College of Natural and Computational Science
Department of Mathematics

Young tableaux and the representations of the symmetric group

By
Blen Belete

Advised by
Samuel Asefa (PHD)

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Addis Ababa, Ethiopia

Approved by the Board Examiner

Advisor : Samuel Asefa(PHD)

Signature:_______________ Date:_______________

Examiners

1. **Dr.Yirgalem Tsegaye** : Signature:_______________ Date:_______________

2. **Dr.Tilahun Abebaw** : Signature:_______________ Date:_______________

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Abstract

In this work, we look at the connection between Young tableaux and symmetric group representations. The design of Specht is discussed in detail. Two noteworthy findings presented and followed by instances are modules that are irreducible representations of $\mathcal{S}_n$, as well as many of the branching rule and Young's rule.

Acknowledgement

First and foremost, I want to express my eternal appreciation to **God** for allowing me to successfully complete the task. I was also extremely fortunate to have **Samuel Asefa (PHD)** as an advisor. Without his counsel and motivation, his endless endurance and his sure-handed guidance, I would not have been able to write my paper. I am grateful to **Fufa Beyene's** interesting conversations and useful comments, as well as his meticulous attention to detail in reading and editing the previous version of the paper. Special thanks to **Manalebish Debalke** for her tireless latex assistance.
Finally, I really would like to express my gratitude to all of my **friends** for their love and assistance.

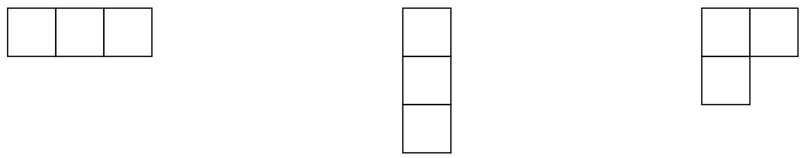
Chapter 1

Introduction

According to [8], in this project, we investigate a close relationship between two largely disparate objects: on the one hand, we have representations of $\mathbf{S}_n$, and on the other, we have *young tableaux*, which are fillings of a certain configuration of boxes with entries from $\{1, 2, ..., n\}$, an example of which is shown below. We provide a more detailed description of the term "young tableau" in chapter 2.

1	2	4
3	5	6
7	8	
9		

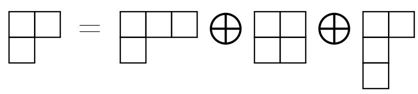
So, how are $\mathcal{S}_n$ representations connected to *young tableaux*? It turns out that Young tableaux may be used to describe irreducible representations of $\mathcal{S}_n$ in a fairly elegant way. Let's have a look at the findings. Keep in mind that $\mathcal{S}_3$ has three irreducible representations. It turns out that they can be explained using a series of three-box Young diagrams. Below is a diagram of the interaction.



trivial representation sign representation standard representation

In general, irreducible representations of $\mathcal{S}_n$ may be explained using Young diagrams of n boxes. Standard Young tableaux, on the other hand, are the box numberings of a Young diagram with $\{1, 2, ..., n\}$ such that the The number of rows and columns increase, may be used to explain a basis of any irreducible representation. The two standard Young tableaux, for example, relate to the bases of the standard representation of $\mathcal{S}_3$.

The dimension of irreducible representations may be easily determined from its Young diagram using the hook-length method, as we shall see in Chapter 4.
There are a number of additional unexpected links between Young tableaux and representations of $\mathcal{S}_n$, one of which is as follows. Consider the case when we have an irreducible representation in $\mathcal{S}_n$ and desire to determine its induced representation in $\mathcal{S}_{n+1}$.
The induced representation turns out to be nothing more than the direct sum of all the representations corresponding to the Young diagrams generated by adding a new square to the original Young diagram! The induced representation of the standard representation from $\mathcal{S}_3$ to $\mathcal{S}_4$, for example, is simply

$Ind_{s_3}^{s_4}$ 

Similarly, deleting a square from the Young diagram yields the restricted representation:

$$Res_{s_2}\ \ydiagram{2,1} = \ydiagram{2} \bigoplus \ydiagram{1,1}$$

Young tableaux and representations of $\mathcal{S}_n$ are discussed in this work. We will assume that you are familiar with the fundamentals of group representations, such as irreducible representations and characters. Chapter 4 makes use of induced representations. See [2], [4], or [7] for further information on group representations. Young diagrams and Young tableaux are introduced in Chapter 2. In Chapter 3, we introduce tabloids and show how to utilize them to create the **permutation module** M^λ representation of $\mathcal{S}_n$. Permutation modules, on the other hand, are usually reducible. In Chapter 4, we create **Specht modules** S^λ, which are irreducible representations of $\mathcal{S}_n$.
Specht modules S^λ create a full collection of irreducible representations by bijectively corresponding to Young diagrams. We discussed the Young lattice and the branching rule in Chapter 4 to derive the induced and limited representations of S^λ , and in this chapter we introduce Kostka numbers and assert a conclusion about the decomposition of permutation modules into irreducible Specht modules.

Chapter 2

Preliminaries

2.1 Representation theory

Definition 2.1.1. *A* ***representation*** φ *of a group* G *is a homomorphism* $\varphi : G \to GL(n, \mathbb{C})$*, where* $GL(n, \mathbb{C})$ *is the complex general linear group.*

The following step is to create a module. We can use modules to convert linear map issues into vector space problems. Working with vector spaces may be quite useful since it allows you to see some outcomes right away.

2.1.1 Module

Definition 2.1.2. *A G-module, or V, is a vector space for which a homomorphism exists.* $\varphi : G \to GL(V)$*, where* $GL(V)$ *denotes the collection of linear mappings* $V \to V$*.* V *is generally referred to as a module if the group is obvious.*

The representation φ induces a multiplication of G in V . Given $g \in G$ and $v \in V$, we define the operation $g * v$ as $\varphi(g)v$, which is a vector. This operation is linear. If $g, h, e \in G$, where e is the identity element, $\alpha \in \mathbb{C}$ and $v, w \in V$, then,
(1) $g * (v + w) = \varphi(g)(v + w) = \varphi(g)v + \varphi(g)w = g * v + g * w,$

(2) $\alpha(g * v) = \alpha(\varphi(g)v) = \varphi(g)(\alpha v) = g * (\alpha v)$,
(3) $(gh) * v = \varphi(gh)v = \varphi(g)\varphi(h)v = \varphi(g)(h * v) = g * h * v$,
(4) $e * v = \varphi(e)v = Iv = v$.

2.1.2 Group algebra

The set among all linear combinations of elements of G, denoted $\mathbb{C}$[G], is known as the **group algebra**. $\mathbb{C}[G] = \{c_1 g_1 + ... + c_n g_n \mid c_1, ..., c_n \in \mathbb{C}\}$ and $h * (c_1 g_1 + ... + c_n g_n) = c_1(hg_1) + ... + c_n(hg_n)$ for every $h \in G$.

2.2 The Symmetric Group

A finite group is formed by the set of all bijections $\{1, ..., n\} \to \{1, ..., n\}$ with composition of maps. This group is known as the symmetric group of degree n and is represented by the symbol $\mathcal{S}_n$. $\sigma \in \mathcal{S}_n$ can be represented by $\begin{pmatrix} 1 & ... & n \\ \sigma(1) & ... & \sigma(n) \end{pmatrix}$. We will take the convention of composing permutations from right to left (this is most natural as we will think of functions as acting from the left) and so taking $\Pi, \sigma \in \mathcal{S}_n$, we have $\Pi \cdot \sigma = \begin{pmatrix} 1 & ... & n \\ \pi(\sigma(1)) & ... & \pi(\sigma(n)) \end{pmatrix}$. Every $\sigma \in \mathcal{S}_n$ can be written as the product of disjoint cycles. We call these $k - cycles$. A 2-cycle is also called a $transposition$.

Cycle type

Definition 2.2.1. *Take $\sigma \in \mathcal{S}_n$ and replace $\sigma = \sigma_1, ..., \sigma_k$ with $\sigma_1, ..., \sigma_k$, where $\sigma_1, ..., \sigma_k$ are disjoint cycles with lengths $\lambda_1, ..., \lambda_k$. We can assume that $\lambda_1 \geq ... \geq \lambda_k$ is correct. Then $\lambda(\sigma) := (\lambda_1, ..., \lambda_k)$ is known as* ***the cycle type*** *of σ .*

Definition 2.2.2. *A composition on n is a non-negative integer sequence $\lambda = (\lambda_1, ..., \lambda_k)$ such that $\lambda_1 + ... + \lambda_k = n$. $(\lambda_1, ..., \lambda_k, 0)$ is how we identify λ. The integers $\lambda_1, ..., \lambda_k$ are referred to as* ***parts*** *of λ. C_n is the set of all n compositions.*

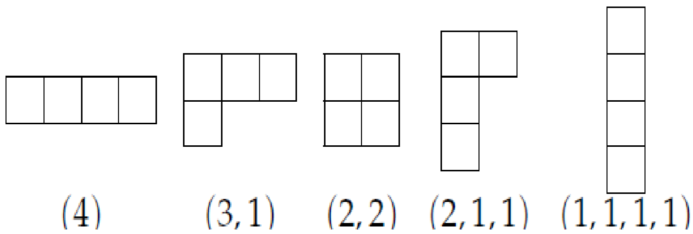
2.3 Partition and young diagram

Definition 2.3.1. *A partition of a positive integer n is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$ satisfies $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_l > 0$ and $n = \lambda_1 + \lambda_2 + ... + \lambda_l$. To signify that λ is a partition of n, we write $\lambda \vdash n$.*

For instance, the number 4 has five partitions: $(4), (3,1), (2,2), (2,1,1)$, $(1,1,1,1)$. Partitions can also be represented graphically using Young diagrams, as seen below.

Definition 2.3.2. *A Young diagram is a finite set of boxes organized in left-justified rows with weakly diminishing row widths. The Young diagram for the partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$ has l rows and λ_i boxes on the ith row.*

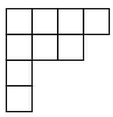
For instance, the Young diagrams corresponding to the partitions of 4 are



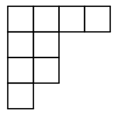
We use partitions and Young diagrams interchangeably since there is a clear one-to-one correspondence between the two words.

Definition 2.3.3. *If $\lambda \vdash n$, the conjugate partition λ^T of n is the one whose young diagram is the transpose of λ's young diagram.*

Example 2.3.1. *If $\lambda = (4,3,1,1)$, then $\lambda \vdash n$ and has young diagram*

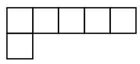
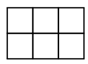
$\lambda =$ 

The transpose of the above diagram is

$\lambda^T =$ 

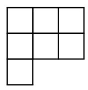
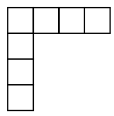
Definition 2.3.4. *Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$ and $\mu = (\mu_1, \mu_1, ..., \mu_m)$ be two partitions of n. We say that λ dominates μ, written $\lambda \trianglerighteq \mu$, if $\lambda_1+\lambda_2+...+\lambda_i \geq \mu_1 + \mu_2 + ... + \mu_i$ for all $i \geq 1$. If $i > l$ (respectively, $i > m$), then we take λ_i (respectively, μ_i) to be zero.*

Example 2.3.2. 1· λ = *(5,1)* *and* μ = *(3,3)*

$\lambda =$  $\mu =$ 

$$\lambda \trianglerighteq \mu$$

Example 2.3.3. 2· λ = *(3,3,1)* *and* μ = *(4,1,1,1)*

$\lambda =$  $\mu =$ 

$$\lambda \ntrianglerighteq \mu$$

In other words, $\lambda \trianglerighteq \mu$ if, for every k, the first k rows of the Young diagram of λ contains more squares than that of μ. Intuitively, this means that diagram for λ is short and fat and the diagram for μ is long and skinny. For example, when $n = 6$, we have $(3, 3) \trianglerighteq (2, 2, 1, 1)$. However, $(3, 3)$ and $(4, 1, 1)$ are incomparable, as neither dominates the other. The dominance relations for partitions of 6 is depicted using the figure 2.1. Such diagrams are known as **Hasse diagrams** which are used for representing partially ordered sets.

Proposition 2.3.1. [8] *M^μ contains S^λ as a sub representation if and only if $\lambda \trianglerighteq \mu$. Also, M^μ contains exactly one copy of S^μ.*

We might inquire as to how many copies of S^λ are included in M^μ. This solution has a good combinatorial meaning, it turns out. We'll need a few additional definitions to explain it.

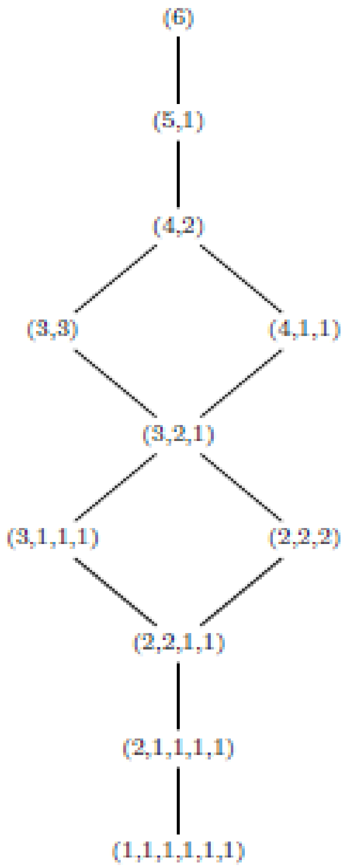


Figure 2.1: Hasse diagram - Dominance relations

2.4 Partition identities

We defined a partition λ of $n \in \mathbb{N}$ and presented its Young diagram in the preceding section. We will expand on the theory of partitions in this section, focusing on the intriguing relationship between generating function identities and bijective proofs.

Let's start with a description of a basic involution on the set of n partitions. If $\lambda \vdash n$, then define the conjugate partition λ' as the partition whose Young diagram is derived by interchanging rows and columns from that of λ'. Alternatively, the Young diagram of λ is a reflection of the λ diagram about the main diagonal. If $\lambda = (\lambda_1, \lambda_2, ...)$, then the number of parts of λ' that equal i

is $\lambda_i - \lambda_{i+1}$. This description of λ' provides a convenient method of computing λ' from λ without drawing a diagram. For instance, if $\lambda = (4, 3, 1, 1, 1)$ then $\lambda' = (5, 2, 2, 1)$.
Remember that the number of partitions of n into k parts is denoted by $p_k(n)$. Let $p_{\leq k}(n)$ represent the number of partitions of n into at most k parts, i.e. $p_{\leq k}(n) = p_0(n) + p_1(n) + ... + p_k(n)$. Now, λ has at most k parts if and only if λ' has at most k largest part.

The generating function $\sum_{n\geq 0} p_{\leq k}(n)q^n$ may be computed using this observation. A partition of n with a largest component of at most k can be considered a nonnegative integer solution to $m_1 + 2m_2 + ... + km_k = n$. Here, m_1 represents the number of times the component i appears in the partition $\lambda, i.e, \lambda = \langle 1^{m_1} 2^{m_2} ... k^{m_k} \rangle$. Hence

$$\begin{aligned}\sum_{n\geq 0} p_{\leq k}(n)q^n &= \sum_{n\geq 0} \sum_{m_1+2m_2+...+km_k=n} q^n \\ &= \sum_{m_1\geq 0}\sum_{m_2\geq 0} ... \sum_{m_k\geq 0} q^{m_1+2m_2+...+km_k} \\ &= \Big(\sum_{m_1\geq 0} q^{m_1}\Big)\Big(\sum_{m_2\geq 0} q^{2m_2}\Big)...\Big(\sum_{m_k\geq 0} q^{km_k}\Big) \\ &= \frac{1}{(1-q)(1-q^2)...(1-q^k)}\end{aligned}$$

The previous computation is simply a more exact expression of the intuitive truth that the most obvious approach of determining the coefficient of q^n in $1/(1-q)(1-q^2)...(1-q^k)$ involves calculating all n partitions with the biggest component being no more than k. We get the renowned generating function if we allow $k \to \infty$

$$\sum_{n\geq 0} p(n)q^n = \prod_{i\geq 1} \frac{1}{1-q^i}$$

The two equations above can be greatly generalized. Although not the most broad conceivable, the following result will work for our purposes.

Proposition 2.4.1. [5] *For each $i \in \mathbb{P}$, fix a set $S_i \subseteq \mathbb{N}$. Let $S = (S_1, S_2, ...)$, and define $\mathcal{P}(\mathcal{S})$ to be the set of all partitions λ such that if the part i occurs*

$m_i = m_i(\lambda)$ times, then $m_i \in \mathcal{S}_i$. Define the generating function in the variables $\boldsymbol{q} = (q_1, q_2, ...)$,

$$F(s,q) = \sum_{\lambda \in p(s)} q_1^{m_1(\lambda)} q_2^{m_2(\lambda)} ....$$

then

$$F(S,q) = \prod_{i \geq 1} \Big(\sum_{j \in S_i} q_i^j \Big).$$

Proof. To proof this proposition we have see the validity of this result by "inspection". The coefficient of $q_1^{m_1} q_2^{m_2}$... in

$$\prod_{i \geq 1} \Big(\sum_{j \in s_i} q_i^j \Big)$$

is 1 if each $m_i \in s_j$, and 0 otherwise, which yields the desired result. □

Corollary 2.4.1. *Preserve the notation of the previous proposition, and let $p(S,n)$ denote the number of partitions of n belonging to $P(S)$, that is,*

$$p(s,n) = \{\lambda \vdash n : \lambda \in p(s)\}.$$

Then

$$\sum_{n \geq 0} p(S,n) q^n = \prod_{i \geq 1} \Big(\sum_{j \in S_i} q^{ij} \Big).$$

Proof. Again to proof this corollary we have see the result by "inspection" like the previous proposition.

The coefficient of $q^{1m_1} q^{2m_2}$...in

$$\prod_{i \geq 1} \Big(\sum_{j \in s_i} q^{ij} \Big)$$

is 1 if each $m_i \in s_j$, and 0 otherwise which yields the desired result. □

Let us now have a look at some of the approaches and outcomes from

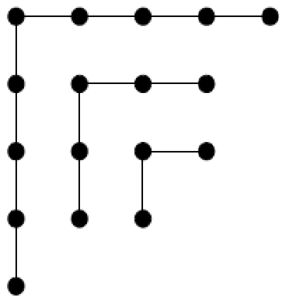


Figure 2.2: The diagonal hooks of the self-conjugate partition 54431

partition theory. First, we'll discuss why Young diagrams are useful.

Proposition 2.4.2. [5] *For any partition* $\lambda = (\lambda_1, \lambda_2, ...)$ *we have*

$$\sum_{i\geq 1}(i-1)\lambda_i \quad = \quad \sum_{i\geq 1}\binom{\lambda_i^{'}}{2}.$$

Proof. In the Young diagram of λ, place an $i-1$ in each square of row i. If $\lambda = 5322$, for example, we obtain

0	0	0	0	0
1	1	1		
2	2			
3	3			

The left-hand side of the above equation is obtained by adding all the numbers in the diagram by rows. We get the right-hand side if we add up the columns. □

Proposition 2.4.3. [5] *Let* $c(n)$ *denote the number of self-conjugate partitions* λ *of* $n, i.e., \lambda = \lambda^{'}$. *Then*

$$\sum_{n\geq 0} c(n)q^n \quad = \quad (1+q)(1+q^3)(1+q^5)....$$

Proof. Let λ be a partition that is self-conjugate. Consider the "diagonal hooks" of the $\lambda \vdash n$, Ferrers diagram for the partition $\lambda = 54431$. The number of dots in each hook partitions n into distinct odd parts, forming a partition μ. Figure 2.2: $\mu = 953$.

The bijection from self-conjugate partitions of n to partitions of n into separate odd portions is clearly observed in the map $\lambda \longmapsto \mu$. The proof now comes from Corollary 2.4.1's special case $S_i = \{0, 1\}$ if i is odd, and $S_i = \{0\}$ if i is even (though it should be apparent from examination that the right-hand side of proposition 2.4.3 is the generating function for the number of distinct odd parts of n). In the theory of partitions, several findings state the equicardinality of two classes of partitions. The following outcome serves as the quintessential illustration. □

Proposition 2.4.4. [5] *Let $q(n)$ represent the number of unique partitions of n and $p_{odd}(n)$ represent the number of odd partitions of n. For every $n \geq 0$, $q(n) = p_{odd}(n)$.*

Proof. (generating functions) Setting each $S_i = \{0, 1\}$ in Corollary 2.4.1 (or by direct inspection), we have

$$\begin{aligned}\sum_{n\geq 0} q(n)q^n &= (1+q)(1+q^2)(1+q^3)... \\ &= \frac{1-q^2}{1-q}\cdot\frac{1-q^4}{1-q^2}\cdot\frac{1-q^6}{1+q^3}... \\ &= \frac{\prod_{n\geq 1}(1-q^{2n})}{\prod_{n\geq 1}(1-q^n)} \\ &= \frac{1}{(1-q)(1-q^3)(1-q^5)...}.\end{aligned}$$

□

Again by Corollary 2.4.1 or by inspection, we have

$$\frac{1}{(1-q)(1-q^3)(1-q^5)...} = \sum_{n\geq 0} p_{odd}(n)q^n,$$

and the proof follows.

Proof. (First bijective) A combinatorial proof of such a simple and elegant result is, of obviously, desirable. The following is maybe the most basic.

Assume that λ is a partition of n into odd parts, with r_j occurrences of the

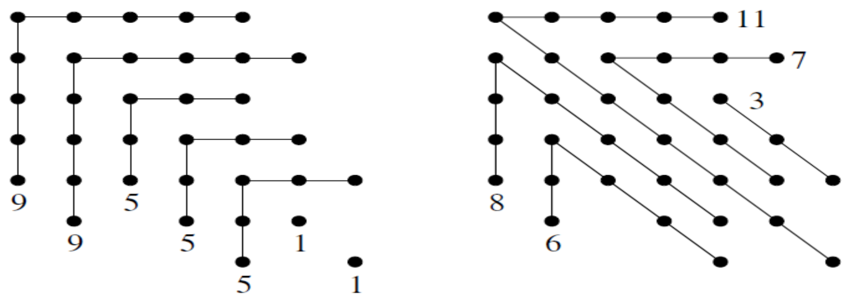


Figure 2.3: A second bijective proof that q(n) = $p_{odd}(n)$

part $2j-1$ Define a μ partition of n into distinct parts by ensuring that the part $(2j-1)2^k, k \geq 0$, occurs in μ only if and only if the binary expansion of r^j contains the term 2^k. The correctness of this bijection, which is based on the fact that every positive integer may be represented uniquely as the product of an odd positive integer and a power of 2. for example, $\lambda = \langle 9^5, 5^{12}, 3^2, 1^3 \rangle \vdash 114$, then

$$\begin{aligned} 114 &= 9(1+4) + 5(4+8) + 3(2) + 1(1+2) \\ &= 9 + 36 + 20 + 40 + 6 + 1 + 2 \end{aligned}$$

so $\mu = (40, 36, 20, 9, 6, 2, 1)$ □

Proof. (Second bijective) Another bijective proof, which is an excellent example of "diagram cutting," is totally different. Using the Ferrers diagram, divide a partition λ into odd parts. Convert each row of λ into a self-conjugate hook, and arrange these hooks in decreasing order diagonally. Now connect the upper left-hand corner u with all dots in the “shifted hook” of u, consisting of all dots directly to the right of u and directly to the southeast of u. For the dot v directly below u (when $|\lambda| \geq 1$), connect it to all the dots in the conjugate shifted hook of u. Connect the northwest-most remaining dot above the main diagonal to its shifted hook, and the northwest-most remaining dot below the main diagonal to its conjugate shifted hook. Continue partitioning the diagram into shifted hooks and conjugate shifted hooks until it's all shifted hooks and conjugate shifted hooks. The number of dots in these hooks determines which parts of a partition λ of n are distinct. $\lambda = 9955511$ and $\mu(11, 8, 7, 6, 3)$. We hope that this diagram clarifies the preceding rather vague description of the map $\lambda \mapsto \mu$. It's simple to verify that this map is a bijection from n partitions into odd parts to n partitions into distinct parts. There are several combinatorial identities that state that a product equals a sum that may be expressed in terms of partitions. Three of the most basic are listed here, with several more interesting and delicate identities relegated to the exercises. The second identity is linked to the idea of the *rank* rank(λ) of a partition $\lambda = (\lambda_1, \lambda_2, ...)$, defined as the largest i for which $\lambda_i \geq i$. In the (Ferrers or Young) diagram of λ, rank(λ) is the length of the main diagonal. It's also the length of the largest square in the λ diagram.

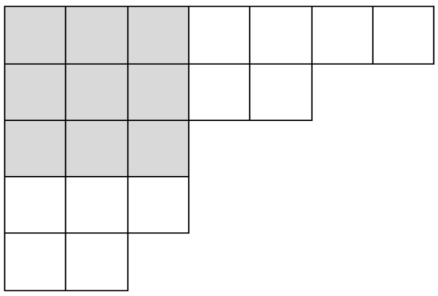


Figure 2.4: The durfee square of the partition 75332

This square can be used to include the first dot or box in the diagram's first row, in which case it is known as the *Durfee square* of λ. The Young diagram of the partition $\lambda = 75332$ of rank 3 is shown in Figure 2.4, with the Durfee square shaded. □

2.5 Young tableaux

Filling the boxes of a Young diagram with numbers yields a Young tableau.

Definition 2.5.1. *Assume that $\lambda \vdash n$. Filling up the boxes of a Young diagram of λ with $1, 2, ..., n$, with each number occurring precisely once yields*

a (Young) tableau t of form λ.
In this situation, t is referred to as a λ-tableau.

Here are all the tableaux matching to the partition $(2, 1)$, for example. :

1	2
3	

2	1
3	

1	3
2	

3	1
2	

2	3
1	

3	2
1	

Standard young tableaux

Definition 2.5.2. *A standard (Young) tableau is one in which the number of items increases in each row and column. The only standard tableaux for* $(2, 1)$ *are*

1	2
3	

and

1	3
2	

.

Two Young tableaux are shown below. The one on the right is standard, but not the one on the left.

4	1	2
3	5	

1	2	4
3	5	

The second is clearly a reordering of the first. In reality, $\mathbf{S}_n$ has the following effect on the set of all Young tableaux with the shape λ:

$(124) \cdot$

4	1	2
3	5	

$=$

1	2	4
3	5	

Another example of a standard tableau is as follows:

1	2	4
3	5	6
7	8	
9		

The concepts we employ here are based on those of Sagan [2], while other writers may use other terminology. A Young tableau, for example, is a filling that is weakly increasing across each row and strictly increasing down each column, but may contain repeated entries, according to Fulton [6]. Such tableaux are referred to as semi-standard tableaux, and they are used in Chapter 4.
Let us review some fundamental permutation facts before moving on. There is a decomposition into disjoint cycles for any permutation $\pi \in \mathcal{S}_n$. For instance, the permutation (123)(45) sends $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 1$ and swaps 4 and 5 (if n $>$ 5, the other elements are fixed by π). The partition whose parts are the lengths of the cycles in the decomposition is the cycle type of π. As a result, $(123)(45) \in \mathcal{S}_n$ has the cycle type $(3, 2)$. The fact that two elements of $\mathcal{S}_n$ are conjugates if and only if they have the same cycle type is a basic result. When the permutation is expressed in cycle notation, conjugation is just a relabeling of the elements. Indeed, if $\pi = (a_1 a_2 ... a_k)(b_1 b_2 ... b_l)...$, and σ sends x to $x^{'}$, then $\sigma\pi\sigma^{-1} = (a_1^{'} a_2^{'} ... a_k^{'})(b_1^{'} b_2^{'} ... b_l^{'})....$
This indicates that the cycle types characterize the conjugacy classes of $\mathcal{S}_n$, and that they correspond to partitions of n, which are equal to Young diagrams of size n. Remember that the number of irreducible representations of a finite group equals the number of its conjugacy classes from representation theory.As a result, the objective of the next two sections is to construct an irreducible representation of $\mathcal{S}_n$ for each Young diagram.

Semi-standard tableaux

Definition 2.5.3. *A shape-based tableau that's a little different from the norm. λ is an array T created by filling up the boxes of λ with positive integers, with repeats permitted, and in such a way that the rows weakly increase while the columns rigorously increase. The composition $\mu = (\mu_1, \mu_2, ..., \mu_m)$ is the content of T, with μ_i matching the number of i in T.*

The following semi-standard tableau, for example, has shape (4, 2) and content (2, 2, 1, 0, 1, 1):

1	1	2	5
2	3		
6			

The number of semi-standard tableau of a given type and content is known as the *Kostka* number.

Definition 2.5.4. *Let R be a ring. If there is a least positive integer n such that $na = 0$ for all $a \in R$, then R is said to have characteristic n. If no such n exists R is said to have characteristic 0.*

Chapter 3

Permutation module M^{λ}

3.1 Tabloid

We would like to have a look at several permutation representations of $\mathcal{S}_n$. The most obvious is $\mathcal{S}_n$ permutation action on the elements $1, 2, ..., n$, which extends to the defining representation. In this section, we use equivalence classes of tableaux, commonly known as *tabloids* to create alternative representations of $\mathcal{S}_n$.

Definition 3.1.1. *If the corresponding rows of the two λ-tableaux t_1 and t_2 have the identical items, they are* row-equivalent*, denoted $t_1 \sim t_2$. A* tabloid *of shapeλ, also known as a λ-tabloid, is an equivalence class represented by $\{t\} = \{t_1 \mid t_1 \sim t\}$, where t is a λ-tabloid. The tabloid t is written without vertical bars between the items inside each row, as in the tableaux t.*

For instance, if

$$\{t\} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

then $\{t\}$ is the tabloid drawn as

$$\begin{array}{ll} \hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}$$

which denotes the equivalence class including the two tableaux below :

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 3 \\ \cline{1-1} \end{array}$$

The notation is provocative in that it highlights that the order of the entries inside each row is irrelevant, allowing each row to be shuffled at will. As an instance,

$$\begin{array}{lll} \hline 1 & 7 & 9 \\ \hline 5 & 8 & \\ \cline{1-2} 2 & 6 & \\ \cline{1-2} \end{array} = \begin{array}{lll} \hline 7 & 9 & 1 \\ \hline 8 & 5 & \\ \cline{1-2} 6 & 2 & \\ \cline{1-2} \end{array} \neq \begin{array}{lll} \hline 7 & 9 & 1 \\ \hline 8 & 2 & \\ \cline{1-2} 6 & 5 & \\ \cline{1-2} \end{array} \neq \begin{array}{lll} \hline 7 & 9 & 1 \\ \hline 6 & 5 & \\ \cline{1-2} 8 & 2 & \\ \cline{1-2} \end{array}$$

We aim to create a representation of $\mathcal{S}_n$ on a vector space with the set of tabloids of a particular shape as its basis. We need to figure out how elements of $\mathcal{S}_n$ can react with tabloids. We can achieve this in the most apparent way by allowing the permutations to permute the tabloid's entries. For instance, the cycle $(123) \in \mathcal{S}_3$ acts on a tabloid by replacing its "1" with a "2," its "2" with a "3," and its "3" with a "1," as illustrated below:

$$(123)\begin{array}{ll} \hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array} = \begin{array}{ll} \hline 2 & 3 \\ \hline 1 & \\ \cline{1-1} \end{array}$$

If t includes a standard λ-tableau, we call it a standard λ-tabloid.

Example 3.1.1. *Consider $\lambda \vdash n$ and $\lambda = (3,2)$*

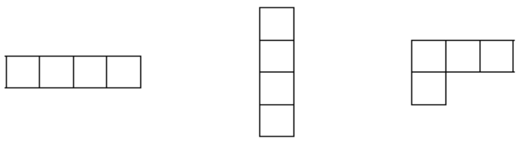
If $t = \begin{array}{|c|c|c|} \hline 2 & 3 & 5 \\ \hline 1 & 4 \\ \cline{1-2} \end{array}$, *then* $\{t\} = \begin{array}{lll} \hline 2 & 3 & 5 \\ \hline 1 & 4 & \\ \cline{1-2} \end{array} = \begin{array}{lll} \hline 3 & 2 & 5 \\ \hline 4 & 1 & \\ \cline{1-2} \end{array}$. *Here,* $\{t\}$ *is not a standard* $\lambda-$*tabloid.*

If $t = \begin{array}{lll} \hline 3 & 1 & 5 \\ \hline 4 & 2 & \\ \cline{1-2} \end{array}$, *then* $\{t\} = \begin{array}{lll} \hline 3 & 1 & 5 \\ \hline 4 & 2 & \\ \cline{1-2} \end{array} = \begin{array}{lll} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \cline{1-2} \end{array}$. *Here,* $\{t\}$ *is a standard* $\lambda-$*tabloid.*

We should check that this action is properly defined, that is, if t_1 and t_2 are row-equivalent, so that $\{t_1\} = \{t_2\}$, then the permutation outcome should be the same, that is, $\pi\{t_1\} = \pi\{t_2\}$. This is obvious since π simply instructs the user to move a number from one row to the next.
Now that we have defined a way for S_n to act on tabloids, we are ready to define a representation of $\mathcal{M}^\lambda$. Recall that a representation of a group G on a complex vector space V is equivalent to extending V to a $\mathbb{C}[G]$-module, so we often use the term *module* to describe representations.

Definition 3.1.2. *Assume that $\lambda \vdash n$. Let M^λ represent the vector space with the set of λ-tabloids as its basis. Then M^λ is the permutation module corresponding to λ, which is a representation of S_n.*

Let's look at some permutation modules in action. The M^λ corresponding to the following Young diagrams are, as we can see, common representations.



Example 3.1.2. *Consider $\lambda = (n)$. We see that M^λ is the vector space generated by the single tabloid*

$$\boxed{1 \;\; 2 \;\; \cdots \;\; n}\,.$$

We can see that $M^{(n)}$ is the one-dimensional trivial representation since this tabloid is fixed by $\mathcal{S}_n$.

Example 3.1.3. *Consider $\lambda = (1^n) = (1, 1, ..., 1)$. Then a λ-tabloid is simply a permutation of $\{1, 2, ..., \}$ into n rows and $\mathcal{S}_n$ acts on the tabloids by acting on the corresponding permutation. It follows that $M^{(1^n)}$ is isomorphic to the regular representation $\mathbb{C}[\mathcal{S}_n]$.*

Example 3.1.4. *Consider $\lambda = (n-1, 1)$. Let $\{t_i\}$ be the λ-tabloid with i on the second row. Then M^λ has basis $\{t_1\}, \{t_2\}, ..., \{t_n\}$. Also, note that the action of $\pi \in \mathcal{S}_n$ sends t_i to $t_\pi(i)$. And so $M^{(n-1,1)}$ is isomorphic to the defining representation $\mathbb{C}\{1, 2, ..., n\}$. For example, in the $n = 4$ case, the representation $M^{(3,1)}$ has the following basis:*

$$\{t_1\} = \begin{array}{lll} \hline 2 & 3 & 4 \\ \hline 1 & & \\ \cline{1-1} \end{array}, \ \{t_2\} = \begin{array}{lll} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \cline{1-1} \end{array}, \ \{t_3\} = \begin{array}{lll} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \cline{1-1} \end{array}, \ \{t_4\} = \begin{array}{lll} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \cline{1-1} \end{array}$$

Now we will look at the dimension and characters of the M^λ representation. First, we will calculate the number of tabloids in each shape using a formula.

Proposition 3.1.1. [8] *If $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$,*

$$dim M^\lambda = \frac{n!}{\lambda_1!\lambda_2!...\lambda_l!}$$

Proof. Since the basis for M^λ is the set of λ-tabloids, the dimension of M^λ is equal to the number of distinct λ-tabloids.
So let us count the number of λ-tabloids.
Since there are $\lambda_i!$ ways to permute the i^{th} row, the number of tableaux in each row equivalence class is $\lambda_1!\lambda_2!...\lambda_l!$. Since there are $n!$ tableaux in total, the number of equivalence classes is given by $\frac{n!}{\lambda_1!\lambda_2!...\lambda_l!}$ □

Example 3.1.5. *n=3,λ =(2,1)*

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 3 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 2 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 3 & 2 \\ \hline 1 \\ \cline{1-1} \end{array}$$

are tableau.
Tabloid of this are

$$\begin{array}{ll} \hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}, \begin{array}{ll} \hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array}, \begin{array}{ll} \hline 2 & 3 \\ \hline 1 & \\ \cline{1-1} \end{array}, \begin{array}{ll} \hline 2 & 1 \\ \hline 3 & \\ \cline{1-1} \end{array}, \begin{array}{ll} \hline 3 & 1 \\ \hline 2 & \\ \cline{1-1} \end{array}, \begin{array}{ll} \hline 3 & 2 \\ \hline 1 & \\ \cline{1-1} \end{array}.$$

From this 1^{st} and 4^{nd}, 2^{nd} and 5^{th} and 3^{rd} and 6^{th} are equivalent. We observe that there are 3 distinct tabloid corresponding to the tableau.

Using the formula,

$$\frac{n!}{\lambda_1!, \lambda_2!, ...\lambda_l!} = \frac{3!}{2!1!} = \frac{3 \cdot 2!}{2!} = 3.$$

Hence there are 3 distinct tabloids.

Proposition 3.1.2. [8] *Suppose $\lambda = (\lambda_1, \lambda_2, ...\lambda_l), \mu = (\mu_1, \mu_2, ...\mu_m)$ are partitions of n. The character of M^λ evaluated at an element of $\mathcal{S}_n$ with cycle type μ is equal to the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} ... x_l^{\lambda_l}$ in*

$$\prod_{i=1}^{m}(x_1^{\mu_i} + x_2^{\mu_i} + ... + x_l^{\mu_i}).$$

Example 3.1.6. *Let's compute the whole set of permutation module characters for $\mathcal{S}_4$. The dimension is equal to the character at the identity element, which can be determined using Proposition 3.1.1. For instance, the character of $M^{(2,1,1)}$ at e $\in \mathcal{S}_4$ is $4!/2! = 12$. we want to compute the character of $M^{(2,2)}$ at the permutation (12), which has cycle type (2, 1, 1). Using Proposition 3.1.2, we see that the character is equal to the coefficient of $x_1^2 x_2^2$ in $(x_1^2 + x_2^2)(x_1 + x_2)^2$, which is 2. Other characters may be calculated in the same way, and the results are presented in the table below.*

Permutation cycle type	e (1,1,1,1)	(12) (2,1,1)	(12)(34) (2,2)	(123) (3,1)	(1234) (4)
$M^{(4)}$	1	1	1	1	1
$M^{(3,1)}$	4	2	0	1	0
$M^{(2,2)}$	6	2	2	0	0
$M^{(2,1,1)}$	12	2	0	0	0
$M^{(1,1,1,1)}$	24	0	0	0	0

Lemma 3.1.1. *(Dominance Lemma for Partitions) Let t^λ and s^μ be tableaux of shape λ and μ, respectively. If, for each index i, the elements of row i of s^μ are all in different columns in t^λ , then $\lambda \trianglerighteq \mu$.*

Proof. By hypothesis, we can sort the entries in each column of t^λ so that the elements of rows 1, 2, ... , i of s^μ all occur in the first i rows of t^λ. Thus

$$\begin{aligned} \lambda_1 + \lambda_2 + ... + \lambda_i &= number\ of\ elements\ in\ the\ first\ i\ rows\ of\ t^\lambda. \\ &\geq number\ of\ elements\ of\ s^\mu\ in\ the\ first\ i\ rows\ of\ t^\mu \\ &= \mu_1 + \mu_2 + ... + \mu_i. \end{aligned}$$

□

Lemma 3.1.2. *(Dominance Lemma for Tabloids) If $k < l$ and k appears in a lower row than l in $\{t\}$, then*

$$\{t\} \quad \triangleleft \quad (k,l)\{t\}.$$

Proof. Suppose that $\{t\}$ and (k,l)$\{t\}$ have composition sequences λ^i and μ^i respectively. Then for $i < k$ or $i \geq l$ we have $\lambda^i = \mu^i$. Now consider the case where $k \leq i < l$. If r and q are the rows of $\{t\}$ in which k and l appear respectively, then
$\lambda^i = \mu^i$ with the q^{th} part decreased by 1 and the r^{th} part increased by 1. Since $q < r$ by assumption, $\lambda^i \triangleleft \mu^i$.
If $v = \sum_i C_i\{t_i\} \in M^\mu$, then we say that $\{t_i\}$ appears in v if $C_i \neq 0$.
In a typical polytabloid, the dominance lemma limits the number of tableaux that can occur.

□

Chapter 4

Specht Modules $\mathcal{S}^\lambda$

In the previous chapter, we constructed permutation modules, which are representations of M^λ of $\mathcal{S}_n$. We explore at an irreducible sub representation of M^λ that only relates to λ in this chapter.
The group $\mathcal{S}_n$ has the following effect on the set of Young tableaux: for a tableaux t of size n and a permutation $\pi \in \mathcal{S}_n$, the tableaux πt is the tableaux that places the integer $\pi(i)$ in the box where t places i. As an example,

$$(123)(45)\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 & 6 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} \quad = \quad \begin{array}{|c|c|c|c|}\hline 2 & 3 & 5 & 4 \\ \hline 1 & 6 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array}$$

Observe that the permutations that simply permute the entries of the rows among themselves fix a tabloid. The row group is a subgroup of $\mathcal{S}_n$ that contains these permutations. The column group can be defined in the same way.

4.1 Poly-tabloid

Definition 4.1.1. *The row stabilizer (group) of a tableau t of size n, represented by the row group of t, termed $\mathcal{R}_t$, is a subgroup of $\mathcal{S}_n$ consisting of permutations that exclusively permutes the elements inside each row of t for a tableau t of size n. Similarly, the column stabilizer (group) $\mathcal{C}_t$ is a* *permutation-only subgroup of $\mathcal{S}_n$ that only permutes the elements inside each column of t.*

If, for example,

$$t = \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}$$

then
$\mathcal{R}_t = S_{\{1,2,4\}} \times S_{\{3,5\}}$ *and* $\mathcal{C}_t = S_{\{3,4\}} \times S_{\{1,5\}} \times S_{\{2\}}$. Let us select certain elements from the space M^λ that we use to span a subspace.

Definition 4.1.2. *Let $\lambda \vdash n$, and let t be a λ-tableau with column stabilizer $\mathcal{C}_t$. Set*

$$\mathrm{k}_t := \sum_{\pi \in \mathcal{C}_t} sgn(\pi)\pi.$$

$$\mathrm{e}_t := k_t\{t\}$$

We call e_t *a λ- polytabloid.*

For the tableau t,

$$t = \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}$$

we have

$$\begin{aligned} c_t &= \{(1),(15),(34),(34)(15)\} \\ k_t &= \{(1)-(15)-(34)+(34)(15)\} \end{aligned}$$

Then the poly-tabloid for t is calculated to be

$$\begin{aligned} e_t &= k_t\{t\} = [(1)-(15)-(34)+(34)(15)]\begin{array}{ccc}\hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \\ &= \begin{array}{ccc}\hline 4 & 1 & 2 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} - \begin{array}{ccc}\hline 4 & 5 & 2 \\ \hline 3 & 1 \\ \cline{1-2}\end{array} - \begin{array}{ccc}\hline 3 & 1 & 2 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} + \begin{array}{ccc}\hline 3 & 5 & 2 \\ \hline 4 & 1 \\ \cline{1-2}\end{array} \end{aligned}$$

Example 4.1.1. $t = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & & \end{array}$

$e_t = sgn((1))(1) * \left\{\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & & \end{array}\right\} + sgn((15))(15) * \left\{\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & & \end{array}\right\} +$
$sgn((15)(26))(15)(26) * \left\{\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & & \end{array}\right\} + sgn((26))(26) * \left\{\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & & \end{array}\right\}$
$= \left\{\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & & \end{array}\right\} - \left\{\begin{array}{cccc} 5 & 2 & 3 & 4 \\ 1 & 6 & & \end{array}\right\} + \left\{\begin{array}{cccc} 5 & 6 & 3 & 4 \\ 1 & 2 & & \end{array}\right\} - \left\{\begin{array}{cccc} 1 & 6 & 3 & 4 \\ 5 & 2 & & \end{array}\right\}$

We can now demonstrate that $\mathcal{S}_n$ permutes the set of poly-tabloids using the following technical lemma.

Lemma 4.1.1. *Consider the tableau t and the permutation π. Then $\mathrm{e}_{\pi t} = \pi \mathrm{e}_t$ occurs.*

Proof.

$$\begin{aligned} \sigma \in k_{\pi t} &\Leftrightarrow \sigma\{\pi t\} = \{\pi t\} \\ &\Leftrightarrow \pi^{-1}\sigma\pi\{t\} = \{t\} \\ &\Leftrightarrow \pi^{-1}\sigma\pi \in k_t \\ &\Leftrightarrow \sigma \in \pi k_t \pi^{-1} \end{aligned}$$

from this

$$\begin{aligned} e_{\pi t} &= k_{\pi t}\{\pi t\} \\ &= \pi k_t \pi^{-1}\{\pi t\} \\ &= \pi k_t \pi^{-1}\pi\{t\} \\ &= \pi k_t\{t\} \\ &= \pi e_t \end{aligned}$$

□

Definition 4.1.3. *The equivalent Specht module, denoted S^λ, for any partition λ, is the submodule of M^λ spanned by the poly-tabloids e_t, where t is taken across all tableaux of form λ.*

Example 4.1.2. *Take the equation $\lambda = (n)$ for example. There is just one form of poly-tabloid, which is*

$$\overline{\underline{1 \quad 2 \quad . \quad . \quad . \quad n}}$$

We can see that $S^{(n)}$ is the one-dimensional trivial representation since this poly-tabloid is fixed by $\mathcal{S}_n$.

Example 4.1.3. *Consider $\lambda = 1^{(n)} = (1, 1, ..., 1)$. Let*

1
2
...
n

Note that e_t is equal to the total of all λ-tabloids multiplied by the sign of permutation used to get there. For any other λ-tableau, use t'. If t' is acquired from t by an even permutation, we have $e_t = e_{t'}$, and if t' is gotten from t through an odd permutation, we have $e_t = -e_{t'}$. So S^λ is a one-dimensional representation. From Lemma 4.1.1 we have $\pi e_t = e_{\pi t} = sgn(\pi)e_t$. From this we see that $S^{(1^n)}$ is the sign representation.

Example 4.1.4. *Consider the case where $\lambda = (n-1, 1)$. We can see that the poly-tabloids take the form $\{t_i\} - \{t_j\}$ by continuing the notation from Example 3.1.4, where we used $\{t_i\}$ to represent the $\lambda - tabloid$ with i in the second row. In fact, the tableau was used to create a poly tabloid*

i	a	b	$\cdots$
j			

is equal to $\{t_i\} - \{t_j\}$. Let us temporarily use e_i to denote the tabloid $\{t_i\}$. Then S^λ is spanned by elements of the form $e_i - e_j$, and it follows that

$$s^{(n-1,1)} = \{c_1e_1 + c_2e_2 + ... + c_ne_n \,|c_1 + c_2 + ... + c_n = 0\}$$

This is the standard representation, which is an irreducible representation. The defining representation is equal to the direct sum of the standard and *trivial representations, which is* $S^{(n-1,1)} \oplus S^{(n)} = M^{(n-1,1)}$. *We know there are three irreducible representations of the* S_3*: trivial, sign, and standard. These are the same as the ones mentioned previously. In addition, there are precisely three* 3 *partitions:*$(3), (1,1,1)$, *and* $(2,1)$. *As a result, the irreducible representations in this situation are the Specht modules. Surprisingly, this holds true in every case.*

Theorem 4.1.1. [8] *A full set of irreducible representations of* S_n *over* $\mathbb{C}$ *is formed by the Specht modules* S^λ *for* $\lambda \vdash n$.

Proof. It is sufficient to show that the Specht modules are all unique since they are indexed by S_n conjugacy classes and the number of irreducible S_n -modules is the same. Assume that $S^\lambda \cong S^\mu$. Then $\mathrm{Hom}(S^\lambda, S^\mu) \subseteq Hom(S^\lambda, M^\mu)$. Because $Hom(S^\lambda, S^\mu)$ is nonempty because it contains an isomorphism, $Hom(S^\lambda, M^\mu)$ is also nonempty. Then, according to Dominance Lemma for partition, $\lambda \trianglerighteq \mu$. When we reverse the roles of λ and μ, we get $\mu \trianglerighteq \lambda$, therefore $\lambda = \mu$. As a consequence, all n partitions' Specht modules are distinct. □

Theorem 4.1.2. [8] *Assume that* λ *is any partition. The set*

$$\{e_t : t \, is \, a \, standard \, \lambda - tableau\}$$

forms a basis for S^λ *as a vector space.*

Proof. Let us show standard polytabloid spans S^λ to demonstrate the standard polytabloid form as a basis for S^λ.

If e_t is in the span of the set of standard polytabloid, then e_s is in the span of any $s \in [t]$, according to the statements at the beginning of this section. As a consequence, we may assume that t has growing columns at all times. The poset of column tabloids has a maximum element of $[t]$, which is determined by counting the cells of each column successively from top to bottom, starting with the leftmost column and working right. Because $[t]$ is a standard, we're done with this equivalence class.

Select any tableau t now. We may infer that every tableau s with [s] $\rhd$ [t] is in the span of the standard polytabloid set via induction. We have finished if t is standard. If there isn't standard, then one must be one in some row i except row 1 and column 1. (since columns increase). Take A $= \{a_i, ..., a_p\}$ and B $= \{b_1, ..., b_i\}$ with associated Garnir element $g_{A,B} = \sum_{\pi} sgn(\pi)\pi$, By proposition 2.6.3 in[2]. We have $g_{A,B}e_t = 0$, so that

$$e_t \;=\; -\sum_{\pi \neq \epsilon} sgn(\pi)e_{\pi t}$$

But $b_1 < ... < b_i < a_i < ... < a_p$ implies $[\pi t] \trianglerighteq$ [t] for $\pi \neq \epsilon$. by the column analogue of the dominance lemma for tabloids.
Thus all terms on the right side of the above equation and hence e_t itself, are in the span of the standard polytabloids. Hence e_t, if t is standard of λ-tableau forms a basis for S^{λ}.

□

Corollary 4.1.1. *Let F be an (algebraically closed) field with char$(F) \nmid |\ \mathcal{S}_n\ |$. Then, for the isomorphism classes of simple $F\mathcal{S}_n$-modules, $\{S_F^{\lambda} : \lambda \vdash n\}$ is a transversal.*

Theorem 4.1.3. [7] *(Wedderburn).Let G be a finite group, and let F be a group of char$(F) \nmid |\ G\ |$. Let $\{s_1, ..., s_n\}$ be a transversal for the isomorphism classes of simple FG-modules. Then*

$$|\ G\ | \;=\; \sum_{i=1}^{n} (dim(s_i))^2.$$

Theorem 4.1.4. [8] *If $\lambda \vdash n$, then an F-basis of S^{λ} is formed by all standard λ-polytabloids. $dim(S^{\lambda}) = f^{\lambda}$, in particular.*

Proof. We first prove linear independence. Suppose for a contradiction that $\sum \alpha_t e_t = 0$ (call this equation $(*)$),$\alpha_t \in F$, not all zero, where the sum runs over all standard λ-talbeaux t. Let $\{\bar{t}$ be maximal with respect to " $<$ " such that $\bar{t}$ is standard and $\alpha_{\bar{t}} \neq 0$ By the above remark, the coefficient of $\{\bar{t}\}$ in $(*)$ is $\alpha_{\bar{t}}$. But the λ-tabloids are an F-basis of M^{λ}, and hence $\alpha_{\bar{t}} = 0$.

As a result of this contradiction, we have linear independence. We can now show that this set is a generating set. Wedderburn's theorem, along with the preceding statement (corollary) that the Specht modules form a transversal for the isomorphism classes of simple $char(F) \nmid \mid \mathfrak{S}_n \mid$,-modules, create a transversal for the isomorphism classes of simple $F\mathfrak{S}_n$-modules. we have

$$n! \;=\; \mid \mathcal{S}_n \mid = \sum_{\lambda \vdash n} (dim(S^\lambda))^2.$$

From the linear independence, we know that $dim(S^\lambda) \geq f^\lambda$. What remains to be shown is that

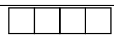
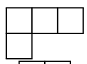
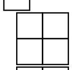
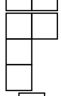
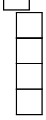
$$n! \;=\; \sum_{\lambda \vdash n} (f^\lambda)^2.$$

Therefore,

$$dim(S^\lambda) \;=\; f^\lambda = \sum_{i=1}^{m} (f^{\lambda_i}) = \sum_{i=1}^{m} dim(S^{\lambda_i}).$$

□

Example 4.1.5. *The table below shows the $n = 4$ partition, as well as the young diagram and f^λ for each.*

λ	l	f^λ
(4)		1
(3,1)		3
(2,2)		2
(2,1,1)		3
(1,1,1,1)		1

Definition 4.1.4. *Consider the Young diagram λ. We define the hook of u (or at u) for a square u on the diagram (denoted by $u \in \lambda$) as the set of all*

squares immediately to the right of u or directly below u, including u itself. The number of squares in the hook is known as the hook − length of u (or at u), and it is represented by $h_{\lambda}(u)$.

Take, for example, the partition $\lambda = (5,5,4,2,1)$. A typical hook is depicted on the left, while all hook lengths are depicted on the right.

	u	•	•	•
	•			
	•			

9	7	5	4	2
8	6	4	3	1
6	4	2	1	
3	1			
1				

Theorem 4.1.5. *(Hook-length formula).* [8] *Let* $\lambda \vdash n$ *be a Young diagram. Then*

$$dim S^{\lambda} = f^{\lambda} = \frac{n!}{\prod_{u\in\lambda} h_{\lambda}(u)}$$

For instance, from the above example, we get

$$dim S^{(5,5,4,2,1)} = f^{(5,5,4,2,1)} = \frac{17!}{9\cdot7\cdot5\cdot4\cdot2\cdot8\cdot6\cdot4\cdot3\cdot1\cdot6\cdot4\cdot2\cdot1\cdot3\cdot1\cdot1}$$
$$= 1,707,400.$$

Example 4.1.6. *For* $n = 5$ *and* $\lambda = (2,2,1)$, $t =$

4	2
3	1
1	

$$f^{\lambda} = \frac{5!}{4\cdot3\cdot2\cdot1\cdot1} = 5$$

Thus, dim $s^{\lambda} = 5$ *and there are 5 standard* λ*-tableaux. Because n is so tiny in this example, writing down all the standard tableau of form (2,2,1) is simple;*

1	4
2	5
3	

,

1	3
2	5
4	

,

1	3
2	4
5	

,

1	2
3	5
4	

and

1	2
3	4
5	

Theorem 4.1.6. [6] *(Frobenius Reciprocity) Let's call the groups $H \leq G$ finite. Let χ be a H character with a complex character. Let χ be a G character that is complex. Then*

$$(Ind_H^G(\chi)|\psi)_G = (\chi|Res_H^G(\psi))_H.$$

Proof.

$$\begin{aligned}
(Ind_H^G(\chi)|\psi)_G &= \frac{1}{|G|}\sum_{g\in G} Ind_H^G(\chi)(g)(|\psi)(g^{-1}) \\
&= \frac{1}{|G|}\frac{1}{|H|}\sum_{g\in G}\sum_{x\in G}\chi(xgx^{-1})\psi(g^{-1}) \\
&= \frac{1}{|G|}\frac{1}{|H|}\sum_{x\in G}\sum_{y\in G}\chi(y)\psi(xy^{-1}x^{-1}) \\
&= \frac{1}{|G|}\frac{1}{|H|}\sum_{x\in G}\sum_{y\in G}\chi(y)\psi(y^{-1}) \\
&= \frac{1}{|H|}\sum_{y\in G}\chi(y)\psi(y^{-1}) \\
&= \frac{1}{|H|}\sum_{y\in H}\chi(y)\chi(y^{-1}) \\
&= (\chi|Res_H^G(\psi))_H.
\end{aligned}$$

□

Theorem 4.1.7. [7]*(Frobenius formula).Suppose $\lambda = (\lambda_1, ..., \lambda_l), \mu = (\mu_1, ..., \mu_m)$ are partitions of n. The character of S^λ evaluated at an element of S_n with cycle type μ is equal to the coefficient of $x_l^{\lambda_1+l-1}x_2^{\lambda_2+l-2}, ..., x_l^{\lambda_l}$ in*

$$\prod_{1\leq i\leq j\leq l}(x_i - x_j) = \prod_{i=1}^{m}(x_1^{\mu_i} + x_2^{\mu_i} + ... + x_l^{\mu_i})$$

Proof. For any partition λ of n, we have a Young subgroup,

$$S_\lambda = S_{\lambda_l} \times ...S_{\lambda_k} \hookrightarrow S_n \tag{4.1}$$

Let U_λ be the representation of S_n induced from the trivial representation of S_λ. Equivalently, $U_\lambda = A \cdot a_\lambda$, with a_λ as in the preceding section. Let

$$\psi_\lambda \quad = \quad \chi_{U_\lambda} \quad = \quad \textit{character of } U_\lambda \tag{4.2}$$

Key to this investigation is the relation between U_λ and V_λ i.e., between ψ_λ and the character χ_λ of V_λ. Note first thatV_λ appears in U_λ, since there is a surjection

$$U_\lambda \quad = \quad Aa_\lambda \twoheadrightarrow V_\lambda \quad = \quad Aa_\lambda b_\lambda, x \mapsto xb_\lambda \tag{4.3}$$

Alternatively,

$$V_\lambda \;=\; Aa_\lambda b_\lambda \;\cong\; Ab_\lambda a_\lambda \subset Aa_\lambda \;=\; U_\lambda.$$

The character of U_λ is easy to compute directly since U_λ is an induced representation, and we do this next.
For $i = (i_1, ..., i_n)$ a n-tuple of non-negative integers with $\sum \alpha i_\alpha = n$, denote by

$$C_i \quad \subset \quad S_n$$

the conjugacy class consisting of elements made up of i_1 1-cycles, i_2 2-cycles,...,i_n n-cycles. The number of elements in C_i is easily counted to be

$$|C_i| \quad = \quad \frac{n!}{1^{i_1} i_1! 2^{i_2} i_2! \cdot ... \cdot n^{i_n} i_n!} \tag{4.4}$$

By the formula for characters of induced representations

$$\begin{aligned}
\psi_\lambda(C_i) \quad &= \quad \frac{1}{|C_i|}[S_n : S_\lambda] \cdot |C_i \cap S_\lambda| \\
&= \quad \frac{1^{i_1} i_1! \cdot ... \cdot n^{i_n} i_n!}{n!} \cdot \frac{n!}{\lambda_1 \cdot ... \cdot \lambda_k} \cdot \sum \prod_{p=1}^{k} \frac{\lambda_p!}{1^{r_{p1}} r_{p1}! \cdot ... \cdot n^{r_{pn}} r_{pn}!}
\end{aligned}$$

where the sum is over all collections $\{r_{pq} : 1 \leq p \leq k, 1 \leq q \leq n\}$ of nonnegative integers satisfying

$$\begin{aligned} i_q &= r_{1q} + r_{2q} + ... + r_{kq}, \\ \lambda_p &= r_{p1} + 2r_{p2} + ... + nr_{pn}. \end{aligned}$$

(To count $C_i \cap S_\lambda$, write the p^{th} component of an element of S_λ as a product of r_{p1} 1-cycles, r_{p2} 2-cycles,...) Simplifying,

$$\psi_\lambda(c_i) = \sum \prod_{q=1}^{n} \frac{i_q!}{r_{1q}!r_{2q}! \cdot ... \cdot r_{kq}!} \tag{4.5}$$

the sum over the same collections of integers $\{r_{pq}\}$. This sum is exactly the coefficient of the monomial $X^\lambda = x_1^{\lambda l}...x_1^{\lambda k}$ in the power sum symmetric polynomial

$$p^{(i)} = (X_l + ... + X_k)^{i_l} \cdot (X_l^2 + ... + X_k^2)^{i_2} \cdot ... \cdot (X_l^n + ... + X_k^n)^{i_n}. \tag{4.6}$$

So we have the formula

$$\psi_\lambda(C_i) = [p^{(i)}]_\lambda = coefficient\, of\, X^\lambda\, in\, p^{(i)}. \tag{4.7}$$

To prove Frobenius's formula, we need to compare these coefficients with the coefficients $\omega_\lambda(i)$ defined by

$$\omega_\lambda(i) = [\Delta \cdot p^{(i)}]_l, l = (\lambda_1 + k - 1, \lambda_2 + k - 2, ..., \lambda_k). \tag{4.8}$$

Our goal, Frobenius's formula, is the assertion that $\chi_\lambda(C_i) = \omega_\lambda(i)$.
There is a general identity, valid for any symmetric polynomial P, relating such coefficients:

$$[p]_\lambda = \sum_\mu K_{\mu\lambda}[\Delta \cdot p](\mu_1 + k - 1, \mu_2 + k - 2, ..., \mu_k),$$

where the coefficients $K_{\mu\lambda}$ are certain universally defined integers, called

Kostka numbers. For any partitions λ and μ of n, the integer $K_{\mu\lambda}$ may be defined combinatorially as the number of ways to fill the boxes of the Young diagram for μ with λ_1 1's, λ_2 2's, up to λ_k k's, in such a way that the entries in each row are nondecreasing, and those in each column are strictly increasing; such are called semistandard tableaux on μ of type λ. In particular,
$K_{\lambda\lambda} = 1$, and $K_{\mu\lambda} = 0$ for $\mu < \lambda$.
The integer $K_{\mu\lambda}$ may be also be defined to be the coefficient of the monomial $X^\lambda = x_1^{\lambda_1} \cdot ... \cdot x_k^{\lambda_k}$ in the Schur polynomial S_μ corresponding to μ. For the proof that these are equivalent definitions, see (A.9) and (A.19) of Appendix A in[7]. In the present case, applying Lemma (A.26) in [7] to the polynomial $P = p^{(i)}$, we deduce

$$\psi_\lambda(C_i) \;=\; \sum_{\mu} K_{\mu\lambda}\omega_\mu(i) = \omega_\lambda(i) + \sum_{\mu>\lambda} K_{\mu\lambda}\omega_\mu(i). \tag{4.9}$$

The result of Lemma A.28 in [7] can be written, using (4.4), in the form

$$\frac{1}{n!}\sum_{i} |C_i|\omega_\lambda(i)\omega_\mu(i) \;=\; \delta_{\lambda\mu}. \tag{4.10}$$

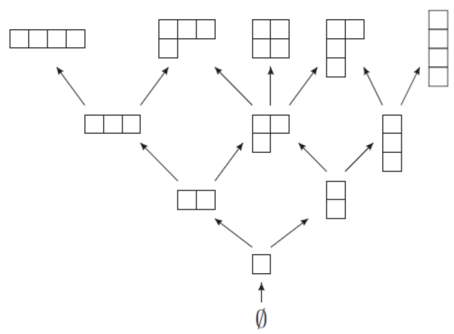
This indicates that the functions ω_λ regarded as functions on the conjugacy classes of S_n, satisfy the same orthogonality relations as the irreducible characters of S_n.
In fact, one can deduce formally from these equations that the ω_λ must be the irreducible characters of S_n, which is what Frobenius proved. □

4.2 Young lattice and branching rule

Now let us consider the relationships between the irreducible representations of S_n and those of S_{n+1}. Consider the entire collection of Young diagrams. Inclusion may be used to partially order these diagrams. $Young's\, lattice$ is the partially ordered set that results. Young's lattice can be represented visually as follows. Let $\lambda \nearrow \mu$ denote that μ can be obtained by multiplying

λ by a single square. All of the Young diagrams with n boxes are drawn at the n^{th} level. In addition,λ to connected to μ if $\lambda \nearrow \mu$. Here is a figure showing the lowest section of Young's lattice (of course, it extends infinitely upwards).

YOUNG TABLEAUX AND THE REPRESENTATIONS OF THE SYMMETRIC GROUP



Lemma 4.2.1. *we have*

$$f^\lambda = \sum_{\lambda^-} f^{\lambda^-}.$$

Proof. Every standard tableau of shape $\lambda \vdash n$ consists of n in some inner corner together with a standard tableau of shape $\lambda^- \vdash n-1$. The result follows. □

Theorem 4.2.1. [6] *(Branching Rule). Suppose $\lambda \vdash n$, then*

$$1.\ S^\lambda \downarrow_{S_{n-1}} \cong \bigoplus_{\lambda^-} S^{\lambda^-},\ \textit{and}$$

$$2.\ S^\lambda \uparrow^{S_{n+1}} \cong \bigoplus_{\lambda^+} S^{\lambda^+}.$$

Proof. 1. Let the inner corners of λ appear in rows $r_1 < r_2 < ... < r_k$. For each i, let λ^i denote the partition λ^- obtained by removing the corner cell in row r_i. In addition, if n is at the end of row r_i of tableau t (respectively, in row r_i of tabloid $\{t\}$), then t^i (respectively, $\{t^i\}$) will be the array obtained by removing the n. Now given any group G with module V and submodule W, it is easy to see that

$$V \cong W \oplus (V/W),$$

where V/W is the quotient space. Thus it suffices to find a chain of subspaces

$$\{0\} = V^{(0)} \subset V^{(1)} \subset V^{(2)} \subset ... \subset V^{(k)} = S^\lambda$$

such that $V^{(i)}/V^{(i-l)} \cong s^{\lambda^i}$ as S_{n-1}-modules for $1 \leq i \leq k$. Let $V^{(i)}$ be the vector space spanned by the standard polytabloids e_t, where n appears in t at the end of one of rows r_1 through r_i. We show that the $V^{(i)}$ are our desired modules as follows. Define maps $S_{n-1} : M^\lambda \to M^{\lambda^i}$ by linearly extending

$$\{t\} \xrightarrow{S_{n-1}} \begin{cases} \{t_i\} & if\, n\, is\, in\, row\, r_i\, of\, \{t\}, \\ 0 & otherwise. \end{cases}$$

Furthermore, for standard t we have

$$e_t \xrightarrow{S_{n-1}} \begin{cases} \{e_{t^i}\} & if\, n\, is\, in\, row\, r_i\, of\, t, \\ 0 & if\, n\, is\, in\, row\, r_j\, of\, t,\ where\, j < i. \end{cases} \tag{4.11}$$

This is because any tabloid appearing in e_t, t standard, has n in the same row or higher than in t. Since the standard polytabloids form a basis for the corresponding Specht module, the two parts of (4.1) show that

$$S_{n-1}V^{(i)} = S^{\lambda^i} \tag{4.12}$$

and

$$V^{(i-1)} \subseteq ker S_{n-1} \tag{4.13}$$

From equation (4.13), we can construct the chain

$$0 \;=\; V^{(0)} \subseteq V^{(l)} \cap ker\theta_1 \subseteq V^{(l)} \subseteq ... \subseteq V^{(k)} = s^\lambda. \tag{4.14}$$

But from equation (4.2)

$$dim\frac{V^{(i)}}{V^{(i)} \cap kerS_{n-1}} \;=\; dimS_{n-1}V^{(i)} = f^{\lambda^i}.$$

By the preceding lemma, the dimensions of these quotients add up to dim S^λ. Since this leaves no space to insert extra modules, the chain (4.4) must have equality for the first, third, etc. containments. Furthermore,

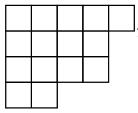
$$\frac{V^{(i)}}{V^{(i-1)}} \;\cong\; \frac{V^{(i)}}{V^{(i)} \cap kerS_{n-1}} \cong S^{\lambda^i}$$

as desired.
2. We show that this part follows from the first by Frobenius reciprocity (Theorem 4.1.6). In fact, parts 1 and 2 can be shown to be equivalent by the same method. Let χ^λ be the character of S^λ.
If $S^\lambda \uparrow S_{n+l} \cong \oplus_{\mu \vdash n+l} m_\mu S^\mu$, then by taking characters, $\chi^\lambda \uparrow S_{n+l} \cong \sum_{\mu \vdash n+l} m_\mu \chi^\mu$. The multiplicities are given by

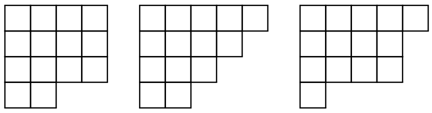
$$\begin{aligned} m_\mu &= \langle \chi^\lambda \uparrow S_{n+l}, \chi^\mu \rangle \\ &= \langle \chi^\lambda, \chi^\mu \downarrow s_n \rangle \\ &= \langle \chi^\lambda, \sum_{\mu^-} \chi^{\mu^-} \rangle \\ &= \begin{cases} 1 & if\, \lambda \;=\; \mu^-, \\ 0 & \quad otherwise \end{cases} \\ &= \begin{cases} 1 & if\, \mu \;=\; \lambda^+, \\ 0 & \quad otherwise \end{cases} \end{aligned}$$

□

For instance, if $\lambda = (5, 4, 4, 2)$, so that

$\lambda =$  ,

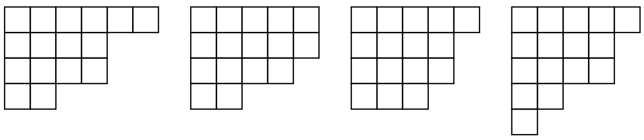
then the diagrams that can be obtained by removing a square are



So

$$Res_{s_{14}} S^{(5,4,4,2)} = S^{(4,4,4,2)} \oplus S^{(5,4,3,2)} \oplus S^{(5,4,4,1)}.$$

Similarly, the diagrams that can be obtained by adding a square are



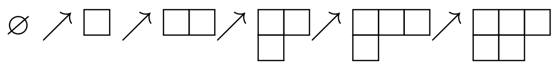
So

$$Ind_{s_{15}}^{s_{16}} S^{(5,4,4,2)} = S^{(6,4,4,2)} \oplus S^{(5,5,4,2)} \oplus S^{(5,4,4,3)} \oplus S^{(5,4,4,2,1)}.$$

As we state that the Frobenius reciprocity theorem equates the two parts of the branching rules. This outcome can be seen in a variety of ways. If we consider S^{λ} only as a vector space, then the branching rule implies that

$$S^{\lambda} \cong \bigoplus_{\mu:\mu\nearrow\lambda} S^{\mu} \cong \bigoplus_{v:v\nearrow\mu\nearrow\lambda} S^{v} \cong ... \cong \bigoplus_{\emptyset=\lambda^{(0)}\nearrow\lambda^{(1)}\nearrow...\nearrow\lambda^{(n)}=\lambda} S^{\emptyset}$$

In Young's lattice, the final sum is indexed over all upward routes from $\emptyset$ to λ. Given that $S^{\emptyset}$ is a one-dimensional vector space, we may construct a basis for S^{λ} in which each basis vector corresponds to an upward path in the Young lattice from $\emptyset$ to λ. However, upward paths in the Young lattice from $\emptyset$ to λ correspond to standard λ-tableaux. Indeed, we can connect each standard λ-tableaux with a path in the Young lattice created by adding the boxes in the sequence designated in the standard tableaux. The reverse construction is similar.



corresponds to the following standard tableau.

1	2	4
3	5	

Now, one may object that this argument contains some circular reasoning, namely because the proof of the branching rule [2] uses Theorem 3.1.2, that a basis of S^λ can be found through standard tableaux. This is indeed the case. However, there is an alternative view on the subject, given recently by Okounkov and Vershik [1], in which we start in an abstract algebraic setting with some generalized form of the Young lattice. Then, we can form a basis known as the Gelfand-Tsetlin basis by taking upward paths as we did above. We then specialize to the symmetric group and "discover" the standard tableaux. This means that the standard tableaux in some sense form a "natural" basis for S^λ .

4.3 Decomposition of M^μ

First, we constructed the permutation modules M^μ, and from it we extracted irreducible sub representations S^λ, such that S^λ forms a complete list of irreducible representations of S_n as λ varies over all partitions of n.
Let us revisit M^μ and ask, how does M^μ decompose into irreducible representations. It turns out that M^μ only contains the irreducible s^λ if λ is, in some sense, "greater" than μ! To make this notation more precise, let us define a partial order on partitions of n. (Note that this is not the same as the one used to define Young's lattice!)

4.4 Kostka number and young's rule

Definition 4.4.1. *If $\lambda, \mu \vdash n$, is the number of semi-standard tableaux of shape λ and content μ, then the Kostka number $K_{\lambda\mu}$ is the number of semi-*

standard tableaux of shape λ and content μ.

For example, if $\lambda = (3, 2)$ and $\mu = (2, 2, 1)$,

1	1	2
2	3	

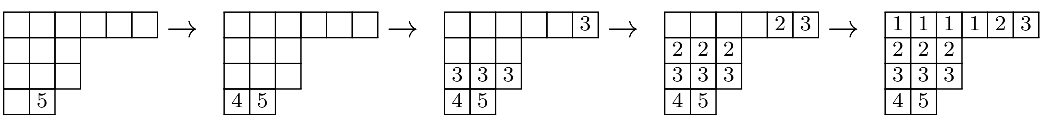
and

1	1	3
2	2	

then $K_{\lambda\mu} = 2$, since there are precisely two semi-standard tableaux of form λ and content μ: first, note the following.

Proposition 4.4.1. [8] *If $\lambda, \mu \vdash n.$, $K_{\lambda\mu} \neq 0$ if and only if $\lambda \trianglerighteq \mu$. Moreover, $K_{\lambda\lambda} = 1$.*

Proof. Since the columns of a semi-standard tableau are strictly increasing, an entry k can only appear in one of the first k rows. So, if $K_{\lambda\mu} \neq 0$, so that there exists some semi-standard tableau T of shape λ and content μ, then the first k rows of T must contain all the entries from the set $\{1, 2, ..., \}$. It follows that $\lambda_1 + \lambda_2 + ... + \lambda_k \geq \mu_1 + \mu_2 + ... + \mu_k$. Therefore, $\lambda \trianglerighteq \mu$.
Conversely, suppose $\lambda \trianglerighteq \mu$. We give a procedure for constructing a λ-semistandard tableau. Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ and $\mu = (\mu_1, \mu_2, ..., \mu_m)$.
First, fill in μ_m copies of the number m as follows: starting from the last (bottom most) row, start filling in m from the right. If we run out of space, then move to the previous row and fill the number in all the legal positions starting from the right (so that no two copies of m lie on the same column), and repeat for the previous rows as many times if necessary. After we finish with m, repeat the same procedure for $m - 1$ on the remaining empty boxes, and so on.
For instance, if $\lambda = (6, 3, 3, 2)$ and $\mu = (4, 4, 4, 1, 1)$ then the construction yields the following:



To show that this works, observe that after each step, the set of empty squares form another Young diagram. So by induction it suffices to prove that (1) the first step is possible and (2) if λ' is the diagram formed by the remaining empty squares of λ, and $\mu^{'}$ is the partition μ without the last part, then $\lambda^{'} \trianglerighteq \mu^{'}$.

For (1), observe that $\mu_m \leq \mu_1 \leq \lambda_1$, and so there are at least as many columns in λ as λ_k, and so there is enough space for all the copies of k to be filled in.

For (2), we need to prove that $\lambda_1^{'} + ... + \lambda_k^{'} \geq \mu_1^{'} + ... + \mu_k^{'}$ for all k. If none of the copies of m are filled in the first k rows of λ, then $\lambda_i = \lambda_i^{'}$ for $1 \leq i \leq k$, and the inequality follows from the hypothesis $\lambda \trianglerighteq \mu$.

Otherwise, note that it suffices to prove that $\lambda_{k+1}^{'} + \lambda_{k+2}^{'} ... + \lambda_l^{'} \geq \mu_{k+1}^{'} + \mu_{k+2}^{'} ... + \mu_l^{'}$ (note that $\lambda \leq m$ follows from $\lambda \trianglerighteq \mu$). But this is true since the algorithm guarantees that $\lambda_i \leq \mu_m \leq m_i$ for $k < i \leq m$, and so $\lambda_i \leq m_i = \mu_i$ for $k < i < l \leq m$. It follows that $\lambda^{'} \trianglerighteq \mu^{'}$.

Finally, if $\lambda = \mu$, then since the numbers $1tok$ must only appear in the first k rows, the number k can only appear in the kth row, which gives a unique semi-standard tableau.

Thus, $K_{\lambda\lambda} = 1$.

Theorem 4.4.1. *(Young's rule)* [8] *The multiplicity of S^λ in M^μ is equal to the number of semistandard tableaux of shape λ and content μ, i.e,*

$$M^\mu \cong \bigoplus_\lambda K_{\lambda\mu} S^\lambda.$$

Example 4.4.1. *suppose $\mu = (2,2,1)$. Then the positive $\lambda \trianglerighteq \mu$ and the associated λ-tableaux of type μ are as follows:*

$\lambda^1 = (2,2,1) = \begin{ytableau} 1 & 1 \\ 2 & 2 \\ 3 \end{ytableau}$ $\lambda^2 = (3,1,1) = \begin{ytableau} 1 & 1 & 2 \\ 2 \\ 3 \end{ytableau}$

$\lambda^3 = (3,2) = \begin{ytableau} 1 & 1 & 2 \\ 2 & 3 \end{ytableau}$, $\begin{ytableau} 1 & 1 & 3 \\ 2 & 2 \end{ytableau}$

$\lambda^4 = (4,1) = \begin{ytableau} 1 & 1 & 2 & 2 \\ 3 \end{ytableau}$, $\begin{ytableau} 1 & 1 & 2 & 3 \\ 2 \end{ytableau}$

$\lambda^5 = (5) = \begin{ytableau} 1 & 1 & 2 & 2 & 3 \end{ytableau}$

Thus

$$M^{(2,2,1)} \cong S^{(2,2,1)} \bigoplus S^{(3,1,1)} \bigoplus 2S^{(3,2)} \bigoplus 2S^{(4,1)} \bigoplus S^{(5)}$$

□

Summary and Conclusion

This study demonstrated how partitions may be linked to young diagrams and tableaux. It also expands on the notion of division identities. It's worth noting that these words are used in Chapters 3 and 4. The tabloid is associated with the young tableau, which is significant for the formulation of the permutation module in Chapter 3 and, as we mentioned in Chapter 4, for the construction of irreducible representation (specht module). The unique module, as well as its dimensions and character, were briefly discussed in Chapter 4.

Finally, this paper is based on the title 'young tableau and representation of the symmetric group', which explains how to relate algebra with combinatorics.

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