Q-IORDAN MATRIX AND THEIR COMBINATORIAL SIGNIFICANCE

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Abstract

In this project paper, works on q-Riordan matrices, their algebraic structures and applications on some famous numbers like the q-Striling and q-Bessel numbers have been studied and reported.

Introduction

The Riordan group (R,*) is the set of all lower triangular matrices $R = (e_{n,k})_{,n,k\geq 0}$, defined by the relation $e_{n,k} = [x^n]l(x)h(x)^k$, in the case of ordinary Riordan matrix; or $e_{n,k} = n![x^n]l(x)\frac{h(x)^k}{k!}$, in the case of exponential Riordan matrix; where $l(x) = l_0 + l_1x + l_2x^2 + \cdots, l_0 \neq 0$, and $h(x) = h_1x + h_2x^2 + \cdots, h_1 \neq 0$, are two formal power series.

Consequently, the Riordan matrix $R = (e_{n,k})_{,n,k\geq 0}$, could also be denoted simply as a pair (l,h) of formal power series generating it.

So, the Riordan group (R,*) could be defined as the set of ordered pairs of formal power series $l(x) = l_0 + l_1(x) + l_2x^2 + \cdots$, $l_0 \neq 0$ and $h(x) = h_1x + h_2x^2 + \cdots$, $h_1 \neq 0$, on which the binary operation $*$ is defined by: $(l.h) * (r, s) = (l(x)r(h(x)), r(h(x)))$, where (l,h), (r,s) are arbitrary elements of R . The identity matrix is I=(1,x), and the inverse of an arbitrary Riordan matrix (l,h) is the pair of formal power series , $(l, h)^{-1} = (\frac{1}{l(\overline{h})}, \overline{h})$. The conditions $h_1 \neq 0$, $l_0 \neq 0$ are there to guarantee the existence of $\overline{h}$, the compositional inverse of h, and the composition $l(\overline{h})$.

Example 0.0.1. *(Ordinary Riordan matrix) Consider the pair of formal power series* $R = (\frac{1}{1-t}, \frac{t}{1-t})$. *The entries of the ordinary Riordan matrix whose* k^{th} *column are generated by* $\frac{1}{1-t}(\frac{t}{1-t})^k = \frac{t^k}{(1-t)^{k+1}} = t^k(1-t)^{-k-1}$ *are given by,*

$$
\begin{aligned}
e_{n,k} &= [t^n]t^k(1-t)^{-k-1} \\
&= [t^{n-k}](1-t)^{-k-1} \\
&= [t^{n-k}]\sum_{j=0}^{\infty}\binom{-k-1}{j}(-1)^j x^j \\
&= [t^{n-k}]\sum_{j=0}^{\infty}\binom{j+k+1-1}{j}(-1)^{2j}t^j \\
&= \binom{n}{n-k} \\
&= \binom{n}{k}
\end{aligned}
$$

these are the so called Binomial coefficients, and are the entries of the famous Pascal's triangle, which is a typical example of Riordan matrices.

$$
(e_{n,k})_{n,k\geq 0} = (\binom{n}{k})_{n,k\geq 0}
$$

The first few entries of Pascals triangle are as shown below, and this matrix is a typical example of Ordinary Riordan Arrays.

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 \\
1 & 3 & 3 & 1 & 0 \\
1 & 4 & 6 & 4 & 1
\end{pmatrix}
$$

Example 0.0.2. *(Exponential Riordan matrix) Consider the same pair of formal power series* $R = (\frac{1}{1-t}, \frac{t}{1-t})$ *we used in the example above. Here, we need an example of Exponential Riordan matrix. This means one whose* k^{th}

columns are generated by $l(x)\frac{h(x)^k}{k!}$*. For this particular pair of formal power series, it reduces to*

$$\begin{aligned}
e_{n,k} &= n![t^n]l(x)\frac{h(x)^k}{k!} \\
&= n![t^n]\frac{1}{k!}\frac{1}{1-t}(\frac{t}{1-t})^k \\
&= n![t^n]\frac{1}{k!}\frac{t^k}{(1-t)^{k+1}} \\
&= n![t^n]\frac{1}{k!}t^k(1-t)^{-k-1} \\
&= \frac{n!}{k!}[t^n]t^k(1-t)^{-k-1} \\
&= \frac{n!}{k!}[t^{n-k}](1-t)^{-k-1} \\
&= \frac{n!}{k!}[t^{n-k}]\sum_{j=0}^{\infty}\binom{-k-1}{j}(-1)^j x^j \\
&= \frac{n!}{k!}[t^{n-k}]\sum_{j=0}^{\infty}\binom{j+k+1-1}{j}(-1)^{2j}t^j \\
&= \frac{n!}{k!}\binom{n}{n-k} \\
&= \frac{n!}{k!}\binom{n}{k} \\
(e_{n,k})_{n,k\geq 0} &= (\frac{n!}{k!}\binom{n}{k})_{n,k\geq 0}
\end{aligned}$$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
2 & 4 & 1 & 0 & 0 \\
6 & 18 & 9 & 1 & 0 \\
24 & 96 & 72 & 16 & 1
\end{pmatrix}$$

Applications of Riordan arrays, both ordinary and exponential, in solving enumeration problems in various fields have been studied by various mathematicians. See[9].

These works have inspired many to study similar properties of the generalizations, in particular properties of q-Riordan matrices which are q-analogues of the exponential Riordan matrices, and their combinatorial interpretations.

That is why the student who did this project work was interested in studying and also informing others who could be interested about the works done in this interesting area of combinatorics.

The authors in the papers studied, function of the form; $l(z) = \sum_{n\geq 0} l_n \frac{z^n}{n!_q}$, known as Eulerian generating functions.

Where $n!_q = 1_q 2_q \cdots n_q$ with the q-numbers $n_q = \frac{1-q^n}{1-q} = 1+q+\cdots+q^{n-1}$ and $0!_q = 1$ and l_n is a polynomial in q. This function is a q-analogue of the exponential generating function. It arises in several combinatorial applications [13, 20] such as finite vector spaces, partitions and counting permutations by inversions.

In this paper, the author also applies ,the concept of q-Riordan matrices to well known numbers known as (r,s)-Bessel numbers $B_{r,s}(n,k)$, of the second type and of the first type $A_{r,s}(n,k)$.

Throughout this paper, the set $\{1, 2, \cdots, n\}$ is denoted by $\langle n\rangle$. The numbers $B_{r,s}(n,k)$ count the number of (k+r)-partitions $\{B_1, \cdots B_{k+r}\}$ of $\langle n+r\rangle$ in which each block has size at most s and no two of the element $1, 2, \cdots, r$ are in the same block. Similarly, the numbers $A_{r,s}(n,k)$ count the number of (k+r)-cyclic permutations $C_1 \cdots C_{k+r}$ of $\langle n+r\rangle$ in which each cycle has length at most s and no two of the element $1, 2, \cdots, r$ are in the same cycle. when r=0 it means there is no restriction on the elements of blocks or cycles. Conversely, by letting $s \to \infty$, these numbers reduce to the r-Stirling numbers of the second kind $S_r(n,k)(= B_{r,\infty}(n,k))$ and of the first kind $s_r(n,k)(= A_{r,\infty}(n,k))$ [2]. By setting r=0 and s=2, the $B_{r,s}(n,k)$ coincide with classical Bessel numbers of the second kind B(n,k) [10].

The purpose of this paper is to introduce a q-analogue of the Riordan group and its combinatorial significance.In section 2 q-Riordan matrices are defined to be a q-analogue of an exponential Riordan matrices by using the Eulerian generating functions. In section 3 algebraic structure of a q-Riordan array are introduced and discused. In Section 4, q-Riordan array associated to counting function could be applied to enumeration formula this shows their combinatorial significance, consequently q-analogues of the Striling numbers of both kind and their combinatorial interpretation are introduced and discussed. In section 5,the q-analogue of (r,s)-Bessel numbers of both types along with some algebraic formulas are introduced and discussed: in particular examples of application, matching and covering, using the concept of q-Bessel numbers of both kind.

Chapter 1

Q-RIORDAN MATRICES

In this section, the q-Riordan matrices will be introduced, and some of their properties will also be given and discussed, often in comparison with the relevant properties of the ordinary and exponential Riordan arrays.

Consider the relation $x^n = \sum_{k=0}^{n} \binom{n}{k}(x-1)^k$; it is easy to verify using the Binomial Theorem that:

$$x^n = (x-1+1)^n = \sum_{k=0}^{n} \binom{n}{k}(x-1)^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}(x-1)^k$$

We would like to see what coefficients(generalized form of the Binomial coefficients) connect the two polynomial sequences $g_n(x)$ and x^n where $g_n(x)$, a generalized form of $(x-1)^n$ is given by $g_n(x) = \prod_{k=0}^{n-1}(x-q^k)$.

Denoting these generalized coefficients by $\binom{n}{k}_q$, and expecting them to connect the two sequences of polynomials, x^n and $g_n(x)$, the way the binomial coefficients $\binom{n}{k}$ connect the two polynomial sequences in x^n and $(x-1)^k$, we get the relation $x^n = \sum_{k=0}^{n} \binom{n}{k}_q g_k(x)$.

It seems that the latter one is a generalization of the former, and it indeed is, and everything has to do with the q. From the polynomial sequence $g_n(x) = \prod_{k=0}^{n-1}(x - q^k)$, it is easy to see this relation.

$$\begin{aligned}
g_n(x) &= \prod_{k=0}^{n-1}(x - q^k) \\
&= \underline{(x-1)(x-q)(x-q^2)\cdots(x-q^{n-2})}(x - q^{n-1}) \\
&= (x - q^{n-1})(\prod_{k=0}^{n-2}(x - q^k)) \\
&= (x - q^{n-1})g_{n-1}(x) \\
&= xg_{n-1}(x) - q^{n-1}g_{n-1}(x)
\end{aligned}$$

leading to the relation:

$$\begin{aligned}
x^n &= xx^{n-1} \\
&= x\sum_{k=0}^{n-1}\binom{n}{k}_q g_k(x) \\
&= x\sum_{k=1}^{n-1}\binom{n-1}{k-1}_q g_{k-1}(x) \\
&= \sum_{k=1}^{n-1}\binom{n-1}{k-1}_q xg_{k-1}(x)
\end{aligned}$$

Since

$$\begin{aligned}
g_n(x) &= (x - q^{n-1})g_{n-1}(x) \\
&= xg_{n-1}(x) - q^{n-1}g_{n-1}(x)
\end{aligned}$$

From $g_n(x) = (x-q^{n-1})g_{n-1}(x)$, it follows that $xg_{n-1}(x) = g_n(x)+q^{n-1}g_{n-1}(x)$, and putting this in:
$\sum_{k=1}^{n-1}\binom{n-1}{k-1}_q xg_{k-1}(x)$, we get $\sum_{k=1}^{n-1}\binom{n-1}{k-1}_q(g_k(x) + q^{n-1}g_{k-1}(x))$

Now we are arrived at the conclusion that

$$\begin{aligned}
x^n &= \sum_{k=0}^{n-1} \binom{n}{k}_q g_k(x) \\
&= \sum_{k=1}^{n-1} \binom{n-1}{k-1}_q (g_k(x) + q^{k-1} g_{k-1}(x)) \\
&= \sum_{k=1}^{n-1} \binom{n-1}{k-1}_q g_k(x) + q^{k-1} \sum_{k=1}^{n-1} \binom{n-1}{k-1}_q g_{k-1}(x) \\
\sum_{k=0}^{n-1} \binom{n}{k}_q g_k(x) &= \sum_{k=0}^{n-1} (\binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q) g_k(x)
\end{aligned}$$

Which leads to a q-version of the addition formula for the binomial coefficients:

it then follows that

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q, n \geq k \geq 1, \binom{0}{0}_q = 1$$

The addition formula for the binomial coefficients,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, n \geq k \geq 1$$

is useful to construct the entries of the Pascals triangle, given the initial condition, $\binom{0}{0} = 1$.

These coefficients are known as the q-Binomial coefficients and are defined as:

$\binom{n}{k}_q = \frac{n!_q}{k!_q (n-k)_q!}$, and it is easy to prove it by induction on n, using the q-version of the addition formula we obtained earlier and the fact that $\binom{n}{k}_q = \binom{n}{k}$ if q=1.

It was introduced earlier that $[n]_q = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$.

This in turn implies that $[n]!_q = [1]_q[2]_q \cdots [n]_q = \frac{\prod_{j=1}^{n}(1-q^j)}{(1-q)^n}$, and hence, another way to express $\binom{n}{k}_q$ will be:

$$\begin{aligned}\binom{n}{k}_q &= \frac{n!_q}{k!_q(n-k)_q!}\\ &= \frac{\prod_{j=1}^{n}(1-q^j)(1-q)^k(1-q)^{n-k}}{(1-q)^n(\prod_{j=1}^{k}(1-q^j)\prod_{j=1}^{n-k}(1-q^j))}\\ &= \frac{\prod_{j=n-k+1}^{n}(1-q^j)}{\prod_{j=1}^{k}(1-q^j)}\end{aligned}$$

and adding the relation $\binom{n}{k}_q = \binom{n}{n-k}_q$ to it we end up with
$\binom{n}{k}_q = \frac{n!_q}{k!_q(n-k)_q!} = \binom{n}{n-k}_q = \frac{\prod_{j=n-k+1}^{n}(1-q^j)}{\prod_{j=1}^{k}(1-q^j)}$
For example,
$\binom{n}{1}_q = \binom{n}{n-1}_q \frac{1-q^n}{1-q} = [n]_q$
and just to see the polynomial for some fixed n and k, take n=4 and k=2.

$$\begin{aligned}\binom{4}{2}_q &= \frac{(1-q^3)(1-q^4)}{(1-q)(1-q^2)}\\ &= (1+q+q^2)(1+q^2)\end{aligned}$$

We used the definition to evaluate q-binomial s, but with the initial value $\binom{0}{0} = 1$.
We can use the q-version of the addition formula to determine $\binom{n}{k}_q$.
It is interesting to see that $\binom{n}{1}_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$ as $\binom{n}{1} = n$ same is true with $\binom{n}{n-1}_q$ for $\binom{n}{k}_q = \binom{n}{n-k}_q$.

Which can also be obtained using the recurrence $\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q$

$$\begin{aligned} \binom{4}{2}_q &= \binom{4-1}{2-1}_q + q^2 \binom{4-1}{2}_q. \\ &= 1 + q + q^2 + q^2(1 + q + q^2) \\ &= (1 + q + q^2)(1 + q^2) \\ &= 1 + q + 2q^2 + q^3 + q^4 \end{aligned}$$

Since our discussion of q-Riordan matrices involves Eulerian generating functions generating the matrix; and one of them is the k^{th}-symbolic power, the q-derivative need to be introduced as a tool to define the symbolic k^{th} power.

Consider an Eulerian generating functions of the form

$\sum_{n\geq 0} h_n \frac{z^n}{n!_q}$

The q-derivative $D_q h(z)$ of h(z) is defined by;

$$D_q h(z) = \frac{h(z) - h(qz)}{z - qz}$$

and is often denoted by $h^{'}(z)$. The q-product rule is given by

$$D_q h(z)l(z) = h(z)l'(z) + h'(z)l(qz),$$

For $k \in N_0$, the k^{th} symbolic power $h^{[k]}$ of h(z) with $h(0) = 0$ is inductively defined as

$$D_q h^{[k]}(z) = [k]_q h^{[k-1]}(z)h'(z)$$

for $k \geq 1$ and $h^{[0]}(z) = 1$;

$h^{[k]}(0) = 0$ for $k \geq 1$ and, $h^{[1]}(z) = h(z)$.

If $l(z) = \sum_{n\geq 0} l_n \frac{z^n}{n!_q}$ then the q-composition $lo_q h$ is defined as

$$(lo_q h)(z) = l[h(z)] = \sum_{n\geq 0} l_n \frac{h^{[n]}(z)}{n!_q}$$

It is not difficult to show that $z^{[n]} = z^n$, $l[z] = l(z)$. So have $l[z] = l(z)$.

For $n \in N_0$ let $\varepsilon_q(n)$ be the set of Eulerian generating functions of the form

$$l(z) = l_n \frac{z^n}{n!_q} + l_{n+1} \frac{z^{n+1}}{(n+1)!_q} + l_{n+2} \frac{z^{n+2}}{(n+2)!_q} + \cdots, l_n = 1.$$

l_n stands for the coefficient of $\frac{z^n}{n!_q}$ in l(z). Given a pair of functions $l(z) \in \varepsilon_q(0)$ and $h(z) \in \varepsilon_q(1)$. A q-Riordan matrix R is defined to be the matrix whose $(n,k)^{th}$ entries are the coefficients of an Eulerian generating function of the form, $l(z)\frac{h^{[k]}(z)}{k!}$ which can be given as $\sum_{n \geq k} t_{n,k} \frac{z^n}{n!_q}$.
where $t_{n,k} = \sum_{j=k}^{n} \binom{n}{j}_q l_{n-j} h_{j,k}$, by convolution, and $h_{n,k}$ is the coefficient of $\frac{z^n}{n!_q}$ in the exponential of $\frac{h^{[k]}(z)}{k!_q}$.
Being a generalization of Riordan arrays, a q-Riordan array R is denoted by $R = (l,h)_q$.

Since $t_{n,k} = 0$ for $n < k$ and $l_{n,n} = 1(l(z) \in \varepsilon_q(0))$ for $n \in N_0$, every q-Riordan matrix is an infinite lower triangular matrix with unit diagonal elements.
For q = 0 and q = 1 $(l(z), h(z))_q$ reduces to the usual Riordan matrix and the exponential Riordan matrix, respectively.
For example $l(z) = e_q(z)$ and h(z)=z, then since $l_{n-j} = 1, h_{j,k} = \delta_{j,k}$ it follows that $t_{n,k} = \binom{n}{k}_q$. so the q-Riordan matrix $(l(z), h(z))_q$ has the entries:-

- n=0,k=0

$$\begin{aligned} t_{0,0} &= \binom{0}{0}_q \\ &= 1 \end{aligned}$$

- n=1,k=0

$$\begin{aligned} t_{1,0} &= \binom{1}{0}_q \\ &= 1 \end{aligned}$$

- n=2, k=1

$$\begin{aligned} t_{2,1} &= \binom{2}{1} \\ &= 1+q \end{aligned}$$

We can apply the formula $\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q$ to determine the remaining entries shown bellow.

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1+q & 1 & 0 & 0 \\ 1 & 1+q+q^2 & 1+q+q^2 & 1 & 0 \\ 1 & 1+q+q^2+q^3 & (1+q+q^2)(1+q^2) & 1+q+q^2+q^3 & 1 \end{pmatrix}$$

The fundamental theorem of Riordan arrays could be generalized to the case that

Theorem 1. *Let $a, b \in \varepsilon_q(0)$. Then*
$(l,h)_q(a_0, a_1, \cdots)^T = (b_0, b_1, \cdots)^T$ *if and only if* $la[h] = b$.

We call Theorem 1 the fundamental theorem of q-Riordan matrix, and write as $(l,h)_q a = la[h]$. This simple observation reduces many computations to merely q-composition of functions.

let $k^j(x) = m(x).n(x)^j$. Then using operations of $*_{\frac{1}{q}}$ and $*_q$, we have

1.1 Loop structure of q-riordan matrices

In the introduction we have seen that the set of all infinite lower triangular matrix whose k^{th} columns are generated by a formal power series $l(z)f(z)^k$, where l and f are formal power series with properties; $l(0) \neq 0$ $f(0) = 0$, $f'(0) \neq 0$, forms what we call the Riordan group. We will see algebric structure of q-Riordan arrays.

First let us introduce some important ideas we need to describe R_q. Let S be a set and $* : S \times S \to$ S be a binary operation. $(S, *)$ is a groupoid if it contains an identity element. A groupoid (S,*) is a loop if the equation $a * x = b$ has a unique solution $x \in S$ for every $a, b \in S$.

Definition 1.1.1. *An Eulerian generating function l is called a left q-compositional inverse of h if* $(lo_q h) = z$.

Theorem 2. *let* $h(z) = \sum_{k\geq 1} h_k \frac{z^k}{k!_q}$. *then h has a compositional inverse* $\overline{h}$ *if and only if* $h_1 \neq 0$.

Proof. suppose h has a compositional inverse $l(z) = \sum_{k\geq 1} l_k \frac{z^k}{k!_q}$.

$(l_0 h)(z) = z$

$$
\begin{aligned}
(l_0 h)(z) &= \sum_{k\geq 1} l_k \frac{z^k}{k!_q} \\
&= \sum_{k\geq 1} (\sum_{k=1}^{n} l_k h_{n,k}) \frac{z^n}{n!_q} \\
&= l_1 h_1 z + \sum_{n\geq 2} (l_n h_1^n + \sum_{k=1}^{n-1} l_k h_{n,k}) \frac{z^n}{n!_q} \\
&= z
\end{aligned}
$$

we have $l_1 h_1 = 1$ $l_n h_1^n + \sum_{k=1}^{n-1} l_k h_{n,k} = 0$, $n \geq 2$.
We can solve the first equation uniquely for l_1 if and only if $h_1 \neq 0$. We can then solve the second equation uniquely for l_2, etc. Hence l exists if and only if $h_1 \neq 0$.
In a similar way, an analogues result holds for the right q-compositional inverse of f. □

Let R_q be the set of q-Riordan matrices. Let "x" be a binary operation defined on R_q as:

$$(l, h)_q \times (a, b)_q = (la[h], b[h])_q.$$

Note that (R_q, x) is a groupoid with the identity matrix $(1, z)_q$, the infinite lower triangular matrix with each diagonal element equal to 1, and every other entry equal to 0.
$(R_q, \times)$ is a loop.
For $l \in \varepsilon_q(0)$ and $h \in \varepsilon_q(1)$, it is not difficult to show that $\{(1, h)_q$, $(l, z)_q$, $(h', h)_q\}$ are also loops; in fact subloop of R_q.

Theorem 3. $(R_q, \times)$ *is a group if and only if q=0 or q=1.*

Proof. The converse is easy for the case q=0 or q=1 reduce to the ordinary or the exponential Riordan groups respectively.

Can be prooved by contradiction [9]. Assume $q \neq 0$ and $q \neq 1$, then this would contradict the associative property .

Therefore $(R_q, \times)$ is a group if and only if q=0 or q=1. □

Chapter 2

APPLICATION OF Q-RIORDAN ARRAYS; THEIR COMBINATORIAL SIGNIFICANCE

If L and H are the Eulerian generating functions whose coefficients are associated to the counting functions $N_0 \to C[[_q]]$, then the q-Riordan matrix$(L, H)_q$ can be applied to some enumeration problem related to q-Striling numbers and q-Bessel numbers.

Exponential formula

It is important that the ordinary exponential formula is introduced prior to the q-exponential formula, to make latter easier for the rider to understand.

Lemma 2.0.1. *Let $h, l : N_0 \to \mathbb{C}$ and $k : N_0 \to \mathbb{C}$ be defined by*

$$k(|X|) = \sum_{(A,B)} h(|A|)l(|B|),$$

where (A,B) runs through all ordered pairs (A,B) with $A \bigcup B = X$, $A \bigcap B = \emptyset$,

If H(z) and L(z) are the exponential generating functions of h(n) and l(n), Then

$$K(z) = H(z)L(z)$$

Proof. Let $|X| = m$. There are $\binom{m}{j}$ such pairs (S, T) with $|S| = j, |T| = mj$; hence

$$k(m) = \sum_{j=0}^{m} \binom{i}{j} h(j)l(m - j),$$

□

This could be extended to the case of j factors. Let $h_1, h_2, \cdots, h_m$: $N_0 \to C$, and let $H_i(z)$ be the exponential generating function of $h_i(n)$, $i = 1, 2, \cdots, j$. If

$$k(|X|) = \sum_{(T_1,\cdots T_j)} h_1(|T_1|), \cdots h_j(|T_j|)$$

where the sum is overall $(T_1, \cdots, T_j)$ with $\bigcup_{j=1} T_j = X$ and $T_n \bigcap T_m = \emptyset, n \neq m$. Then,

$$K(z) = \prod_{j=1} H_j(z)$$

Theorem 4. *(**Composition Formula**) Let $\prod$ be the set of all j-partitions of $\times$. Let $h, l : N_0 \to \mathbb{C}$ with non empty h(0)=0, and let $k : N_0 \to \mathbb{C}$ be given by*

$$k(|X|) = \sum_{j\geq 1} \sum_{\{B_1,\ldots,B_j\}\in\prod(X)} h(|B_1|) \ldots h(|B_j|)l(j),$$

$(|X| > 0)$ *and k(0)=l(0),*

assuming that the inner sum runs over all j-partitions of X. Then

$$K(z) = L(H(z)).$$

$K(z) = \sum_{n\geq 1} k(n)\frac{z^n}{n!}$, $L(z) = \sum_{n\geq 1} l(n)\frac{z^n}{n!}$, $H(z) = \sum_{n\geq 1} h(n)\frac{z^n}{n!}$

Corollary 2.0.1. *(***The exponential Formula***). Let $h : N_0 \to C$ with $h(0) = 0$ and $k : N_0 \to C$ be given by*

$$k(|X|) = \sum_{j\geq 1} \sum_{\{B_1,\ldots,B_j\}\in\prod(X)} h(|B_1|)\ldots h(|B_j|),$$

$(|X| > 0)$ *and k(0)=1.*

Then $K(z) = e^{H(z)}$

This is the result of theorem 3 for l(n)=1 for all n, making $L(z) = e^z$.

Example 2.0.1. *In how many ways can we partition an n-set into nonempty blocks and order each block linearly? here l(n) = 1 for all n, h(n) = n!; hence* $L(z) = e^z$*, let l(n) count* $\sum_{n\geq 1} n!\frac{z^n}{n!} = \frac{1}{1-z} - 1 = \frac{z}{1-z}$ *and we obtain the result and* $H(z) = \sum_{n\geq 1} n!\frac{z^n}{n!} = \frac{z}{1-z}$,

$$K(z) = e^{\frac{z}{1-z}} = \sum_{n\geq 0}\sum_{k=0}^{n} \frac{n!}{k!}\binom{n-1}{k-1}\frac{z^n}{n!}$$

Let $\Pi = \{B_1, B_2, \cdots, B_k\}$ be a set partition. It is possible to represent Π as $\Pi = B_1|B_2|\cdots|B_k$, by dropping the set brackets and using a slash as a boundary between consecutive blocks. This notation makes it easier to see the inversions, which are introduced as follows.

Definition 2.0.2. *let $\pi = B_1/B_2/\cdots/B_k$ be any set partition and let $b \in B_i$ and $a \in B_j$ (b, a) is said to be an inversion if $b > a$ and $i < j$. The inversion number of π, written $inv(\pi)$ is defined to be the number of inversions of π.*

The number of inversions of a k-partition $\pi = \{B_1, \dots, B_k\} \in \prod_{n,k}$ is denoted by $inv(\pi)$ and is defined by;

$$inv(\pi) = \sum_{1\le i<j\le k} inv(B_i, B_j)(k \ge 2)$$

where $inv(\pi) = 0$ if k = 1.

Example 2.0.2. *Let* $\pi = 137/26/45$ *be a partition of [7]. Then* , $(3,2),(7,2),(7,6),(7,4),(7,5),(6,4$ *and* $(6,5)$ *are inversions in* π. *Thus* $inv(\pi) = 7$.

Theorem 5. *Let* $l, h : N_0 \to C[[_q]]$ *be counting functions with l(0) = 1, h (0) = 0 and h (1) = 1. If* $f_k : N_0 \to C[[_q]]$ *for fixed k is defined by*

$$f_k(n) = \sum_{\Pi=\{B_1,B_2,\cdots,B_k\}\in\Pi_{n+1,k+1}} l(|B_1|-1)f(|B_2|)\cdots f(|B_{k+1}|)q^{inv(\pi)},$$

then the array $(f_k(n))_{n,k\in N_0}$ *may be expressed as the q-Riordan matrix given by* $(L, H)_q$.

Corollary 2.0.2. *Let H(z) be an Eulerian generating functions associated to the counting functions* $h : N_0 \to \mathbb{C}[[_q]]$ *with f(0)=0, f(1)=1. If* $(1, F(z))_q = (h_{n,k})_{n,k\in N_0}$, *then*

$$f_{n,k} = \sum_{\Pi=\{B_1,B_2,\cdots,B_k\}\in\Pi_{n,k}} h(|B_1|)\cdots h(|B_k|)q^{inv(\pi)}, n \ge k \ge 1$$

with $f_{0,0} = 1$, $f_{n,0} = 0$ *for* $n \ge 1$.
The q-analogue of exponential formula : Let $h : N_0 \to \mathbb{C}[[_q]]$ *be the counting function with h(0)=0, h(1)=1, and let* $f : N_0 \to \mathbb{C}[[_q]]$ *be defined as*

$$f(a) = \sum_{k=1}^{n} \sum_{\Pi=\{B_1,B_2,\cdots,B_k\}\in\Pi_{n,k}} h(|B_1|h(|B_2|))\cdots h(|B_k|)q^{inv(\pi)}$$

Then, $F(z) = e_q[H(z)]$
$(1, e^z - 1)$, $e^z - 1 = \sum_{k=1}^{\infty} \frac{z^k}{k!_q}$, $f(n) \geq 1$, *for all* $n \geq 1$.

Corollary 2.0.3. *(The q-analogue of exponential formula). Let* $h : N_0 \to \mathbb{C}[[_q]]$ *be the counting function with h(0)=0 and h(1)=1, and let* $f : N_0 \to \mathbb{C}[[_q]]$ *be defined through*

$$f(a) = \sum_{k=1}^{n} \sum_{\Pi=\{B_1,B_2,\cdots,B_k\}\in\Pi_{n,k}} h(|B_1|h(|B_2|))\cdots h(|B_k|)q^{inv(\pi)}(n \geq 1),$$

Then $F(z) = e_q[H(z)]$.

are (5, 3), (5, 4), which are the same as inversions of the permutation = 1 2 5 3 4 6 7.

2.1 q-Stiriling numbers

Let Z and N be the integers and non-negative integers, respectively.
Let $\rho_{n,k}$ denote the set of all permutations of n in to k disjoint cycles. Now we are ready to see two Striling numbers and combinatorial interpretation are introduced and discussed.

2.1.1 q-Striling numbers of first kind

The (ordinary)Stirling numbers of the first kind, s(n, k), are defined by

$$s(n, k) = (-1)^{n-k}|\rho_{n,k}|$$

where |.| denotes cardinality.

The recurrence relation for this number is

$$s(n,k) = \begin{cases} s(n-1,k-1) + (n-1)s(n-1,k), & \text{if } 0 < k \leq n \\ 1, & \text{if } n = k = 0. \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

k and n running over Z and N, respectively. A q-analogue of this recurrence relation is .

$$s_q[n,k] = \begin{cases} s_q[n-1,k-1] + [n-1]_q s_q[n-1,k], & \text{if } 0 < k \leq n \\ 1, & \text{if } n = k = 0. \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

2.1.2 q-Striling numbers of second kind

The set of all partitions of n into disjoint subsets $B_1, B_2, \cdots, B_k$ where the B's are called block of partition is denoted by $\prod_{n,k}$. The (ordinary) Stirling numbers of the second kind S(n,k) are defined to be:

$$S(n,k) = |\Pi_{n,k}|.$$

The recurrence for these numbers is

$$S(n,k) = \begin{cases} S(n-1,k-1) + kS(n-1,k), & \text{if } 0 < k \leq n \\ 1, & \text{if } n = k = 0. \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

$n, k \in N_0$. The q-analog of these numbers are known as the q-Stirling numbers of second kind, and the recurrence relation for these numbers is:

$$S_q[n,k] = \begin{cases} S_q[n-1,k-1] + [k]_q S_q[n-1,k], & \text{if } 0 < k \leq n \\ 1, & \text{if } n = k = 0. \\ 0, & \text{otherwise} \end{cases} \tag{2.4}$$

From the relation $\sum_k \left\{ {n \atop k} \right\}_q \frac{z^n}{n!_q} = \frac{(e_q(z)-1)^{[k]}}{k!_q}$, it follows that $(\binom{n}{k})_q) = (1, e_q(z)-1)_q$, and using the corollary given above, and the fact that $e_q(z) - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!_q}$, it follows that $h(|B_1|)h(|B_2|) \cdots h(|B_k|) = 1$ in the corollary, and this leads to the new combinatorial significance of the q-Stirling numbers:

$$\left\{ {n \atop k} \right\}_q = \sum_{\pi = B_1, B_2, \cdots B_k \in \Pi_{n,k}} q^{inv(\pi)}$$

Example 2.1.1. *As shown in the table below, all the partitions of the given set into two blocks are given, and to each partition , there corresponds the inversion number* $inv(\pi)$ *, which is used to express the q-Stirling numbers of the second kind combinatorially; which is different from the previous interpretation:*

combinatorial interpretation for the q-Stirling numbers $\left\{ {n \atop k} \right\}_q$ of the second kind as given in [**?**]su+

$$\left\{ {n \atop k} \right\}_q = \sum_{\pi \in \prod_{n,k}} q^{inv(\pi)} \tag{2.5}$$

Example 2.1.2. *let us consider the 2-partitions of* $\{1, 2, 3, 4\}$.
As shown in the table, all the partitions of this set into two blocks are given, and to each partition π*, there corresponds the inversion number* $inv(\pi)$*, used*

to express the q-Stirling numbers of the second kind combinatorially, as polynomials in q.

$\pi \in \prod_{4,2}$	$inv(\pi)$
{{1},{2, 3, 4}}	0
{{1, 2},{3, 4}}	0
{{1, 2, 3},{4}}	0
{{1, 3},{2, 4}}	1
{{1, 2, 4},{3}}	1
{{1, 4},{2, 3}}	2
{{1,3,4},{2}}	2

Thus

$$\begin{Bmatrix} 4 \\ 2 \end{Bmatrix}_q = \sum_{\pi \in \prod_{4,2}} q^{inv(\pi)} = 3 + 2q + 2q^2$$

2.2 The q-analogue of (r,s)-Bessel numbers

A(n,k), the Bessel number of first kind is the number of k-cyclic permutations q[n], where each cycle is q length ≤ 2. the 2^{nd} kind and Bessel number of the numbers of k-partitions of [n], where each block is q length ≤ 2. The Bessel number of the first kind A(n,k) and second kind B(n, k), is the number of ways in which an n-set into k cycle permutation and partitioned into k blocks respectively. These Bessel numbers have many properties similar to those of the Stirling numbers of the first and second kind respectively. Let B_i represent block and let C_i represent cycle, where $i = \{1, \cdots, k+1\}$.

Let $\prod_{n+r,k+r}^{(r,s)}$ be the collection of (k+r)-partitions of the set $\langle n+r \rangle$ where each block in q size at most s, and the numbers $1, 2, \cdots, r$ are in the distnict blocks. let $\rho_{n+r,k+r}^{(r,s)}$ be the collection of (k+r)-cyclic permutations of $\langle n+r \rangle$

where each cycle has length at most s, and $1, 2, \cdots, r$ are in the same cycles. Then

$$|\Pi^{(r,s)}_{n+r,k+r}| = B^q_{r,s}(n,k)$$

and

$$|\rho^{(r,s)}_{n+r,k+r}| = A^q_{r,s}(n,k).$$

If $\Pi^{(r,s)}_{n+r,k+r}$ and $\rho^{(r,s)}_{n+r,k+r}$ are as introduced above, then q-analogue of (r,s)-Bessel numbers of both kinds, $B_{r,s}(n,k)$ and $A_{r,s}(n,k)$, are defined as:

$$B^q_{r,s}(n,k) = \sum_{\pi \in \Pi^{(r,s)}_{n+r,k+r}} q^{inv(\pi)} A^q_{r,s}(n,k) = \sum_{\delta \in \rho^{(r,s)}_{n+r,k+r}} q^{inv(\delta)}$$

By setting r=0 and s=2, we obtain $A^q_{0,2}(n,k) = B^q_{0,2}(n,k)$ which reduces to the classical Bessel numbers of the second kind when q=1.

The numbers $B^q_{0,2}(n,k)(or A^q_{0,2}(n,k))$ refer to the q-Bessel numbers of the second kind and are simply denoted by $B_q(n,k)$. thus have

$$B_q(n,k)^q = \sum_{\pi \in \prod^{(0,2)}_{n,k}} q^{inv(\pi)}, (n-k \leq \frac{n}{2}) \tag{2.6}$$

$inv(\pi)$ and $inv(\delta)$ stands for the number of inversion of the partition π and the cyclic permutation δ of [n+r].

2.3 Combinatorial interpretation of q-Bessel numbers

What the authors of [9] and is that they have shown that the q-analogue of the classical (r,s)-Bessel numbers, $B^q_{r,s}(n,k)$ and $A^q_{r,s}(n,k)$ are the entries of some q-Riordan matrices, as given bellow:

$$(B^q_{r,s}(n,k))_{n,k\in N_0} = ((\psi)) = ((\psi_s^{'}))(A^q_{r,s}(n,k))_{n,k\in N_0} = ((\varphi_s^{'})^r, \varphi_s)_q$$

Where $\psi_s = z + \frac{z^2}{2!_q} + \cdots + \frac{z^s}{s!_q}$ and $\varphi_s = z + \frac{z^2}{[2]_q} + \cdots + \frac{z^s}{[s]_q}$, $s \geq 2$

Now we turn to combinatorial interpretations for the q-Bessel numbers of both kinds.

Theorem 6. *The q-Bessel numbers of both kind satisfy the recurrence relations:*

1. $B_q(n+1,k+1) = B_q(n,k) + [n]_q B_q(n-1,k)$;
2. $b_q(n+1,k+1) = b_q(n,k) + [k+1]_q b_q(n+1,k+2)$

where $B_q(0,0) = b_q(0,0) = 1$,$B_q(1,1) = b_q(1,1) = 1$ *and* $B_q(n,0) = b_q(n,0) = 0$ *for* $n \geq 1$.

Example 2.3.1. *construct few rows of the q-Bessel numbers of first kind by using recurrence relation defined in the theorem given above.*

$b_q(n+1,k+1) = b_q(n,k) + [k+1]_q b_q(n+1,k+2)$

$b_q(n,k) = b_q(n-1,k-1) + [k]_q b_q(n,k+1)$

First let get some entry ;

$b_q(0,0) = 1$, *since for every q-Riordan matrix the diagonal elements are 1.*

$b_q(1,0) = 0$, *since* $b_q(n,0) = 0$ *for* $n \geq 1$

$$\begin{aligned} b_q(2,1) &= b_q(1,0) + [1]_q b_q(2,2) \\ &= 0 + 1 \\ &= 1 \end{aligned}$$

$$
\begin{aligned}
b_q(3,2) &= b_q(2,1) + [2]_q b_q(3,3) \\
&= 1 + 1 + q \\
&= 2 + q
\end{aligned}
$$

$$b_q(3,1) = b_q(2,0) + [1]_q b_q(3,2) = 0 + b_q(3,2) = 2 + q$$

$$
\begin{aligned}
b_q(4,3) &= b_q(3,2) + [3]_q b_q(4,4) \\
&= q + 2 + 1 + q + q^2 \\
&= 3 + 2q + q^2
\end{aligned}
$$

$$
\begin{aligned}
b_q(4,2) &= b_q(3,1) + [2]_q b_q(4,3) \\
&= 2 + q + (1+q)(3 + 2q + q^2) = 5 + 6q + 3q^2 + q^3
\end{aligned}
$$

$b_q(4,1) = b_q(3,0) + [1]_q b_q(4,2) = 5 + 6q + 3q^2 + q^3$ *then , the matrix is;*

$$
(b_q(n,k))_{n,k\in N_0} = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & & \\
0 & 1 & 0 & 0 & \cdots & & \\
0 & 1 & 1 & 0 & \cdots & & \\
0 & 2+q & 2+q & 1 & 0\cdots & & \\
0 & 5+6q+3q^2+q^3 & 5+6q+3q^2+q^3 & 3+2q+q^2 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & & &
\end{pmatrix}
$$

2.3.1 Matching and covering

Definition 2.3.1. *A set of non loop edges which are mutually non adjacent is called a* **matching** *of a graph.*

As defined earlier let $\langle n \rangle$ be the set of vertices. A matching M of $\langle n \rangle$ is a set of pairs (i,j) with no common vertices in which $i < j$ for $i, j \in \langle n \rangle$. Each pair (i,j) is called an edge of the matching M and is denoted by $E_{i,j}$.

Two edges $E_{i,j}$ and $E_{k,l}$ of M have a crossing if $i < k < j < l$ or $k < i < l < j$.

Definition 2.3.2. *A set of vertices K in a graph G is said to be a* **vertex covering** *(or simply covering) if every edge in G has at least one end in K.*
The edge E_{i_j} of M covers $v \in \langle n \rangle$ if $i < v < j$.

A vertex $v \in \langle n \rangle$ is matched at M if v is incident to an edge in the matching M; otherwise the vertex is unmatched at M.
In this section, let cro(M) denote the number of crossings of M, $cov(M) = \sum_{v \in \langle n \rangle} cov(v)$ where cov(v) is the number of edges of M that cover v, and $\mathcal{M}_{n,k}$ the set of matchings of $\langle n \rangle$ with k edges.

Theorem 7. *The q-Bessel numbers $B_q(n,k)$ of the second kind may be combinatorially expressed as follows:*

$$B_q(n,k) = \sum_{M \in \mathcal{M}_{n,n-k}} q^{cov(M) - cro(M)}, (n - k \leq \frac{n}{2})$$

Proof. For integers $n, k \geq 0$ with $n - k \leq \frac{n}{2}$, let $\gamma : \prod_{n.k}^{(0,2)} \to \mathcal{M}_{n,n-k}$ be a map defined by $\gamma(\pi) = \{E_{i,j} : \{i,j\} \in \pi\}$. Since every partition $\pi \in \prod_{n,k}^{(0,2)}$ has exactly n-k blocks of size 2, obviously the map γ is bijective. If (a,b) is an inversion of $\pi = \{B_1 \cdots B_k\} \in \prod_{n,k}^{(0,2)}$, then there are i, j with $i < j$ such that $a \in B_i$ and $b \in B_j$. Since $minB_i < minB_j$, the block B_i contains an element $c \in \langle n \rangle$ with $c < b < a$ while B_j does not contain an element $d \in \langle n \rangle$ with $d < c$. Thus the edge $E_{c,a}$ of the matching $\gamma(\pi)$ covers b and there is no edge $E_{d,b}$ crossing $E_{c,a}$. It follows $inv(\pi) = cov(\gamma(\pi)) - cro(\gamma(\pi))$. By (2.7) we thus have

$$B_q(n,k) = \sum_{\pi \in \prod_{n.k}^{(0,2)}} q^{inv(\pi)} = \sum_{\gamma(\pi)\in\mathcal{M}_{n,n-k}} q^{cov(\gamma(\pi))-cro(\gamma(\pi))}, (n-k \leq \frac{n}{2})$$

as desired. □

To give a combinatorical interpretation for the q-Bessel numbers of the first kind, let $ucov(M) = \sum_{v\in U(M)} cov(v)$ where U(M) is the set of vertices unmatched at M.

Lemma 2.3.1. *Let $\overline{\mathcal{M}}_{n,k}$ be the collection of the matchings $\overline{M}$ in $\mathcal{M}_{n,k}$ in which n is among the matched vertices. Then*

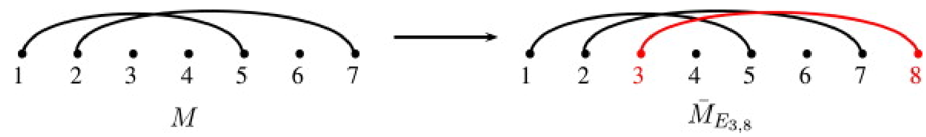
$$\sum_{\overline{M}\in\overline{\mathcal{M}}_{n,k}} q^{ucov(\overline{M})-cro(\overline{M})} = [n-2k+1]_q \sum_{M\in\mathcal{M}_{n-1,k-1}} q^{ucov(M)-cro(M)}, (n \geq 2k).$$

Proof 1. *Let M be a matching in $\mathcal{M}_{n-1,k-1}$. Since there are n-2k+1 unmatched vertices in M, say $v_1, \cdots, v_{n-2k+1}$ where $v_i < v_j$ for $i < j$, every matching in $\overline{\mathcal{M}}_{n,k}$ may be obtained from M by adding the edge $E_{v_l,n}$ for some $l \in \{1, \cdots, n-2k+1\}$. Let $\overline{M}_{E_{v_l,n}} = M \bigcup \{E_{v_l,n}\} \in \overline{\mathcal{M}}_{n,k}$. Since the number of crossings with the edge $E_{v_l,n}$ of $\overline{M}_{E_{v_l,n}}$ is equal to $cov(v_l)$ at M, we have $cro(\overline{M}_{E_{v_l,n}}) = cro(M) + cov(v_l)$. Also since the vertex v_l is matched at $\overline{M}_{E_{v_l,n}}$ and $E_{v_l,n}$ covers vertices $v_{l+1}, \cdots, v_{n-2k+1}$ unmatched at $\overline{M}_{E_{v_l,n}}$, we have $ucov(\overline{M}_{E_{v_l,n}}) = ucov(M) - cov(v_l) + n - 2k + 1 - l$. Thus*

$$ucov(\overline{M}_{E_{v_l,n}}) + cro(\overline{M}_{E_{v_l,n}}) = ucov(M) - cov(M) + n - 2k + 1 - l. \quad (2.7)$$

For instance, consider two matchings M and $\overline{M}_{E_{v_l,n}}$ with n=8,k=3 and l=1 as shown in the following figure. Note that $v_1 = 3, v_2 = 4, v_3 = 6$ and $E_{v_1,n} = E_{3,8}$. The vertex v_1 is covered by $E_{1,5}$ and $E_{2,7}$ in M and the edge $E_{3,8}$

is crossing with $E_{1,5}$ *and* $E_{2,7}$ *in* $\overline{M}_{E_{3,8}}$. *Also we have* $ucov(M) = cov(3) + cov(4) + cov(6) = 5, cro(M) = 1, ucov(\overline{M}_{E_{3,8}}) = cov(4) + cov(6) = 5$ *and* $cro(\overline{M}_{E_{3,8}}) = 3$.



Hence, from (2.8) we obtain

$$\begin{aligned}
\sum_{\overline{M}\in\overline{\mathcal{M}}_{n,k}} q^{ucov(\overline{M})-cro(\overline{M})} &= \sum_{l=1}^{n-2k+1} \sum_{\overline{M}_{E_{v_l,n}}\in\overline{\mathcal{M}}_{n,k}} q^{ucov(\overline{M}_{E_{v_l,n}})} - cro(\overline{M}_{E_{v_l,n}}) \\
&= \sum_{l=1}^{n-2k+1} q^{n-2k+1-l} \sum_{\overline{M}\in\overline{\mathcal{M}}_{n-1,k-1}} q^{ucov(M)-cro(M)} \\
&= [n-2k+1]_q \sum_{\overline{M}\in\overline{\mathcal{M}}_{n-1,k-1}} q^{ucov(M)-cro(M)}
\end{aligned}$$

as desired.

Theorem 8. *The q-Bessel numbers of the first kind $b_q(n,k)$ may be combinatorially expressed as follows:*

$$b_q(n,k) = \sum_{M\in\mathcal{M}_{2n-k-1,n-k}} q^{ucov(M)+cro(M)}, (n \geq k \geq 1) \tag{2.8}$$

where $b_q(1,1) = 1$.

Proof 2. *We proceed by induction on $n \geq 2$ for each $k = 1, \cdots, n$. Let n=2. Since $\mathcal{M}_{2,1} = \{\{E_{1,2}\}\}$ and $\mathcal{M}_{1,0} = \{0\}$, we have $\sum_{M\in\mathcal{M}_{2,1}} q^{ucov(M)+cro(M)} = 1 = b_q(2,1)$ and $\sum_{M\in\mathcal{M}_{1,0}} q^{ucov(M)+cro(M)} = 1 = b_q(2,2)$, which completes the case n=2. Let $n \geq 3$ and assume that (2.8) holds for each $k = 1, \cdots, n$. We note that every matching of $\mathcal{M}_{2n-k,n-k}$ may be obtained from one of the following ways:*

1. *by adding the vertex n unmatched at a matching M of $\mathcal{M}_{2n-k-1,n-k}$;*

2. *by adding the edge* $E_{u,2n-k}$ *to a matching of* $\mathcal{M}_{2n-k-1,n-k-1}$ *if* u *is a vertex unmatched.*

Thus

$$\begin{aligned} b_q(n+1,k+1) &= \sum_{M\in\mathcal{M}_{2n-k,n-k}} q^{ucov(M)+cov(M)} \qquad (2.9) \\ &= \sum_{M\in\mathcal{M}_{2n-k,n-k}} q^{ucov(M)+cro(M)} + \sum_{\overline{M}\in\overline{\mathcal{M}}_{2n-k,n-k}} q^{ucov(\overline{M})+cro(\overline{M})} \end{aligned}$$

Applying Lemma 2.3.1 to the second sum of the right-hand side of (2.9), we obtain

$$b_q(n+1,k+1) = \sum_{M\in\mathcal{M}_{2n-k,n-k}} q^{ucov(M)+cro(M)} + [k+1]_q \sum_{M\in\mathcal{M}_{2n-k-1,n-k-1}} q^{ucov(M)+cro(M)}$$

By induction, we thus have

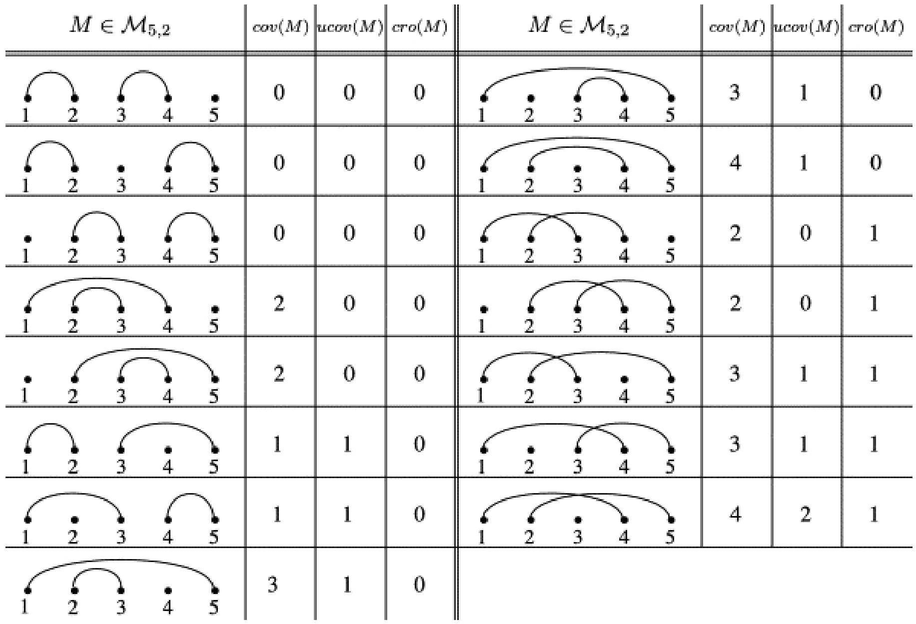
$$b_q(n+1,k+1) = b_q(n,k) + [k+1]_q b_q(n+1,k+2),$$

which implies that $b_q(n,k)$ *are the Bessel numbers of the first kind from (ii) of Theorem 6. Hence the proof follows.*

It is known [17] that noncrossing matchings of $\mathcal{M}_{2n,n}$ can be counted by the n^{th} Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. More generally, we obtain the following corollary.

Corollary 2.3.1. *The number of non crossing matchings of* $\mathcal{M}_{2n-k,n-k}, k = 0,\cdots n$, *with no edge covering unmatched vertices is* $\frac{k+1}{n+1}\binom{2n-k}{n-k}$.

Example 2.3.2. *Consider the set* $\mathcal{M}_{5,2}$ *of matchings of* $\langle 5\rangle$ *with 2 edges. All the matchings and corresponding numbers cov(M),ucov(M) and cro(M) are displayed in the following table.*



$M \in \mathcal{M}_{5,2}$	$cov(M)$	$ucov(M)$	$cro(M)$	$M \in \mathcal{M}_{5,2}$	$cov(M)$	$ucov(M)$	$cro(M)$
	0	0	0		3	1	0
	0	0	0		4	1	0
	0	0	0		2	0	1
	2	0	0		2	0	1
	2	0	0		3	1	1
	1	1	0		3	1	1
	1	1	0		4	2	1
	3	1	0				

thus we have

$$\begin{aligned} B_q(5,3) &= \sum_{M\in\mathcal{M}_{5,2}} q^{cov(M)-cro(M)} \\ &= 3+4q+4q^2+3q^3+q^4 \end{aligned}$$

$$\begin{aligned} b_q(4,2) &= \sum_{M\in\mathcal{M}_{5,2}} q^{ucov(M)+cro(M)} \\ &= 5+6q+3q^2+q^3 \end{aligned}$$

In particular,$b_0(4,2) = C_{3,1} = 5$ *counts noncrossing matchings of* $\mathcal{M}_{5,2}$ *with no edge covering unmatched vertices. See the first 5 matchings of the table.*

Chapter 3

CONCLUSION

In this project work, a generalization of Riordan arrays: q-Riordan arrays and combinatorial interpretations of generalized forms of some well known numbers: q-Stirling numbers and q-Bessel numbers, based on some published studies, have been introduced and discussed.

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