s}p!r!s!}\frac{\partial^{2n}}{\partial\gamma^{*n}\partial\gamma^n}\big[\gamma^{*(p+2r)}\gamma^{p+2s}\big]_{\gamma^*=\gamma=0}, \tag{4.5.15}$$

so that on carrying out the differentiation and applying the condition $\gamma^* = \gamma = 0$, there follows

$$P(n,t) = \frac{1}{n!}[U^2 - V^2]^{\frac{1}{2}} \sum_{prs} \frac{(-1)^{r+s}(1-U)^p (V)^{r+s}}{2^{r+s} p! r! s!} \times \frac{(p+2r)!(p+2s)!}{(p+2r-n)!(p+2s-n)!} \delta_{p+2r,n} \delta_{p+2s,n}. \tag{4.5.16}$$

Furthermore, we observe that $p + 2r = n$ and $p + 2s = n$ we get

$$s = r, \tag{4.5.17}$$

$$p = n - 2r. \tag{4.5.18}$$

Hence on account of these results, the photon number distribution can be put in the form

$$P(n,t) = \frac{1}{n!}[U^2 - V^2]^{\frac{1}{2}} \sum_{r}^{[n]} n! \frac{(1-U)^{n-2r} V^{2r}}{2^{2r} r!^2 (n-2r)!}, \tag{4.5.19}$$

where $[n] = \frac{n}{2}$ for even n and $[n] = \frac{n-1}{2}$ for odd n.

5. CONCLUSION

In this thesis we have seen the simplicity with which the squeezing and the statistical properties of the light, produced by a nonodegenerate parametric oscillator with the cavity modes driven by coherent light and coupled to a two-mode squeezed vacuum reservoir, could be analyzed with the aid of stochastic differential equations.

Applying these equations, we have calculated the quadrature variance and the squeezing spectrum. Our results show that the quadrature variance and the squeezing spectrum for the cavity radiation have the same forms as that of the degenerate parametric oscillator with the cavity mode driven by coherent light and coupled to a squeezed vacuum reservoir. We have seen one effect of the two-mode squeezed vacuum is to increase the degree of squeezing of the radiation produced by the quantum optical system under consideration. In particular, at threshold and at steady state, the variance of the plus quadrature turns out to be the product of the variance pertaining to the nondegenerate parametric oscillator and the vacuum reservoir. This result is in complete agreement with the one found by K. Fesseha [3], obtained applying quantum Langevin equations. Hence at threshold and at steady state, the squeezing of the cavity radiation is greater than 50 %. We have also observed that at threshold and at steady state there is 100 % squeezing in the out put radiation at zero frequency. However, for $\omega \neq 0$ the squeezing spectrum depends also on the squeeze parameter r. In addition, we have found that the two-mode driving coherent light has no effect on the quadrature variance and squeezing spectrum.

On the other hand, employing the solution of the stochastic differential equations, we have determined the mean and the variance of the photon number. From the results obtained, we realized that the two-mode coherent light and the two-mode squeezed vacuum reservoir increase the mean and the variance of the photon number. Furthermore, applying the same solutions, we have obtained the antinormally ordered characteristic function. Using this characteristic function, we have determined the Q-function which is then used to calculate the photon number distribution. The special cases (4.121) and (4.125) are found to be in complete agreement with the results obtained by K. Fesseha [3]. Moreover, employing the same solutions we have calculated the mean and the variance for the photon number difference.

Finally we have determined the mean, the variance, and the photon number distribution for the number of photon pairs, in the absence of the two-mode driving light. It so turns out that our results have the same form as that of the signal-mode produced by a degenerate parametric oscillator coupled to a squeezed vacuum reservoir.

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DECLARATION

I hereby declare that this thesis is my original work and has not been presented for a degree in any other university. All sources of material used for the thesis have been duly acknowledged.

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