The FBI Transform and Microlocal Analysis in Ultradifferentiable Classes

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Declaration

I, Jemal Yesuf, with student number GSR/6062/09 hereby declare that this dissertation is my original work and has not been presented for a degree in any other University and that all sources of information used for the thesis have been fully acknowledged.

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Abstract

The FBI transform is a nonlinear Fourier transform that characterizes the local/ microlocal smoothness and analyticity of functions (or distributions) in terms of appropriate decays. This characterization is very useful in studying the local and microlocal regularity of solutions of partial differential equations.

The ultradifferentiable classes play an important role in the theory of differential equations as they provide an intermediate scale of spaces between C^∞ and real analytic functions.

In this thesis, we establish the boundedness of a class of FBI transforms in Sobolev spaces. We characterize the ultradifferentiable wave front set by a class of FBI transforms. We also provide an application that shows how powerful are these generalized class of FBI transforms by exhibiting a result on microlocal regularity for solutions of first order nonlinear partial differential equations in these classes, which can not be solved by the classical FBI transforms. Finally, we use the FBI transform to characterize microlocal smoothness and microlocal ultradifferentiablity on maximally real submanifolds.

List of Papers

I. S. Berhanu and Jemal Yesuf, Continuity of a class of FBI transforms on Sobolev spaces, Accepted for publication in Houston Journal of Mathematics.

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Addis Ababa, Ethiopia

Contents

Notations

We will use the following notations. Let $\Omega \subset \mathbb{R}^m$ open.

$\Re \equiv$ The real part

$\Im \equiv$ The imaginary part

$\mathbb{N}_0 \equiv \{0, 1, 2, \ldots\}$

$\mathbb{N}_0^m \equiv \{(\alpha_1, \ldots, \alpha_m) : \alpha_i \in \mathbb{N}_0\}$

$C^k(\Omega) \equiv \{f : \partial_x^\alpha f \in C(\Omega)\ \forall \alpha \text{ with } |\alpha| \leq k\}$

$C^\infty(\Omega) \equiv \{f : f \in C^k(\Omega)\ \forall k\}$

$C_0^k(\Omega) \equiv \{f \in C^k(\Omega) : supp(f) \text{ is compact}\}$

$C_0^\infty(\Omega) \equiv \{f \in C^\infty(\Omega) : supp(f) \text{ is compact}\}$.

$C^\omega(\Omega) \equiv$ The space of real analytic functions on Ω.

$D'(\Omega) \equiv$ The space of distributions on $C_0^\infty(\Omega)$.

$S(\mathbb{R}^m) \equiv \{f \in C^\infty(\mathbb{R}^m) : \sup_{x \in \mathbb{R}^m} |x^\alpha \partial_x^\beta f(x)| < \infty\ \forall \alpha, \beta\}$

$S'(\mathbb{R}^m) \equiv$ The space of tempered distributions on $\mathbb{S}(\mathbb{R}^m)$.

$\mathcal{E}'(\Omega) \equiv \{u \in D'(\Omega) : supp(u) \text{ is compact}\}$

$\mathcal{O}(\Omega') \equiv \{f : f \text{ is holomorphic on } \Omega'\}$, $\Omega' \subset \mathbb{C}^m$ open.

Introduction

The Fourier transform of a function $u \in C_c^0(\mathbb{R}^m)$ or a distribution $u \in \mathcal{E}'(\mathbb{R}^m)$ is defined by

$$\hat{u}(\xi) = \int_{\mathbb{R}^m} e^{i\xi.x} u(x)dx, \ \xi \in \mathbb{R}^m$$

where $\xi.x = \sum_{j=1}^m x_j\xi_j$, in the latter case the integral is understood in duality sense. Besides the applications to solve differential equations, the Fourier transform is used to characterize smoothness of a function. For instance, the Paley-Wiener theorem states that a function u is C^∞ on $\mathbb{R}^m$ if and only if for every $k = 1, 2, \ldots$, there exist $C_k > 0$ such that

$$|\hat{u}(\xi)| \leq \frac{C_k}{(1 + |\xi|)^k}, \ \ \forall \xi \in \mathbb{R}^m.$$

However the regular Fourier transform is not sufficient for many purposes, in particular, in microlocal analysis when the locality is not only with respect to position in space but also cotangent space directions at a given point. The FBI (Fourier-Bros-Iagolnitzer) transform is a nonlinear transform developed by the French mathematical physicists Joseph Fourier, Jacques Bros and Daniel Iagolnitzer in order to characterize the local and microlocal analyticity of functions (or distributions).

The classical FBI (Fourier-Bros-Iagolintzer) transform is of the form

$$\mathcal{F}u(x, \xi) = \int_{\mathbb{R}^m} e^{i\xi.(x-x') - |\xi||x-x'|^2} u(x')dx', \ x, \xi \in \mathbb{R}^m \qquad (0.0.1)$$

where $u \in C_c^0(\mathbb{R}^m)$ or $u \in \mathcal{E}'(\mathbb{R}^m)$ in which case the integral is understood in the duality sense. This transform characterizes microlocal analyticity and microlocal smoothness, has been used to study the regularity of solutions of linear and nonlinear partial differential equations (see [3, 7, 16, 19, 21, 34]).

In [11] S. Berhanu and J. Hounie introduced the following more general class of FBI transforms. Let $\psi \in \mathcal{S}(\mathbb{R}^m)$ such that $0 \neq \int |\psi(x)|dx < \infty$. They define the FBI transform of a compactly supported distribution $u \in \mathcal{E}'(\mathbb{R}^m)$ with generating function ψ and parameter λ $(0 < \lambda < 1)$, by

$$\mathcal{F}_{\psi,\lambda}u(x,\xi) = \langle u(x'), e^{i\xi.(x-x')}\psi\Big(|\xi|^{\lambda}(x-x')\Big)\rangle, \tag{0.0.2}$$

A simple example of this generalized transform is

$$\mathcal{F}_k u(x,\xi) = \int_{\mathbb{R}^m} e^{i\xi.(x-x')-|\xi||x-x'|^{2k}} u(x')dx', \; x,\xi \in \mathbb{R}^m, \; k = 1,2,\dots \tag{0.0.3}$$

where the $k = 1$ case gives the classical FBI transform.
We establish the boundedness of the FBI transform (0.0.3) in Sobolev spaces.

S. Berhanu and J. Hounie in [11] characterized microlocal smoothness and microlocal analyticity by taking $\psi(x) = c_p e^{-p(x)}$, where $p(x)$ is real, homogenoues, positive elliptic polynomial, and $c_p = \left(\int e^{-p(x)}dx\right)^{-1}$. Later, S. Berhanu and Abraham Hailu [10] used a variant of these transforms (taking a sum of two polynomials) to characterize microlocal Gevrey regularity.

A natural extension is the ultradifferentiable class $\mathcal{E}^M$, which is obtained by considering a sequence of positive real numbers $M = (M_j)$ satisfying some properties (see Chapter 1 section 1.1). Under suitable assumption on the sequence M, one obtains for $\mathcal{E}^M$ results similar to those valid for G^s (see [22, 27, 29]). Recently, G. Hoepfner and R. Medrado [22] used these class of FBI transforms to characterize local and microlocal regularity of ultradistributions. Here we use a more general class of the FBI transform of [11] to characterize the ultradifferentiable wavefront set, which generalizes the work of S. Berhanu and Abraham

Hailu [10].

Let U be an open neighborhood of 0 in $\mathbb{R}^m$ with coordinates $x_1, \ldots, x_m$ and $Z = (Z_1, \ldots, Z_m) : U \to \mathbb{C}^m$ is a C^∞ map, $dZ_1, \ldots, dZ_m$ linearly independent on U. Write $Z(x) = x + i\phi(x)$, where ϕ is real valued, smooth, $\phi(0) = 0$ and $d\phi(0) = 0$ so that $\mathcal{X} = Z(U)$ is maximally real submanifold. Let u be a compactly supported distribution in the manifold $\mathcal{X}$. Define the FBI transform of u by

$$\mathcal{F}u(z,\zeta) = \int_{\mathcal{X}} e^{i\zeta.(z-z')-\langle\zeta\rangle[z-z']^2} u(z')\Delta(z-z',\zeta)dz'$$

in which $z \in \mathcal{X}$, $\zeta \in \mathcal{C}_1$, where $\mathcal{C}_1 = \{\zeta \in \mathbb{C}^m : |\Im\zeta| < |\Re\zeta|\}$, $\Delta(z,\zeta)$ is the Jacobian determinant of the map $\zeta \to \zeta + i\langle\zeta\rangle z$ (where $\zeta \in \mathcal{C}_1, z \in \mathbb{C}^m$). Here $\langle\zeta\rangle = (\zeta.\zeta)^{\frac{1}{2}}$ (main branch of the square root). In [35], the author used the above FBI transform to characterize smoothness locally.

We will use the above FBI transform to characterize microlocal smoothness of a distribution. When the Z_j are $\mathcal{E}^M$, we obtain a $\mathcal{E}^M$ maximally real submanifold and we use the corresponding FBI transform to characterize microlocal ultradifferentiablity. When $\phi(x) \equiv 0$, we recover the FBI transform (0.0.1) and hence our result generalizes characterizations using (0.0.1).

In chapter one we review some notions, definitions and results that are useful for the later chapters. In chapter two we establish boundedness of a class of FBI transforms (0.0.3) on Sobolev spaces. In the third chapter we chracterize the ultradifferentiable wave front of a distrbution by a class of FBI transforms. The last chapter gives FBI transform characterization of microlocal smoothness and microlocal ultradifferentiablity on maximally real submnifolds.

Chapter 1

Some Preliminary Concepts

In this chapter, we review ultradifferentiable classes, conic sets, boundary value of holomorphic functions, wave front sets and FBI transforms, integrable structures, maximally real submanifolds and pseudo-differential operators for later use.

1.1 Ultradifferentiable Functions and Some Properties

The ultradifferentiable class play an important role in the theory of differential equations as they provide an intermediate scale of spaces between C^∞ and real analytic functions.

Definition 1.1.1. Let $\Omega \subset \mathbb{R}^m$ an open subset and $(M_j)_{j\in\mathbb{N}}$ be an increasing sequence of positive real numbers satisfying some properties (see below). The ultradifferentiable (Denjoy-Carleman) spaces in Ω, $\mathcal{E}^M(\Omega)$, is defined as the set of all functions f in $C^\infty(\Omega)$ that satisfies the following property: for each $K \subset\subset \Omega$ there exist a constant $C > 0$, depending on K and f, such that

$$|\partial^\alpha f(x)| \leq C^{|\alpha|+1} M_{|\alpha|}, \ \ \forall \alpha \in \mathbb{N}_0^m, \ \forall x \in K.$$

We will assume that the sequence $M = (M_j)$ of positive real numbers to satisfy the following conditions:

(1)(Initial Conditions)

$$(P1) \qquad M_0 = M_1 = 1.$$

(2)(Non-quasianalyticity)

$$(P2) \qquad \sum_{j=0}^{\infty} \frac{M_j}{M_{j+1}} < \infty.$$

This condition insures the existence of no-trivial $\mathcal{E}^M$ functions of compact support.

(3)(Stability under ultradifferentiable operators) There exists a constant $D > 1$, independent of j, such that for all $j \geq 0$,

$$(P3) \qquad M_{j+1}^{j} \leq D^j M_j^{j+1}.$$

Condition (P3) implies the following condition: There exists a constant $D > 1$, independent of j, k, such that for all $j \leq k, k, j \in \mathbb{N}$

$$(P3') \qquad M_k^{\frac{1}{k}} \leq D^{\frac{k}{j}} M_j^{\frac{1}{j}}.$$

Also, condition (P3) implies the stability under differential operators condition; i.e., for all $j, k \in \mathbb{N}$

$$(P3'') \qquad M_{j+k} \leq AH^{j+k} M_j M_k,$$

where $AH^j = D^j,\ A, H > 0$ and D is the same constant appearing in (P3) and is independent of j, k.

(4)(Invariance under composition) For all $j, k \in \mathbb{N}$ with $0 \leq j \leq k$, we have

$$(P4) \qquad \binom{k}{j} M_{k-j} M_j \leq M_k.$$

(5)(Strong logarithmic convexity) For fixed $A > 1$ and for any $r, 0 \leq r < \frac{1}{A} < 1$, the sequence

$$(P5) \qquad \frac{P_j}{j P_{j-1}}.$$

is increasing, where $P_j = \frac{M_j}{(j!)^r}$. This implies, in particular, that M is logarithimically convex;i.e., for all $j \in \mathbb{N}$

$$(P5') \qquad M_j^2 \leq M_{j-1} M_{j+1}.$$

and this implies, in particular, that for all $j, k \in \mathbb{N}$

$$(P5'') \qquad M_j M_k \leq M_{j+k}.$$

(6)(Invariance under division) The sequence $Q = (Q_j)$ where $Q_0 = 1$ and for $j \geq 1$

$$Q_j = \left(\frac{M_j}{j!} \right)^{\frac{1}{j}}$$

is increasing, that is, for all $j < k$

$$(P6) \qquad Q_j \leq Q_k.$$

This condition insures that the class $\mathcal{E}^M(\Omega)$ is inverse closed; i.e., if $f \in \mathcal{E}^M(\Omega)$ and $\inf_{x \in \Omega} |f(x)| > 0$, then $\frac{1}{f} \in \mathcal{E}^M(\Omega)$.
(7)(Faá di Bruno) For all $j, k \in \mathbb{N}$, if $n = jk$, there is a constant $C > 1$, independent of n, so that

$$(P7) \qquad M_j^k \leq C^n M_{n-k}.$$

This condition is used when we need to apply multi-variable Faá di Bruno formula for computing the derivatives of compositions of functions.

Remark 1.1.2. i). If M satisfies (P1) and (P6), then it satisfies the following: For all $j = 1, 2, \ldots$

$$(P8) \qquad\qquad M_j \geq j!.$$

ii). The sequence M is quasianalytic if

$$\sum_{j=0}^{\infty} \frac{M_j}{M_{j+1}} = \infty.$$

Example 1.1.3. Let $s > 1$ be a real number and choose

$$M_j = (j!)^s.$$

Then $M = (M_j)$ satisfies (P1)-(P7) and $\mathcal{E}^M(\Omega) = G^s(\Omega)$ denotes the $s-$ Gevrey space. If $M_j = j!$, then $M = (M_j)$ satisfies all conditions except $(P2)$, and $\mathcal{E}^M(\Omega) = C^\omega(\Omega)$ (the space of real analytic functions).

Proof. (P1): Clearly $M_0 = M_1 = 1$.
(P2): Since a $p-$series with $p > 1$ converges, we have

$$\sum_{j=0}^{\infty} \frac{M_j}{M_{j+1}} = \sum_{j=0}^{\infty} \frac{1}{(j+1)^s} < +\infty.$$

$$
\begin{aligned}
(P3): M_{j+1}^{j} = (j+1)!^{sj} &= (j+1)^{sj}(j!)^{sj} \\
&= \frac{(j+1)^{sj}}{(j!)^{s}}(j!)^{s(j+1)} = \frac{(j+1)^{sj}}{(j!)^{s}} M_j^{j+1} \\
&= \left(\frac{(j+1)}{(j!)^{\frac{1}{j}}}\right)^{js} M_j^{j+1} \\
&\le \left(\frac{e(j+1)}{j}\right)^{js} M_j^{j+1} \text{ since } \frac{1}{(j!)^{\frac{1}{j}}} \le \frac{e}{j} \\
&\le \left((2e)^{s}\right)^{j} M_j^{j+1}
\end{aligned}
$$

Thus the condition follows with $D = (2e)^s$.

$$
\begin{aligned}
(P4): \binom{k}{j} M_{k-j} M_j &= k!(k-j)!^{s-1}(j!)^{s-1} \\
&\le k!(k-j+j)!^{s-1} \text{ since } a!b! \le (a+b)! \\
&= (k!)^{s} = M_k
\end{aligned}
$$

Thus the condition holds.

(P5): $\frac{P_j}{jP_{j-1}} = j^{s-r-1}$. Fix $A > 1$ such that $s \ge 1 + \frac{1}{A}$. Then we have $s - r - 1 \ge 0$ and hence the condition follows.

(P6): Now, for $s > 1, j \ge 1$ we have

$$
\frac{Q_{j+1}}{Q_j} = \left(\frac{((j+1)!)^{\frac{1}{j+1}}}{(j!)^{\frac{1}{j}}}\right)^{s-1} = \left(\frac{((j+1)!)^{j}}{(j!)^{j+1}}\right)^{\frac{s-1}{j(j+1)}} = \left(\frac{(j+1)^{j}}{j!}\right)^{\frac{s-1}{j(j+1)}} > 1.
$$

(P7): Let $j, k \in \mathbb{N}$ and $n = jk$. Then

$$
\begin{aligned}
\frac{M_j^k}{M_{n-k}} &= \frac{(j!)^{sk}}{(k(j-1))!^{s}} \\
&\le \frac{(j!)^{sk}}{(j-1)!^{sk}} \text{ since } (a!)^{b} \le (ab)! \\
&= j^{sk} \le 2^{sn} = (2^{s})^{n}.
\end{aligned}
$$

Hence, condition (P7) is satisfied with $C = 2^s$. □

Example 1.1.4 (More Examples)**.** (See [32]) a) Let $q > 1$. Put $M_j = q^{j^2},\ j \in \mathbb{N}$. The corresponding $\mathcal{E}^M$ functions are called $q-$Gevrey regular. Then $M = (M_j)$ is non-quasianalytic.
b) Let $\delta > 0$ and $M_j = \Big(\log(j+e)\Big)^{\delta j}$ for $j \in \mathbb{N}$. Then $M = (M_j)$ is quasianalytic for $0 < \delta \leq 1$ and non-quasianalytic for $\delta > 1$.

Note that if $M = (M_j)$ and $N = (N_j)$ satisfy $M_j \leq C^j N_j$, $\forall j$ and a constant C, then $\mathcal{E}^M(\Omega) \subset \mathcal{E}^N(\Omega)$. The converse is true as well by the logarithimic convexity assumption. In particular, if $f \in G^s(\Omega)$ and $s \leq t$, then $f \in G^t(\Omega)$. Thus $G^1 \subset G^s$, $\forall s \geq 1$.
Setting $N_j = j!$ in the above inequality yields $C^\omega(\Omega) = \mathcal{E}^M(\Omega)$ if and only if $\sup_{j\in\mathbb{N}}(\frac{M_j}{j!})^{\frac{1}{j}} < \infty$. Since $(\frac{M_j}{j!})^{\frac{1}{j}}$ is almost increasing the strict inclusion $C^\omega(\Omega) \subset \mathcal{E}^M(\Omega)$ is equivalent to $\sup_{j\in\mathbb{N}}(\frac{M_j}{j!})^{\frac{1}{j}} = \infty$.

Lemma 1.1.5. *(E. Borel)[[27], Theorem 3.1.1] For each multi-index α of length N, let a_α be a real number. Then there is a C^∞ function on the unit ball $\mathbb{B}(0,1) \subset \mathbb{R}^N$ with*

$$\frac{\partial^\alpha f}{\partial x^\alpha}(0) = a_\alpha \ \text{ for every multi-index } \alpha.$$

Example 1.1.6. By the Borel's lemma there is a smooth function $f(x)$ on $\mathbb{R}$ such that $f^{(n)}(0) = n!(n+1)^{(n+1)}(n!)^n$, $\forall n \geq 0$. One can prove that $f \notin G^s(\mathbb{R})\ \forall s \geq 1$.

As the following example shows there are functions in G^s for $s > 1$, which are not in C^ω.

Example 1.1.7. For $s > 1$ define $f_s(t)$ by

$$f_s(t) = \begin{cases} e^{-t^{-\frac{1}{s-1}}},\ t > 0 \\ 0,\ t \leq 0 \end{cases}$$

Then $f_s \in G^s(\mathbb{R})$.But $f \notin\ C^\omega(\mathbb{R})$.

Proof. Clearly $f_s \notin C^\omega(\mathbb{R})$. Since $e^{-z^{-\frac{1}{s-1}}} \in \mathcal{O}(\mathbb{C} \setminus \{0\})$, for each $t > 0$ using the

Cauchy integral formula, we have

$$f_s^{(n)}(t) = \frac{n!}{2\pi i}\int_{|z-t|=\frac{1}{2}t} \frac{e^{-z^{-\frac{1}{s-1}}}}{(z-t)^{n+1}}dz.$$

Thus it follows that

$$\begin{aligned} |f_s^{(n)}(t)| &\leq n!(\frac{1}{2}t)^{-n} \sup_{|z-t|=\frac{1}{2}t} \left| e^{-z^{-\frac{1}{s-1}}} \right| \\ &\leq n!(\frac{1}{2}t)^{-n} \sup_{|z-t|=\frac{1}{2}t} e^{-\Re z^{-\frac{1}{s-1}}} \\ &\leq n!(\frac{1}{2}t)^{-n} e^{-\frac{1}{(\frac{3}{2}t)^{\frac{1}{s-1}}}}, \text{ since } |z| \leq \frac{1}{2}t \end{aligned}$$

Thus for every $t > 0$, we have

$$|f_s^{(n)}(t)| \leq n!(\frac{1}{2}t)^{-n} e^{-\frac{1}{(\frac{3}{2}t)^{\frac{1}{s-1}}}} \tag{1.1.1}$$

The right hand side of the inequality in (1.1.1) tends to 0 as $t \to 0^+$ so that $f_s(t)$ is infinitely differentiable in $\mathbb{R}$. Moreover, if $L > 0$ then by Stirling's formula, we have

$$\sup_{t\in(0,\infty)} t^{-n}e^{-Lt^{-\frac{1}{s-1}}} = \left(\frac{s-1}{Le}\right)^{(s-1)n} n^{(s-1)n} \leq \left(\frac{s-1}{L}\right)^{(s-1)n} n!^{s-1}$$

Thus if we take the suprimum for $t > 0$ in (1.1.1), we have

$$\begin{aligned} \sup_{t\in\mathbb{R}} |f_s^{(n)}(t)| &\leq n!2^n \sup_{t\in(0,\infty)} \left\{ t^{-n} e^{-\frac{1}{(\frac{3}{2}t)^{\frac{1}{s-1}}}} \right\} \\ &\leq n!2^n \left(\frac{3}{2}(s-1)^{s-1}\right)^n n!^{s-1} \\ &= (3(s-1)^{s-1})^n n!^s \\ &\leq C^{n+1} n!^s \end{aligned}$$

Therefore, $f_s \in G^s(\mathbb{R}),\ s > 1$. □

Definition 1.1.8. Let $\Omega \subset \mathbb{R}^m$ be an open set. We denote by $D^M(\Omega)$ the vector space of all $\phi \in \mathcal{E}^M(\Omega)$ with compact support in Ω. The space $D^{M'}(\Omega)$ of $M-$ultradistributions is defined to be the dual of $D^M(\Omega)$; more precisely, $D^{M'}(\Omega)$ is the space of all linear form u on $D^M(\Omega)$ such that for each $K \subset\subset \Omega$ and for all $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that

$$|u(\phi)| \leq C_\epsilon \sup_{\alpha \in \mathbb{N}_0^m} \left\{ \frac{\epsilon^{|\alpha|}}{M_{|\alpha|}} \sup_{x \in K} |\partial^\alpha \phi(x)| \right\}$$

$\forall \phi \in D^M(K) = D^M(\Omega) \cap C_0^\infty(K)$.

Finally, we define $\mathcal{E}^{M'}(\Omega)$ to be the space of all linear form u on $\mathcal{E}^M(\Omega)$ such that for each $K \subset\subset \Omega$ and, for all $\epsilon > 0$, a constant $C_\epsilon > 0$ such that

$$|u(\phi)| \leq C_\epsilon \sup_{\alpha \in \mathbb{N}_0^m} \left\{ \frac{\epsilon^{|\alpha|}}{M_{|\alpha|}} \sup_{x \in K} |\partial^\alpha \phi(x)| \right\}, \ \forall \phi \in \mathcal{E}^M(\Omega).$$

The restriction of $u \in \mathcal{E}^{M'}(\Omega)$ to $D^M(\Omega)$ define an $M-$ultradistribution $u \in D^{M'}(\Omega)$ with compact support in K.

Definition 1.1.9. For each sequence $M = (M_j)$ of positive numbers we define its associate function $M(t)$ on $[0, \infty)$ by

$$M(t) = \sup_j \log \frac{t^j}{M_j}, \ t \in (0, \infty), M(0) = 0.$$

Note that if $M_j = j!^s$, then $M(t)$ is equivalent to $t^{\frac{1}{s}}$ (see [27]). Also if (M_j) satisfies (P1) and (P5′), the original sequence is retrieved by the formula

$$M_j = \sup_{t > 0} \frac{t^j}{e^{M(t)}}.$$

The following Lemma (from [22] and [27]) summarizes some properties the associated function.

Lemma 1.1.10. *Let $M = (M_j)$ be a sequence of positive numbers satisfying $(P1)$ and let $M(t)$ be its associated function. It follows that* *a) If M satisfies (P8), then for all $t > 0$,*

$$\log t \leq M(t) \leq t. \tag{1.1.2}$$

b) If M satisfies (P8), then $M(t)$ is increasing convex function in $\log t$ which vanishes for sufficiently small $t > 0$ and increases more rapidly than $\log t$ as $t \to +\infty$.

c) If M satisfies (P3$''$), then for each $k > 0$ and $t > 0$ we have

$$M(kt) - M(t) \geq \frac{\log(t/A)\log k}{\log H}, \tag{1.1.3}$$

where A and H were defined by (P3$''$).

d) The property (P3$''$) is equivalent to

$$M\left(\frac{t}{H}\right) \leq \frac{1}{2}M(t) + \log(\sqrt{A}). \tag{1.1.4}$$

e) If M satisfies (P3$''$) and (P8), then for $k > 0$ fixed there exist $c = c(k)$ such that

$$\frac{3}{2}M(kt) - M(t) \geq 0,\ t > c. \tag{1.1.5}$$

f) If M satisfies (P3$''$) and (P4), then for each $L > 0$ and $k, r \in \mathbb{N}$ such that $k \geq r \geq 0$, we have

$$t^r M_{k-r} \leq \sqrt{A}\frac{H^r}{\theta^r} M_k e^{\frac{1}{2}M(Lt)},\ t > 0 \tag{1.1.6}$$

where A and H were defined by (P3$''$). When $k = r$ the sequence does not need to satisfy (P4).

g) Let M be a sequence satisfying (P8). If $c, \gamma > 0$ then

$$\int_{\mathbb{R}^m} e^{-cM(\gamma|\xi|)} d\xi < +\infty \tag{1.1.7}$$

h) Let M be a sequence satisfying (P3$''$). For each $c > 0$ there exist $c' > 0$

(depending on the sequence M and c) such that

$$cM(t) \geq M(c't), \ \forall t > 0 \tag{1.1.8}$$

i) Let M be a sequence satisfying ($P3''$) and (P8). If $c, \gamma > 0$ and $r \in \mathbb{R}_+$ then

$$\int_{\mathbb{R}^m} |\xi|^r e^{-cM(\gamma|\xi|)} d\xi < +\infty \tag{1.1.9}$$

Proof. a) From the definition of $M(t)$ and (P1) we have

$$\log t = \log \frac{t^1}{M_1} \leq \sup_j \log \frac{t^j}{M_j} = M(t) \leq \sup_j \log \frac{t^j}{j!} \ \text{ from } (P8)$$
$$\leq t, \ t > 0 \text{ since } \frac{t^j}{j!} \leq e^t$$

b) For any $t' > t \geq 0$ and any $j \in \mathbb{N}$ we have that $\frac{t'^j}{M_j} \geq \frac{t^j}{M_j}$, so it follows that $M(t)$ is increasing.

For any $j \in \mathbb{N}$ we have

$$M(t) \geq j \log t - \log M_j.$$

So for $t_0 > 0$ such that $\log M_j \leq \frac{j}{2} \log t_0$ we get

$$\forall t > t_0 : \frac{M(t)}{\log t} \geq \frac{j}{2}.$$

Since this holds for any $j \in \mathbb{N}$, it follows that $\lim_{t \to \infty} \frac{M(t)}{\log t} = \infty$.

The convexity $M \circ \exp : x \mapsto \sup_{j \in \mathbb{N}} \Big(jx - \log M_j \Big)$ follows from the fact that for any $x, x' \in \mathbb{R}$ we have, by the subadditivity of sup

$$\frac{1}{2}\left(\sup_{j \in \mathbb{N}} (jx - \log M_j) + \sup_{j \in \mathbb{N}} (jx' - \log M_j) \right) \geq \sup_{j \in \mathbb{N}} \left(j(\frac{x + x'}{2}) - \log M_j \right)$$

c) By Proposition 3.4 (in [27], page 50), M_j satisfies ($P3''$) if and only if there are constants A and $H > 1$ such that

$$m(\lambda) \geq \frac{\log(\lambda/A)}{\log H},\ \lambda > 0. \tag{1.1.10}$$

Thus (1.1.10) implies that

$$\begin{aligned} M(kt) - M(t) &\geq \int_t^{kt} \frac{\log(\lambda/A)}{\log H}\frac{d\lambda}{\lambda} \\ &= \frac{(2\log(t/A) + \log k)\log k}{2\log H} \geq \frac{\log(t/A)\log k}{\log H} \end{aligned}$$

d)([27], Proposition 3.6).

e) ([22],Lemma A.1 (E)).

f)Let $A, H > 0$ be as in $(P3'')$. For arbitrary $L > 0$ and $k, r \in \mathbb{N}$ such that $k \geq r \geq 0$, (P4) implies for $t \geq 0$ that

$$t^r M_{k-r} \leq \frac{H^r}{L^r} M_k \frac{(\frac{Lt}{H})^r}{M_r} \leq \frac{H^r}{L^r} M_k e^{M((Lt)/H)}\ \text{(by definition)}.$$

The conclusion follows from (1.1.4).

g)([22],Lemma A.1 (g)).

h)([22],Lemma A.1 (h)).

i) First suppose $r \in \mathbb{N}$. Then as proved in Lemma A.1(i) of [22], since the sequence M satisfies $(P3'')$ and $(P8)$ we apply (1.1.6),(1.1.7) and (1.1.8). Thus, there exist $c' > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^m} |\xi|^r e^{-cM(\gamma|\xi|)} d\xi &\leq \int_{\mathbb{R}^m} \sqrt{A}\frac{H^r}{(c'\gamma)^r} M_r e^{\frac{1}{2}M(c'\gamma|\xi|)} e^{-M(c'\gamma|\xi|)} d\xi\ \text{ by (1.1.6) and (1.1.8)} \\ &\leq \frac{\sqrt{A}H^r M_r}{(c'\gamma)^r} \int_{\mathbb{R}^m} e^{-M(c'\gamma|\xi|)} d\xi < +\infty\ \text{ by (1.1.7)}. \end{aligned}$$

For $r \in \mathbb{R}_+$, we have

$$\begin{aligned}\int_{\mathbb{R}^m} |\xi|^r e^{-cM(\gamma|\xi|)} d\xi &= \int_{|\xi|\le 1} |\xi|^r e^{-cM(\gamma|\xi|)} d\xi + \int_{|\xi|>1} |\xi|^r e^{-cM(\gamma|\xi|)} d\xi \\ &\le \int_{|\xi|\le 1} e^{-cM(\gamma|\xi|)} d\xi + \int_{|\xi|>1} |\xi|^r e^{-cM(\gamma|\xi|)} d\xi \\ &\le \int_{|\xi|\le 1} e^{-cM(\gamma|\xi|)} d\xi + \int_{|\xi|>1} |\xi|^m e^{-cM(\gamma|\xi|)} d\xi\end{aligned}$$

where m is a positive integer greater than r. Then clearly the first term is finite and the second is finite by (g) above. Thus the whole integral is finite as desired. □

1.2 Conic Sets and Boundary Value of Holomorphic Functions

We consider now the boundary value of holomorphic functions defined on wedges with flat edges, that is, edges that are open subsets of $\mathbb{R}^m$.

Definition 1.2.1. A set $\Gamma \subset \mathbb{R}^m \setminus \{0\}$ is called a conic set if whenever $\xi \in \Gamma$, then $t\xi \in \Gamma$ for all $t > 0$. It is acute if it is contained in strictly convex cone.

A conic set is completely determined by its intersection with the unit sphere in $\mathbb{R}^m$. That is, for $\mathbb{S}^{m-1} = \{x \in \mathbb{R}^m : ||x|| = 1\}$, $\Gamma = \{t\eta : t > 0, \eta \in \Gamma \cap \mathbb{S}^{m-1}\}$. If $\Gamma' \subset \mathbb{R}^m \setminus \{0\}$ is another cone, then we write $\Gamma' \subset\subset \Gamma$ if $\overline{\Gamma'} \cap \mathbb{S}^{m-1} \subset \Gamma \cap \mathbb{S}^{m-1}$. A conic neighborhood of a point is an open conic set containing the point.

A set $\Gamma_\delta \in \mathbb{R}^m \setminus \{0\}$, $(\delta > 0)$ is called a truncated cone if there exists a cone Γ such that

$$\Gamma_\delta = \Gamma \cap \{|\xi| < \delta\}.$$

An open truncated cone is a truncated cone which is open. A wedge with edge V is an open set $W = V + i\Gamma$, where V is an open set in $\mathbb{R}^m$ and Γ is a conic set in $\mathbb{R}^m$. If $W = V + i\Gamma_\delta$, for some $\delta > 0$, we call W a truncated wedge.

Definition 1.2.2. Let $V \subset \mathbb{R}^m$ be an open set and Γ be a convex conic set. A function $f \in \mathcal{O}(V + i\Gamma_\delta)$ is said to be of tempered growth if there is an integer $k \geq 1$ and a constant $c > 0$ such that

$$|f(x+iy)| \leq \frac{c}{|y|^k}, \ \forall x \in V, \ \forall y \in \Gamma_\delta.$$

For $f \in \mathcal{O}(V + i\Gamma_\delta)$, $\phi \in C_0^\infty(V)$ and $y \in \Gamma_\delta$ we set

$$\langle f_y, \phi \rangle = \int_V f(x+iy)\phi(x)\, dx.$$

The following theorem (from [9]) shows the existence of a boundary value for a holomorphic function of tempered growth defined on wedges.

Theorem 1.2.3. *Suppose $f \in \mathcal{O}(V + i\Gamma_\delta)$ is of tempered growth. Then the boundary value of the holomorphic function f, denoted by bf, defined by*

$$bf = \lim_{y \to 0, y \in \Gamma_\delta} f_y$$

exists in $D'(V)$ and is of order $k+1$, where k is as in the definition.

Example 1.2.4. Let $f(x,y) = \frac{1}{x+iy}$. Then f is holomorphic and of tempered growth in the upper half-plane $y > 0$. By the theorem f has boundary value $bf \in D'(\mathbb{R})$. Moreover, one can show that

$$bf = Pv\left(\frac{1}{x}\right) - i\pi\delta_0,$$

where Pv denotes the Cauchy principal value, and δ_0 is the Dirac distribution.

Lemma 1.2.5. *Let $\mathcal{C}_0 = \Gamma$, $\mathcal{C}_j$, $1 \leq j \leq n$ be open acute cones such that*

$$\mathbb{R}^m = \cup_{j=0}^n \overline{\mathcal{C}_j}, \ \overline{\mathcal{C}_j} \cap \overline{\mathcal{C}_k}$$

has measure zero when $j \neq k$ and $\xi^0 \notin \overline{\mathcal{C}_j}$ for $j \geq 1$. Then there exists acute, *open cones* $\Gamma^j,\ 1 \leq j \leq n$ *and a constant* $c > 0$ *such that*

$$\xi^0.\Gamma^j < 0 \textit{ and } y.\xi \geq c|y||\xi| \ \ \forall y \in \Gamma^j,\ \forall \xi \in \mathcal{C}_j.$$

Proof. Fix $j = 1, \dots, n$. Since $\xi^0 \notin \overline{\mathcal{C}_j}$ and $\mathcal{C}_j$ are acute there is a vector $y^j \in \mathbb{R}^m \setminus \{0\}$ such that

$$\xi^0.y^j < 0 \text{ and } (\overline{\mathcal{C}_j} \setminus 0).y^j > 0,$$

which implies that

$$\xi^0.\frac{y^j}{|y^j|} < 0 \text{ and } \xi.\frac{y^j}{|y^j|} > 0 \ \ \forall \xi \in \overline{\mathcal{C}_j} \setminus 0.$$

By continuity, there exists a neighborhood U_j of $\frac{y^j}{|y^j|} \in S^{m-1}$ such that

$$\xi^0.y < 0 \text{ and } \xi.y > 0 \ \ \forall \xi \in \overline{\mathcal{C}_j} \setminus 0, \forall y \in U_j.$$

Let V_j be the cone generated by U_j. Then

$$V_j \subset \{y \in \mathbb{R}^m \setminus \{0\} : \xi^0.y < 0 \text{ and } y.\xi > 0 \, \forall \xi \in \overline{\mathcal{C}_j} \setminus 0\}.$$

For $j = 1, \dots, n$ choose $\Gamma^j \subset\subset V_j$. Since the function $(y, \xi) \mapsto y.\xi > 0$ is continuous on the compact set

$$A_j = \overline{\Gamma}_j \cap S^{m-1} \times \overline{C}_j \cap S^{m-1}, b_j = \min_{A_j} y.\xi > 0,\ \forall j = 1, \dots, n.$$

If $y \in \Gamma^j$ and $\xi \in \mathcal{C}_j$, then

$$(\frac{y}{|y|}, \frac{\xi}{|\xi|}) \in A_j \ \text{ and so } y.\xi \geq b_j|y||\xi| \ \forall j = 1, \dots, n.$$

Let $c = \min\{b_1, \dots, b_n\}$. Then we can get acute, open cones $\Gamma^j,\ 1 \leq j \leq n$ and a constant $c > 0$ such that

$$\xi^0.\Gamma^j < 0 \text{ and } y.\xi \geq c|y||\xi| \ \ \forall y \in \Gamma^j, \ \forall \xi \in \mathcal{C}_j.$$

□

1.3 Wave Front Sets and the FBI Transform

Definition 1.3.1. Let $\Omega \subset \mathbb{R}^m$ be an open set and $u \in D'(\Omega), x_0 \in \Omega, \xi^0 \in \mathbb{R}^m \setminus \{0\}$. u is called microlocally analytic at (x_0, ξ^0) if there exist a neighborhood V of x_0, cones $\Gamma^1, \Gamma^2, ..., \Gamma^n$ in $\mathbb{R}^m \setminus \{0\}$ with $\xi^0.\Gamma^j < 0, \forall j$ and holomorphic functions $f_j \in \mathcal{O}(V + i\Gamma^j_\delta)$ (for some $\delta > 0$) of tempered growth such that

$$u = \sum_{j=1}^{n} bf_j \ \text{ near } x_0.$$

The analytic wave front set of a distribution $u \in D'(\Omega)$, is defined by

$$WF_a(u) = \{(x, \xi) \in \Omega \times \mathbb{R}^m \setminus \{0\} : u \text{ is not microlocally analytic at } (x, \xi)\}.$$

Definition 1.3.2. Let $\Omega \subset \mathbb{R}^m$ be an open set and $u \in D'(\Omega), x_0 \in \Omega, \xi^0 \in \mathbb{R}^m \setminus \{0\}$. u is called microlocally smooth at (x_0, ξ^0) if there exist $\phi \in C_0^\infty, \phi = 1$ near x_0 and a conic neighborhood $\Gamma \subset \mathbb{R}^m \setminus \{0\}$ of ξ^0 such that for each positive integer k there is C_k such that

$$|\widehat{\phi u}(\xi)| \leq \frac{C_k}{(1 + |\xi|)^k}, \forall \xi \in \Gamma,$$

where $\widehat{\phi u}$ denotes the Fourier transform of ϕu.

The C^∞ wave front set of a distribution $u \in D'(\Omega)$, is defined by

$$WF(u) = \{(x, \xi) \in \Omega \times \mathbb{R}^m \setminus \{0\} : u \text{ is not microlocally smooth at } (x, \xi)\}.$$

Definition 1.3.3. Let Ω be open, $x_0 \in \Omega$, $s > 1$. Then u is microlocally Gevrey regular at $(0, \xi^0$, that is $(x_0, \xi^0) \notin WF_s(u)$(Gevrey wave front set), if there exist $\phi \in G_0^s(\Omega), \phi = 1$ near x_0, a conic neighborhood Γ of ξ^0 and constants $c_1, c_2 > 0$ such that

$$|\widehat{\phi u}(\xi)| \leq c_1 \exp\big(-c_2|\xi|^{\frac{1}{s}}\big), \, \forall \xi \in \Gamma.$$

Remark 1.3.4. If $u \in D'(\Omega)$, then $WF(u) \subset WF_s(u) \subset WF_a(u)$.

Definition 1.3.5. Let $u \in \mathcal{E}'(\mathbb{R}^m)$. The FBI (Fourier-Bros-Iagolintzer) transform of u is defined by

$$\mathcal{F}u(x, \xi) = \int_{\mathbb{R}^m} e^{i\xi.(x-x')-|\xi||x-x'|^2} u(x') \, dx', \, (x, \xi) \in \mathbb{R}^m \times \mathbb{R}^m,$$

where the integral is understood in duality sense.

The FBI transform characterizes the smoothness, real analyticity, microlocal smoothness, and microlocal analticity of functions or distributions (See,[9, 19, 23, 28]). The following characterization of analyticity by means of exponential decay of the FBI transform may be viewed as an analogue of the Palay-Wiener theorem.

Theorem 1.3.6 ([9], Theorem V.2.4)**.** *Let $u \in \mathcal{E}'(\mathbb{R}^m)$. The following are equivalent.*

i) u is real analytic at $x_0 \in \mathbb{R}^m$

ii) There exist a neighborhood V of x_0 in $\mathbb{R}^m$ and constants $c_1, c_2 > 0$ such that

$$|\mathcal{F}u(x, \xi)| \leq c_1 e^{-c_2|\xi|}, (x, \xi) \in V \times \mathbb{R}^m.$$

Definition 1.3.7. Let $f \in C^\infty(\Omega), \Omega \subset \mathbb{R}^m$ open, and $\tilde{\Omega}$ is a neighborhood of Ω in $\mathbb{C}^m$. A function $\tilde{f}(x, y) \in C^\infty(\tilde{\Omega})$ is called an almost analytic extension of $f(x)$ if

1) $\tilde{f}(x, 0) = f(x) \; \forall x \in \Omega$ and ,

2) for each $k = 1, 2, \dots$ there is $C_k > 0$ such that

$$|\frac{\partial \tilde{f}}{\partial \bar{z}_j}(x, y)| \leq C_k(|y|^k), \quad j = 1, 2, \dots, m.$$

We recall from [9] that every smooth function has an almost analytic extension. The following results shows us that how the FBI transform is useful to characterize wave front sets of distributions.

Theorem 1.3.8 ([9], Theorem V.2. 14)**.** *Let $u \in \mathcal{E}'(\mathbb{R}^m)$, $x_0 \in \mathbb{R}^m$,$\xi^0 \in \mathbb{R}^m \setminus \{0\}$. Then $(x_0, \xi^0) \notin WF_a(u)$ if and only if there is a neighborhood V of x_0 in $\mathbb{R}^m$, an open cone $\Gamma \subset \mathbb{R}^m \setminus \{0\}, \xi^0 \in \Gamma$ and constants $c_1, c_2 > 0$ such that*

$$|\mathcal{F}u(x, \xi)| \leq c_1 e^{-c_2|\xi|}, \, \forall (x, \xi) \in V \times \Gamma.$$

Theorem 1.3.9 ([9], Theorem V.3.7)**.** *Let $u \in \mathcal{E}'(\mathbb{R}^m)$, $x_0 \in \mathbb{R}^m$,$\xi^0 \in \mathbb{R}^m \setminus \{0\}$. Then $(x_0, \xi^0) \notin WF(u)$ if and only if there is a neighborhood V of x_0, open acute cones $\Gamma^1, \dots, \Gamma^N$ in $\mathbb{R}^m \setminus \{0\}$, and almost analytic functions f_j on $V + i\Gamma^j_\delta$ (for some $\delta > 0$) of tempered growth such that*

$$u = \sum_{j=1}^{N} bf_j \text{ near } x_0 \text{ and } \xi^0.\Gamma^j < 0 \ \ \forall j.$$

The following theorem characterizes Gevrey wave front sets using FBI transforms.

Theorem 1.3.10 ([16], Theorem 2.3)**.** *Let $u \in \mathcal{E}'(\mathbb{R}^m)$, $x_0 \in \mathbb{R}^m$,$\xi^0 \in \mathbb{R}^m \setminus \{0\}$. Then $(x_0, \xi^0) \notin WF_s(u)$ if and only if there is a neighborhood V of x_0, a conic neighborhood of Γ of ξ^0 such that for some $\phi \in C_0^\infty(\mathbb{R}^m), \phi = 1$ near x_0,*

$$|\mathcal{F}(\phi u)(x, \xi)| \leq c_1 e^{-c_2|\xi|^{\frac{1}{s}}}, \, \forall (x, \xi) \in V \times \Gamma$$

for some constants $c_1, c_2 > 0$.

1.4 Integrable Structures

Let $\mathcal{M}$ be a smooth manifold of dimension N.

Let $C^\infty(p)$ be the collection of all C^∞ functions whose domain includes p, identifying those functions which agree on an open set containing p. Recall that a complex tangent vector (to $\mathcal{M}$) at p is a $\mathbb{C}-$linear map

$$v : C^\infty(p) \to \mathbb{C}$$

satisfying

$$v(fg) = f(p)v(g) + g(p)v(f)\ \forall\, f, g \in C^\infty(p).$$

The set of all tangent vectors at p, denoted by $\mathbb{C}T_p\mathcal{M}$ is called the complex tangent space to $\mathcal{M}$ at p . This space has a basis $\{\frac{\partial}{\partial x_j} : j = 1, 2, \ldots, N\}$. The dual space of the complex tangent space denoted by $\mathbb{C}T_p^*\mathcal{M}$ is called the complex cotangent space.

The complexified tangent bundle of $\mathcal{M}$ is defined as a disjoint union

$$\mathbb{C}T\mathcal{M} = \bigcup_{p\in\mathcal{M}} \mathbb{C}T_p\mathcal{M}.$$

Tthe complexified cotangent bundle of $\mathcal{M}$ is defined as a disjoint union

$$\mathbb{C}T^*\mathcal{M} = \bigcup_{p\in\mathcal{M}} \mathbb{C}T_p^*\mathcal{M}.$$

The real tangent space $T_p\mathcal{M}$, the real cotangent space $T_p^*\mathcal{M}$, the real tangent bundle $T\mathcal{M}$ and the real cotangent bundle $T^*\mathcal{M}$ are defined analogously.

Definition 1.4.1. A (smooth) vector field over $\mathcal{M}$ is a $\mathbb{C}-$linear map

$$L : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$$

which satisfies the Leibniz rule

$$L(fg) = fL(g) + gL(f),\ f, g \in C^\infty(\mathcal{M}).$$

The set of all vector fields on $\mathcal{M}$ is denoted by $\mathfrak{X}(\mathcal{M})$.

Definition 1.4.2. A complex vector sub-bundle of $\mathbb{C}T\mathcal{M}$ of rank n and corank $m = N - n$ is a disjoint union

$$\mathcal{V} = \bigcup_{p\in\mathcal{M}} \mathcal{V}_p \subset \mathbb{C}T\mathcal{M}$$

such that

i) for each $p \in \mathcal{M}$, $\mathcal{V}_p$ is a vector space of $\mathbb{C}T_p\mathcal{M}$ of dimension n.

ii) given $p_0 \in \mathcal{M}$ there is an open set U_0 containing p_0, and vector fields $L_1, L_2, ..., L_n \in \mathfrak{X}(U_0)$ such that $L_{1p}, L_{2p}, ..., L_{np}$ span $\mathcal{V}_p$ for every $p \in U_0$. The vector space $\mathcal{V}_p$ is called the fiber of $\mathcal{V}$ at p.

Let $\mathcal{V}$ be a vector sub-bundle of $\mathbb{C}T\mathcal{M}$, and W be an open subset of $\mathcal{M}$. A section of $\mathcal{V}$ over W is an element L of $\mathfrak{X}(W)$ such that $L_p \in \mathcal{V}_p, \forall p \in W$.

Definition 1.4.3. A formally integrable structure over $\mathcal{M}$ is a complex vector sub-bundle $\mathcal{V}$ of $\mathbb{C}T\mathcal{M}$ satisfying the involutivity condition: If $W \subset \mathcal{M}$ open and $L_1, L_2 \in \mathfrak{X}(W)$ are sections of $\mathcal{V}$ over W, then the lie bracket $[L_1, L_2]$ is also a section of $\mathcal{L}$ over W. We call the pair $(\mathcal{M}, \mathcal{V})$ an involutive structure.

Definition 1.4.4. A solution of the formally integrable structure $\mathcal{V}$ over $\mathcal{M}$ is a C^1 function u on $\mathcal{M}$ such that $Lu = 0$ for every section L of $\mathcal{V}$ defined in an open subset of $\mathcal{M}$.

Let $\mathcal{V} = \bigcup_{p\in\mathcal{M}} \mathcal{V}_p$ be a complex vector sub-bundle of $\mathbb{C}T\mathcal{M}$ and set, for each $p \in \mathcal{M}$,

$$\mathcal{V}_p^\perp = \{\lambda \in \mathbb{C}T_p^*\mathcal{M} : \lambda = 0 \text{ on } \mathcal{V}_p\}.$$

Then $\mathcal{V}^\perp = \bigcup\limits_{p\in\mathcal{M}} \mathcal{V}_p^\perp$ is a complex vector sub-bundle of $\mathbb{C}T^*\mathcal{M}$. The above statements can be reversed: If $\mathcal{V}^\perp$ is a vector sub-bundle of $\mathbb{C}T^*\mathcal{M}$, then $\mathcal{V}$ is a vector sub-bundle of $\mathbb{C}T\mathcal{M}$.

When $\mathcal{V}$ is a formally integrable structure over $\mathcal{M}$ of dimension $N = n + m$, where n is the rank of $\mathcal{V}$, we shall always denote the sub-bundle $\mathcal{V}^\perp$ by T' so that m is the rank of T'.

Definition 1.4.5. A complex vector sub-bundle $\mathcal{V}$ of $\mathbb{C}T\mathcal{M}$, of rank n, is said to be a locally integrable structure if given an arbitrary point $p_0 \in \mathcal{M}$ there is an open neighborhood U_0 of p_0 and functions $Z_1, \ldots, Z_m \in C^\infty(U_0)$, with $m = N-n$, such that

$$span\{dZ_{1p}, ..., dZ_{mp}\} = \mathcal{V}_p^\perp,\ \forall p \in U_0.$$

Thus if the vector field L is a section of $\mathcal{V}$, then $LZ_j = \langle dZ_j, L\rangle = 0, \forall j = 1, \ldots, m$. $(Z_1, ..., Z_m)$ are called complete set of first integrals. We remark here that every locally integrable structure defined a formally integrable structure. A formally integrable structure $\mathcal{V}$ is locally integrable if and only if given $p_0 \in \mathcal{M}$ and vector fields $L_1, ..., L_n$ which span $\mathcal{V}$ in an open neighborhood U_0 of p_0, there is an open neighborhood $W_0 \subset U_0$ of p_0 and smooth functions $Z_1, \ldots, Z_m \in C^\infty(W_0)$ such that

$$dZ_1 \wedge \ldots \wedge dZ_m \neq 0 \text{ in } W_0;$$

$$L_j Z_k = 0, j = 1, \ldots, n, k = 1, \ldots, m$$

Example 1.4.6. Let $\mathcal{M} \subset \mathbb{C}^n$ be an open set and $\mathcal{V}$ be the bundle generated by $\frac{\partial}{\partial \bar{z}_j}, 1 \leq i \leq n$. $\mathcal{V}$ is locally integrable since it has the global complete set of first integrals given by the the coordinate functions $z_1, \ldots, z_m$. The solutions are the holomorphic functions.

The following theorem gives appropriate local coordinates and local generators of the sub-bundle T' when the structure $\mathcal{V}$ is locally integrable.

Theorem 1.4.7 ([9], Corollary I.10.2.)**.** *Let $\mathcal{V}$ be a locally integrable structure defined on a manifold $\mathcal{M}$. Let $p \in \mathcal{M}$. Then there is a coordinate system*

$\{x_1, \ldots, x_m, t_1, \ldots, t_n\}$ *vanishing at* p *and smooth, real-valued* $\phi_1, \ldots, \phi_m$ *functions defined in an open neighborhood of the origin, and satisfying*

$$\phi_k(0,0) = 0,\ d_x\phi_k(0,0) = 0, k = 1, \ldots, m$$

such that the differentials of the function

$$Z_k(x,t) = x_k + i\phi_k(x,t), k = 1, \ldots, m$$

span T' *in a neighborhood of the origin.*

Writing $Z(x,t) = (Z_1(x,t), \ldots, Z_m(x,t))$, we see that $Z_x(0,0)$ is the identity $m \times m$ matrix. Hence introduce, in a neighborhood of the origin in $\mathbb{R}^N$, the vector fields

$$M_k = \sum_{l=1}^{m} \mu_{kl}(x,t)\frac{\partial}{\partial x_l},\ k = 1, ..., m$$

characterized by the relations $M_k Z_l = \delta_{kl}$ (Kroncker delta), where $\mu_{kl} = Z_x^{-1}$. Consequently the vector field

$$L_j = \frac{\partial}{\partial t_j} - i\sum_{k=1}^{m} \frac{\partial \phi_k}{\partial t_j}(x,t)M_k,\ j = 1, \ldots, n$$

are linearly independent and satisfy

$$L_j Z_k = 0,\ 1 \le j \le n, 1 \le k \le m.$$

Moreover, $L_1, \ldots, L_n$ span $\mathcal{V}$ in a neighborhood of the origin, and $L_1, \ldots, L_n, M_1, \ldots, M_m$ span $\mathbb{C}T\mathbb{R}^N$ in neighborhood of the origin in $\mathbb{R}^N$. Moreover, $\{dZ_1, \ldots, dZ_m, dt_1, \ldots, dt_n\}$ is a basis of $\mathbb{C}T^*\mathbb{R}^n$.

1.5 Maximally Real Submanifolds

In this section we present definition of maximally real submanifolds and some properties.

Definition 1.5.1. Let $\mathcal{M}$ be an $N-$dimensional manifold and $\mathcal{V}$ be locally integrable bundle on $\mathcal{M}$ of rank n, and let $m = N - n$. A submanifold $X \subset \mathcal{M}$ of dimension m is called maximally real if for each $p \in X$

$$\mathbb{C}T_p\mathcal{M} = \mathbb{C}T_pX \bigoplus \mathcal{V}_p.$$

Note that when the manifold and the bundle are smooth (respectively $\mathcal{E}^M$) we get a smooth (respectively $\mathcal{E}^M$) maximally real submanifolds. The following proposition gives equivalent definitions of maximally real submanifolds.

Proposition 1.5.2. *Let $X \subset \mathcal{M}$ be a submanifold. Then the following are equivalent.*
a) X is maximally real
b) The pull back map $\pi^* : \mathbb{C}T^*\mathcal{M}_{|X} \to \mathbb{C}T^*X$ induces an isomorphism

$$T'_{|X} \cong \mathbb{C}T^*X.$$

Proof. (a) $\Rightarrow$ (b): Suppose $\mathbb{C}T_p\mathcal{M} = \mathbb{C}T_pX \oplus V_p \quad \forall p \in X$. We will show that the pull back map $\pi^* : \mathbb{C}T^*\mathcal{M}_{|X} \to \mathbb{C}T^*\mathcal{X}$ induces an isomorphism $T'_{|X} \cong \mathbb{C}T'X$. Since $\dim_{\mathbb{C}} T'_p = m = \dim_{\mathbb{C}} \mathbb{C}T^*_pX$, it suffice to show that $\pi^* : T'_p \to \mathbb{C}T^*_pX$ is injective for every $p \in X$. So, fix $p \in X$ and let $\lambda \in T'_p$. Suppose that $\pi^*(\lambda) = 0$. Then $\langle \lambda, V_p \rangle = 0$ and

$$0 = \langle \pi^*(\lambda), \mathbb{C}T_pX \rangle = \langle \lambda, \mathbb{C}T_pX \rangle.$$

But then since $\mathbb{C}T_p\mathcal{M} = \mathbb{C}T_pX \bigoplus \mathcal{V}_p$, we have $\langle \lambda, \mathbb{C}T_p\mathcal{M} \rangle = 0$, which in turn implies that $\lambda = 0$. This implies that $\pi^* : T'_p \to \mathbb{C}T^*_pX$ is injective and hence, an isomorphism.

(b) $\Rightarrow$(a): Suppose that the pull back map $\pi^* : \mathbb{C}T^*\mathcal{M}_{|X} \to \mathbb{C}T^*\mathcal{X}$ induces an isomorphism $T'_{|X} \cong \mathbb{C}T^*X$. Let $p \in X$. If $\{\lambda_1, \ldots, \lambda_m\}$ is a basis of T'_p, then $\{\pi^*(\lambda_1), \ldots, \pi^*(\lambda_m)\}$ is a basis of $\mathbb{C}T^*_pX$. Let $v \in \mathbb{C}T_pX \cap \mathcal{V}_p$. Then $v \in \mathcal{V}_p$ implies that $\langle \pi^*(\lambda_j), v\rangle = 0, \quad \forall 1 \leq j \leq m$. Then $\langle \mathbb{C}T^*_pX, v\rangle = 0$ and since $v \in \mathbb{C}T_pX$, $v = 0$. Hence $\mathbb{C}T_pX \cap \mathcal{V}_p = \{0\}$.

Since $\mathbb{C}T_pX \oplus \mathcal{V}_p \subset \mathbb{C}T_p\mathcal{M}$, $\dim_{\mathbb{C}} \mathbb{C}T_pX = m$, $\dim_{\mathbb{C}} \mathcal{V} = n$ and $\dim_{\mathbb{C}} \mathbb{C}T_p\mathcal{M} = m + n$, we obtain

$$\mathbb{C}T_p\mathcal{M} = \mathbb{C}T_pX \oplus \mathcal{V}_p.$$

$\square$

Definition 1.5.3. Let $X \subset \mathcal{M}$ be a maximally real submanifold. The real structure bundle of X, denoted by $\mathbb{R}T'_X$ is the image of the real cotangent bundle of X, T^*X, under the natural isomorphism $T'_{|X} \cong \mathbb{C}T'X$.

The following propositions characterizes $\mathcal{E}^M$ maximally real submanifolds. See [12] for the corresponding statements in the smooth case.

Proposition 1.5.4. *X is $\mathcal{E}^M$ maximally real if and only if near each $p \in X$ and a complete set of $\mathcal{E}^M$ first integrals Z_j, $1 \leq j \leq m$, the restrictions of the Z_j to X have linearly independent differentials.*

Proof. Suppose X is $\mathcal{E}^M$ maximally real. Let $p \in X$. Let $i : X \to \mathcal{M}$ be the inclusion map, and i^* is its pullback. Then we have $i^*(dZ_j(p)) = d(Z_{j|X})(p)$ Suppose $w = \sum_{j=1}^m a_j i^*(dZ_j(p)) = 0$. Then

$$\langle w, v\rangle = 0 \; \forall v \in \mathbb{C}T_pX,$$

which implies that

$$\langle \sum_{j=1}^m a_j(dZ_j(p)), v\rangle = 0.$$

But for $v' \in \mathcal{V}_p$, $\langle \sum_{j=1}^m a_j(dZ_j(p)), v'\rangle = 0$. Thus $\sum_{j=1}^m a_j(dZ_j(p)) \in (\mathbb{C}T_pX \cap \mathcal{V}_p)^\perp = \{0\}$, which implies that $\sum_{j=1}^m a_j(dZ_j(p)) = 0$. But $\{dZ_j(p)\}$ are linearly independent. Therefore, we have $a_j = 0\ \forall j$ and hence $\{d(Z_{j|X})(p)\}$ are linearly independent. Conversely, let $v \in \mathcal{V}_p \cap \mathbb{C}T_pX$. Then $\langle dZ_j(p), v\rangle = 0$. Now $\langle i^*(dZ_j(p)), v\rangle = \langle d(Z_{j|X})(p), i_*v\rangle = \langle dZ_j(p), v\rangle = 0$. Since $\{i^*(dZ_j(p))\}$ spans $\mathbb{C}T_p^*X$, we have $v = 0$. Hence $\mathbb{C}T_pX \cap \mathcal{V}_p = \{0\}$. Since $\mathbb{C}T_pX \bigoplus \mathcal{V}_p \subset \mathbb{C}T_p\mathcal{M}$, $\dim \mathbb{C}T_pX = m, \dim \mathcal{V}_p = n$, and $\dim \mathbb{C}T_p\mathcal{M} = m+n$, we get $\mathbb{C}T_pX \bigoplus \mathcal{V}_p = \mathbb{C}T_p\mathcal{X}$. Therefore, X is $\mathcal{E}^M$ maximally real submanifold. □

Proposition 1.5.5. *Let X be a $\mathcal{E}^M$ maximally real submanifold. In a neighborhood of each $p \in X$, we can find coordinates $x_1, \dots, x_m, t_1, \dots, t_n$ that vanish at p and a complete set of $\mathcal{E}^M$ first integrals $Z_1, \dots, Z_m$ such that in these coordinates,*

$$M = \{t_j = 0 : 1 \le j \le n\}$$

and the Z_j are given by

$$Z_j(x,t) = x_j + i\phi_j(x,t),\ 1 \le j \le m$$

with ϕ_j real-valued, $\mathcal{E}^M$, $\phi_j(0,0) = 0$ and $d_x\phi_j(0,0)$.

Proof. Assume p is the origin in $\mathbb{R}^N$. First flatten X near the origin so that

$$X = \{y : y_j = 0, j \ge m+1\}$$

Let $\{Z_1(y), \dots, Z_m(y)\}$ be complete set of $\mathcal{E}^M$ first integrals on a neighborhood of 0 where $y = (y_1, \dots, y_N)$ denote the coordinates in Ω. By Proposition 1.5.4, since the differentials $dZ_1, \dots, dZ_m$ are linearly independent at 0, without loss of generality, we may assume that the Jacobian matrix

$$A = \frac{\partial(Z_1, \dots, Z_m)}{\partial(y_1, \dots, y_m)}(0)$$

is invertible. After replacing $Z = (Z_1, \dots, Z_m)$ by $A^{-1}Z$, we get firt integrals

$Z_1, \dots, Z_m$ such that
$$\frac{\partial(Z_1, \dots, Z_m)}{\partial(y_1, \dots, y_m)}(0)$$
is the identity matrix. We use new coordinates

$$x_1 = \Re Z_1(y), \dots, x_m = \Re Z_m(y), t_1 = y_{m+1}, \dots, t_n = y_{m+n}$$

near the origin. This is a $\mathcal{E}^M$ change of coordinates and in these new coordinates, we can write

$$Z_j(x,t) = x_j + i\phi_j(x,t),\ j = 1, \dots, m,$$

where $\phi_j \in \mathcal{E}^M$ near the origin, are real-valued, and

$$d_x\phi_j(0,0) = 0 \text{ for } j = 1, \dots, m.$$

Moreover, by replacing $Z(x,t)$ with $Z(x,t) - Z(0,0)$, we may assume that

$$\phi_j(0,0) = 0 \text{ for } j = 1, \dots, m.$$

In these coordinates the first integrals Z_j and X have the asserted form. $\square$

1.6 Pseudo-differential Operators, Symbols and Some Properties

Consider the Fourier inversion formula

$$u(x) = \frac{1}{(2\pi)^n} \int e^{ix.\xi} \hat{u}(\xi) d\xi, \quad u \in S, x \in \mathbb{R}^m \tag{1.6.1}$$

Then differentiating under the integral sign we obtain

$$D^\alpha u(x) = \frac{1}{(2\pi)^n} \int e^{ix.\xi} \xi^\alpha \hat{u}(\xi) d\xi, \tag{1.6.2}$$

where $D_j = \frac{1}{i}\partial_j, D = (D_1, \ldots, D_n)$. Consider the differential operator

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha.$$

If we set $p(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x)\xi^\alpha$ which is a polynomial in ξ with x dependent coefficients, then (1.6.2) implies that

$$P(x, D)u(x) = \frac{1}{(2\pi)^n} \int e^{ix.\xi} p(x, \xi)\hat{u}(\xi) d\xi.$$

The function p is called the symbol of the operator P. Pseudo-differential operators are generalization of differential operators in that they are defined by symbols which are not necessarily polynomials with respect to ξ.

Definition 1.6.1. Let $m, \rho, \delta \in \mathbb{R}; n, N \in \mathbb{N}$ with $0 \leq \rho, \delta \leq 1$. Then Hormanders' class $S^m_{\rho,\delta}(\mathbb{R}^N \times \mathbb{R}^n \times \mathbb{R}^m)$ is the vector space of all smooth functions $p : \mathbb{R}^N \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{C}$ such that

$$|D^\alpha_\xi D^\beta_x D^\gamma_y p(x, \xi, y)| \leq C_{\alpha,\beta,\gamma}(1 + |\xi|)^{m-\rho|\alpha|+\delta(|\beta|+|\gamma|)} \;\; \forall \alpha, \beta, \gamma \in \mathbb{N}^n_0, C_{\alpha,\beta,\gamma}$$

independent of $x \in \mathbb{R}^N, \xi \in \mathbb{R}^n, y \in \mathbb{R}^m$. The function p is called pseudo-differential symbol and m is the order of p. For $p(x, \xi) \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ we define the pseudo-differential operator $P(x, D)$ by

$$P(x, D)u = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix.\xi} p(x, \xi)\hat{u}(\xi) d\xi.$$

In particular, when $\rho = 1, \delta = 0$ we obtain the most simple and most common symbol class $S^m_{1,0}$. If $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ is a symbol, then

$$P(x, D)u = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix.\xi} p(x, \xi)\hat{u}(\xi) d\xi$$

defines the associated pseudo-differential operator, where $u : \mathbb{R}^n \to \mathbb{C}$ is suitable function. For instance, if $u \in S(\mathbb{R}^n)$, then $p(x, \xi)\hat{u}(\xi) \in S(\mathbb{R}^n)$ with respect to ξ

for fixed $x \in \mathbb{R}^n$. Therefore, the integral exists and $P(x, D)u$ is well defined.

Example 1.6.2. Let $p(x,\xi) = \sum_{|\alpha|\le m} c_\alpha(x)\xi^\alpha$ be a polynomial of order $m \in \mathbb{N}_0$ in ξ with $c_\alpha \in C_0^\infty(\mathbb{R}^n)$. Then $p \in S^m_{1,0}$ and $P(x,D)u = \sum_{|\alpha|\le m} c_\alpha(x) D^\alpha u$ $\forall u \in S(\mathbb{R}^n)$. Hence every linear differential operator with smooth and bounded coefficients is a pseudo-differential operator. In particular, the Laplacian

$$\Delta = \partial_1^2 + \ldots + \partial_n^2$$

is a pseudo-differential operator of order 2 with symbol $-|\xi|^2 = -\sum_{j=1}^n \xi_j^2$.

Definition 1.6.3 (Sobolev space H^s:)**.** Let $s \in \mathbb{R}$. Define

$$H^s = \{u \in S'(\mathbb{R}^n) : \hat{u} \in L^2_{loc}(\mathbb{R}^n) \text{ and } ||u||^2_{H^s} < \infty\},$$

where $||u||^2_{H^s} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 (1 + |\xi|^2)^s d\xi$.

From the L^2 boundedness of pseudo-differential operator we can easily derive the continuity of pseudo-differential operators on the Sobolev space $H^s(\mathbb{R}^n)$. An operator $P^*(x, D)$ is formally adjoint to the operator $P(x, D)$ if

$$\int P(x,D)u(x)\overline{v(x)}dx = \int u(x)\overline{P^*(x,D)v(x)}dx, \quad u, v \in C_0^\infty(\mathbb{R}^n).$$

If $P(x, D)$ is a pseudo-differential operator of order m, then $P^*(x, D)$ is also a pseudo-differential operator of the same order.

Lemma 1.6.4 (Gårding inequality)**.** *(See [24] and [25]) If A is a classical pseudo-differential operator of order m with the principal symbol $p_0(x,\xi)$ satisfying the condition $\Re p_0(x,\xi) \ge c_0|\xi|^m, |\xi| \ge 1$, then*

$$\Re(Au, u) \ge c_0||u||^2_{\frac{m}{2}} - c_1||u||^2_{\frac{m-1}{2}}, \ \forall u \in C_0^\infty(K),$$

where $c_1 = C_1(K)$, K compact subset of $\mathbb{R}^m$.

Chapter 2

Continuity of a Class of FBI Transforms on Sobolev Spaces

2.1 Introduction

Consider the class of FBI transform (0.0.3), that is, for each $k = 1, 2, \ldots$

$$\mathcal{F}_k(u, x, \xi) = \int_{\mathbb{R}^m} e^{i\xi\cdot(x-y)-|\xi||x-y|^{2k}} u(y)dy, \ x, \xi \in \mathbb{R}^m, \tag{2.1.1}$$

where the integral is understood in duality sense when u is a distribution. In this chapter we establish the boundedness of the transforms (2.1.1) on Sobolev spaces. The case where $k = 1$ was treated in the work [8].

2.2 Continuity of the FBI transform on Sobolev Spaces

We will use of the following Faá di Bruno generalized formula to prove some results in this section and in the later chapters.

Theorem 2.2.1 ([18], Corollary 2.10). *Let* $\gamma \in \mathbb{N}_0^m$, $|\gamma| \geq 1$ *and* $h(x_1, \ldots, x_d) = f(g(x_1, \ldots, x_d))$ *with* $g \in C^\gamma(U_{x_0})$ *and* $f \in C^{|\gamma|}(V_{y_0})$, *where* $y_0 = g(x_0)$, *and*

$U_{x_0} \subset \mathbb{R}^d$ and $V_{y_0} \subset \mathbb{R}$ open neighborhoods of x_0 and y_0, respectively. Then

$$\partial^\gamma h = \sum_{r=1}^{|\gamma|} \partial^r f \sum_{p(\gamma,r)} (\gamma!) \prod_{j=1}^{|\gamma|} \frac{(\partial^{\alpha_j} g)^{k_j}}{k_j!(\alpha_j!)^{k_j}},$$

where

$p(\gamma, r) = \{(k_1, \dots, k_{|\gamma|}; \alpha_1, \dots, \alpha_{|\gamma|}) :$ *for some* $1 \leq s \leq |\gamma|, k_i = 0$ *and* $\alpha_i = 0$ *for* $1 \leq i \leq |\gamma| - s; k_i > 0$ *for* $|\gamma| - s + 1 \leq i \leq |\gamma|;$ *and* $0 \prec \alpha_{|\gamma|-s+1} \prec \dots \prec \alpha_{|\gamma|}$ *are such that*

$$\sum_{i=1}^{|\gamma|} k_i = r, \ \sum_{i=1}^{|\gamma|} k_i \alpha_i = \gamma\}.$$

In particular, we have (see [18], page 515) that there exist $C > 0$ such that

$$r! \sum_{p(\gamma,r)} \prod_{j=1}^{|\gamma|} \frac{1}{k_j!} = \binom{|\gamma| - 1}{r - 1} \leq C^{|\gamma|}. \tag{2.2.1}$$

Here, for two multi-indices $\nu = (\nu_1, \dots, \nu_d)$ and $\mu = (\mu_1, \dots, \mu_d)$, the linear order $\nu \prec \mu$ means one of the following holds:

(i) $|\nu| < |\mu|$;

(ii) $|\nu| = |\mu|$, and $\nu_1 < \mu_1$, or

(iii) $|\nu| = |\mu|$, $\nu_1 = \mu_1, \dots, \nu_k = \mu_k$, and $\nu_{k+1} < \mu_{k+1}$ for some $1 \leq k < d$.

We write $\nu \leq \mu$ if $\nu_j \leq \mu_j$ for every $1 \leq j \leq d$.

We now state and prove the boundedness result.

Theorem 2.2.2. *Let $\Omega' \subset\subset \Omega \subseteq \mathbb{R}^m$ be open sets, Ω bounded. Then for any $u \in \mathcal{E}'(\Omega')$,*

(a) $||u||^2_{H^t}$

$$\leq C\left(\int_\Omega \int_{\mathbb{R}^m} |\mathcal{F}_k(u, x, \xi)|^2 |\xi|^{\frac{m}{2}} (1 + |\xi|^2)^{t + \frac{m-mk}{4k}} \, d\xi dx + ||u||^2_{H^{t-\frac{1}{4}}}\right);$$

(b) Conversely,

$$\int_\Omega \int_{\mathbb{R}^m} |\mathcal{F}_k(u,x,\xi)|^2 |\xi|^{\frac{m}{2}} (1+|\xi|^2)^{t+\frac{m-mk}{4k}} \, d\xi dx \leq C||u||^2_{H^t},$$

where in both (a) and (b), the constant C is independent of u.

Proof. Observe that

$$|\mathcal{F}_k(u,x,\xi)|^2 = \int_\Omega \int_\Omega e^{i\xi.(s-y)-|\xi|(|x-y|^{2k}+|x-s|^{2k})} u(y)\overline{u(s)} dyds,$$

which leads to

$$\begin{aligned}
&\int_\Omega \int_{\mathbb{R}^m} |\mathcal{F}_k(u,x,\xi)|^2 |\xi|^{\frac{m}{2}} (1+|\xi|^2)^s \, d\xi dx \\
&= \int_\Omega \int_{\mathbb{R}^m} \int_\Omega \int_\Omega e^{i\xi.(s-y)-|\xi|(|x-y|^{2k}+|x-s|^{2k})} |\xi|^{\frac{m}{2}} (1+|\xi|^2)^s u(y)\overline{u(s)} \, dydsd\xi dx \\
&= \int_{\mathbb{R}^m} \int_\Omega \int_\Omega e^{i\xi.(s-y)} q(y,s,\xi) |\xi|^{\frac{m}{2}} (1+|\xi|^2)^s u(y)\overline{u(s)} \, dydsd\xi,
\end{aligned}$$

where

$$q(y,s,\xi) = \int_\Omega e^{-|\xi|(|x-y|^{2k}+|x-s|^{2k})} \, dx.$$

Let $Q(y,s,\xi) = q(y,s,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s$. We will show that $q(y,s,\xi) \in S^{\frac{-m}{2k}}_{1,\frac{1}{2k}}$ and that for any $\Omega' \subseteq \Omega'' \subset\subset \Omega$, there exist $c, b > 0$ such that

$$Q(y,y,\xi) \geq c|\xi|^{\frac{m}{2}-\frac{m}{2k}}(1+|\xi|^2)^s \text{ for } y \in \Omega'', |\xi| \geq b.$$

We have

$$\begin{aligned}
Q(y,y,\xi) &= \left(\int_\Omega e^{-2|\xi||x-y|^{2k}} dx \right) |\xi|^{\frac{m}{2}} (1+|\xi|^2)^s \\
&= \left(\int_{\Omega\setminus y} e^{-2|\xi||t|^{2k}} dt \right) |\xi|^{\frac{m}{2}} (1+|\xi|^2)^s.
\end{aligned}$$

Since $0 \in \Omega \setminus y$ for each $y \in \Omega''$, by compactness, there exists $\delta > 0$ such that the ball $\mathbb{B}_\delta(0) \subseteq \Omega \setminus y$ for every $y \in \Omega''$. Hence $\mathbb{B}_1(0) \subseteq \frac{1}{\delta}(\Omega \setminus y)$ for every $y \in \Omega''$

and therefore, $\mathbb{B}_1(0) \subseteq |\xi|^{\frac{1}{2k}}(\Omega \setminus y)$ for every $y \in \Omega''$ and all $\xi \in \mathbb{R}^m$ satisfying $|\xi| \geq \frac{1}{\delta^{2k}}$. It follows that for any $y \in \Omega'', \xi \in \mathbb{R}^m, |\xi| \geq \frac{1}{\delta^{2k}}$,

$$\begin{aligned}\int_{\Omega\setminus y} e^{-2|\xi||t|^{2k}} dt &= \frac{1}{|\xi|^{\frac{m}{2k}}} \int_{|\xi|^{\frac{1}{2k}}(\Omega\setminus y)} e^{-2|v|^{2k}} dv \\ &\geq \frac{1}{|\xi|^{\frac{m}{2k}}} \int_{\mathbb{B}_1(0)} e^{-2|v|^{2k}} dv \\ &= \frac{c}{|\xi|^{\frac{m}{2k}}},\ c > 0,\end{aligned}$$

and so for such y and ξ,

$$Q(y, y, \xi) \geq c|\xi|^{\frac{m}{2} - \frac{m}{2k}}(1 + |\xi|^2)^s.$$

Set $b = \frac{1}{\delta^{2k}}$, and let α, β, γ be multi-indices. We will show that there is a constant $C_{\alpha,\beta,\gamma} > 0$ such that

$$\left|\partial_\xi^\alpha \partial_y^\beta \partial_s^\gamma q(y, s, \xi)\right| \leq C_{\alpha,\beta,\gamma}(1 + |\xi|)^{\frac{-m}{2k} - |\alpha| + \frac{1}{2k}(|\beta| + |\gamma|)}$$

for y and s in compact subsets and $|\xi| \geq b$.

Consider first $\partial_\xi^\alpha q(y, s, \xi)$:

Let $h(x, y, s) = |x - y|^{2k} + |x - s|^{2k}$, $F(r) = e^{-rh(x,y,s)}$ and $g(\xi) = |\xi|$. Then $e^{-|\xi|h(x,y,s)} = F(g(\xi))$. To estimate $\partial_\xi^\alpha F \circ g(\xi)$, we will use the multivariate version of the formula of Faá di Bruno formula (Theorem 2.2.1).

$$\partial_\xi^\alpha F \circ g(\xi) = \sum_{1 \leq \lambda \leq |\alpha|} D^\lambda F(g(\xi)) \sum_{p(\alpha,\lambda)} (\alpha)! \prod_{j=1}^{|\alpha|} \frac{(D^{l_j} g)^{k_j}}{k_j!(l_j!)^{k_j}},$$

where

$p(\alpha,\lambda) = \{(k_1,\ldots,k_{|\alpha|}; l_1,\ldots,l_{|\alpha|})$ for some $1 \le s \le |\alpha|, k_i = 0$ and $l_i = 0$ for $1 \le i \le |\alpha| - s; k_i > 0$ for $|\alpha| - s + 1 \le i \le |\alpha|;$ and $0 \prec \alpha_{|\alpha|-s+1} \prec \ldots \prec l_{|\alpha|}$ are such that

$$\sum_{i=1}^{|\alpha|} k_i = \lambda,\ \sum_{i=1}^{|\alpha|} k_i l_i = \alpha\}.$$

Fix $(k_1,\ldots,k_{|\alpha|}; l_1,\ldots,l_{|\alpha|}) \in p(\alpha,\lambda)$. Then

$$D^\lambda F(r) = (-1)^\lambda h(x,y,s)^\lambda e^{-rh(x,y,s)},$$

and since $g(\xi)$ is homogeneous of degree 1, the factor $\prod_{j=1}^{s} \frac{(D^{l_j} g)^{k_j}}{k_j!(l_j!)^{k_j}}$ is homogeneous of degree $\sum_{j=1}^{s}(1-|l_j|)k_j = \lambda - |\alpha|$. It follows that $\partial_\xi^\alpha q(y,s,\xi)$ is a finite sum of constant multiples of terms of the type

$$\int_\Omega h(x,y,s)^\lambda e^{-|\xi| h(x,y,s)} q_\lambda(\xi) dx,$$

where $q_\lambda(\xi)$ is homogeneous of degree $\lambda - |\alpha|$.

For a multi-index β, we next consider $\partial_y^\beta \partial_\xi^\alpha q(y,s,\xi)$:

From the form of $\partial_\xi^\alpha q(y,s,\xi)$ that we have seen, we only need to consider terms of the form

$$\int_\Omega \partial_y^\beta \Big\{ h(x,y,s)^\lambda e^{-|\xi| h(x,y,s)} \Big\} q_\lambda(\xi) dx,$$

where $q_\lambda(\xi)$ is homogeneous of degree $\lambda - |\alpha|$ and $1 \le \lambda \le |\alpha|$. We have

$$\int_\Omega \partial_y^\beta \Big\{ h(x,y,s)^\lambda e^{-|\xi| h(x,y,s)} \Big\} q_\lambda(\xi)\, dx = \sum_{\delta \le \beta} \binom{\beta}{\delta} \int_\Omega \partial_y^{\beta-\delta} \Big\{ h(x,y,s)^\lambda \Big\} \partial_y^\delta e^{-|\xi| h(x,y,s)} q_\lambda(\xi)\, dx.$$

In the latter sum, consider a term

$$\partial_y^{\beta-\delta} \Big\{ h(x,y,s)^\lambda \Big\} \partial_y^\delta e^{-|\xi| h(x,y,s)}.$$

Once again we use Faá di Bruno's multivariable formula to compute

$$\partial_t^\delta e^{-|\xi||t|^{2k}} = \partial_t^\delta F(g(t)),$$

where $g(t) = |t|^{2k}$ and $F(r) = e^{-r|\xi|}$. We have

$$\partial_t^\delta e^{-|\xi||t|^{2k}} = \sum_{1\le\lambda'\le|\delta|} (D^{\lambda'}F)(g(t)) \sum_{p(\delta,\lambda')} \delta! \prod_{j=1}^{|\delta|} \frac{(D^{l_j}g)^{k_j}}{k_j!(l_j!)^{k_j}},$$

where $\sum_{i=1}^{|\delta|} k_i = \lambda', \sum_{i=1}^{|\delta|} k_i l_i = \delta$, and $0 < l_1 < \ldots < l_{|\delta|}$.
Fix $\lambda', 1 \le \lambda' \le |\delta|$ and $(k_1, \ldots, k_{|\delta|}; l_1, \ldots, l_{|\delta|}) \in p(\delta, \lambda')$. For each $1 \le j \le s$, $D^{l_j}g(t)$ is either 0 or homogenous of degree $2k - |l_j| \ge 0$. Therefore,

$$\prod_{j=1}^{s} \frac{(D^{l_j}g)^{k_j}}{k_j!(l_j!)^{k_j}}$$

is either 0 or a homogeneous polynomial of degree $\sum_{j=1}^{s} k_j(2k - |l_j|) = 2k\lambda' - |\delta|$. Thus $\partial_y^\delta e^{-|\xi||x-y|^{2k}}$ is a constant linear combination of terms of the form

$$g_{\lambda'}(x-y)|\xi|^{\lambda'} e^{-|\xi||x-y|^{2k}},$$

where $g_{\lambda'}$ is either 0 or a homogeneous polynomial of degree $2k\lambda' - |\delta|$. It follows that

$$\begin{aligned}&\int_\Omega \partial_y^{\beta-\delta}\Big\{h(x,y,s)^\lambda\Big\}\partial_y^\delta e^{-|\xi|h(x,y,s)} q_\lambda(\xi)\, dx \\ &= \int_\Omega \partial_y^{\beta-\delta}\Big\{h(x,y,s)^\lambda\Big\}\Big(\partial_y^\delta e^{-|\xi||x-y|^{2k}}\Big) e^{-|\xi||x-s|^{2k}} q_\lambda(\xi)\, dx\end{aligned}$$

is a constant linear combination of terms of the form

$$\int_\Omega \partial_y^{\beta-\delta}\Big\{h(x,y,s)^\lambda\Big\} g_{\lambda'}(x-y) q_\lambda(\xi)|\xi|^{\lambda'} e^{-|\xi|h(x,y,s)}\, dx,$$

where $g_{\lambda'}$ is either 0 or homogeneous of degree $2k\lambda' - |\delta|, 1 \le \lambda' \le |\delta|, q_\lambda(\xi)$

homogeneous of degree $\lambda - |\alpha|, 1 \leq \lambda \leq |\alpha|$. The same argument shows that for any multi-index γ, $\partial_\xi^\alpha \partial_y^\beta \partial_s^\gamma q(y,s,\xi)$ is a constant linear combination of terms of the form

$$\int_\Omega \partial_s^{\gamma-\delta'} \partial_y^{\beta-\delta} \Big\{ h(x,y,s)^\lambda \Big\} g_{\lambda''}(x-s) g_{\lambda'}(x-y) q_\lambda(\xi) |\xi|^{\lambda'+\lambda''} e^{-|\xi| h(x,y,s)}\, dx,$$

where $g_{\lambda''}$ is either 0 or a homogeneous polynomial of degree $2k\lambda'' - |\delta'| \geq 0, \;\; 1 \leq \lambda'' \leq |\delta'|, |\delta'| \leq |\gamma|, |\delta| \leq |\beta|, 1 \leq \lambda'' \leq |\delta'|$, and $g_{\lambda'}$, and q_λ are as before. Since $h(x,y,s)^\lambda$ is a polynomial of degree $2k\lambda$, we may assume that

$$|\gamma| - |\delta'| + |\beta| - |\delta| \leq 2k\lambda.$$

Clearly, for some constant $C > 0$,

$$\left| \partial_s^{\gamma-\delta'} \partial_y^{\beta-\delta} \Big\{ h(x,y,s)^\lambda \Big\} \right| \leq C h(x,y,s)^{\lambda - \frac{|\gamma|}{2k} - \frac{|\beta|}{2k} + \frac{|\delta'|}{2k} + \frac{|\delta|}{2k}},$$

and

$$\left| g_{\lambda''}(x-s) g_{\lambda'}(x-y) \right| \leq C h^{\lambda' - \frac{|\delta|}{2k} + \lambda'' - \frac{|\delta'|}{2k}}.$$

Thus

$$\left| \partial_s^{\gamma-\delta'} \partial_y^{\beta-\delta} \Big\{ h(x,y,s)^\lambda \Big\} g_{\lambda''}(x-s) g_{\lambda'}(x-y) q_\lambda(\xi) |\xi|^{\lambda'+\lambda''} e^{-|\xi| h(x,y,s)} \right|$$
$$\leq C_1 h(x,y,s)^{\lambda+\lambda'+\lambda'' - \frac{|\beta|}{2k} - \frac{|\gamma|}{2k}} |\xi|^{\lambda+\lambda'+\lambda''} |\xi|^{-|\alpha|} e^{-|\xi| h(x,y,s)}.$$

We claim that we may assume $\lambda + \lambda' + \lambda'' - \frac{|\beta|}{2k} - \frac{|\gamma|}{2k} \geq 0$. Indeed, this follows from the fact that unless

$$|\gamma| - |\delta'| + |\beta| - |\delta| \leq 2k|\delta|, \;\; 2k\lambda' \geq |\delta| \text{ and } 2k\lambda'' \geq |\delta'|,$$

the product

$$\partial_s^{\gamma-\delta'} \partial_y^{\beta-\delta} \Big\{ h(x,y,s)^\lambda \Big\} g_{\lambda''}(x-s) g_{\lambda'}(x-y)$$

would be zero. Thus

$$
\begin{aligned}
&\left|h(x,y,s)^{\lambda+\lambda'+\lambda''-\frac{|\beta|}{2k}-\frac{|\gamma|}{2k}}|\xi|^{\lambda+\lambda'+\lambda''}|\xi|^{-|\alpha|}e^{-|\xi|h(x,y,s)}\right| \\
&=\Big(h(x,y,s)|\xi|\Big)^{\lambda+\lambda'+\lambda''-\frac{|\beta|}{2k}-\frac{|\gamma|}{2k}} e^{-\frac{|\xi|}{2}h(x,y,s)}|\xi|^{\frac{|\beta|+|\gamma|}{2k}-|\alpha|}e^{-|\frac{|\xi|}{2}|h(x,y,s)} \\
&\leq C_2|\xi|^{\frac{|\beta|+|\gamma|}{2k}-|\alpha|}e^{-\frac{|\xi|}{2}h(x,y,s)}
\end{aligned}
$$

for some $C_2 > 0$, where we have used the fact that for any $d \geq 0$, the function $t^d e^{-t}$ is bounded on $[0, \infty)$.

It follows that for some constants $C' > 0, C > 0$,

$$
\begin{aligned}
\left|\partial_\xi^\alpha \partial_y^\beta \partial_s^\gamma q(y,s,\xi)\right| &\leq C'|\xi|^{\frac{|\beta|+|\gamma|}{2k}-|\alpha|}\int_\Omega e^{-\frac{|\xi|}{2}h(x,y,s)}\,dx \\
&\leq C|\xi|^{\frac{|\beta|+|\gamma|}{2k}-|\alpha|-\frac{m}{2k}}.
\end{aligned}
$$

We have shown that $q(y,s,\xi) \in S^{-\frac{m}{2k}}_{1,\frac{1}{2k}}$. Let $\varphi(\xi) \in C_0^\infty(\mathbb{R}^m), \varphi(\xi) \equiv 1$ for $|\xi| \leq \frac{b}{2}$ and $\varphi(\xi) \equiv 0$ for $|\xi| \geq b$. We write

$$
\int_\Omega \int_{\mathbb{R}^m} |\mathcal{F}_k(u,x,\xi)|^2 |\xi|^{\frac{m}{2}}(1+|\xi|^2)^s\,d\xi dx = A_1 + A_2,
$$

where

$$
A_1 = \int_\Omega \int_{\mathbb{R}^m} |\mathcal{F}_k(u,x,\xi)|^2 (1-\varphi(\xi))|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s\,d\xi dx,
$$

and

$$
A_2 = \int_\Omega \int_{\mathbb{R}^m} |\mathcal{F}_k(u,x,\xi)|^2 \varphi(\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s\,d\xi\,dx.
$$

We have

$$
\begin{aligned}
A_1 &= \int_{\mathbb{R}^m}\int_\Omega\int_\Omega e^{i\xi.(s-y)}(1-\varphi(\xi))q(y,s,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s u(y)\overline{u(s)}\,dyds d\xi \\
&= \langle Tu, u\rangle,
\end{aligned}
$$

where

$$Tu(s) = \int_{\mathbb{R}^m} \int_{\Omega} e^{i\xi.(s-y)} (1-\varphi(\xi)) q(y,s,\xi) |\xi|^{\frac{m}{2}} (1+|\xi|^2)^s u(y)\, dy d\xi$$

is a pseudodifferential operator in the class $\Psi^{2s+\frac{mk-m}{2k}}_{1,\frac{1}{2k}}$. By the boundedness of pseudodifferential operators in this class (see [23]), there exists $C_1 > 0$ such that

$$A_1 \leq C_1 ||u||^2_{H^{s+\frac{mk-m}{4k}}}.$$

The integral A_2 is of the form

$$A_2 = \langle Su, u \rangle,$$

where S is a smoothing operator and hence for any $M > 0$ there exists $C_M > 0$ such that

$$A_2 \leq C_M ||u||^2_{H^{-M}}.$$

It follows that for some $C > 0$,

$$\int_{\Omega} \int_{\mathbb{R}^m} |\mathcal{F}_k(u,x,\xi)|^2 |\xi|^{\frac{m}{2}} (1+|\xi|^2)^s \, d\xi dx \leq C ||u||^2_{H^{s+\frac{mk-m}{4k}}},$$

which establishes part (b) of Theorem 2.2.2.

To prove part (a), observe that the amplitude of the operator T is

$$B(y,s,\xi) = (1-\varphi(\xi)) q(y,s,\xi) |\xi|^{\frac{m}{2}} (1+|\xi|^2)^s,$$

and therefore, for any $\Omega' \subseteq \Omega'' \subset\subset \Omega$, as we saw before , for some $C > 0$,

$$\begin{aligned} B(y,y,\xi) &= (1-\varphi(\xi)) q(y,y,\xi) |\xi|^{\frac{m}{2}} (1+|\xi|^2)^s \\ &\geq C(1+|\xi|^2)^{s+\frac{mk-m}{4k}} \text{ for } y \in \Omega'', |\xi| \geq b. \end{aligned}$$

Hence by Garding's inequality, there exists $C > 0$ such that

$$||u||^2_{H^{s+\frac{mk-m}{4k}}} \leq A_1 + C||u||^2_{H^{s+\frac{mk-m}{4k}-\frac{1}{4}}}.$$

Therefore, for some $C > 0$,

$$||u||^2_{H^{s+\frac{mk-m}{4k}}} \leq C\bigg(\int_\Omega \int_{\mathbb{R}^m} |\mathcal{F}_k(u,x,\xi)|^2 |\xi|^{\frac{m}{2}} (1+|\xi|^2)^s \, d\xi dx + ||u||^2_{H^{s+\frac{mk-m}{4}-\frac{1}{4}}}\bigg),$$

which proves part (a) of Theorem 2.2.2. $\square$

We recall that a function $p(x,\xi) \in C^\infty(\Omega \times \mathbb{R}^m)$ ($\Omega \subset \mathbb{R}^m$ open) is said to belong to the symbol class $S^k_{\rho,\delta}$ if for every pair of multi-indices α, β and every compact subset $K \subset \Omega$,

$$|\partial_x^\beta \partial_\xi^\alpha p(x,\xi)| \leq (1+|\xi|)^{k-\rho|\alpha|+\delta|\beta|}, \;\; x \in K, \; \xi \in \mathbb{R}^m.$$

Given a symbol $p(x,\xi) \in S^k_{\rho,\delta}$, the corresponding pseudodifferential operator $P(x,D) \in \Psi^k_{\rho,\delta}$ is defined by

$$P(x,D)u(x) = \int_{\mathbb{R}^m} \int_\Omega e^{i(x-y)\cdot\xi} p(x,\xi) u(y) \, dy d\xi, \quad u \in \mathcal{E}'(\Omega).$$

If $u \in \mathcal{E}'(\Omega)$, one says the point $(x_0,\xi^0) \in \Omega \times \mathbb{R}^m \setminus \{0\}$ is not in the H^s wavefront set of u (denoted $(x_0,\xi^0) \notin \mathrm{WF}_s(u)$) if for some $\varphi(x) \in C_0^\infty(\Omega)$, $\varphi(x_0) \neq 0$, and an open cone $\Gamma \subset \mathbb{R}^m$ with vertex at the origin and containing ξ^0,

$$\int_\Gamma |\widehat{\varphi u}(\xi)|^2 (1+|\xi|^2)^s \, d\xi < \infty.$$

It is well known that $(x_0,\xi^0) \notin \mathrm{WF}_s(u)$ if and only if whenever $P(x,D)$ is an elliptic pseudodifferential operator of order zero whose support is in a conic neighborhood of (x_0,ξ^0), $P(x,D)u \in H^s$. The following theorem is a microlocal version of Theorem 2.2.2.

Theorem 2.2.3. *Let $(x_0, \xi^0) \in \mathbb{R}^m \times \mathbb{R}^m \setminus \{0\}$ and $p(x,\xi) \in S^0_{1,0}$, with support in a conic neighborhood $\Omega_1 \times \Gamma$ of (x_0, ξ^0), $\Omega_1 \subset\subset \Omega$. Then for any $u \in \mathcal{E}'(\Omega')$ $(\Omega_1 \subset\subset \Omega' \subset\subset \Omega)$, there exist constants $C_1, C_2 > 0$ independent of u such that:*

$$(a)\ ||P(x,D)u||^2_{H^t}$$
$$\leq C_1\left(\int_{\mathbb{R}^m}\int_\Omega |\mathcal{F}_k(u,x,\xi)|^2 |p(x,\xi)|^2 |\xi|^{\frac{m}{2}}(1+|\xi|^2)^{t+\frac{m-mk}{4k}}\, dxd\xi + ||u||^2_{H^{t-\frac{1}{4}}}\right);$$

$$(b)\ \int_{\mathbb{R}^m}\int_\Omega |\mathcal{F}_k(u,x,\xi)|^2 |p(x,\xi)|^2 |\xi|^{\frac{m}{2}}(1+|\xi|^2)^{t+\frac{m-mk}{4k}}\, dxd\xi$$
$$\leq C_2\left(||P(x,D)u||^2_{H^t} + ||u||^2_{H^{t-\frac{1}{4}}}\right).$$

Proof. We have

$$\int_{\mathbb{R}^m}\int_\Omega |\mathcal{F}_k(u,x,\xi)|^2 |p(x,\xi)|^2 |\xi|^{\frac{m}{2}}(1+|\xi|^2)^s\, dxd\xi$$
$$= \int_\Omega\int_{\mathbb{R}^m}\int_\Omega\int_\Omega e^{i\xi.(s-y)-|\xi|(|x-y|^{2k}+|x-s|^{2k})}|p(x,\xi)|^2|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s u(y)\overline{u(s)}\, dydsd\xi dx$$
$$= \int_{\mathbb{R}^m}\int_\Omega\int_\Omega e^{i\xi.(s-y)}q_1(y,s,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s u(y)\overline{u(s)}\, dydsd\xi,$$

where

$$q_1(y,s,\xi) = \int_\Omega e^{-|\xi|(|x-y|^{2k}+|x-s|^{2k})}|p(x,\xi)|^2\, dx.$$

For any multi-indices α, β, γ,

$$\partial_s^\gamma \partial_y^\beta \partial_\xi^\alpha q_1(y,s,\xi) = \sum_{\delta\leq\alpha}\binom{\alpha}{\delta}\int_\Omega \partial_s^\gamma\partial_y^\beta\partial_\xi^\delta\left(e^{-|\xi|h(x,y,s)}\right)\partial_\xi^{\alpha-\delta}|p(x,\xi)|^2\, dx.$$

We saw in the proof of Theorem 2.2.2 that for some $C > 0$,

$$\left|\partial_s^\gamma\partial_y^\beta\partial_\xi^\delta\left(e^{-|\xi|h(x,y,s)}\right)\right| \leq C|\xi|^{-\delta+\frac{|\beta|+|\gamma|}{2k}}.$$

Since $|p(x,\xi)|^2 \in \mathbb{S}^0_{1,0}$, for some $C' > 0$,

$$\left|\partial_\xi^{\alpha-\delta}|p(x,\xi)|^2\right| \leq C'|\xi|^{|\delta|-|\alpha|}$$

and hence

$$(1-\varphi(\xi))q_1(y,s,\xi) \in \mathbb{S}^{-\frac{m}{2k}}_{1,\frac{1}{2k}}.$$

Write

$$\int_{\mathbb{R}^m}\int_\Omega |\mathcal{F}_k(u,x,\xi)|^2|p(x,\xi)|^2|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s u(y)\overline{u(s)}\,dxd\xi = A_1 + A_2,$$

where

$$A_1 = \int_{\mathbb{R}^m}\int_\Omega\int_\Omega e^{i\xi.(s-y)}(1-\varphi(\xi))q_1(y,s,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s u(y)\overline{u(s)}\,dydsd\xi,$$

and

$$A_2 = \int_{\mathbb{R}^m}\int_\Omega\int_\Omega e^{i\xi.(s-y)}\varphi(\xi)q_1(y,s,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s u(y)\overline{u(s)}\,dydsd\xi.$$

We have $A_1 = \langle T_1u, u\rangle$, where

$$T_1u(s) = \int_{\mathbb{R}^m}\int_\Omega e^{i\xi.(s-y)}(1-\varphi(\xi))q_1(y,s,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s u(y)\,dyd\xi.$$

We recall from the proof of Theorem 2.2.2 that

$$Tu(s) = \int_{\mathbb{R}^m}\int_\Omega e^{i\xi.(s-y)}(1-\varphi(\xi))q(y,s,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s u(y)\,dyd\xi,$$

where $q(y,s,\xi) = \int_\Omega e^{-|\xi|(|x-y|^{2k}+|x-s|^{2k})}\,dx$. Write

$$P(x,D)u(x) = \int_{\mathbb{R}^m}\int_\Omega e^{i\xi.(s-y)}p(x,\xi)u(y)\,dyd\xi.$$

We observe that if $P^*(x,D)$ denotes the adjoint of $P(x,D)$, then the principal symbol of the composition $P^* \circ T \circ P$ is the same as that of T_1. Indeed, the principal symbol of T_1 is given by

$$(1-\varphi(\xi))q_1(y,y,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s = (1-\varphi(\xi))|p(x,\xi)|^2 q(y,y,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s,$$

while that of $P^* \circ T \circ P$ is

$$\overline{p(x,\xi)}(1-\varphi(\xi))q(y,y,\xi)|\xi|^{\frac{m}{2}}(1+|\xi|^2)^s p(x,\xi).$$

Therefore, the difference $E = T_1 - P^* \circ T \circ P$ is a pseudodifferential operator in the class $\Psi^{2s+\frac{mk-m}{2k}-\frac{1}{2}}_{1,\frac{1}{2k}}$. It follows that

$$\begin{aligned} A_1 &= \langle T_1 u, u\rangle \\ &= \langle P^* \circ T \circ P(u), u\rangle + \langle Eu, u\rangle \\ &= \langle T(Pu), Pu\rangle + \langle Eu, u\rangle. \end{aligned}$$

By Gårding's inequality, there are constants $C_1', C_2' > 0$ such that

$$\mathrm{Re}\Big\{\langle T(Pu), Pu\rangle\Big\} \geq C_1'||Pu||^2_{H^{s+\frac{mk-m}{4k}}} - C_2'||Pu||^2_{H^{s+\frac{mk-m}{4k}-\frac{1}{4}}}.$$

We also have, for some $C_3 > 0$,

$$|\langle Eu, u\rangle| \leq C_3||u||^2_{H^{s+\frac{mk-m}{4k}-\frac{1}{4}}}.$$

Hence for some $C_1, C_2 > 0$, since $P(x, D)$ is of order 0,

$$A_1 \geq C_1||Pu||^2_{H^{s+\frac{mk-m}{4k}}} - C_2||Pu||^2_{H^{s+\frac{mk-m}{4k}-\frac{1}{4}}}.$$

Since A_2 involves a smoothing operator, the proof of (a) is completed.
(b) follows from the continuity of T_1 and the fact that A_2 involves a smoothing operator. □

Chapter 3

FBI Transform Characterization of the Ultradifferentiable Wave Front Set

3.1 Introduction

The ultradiffrentiable classes are a natural generalization of Gevrey classes obtained by considering a sequence of real numbers $M = (M_j)_{j\in\mathbb{N}}$ satisfying some properties. Recall that a tempered growth of holomorphic function implies the existence of boundary values of distributions (Theorem 1.2.3. Z. Adwan and G. Hoepfner [2] recently proved sufficient conditions for the existence of boundary values in the sense of ultradistributions(Lemma 3.3.6). Our goal here to characterize ultradiffrentiable functions in terms of existence of almost analytic extensions (together with the work of Z. Adwan and G. Hopfner ([1], Lemma 17) and to characterizize the ultradifferentiable wave front set by using more general class of transforms (with a sum of finite number of elliptic, homogenous polynomials), which generalizes the work in [10] to ultradifferentiable classes.

3.2 Almost Analyticity in Ultradifferentiable Functions

Let $\Omega \subset \mathbb{R}^m$ an open subset and $(M_j)_{j\in\mathbb{N}}$ be an increasing sequence of positive real numbers with the properties $(P1)-(P7)$ (see chapter one) and $\mathcal{E}^M(\Omega)$ be the corresponding ultradifferentiable space.
For each sequence $M = (M_j)$, let

$$M(t) = \sup_j \log \frac{t^j}{M_j},\ t \in (0,\infty), M(0) = 0.$$

be the associated function.

Definition 3.2.1. Let $\Omega \subset \mathbb{R}^m$ be an open set and $f = f(x) \in \mathcal{E}^M(\Omega)$. A function $F = F(x,y) \in \mathcal{E}^M(\Omega \times (-1,1)^m)$ is called an $M-$ almost analytic extension of f if the following holds:
i) $F(x,0) = f(x)$ for all $x \in \Omega$; and
ii) for all $(x,y) \in \Omega \times (-1,1)^m$ and for all $N = 1,2,\dots$ there exists a constant $C > 0$, independent of N, such that, for all $j = 1,\dots,m$ it holds

$$\left|\frac{\partial F}{\partial \bar{z}_j}(z)\right| \leq \frac{C^{N+1}}{N!} M_N |y|^N.$$

Lemma 3.2.2. *Let $\alpha \in \mathbb{N}_0^m, x \in \mathbb{R}^m\{0\}$ and $h(x) = |x|^{-2m}$. Then there exist $C > 0$ such that*

$$|\partial^\alpha h(x)| \leq C^{|\alpha|+1}(m + |\alpha| - 1)!|x|^{-2m-|\alpha|}.$$

Proof. Let

$$f(t) = t^{-m} \text{ and } g(x) = \sum_{j=1}^m x_j^2$$

so that $h(x) = f(g(x))$. We will use Faá di Bruno formula to compute this derivative, that is,

$$\partial^\alpha h(x) = \sum_{r=1}^{|\alpha|} f^{(r)}(g(x)) \sum_{P(\alpha,r)} \alpha! \prod_{j=1}^{|\alpha|} \frac{(\partial^{\alpha_j} g)^{k_j}}{k_j!(\alpha_j!)^{k_j}},$$

where

$p(\alpha, r) = \{(k_1, \ldots, k_{|\alpha|}; \alpha_1, \ldots, \alpha_{|\alpha|}) :$ for some $1 \leq s \leq |\alpha|, k_j = 0$ and $\alpha_j = 0$ for $1 \leq j \leq |\alpha| - s; k_j > 0$ for $|\alpha| - s + 1 \leq j \leq |\alpha|;$ and $0 \prec \alpha_{|\alpha|-s+1} \prec \ldots \prec \alpha_{|\alpha|}$ are such that

$$\sum_{i=1}^{|\alpha|} k_j = r, \ \sum_{j=1}^{|\alpha|} k_j \alpha_j = \alpha\}.$$

Now $f'(t) = -mt^{-m-1}, f''(t) = -m(-m-1)t^{-m-2}$ and in general

$$f^{(r)}(t) = (-1)^r r! \binom{m+r-1}{r} t^{-m-r} = (-1)^r (m+r-1)(m+r-2) \ldots mt^{-m-r}.$$

Let $\alpha_j = (\alpha_j^1, \ldots, \alpha_j^m), \ j = 1, 2, \ldots, |\alpha|$. Then

$$\partial^{\alpha_j} g(x) = \frac{\partial^{|\alpha_j|}}{\partial x_1^{\alpha_j^1} \ldots \partial x_m^{\alpha_j^m}} g(x) = 0$$

except when $\alpha_j = e_j = (0, \ldots 1, 0 \ldots, 0)$ in that case it is $2x_j$ and when $\alpha_j = 2e_j$ the derivative is 2.

Thus when $\alpha_j = e_j$ from $\sum_{j=1}^{|\alpha|} k_j = r$ and $\sum_{j=1}^{|\alpha|} k_j \alpha_j = \alpha$ we have $|\alpha| = r, \alpha = (k_1, \ldots, k_{|\alpha|})$. When $\alpha_j = 2e_j$, we have $|\alpha| = 2r$ and $\alpha = 2(k_1, \ldots, k_{|\alpha|})$.

Therefore, since there are non zero terms only when $r = |\alpha|$ and $2r = |\alpha|$, and using the fact that

$$(\frac{|\alpha|}{2})! \binom{m + \frac{|\alpha|}{2} - 1}{\frac{|\alpha|}{2}} \leq |\alpha|! \binom{m + |\alpha| - 1}{|\alpha|},$$

we have

$$
\begin{aligned}
|\partial^\alpha h(x)| &\le |\alpha|!\binom{m+|\alpha|-1}{|\alpha|}|x|^{-2m-2|\alpha|}\left|\sum_{k_1+\ldots+k_{|\alpha|}=|\alpha|} k_1!\ldots k_{|\alpha|}!\prod_{j=1}^{|\alpha|}\frac{(2x_j)^{k_j}}{k_j!}\right| \\
&\quad + |\alpha|!\binom{m+|\alpha|-1}{|\alpha|}|x|^{-2m-|\alpha|}\sum_{2k_1+\ldots+2k_{|\alpha|}=|\alpha|}(2k_1)!\ldots(2k_{|\alpha|})!\prod_{j=1}^{|\alpha|}\frac{(2)^{k_j}}{k_j!(2!)^{k_j}} \\
&= |\alpha|!\binom{m+|\alpha|-1}{|\alpha|}|x|^{-2m-2|\alpha|}2^{|\alpha|}\left|x^{(k_1,\ldots,k_{|\alpha|})}\right|\sum_{k_1+\ldots+k_{|\alpha|}=|\alpha|}1 \\
&\quad + |\alpha|!\binom{m+|\alpha|-1}{|\alpha|}|x|^{-2m-|\alpha|}\sum_{2k_1+\ldots+2k_{|\alpha|}=|\alpha|}(2k_1)!\ldots(2k_{|\alpha|})!\prod_{j=1}^{|\alpha|}\frac{1}{k_j!} \\
&\le |\alpha|!\binom{m+|\alpha|-1}{|\alpha|}|x|^{-2m-|\alpha|}\left(2^{|\alpha|}\sum_{k_1+\ldots+k_{|\alpha|}=|\alpha|}1+|\alpha|!\sum_{2k_1+\ldots+2k_{|\alpha|}=|\alpha|}\prod_{j=1}^{|\alpha|}\frac{1}{k_j!}\right) \\
&\le |\alpha|!\binom{m+|\alpha|-1}{|\alpha|}|x|^{-2m-|\alpha|}\left(2^{|\alpha|}|\alpha|+C^{|\alpha|}\right) \text{ from (2.2.1)} \\
&\le |\alpha|!\binom{m+|\alpha|-1}{|\alpha|}|x|^{-2m-|\alpha|}\left(2^{2|\alpha|}+C^{|\alpha|}\right) \\
&\le C^{|\alpha|+1}|\alpha|!\binom{m+|\alpha|-1}{|\alpha|}|x|^{-2m-|\alpha|} \\
&\le C^{|\alpha|+1}(m+|\alpha|+1)!|x|^{-2m-|\alpha|}
\end{aligned}
$$

□

Lemma 3.2.3. *Let $\Omega \subset \mathbb{R}^m$ be an open set. $f \in \mathcal{E}^M(\Omega)$ if and only if there exist $F(x,y) \in \mathcal{E}^M(\Omega \times (-1,1)^m)$ such that*
(1) $F(x,0) = f(x)$ on Ω and
(2)

$$
\left|\frac{\partial F}{\partial \bar{z}_j}(z)\right| \le \frac{C^{N+1}}{N!}M_N|y|^N,\ \forall j = 1,2,\ldots,m
$$

on $\Omega \times (-1,1)^m$ for some constant $C > 0$, where $z_j = x_j + iy_j$.

Proof. Suppose $f \in \mathcal{E}^M(\Omega)$. Then (by [1], Lemma 17) f has an $M-$ almost analytic extension $F = F(x,y) \in \mathcal{E}^M(\Omega \times (-1,1)^m))$.

Conversely suppose there exist $F = F(x,y) \in \mathcal{E}^M(\Omega \times (-1,1)^m))$ such that
(1) $F(x,0) = f(x)$ on Ω and
(2)
$$\left|\frac{\partial F}{\partial \overline{z}_j}(z)\right| \leq \frac{C^{N+1}}{N!} M_N |y|^N,\ \forall j = 1,2,\ldots,m$$
on $\Omega \times (-1,1)^m$ for some constant $C > 0$. We will show that $f \in \mathcal{E}^M(\Omega)$. It suffice to show that $f \in \mathcal{E}^M(B_r)$ for each sufficiently small ball in Ω. Let $B_{2r} = \{x \in \Omega : |x| < 2r\}$ such that $\overline{B_{2r}} \subset \Omega$. Let $F(x,y)$ be as given above on a neighborhood of the closure of $\Omega_r = B_{2r} \times B_r$. We may assume that $F(x,y) = 0$ for $y \in B_r \setminus (-1,1)^m$.
Set $\omega(z) = dz_1 \wedge \ldots \wedge dz_m$. We will identify $\mathbb{C}^m$ with $\mathbb{R}^{2m}$. For $k = 1,\ldots,m$, let

$$\omega_k(\overline{z}) = (-1)^{k-1} d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_{k-1} \wedge \hat{d\overline{z}_k} \wedge d\overline{z}_{k+1} \wedge \ldots \wedge d\overline{z}_m,$$

where $d\overline{z}_k$ is removed. Let σ_n be the area of a unit sphere S^{n-1} in $\mathbb{R}^n$. The for each $x \in B_r$, from the higher dimensional version of the inhomogeneous Cauchy Integral Formula, we have

$$\begin{aligned} f(x) = F(x,0) &= \frac{2(2i)^{-m}}{\sigma_{2m}} \int_{\partial\Omega_r} F(w) \sum_{k=1}^{m} (\overline{w}_k - x_k)|w-x|^{-2m} \omega_k(\overline{w}) \wedge \omega(w) \\ &- \frac{2(2i)^{-m}}{\sigma_{2m}} \int_{\Omega_r} \sum_{k=1}^{m} \frac{\partial F}{\partial \overline{w}_k}(w)(\overline{w}_k - x_k)|w-x|^{-2m} \omega(\overline{w}) \wedge \omega(w) \\ &= f_1(x) + f_2(x) \end{aligned}$$

Since $f_1(x)$ is real analytic on B_r, $f_1 \in \mathcal{E}^M(B_r)$. It remains to show that $f_2 \in \mathcal{E}^M(B_r)$.
Let $\alpha = (\alpha_1,\ldots,\alpha_m)$. Then

$$\partial^\alpha f_2(x) = -\frac{2(2i)^{-m}}{\sigma_{2m}} \int_{\Omega_r} \sum_{k=1}^{m} \frac{\partial F}{\partial \overline{w}_k}(w) \partial_x^\alpha \Big[(\overline{w}_k - x_k)|w-x|^{-2m}\Big] \omega(\overline{w}) \wedge \omega(w).$$

For $x \neq w$,

$$\partial^\alpha\left[(\overline{w}_k - x_k)|w-x|^{-2m}\right] = \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\partial_x^\beta(\overline{w}_k - x_k)\partial_x^{\alpha-\beta}[|w-x|^{-2m}]$$
$$= (\overline{w}_k - x_k)\partial_x^\alpha[|w-x|^{-2m}] - \frac{\alpha!}{(\alpha-e_k)!}\partial_x^{\alpha-e_k}[|w-x|^{-2m}]$$
$$= (\overline{w}_k - x_k)\partial_x^\alpha[|w-x|^{-2m}] - \alpha_k\partial_x^{\alpha-e_k}[|w-x|^{-2m}]$$

Using Lemma 3.2.2, we have

$$|\partial^\alpha\left[(\overline{w}_k - x_k)|w-x|^{-2m}\right]| \leq$$
$$|w-x||\partial_x^\alpha[|w-x|^{-2m}]| + \alpha_k|\partial_x^{\alpha-e_k}[|w-x|^{-2m}]|$$
$$\leq (m+|\alpha|-1)!\left(C^{|\alpha|+1}|w-x||w-x|^{-2m-|\alpha|} + \alpha_k C^{|\alpha|-1}|w-x|^{-2m-|\alpha|+1}\right)$$
$$\leq C^{|\alpha|+1}(m+|\alpha|-1)!|w-x|^{-2m-|\alpha|+1}$$

Therefore,

$$|\partial^\alpha f_2(x)| = -\frac{2^{1-m}}{\sigma_{2m}}\int_{\Omega_r}\sum_{k=1}^m\left|\frac{\partial F}{\partial\overline{w}_k}(w)\right||\partial_x^\alpha\left[(\overline{w}_k - x_k)|w-x|^{-2m}\right]||\omega(\overline{w})\wedge\omega(w)|$$
$$\leq \frac{22^{-m}}{\sigma_{2m}}c^{|\alpha|+1}(m+|\alpha|-1)!C^{N+1}\frac{M_N}{N!}\int_{\Omega_r}\sum_{k=1}^m\frac{|\Im w|^N}{|w-x|^{2m+|\alpha|-1}}\omega(\overline{w})\wedge\omega(w)|$$
$$\leq \frac{22^{-m}}{\sigma_{2m}}c^{|\alpha|+1}(m+|\alpha|-1)!C^{N+1}\frac{M_N}{N!}\int_{\Omega_r}|\Im w|^{N-2m-|\alpha|+1}dv$$
$$\leq \frac{22^{-m}}{\sigma_{2m}}c^{|\alpha|+1}(m+|\alpha|-1)!C^{|\alpha|+2m}\frac{M_{2m+|\alpha|-1}}{(2m+|\alpha|-1)!}\int_{\Omega_r}dv \text{ (let } N = 2m+|\alpha|-1)$$
$$\leq \frac{22^{-m}}{\sigma_{2m}}c^{|\alpha|+1}C^{|\alpha|+2m}M_{2m+|\alpha|-1}$$
$$\leq \frac{22^{-m}}{\sigma_{2m}}c^{|\alpha|+1}C^{|\alpha|+2m}AH^{2m+|\alpha|-1}M_{2m-1}M_{|\alpha|}$$
$$\leq C'^{|\alpha|+1}M_{|\alpha|}$$

Therefore, $f_2 \in \mathcal{E}^M(B_r)$ and hence $f \in \mathcal{E}^M(B_r)$. □

3.3 Characterization of the Ultradifferentiable Wave Front Set

Let $\psi \in \mathcal{E}^M(\mathbb{R}^m)$ such that $0 \neq \int |\psi(x)|dx < \infty$. Following S. Berhanu and J. Hounie [11], define the FBI transform of $u \in \mathcal{E}^{M'}(\Omega)$ with generating function ψ and parameter λ, $0 < \lambda < 1$, by

$$\mathcal{F}_{\psi,\lambda}u(x,\xi) = \langle u(x'), e^{i\xi.(x-x')}\psi\Big(|\xi|^\lambda(x-x')\Big)\rangle.$$

Note that if $\psi(x) = e^{-|x|^2}$, then $\mathcal{F}_{\psi,\frac{1}{2}}$ is the classical Fourier-Bros-Iagolnitzer (FBI) transform.

Let $p(x)$ be a positive polynomial of the form

$$p(x) = \sum_{l=1}^{k} p_l(x) = p_1(x) + \ldots + p_k(x)$$

where

$$p_l(x) = \sum_{|\alpha|=2l} a_\alpha x^\alpha,\ l = 1, \ldots, k;\ a_\alpha, \in \mathbb{R}_+$$

and $p_l(x)$ $(l = 1, \ldots, k)$ satisfies

$$c_l|x|^{2l} \leq \sum_{|\alpha|=2l} a_\alpha x^\alpha \leq b_l|x|^{2l}$$

for some constants $0 < c_l \leq b_l,\ l = 1, \ldots, k$.

Take $\psi(x) = c_p e^{-p(x)}$, where $c_p = (\int_{\mathbb{R}^m} e^{-p(x)}dx)^{-1}$, as a generating function and $\frac{1}{2k}$ as a parameter. Then for $u \in \mathcal{E}^{M'}(\mathbb{R}^m)$ we define the FBI transform as

$$\mathcal{F}u(x,\xi) = c_p\langle u(x'), e^{i\xi.(x-x')}e^{-|\xi|^{\frac{1}{k}}p_1(x-x')-\ldots-|\xi|p_k(x-x')}\rangle,\ \ x, \xi \in \mathbb{R}^m.$$

Let $\chi(x) \in D^M(\mathbb{R}^m)$, such that $\chi \geq 0$ and $\int_{\mathbb{R}^m} \chi(x)dx = 1$. Set

$$\sigma(\xi) = \frac{\tilde{\chi}(\xi)}{(2\pi)^m}.$$

Let $u \in \mathcal{E}^{M'}(\mathbb{R}^m)$ then the inversion formula becomes

$$u(x) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^m \times \mathbb{R}^m} e^{i\xi.(x-x')} \sigma(\epsilon\xi) \mathcal{F}u(x', \xi)|\xi|^{\frac{m}{2k}} dx' d\xi$$

We recall the following theorem, which characterizes Gevrey wave front set of a function in terms of the above generalized FBI transform, where they used positive elliptic polynomial of the form $p_1(x) = \sum_{|\alpha|=2l} a_\alpha x^\alpha, p_2(x) = \sum_{|\beta|=2k} b_\beta x^\beta$ with $l < k$.

Theorem 3.3.1 ([10], Theorem 3.2)**.** *Let $u \in \mathcal{E}'(\mathbb{R}^m), x_0 \in \mathbb{R}^m, \xi^0 \in \mathbb{R}^m$ with $|\xi^0| = 1$. Then $(x_0, \xi^0) \notin WF_s(u), s > 1$ if and only if there exist a neighborhood V of x_0, a conic neighborhood Γ of ξ^0 and constants $a, b > 0$ such that for some $\phi \in C_0^\infty(\mathbb{R}^m), \phi \equiv 1$ near x_0,*

$$|\mathcal{F}(\phi u)(x, \xi)| \leq a e^{-b|\xi|^{\frac{1}{s}}}, \quad (x, \xi) \in V \times \Gamma.$$

We will generalize Theorem 3.3.1 to ultradifferentiable classes. First we will have some lemmas. In the following lemma $\Omega \subset \mathbb{R}^{2m} = \mathbb{R}^m_x \times \mathbb{R}^m_{x'}$ will denote an open subset containing the origin, $x = (x_1, \ldots, x_m) \in \mathbb{R}^m_x$, $x' = (x'_1, \ldots, x'_m) \in \mathbb{R}^m_{x'}$.

Lemma 3.3.2. *Let $\Omega \subset \mathbb{R}^{2m} = \mathbb{R}^m_x \times \mathbb{R}^m_{x'}$ be an open set. Let*

$$Q(x, x', \xi) = i\xi.(x - x') - |\xi|^{\frac{1}{k}} p_1(x - x') - \ldots - |\xi| p_k(x - x').$$

Then for each compact subset set $K \subset \Omega$, there exist $C > 0$ such that for all $\alpha \in \mathbb{N}_0^m$, $L > 0$ and $|\xi| > 1$,

$$|\partial^\alpha_{x'} e^{Q(x,x',\xi)}| \leq e^{\frac{1}{2}M(L|\xi|)} M_{|\alpha|} e^{\frac{H}{L}} C^{|\alpha|+1} \text{ on } K.$$

Proof. In what follows, we may increase the constant C from line to line, a finite number of times. Let $K \subset \Omega$ be compact.

Applying Faá di Bruno formula

$$\partial_{x'}^{\alpha} e^{Q(x,x',\xi)} = \sum_{r=1}^{|\alpha|} e^{Q(x,x',\xi)} \sum_{p(\alpha,r)} \alpha! \prod_{j=1}^{|\alpha|} \frac{(\partial_{x'}^{\alpha_j} Q)^{k_j}}{k_j!(\alpha_j!)^{k_j}}.$$

Thus using the fact that $p_1, \dots, p_m \in \mathcal{E}^M(\mathbb{R}^m)$, we have for $|\xi| > 1$,

$$\begin{aligned}
|\partial_{x'}^{\alpha} e^{Q(x,x',\xi)}| &= |\sum_{r=1}^{|\alpha|} e^{Q(x.x',\xi)} \sum_{p(\alpha,r)} \alpha! \prod_{j=1}^{|\alpha|} \frac{(\partial_{x'}^{\alpha_j} Q(x,x',\xi))^{k_j}}{k_j!(\alpha_j!)^{k_j}}| \\
&\leq \sum_{r=1}^{|\alpha|} e^{\Re Q(x.x',\xi)} \sum_{p(\alpha,r)} |\alpha|! \prod_{j=1}^{|\alpha|} |\frac{(\partial_{x'}^{\alpha_j} Q(x,x',\xi))^{k_j}}{k_j!(\alpha_j!)^{k_j}}| \\
&\leq \sum_{r=1}^{|\alpha|} \sum_{p(\alpha,r)} |\alpha|! \prod_{j=1}^{|\alpha|} \frac{\left[|\xi||\partial_{x'}^{\alpha_j}(x-x')| + |\xi|^{\frac{1}{k}} |\partial_{x'}^{\alpha_j} p_1(x-x')| + \ldots + |\xi||\partial_{x'}^{\alpha_j} p_k(x-x')|\right]^{k_j}}{k_j!(\alpha_j!)^{k_j}} \\
&\leq C \sum_{r=1}^{|\alpha|} \sum_{p(\alpha,r)} |\alpha|! \prod_{j=1}^{|\alpha|} \frac{\left(|\xi| C^{|\alpha_j|+1} M_{|\alpha_j|}\right)^{k_j}}{k_j!(\alpha_j!)^{k_j}}
\end{aligned}$$

Now from $(P7)$ and $(P5'')$, respectively, we get,

$$\prod_{j=1}^{|\alpha|} M_{|\alpha_j|}^{k_j} \leq \prod_{j=1}^{|\alpha|} C^{k_j|\alpha_j|} M_{|\alpha_j|k_j - k_j} \leq C^{|\alpha|} M_{|\alpha|-r}.$$

Also the inequality $(a+b)! \leq 2^{a+b} a! b!$ implies that

$$\prod_{j=1}^{|\alpha|} \left(\frac{|\xi| C^{|\alpha_j|+1}}{\alpha_j!} \right)^{k_j} \leq \frac{|\xi|^r C^{|\alpha|+r} 4^{|\alpha|}}{|\alpha|!}.$$

Therefore, using

$$t^r M_{|\alpha|-r} \leq \sqrt{A} (\frac{H}{L})^r M_{|\alpha|} e^{\frac{1}{2} M(Lt)}, t > 0 \;\; \text{(Lemma 1.1.10 (f))},$$

we obtain

$$
\begin{aligned}
|\partial_{x'}^{\alpha} e^{Q(x,x',\xi)}| &\leq \sum_{r=1}^{|\alpha|} \sum_{p(\alpha,r)} |\alpha|! \prod_{j=1}^{|\alpha|} \frac{\left(|\xi| C^{|\alpha_j|+1} M_{|\alpha_j|}\right)^{k_j}}{k_j!(\alpha_j!)^{k_j}} \\
&\leq \sum_{r=1}^{|\alpha|} \sum_{p(\alpha,r)} |\xi|^r C^{2|\alpha|+r} 4^{|\alpha|} M_{|\alpha|-r} \prod_{j=1}^{|\alpha|} \frac{1}{k_j!} \\
&\leq e^{\frac{1}{2}M(L|\xi|)} M_{|\alpha|} \sum_{r=1}^{|\alpha|} \sum_{p(\alpha,r)} (\frac{H}{L})^r C^{|\alpha|+r} \prod_{j=1}^{|\alpha|} \frac{1}{k_j!} \\
&\leq e^{\frac{1}{2}M(L|\xi|)} M_{|\alpha|} C^{|\alpha|} \sum_{r=1}^{|\alpha|} \sum_{p(\alpha,r)} (\frac{H}{L})^r \prod_{j=1}^{|\alpha|} \frac{1}{k_j!} \\
&= e^{\frac{1}{2}M(L|\xi|)} M_{|\alpha|} C^{|\alpha|} \sum_{r=1}^{|\alpha|} r! \sum_{p(\alpha,r)} \frac{(\frac{H}{L})^r}{r!} \prod_{j=1}^{|\alpha|} \frac{1}{k_j!} \\
&\leq e^{\frac{1}{2}M(L|\xi|)} M_{|\alpha|} e^{\frac{H}{L}} C^{|\alpha|} \sum_{r=1}^{|\alpha|} r! \sum_{p(\alpha,r)} \prod_{j=1}^{|\alpha|} \frac{1}{k_j!} \\
&\leq e^{\frac{1}{2}M(L|\xi|)} M_{|\alpha|} e^{\frac{H}{L}} C^{|\alpha|} C^{|\alpha|} \text{ from 2.2.1} \\
&\leq e^{\frac{1}{2}M(L|\xi|)} M_{|\alpha|} e^{\frac{H}{L}} C^{|\alpha|+1}
\end{aligned}
$$

□

Theorem 3.3.3. *Let* Ω *be as in Lemma 3.3.2 and* $u \in \mathcal{E}^{M'}(\Omega)$*. Then for each* $K \subset\subset \Omega$ *and for every* $L > 0$*, there exist* $C_L > 0$ *such that*

$$|\mathcal{F}u(x,\xi)| \leq C_L e^{M(L|\xi|)},\ x \in K,\ \xi \in \mathbb{R}^m.$$

Proof. Let $u \in \mathcal{E}^{M'}(\Omega)$. Then by definition $\mathcal{F}u(x,\xi) = \langle u(x'), \varphi(x,x',\xi)\rangle$, where $\varphi(x,x',\xi) = e^{i\xi.(x-x')-|\xi|^{\frac{1}{k}}p_1(x-x')-\ldots-|\xi|p_k(x-x')}$. Fix $K \subset\subset \Omega$. Then from definition

1.1.8 , for any $\epsilon_1 > 0$ and suitable $C_1 > 0$ depending on ϵ_1 we get

$$
\begin{aligned}
|\mathcal{F}u(x,\xi)| &= |\langle u(x'), \varphi(x,x',\xi)\rangle| \\
&\le C_1 \sup_{\alpha\in\mathbb{N}_0^m} \left\{ \frac{\epsilon_1^{|\alpha|}}{M_{|\alpha|}} \sup_{x'\in K} |\partial_{x'}^{\alpha}\varphi(x,x',\xi)| \right\} \\
&= C_1 \sup_{\alpha\in\mathbb{N}_0^m} \left\{ \frac{\epsilon_1^{|\alpha|}}{M_{|\alpha|}} \sup_{x'\in K} \left|\partial_{x'}^{\alpha}\left(e^{i\xi.(x-x')-|\xi|^{\frac{1}{k}}p_1(x-x')-\ldots-|\xi|p_k(x-x')}\right)\right| \right\}
\end{aligned}
$$

By Lemma 3.3.2, for such a compact set K there exist $C > 0$ such that for $\alpha \in \mathbb{N}_0^m$, $L > 0$ and $|\xi| > 1$,

$$
\left|\partial_{x'}^{\alpha}\left(e^{i\xi.(x-x')-|\xi|^{\frac{1}{k}}p_1(x-x')-\ldots-|\xi|p_k(x-x')}\right)\right| \le e^{\frac{1}{2}M(L|\xi|)}e^{\frac{H}{L}}C^{|\alpha|+1}M_{|\alpha|} \text{ on } K.
$$

Thus for $|\xi| > 1$,

$$
\begin{aligned}
&\sup_{x'\in K}\left|\partial_{x'}^{\alpha}\left(e^{i\xi.(x-x')-|\xi|^{\frac{1}{k}}p_1(x-x')-\ldots-|\xi|p_k(x-x')}\right)\right| \\
&\le e^{\frac{1}{2}M(L|\xi|)}e^{\frac{H}{L}}C^{|\alpha|+1}M_{|\alpha|} \\
&\le e^{M(L|\xi|)}e^{\frac{H}{L}}C^{|\alpha|+1}M_{|\alpha|}
\end{aligned}
$$

Thus

$$
\begin{aligned}
|\mathcal{F}u(x,\xi)| &\le C_1 \sup_{\alpha\in\mathbb{N}_0^m} \left\{ \frac{\epsilon_1^{|\alpha|}}{M_{|\alpha|}} C^{|\alpha|+1} e^{M(L|\xi|)} M_{|\alpha|} e^{\frac{H}{L}} \right\} \\
&= C_1 C e^{\frac{H}{L}} e^{M(L|\xi|)} \sup_{\alpha\in\mathbb{N}_0^m} \left\{ (C\epsilon_1)^{|\alpha|} \right\},\ |\xi| \ge 1 \\
&\le C_L e^{M(L|\xi|)}
\end{aligned}
$$

where we choose $\epsilon_1 < \frac{1}{C}$. For $|\xi| \le 1$, the estimate holds by continuity. □

Let $M(t)$ be the associated function to the sequence (M_j). The Young conjugate of the associated function, $w^* : [0, +\infty) \to [0, +\infty]$ is defined by

$$w^*(r) = \sup_{t\geq 0}\{M(t) - rt\},$$

which is comparable with w^* (see [31]) in the sense that for every $H > 1$ there exists $C > 0$ so that

$$M^*(Hs) - C \leq w^*(s) \leq M^*(s),\ \forall s > 0.$$

Another important function is

$$M^*(s) = -\log \inf_{j\in\mathbb{N}} \left\{\frac{s^j M_j}{j!}\right\}.$$

Note that from the definition of M^* and M one has

$$M(t) \leq \inf_{s}\{M^*(s) + st\},\ \forall t \geq 0.$$

Note also that if $M_j = j!^s$ $(s > 1)$, then $M^*(t)$ is equivalent to $\frac{1}{t^{\frac{1}{s-1}}}$ (see [26], pp. 198).

Remark 3.3.4. Condition (ii) in Definition 3.2.1 is equivalent to the condition

$$\left|\frac{\partial F}{\partial \overline{z}_j}(z)\right| \leq Ce^{-M^*(C|y|)}.$$

Definition 3.3.5. A complex vector sub-bundle $\mathcal{V}$ of $\mathbb{C}T(U \times V)$ (where U and V are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively) of rank n is called $M-$locally integrable structure if for each $p \in U\times V$, there are open neighborhood $\Omega_p \subset U\times V$ of p, functions $a_{k,j}, Z_k \in \mathcal{E}^M(\Omega_p)$, $j \in \{1, \ldots, m\}$ such that

$$\mathcal{V}_q = span\{(L_1)_q, \ldots, (L_n)_q\} \text{ for each } q \in \Omega_p,$$

where

$$L_j = \frac{\partial}{\partial t} + \sum_{k=1}^{m} a_{kj}(x,t)\frac{\partial}{\partial x_k}, \quad (x,t) \in \Omega_p,\ j = 1, \dots, n$$

and

$$span\{(dZ_1)_q, \dots, (dZ_m)_q\} = \mathcal{V}_q^{\perp},\ \forall q \in \Omega_p.$$

The following lemma [Theorem 3.1, [2]] gives sufficient conditions for the existence of boundary values in the sense of ultradistributions for certain solutions of $M-$locally integrable structures.

Lemma 3.3.6. *Let V be a neighborhood of $0 \in \mathbb{R}^m$, $\Gamma \subset \mathbb{R}^n$ be an open acute cone with vertex at the origin, $\delta > 0, \Gamma_\delta = \Gamma \cap \{v : |v| < \delta\}$, $f \in C^0(V \times \Gamma)$ and $\mathcal{V} = span\{L_1, \dots, L_n\}$ be $M-$ locally integrable structure of class $\mathcal{E}^M(\Omega)$. If*
(1) $L_j f \in L^\infty(V \times \Gamma),\ 1 \le j \le n$;
(2)f increases M^*-exponentially, that is, for all $\varrho > 0$ we have

$$|f(x,t)|e^{-\varrho M^*(|t|/\varrho)} < \infty.$$

Then, there exist the boundary value of f in $D^{M'}(V)$, i.e., there exist $bf \in D^{M'}(V)$ such that

$$\langle bf, \varphi \rangle = \lim_{\Gamma \ni t \to 0} \int f(x,t)\varphi(x)dx, \quad \forall \varphi \in D^M(V).$$

For future reference, we will use Lemma 3.3.6, under the hypothesis in which $m = n$ and $L_j = \partial_{\bar{z}_j}$.

Definition 3.3.7. Let $u \in D^{M'}(\Omega)$, $x_0 \in \Omega$ and $\xi^0 \in \mathbb{R}^m \setminus \{0\}$. We say that u is $M-$micro-regular at (x_0, ξ^0) if there is a neighborhood V of x_0, acute open cones $\Gamma_1, \dots, \Gamma_n \subset \mathbb{R}^m \setminus \{0\}$ and $f_j \in \mathcal{E}^M(V + i\Gamma_j^\delta)$ (for some $\delta > 0$) that increases M^*-exponentially such that
i) $\xi^0.\Gamma_j < 0\ \forall j$
ii) $u = \sum_{j=1}^n bf_j$ near x_0
iii)$\left|\frac{\partial f_j}{\partial \bar{z}_k}(x,y)\right| \le C^{N+1}\frac{M_N}{N!}|y|^N\ \forall j = 1, 2, \dots, n; \forall k = 1, 2, \dots m,\ \forall N \in \mathbb{N}.$

The ultradifferentiable ($M-$) wave front set of u is defined by

$$WF_M(u) = \{(x,\xi) : u \text{ is not } M- \text{ micro-regular at } (x,\xi)\}.$$

Our main theorem is the following which generalizes Theorem 3.3.1 to ultra-distributions.

Theorem 3.3.8. *Let* $u \in \mathcal{E}^{M'}(\mathbb{R}^m), x_0 \in \mathbb{R}^m, \xi^0 \in \mathbb{R}^m$ *with* $|\xi^0| = 1$. *Then* $(x_0,\xi^0) \notin WF_M(u)$ *if and only if there is a neighborhood* V *of* x_0*, a conic neighborhood* Γ *of* ξ^0 *and constants* $a, b > 0$ *such that for some* $\phi \in D^M(\mathbb{R}^m), \phi \equiv 1$ *near* x_0*,*

$$|\mathcal{F}(\phi u)(x,\xi)| \leq a e^{-M(b|\xi|)}, \ \ (x,\xi) \in V \times \Gamma.$$

Proof. Without loss of generality take $x_0 = 0$. Suppose $(0,\xi^0) \notin WF_M(u)$. Then by definition there exist $f_j \in \mathcal{E}^M$ in some truncated wedge $V + i\Gamma_j^\delta$ $(\delta > 0)$, V a neighborhood of 0 and Γ an open cone such that

i) $u = \sum_{j=1}^n bf_j$ on V

ii) $\xi^0.\Gamma_j < 0$ and

iii)$|\frac{\partial f_j}{\partial \bar{z}_k}| \leq C^{N+1}\frac{M_N}{N!}|y|^N \ \forall k = 1,2,\dots m \ j = 1,\dots,m; \ \forall N \in \mathbb{N}$ with

$$|f_(x+iy)|e^{-\varrho M^*(\frac{|y|}{\varrho})} < \infty, \ \forall \varrho > 0.$$

Without loss of generality we may assume that $u = bf$. Let $r > 0$ such that $B_{2r} \subset\subset V$ and $\phi \in D^M(B_{2r})$ such that $\phi_{|B_r} \equiv 1$. Let $v \in \Gamma_\delta$ and

$$Q(x,\xi,x') = i\xi.(x-x') - |\xi|^{\frac{1}{k}} p_1(x-x') - \dots - |\xi| p_k(x-x').$$

Then

$$\mathcal{F}(\phi u)(x,\xi) = c_p \lim_{\lambda \to 0^+} \int_{B_{2r}} e^{Q(x,\xi,x')} \phi(x') f(x' + i\lambda v) dx'.$$

Since $\phi \in D^M(\mathbb{R}^m)$, it has an almost holomorphic extension $\tilde{\phi}(x+iy)$ on $V + i\mathbb{R}^m$

with $x-$ support in B_{2r}. Then

$$\mathcal{F}(\phi u)(x,\xi) = c_p \lim_{\lambda\to 0^+} \int_{B_{2r}} e^{Q(x,\xi,x'+i\lambda v)}\tilde{\phi}(x'+i\lambda v)f(x'+i\lambda v)dx'.$$

For $0 < t < 1$, let

$$D_t = \{x'+i\lambda v \in \mathbb{C}^m : x' \in B_{2r}, t \le \lambda \le 1\}.$$

Consider the $m-$ form

$$\omega(z) = e^{Q(x,\xi,z)}\tilde{\phi}(z)f(z)dz,$$

where $dz = dz_1 \wedge \ldots \wedge dz_m,\ z = x'+iy'$. Since $\tilde{\phi}(z) = 0$ for $|x'| \ge 2r$ and since $e^{Q(x,\xi,z)}$ is holomorphic in z, by Stokes theorem we have

$$\begin{aligned}
\mathcal{F}(\phi u)(x,\xi) &= c_p \lim_{t\to 0^+} \int_{B_{2r}} e^{Q(x,\xi,x'+itv)}\tilde{\phi}(x'+itv)f(x'+itv)dx' \\
&= c_p \int_{B_{2r}} e^{Q(x,\xi,x'+iv)}\tilde{\phi}(x'+iv)f(x'+iv)dx' \\
&+ c_p \lim_{t\to 0^+} \sum_{j=1}^{m} \int\int_{D_t} e^{Q(x,\xi,x'+i\lambda v)}\tilde{\phi}(x'+i\lambda v)\frac{\partial f}{\partial \overline{z}_j}(x'+i\lambda v)d\overline{z}_j \wedge dz \\
&+ c_p \lim_{t\to 0^+} \sum_{j=1}^{m} \int\int_{D_t} e^{Q(x,\xi,x'+i\lambda v)}f(x'+i\lambda v)\frac{\partial \tilde{\phi}}{\partial \overline{z}_j}(x'+i\lambda v)d\overline{z}_j \wedge dz \\
&= c_p[I_0(x,\xi) + \lim_{t\to 0^+}\Big(I_1^t(x,\xi) + I_2^t(x,\xi)\Big)]
\end{aligned}$$

Since $v \in \Gamma$ and $\xi^0.\Gamma < 0$, there is a conic neighborhood Γ_1 of ξ^0 and a constant $c > 0$ such that

$$\xi.v \le -c|\xi||v|,\ \forall \xi \in \Gamma_1.$$

Consider $I_0(x,\xi)$:

$$|I_0(x,\xi)| \le \sup_{x'\in\overline{B_{2r}}} |\tilde{\phi}(x'+iv)f(x'+iv)| \int_{B_{2r}} e^{\Re Q(x,\xi,x'+iv)}dx'$$

Recall that if $a, b \in \mathbb{R}^m$ are such that $|a| + |b| \leq M$ for some $M > 0$ and $\alpha \in \mathbb{N}_0^m$, we have

$$-\Re(a+ib)^\alpha = -a^\alpha + O(|b|^2).$$

Thus

$$\begin{aligned}
-\Re p_l(x - x' - iv) &= -\Re \sum_{|\alpha|=2l} a_\alpha (x - x' - iv)^\alpha \\
&= -\sum_{|\alpha|=2l} a_\alpha (x - x')^\alpha + O(|v|^2) \\
&= -p_l(x - x') + O(|v|^2),\ l = 1, 2, \ldots, k.
\end{aligned}$$

Therefore, for $\xi \in \Gamma_1, |\xi| \geq 1$, since $l = 1, \ldots, k-1 < k$,

$$\begin{aligned}
&\Re Q(x, \xi, x' + iv) \\
&\quad = \Re\Big(i\xi.(x - x' - iv)\Big) - |\xi|^{\frac{1}{k}} p_1(x - x' - iv) - \ldots - |\xi| p_k(x - x' - iv) \\
&\quad = \xi.v - |\xi|^{\frac{1}{k}} \Re p_1(x - x' - iv) - \ldots - |\xi| \Re p_k(x - x' - iv) \\
&\quad = \xi.v - |\xi|^{\frac{1}{k}} p_1(x - x') + O(|v|^2)|\xi|^{\frac{1}{k}} - \ldots - |\xi| p_k(x - x') + O(|v|^2)|\xi| \\
&\quad \leq -c|v||\xi| - c_1|x - x'|^2 + O(|v|^2)|\xi|^{\frac{1}{k}} - \ldots - c_k|\xi||x - x'|^{2k} + O(|v|^2)|\xi| \\
&\quad \leq -c|v||\xi| + O(|v|^2)|\xi|
\end{aligned}$$

Choosing $|v|$ small such that $O(|v|^2) \leq \frac{c|v|}{2} = c'$, we then have

$$\Re Q(x, \xi, x' + iv) \leq -c'|\xi|,\ \xi \in \Gamma_1, x \in \mathbb{R}^m.$$

Thus, for $\xi \in \Gamma_1, |\xi| \geq 1$,

$$|I_0(x, \xi)| \leq c'' e^{-c'|\xi|} \leq c'' e^{-M(c'|\xi|)}$$

for some $c'' > 0$, because $M(t) \leq t$. Since

$$\frac{I_0(x, \xi)}{e^{-M(c'|\xi|)}}$$

is bounded on $\overline{B_{2r}} \times \{\xi : |\xi| \leq 1\}$, there are constants $A_0, B_0 > 0$ such that

$$|I_0(x, \xi)| \leq A_0 e^{-M(B_0|\xi|)}, \ \forall \xi \in \Gamma_1, |x| < 2r \tag{3.3.1}$$

Consider

$$I_1^t(x, \xi) = \sum_{j=1}^m \int \int_{D_t} e^{Q(x,\xi,x'+i\lambda v)} \tilde{\phi}(x' + i\lambda v) \frac{\partial f}{\partial \overline{z}_j}(x' + i\lambda v) d\overline{z}_j \wedge dz.$$

Now

$$\begin{aligned}
\Re Q(x, \xi, x' + i\lambda v) &\leq -c\lambda|v||\xi| - c_k|x - x'|^{2k}|\xi| + O(|\lambda v|^2)|\xi| \\
&\leq -c\lambda|v||\xi| - c_k|x - x'|^{2k}|\xi| + A''\lambda^2|v|^2|\xi| \\
&\leq -c\lambda|v||\xi| + A''\lambda|v|^2|\xi| \\
&\leq -c'\lambda|v||\xi| \ \ \text{(taking } |v| \text{ small such that } A''|v|^2 \leq \frac{c|v|}{2} = c'|v|).
\end{aligned}$$

Thus for $\xi \in \Gamma_1, |\xi| \geq 1$, letting

$$C' = \sup_{(x',\lambda) \in \overline{B_{2r}} \times [0,1]} |\tilde{\phi}(x' + i\lambda v)|),$$

we obtain

$$\begin{aligned}
|e^{Q(x,\xi,x'+i\lambda v)} \tilde{\phi}(x' + i\lambda v) \frac{\partial f}{\partial \overline{z}_j}(x' + i\lambda v| &\leq C' e^{\Re Q(x,\xi,x'+i\lambda v)} |\frac{\partial f}{\partial \overline{z}_j}(x' + i\lambda v|, \\
&\leq A' e^{-c'\lambda|v||\xi|} C^{N+1} \frac{M_N}{N!} |v|^N \lambda^N
\end{aligned}$$

Now using $e^{-t} \leq \frac{k!}{t^k}, \forall k, t > 0$ we have

$$e^{-c'\lambda|v||\xi|} \leq \frac{k!}{(c')^k} \cdot \frac{1}{\lambda^k |v|^k |\xi|^k} \ \forall k.$$

Thus letting $k = N$ we obtain

$$\begin{aligned}
&|e^{Q(x,\xi,x'+i\lambda v)} \tilde{\phi}(x'+i\lambda v) \frac{\partial f}{\partial \overline{z}_j}(x'+i\lambda v| \\
&\qquad \leq A' e^{-c'\lambda|v||\xi|} C^{N+1} \frac{M_N}{N!} |v|^N \lambda^N \\
&\qquad \leq A' \frac{N!}{(c')^N} \cdot \frac{1}{\lambda^N |v|^N |\xi|^N} C^{N+1} \frac{M_N}{N!} |v|^N \lambda^N \\
&\qquad \leq C^{N+1} \frac{M_N}{|\xi|^N}
\end{aligned}$$

Since the last condition is equivalent to the decay condition we then have

$$\begin{aligned}
\lim_{t \to 0^+} |I_1^t(x,\xi)| &\leq A' e^{-M(c''|\xi|)} \sum_{j=1}^{m} \int_0^1 \int_{B_{2r}} d\overline{z}_j \wedge dz_j \\
&\leq a_1 e^{-M(c''|\xi|)}
\end{aligned}$$

Therefore,

$$\lim_{t \to 0^+} |I_1^t(x,\xi)| \leq a_1 e^{-M(b_1|\xi|)}, \ \forall \xi \in \Gamma_1, |\xi| \geq 1, x \in B_{2r}$$

for some $a_1, b_1 > 0$ independent of t.
But $\frac{|I_1^t(x,\xi)|}{e^{-M(b_1|\xi|)}}$ is uniformly bounded on $\overline{B_{2r}} \times \{\xi : |\xi| \leq 1\}$. Thus, there are $A_1, B_1 > 0$ such that

$$\lim_{t \to 0^+} |I_1^t(x,\xi)| \leq A_1 e^{-M(B_1|\xi|)}, \ \forall \xi \in \Gamma_1, |x| < B_{2r}. \tag{3.3.2}$$

Consider

$$I_2^t(x,\xi) = \sum_{j=1}^{m} \int \int_{D_t} e^{Q(x,\xi,x'+i\lambda v)} f(x'+i\lambda v) \frac{\partial \tilde{\phi}}{\partial \overline{z}_j}(x'+i\lambda v) d\overline{z}_j \wedge dz.$$

For $\xi \in \Gamma_1, |\xi| \geq 1$,

$$\begin{aligned}
\Re Q(x,\xi,x'+i\lambda v) &\leq -c\lambda|v||\xi| + O(\lambda^2|v|^2)|\xi| - c_k|\xi||x-x'|^{2k} \\
&\leq O(|v|^2)|\xi| - c_k|\xi||x-x'|^{2k} \text{ since } \lambda \leq 1 \\
&\leq a'|v|^2|\xi| - c_k|\xi||x-x'|^{2k}
\end{aligned}$$

Since $\frac{\partial\tilde{\phi}}{\partial\bar{z}_j} \equiv 0$ for $|x'| \leq r$, the integral over $|x'| \leq r$ is zero. Then for $|x| < \frac{r}{2}$ and $|x'| \geq r$,

$$\Re Q(x,\xi,x'+i\lambda v) \leq a'|v|^2|\xi| - c_k\frac{r^{2k}}{2^{2k}}|\xi|.$$

Choosing $|v|$ small such that $a'|v|^2 \leq c_k\frac{r^{2k}}{2^{2k+1}} = c''$, we obtain

$$\Re Q(x,\xi,x'+i\lambda v) \leq -c''|\xi|,\ \xi \in \Gamma_1, |\xi| \geq 1.$$

Since f increases M^*-exponentially, for all $\varrho > 0$, there is a constant $d > 0$ such that

$$|f(x'+i\lambda v)| \leq de^{\varrho M^*(\frac{\lambda|v|}{\varrho})}.$$

Since $\tilde{\phi}$ is almost holomorphic, for each compact set $K \subset B_{2r}(0)$, there is $C_K = C > 0$ such that

$$|\frac{\partial\tilde{\phi}}{\partial\bar{z}_j}(x+i\lambda v)| \leq C^{N+1}\frac{M_N}{N!}\lambda^N|v|^N \ \forall N = 1,2,\ldots.$$

Equivalently,

$$|\frac{\partial\tilde{\phi}}{\partial\bar{z}_j}(x+i\lambda v)| \leq Ce^{-M^*(C\lambda|v|)}.$$

Now taking $C > 1$ and $\varrho = \frac{1}{C}$, we have that

$$\begin{aligned}
|f(x'+i\lambda v)||\frac{\partial\tilde{\phi}}{\partial\bar{z}_j}(x+i\lambda v)| &\leq dCe^{\varrho M^*(\frac{\lambda|v|}{\varrho}) - \frac{1}{C}M^*(c\lambda|v|)} \\
&\leq \frac{d}{\varrho}e^{-\varrho M^*(\frac{1}{\varrho}\lambda|v|) + \varrho M^*(\frac{1}{\varrho}\lambda|v|)} = C'
\end{aligned}$$

Thus we can get $A_2, B_2 > 0$ independent of t such that

$$\lim_{t \to 0^+} |I_2^t(x,\xi)| \leq A_2 e^{-M(B_2|\xi|)}, \ \forall \xi \in \Gamma_1, |x| < \frac{r}{2}. \tag{3.3.3}$$

Therefore, from (3.3.1), (3.3.2) and (3.3.3), we can find constants $a, b > 0$ such that

$$|\mathcal{F}\phi u(x,\xi)| \leq a e^{-M(b|\xi|)}, \ \forall (x,\xi) \in B_{\frac{r}{2}} \times \Gamma_1,$$

where Γ_1 is a conic neighborhood of ξ^0.

Conversely, suppose

$$|\mathcal{F}(\phi u)(x,\xi)| \leq a e^{-M(b|\xi|)}, \ \forall (x,\xi) \in V \times \Gamma,$$

where V is some neighborhood of $0, \Gamma$ a conic neighborhood of ξ^0 and, $A, B > 0$ are some constants and $\phi \in D^M(\mathbb{R}^m), \phi \equiv 1$ near 0. We will show that $(0,\xi^0) \notin WF_M(u)$. Let $\sigma(\xi) = e^{-|\xi|^2}$ (so, $\chi(x) = (4\pi)^{\frac{m}{2}} e^{-\frac{1}{4}|x|^2}$). We apply the inversion formula

$$\phi(x)u(x) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^m \times \mathbb{R}^m} e^{i\xi.(x-x') - \epsilon^2 |\xi^2|} \mathcal{F}(\phi u)(x',\xi) |\xi|^{\frac{m}{2k}} dx' d\xi.$$

Let

$$u_\epsilon(z) = \int_{\mathbb{R}^m \times \mathbb{R}^m} e^{i\xi.(z-x') - \epsilon^2 |\xi^2|} \mathcal{F}(\phi u)(x',\xi) |\xi|^{\frac{m}{2k}} dx' d\xi, \ \ z = x + iy \in \mathbb{C}^m.$$

Clearly $u_\epsilon(z)$ is an entire function of z for each $\epsilon > 0$. For some $a > 0$, write $u_\epsilon(z)$ as

$$u_\epsilon(z) = u_0^\epsilon(z) + u_1^\epsilon(z),$$

where

$$u_0^\epsilon(z) = \int_{\mathbb{R}^m} \int_{|x'| \leq a} e^{i\xi.(z-x')} \sigma(\epsilon\xi) \mathcal{F}(\phi u)(x',\xi) |\xi|^{\frac{m}{2k}} dx' d\xi$$

and

$$u_1^\epsilon(z) = \int_{\mathbb{R}^m} \int_{|x'|\geq a} e^{i\xi.(z-x')}\sigma(\epsilon\xi)\mathcal{F}(\phi u)(x',\xi)|\xi|^{\frac{m}{2k}} dx' d\xi.$$

Consider $u_0^\epsilon(z)$: Choose $a > 0$ such that $\{x' : |x'| \leq a\} \subset V$. Let $\mathcal{C}_0 = \Gamma, \mathcal{C}_j,\ 1 \leq j \leq n$ be open acute cones (we may take Γ to be acute) such that

$$\mathbb{R}^m = \cup_{j=0}^n \overline{\mathcal{C}_j}, \quad \overline{\mathcal{C}_j} \cap \overline{\mathcal{C}_k}$$

has measure zero when $j \neq k$ and $\xi^0 \notin \overline{\mathcal{C}_j}$ for $j \geq 1$.

Since $\xi^0 \notin \overline{\mathcal{C}_j}$ and $\mathcal{C}_j$ is acute we can get acute, open cones $\Gamma^j,\ 1 \leq j \leq n$ and a constant $c > 0$ such that

$$\xi^0.\Gamma^j < 0 \text{ and } y.\xi \geq c|y||\xi|,\ \forall y \in \Gamma^j, \forall \xi \in \mathcal{C}_j.$$

We then have

$$u_0^\epsilon(z) = \sum_{j=1}^n \int_{\mathcal{C}_j} \int_{|x'|\leq a} e^{i\xi.(z-x')-\epsilon^2|\xi|^2} \mathcal{F}(\phi u)(x',\xi)|\xi|^{\frac{m}{2k}} dx' d\xi.$$

For $j = 0, 1, \ldots, n$, and $z = x + iy \in \mathbb{R}^m + i\Gamma^j$, define

$$f_j^\epsilon(x+iy) = \int_{\mathcal{C}_j} \int_{|x'|\leq a} e^{i\xi.(x+iy-x')-\epsilon^2|\xi|^2} \mathcal{F}(\phi u)(x',\xi)|\xi|^{\frac{m}{2k}} dx' d\xi.$$

Then $f_j^\epsilon(z)$ are entire for $j \geq 1$ and converges uniformly on compact subsets of $\mathbb{R}^m + i\Gamma^j$ to the function

$$f_j(x+iy) = \int_{\mathcal{C}_j} \int_{|x'|\leq a} e^{i\xi.(x+iy-x')} \mathcal{F}(\phi u)(x',\xi)|\xi|^{\frac{m}{2k}} dx' d\xi,$$

which is holomorphic.

Moreover, f_j increases M^*- exponentially on $\mathbb{R}^m + i\Gamma_\delta^j$ for some $0 < \delta \leq 1$.

Indeed,

$$
\begin{aligned}
|f_j(x+iy)| &= |\int_{\mathcal{C}_j}\int_{|x'|\le a} e^{i\xi.(x+iy-x')}\mathcal{F}(\phi u)(x',\xi)|\xi|^{\frac{m}{2k}}dx'd\xi| \\
&\le \int_{\mathcal{C}_j}\int_{|x'|\le a} |e^{i\xi.(x+iy-x')}\mathcal{F}(\phi u)(x',\xi)|\xi|^{\frac{m}{2k}}|dx'd\xi \\
&\le \int_{\mathcal{C}_j}\int_{|x'|\le a} e^{-\xi.y}|\mathcal{F}(\phi u)(x',\xi)||\xi|^{\frac{m}{2k}}dx'd\xi \\
&\le C\int_{\mathcal{C}_j} e^{-c|y||\xi|}e^{M(L|\xi|)}|\xi|^{\frac{m}{2k}}d\xi \\
&= C\int_{\mathcal{C}_j} e^{M(|L\xi|)-c|y||\xi|}|\xi|^{\frac{m}{2k}}d\xi
\end{aligned}
$$

Now since for each $\varrho>0$ there exist $c_\varrho>0$ such that (see Lemma 1.1.10 (h))

$$
M(L|\xi|) = 2M(L|\xi|) - M(L|\xi|) \le \varrho M(c_\varrho L|\xi|) - M(L|\xi|),
$$

choosing $L = L_\varrho \doteq \frac{c}{c_\varrho}$, we have

$$
\begin{aligned}
|f_j(x+iy)| &\le C\int_{\mathcal{C}_j} e^{\varrho M(c_\varrho L|\xi|)-c|y||\xi|}e^{-M(L|\xi|)}|\xi|^{\frac{m}{2k}}d\xi \\
&= C\int_{\mathcal{C}_j} e^{\varrho M(c_\varrho L_\varrho|\xi|)-L_\varrho c_\varrho|y||\xi|}e^{-M(L|\xi|)}|\xi|^{\frac{m}{2k}}d\xi \\
&= C\int_{\mathcal{C}_j} e^{\varrho[M(c_\varrho L_\varrho|\xi|)-\frac{L_\varrho c_\varrho}{\varrho}|y||\xi|]}e^{-M(L|\xi|)}|\xi|^{\frac{m}{2k}}d\xi \\
&\le Ce^{\varrho\sup_{r>0}[M(r)-r\frac{|y|}{\varrho}]}\int_{\mathcal{C}_j} e^{-M(L|\xi|)}|\xi|^{\frac{m}{2k}}d\xi \\
&= Ce^{\varrho\omega^*(\frac{|y|}{\varrho})}\int_{\mathcal{C}_j} e^{-M(L|\xi|)}|\xi|^{\frac{m}{2k}}d\xi \\
&\le C_1e^{\varrho M^*(\frac{|y|}{\varrho})}.
\end{aligned}
$$

Thus each $f_j,\ j=1,\dots,n$ has boundary value $bf_j \in D^{M'}(\mathbb{R}^m)$.

Let

$$
g_0^\epsilon(x) = \int_\Gamma\int_{|x'|\le a} e^{i\xi.(x-x')-\epsilon^2|\xi|^2}\mathcal{F}(\phi u)(x',\xi)|\xi|^{\frac{m}{2k}}dx'd\xi.
$$

By the estimate for $\mathcal{F}(\phi u)(x',\xi)$ on the set $\{x' : |x'|\le a\}\times\Gamma$, $g_0^\epsilon(x)$ are smooth for all $\epsilon > 0$. And since $e^{-M(t)}$ is decreasing, and

$$\begin{aligned}|g_0(x) - g_0^\epsilon(x)| &= \int_\Gamma \int_{|x'|\leq a} |\mathcal{F}(\phi u)(x',\xi)||\xi|^{\frac{m}{2k}}|1-e^{-\epsilon^2|\xi|^2}|dx'd\xi \\ &\leq C\int_\Gamma |\xi|^{\frac{m}{2k}} e^{-M(b|\xi|)}|1-e^{-\epsilon^2|\xi|^2}|d\xi \to 0\end{aligned}$$

by monotone convergence theorem, $g_0^\epsilon(x)$ converge uniformly on $\mathbb{R}^m$ to the function

$$g_0(x) = \int_\Gamma \int_{|x'|\leq a} e^{i\xi.(x-x')}\mathcal{F}(\phi u)(x',\xi)|\xi|^{\frac{m}{2k}}dx'd\xi.$$

Clearly g_0 is smooth on $\mathbb{R}^m$. In fact, $g_0 \in \mathcal{E}^M$ since

$$\begin{aligned}|\partial^\alpha g_0(x)| &\leq \int_\Gamma |\xi|^{\frac{m}{2k}+|\alpha|}|\mathcal{F}(\phi u)(x',\xi)|dx'd\xi \\ &\leq C\int_\Gamma |\xi|^{\frac{m}{2k}+|\alpha|}e^{-M(b|\xi|)}d\xi \\ &= \int_\Gamma |\xi|^{\frac{m}{2k}+|\alpha|}e^{-\frac{1}{2}M(b|\xi|)}e^{-\frac{1}{2}M(b|\xi|)}d\xi \\ &\leq C\int_\Gamma |\xi|^{\frac{m}{2k}+|\alpha|}\frac{(M_{2|\alpha|})^{1/2}}{(b|\xi|)^{|\alpha|}}e^{-\frac{1}{2}M(b|\xi|)}d\xi \quad \text{since } e^{-M(t)} \leq \frac{M_j}{t^j}\ \forall j \\ &\leq C\int_\Gamma |\xi|^{\frac{m}{2k}+|\alpha|}\frac{\left(AH^{2|\alpha|}M_{|\alpha|}^2\right)^{1/2}}{(b|\xi|)^{|\alpha|}}e^{-\frac{1}{2}M(b|\xi|)}d\xi \text{ by } P3'' \\ &= C''\int_\Gamma |\xi|^{\frac{m}{2k}}M_{|\alpha|}\frac{H^{|\alpha|}}{b^{|\alpha|}}e^{-\frac{1}{2}M(b|\xi|)}d\xi \\ &= C''\left(\frac{H}{b}\right)^{|\alpha|}M_{|\alpha|}\int_\Gamma |\xi|^{\frac{m}{2k}}e^{-\frac{1}{2}M(b|\xi|)}d\xi \\ &\leq C^{|\alpha|+1}M_{|\alpha|} \text{ by Lemma 1.1.10}(i).\end{aligned}$$

Thus there is $f_0(x,y) \in \mathcal{E}^M(V\times\mathbb{R}^m)$ such that $f_0(x,0) = g_0(x)$ and

$$|\frac{\partial f_0}{\partial \bar{z}_j}(x,y)| \leq C^{N+1}\frac{M_N}{N!}|y|^N.$$

Choose Γ_0 an open cone such that $\xi^0.\Gamma_0 < 0$. Thus we have found open cones $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$ and functions $f_j, j \geq 1$ holomorphic on $\mathbb{R}^m + i\Gamma_j^\delta$ (for some $0 < \delta <$

1), which increases M^*- exponentially and $f_0(x,y)$ in $\mathcal{E}^M(\mathbb{R}^m + i\Gamma_0^\delta)$ (for some $0<\delta<1$) which also increases M^*- exponentially and such that

$$\xi^0.\Gamma_j < 0,\ 0 \le j \le n$$

and

$$|\frac{\partial f_j}{\partial \bar{z}_k}(x,y)| \le C^{N+1}\frac{M_N}{N!}|y|^N,\ \forall j=1,\dots,n,\ \forall k=0,1,\dots,m,\ \forall N\in\mathbb{N}.$$

In the sense of distributions, for $j=1,\dots,n$, we have

$$\lim_{\Gamma_j\ni y\to 0} f_j(x+iy) = \lim_{\epsilon\to 0^+} f_j^\epsilon(x)$$

and

$$\lim_{\Gamma_0\ni y\to 0} f_0(x+iy) = \lim_{\epsilon\to 0^+} g_0^\epsilon(x).$$

Hence

$$u_0(x) = \sum_{j=0}^{n} bf_j \text{ in } D^{M'}(\mathbb{R}^m).$$

This implies that $(0,\xi^0)\notin WF_M(u_0)$. To complete the proof it remains to show that $u_1^\epsilon(z)$ converges to a holomorphic function $u_1(z)$, which is proved below. In that case $(0,\xi^0)\notin WF_a(u_1)$ and so $(0,\xi^0)\notin WF_M(u_1)$. Since $WF_M(u)\subset WF_M(u_0)\cup WF_M(u_1)$, we conclude that $(0,\xi^0)\notin WF_M(u)$. □

To prove the lemma we follow the lines of proofs of Lemma 4.1 of [11] and Theorem 3.2 of [10].

Lemma 3.3.9. *There is $\delta>0$ and a holomorphic function $u_1(z)\in B_\delta\subset\mathbb{C}^m$ such that* $\lim_{\epsilon\to 0^+} u_1^\epsilon(z) = u_1(z)$, $z\in B_\delta$.

Proof. It suffice to show that there exist $\delta>0$ and $L>0$ such that $|u_1^\epsilon(z)|\le L$ for any $|z|<\delta$ and $0<\epsilon\le 1$. In this case $\{u_1^\epsilon\}_{0<\epsilon\le 1}$ will be a normal family on the ball $B_\delta\subset\mathbb{C}^m$ and hence there exist a subsequence $\epsilon_k>0$ such that for some

$0 < \delta' < \delta$,

$$u_1^{\epsilon_k}(x+iy) \to u_1(x+iy)$$

uniformly on $|x+iy| \le \delta'$. In particular, $u_1(z)$ is holomorphic on $|z| < \delta$.
Write $u_1^\epsilon(z)$ as a sum of functions

$$u_1^\epsilon(z) = I_1^\epsilon(z) + I_2^\epsilon(z) + I_3^\epsilon(z),$$

where

$$I_j^\epsilon(z) = \int_{X_j} e^{i\xi.(z-x')-\epsilon^2|\xi^2|}\mathcal{F}(\phi u)(x',\xi)|\xi|^{\frac{m}{2k}}dx'd\xi,\ j=1,2,3$$

where

$$X_1 = \{(x',\xi) : a \le |x'| \le A,\ |\xi| \le 1\},$$

$$X_2 = \{(x',\xi) : |x'| \ge A,\ \xi \in \mathbb{R}^m\},$$

$$X_3 = \{(x',\xi) : a \le |x'| \le A,\ |\xi| \ge 1\},$$

where we will choose the constant A later. We will show that $|I_j^\epsilon(z)|,\ j=1,2,3$ remain bounded for $0 < \epsilon < 1$ and $|z| < \delta$ if δ is small.

Since X_1 is a bounded set and $\mathcal{F}(\phi u)$ is continuous function, it is clear that there is a constant $L_1 > 0$ independent of ϵ such that $|I_1^\epsilon(z)| \le L_1$ for, say, $|y| \le 1$
Consider $I_2^\epsilon(z)$: Let $r > 0$ such that

$$supp(\phi) \subset \{x : |x| \le r\} = B_r.$$

Choose $A = 2r$. Then for $|x| \le r$ and $|x'| \ge A$,

$$\begin{aligned}|x'-x| \ge |x'| - |x| \ge |x'| - r = |x'| - \frac{A}{2} &\ge |x'| - \frac{|x'|}{2} \\ &= \frac{|x'|}{2} = \frac{|x'|}{4} + \frac{|x'|}{4}\end{aligned}$$

Thus $|x'-x| \geq \frac{|x'|}{4} + \frac{A}{4}$ and hence

$$|x'-x|^{2k} \geq \left(\frac{|x'|}{4} + \frac{A}{4}\right)^{2k} \geq \frac{|x'|^{2k}}{4^{2k}} + \frac{A^{2k}}{4^{2k}}$$

Then we have

$$\begin{aligned}
|\mathcal{F}(\phi u)(x',\xi)| &= \left| \int_{|x|\leq r} e^{i\xi.(x'-x)-|\xi|^{\frac{1}{k}}p_1(x'-x)-\ldots-|\xi|p_k(x'-x)}\phi(x)u(x)dx \right| \\
&\leq C \sup_{|x|\leq r, |\alpha|\leq N_1} \left| \partial_x^\alpha \left(e^{i\xi.(x'-x)-|\xi|^{\frac{1}{k}}p_1(x'-x)-\ldots-|\xi|p_k(x'-x)} \right) \right| \quad N_1 = \text{ the order of } u \\
&\leq C' e^{-A_1|\xi||x'|^{2k}-B_1|\xi|}, \ |x'| \geq A, \xi \in \mathbb{R}^m,
\end{aligned}$$

for some constants $C', A_1, B_1 > 0$ independent of $\epsilon > 0$.
To obtain the above estimate we use the observation that if c is a constant and $A(x)$ is a smooth function, for any multi index β, the derivative $\partial_x^\beta e^{cA(x)}$ is a sum of terms of the form $c^{l_1+\ldots+l_n}(\partial^{m_1}p)^{l_1}\ldots(\partial^{m_n}p)^{l_n}$, where $\sum_{j=1}^n m_j l_j = |\beta|$ together with the fact that $e^{-c} \leq \frac{k!}{c^k}$ for any $c > 0$.
Therefore,

$$\begin{aligned}
|I_2^\epsilon(z)| &= \left| \int_{\mathbb{R}^m} \int_{|x'|\geq A} e^{i\xi.(z-x')-\epsilon^2|\xi|^2} \mathcal{F}(\phi u)(x',\xi)|\xi|^{\frac{m}{2k}} dx' d\xi \right| \\
&\leq C' \int_{\mathbb{R}^m} \int_{|x'|\geq A} e^{|y||\xi|} e^{-A_1|\xi||x'|^{2k}-B_1|\xi|} |\xi|^{\frac{m}{2k}} dx' d\xi \\
&= C' \int_{\mathbb{R}^m} e^{|y||\xi|} e^{-B_1|\xi|} |\xi|^{\frac{m}{2k}} \left(\int_{|x'|\geq A} e^{-A_1|\xi||x'|^{2k}} dx' \right) d\xi \\
&= C'' \int_{\mathbb{R}^m} e^{|y||\xi|} e^{-B_1|\xi|} d\xi \\
&\leq C'' \int_{\mathbb{R}^m} e^{\frac{-B_1}{2}|\xi|} d\xi, \ \forall z = x+iy, |y| < \frac{B_1}{2} \\
&\leq C'' \int_{\mathbb{R}^m} e^{-M(\frac{B_1}{2}|\xi|)} d\xi \\
&\leq L_2
\end{aligned}$$

Thus there is $L_2 > 0$ independent of $0 < \epsilon \leq 1$ such that

$$|I_2^\epsilon(z)| \leq L_2, \ \forall |z| < \delta_2 = \frac{B_1}{2}, \ \forall \, 0 < \epsilon \leq 1.$$

Consider

$$I_3^\epsilon(z) = \int\int\int_R e^{i\xi.(z-t)-|\xi|^{\frac{1}{k}}p_1(x'-t)-\ldots-|\xi|p_k(x'-t)-\epsilon^2|\xi|^2}\phi(t)u(t)|\xi|^{\frac{m}{2k}}d\xi dt dx',$$

where

$$R = \{(\xi, x', t) : |\xi| \geq 1, |t| \leq r, a \leq |x'| \leq A\}.$$

Recall that the function $\xi \longmapsto |\xi|$ has a holomorphic extension

$$\langle\zeta\rangle = \left(\sum_{j=1}^{m} \zeta_j^2\right)^{\frac{1}{2}}.$$

In particular the functions $\zeta \longmapsto \langle\zeta\rangle$ and $\zeta \longmapsto \langle\zeta\rangle^{\frac{m}{2k}}$ are holomorphic on the set

$$S = \{\zeta = \xi + i\eta \in \mathbb{R}^m : |\eta| < |\xi|\}.$$

Fix x, t. Then we will change the contour of integration in ξ from the $m-$cycle $\{\xi : |\xi| \geq 1\} \subset \mathbb{R}^m$ to the image under the map

$$\zeta(\xi) = \xi + ib|\xi|(x-t)$$

where $b > 0$ is chosen small so that

$$|\Im\zeta(\xi)| = b|\xi||x-t| < |\Re\zeta(\xi)| = |\xi|.$$

Let

$$D = \{\xi + i\sigma b|\xi|(x-t) : |\xi| \geq 1, 0 \leq \sigma \leq 1\}.$$

Then

$$\partial D=\{\xi:|\xi|\geq 1\}\cup\{\xi+ib|\xi|(x-t):|\xi|\geq 1\}\cup\{\xi+i\sigma b|\xi|(x-t):|\xi|=1,0\leq\sigma\leq 1\}.$$

Consider the $m-$form

$$\omega(z,x',t,\zeta,\epsilon)=e^{i\zeta(z-t)-\langle\zeta\rangle^{\frac{1}{k}}p_1(x'-t)-\ldots-\langle\zeta\rangle p_k(x'-t)-\epsilon^2\langle\zeta\rangle^2}\phi(t)u(t)\langle\zeta\rangle^{\frac{m}{2k}}d\zeta,$$

where $\zeta=\xi+i\eta\in\mathbb{C}^m, d\zeta=d\zeta_1\wedge\ldots\wedge d\zeta_m$. Since

$$g(\zeta)=e^{i\zeta(z-t)-\langle\zeta\rangle^{\frac{1}{k}}p_1(x'-t)-\ldots-\langle\zeta\rangle p_k(x'-t)-\epsilon^2|\xi|^2}\phi(t)u(t)\langle\zeta\rangle^{\frac{m}{2k}}$$

is a holomorphic function of ζ, ω is a closed form. Thus by the Stokes theorem we have

$$\int_{\partial D}\omega d\zeta=\int_D d\omega\wedge d\zeta=0.$$

Therefore,

$$\begin{aligned}&\int_{|\xi|\geq 1}e^{i\xi.(z-t)-|\xi|^{\frac{1}{k}}p_1(x'-t)-\ldots-|\xi|p_k(x'-t)-\epsilon^2\langle\zeta\rangle^2}\phi(t)u(t)|\xi|^{\frac{m}{2k}}d\xi\\&=\int_{|\xi|\geq 1}\omega(z,x',\xi+ib|\xi|(x-t))d\xi-\int_0^1\int_{|\xi|=1}\omega(z,x',\xi+i\sigma b|\xi|(x-t))d\xi d\sigma\end{aligned}$$

Clearly there is $B_2>0$ independent of ϵ such that

$$\left|\int_0^1\int_{|\xi|=1}\omega(z,x',\xi+i\sigma b|\xi|(x-t))d\xi d\sigma\right|\leq B_2.$$

To estimate the other integrals let

$$Q(z,x',t,\xi,\epsilon)=i\zeta(\xi).(z-t)-\langle\zeta(\xi)\rangle^{\frac{1}{k}}p_1(x'-t)-\ldots-\langle\zeta(\xi)\rangle p_k(x'-t)-\epsilon^2\langle\zeta(\xi)\rangle^2,$$

where

$$\zeta(\xi)=\xi+ib|\xi|(x-t), z=x+iy.$$

Then

$$\Re Q(z,x',t,\xi,\epsilon) = -b|\xi||x-t|^2 - y.\xi - \Re\langle\zeta(\xi)\rangle^{\frac{1}{k}} p_1(x'-t) - \ldots - \Re\langle\zeta(\xi)\rangle p_k(x'-t) - \epsilon^2\Re\langle\zeta(\xi)\rangle^2$$

Now

$$\langle\zeta(\xi)\rangle^2 = \sum_{j=1}^{m}(\xi_j + ib|\xi|(x_j - t_j))^2 = |\xi|^2 - b^2|\xi|^2|x-t|^2 + i2b|\xi|\xi.(x-t).$$

Let $|x| \leq 1$. Then since $|t| \leq r$,

$$b^2|\xi|^2|x-t|^2 \leq b^2B|\xi|^2$$

for some $B > 0$. Then we can choose $b > 0$ small enough such that

$$\Re\langle\zeta(\xi)\rangle^2 = |\xi|^2 - b^2|\xi|^2|x-t|^2 \geq \frac{|\xi|^2}{2}$$

and

$$\arg\langle\zeta(\xi)\rangle^2 \in [\frac{-\pi}{4}, \frac{\pi}{4}].$$

Hence

$$\begin{aligned}
\Re\langle\zeta(\xi)\rangle^{\frac{l}{k}} = \Re\left(\sum_{j=1}^{m}\zeta_j^2(\xi)\right)^{\frac{l}{2k}} &= \Re(\langle\zeta(\xi)\rangle^{\frac{l}{2k}} \\
&= \Re e^{\frac{l}{2k}\log(\langle\zeta(\xi)\rangle^2)} \\
&= |\langle\zeta(\xi)\rangle^2|^{\frac{l}{2k}}\cos(\frac{l}{2k}\arg\langle\zeta(\xi)\rangle^2) > 0, l = 1, \ldots m-1
\end{aligned}$$

and

$$\begin{aligned}
\Re\langle\zeta(\xi)\rangle &= |\langle\zeta(\xi)\rangle^2|^{\frac{1}{2}}\cos(\frac{1}{2}\arg\langle\zeta(\xi)\rangle^2) \\
&\geq (\Re\langle\zeta(\xi)\rangle^2)^{\frac{1}{2}}\cos(\frac{1}{2}\arg\langle\zeta(\xi)\rangle^2) \\
&= B'|\xi|,\ B' > 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Re Q(z,x',t,\xi,\epsilon) \\
&\qquad = -b|\xi||x-t|^2 - y.\xi - \Re\langle\zeta(\xi)\rangle^{\frac{1}{k}} p_1(x'-t) - \ldots - \Re\langle\zeta(\xi)\rangle p_k(x'-t) - \epsilon^2\Re\langle\zeta(\xi)\rangle^2 \\
&\qquad \le -b|\xi||x-t|^2 + |y||\xi| - B'c_k|\xi||x'-t|^{2k}
\end{aligned}$$

Let $z = x + iy = 0$. Then

$$\Re Q(0,x',x,\xi,\epsilon) \le -b|\xi||t|^2 - B'c_k|\xi||x'-t|^{2k}.$$

If $|t| \ge \frac{a}{2}$, then

$$\Re Q(0,x',t,\xi,\epsilon) \le -b|\xi||t|^2 \le -b\frac{a^2}{4}|\xi|.$$

If $|t| \le \frac{a}{2}$, then since $|x'| \ge a, |x'-t| \ge \frac{a}{2}$ and so

$$\Re Q(0,x',t,\xi,\epsilon) \le -B'c_k|\xi||x'-t|^{2k} \le \frac{-B'c_k a^{2k}}{2^{2k}}|\xi|.$$

Thus there is $A_1 > 0$ independent of $\epsilon > 0$ such that

$$\Re Q(0,x',t,\xi,\epsilon) \le -A_1|\xi|, \quad \forall|\xi| \ge 1.$$

By continuity and homogeneity in ξ, there is $\delta_3 > 0$ such that for some $A_2 > 0$

$$\Re Q(z,x',t,\xi,\epsilon) \le -A_2|\xi|, \quad \forall|\xi| \ge 1, |z| \le \delta_3.$$

Therefore,

$$\begin{aligned}
\left|\int_{|\xi|\ge 1} \omega(z,x',t,\zeta(\xi),\epsilon)\right| &\le C'\int_{|\xi|\ge 1} e^{-A_2|\xi|}\left|\langle\zeta(\xi)\rangle^{\frac{m}{2k}}\right| d\xi \\
&\le C'\int_{|\xi|\ge 1} e^{-M(A_2|\xi|)}\left|\langle\zeta(\xi)\rangle^{\frac{m}{2k}}\right| d\xi \\
&\le L_3
\end{aligned}$$

for some $L_3 > 0$ independent of $\epsilon > 0$ for all $|z| < \delta_3$.

Let $\delta = \min\{1, \delta_1, \delta_2, \delta_3\}$. Then there is $0 < L < \infty$ such that

$$\sup_{0<\epsilon\leq 1} |u_1^\epsilon(z)| \leq L, \quad \forall |z| < \delta.$$

□

3.4 Applications

In [11](Theorem 5.1) the authors gave an application where the classic FBI transform can not be used but one with a power of four can be used to characterize the analytic wave front set. Here we will present an application where the FBI transform with phase power of four can be used to characterize the $M-$ wave front set of a distribution but the quadratic phase FBI transform can not be used.

Let $I \subset \mathbb{R}$ be an open interval with $0 \in I$ and $t^* \in I^+ = I \cap (0, +\infty)$. Consider $\phi := (\phi_1, \dots, \phi_m) \in (\mathcal{E}^M(I))^m$, $\phi(0) = 0$ and the vector field L defined by

$$L \doteq \frac{\partial}{\partial t} - i \sum_{k=1}^{m} \frac{\partial \phi_k}{\partial t}(t) \frac{\partial}{\partial y_k}, \quad k = 1, \dots, m.$$

If $Z_k(y,t) = y_k + i\phi_k(t)$, then $LZ_k = 0$ and $Z_k(y,0) = y_k$ $(k = 1, \dots, m)$. Let $Z \doteq (Z_1, \dots, Z_m)$.

Let $B_r(0) = \{x \in \mathbb{R}^m : |x| < r\}$ for some $r > 0$. Set $\Omega = B_r(0) \times I$. Let $h(x,t)$ be a solution in Ω of the equation

$$Lh = 0. \tag{3.4.1}$$

We will study the M-wave front set of the solution $h_0(x) = h(x,0)$.

Theorem 3.4.1. *Let I, I^+ and ϕ be as above. Let $\xi^0 \in \mathbb{R}^m \setminus \{0\}$ with $|\xi^0| = 1$. For some $\epsilon > 0$, assume that*

$$1) \ -\phi(t^*).\xi^0 \geq \epsilon^7$$

$$2) \ |\phi(t)| \leq \epsilon^2, \ \forall t \in [0, t^*]$$

Then there exists $\epsilon_0 > 0$ such that if $0 < \epsilon \leq \epsilon_0$ and h is a solution of (3.4.1) in $\Omega = B_r(0) \times I$, then $(0, \xi) \notin WF_M(h_0)$.

Example 3.4.2. Let

$$\phi : (-1, 1) \to \mathbb{R}, \ \phi(t) = \epsilon^7 \frac{t}{t^*}, \ 0 < t^* < 1 \text{ for some } \ 0 < \epsilon \leq 1.$$

Then $\phi(t)$ satisfies both conditions of the theorem with $\xi^0 = -1$.

proof of Theorem 3.4.1: Shrinking $B_r(0)$ and taking $\epsilon > 0$ small, without loss of generality assume $r^2 = \epsilon$. Let $g \in D^M(B_r(0)), g(y) \equiv 1$ for $|y| \leq \frac{r}{2}$. Let

$$F(x, t, \xi) = \int_{\mathbb{R}^m} e^{Q(x,y,t,\xi)} g(y) h(y, t) dy,$$

where

$$Q(x, y, t, \xi) = i\xi.(x - Z(y, t)) - K|\xi|(x - Z(y, t))^4,$$

where $K > 0$ is to be determined.

For $z \in \mathbb{C}^m$ we used the notation

$$z^4 = \left(\sum_{j=1}^{m} z_j^2 \right)^2.$$

Let

$$I(x,\xi) = \int_{\mathbb{R}^m}\int_0^{t^*} e^{Q(x,y,t,\xi)} L(g(y)h(y,t))dtdy$$
$$= \int_{\mathbb{R}^m}\int_0^{t^*} e^{Q(x,y,t,\xi)} [h(y,t)L(g(y)) + g(y)L(h(y,t))]dtdy$$
$$= \int_{\mathbb{R}^m}\int_0^{t^*} e^{Q(x,y,t,\xi)} h(y,t)L(g(y))dtdy \tag{3.4.2}$$

Integration by parts and the fact $LZ_k = 0$ gives

$$I(x,\xi) = F(x,t^*,\xi) - F(x,0,\xi). \tag{3.4.3}$$

Now choosing $p(x) = |x|^4$ and $\Psi(x) = e^{-|\xi|p(x)}$, we obtain

$$\mathcal{F}(gh_0)(x,\xi) = \int_{\mathbb{R}^m} e^{i\xi.(x-y)} e^{-|\xi|p(x-y)} g(y)h(y,0)dy = F(x,0,\xi).$$

Thus we are done if we show that for some $C_1, C_2 > 0$,

$$|F(x,0,\xi)| \leq C_1 e^{-M(C_2|\xi|)},\ (x,\xi) \text{ in a conic neighborhood of } (0,\xi^0).$$

Let $E(x,y,t,\xi) = -\Re Q(x,y,t,\xi)$. For $x = 0$ and $\xi = \xi^0$, $|\xi^0| = 1$, we get

$$E(0,y,t,\xi^0) = -\Re Q(0,x,y,t,\xi^0)$$
$$= -\Re\Big(i\xi^0.(-Z(y,t)) - K|\xi^0|(-Z(y,t))^4\Big)$$
$$= -\xi^0.\phi(t) + K\Re\Big(Z(y,t)\Big)^4$$

Now

$$\Re z^4 = (|x|^2 - |y|^2)^2 - 4(x.y)^2$$
$$\geq (|x|^2 - |y|^2)^2 - 4|x|^2|y|^2$$
$$\geq |x|^4 - 6|x|^2|y|^2 \tag{3.4.4}$$

Therefore,

$$\Re\bigg(Z(y,t)\bigg)^4 \geq |y|^4 - 6|y|^2|\phi(t)|^2 \tag{3.4.5}$$

and hence

$$E(0,y,t,\xi^0) \geq -\xi^0.\phi(t) + K(|y|^4 - 6|y|^2|\phi(t)|^2). \tag{3.4.6}$$

In particular, taking $t = t^*$ and $|y| \leq r$, $r^2 = \epsilon$ by assumptions of the theorem, we obtain

$$\begin{aligned}
E(0,y,t^*,\xi^0) &\geq \epsilon^7 + K(|y|^4 - 6|y|^2|\phi(t^*)|^2) \\
&= \epsilon^7 + K(|y|^4 - 6\epsilon^4|y|^2) \\
&\geq \epsilon^7 + K(-9\epsilon^8) \\
&= \epsilon^7(1 - 9K\epsilon)
\end{aligned} \tag{3.4.7}$$

since $|y|^4 - 6\epsilon^4|y|^2$ attains its minimum $-9\epsilon^8$.

For $r^2 = \epsilon$, $\frac{r}{2} \leq |y| \leq r$ using (3.4.4) and the assumptions, we get for any $\epsilon > 0$ such that $\epsilon < \frac{1}{(192)^{\frac{1}{3}}}$,

$$\begin{aligned}
E(0,y,t,\xi^0) &\geq -\xi^0.\phi(t) + K(|y|^4 - 6|y|^2|\phi(t)|^2) \\
&\geq -\epsilon^2 + K(\frac{r^4}{4} - 6r^2\epsilon^4) \\
&= -\epsilon^2 + K(\frac{\epsilon^2}{2^4} - 6\epsilon^5) \\
&= -\epsilon^2 + \epsilon^2 K(\frac{1}{2^4} - 6\epsilon^3) \\
&\geq \epsilon^2(\frac{K}{32} - 1)
\end{aligned}$$

Thus for $K \geq 64$, we have

$$E(0,y,t,\xi^0) \geq \epsilon^2 \tag{3.4.8}$$

Choose $K = 64$ to grant (3.4.8) and pick $\epsilon_0 < \frac{1}{576}$. Then for any $0 < \epsilon < \epsilon_0$ the right hand side of (3.4.7) will be positive. Since $E(x,t,y,\xi)$ is homogeneous function of degree one in ξ, (3.4.7) and (3.4.8) imply that there is a neighborhood

$V \subset \mathbb{R}^m$ of the origin, and $\Gamma \subset \mathbb{R}^m$ an open cone centered ξ^0 such that for some $C > 0$

$$E(x, y, 0, \xi) \geq C|\xi|, \ \forall (x, \xi) \in V \times \Gamma, y \in \text{ supp } g \tag{3.4.9}$$

and

$$E(x, y, t, \xi) \geq C|\xi|, \ \forall (x, \xi) \in V \times \Gamma, 0 \leq t \leq t^*, \frac{\epsilon^{\frac{1}{2}}}{2} \leq |y| \leq \epsilon^{\frac{1}{2}}. \tag{3.4.10}$$

Therefore from (3.4.2),(3.4.3), (3.4.9) and (3.4.10), we obtain

$$\begin{aligned}
&|\mathcal{F}(gh_0)(x, \xi)| \\
&\qquad = |F(x, 0, \xi)| = |F(x, t^*, \xi) - I(x, \xi)| \\
&\qquad \leq |F(x, t^*, \xi)| + |I(x, \xi)| \\
&\qquad = |\int_{\mathbb{R}^m} \int_0^{t^*} e^{Q(x,y,t,\xi)} h(y, t) Lg(y) dt dy| + |\int_{\mathbb{R}^m} e^{Q(x,y,t^*,\xi)} g(y) h(y, t) dt dy| \\
&\qquad \leq \int_{\mathbb{R}^m} \int_0^{t^*} e^{-C|\xi|} |h(y, t) Lg(y)| dy dt + \int_{\mathbb{R}^m} e^{-C|\xi|} |g(y) h(y, t)| dt dy \\
&\qquad \leq Be^{-C|\xi|} \leq Be^{-M(C|\xi|)}, \ \ (x, \xi) \in V \times \Gamma.
\end{aligned}$$

□

Remark 3.4.3. We indicate here that it is not possible to prove the preceding theorem with the classical FBI transform (quadratic phase).
Let

$$F_1 = \int_{\mathbb{R}^m} e^{Q_1(x,y,t,\xi)} g(y) h(y, t) dy,$$

where

$$Q_1(x, y, t, \xi) = i\xi.(x - Z(y, t)) - K|\xi|(x - Z(y, t))^2$$

for some $K > 0$ fixed. For $z \in \mathbb{C}^m$, $z^2 = \sum_{j=1}^m z_j^2$. Observe that

$$Q_1(x, y, 0, \xi) = i\xi.(x - y) - K|\xi|(x - y)^2$$

is the phase of the classical FBI transform. Assuming $|\xi^0| = 1$ and letting $E_1(x, y, t, \xi) = -\Re Q_1(x, y, t, \xi)$ we get

$$E_1(x, y, t^*, \xi^0) = -\xi^0.\phi(t^*) + K|\xi^0|(|x-y|^2 - |\phi(t^*|^2).$$

Assume that $-\xi^0.\phi(t^*) = \epsilon^7$ and $|\phi(t^*| = \epsilon^2$. Then

$$E_1(x, y, t^*, \xi^0) = \epsilon^7 + K(|x-y|^2 - \epsilon^4).$$

In particular,

$$E_1(x, x, t^*, \xi^0) = \epsilon^7 - K\epsilon^4 = \epsilon^4(\epsilon^3 - K)$$

which is negative if $K > \epsilon^3$.
Suppose $K < \epsilon^3$. Since $E_1(x, y, t, \xi^0) = -\phi(t) \cdot \xi^0 + K[|x-y|^2 - |\phi(t)|^2]$, we have

$$\begin{aligned} E_1(0, y, t, \xi^0) &= -\phi(t) \cdot \xi^0 + K[|y|^2 - |\phi(t)|^2] \\ &\leq -\phi(t) \cdot \xi^0 + K|y|^2 \end{aligned}$$

Thus for a point t such that $-\phi(t) \cdot \xi^0 = -\epsilon^2$ and for $|y| \geq \frac{r}{2}$ we get

$$\begin{aligned} E_1(0, y, t, \xi^0) &\leq -\phi(t) \cdot \xi^0 + K|y|^2 \\ &< -\epsilon^2 + \epsilon^3 r^2 \end{aligned}$$

But then since ϵ is small, $-\epsilon^2 + \epsilon^3 r^2$ is negative unless $r > \frac{1}{\epsilon}$ is large. Thus the quantity $E_1(x, y, t, \xi^0)$ may be negative in this case also.

Chapter 4

Microlocal Analysis on Maximally Real Complex Submanifolds

4.1 Microlocal Smoothness in a Maximally Real Submanifold of $\mathbb{C}^m$

4.1.1 Introduction

Consider an involutive structure on a C^∞ manifold $\mathcal{M}$; denote the tangent and cotangent bundles by $\mathcal{V}$ and T' and their fiber dimensions by n and m, respectively. Let $\mathcal{X} \subset \mathcal{M}$ maximally real submanifold. Then the pullback map

$$\pi^* : \mathbb{C}T^*\mathcal{M}|_{\mathcal{X}} \to \mathbb{C}T^*\mathcal{X}$$

induces an isomorphism

$$T'|_{\mathcal{X}} \cong \mathbb{C}T^*\mathcal{X}.$$

Let $\mathbb{R}T'_{\mathcal{X}}$ be the real structure bundle of $\mathcal{X}$. Note that, therefore, $\dim_{\mathbb{R}} \mathcal{X} = m$ and $\mathbb{R}T'_{\mathcal{X}}$ is a real vector bundle over $\mathcal{X}$ of fiber dimension equal to m.

Remark 4.1.1. (Description of the real structure bundle $\mathbb{R}T'_{\mathcal{X}}$ near 0)
Let $\mathcal{X} \subset \mathbb{C}^m$ be a maximally real submanifold. After a translation and a $\mathbb{C}-$linear transformation in $\mathbb{C}^m$, we may assume that $0 \in \mathcal{X}$ and that $T_0\mathcal{X} = \mathbb{R}^m$. Then in a small enough neighborhood Ω of 0 in $\mathcal{X}$, Ω is the image of some open neighborhood U of 0 in $\mathbb{R}^m$ under the map $x \mapsto Z(x)$ with $Z(x) = x + i\phi(x)$; where $\phi : U \to \mathbb{R}^m, \phi(0) = 0$, and $d\phi(0) = 0$. Then a point $(z, \zeta) \in \mathbb{R}T'_{\mathcal{X}}$, with $z \in Z(U)$, if there is $x \in U$ and $\xi \in \mathbb{R}^m$ such that

$$z = Z(x) \text{ and } \zeta = {}^tZ_x(x)^{-1}\xi,$$

where ${}^tZ_x(x)^{-1}$ denotes the transpose of the matrix $Z_x(x)^{-1}$.

We denote the variable points in $\mathbb{C}^m$ by z or z' and dual coordinates by $\zeta_j(1 \leq j \leq m)$. For any number $\kappa > 0$ we shall write

$$\mathcal{C}_\kappa = \{\zeta \in \mathbb{C}^m : |\Im\zeta| < \kappa|\Re\zeta|\}.$$

For any $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ we write $[z]^2 = z_1^2 + \ldots + z_m^2$ and $\zeta \in \mathcal{C}_1$ we write $\langle\zeta\rangle = (\zeta.\zeta)^{\frac{1}{2}}$ (main branch of the square root).
Let

$$\mathcal{M}_k = \sum_{j=1}^{m} \mu_{kj} \frac{\partial}{\partial x_j}, 1 \leq k \leq m \tag{4.1.1}$$

be the vector fields characterized by the relations

$$\mathcal{M}_k(z_j|_{\mathcal{X}}) = \delta_{kj}.$$

Then $\mathcal{M}_1, \ldots, \mathcal{M}_m$ form a smooth basis of $\mathbb{C}T\mathcal{X}$.

Definition 4.1.2. A maximally real submanifold $\mathcal{X}$ of $\mathbb{C}^m$ is called well positioned at one of its points, z_0, if there is a number $\kappa, 0 < \kappa < 1$, and an open neighborhood Ω of z_0 such that:
i. $|\Im\zeta| < \kappa|\Re\zeta|; \forall z \in \Omega, \zeta \in \mathbb{R}T'_{\mathcal{X}}|_z$;

ii. $\Im\{\zeta.(z-z')+i\langle\zeta\rangle[z-z']^2\} \geq (1-\kappa)|\zeta||z-z'|^2,\ \forall z,z' \in \Omega,$
$\forall\zeta \in \left(\mathbb{R}T'_{\mathcal{X}}|_z\right) \cup \left(\mathbb{R}T'_{\mathcal{X}}|_{z'}\right)$
We say $\mathcal{X}$ is very well positioned at z_0 if for any number $\kappa, 0 \leq \kappa < 1$, there is an open neighborhood Ω of z_0 in $\mathcal{X}$ such that the above holds.

Remark 4.1.3. The conditions in (i) and (ii) above are unchanged if we exchange z and z' and replace ζ by $-\zeta$. It would therefore have sufficed to require that ζ belongs to $\mathbb{R}T'_{\mathcal{X}}|_z$.

4.1.2 FBI Transform in a Maximally Real Submanifold of $\mathbb{C}^m$

Let U be an open neighborhood of 0 in $\mathbb{R}^m$ and $Z : U \to \mathbb{C}^m$ with $Z(x) = x+i\phi(x)$, where map $\phi : U \to \mathbb{R}^m$ is a C^∞, $\phi(0) = 0, d\phi(0) = 0$.
Then $\mathcal{X} = Z(U)$ is a maximally real smooth submanifold of $\mathbb{C}^m$.
It was shown in [35] (Proposition IX.2.2) that $\mathcal{X}$ defined above is very well positioned at 0, that is, given any number $\kappa, 0 \leq \kappa < 1$, there is an open neighborhood Ω of 0 in $\mathcal{X}$ such that the following holds true:

$$1.|\Im\zeta| < \kappa|\Re\zeta|, 2.\Im\left[\zeta.(z-z')+i\langle\zeta\rangle[z-z']^2\right] \geq (1-\kappa)|\zeta||z-z'|^2 \qquad (4.1.2)$$

whatever $z, z' \in \Omega$ and $\zeta \in (\mathbb{R}T'_{\mathcal{X}}|_z) \cup (\mathbb{R}T'_{\mathcal{X}}|_{z'})$.
Let $\Delta(z,\zeta)$ be the Jacobian determinant of the map $\zeta \to \zeta + i\langle\zeta\rangle z$ (where $\zeta \in \mathcal{C}_1, z \in \mathbb{C}^m$), that is,

$$\Delta(z,\zeta) = \det\{I + i(z \odot \zeta)/\langle\zeta\rangle\},$$

where $z \odot \zeta$ denotes the matrix $(z_i\zeta_j)_{1\leq i,j\leq m}$.

Definition 4.1.4. Let u be a compactly supported distribution in the manifold

$\mathcal{X}$. For $(z,\zeta) \in \mathbb{C}^m \times \mathcal{C}_1$, define the FBI transform of u as a duality bracket

$$\mathcal{F}u(z,\zeta) = \int_{\mathcal{X}} e^{i\zeta.(z-z')-\langle\zeta\rangle[z-z']^2} u(z')\Delta(z-z',\zeta)dz'.$$

Proposition 4.1.5. $\mathcal{F}u(z,\zeta) \in \mathcal{O}(\mathbb{C}^m \times \mathcal{C}_1)$.

Proposition 4.1.6 ([35],Theorem IX.2.1)**.** *Le $\mathcal{X}$ be a maximally real submanifold of $\mathbb{C}^m$ that is well-positioned at 0. There exist a neighborhood Ω of 0 in $\mathcal{X}$ with the following property. For all $u \in \mathcal{E}'(\mathcal{X})$ there exist an integer $k > 0$ and a number $C > 0$ such that*

$$|\mathcal{F}u(z,\zeta)| \leq C(1+|\zeta|)^k \; \forall (z,\zeta) \in \mathbb{R}T'_X|_\Omega.$$

Definition 4.1.7. Define, for any $\epsilon > 0$ and $z \in \mathbb{C}^m$,

$$\begin{aligned} u^\epsilon(z) &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{-\epsilon\langle\zeta\rangle^2} \mathcal{F}u(z,\zeta)d\zeta \\ &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \int_{\mathcal{X}} e^{i\zeta.(z-z')-\langle\zeta\rangle[z-z']^2-\epsilon\langle\zeta\rangle^2} u(z')\Delta(z-z',\zeta)dz'd\zeta \end{aligned}$$

(Of course, since $\zeta \in \mathbb{R}^m$, we have $\langle\zeta\rangle = |\zeta|$). Observe that for each fixed $\epsilon > 0, u^\epsilon(z) \in \mathcal{O}(\mathbb{C}^m)$.

Theorem 4.1.8. *(FBI Inversion Formula) Let $\mathcal{X} \subset \mathbb{C}^m$ be maximally real submanifold, $0 \in \mathcal{X}$ and $\mathcal{X}$ is well-positioned at the origin. There is a neighborhood Ω of 0 in $\mathcal{X}$ such that whatever $u \in \mathcal{E}'(\Omega)$,*

$$u(z) = \lim_{\epsilon \to 0^+} u^\epsilon(z) \text{ in } D'(\Omega).$$

Remark 4.1.9. Suppose $\mathcal{X} \subset \mathbb{C}^m$ is a maximally real submanifold, and $\mathcal{X}$ is well-positioned at the origin. Using the property that $|\Im\zeta| < \kappa|\Re\zeta|$ we can, for each $z, z' \in \Omega$, deform the domain of $\zeta-$ integration in the integrand at the right in Definition 4.1.7 from $\mathbb{R}^m$ to $\mathbb{R}T'_{\mathcal{X}}|_{z'}$ within the cone $\mathcal{C}_k$. We conclude that the integration with respect to (z',ζ) in that same integral can be carried out over

$\mathbb{R}T'_{\mathcal{X}}$.

The following theorem characterizes smoothness of a distribution near the origin in terms of the rapid decay of its FBI transform. We give a different proof for the necessity part.

Theorem 4.1.10 ([35], Theorem IX.4.1)**.** *Let $\mathcal{X}$ be a maximally real submaniofold of $\mathbb{C}^m$ passing through, and well positioned at, the origin, and Ω be an open neighborhood of 0 in $\mathcal{X}$ that is sufficiently small. Then $u \in \mathcal{E}'(\Omega)$ is C^∞ in some open neighborhood of the origin in Ω if and only if there is a compact neighborhood K of the origin in Ω such that the following is true:*
for every integer $N \geq 0$ there is a constant $C_N \geq 0$ such that, for all $(z,\zeta) \in \mathbb{R}T'_{\mathcal{X}}|_K$,

$$|\mathcal{F}u(z,\zeta)| \leq C_N(1+|\zeta|)^{-N}$$

.

Proof. Necessity: We can replace u by any other distribution equal to u in an open neighborhood of 0 in Ω and whose support is as small as we wish. Thus, if we assume that u is C^∞ in some open neighborhood of 0 and we may as well take $u \in C_c^\infty(\Omega)$. Integration by parts shows that

$$\begin{aligned}&(1+\langle\zeta\rangle^2)^N \mathcal{F}u(z,\zeta) = \\ &\int_{\mathcal{X}} e^{i(z-z')\cdot\zeta}(1+\Delta_{M'})^N\Big(e^{-\langle\zeta\rangle[z-z']^2}u(z')\Delta(z-z',\zeta)\Big)dz'.\end{aligned}$$

where $\Delta'_M = M_1'^2 + \ldots + M_m'^2$ and M'_k is the vector field on $\mathcal{X}$ denoted by M_k in equation (4.1.1) but now acting in the variables z'.
Now $(1+\Delta_{M'})^N = 1 + \sum_{|\alpha|\leq 2N} C_\alpha (M')^\alpha$. Let $|\alpha| \leq 2N$. Then

$$\begin{aligned}&M'^\alpha\Big(e^{-\langle\zeta\rangle[z-z']^2}u(z')\Delta(z-z',\zeta)\Big) = \\ &\qquad\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}M'^\beta\Big(e^{-\langle\zeta\rangle[z-z']^2}\Big)M'^{\alpha-\beta}\Big(u(z')\Delta(z-z',\zeta)\Big)\end{aligned}$$

Now $M'^{\beta}\left(e^{-\langle\zeta\rangle[z-z']^2}\right) = M'^{\beta_m}_m \ldots M'^{\beta_1}_1\left(e^{-\langle\zeta\rangle[z-z']^2}\right)$

Consider $M'^{\beta_1}_1\left(e^{-\langle\zeta\rangle[z-z']^2}\right)$; to compute this we will use Faá di Bruno formula, which is chain rule for higher order differentiation. If $F = F(r)$ and $g = g(t)$ are functions of one variable, then

$$\left(\frac{d}{dt}\right)^n F(g(t)) = \sum_{n_1+2n_2+3n_3+\ldots+nn_n=n} F^{(n_1+n_2+\ldots+n_n)}(g(t))\prod_{j=1}^{n}\left(\frac{g^{(j)}(t)}{j!}\right)^{n_j}.$$

Applying this by setting $F(r) = e^r, g(t) = -\langle\zeta\rangle[Z(x) - Z(t)]^2$ we claim that for each positive integer k

$$M'^{k}_1\left(e^{-\langle\zeta\rangle[z-z']^2}\right) = e^{-\langle\zeta\rangle[z-z']^2}\sum_{n_1+2n_2=k}\frac{k!}{n_1!n_2!}2^{n_1}(-1)^{n_2}\langle\zeta\rangle^{n_1+n_2}\left(z_1 - z_1'\right)^{n_1}$$

It is true for $k = 1$ since $M_1'\left\{e^{-\langle\zeta\rangle[z-z']^2}\right\} = 2\langle\zeta\rangle\left(z_1 - z_1'\right)e^{-\langle\zeta\rangle[z_1-z_1']^2}$. Assume it is true for k and we will show that it is true for $k + 1$.

Now

$$\begin{aligned}
M'^{k+1}_1\left(e^{-\langle\zeta\rangle[z-z']^2}\right) &= M_1'M'^{k}_1\left(e^{-\langle\zeta\rangle[z-z']^2}\right)\\
&= M_1'\left[\sum_{n_1+2n_2=k}\frac{k!}{n_1!n_2!}2^{n_1}(-1)^{n_2}\langle\zeta\rangle^{n_1+n_2}\left(z_1 - z_1'\right)^{n_1}\right]e^{-\langle\zeta\rangle[z-z']^2}\\
&= e^{\langle\zeta\rangle[z-z']^2}\left[\sum_{n_1+2n_2=k}\frac{k!}{n_1!n_2!}2^{n_1+1}(-1)^{n_2}\langle\zeta\rangle^{n_1+n_2+1}\left(z_1 - z_1'\right)^{n_1+1}\right.\\
&\left.+\sum_{n_1+2n_2=k}\frac{k!n_1}{n_1!n_2!}2^{n_1}(-1)^{n_2+1}\langle\zeta\rangle^{n_1+n_2}\left(z_1 - z_1'\right)^{n_1-1}\right]\\
&= e^{\langle\zeta\rangle[z-z']^2}\left[\Sigma_1 + \Sigma_2\right]
\end{aligned}$$

Our goal is to show that

$$\Sigma_1 + \Sigma_2 = \sum_{m_1+2m_2=k+1}\frac{(k+1)!}{m_1!m_2!}2^{m_1}(-1)^{m_2}\langle\zeta\rangle^{m_1+m_2}\left(z_1 - z_1'\right)^{m_1}.$$

Consider Σ_1: Let $m_1 = n_1 + 1$ and $m_2 = n_2$. Then $n_1 + 2n_2 = k$ implies that $m_1 + 2m_2 = k + 1$ and $\frac{k!}{n_1!n_2!} = \frac{k!m_1}{m_1!m_2!}$. Thus

$$\Sigma_1 = \sum_{m_1+2m_2=k+1} \frac{k!m_1}{m_1!m_2!} 2^{m_1}(-1)^{m_2}\langle\zeta\rangle^{m_1+m_2}\left(z_1 - z_1'\right)^{m_1} \tag{4.1.3}$$

Consider Σ_2: Let $m_1 = n_1 - 1$ and $m_2 = n_2 + 1$. Then $n_1 + 2n_2 = k$ implies that $m_1 + 2m_2 = k + 1$ and $\frac{k!n_1}{n_1!n_2!}2^{n_1} = \frac{k!2m_2}{m_1!m_2!}2^{m_1}$. Thus

$$\Sigma_2 = \sum_{m_1+2m_2=k+1} \frac{2m_2k!}{m_1!m_2!} 2^{m_1}(-1)^{m_2}\langle\zeta\rangle^{m_1+m_2}\left(z_1 - z_1'\right)^{m_1} \tag{4.1.4}$$

Adding (4.1.3) and (4.1.4), we get

$$\begin{aligned}
\Sigma_1 + \Sigma_2 &= \sum_{m_1+2m_2=k+1} \frac{k!}{m_1!m_2!}(m_1 + 2m_2)2^{m_1}(-1)^{m_2}\langle\zeta\rangle^{m_1+m_2}\left(z_1 - z_1'\right)^{m_1} \\
&= \sum_{m_1+2m_2=k+1} \frac{(k+1)!}{m_1!m_2!} 2^{m_1}(-1)^{m_2}\langle\zeta\rangle^{m_1+m_2}\left(z_1 - z_1'\right)^{m_1}
\end{aligned}$$

and hence the claim is proved.

Thus

$$\begin{aligned}
M'^{\beta}\left(e^{-\langle\zeta\rangle[z-z']^2}\right) &= M_m'^{\beta_m} \dots M_1'^{\beta_1}\left(e^{-\langle\zeta\rangle[z-z']^2}\right) \\
&= \prod_{j=1}^{m}\Bigg[\sum_{n_1^j+2n_2^j=\beta_j} \frac{\beta_j!}{n_1^j!n_2^j!}2^{n_1^j}(-1)^{n_2^j}\langle\zeta\rangle^{n_1^j+n_2^j}\left(z_1 - z_1'\right)^{n_1^j}\Bigg]e^{-\langle\zeta\rangle[z-z']^2}
\end{aligned}$$

which is a sum of terms with constant coefficients where each terms of the form

$$\langle\zeta\rangle^{n_1^1+\dots+n_1^m+n_2^1+\dots n_2^m}\left(z_1 - z_1'\right)^{n_1^1}\dots\left(z_m - z_m'\right)^{n_1^m} e^{-\langle\zeta\rangle[z-z']^2}$$

and hence each term is dominated by a constant times

$$|\zeta|^{n_1^1+\dots+n_1^m+n_2^1+\dots n_2^m}|z - z'|^{n_1^1+\dots n_1^m}e^{-|\zeta||z-z'|^2}.$$

Here $n_1^j + 2n_2^j = \beta_j$. Now

$$|\zeta|^{n_1^1+\ldots n_1^m+n_2^1+\ldots n_2^m}|z-z'|^{n_1^1+\ldots n_1^m}e^{-|\zeta||z-z'|^2} =$$
$$|\zeta|^{\frac{n_1^1+\ldots n_1^m}{2}+n_2^1+\ldots n_2^m}\Big(|\zeta||z-z'|^2\Big)^{\frac{n_1^1+\ldots n_1^m}{2}+n_2^1+\ldots n_2^m}e^{-|\zeta||z-z'|^2}$$

Thus using the fact that for $d>0$ there exist $C_1>0$ such that $t^d e^{-t} \leq C\ \forall t \geq 0$, we have

$$\left|M'^{\beta}\Big(e^{-\langle\zeta\rangle[z-z']^2}\Big)\right| \leq C_1|\zeta|^{\frac{n_1^1+\ldots n_1^m}{2}+n_2^1+\ldots n_2^m} = C_1|\zeta|^{\frac{\beta_1+\ldots+\beta_m}{2}} = C_1|\zeta|^{\frac{|\beta|}{2}}$$

Moreover,

$$|M'^{\alpha-\beta}\Big(u(z')\Delta(z-z',\zeta)\Big)| \leq C_2.$$

Let $|\alpha| \leq 2N$. Then

$$\begin{aligned}\left|M'^{\alpha}\Big(e^{-\langle\zeta\rangle[z-z']^2}u(z')\Delta(z-z',\zeta)\Big)\right| &\leq D\sum_{\beta\leq\alpha}|\zeta|^{\frac{|\beta|}{2}}\\ &\leq D|\zeta|^{\frac{|\alpha|}{2}} \text{ for } |\zeta|\geq 1\\ &\leq D_N|\zeta|^N \text{ for } |\zeta|\geq 1\end{aligned}$$

Therefore, for $|\zeta| \geq 1$,

$$\begin{aligned}|(1+\langle\zeta\rangle)^2)^N||\mathcal{F}u(z,\zeta)| &= |\int_{\mathcal{X}} e^{i(z-z').\zeta}(1+\Delta_{M'})^N\Big(e^{-\langle\zeta\rangle[z-z']^2}u(z')\Delta(z-z',\zeta)\Big)dz'|\\ &= |\int_{\mathcal{X}} e^{i\zeta.(z-z')-\langle\zeta\rangle[z-z']^2}\\ &\qquad\Big(1+\sum_{|\alpha|\leq 2N}C_\alpha(M')^\alpha\Big)\Big(e^{-\langle\zeta\rangle[z-z']^2}u(z')\Delta(z-z',\zeta)\Big)dz'|\\ &\leq C_3(1+|\zeta|)^k + D_N|\zeta|^N\int_{\mathcal{X}} e^{\Re(i\zeta.(z-z')-\langle\zeta\rangle[z-z']^2)}dz' \text{ by Lemma 4.1.11}\\ &\leq C_N(1+|\zeta|)^{k+N}\end{aligned}$$

which implies that

$$|\mathcal{F}u(z,\zeta)| \leq \frac{C_N}{(1+|\zeta|)^{N-k}}$$

by virtue of 4.1.2 (1). By continuity $\mathcal{F}u(z,\zeta)$ is bounded for $|\zeta| \leq 1$ and hence we derive at once the result.

Sufficiency: We may assume that *supp* $u \subset\subset \Omega$ (Ω is the neighborhood of 0 as in Definition 4.1.8). Let N be any integer $\geq 2m$ and let $(\frac{\partial}{\partial z})^{\alpha}, |\alpha| \leq N - 2m$, act under the integral sign, then the hypothesis and the well positioned condition gives that the absolute value of the differentiated integrand does not exceed $const.(1+|\zeta|)^{-2m}$. Then the restriction to $\mathcal{X}$ of $\frac{\partial^\alpha}{\partial z^\alpha} u^\epsilon$ converges as $\epsilon \to 0^+$ in some open neighborhood of 0 independent of α, whence the sought conclusion, by Theorem 4.1.8. □

4.1.3 Characterization of Microlocal Smoothness

As before, let $U \subset \mathbb{R}^m$ be a neighborhood of 0, $Z_j(x) \in C^\infty(U)$ with $dZ_1, \ldots, dZ_m$ linearly independent on U so that $\mathcal{X} = Z(U) \subset \mathbb{C}^m$ is m dimensional maximally real submanifold.

Let $Z_\#$ be an almost-analytic extensions of the map Z, that is,

$$Z_\# : U + i\mathbb{R}^m \to \mathbb{C}^m$$

is a C^∞ map such that

i. $Z_\#(x) = Z(x)$ for every $x \in U$

ii. $\frac{\partial Z^l_\#}{\partial \bar{z}_j}(x+iy) = O(|y|^k), \ \forall j, \forall k, 1 \leq l \leq m$

Recalling that u is C^∞ on $\mathcal{X}$ means $u(Z(x))$ is C^∞ on U, the following proposition characterizes smoothness locally in terms of almost analytic extensions..

Proposition 4.1.11. *Let $u : \mathcal{X} \to \mathbb{C}$. Then u is C^∞ near $0 \in \mathcal{X}$ if and only if there exist F defined near 0 in $\mathbb{C}^m$ such that $F|_{\mathcal{X}} = u$ near 0 and F is almost analytic near 0.*

Proof. Suppose u is C^∞ near $0 \in \mathcal{X}$. Then $\tilde{u}(x) = u(Z(x))$ is C^∞ near 0 in U.

Then there exist $\tilde{F}$ almost analytic on $V+iW$ $(0\in V, 0\in W)$ open in $\mathbb{C}^m$ such that $\tilde{F}(x,0)=\tilde{u}(x)$ on V. There exist a neighborhood U_1+iV_1 $(U_1\subset V, V_1\subset W)$ of 0 in $\mathbb{C}^m$ such that

$$Z_\#: U_1+iV_1 \to \mathbb{C}^m$$

is a C^∞ diffeomorphism.
Let $F: Z_\#(U_1+iV_1)\to\mathbb{C}$ be $F(z)=\tilde{F}(Z_\#^{-1}(z))$. Thus $F(Z_\#(x+iy))=\tilde{F}(x+iy)$ on U_1+iV_1. Clearly $\tilde{F}|_{U_1}=u$.
Claim: $\frac{\partial F}{\partial \overline{z}_j}(z)=O\Big((dist(z,Z(U_1)))^k\Big)$, $\forall k, z\in Z_\#(U_1+iV_1)$.
Now since $Z_\#: U_1+iV_1\to Z_\#(U_1+iV_1)$ is diffeomorphism, there are constants $C_1, C_2>0$ such that

$$C_1|y|\le dist\Big(Z_\#(x+iy), Z(U_1)\Big)\le C_2|y|.$$

Using the fact that $\tilde{F}$ and $Z_\#$ are almost analytic extensions and applying the chain rule we have

$$\begin{aligned}
O(|y|^k) &= \frac{\partial}{\partial \overline{z}_j}\tilde{F}(x+iy)\\
&= \frac{\partial}{\partial \overline{z}_j}F\circ Z_\#(x+iy)\\
&= \sum_{l=1}^{m}\frac{\partial F}{\partial w_l}\Big(Z_\#(x+iy)\Big)\frac{\partial Z_\#^l}{\partial \overline{z}_j}(x+iy)+\sum_{l=1}^{m}\frac{\partial F}{\partial \overline{w}_l}\Big(Z_\#(x+iy)\Big)\frac{\partial \overline{Z_\#^l}}{\partial \overline{z}_j}(x+iy)
\end{aligned}$$

Thus $\forall k$ there is $C_k>0$ such that for all j

$$\Big|\sum_{l=1}^{m}\frac{\partial F}{\partial \overline{w}_l}\Big(Z_\#(x+iy)\Big)\frac{\partial \overline{Z_\#^l}}{\partial \overline{z}_j}(x+iy)\Big|\le C_k|y|^k.$$

Now let

$$\begin{pmatrix} \frac{\partial \overline{Z_\#^1}}{\partial \overline{z}_1} & \frac{\partial \overline{Z_\#^2}}{\partial \overline{z}_1} & \cdots & \frac{\partial \overline{Z_\#^m}}{\partial \overline{z}_1}\\ \vdots & & & \\ \frac{\partial \overline{Z_\#^1}}{\partial \overline{z}_m} & \frac{\partial \overline{Z_\#^2}}{\partial \overline{z}_m} & \cdots & \frac{\partial \overline{Z_\#^m}}{\partial \overline{z}_m}\end{pmatrix}\begin{pmatrix}\frac{\partial F}{\partial \overline{w}_1}\\ \vdots \\ \frac{\partial F}{\partial \overline{w}_m}\end{pmatrix}_{|Z_\#(x+iy)} = \begin{pmatrix} h_1(x+iy)\\ \vdots \\ h_m(x+iy)\end{pmatrix} = h(x+iy)$$

where
$$h_j(x+iy) = \sum_{l=1}^{m} \frac{\partial F}{\partial \overline{w}_l}(Z_{\#}(x+iy))\frac{\partial \overline{Z^l_{\#}}}{\partial \overline{z}_j}(x+iy).$$
We will show that the matrix
$$A = \left(\frac{\partial \overline{Z^l_{\#}}}{\partial \overline{z}_j}(x+iy)\right)_{1\le l,j\le m}$$
is invertible near 0. Now recalling that $Z(x) = x + i\phi(x), \phi(0) = 0, d\phi(0) = 0$, and $\frac{\partial}{\partial \overline{z}_j} Z^l_{\#}(x+iy) = O(|y|^k)$ we have
$$\frac{\partial}{\partial \overline{z}_j} Z^l_{\#}(x) = \frac{1}{2}\left(\frac{\partial}{\partial x_j} Z^l_{\#}(x) + i\frac{\partial}{\partial y_j} Z^l_{\#}(x)\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_j} Z^l(x) + i\frac{\partial}{\partial y_j} Z^l_{\#}(x)\right) = 0,$$
which implies that
$$\delta^l_j + i\frac{\partial}{\partial y_j} Z^l_{\#}(0) = 0.$$
Thus
$$\frac{\partial}{\partial y_j} Z^l_{\#}(0) = i\delta^l_j \tag{4.1.5}$$
Again
$$\begin{aligned}\frac{\partial}{\partial \overline{z}_j}\overline{Z^l_{\#}}|_{x=0} &= \frac{1}{2}\left(\frac{\partial}{x_j}\overline{Z^l_{\#}}(x) + i\frac{\partial}{y_j}\overline{Z^l_{\#}}(x)\right)|_{x=0}\\ &= \frac{1}{2}\left(\delta^l_j + i\frac{\partial}{y_j}\overline{Z^l_{\#}}(0)\right)\\ &= \frac{1}{2}\left(\delta^l_j + \delta^l_j\right) \text{ from (4.1.5)}\\ &= \delta^l_j\end{aligned}$$
Thus $A(0) = Id$ and hence after contracting $U_1 + iV_1$, we may assume that $A(x+iy)$ is invertible for all $x+iy$ in $U_1 + iV_1$.

Therefore,

$$\begin{pmatrix} \frac{\partial F}{\partial \overline{w}_1} \\ \vdots \\ \frac{\partial F}{\partial \overline{w}_m} \end{pmatrix} = A^{-1}(x+iy) \begin{pmatrix} h_1(x+iy) \\ \vdots \\ h_m(x+iy) \end{pmatrix}$$

Then since $A^{-1}(x+iy)$ is bounded, we have $\forall j$,

$$\frac{\partial F}{\partial \overline{w}_j}(Z_{\#}(x+iy)) = O(|y|^k).$$

Fix k. There is $C'_k > 0$ such that for all j,

$$|\frac{\partial F}{\partial \overline{w}_j}(Z_{\#}(x+iy))| \leq C'_k |y|^k$$

which implies that

$$|\frac{\partial F}{\partial \overline{w}_j}(Z_{\#}(x+iy))| \leq C_k \bigg(d(Z_{\#}(x+iy), Z(U_1)) \bigg)^k.$$

Let $z = Z_{\#}(x+iy)$. Thus we have shown that F is almost analytic on $Z_{\#}(U_1+iV_1)$. Conversely, let $0 \in U_1, 0 \in V_1$ with U_1+iV_1 open in $\mathbb{C}^m$ such that $F : U_1+iV_1 \to \mathbb{C}$ is almost analytic and $F|_{U_1} = u$. Let $\tilde{u}(x) = u(Z(x))$. Then

$$\tilde{u}(x) = u(Z(x)) = F(Z(x)) = (F \circ Z)(x) = (F \circ Z_{\#})(x) := \tilde{F}(x).$$

Clearly $\tilde{F}(x) = F \circ Z_{\#}(x) = F \circ Z(x) = u(x)$. Moreover, using the same procedure as we did in one direction above we have

$$\begin{aligned} \frac{\partial}{\partial \overline{z}_j} \tilde{F}(x+iy) &= \frac{\partial}{\partial \overline{z}_j}(F \circ Z_{\#}(x+iy)) \\ &= \sum_{l=1}^{m} \frac{\partial F}{\partial w_l}\bigg(Z_{\#}(x+iy) \bigg) \frac{\partial Z^l_{\#}}{\partial \overline{z}_j}(x+iy) + \sum_{l=1}^{m} \frac{\partial F}{\partial \overline{w}_l}\bigg(Z_{\#}(x+iy) \bigg) \frac{\partial \overline{Z^l_{\#}}}{\partial \overline{z}_j}(x+iy) \\ &= O(|y|^k) \end{aligned}$$

Thus, $\tilde{F}(x+iy) = F \circ Z_{\#}(x+iy)$ is almost analytic.

Therefore, $\tilde{u}(x) = \tilde{F}(x)$ and $\tilde{F}$ is almost analytic, which implies that $\tilde{u}_{|U_1} = u$ is C^∞. $\square$

Let Γ be a cone in $\mathbb{R}^m \setminus \{0\}$, A be any open subset of U, and $\mathcal{O}$ any open subset in $U + i\mathbb{R}^m$ which contains A. We shall use the following notations:

$$\mathcal{N}_{\mathcal{O}}(A, \Gamma) = \{Z_\#(x + iv) \in \mathbb{C}^m : x \in A, v \in \Gamma \cup \{0\}, x + iv \in \mathcal{O}\},$$

where $Z_\#$ is a given almost analytic extension of Z. Of course $\mathcal{N}_{\mathcal{O}}(A, \Gamma)$ depends on the choice of $Z_\#$. We shall refer sets $\mathcal{N}_{\mathcal{O}}(A, \Gamma)$ as conoids. Conic neighborhood of $(0, \xi^0)$ in $\mathbb{R}T'_{\mathcal{X}}$ is a set of the form

$$\{(z, \zeta) : \zeta = {}^t Z_x(x)^{-1}\xi, z = Z(x) \text{ for } x \in W, \xi \in \Gamma\},$$

where W is a neighborhood of 0 and Γ is a cone, $\xi^0 \in \Gamma$.

Remark 4.1.12. [[4], Lemma 1.3] Let $0 \in V$. A set $Z_\#(V + i\Gamma_\delta)$ contains

$$\{Z(x) + iZ_x(x).v : x \in V_1, v \in \Gamma^1_{\delta_1}\}$$

for some $0 \in V_1$ open subset of V, $\Gamma^1 \subset\subset \Gamma$ and $0 < \delta_1 < \delta$.
Conversely, for $0 \in V_1$ a set

$$\{Z(x) + iZ_x(x).v : x \in V_1, v \in \Gamma^1_{\delta_1}\}$$

contains $Z_\#(V_2 + i\Gamma^2_{\delta_2})$ where $0 \in V_2 \subset V_1, \Gamma^2 \subset\subset \Gamma^1$, $0 < \delta_2 < \delta_1$.

Let $u : \mathcal{X} = Z(U) \to \mathbb{C}$ be a distribution of compact support. For $z = Z(x), z' = Z(x')$ define the FBI transform as

$$\mathcal{F}u(z, \zeta) = \int e^{i\zeta.(z-z') - \langle\zeta\rangle[z-z']^2} u(z')\Delta(z - z', \zeta)dz'.$$

We state and prove the microlocal version of Theorem 4.1.10 as follows.

Theorem 4.1.13. *Let $u \in \mathcal{E}'(\mathcal{X})$, and $\xi^0 \in \mathbb{R}^m \setminus \{0\}$. Then*

$$|\mathcal{F}u(z,\zeta)| \le C_k(1+|\zeta|)^{-k}\ \forall k$$

for (z,ζ) in a conic neighborhood of $(0,\xi^0)$ in $\mathbb{R}T'_{\mathcal{X}}$ if and only if there is a neighborhood V of 0, open acute cones $\Gamma^1,\dots,\Gamma^N$ in $\mathbb{R}^m \setminus \{0\}$ and almost analytic functions f_j on $Z_\#(V+i\Gamma^j_\delta)$ $(\delta>0)$ of tempered growth such that $u=\sum_{j=1}^N bf_j$ and $\xi^0.\Gamma^j<0$, $\forall j$.

Proof. Suppose $u = bf$ on $Z(V)$, where f is almost analytic function on $Z_\#(V + i\Gamma_\delta)$ $(\delta > 0)$ of tempered growth with $\xi^0.\Gamma < 0$. (If $u = \sum_{j=1}^N bf_j$, then the result holds for each bf_j and using linearity of the FBI transform, we get our result.) We claim that

$$|\mathcal{F}u(z,\zeta)| \le C_k(1+|\zeta|)^{-k}\ \forall k$$

for (z,ζ) in a conic neighborhood of $(0,\xi^0)$, that is, on a set of the form $\{(z,\zeta) : z = Z(x), \zeta = {}^t Z_x(x)^{-1}\Gamma, x \in W\}$; W a neighborhood of 0. By Remark 4.1.12, without loss of generality we will work on the conoid

$$\{Z(x) + iZ_x(x).v\ : x \in V, v \in \Gamma_\delta\}.$$

Let $r > 0$ such that $B_{2r} = \{x : |x| < 2r\} \subset\subset V$, $\Omega = Z(B_{2r})$. Let $g \in C_0^\infty(\mathcal{X}), g \equiv 1$ in $Z(B_r)$ and $supp(g) \subset \Omega$. Let $v \in \Gamma_\delta$ and $u \in \mathcal{E}'(\mathcal{X})$. Then

$$\begin{aligned}\mathcal{F}u(z,\zeta) &= \int_{\mathcal{X}} e^{i(z-z')-\langle\zeta\rangle[z-z']^2}u(z')\Delta(z-z',\zeta)dz' \\ &= \lim_{\Gamma\ni v\to 0}\int_{\Omega} e^{i(z-z')-\langle\zeta\rangle[z-z']^2} f(Z(x')+iZ_{x'}(x')v)g(Z(x'))\Delta(z-z',\zeta)dz'\end{aligned}$$

Since $g(Z(x)) \in C^\infty$, it has almost analytic extension $\tilde{g}$ on $Z_\#(V + i\Gamma_\delta)$. Then

$$\mathcal{F}u(Z(x),\zeta) =$$
$$\lim_{\Gamma\ni v\to 0}\int_{B_{2r}} e^{i\zeta.(Z(x)-\tilde{Z}(x'))-\langle\zeta\rangle[Z(x)-\tilde{Z}(x')]^2} f(\tilde{Z}(x'))\tilde{g}(\tilde{Z}(x'))\Delta(Z(x)-\tilde{Z}(x'),\zeta)d\tilde{Z}(x')$$

where $\tilde{Z}(x', v) = Z(x') + iZ_{x'}(x')v$

Let $v_0 \in \Gamma_\delta$. Let

$$D = \{Z(x') + iZ_{x'}(x')v_0 t : x' \in B_{2r}, 0 \le t \le 1\}.$$

Let

$$\begin{aligned} Q(x, x', \zeta, t) &= i\zeta.(Z(x) - Z(x') - itZ_{x'}(x')v_0) - \langle\zeta\rangle[Z(x) - Z(x') - itZ_{x'}(x')v_0]^2 \\ &= i\zeta.(Z(x) - Z(x')) - \langle\zeta\rangle[Z(x) - Z(x')]^2 + tZ_{x'}(x')v_0.\zeta \\ &- \langle\zeta\rangle\Big(- 2itZ_{x'}(x')v_0.(Z(x) - Z(x')) - (tZ_{x'}(x')v_0).(tZ_{x'}(x')v_0)\Big) \end{aligned}$$

Let $w(\tilde{z}) = e^{Q(x,\tilde{z},\zeta)}\phi(\tilde{z})f(\tilde{z})d\tilde{z}_1 \wedge \ldots \wedge d\tilde{z}_m$ $\tilde{z} = z' + iZ_{x'}(x')y$. Then By Stokes theorem we have

$$\begin{aligned} &\mathcal{F}u(Z(x), \zeta) \\ &= \int_{B_{2r}} e^{Q_0} f(Z(x') + iZ_{x'}(x')v_0)\tilde{g}(Z(x') + iZ_{x'}(x')v_0) \\ &\qquad \Delta(Z(x) - Z(x') - iZ_{x'}(x')v_0)dZ(x') \\ &+ \sum_{j=1}^{m} \int_D e^{Q_1} \frac{\partial}{\partial \overline{\tilde{z}_j}}(\tilde{g}f)(Z(x') + iZ_{x'}(x')v_0 t)\Delta(Z(x') - Z(x') - iZ_{x'}(x')v_0 t)d\overline{\tilde{z}_j} \wedge d\tilde{z} \end{aligned}$$

where

$$Q_0(x, x', \zeta) = i\zeta.(Z(x) - Z(x') - iZ_{x'}(x')v_0) - \langle\zeta\rangle[Z(x) - Z(x') - iZ_{x'}(x')v_0]^2$$

and

$$Q_1(x, x', \zeta, t) = i\zeta.(Z(x) - Z(x') - iZ_{x'}(x')v_0 t) - \langle\zeta\rangle[Z(x) - Z(x') - iZ_{x'}(x')v_0 t]^2$$

Now since $v_0 \in \Gamma$ and $\xi^0.\Gamma < 0$ there exist a conic neighborhood Γ_1 of ξ^0 and a constant $c > 0$ such that

$$\xi.v_0 \leq -c|\xi||v_0| \;\forall \xi \in \Gamma_1.$$

Since $\mathcal{X}$ is well positioned at the origin, we have $\forall x, x' \in B_{2r}$, $\xi \in \Gamma$

$$\Re\bigg(i\zeta.(Z(x) - Z(x')) - \langle\zeta\rangle[Z(x) - Z(x')]^2\bigg) \leq -(1-\kappa)|\zeta||Z(x) - Z(x')|^2.$$

$$\begin{aligned}
|(tZ_{x'}(x')v_0).\zeta| &= |(tZ_{x'}(x')v_0).({}^tZ_{x'}^{-1}(x')\xi)| \\
&= |tZ_x(x)Z_{x'}(x')v_0.\xi| \\
&\leq t(1 + c_1|x - x'||v_0||\xi)| \text{ since } Z_x(x)Z_{x'}(x') = Id + O(|x - x'|) \\
&\leq tc_1'|v_0||\xi)| \;, c_1' = 1 + 4c_1 r
\end{aligned}$$

Thus after shrinking Ω if needed so that $|\zeta| \leq 2|\xi|$, we have

$$\Re(tZ_{x'}(x')v_0.\zeta) \leq -\frac{1}{2}cc_1't|v_0||\zeta| = -\frac{1}{2}c't|v_0||\zeta|.$$

Also shrink Ω further if necessary so that $|z - z'| \leq \frac{c}{16}$ for $z, z' \in \Omega$ and $||\phi_x|| \leq \frac{1}{2}\;\forall x \in B_{2r}$, since $|\langle\zeta\rangle| \leq |\zeta|$, we have for $\delta < \frac{c}{36}$,

$$\begin{aligned}
&\Re\bigg(-\langle\zeta\rangle\bigg(-2itZ_{x'}(x')v_0.(Z(x) - Z(x')) - (tZ_{x'}(x')v_0).(tZ_{x'}(x')v_0)\bigg)\bigg) \\
&\leq |\zeta|\bigg(3t|v_0||Z(x) - Z(x')| + \frac{9}{4}|v_0|^2t^2\bigg) \\
&\leq |\zeta||v_0|t(3|Z(x) - Z(x')| + \frac{9}{4}|v_0|) \\
&\leq |\zeta||v_0|t(\frac{3c}{16} + \frac{9}{4}\frac{c}{36}) \text{ since } v_0 \in \Gamma_\delta, \delta < \frac{c}{36} \\
&= \frac{1}{4}tc|v_0||\zeta| \\
&\leq \frac{1}{4}tc_1'c|v_0||\zeta| = \frac{1}{4}c't|v_0||\zeta|
\end{aligned}$$

Therefore, $\Re Q(x, x', \zeta, t) \leq -(1-\kappa)|\zeta||Z(x) - Z(x')|^2 - tc''|v_0||\zeta|,\; c'' = \frac{c'}{2}$. From this estimate and using the almost holomorphy of f and $\tilde{g}$, and since f is of tempered growth, we have

$$\begin{aligned}&\left|\int_D e^{Q_1}\frac{\partial}{\partial\overline{\tilde{z}_j}}(\tilde{g}f)\Delta d\overline{\tilde{z}_j}\wedge d\tilde{z}_1\wedge\ldots\wedge d\tilde{z}_j\right|\\&\leq B_k|v_0t|^k e^{-c''t|v_0||\zeta|}\\&\leq C_k|\zeta|^{-k}\end{aligned}$$

Again using the inequality $v_0.\xi \leq -c|v_0||\xi|$, we also have that

$$\begin{aligned}&\left|\int_{B_{2r}} e^{Q_0}(f\tilde{g})(Z(x')+iZ_{x'}(x')v_0)\Delta(Z(x)-Z(x')-iZ_{x'}(x')v_0,\zeta)dZ(x')\right|\\&\leq \sup_{x\in\overline{B_{2r}}}|(f\tilde{g})(Z(x')+iZ_{x'}(x')v_0)|e^{-c''|v_0||\zeta|}\\&\qquad\int_{B_{2r}} e^{-(1-\kappa)|\zeta||Z(x)-Z(x')|^2}|\Delta(Z(x)-Z(x')-iZ_{x'}(x')v_0)|dZ(x')\\&\leq de^{-c''|v_0||\zeta|}\\&\leq d'e^{-a|\zeta|},\ a>0\end{aligned}$$

Therefore,

$$|\mathcal{F}u(z,\zeta)| \leq C_k(1+|\zeta|)^{-k}\ \forall k$$

for (z,ζ) in a conic neighborhood of $(0,\xi^0)$.

For the converse let $u\in\mathcal{E}'(\mathcal{X})$. Suppose that

$$|\mathcal{F}u(z,\zeta)| \leq C_k(1+|\zeta|)^{-k}\ \forall k$$

for (z,ζ) in a conic neighborhood of $(0,\xi^0)$. Applying the inversion formula we have

$$u(z)=\frac{1}{(2\pi)^m}\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^m}\mathcal{F}u(z,\zeta)e^{-\epsilon|\zeta|^2}d\zeta.$$

Let $\mathcal{C}_j$, $1\leq j\leq N$ be open, acute cones such that

$$\mathbb{R}^m=\cup_{j=1}^N\overline{\mathcal{C}_j}$$

and $\overline{\mathcal{C}_j} \cap \overline{\mathcal{C}_k}$ has measure zero when $j \neq k$. We may assume that $\xi^0 \in \mathcal{C}_1$ and $\xi^0 \notin \overline{\mathcal{C}_j}$ for $j \geq 2$. This implies that we can get acute, open cones Γ^j $, 2 \leq j \leq N$ and a constant $C > 0$ such that

$$\xi^0.\Gamma^j < 0 \text{ and } y.\xi \geq C|y||\xi| \; \forall y \in \Gamma^j, \; \forall \xi \in \mathcal{C}_j.$$

For each $j = 2, \ldots, N, \; z = Z_{\#}(x+iy) \in Z_{\#}(V + i\Gamma^j_\delta)$ define

$$f_j^\epsilon(z) = \frac{1}{(2\pi)^m} \int_{\mathcal{C}_j} e^{-\epsilon|\zeta|^2} \mathcal{F}u(z,\zeta) d\zeta,$$

then f_j^ϵ is entire and as $\epsilon \to 0^+$, the f_j^ϵ converges uniformly on compact subsets of $Z_{\#}(\mathbb{R}^m + i\Gamma^j)$ to a holomorphic function $f_j(z)$. Assuming without loss of generality that $Z_{\#}(V + i\Gamma^j_\delta) = \{Z(x) + iZ_x(x)y : x \in V, y \in \Gamma^j_\delta\}$, we have

$$\begin{aligned}
&|f_j(Z(x) + iZ_x(x)y)| \\
&\qquad = |\frac{1}{(2\pi)^m} \int_{\mathcal{C}_j} \mathcal{F}u(Z(x) + iZ_x(x)y, \zeta) d\zeta| \\
&\qquad = |\frac{1}{(2\pi)^m} \int_\Omega \int_{\mathcal{C}_j} e^{i\zeta.(Z(x)+iZ_x(x)y - Z(x')) - \langle\zeta\rangle[Z(x)+iZ_x(x)y-Z(x')]^2} \\
&\qquad\qquad u(Z(x'))\Delta(Z(x) + iZ_x(x)y - Z(x'), \zeta) dz' d\zeta| \\
&\qquad \leq C \int_\Omega \int_{\mathcal{C}_j} e^{-c''|y||\zeta|}(1+|\zeta|)^l dZ(x') d\zeta \\
&\qquad \leq \frac{c}{|y|^k}
\end{aligned}$$

Thus f_j is of tempered growth on $Z_{\#}(V + i\Gamma^j_\delta)$ and hence has a boundary value $bf_j \in D'(Z(V))$. Let

$$g_1^\epsilon(Z(x)) = \frac{1}{(2\pi)^m} \int_{\mathcal{C}_1} e^{-\epsilon|\zeta|^2} \mathcal{F}u(Z(x), \zeta) d\zeta.$$

Then by hypothesis we may assume that $\mathcal{F}_u(z,\zeta)$ decays rapidly in a conic neighborhood $\{(z,\xi) : z = Z(x), \zeta = \; {}^tZ_x(x)^{-1}\mathcal{C}_1\}$, and g_1^ϵ converges uniformly as $\epsilon \to 0^+$, to a function $g_1(Z(x))$. It is then easy to see that g_1 is C^∞ on Ω. Thus

g_1 has an almost analytic extension f_1 such that $f_{1|Z(V)} = g_1$ and

$$\frac{\partial f_1}{\partial \bar{z}_j} = O\Big((dist(z, Z(V)))^k \Big) \ , z \in Z_{\#}(V + i\Gamma^1_\delta)$$

Thus $u = \sum_{j=1}^{N} bf_j$ near 0 as asserted. □

4.2 Characterization of Microlocal Ultradifferentiablity on Maximally Real Submanifolds

4.2.1 FBI Transform in Maximally Real $\mathcal{E}^M$ Submanifolds

Following the notations and definitions of section 4.1.1, let $(\mathcal{M}, \mathcal{V})$ be $\mathcal{E}^M$ involutive structure and $\mathcal{X} \subset \mathcal{M}$ is maximally real $\mathcal{E}^M$ submanifold so that $\mathbb{R}T'_{\mathcal{X}}$ is a real vector bundle over $\mathcal{X}$. Then in a small enough neighborhood Ω of 0 in $\mathcal{X}$, Ω is the image of some open neighborhood U of 0 in $\mathbb{R}^m$ under the map $x \mapsto Z(x)$ with $Z(x) = x + i\phi(x)$; where $\phi : U \to \mathbb{R}^m, \phi(0) = 0$ and $d\phi(0) = 0$. Then a point $(z, \zeta) \in \mathbb{R}T'_{\mathcal{X}}$, with $z \in Z(U)$, if there is $x \in U$ and $\xi \in \mathbb{R}^m$ such that

$$z = Z(x) \text{ and } \zeta = \ {}^t Z_x(x)^{-1}\xi.$$

Let

$$\mathcal{M}_k = \sum_{j=1}^{m} \mu_{kj} \frac{\partial}{\partial x_j}, 1 \leq k \leq m \qquad (4.2.1)$$

be the vector fields characterized by the relations

$$\mathcal{M}_k(z_j|_{\mathcal{X}}) = \delta_{kj}.$$

Then the vector fields $\mathcal{M}_1, \ldots, \mathcal{M}_m$ form a basis of $\mathbb{C}T\mathcal{X}$.
Let U be an open neighborhood of 0 in $\mathbb{R}^m$ and $Z : U \to \mathbb{C}^m$ with $Z(x) = x + i\phi(x)$, where $\phi : U \to \mathbb{R}^m$ is a $\mathcal{E}^M$ map , $\phi(0) = 0, d\phi(0) = 0$.
Then $\mathcal{X} = Z(U)$ is a $\mathcal{E}^M$ maximally real submanifold of $\mathbb{C}^m$ and it is very well

positioned at 0, that is, given any number $\kappa, 0 \leq \kappa < 1$, there is an open neighborhood Ω of 0 in $\mathcal{X}$ such that the following holds:

$$(1)\ \ |\Im\zeta| < \kappa|\Re\zeta|, (2)\ \ \Im\Big[\zeta.(z-z') + i\langle\zeta\rangle[z-z']^2\Big] \geq (1-\kappa)|\zeta||z-z'|^2 \quad (4.2.2)$$

whatever $z, z' \in \Omega$ and $\zeta \in (\mathbb{R}T'_{\mathcal{X}}|_z) \cup (\mathbb{R}T'_{\mathcal{X}}|_{z'})$.

Definition 4.2.1. Let u be a compactly supported distribution in the manifold $\mathcal{X}$. For $(z,\zeta) \in \mathbb{C}^m \times \mathcal{C}_1$, we define the FBI transform of u as a duality bracket

$$\mathcal{F}_u(z,\zeta) = \int_{\mathcal{X}} e^{i\zeta.(z-z') - \langle\zeta\rangle[z-z']^2} u(z')\Delta(z-z',\zeta)dz'.$$

Note that $\mathcal{F}_u(z,\zeta) \in \mathcal{O}(\mathbb{C}^m \times \mathcal{C}_1)$.

Definition 4.2.2. Let Ω be an open neighborhood of 0 in $\mathcal{X}$ and let K be a compact subset of 0 in Ω. Define, for any $\epsilon > 0$, the modified inversion formula as

$$u_K^{\epsilon}(z) = (2\pi^3)^{-m/2} \int\int e^{i\zeta.(z-z') - \langle\zeta\rangle[z-z']^2 - \epsilon\langle\zeta\rangle^2} \mathcal{F}_u(z',\zeta)\langle\zeta\rangle^{\frac{m}{2}} dz' d\zeta \qquad (4.2.3)$$

where the integration with respect to (z',ζ) is carried out over $\mathbb{R}T'_{\mathcal{X}}|_K$. Property (1) of condition (4.2.2) ensures that $u_K^{\epsilon}(z)$ is an entire holomorphic function in $\mathbb{C}^m$.

In the following lemma $\Omega \subset \mathbb{R}^{2m} = \mathbb{R}^m_x \times \mathbb{R}^m_{x'}$ will denote an open subset containing the origin, $x = (x_1, \ldots, x_m) \in \mathbb{R}^m_x$, $x' = (x'_1, \ldots, x'_m) \in \mathbb{R}^m_{x'}$.

Lemma 4.2.3. *Let $\Omega \subset \mathbb{R}^{2m} = \mathbb{R}^m_x \times \mathbb{R}^m_{x'}$ be an open set. Let $y \in \mathbb{R}^m$ be fixed and close to the origin. Let $\zeta \in \mathcal{C}_1$ and $f(x,x') = (f_1(x,x'), \ldots, f_m(x,x')) \in \mathcal{E}^M(\Omega)$ with $f(0,0) = 0$. Set*

$$Q(x,x') = -i\zeta.f(x,x') - \langle\zeta\rangle[y - f(x,x')]^2.$$

There exist $C > 0$ (independent of x, x', y, ζ, and α) such that for all multi-index

$\alpha \in \mathbb{N}_0^m$, *for all* $L > 0$ *and all* $(x, x') \in \Omega$

$$\left|\partial_x^\alpha \left(e^{Q(x,x')}\right)\right| \le e^{\Re Q(x,x')} C^{|\alpha|+1} M_{|\alpha|} e^{\frac{H}{L}} e^{\frac{1}{2}M(|\zeta|)}.$$

Proof. Since $f(x, x') \in \mathcal{E}^M(\Omega)$, there is a constant $C > 0$ such that for all $(x, x') \in \Omega$ (possibly after shrinking Ω) and all $\alpha \in \mathbb{N}_0^m$,

$$|\partial_x^\alpha f(x)| \le C^{|\alpha|+1} M_{|\alpha|}.$$

Applying Faá di Bruno formula

$$\partial_x^\alpha e^{Q(x,x')} = \sum_{r=1}^{|\alpha|} e^{Q(x,x')} \sum_{p(\alpha,r)} \alpha! \prod_{j=1}^{|\alpha|} \frac{(\partial_x^{\alpha_j} Q)^{k_j}}{k_j!(\alpha_j!)^{k_j}},$$

where

$p(\alpha, r) = \{(k_1, \ldots, k_{|\alpha|}; \alpha_1, \ldots, \alpha_{|\alpha|})$ for some $1 \le s \le |\alpha|, k_i = 0$ and $\alpha_i = 0$ for $1 \le i \le |\alpha| - s; k_i > 0$ for $|\alpha| - s + 1 \le i \le |\alpha|;$ and $0 \prec \alpha_{|\alpha|-s+1} \prec \ldots \prec \alpha_{|\alpha|}$ are such that

$$\sum_{i=1}^{|\alpha|} k_i = r, \ \sum_{i=1}^{|\alpha|} k_i \alpha_i = \alpha\}.$$

Thus increasing the constant C, since $f(x) \in \mathcal{E}^M$ and $|\langle\zeta\rangle| \le |\zeta|$,

$$\left|\partial_x^\alpha \left(e^{Q(x,x')}\right)\right| \le \sum_{r=1}^{|\alpha|} e^{\Re Q(x,x')} \sum_{p(\alpha,r)} |\alpha|! \prod_{j=1}^{|\alpha|} \frac{\left(|\zeta| C^{|\alpha_j|+1} M_{|\alpha_j|}\right)^{k_j}}{k_j!(\alpha_j!)^{k_j}}.$$

Using the inequality $(a + b)! \le 2^{a+b} a!b!$, we have

$$\prod_{j=1}^{|\alpha|} \left(\frac{|\zeta| C^{|\alpha_j|+1}}{\alpha_j!}\right)^{k_j} \le \frac{|\zeta|^r C^{|\alpha|+r} 4^{|\alpha|}}{|\alpha|!}.$$

Also

$$\prod_{j=1}^{|\alpha|} M_{|\alpha_j|}^{k_j} \leq \prod_{j=1}^{|\alpha|} C^{k_j|\alpha_j|} M_{k_j|\alpha_j|-k_j} \text{ by } (P7)$$
$$\leq C^{|\alpha|} M_{|\alpha|-r} \text{ by } (P5'')$$

and from Lemma 1.1.10 we have that

$$|\zeta|^r M_{|\alpha|-r} \leq \sqrt{A}(\frac{H}{L})^r M_{|\alpha|} e^{\frac{1}{2}M(L|\zeta|)}.$$

Therefore,

$$\begin{aligned}
|\partial_x^\alpha e^{Q(x,x')}| &\leq \sum_{r=1}^{|\alpha|} \sum_{p(\alpha,r)} |\alpha|! \prod_{j=1}^{|\alpha|} \frac{\left(|\zeta| C^{|\alpha_j|+1} M_{|\alpha_j|}\right)^{k_j}}{k_j!(\alpha_j!)^{k_j}} \\
&\leq \sum_{r=1}^{|\alpha|} \sum_{p(\alpha,r)} |\zeta|^r C^{2|\alpha|+r} 4^{|\alpha|} M_{|\alpha|-r} \prod_{j=1}^{|\alpha|} \frac{1}{k_j!} \\
&\leq e^{\frac{1}{2}M(L|\zeta|)} M_{|\alpha|} \sum_{r=1}^{|\alpha|} \sum_{p(\alpha,r)} (\frac{H}{L})^r C^{|\alpha|+r} \prod_{j=1}^{|\alpha|} \frac{1}{k_j!} \\
&\leq e^{\frac{1}{2}M(L|\zeta|)} M_{|\alpha|} C^{|\alpha|} \sum_{r=1}^{|\alpha|} \sum_{p(\alpha,r)} (\frac{H}{L})^r \prod_{j=1}^{|\alpha|} \frac{1}{k_j!} \\
&= e^{\frac{1}{2}M(L|\zeta|)} M_{|\alpha|} C^{|\alpha|} \sum_{r=1}^{|\alpha|} r! \sum_{p(\alpha,r)} \frac{(\frac{H}{L})^r}{r!} \prod_{j=1}^{|\alpha|} \frac{1}{k_j!} \\
&\leq e^{\frac{1}{2}M(L|\zeta|)} M_{|\alpha|} e^{\frac{H}{L}} C^{|\alpha|} \sum_{r=1}^{|\alpha|} r! \sum_{p(\alpha,r)} \prod_{j=1}^{|\alpha|} \frac{1}{k_j!} \\
&\leq e^{\frac{1}{2}M(L|\zeta|)} M_{|\alpha|} e^{\frac{H}{L}} C^{|\alpha|} C^{|\alpha|} \\
&\leq e^{\frac{1}{2}M(L|\zeta|)} M_{|\alpha|} e^{\frac{H}{L}} C^{|\alpha|+1}
\end{aligned}$$

where the last inequality follows by definition of $p(\alpha, r)$ (see 2.2.1) □

Recall that u is $\mathcal{E}^M$ on $\mathcal{X}$ means $u(Z(x))$ is $\mathcal{E}^M$ on U.

Theorem 4.2.4. *Let Ω be as in Lemma 4.2.3 and $u \in \mathcal{E}^{M'}(\mathcal{X})$ (a distribution of compact support on $\mathcal{X}$). Then for each $K \subset\subset \Omega$ and for every $L > 0$, there exist $C_L > 0$ such that*

$$|\mathcal{F}u(z,\zeta)| \leq C_L e^{M(L|\zeta|)},\ z \in Z(K), \zeta \in \mathcal{C}_1.$$

Proof. Let $u \in \mathcal{E}^{M'}(\mathcal{X})$. Recall that $z = Z(x)$ and $z' = Z(x')$. Then by definition

$$\mathcal{F}u(Z(x),\zeta) = \langle u(Z(x')), \varphi(x,x',\zeta)\rangle,$$

where

$$\varphi(x,x',\zeta) = e^{i\zeta.(Z(x)-Z(x'))-\langle\zeta\rangle[Z(x)-Z(x')]^2}.$$

Then from Definition 1.1.8 , for any $\epsilon_1 > 0$ and suitable $C_1 > 0$ depending on ϵ_1 we have

$$\begin{aligned}
|\mathcal{F}u(Z(x),\zeta)| &= |\langle u(Z(x')), \varphi(x,x',\zeta)\rangle| \\
&\leq C_1 \sup_{\alpha\in\mathbb{N}_0^m} \left\{ \frac{\epsilon_1^{|\alpha|}}{M_{|\alpha|}} \sup_{x'\in K} |\partial_{x'}^\alpha \varphi(x,x',\zeta)| \right\} \\
&= C_1 \sup_{\alpha\in\mathbb{N}_0^m} \left\{ \frac{\epsilon_1^{|\alpha|}}{M_{|\alpha|}} \sup_{x'\in K} \left|\partial_{x'}^\alpha \left(e^{i\zeta.(Z(x)-Z(x'))-\langle\zeta\rangle[Z(x)-Z(x')]^2} \right)\right| \right\}
\end{aligned}$$

By Lemma 4.2.3, for each K compact there exist $C > 0$ such that for $\alpha \in \mathbb{N}_0^m$, $L > 0$

$$\left|\partial_{x'}^\alpha \left(e^{i\zeta.(Z(x)-Z(x'))-\langle\zeta\rangle[Z(x)-Z(x')]^2} \right)\right| \leq e^{\frac{1}{2}M(L|\zeta|)} e^{\frac{H}{L}} C^{|\alpha|+1} M_{|\alpha|} \text{ on } K.$$

Then

$$\sup_{x'\in K}\left|\partial^{\alpha}_{x'}\left(e^{i\zeta.(Z(x)-Z(x'))-\langle\zeta\rangle[Z(x)-Z(x')]^2}\right)\right| \le e^{\frac{1}{2}M(L|\zeta|)}e^{\frac{H}{L}}C^{|\alpha|+1}M_{|\alpha|}$$
$$\le e^{M(L|\zeta|)}e^{\frac{H}{L}}C^{|\alpha|+1}M_{|\alpha|}$$

Thus

$$\begin{aligned}|\mathcal{F}u(z,\zeta)| &\le C_1 \sup_{\alpha\in\mathbb{N}_0^m}\left\{\frac{\epsilon_1^{|\alpha|}}{M_{|\alpha|}}C^{|\alpha|+1}e^{M(L|\zeta|)}M_{|\alpha|}e^{\frac{H}{L}}\right\}\\ &= C_1Ce^{\frac{H}{L}}e^{M(L|\zeta|)}\sup_{\alpha\in\mathbb{N}_0^m}\left\{(C\epsilon_1)^{|\alpha|}\right\}\\ &\le C_Le^{M(L|\zeta|)}, \;\text{ where we choose } \epsilon_1 < \frac{1}{C}.\end{aligned}$$

□

4.2.2 Characterization of Microlocal Ultradifferentiablity on Maximally Real Submanifolds

As before, let $U \subset \mathbb{R}^m$ be a neighborhood of 0, $Z_j(x) \in \mathcal{E}^M(U)$ with $dZ_1, \ldots, dZ_m$ linearly independent on U so that $\mathcal{X} = Z(U) \subset \mathbb{C}^m$ is m dimensional maximally real $\mathcal{E}^M$ submanifold.

We deal with almost-analytic extensions of the map Z. Such an extension is a $\mathcal{E}^M$ mapping $Z_\# : U + i(-1,1)^m \to \mathbb{C}^m$ such that

i. $Z_\#(x) = Z(x)$ for every $x \in U$

ii. there exist $C > 0$ such that $\frac{\partial Z^l_\#}{\partial \bar{z}_j}(x+iy) \le C^{N+1}\frac{M_N}{N!}|y|^N$, $\forall j, \forall N, 1 \le l \le m$

Recall also that conic neighborhood of $(0,\xi^0)$ in $\mathbb{R}T'_{\mathcal{X}}$ is a set of the form

$$\{(z,\zeta) : \zeta = {}^tZ_x(x)^{-1}\xi, z = Z(x) \text{ for } x \in W, \xi \in \Gamma\},$$

where W is a neighborhood of 0 and Γ is a cone, $\xi^0 \in \Gamma$.

Let $u : \mathcal{X} = Z(U) \to \mathbb{C}$ be a distribution of compact support. For $z = Z(x), z' = Z(x')$ define the FBI transform as

$$\mathcal{F}_u(z,\zeta) = \int e^{i\zeta.(z-z')-\langle\zeta\rangle[z-z']^2} u(z')\Delta(z-z',\zeta)dz'.$$

We now state our main result as follows.

Theorem 4.2.5. *Let $u \in \mathcal{E}^{M'}(\mathcal{X})$, $\xi^0 \in \mathbb{R}^m \setminus \{0\}$. Then there are constants $a, b > 0$ such that*

$$|\mathcal{F}_u(z,\zeta)| \leq ae^{-M(b|\zeta|)}$$

for (z,ζ) in a conic neighborhood of $(0,\xi^0)$ in $\mathbb{R}T'_{\mathcal{X}}$ if and only if there is a neighborhood V of 0, open acute cones $\Gamma^1,\ldots,\Gamma^n$ in $\mathbb{R}^m \setminus \{0\}$ and $\mathcal{E}^M$ functions f_j on $Z_{\#}(V+i\Gamma^j_\delta)$ $(0<\delta<1)$ that increases M^*- exponentially such that
i. $u = \sum_{j=1}^{N} bf_j$ *near* 0,
ii. $\xi^0.\Gamma^j < 0$, $\forall j$,
iii. $\left|\frac{\partial f_j}{\partial \bar{z}_k}(z)\right| \leq C^{N+1}\frac{M_N}{N!}\left(dist(Z_{\#}(x+iy), Z(V))\right)^N$, $\forall j = 1,\ldots,n$, $\forall k = 1,\ldots,m.$

Proof. Without loss of generality assume there exist $f \in \mathcal{E}^M(Z_{\#}(V+i\Gamma_\delta)$ $(\delta > 0)$ that increases M^*- exponentially such that
i. $u = bf$ near 0,
ii. $\xi^0.\Gamma < 0$,
iii. $\left|\frac{\partial f}{\partial \bar{z}_k}(z)\right| \leq C^{N+1}\frac{M_N}{N!}\left(dist(Z_{\#}(x+iy), Z(V))\right)^N$, , $\forall k = 1,\ldots,m.$
By Remark 4.1.12 we will work as well on the conoid

$$\{Z(x) + iZ_x(x)v : x \in V, v \in \Gamma_\delta\}.$$

Let $r > 0$ such that $B_{2r} = \{x : |x| < 2r\} \subset\subset V$, $\Omega_r = Z(B_{2r})$. Let $g \in D_0^M(\mathcal{X})$,

$g \equiv 1$ in $Z(B_r)$ and $supp(g) \subset \Omega$. Let $v \in \Gamma_\delta$ and $u \in \mathcal{E}'^M(\mathcal{X})$. Then

$$\begin{aligned}
&\mathcal{F}_u(z,\zeta)\\
&\qquad = \int_{\mathcal{X}} e^{i\zeta.(z-z')-\langle\zeta\rangle[z-z']^2} u(z')\Delta(z-z',\zeta)dz'\\
&\qquad = \lim_{t\to 0^+}\int_{B_{2r}} e^{i\zeta.(Z(x)-Z(x'))-\langle\zeta\rangle[Z(x)-Z(x')]^2}\\
&\qquad\qquad\qquad f(Z(x')+iZ_{x'}(x')tv)g(Z(x'))\Delta(Z(x)-Z(x'),\zeta)dZ(x')
\end{aligned}$$

Since $g(Z(x)) \in \mathcal{E}^M$, it has almost analytic extension $\tilde{g}$ on $Z_\#(V+i\Gamma_\delta), \delta < 1$. Then

$$\begin{aligned}
&\mathcal{F}_u(Z(x),\zeta) =\\
&\lim_{t\to 0^+}\int_{B_{2r}} e^{i\zeta.(Z(x)-\tilde{Z}(x'))-\langle\zeta\rangle[Z(x)-\tilde{Z}(x')]^2} f(\tilde{Z}(x'))\tilde{g}(\tilde{Z}(x'))\Delta(Z(x)-\tilde{Z}(x'),\zeta)d\tilde{Z}(x')
\end{aligned}$$

where $\tilde{Z}(x',v,t) = Z(x') + iZ_{x'}(x')tv$.

For $0 < \lambda < 1$, let

$$D_\lambda = \{Z(x') + iZ_{x'}(x')tv \in \mathbb{C}^m : x' \in B_{2r}, \lambda \le t \le 1\}.$$

Let $Q(x,x',\zeta) = i\zeta.(Z(x)-Z(x')) - \langle\zeta\rangle[Z(x)-Z(x')]^2$.

Consider the $m-$ form

$$w(\tilde{z}) = e^{Q(x,\tilde{z},\zeta)}\tilde{g}(\tilde{z})f(\tilde{z})d\tilde{z}_1 \wedge \ldots \wedge d\tilde{z}_m \ , \tilde{z} = z' + iZ_{x'}(x')y.$$

Then by Stokes theorem we have,

$$\mathcal{F}_u(Z(x),\zeta) =$$

$$\lim_{\lambda\to 0^+}\int_{B_{2r}} e^{Q_0} f(Z(x')+iZ_{x'}(x')\lambda v)\tilde{g}(Z(x')+iZ_{x'}(x')\lambda v)\Delta(Z(x)-Z(x')-iZ_{x'}(x')\lambda v,\zeta)dZ(x')$$

$$= \int_{B_{2r}} e^{Q_1} f(Z(x')+iZ_{x'}(x')v)\tilde{g}(Z(x')+iZ_{x'}(x')v)\Delta(Z(x)-Z(x')-iZ_{x'}(x')v,\zeta)dZ(x')$$

$$+ \lim_{\lambda\to 0^+}\sum_{j=1}^{m}\int_{D_\lambda} e^{Q_2}\tilde{g}(Z(x')+iZ_{x'}(x')tv)\frac{\partial f}{\partial \overline{\tilde{z}_j}}(Z(x')+iZ_{x'}(x')tv)$$

$$\Delta(Z(x)-Z(x')-iZ_{x'}(x')tv,\zeta)d\overline{\tilde{z}_j}\wedge d\tilde{z}$$

$$+ \lim_{\lambda\to 0^+}\sum_{j=1}^{m}\int_{D_\lambda} e^{Q_2} f(Z(x')+iZ_{x'}(x')tv)\frac{\partial \tilde{g}}{\partial \overline{\tilde{z}_j}}(Z(x')+iZ_{x'}(x')tv)$$

$$\Delta(Z(x)-Z(x')-iZ_{x'}(x')tv,\zeta)d\overline{\tilde{z}_j}\wedge d\tilde{z}$$

$$= I_0 + I_1^\lambda + I_2^\lambda$$

where

$$Q_0(x,x',\zeta) = i\zeta.(Z(x)-Z(x')-iZ_{x'}(x')\lambda v) - \langle\zeta\rangle[Z(x)-Z(x')-iZ_{x'}(x')\lambda v]^2,$$

$$Q_1(x,x',\zeta) = i\zeta.(Z(x)-Z(x')-iZ_{x'}(x')v) - \langle\zeta\rangle[Z(x)-Z(x')-iZ_{x'}(x')v]^2$$

and

$$Q_2(x,x',\zeta,t) = i\zeta.(Z(x)-Z(x')-iZ_{x'}(x')tv) - \langle\zeta\rangle[Z(x)-Z(x')-iZ_{x'}(x')tv]^2.$$

Since $v\in\Gamma$ and $\xi^0.\Gamma<0$, there is a conic neighborhood Γ_1 of ξ^0 and a constant $c>0$ such that

$$\xi.v \le -c|v||\xi| \;\forall \xi\in\Gamma_1.$$

Now

$$
\begin{aligned}
Q_2(x,x',\zeta,t) &= i\zeta.(Z(x)-Z(x')-itZ_{x'}(x')v) - \langle\zeta\rangle[Z(x)-Z(x')-itZ_{x'}(x')v]^2 \\
&= i\zeta.(Z(x)-Z(x')) - \langle\zeta\rangle[Z(x)-Z(x')]^2 + tZ_{x'}(x')v.\zeta \\
&- \langle\zeta\rangle\bigg(-2itZ_{x'}(x')v.(Z(x)-Z(x')) - (tZ_{x'}(x')v).(tZ_{x'}(x')v)\bigg)
\end{aligned}
$$

Since $\mathcal{X}$ is well positioned at the origin, we have $\forall x, x' \in B_{2r}$, $\xi \in \Gamma$

$$
\Re\bigg(i\zeta.(Z(x)-Z(x')) - \langle\zeta\rangle[Z(x)-Z(x')]^2\bigg) \leq -(1-\kappa)|\zeta||Z(x)-Z(x')|^2.
$$

Also

$$
\begin{aligned}
|(tZ_{x'}(x')v).\zeta| &= |(tZ_{x'}(x')v).({}^tZ_{x'}^{-1}(x')\xi)| \\
&= |tZ_x(x)Z_{x'}^{-1}(x')v.\xi| \\
&\leq t(1+c_1|x-x'||v||\xi|) \text{ since } Z_x(x)Z_{x'}^{-1}(x') = Id + O(|x-x'|) \\
&\leq tc_1'|v||\xi| \text{ , } c_1' = 1+4c_1r
\end{aligned}
$$

Thus after shrinking Ω if needed so that $|\zeta| \leq 2|\xi|$, we have

$$
\Re(tZ_{x'}(x')v.\zeta) \leq -\frac{1}{2}cc_1't|v||\zeta| = -\frac{1}{2}c't|v||\zeta|.
$$

Also shrink Ω further if necessary so that $|z-z'| \leq \frac{c}{16}$ for $z, z' \in \Omega$ and $||\phi_x|| \leq \frac{1}{2}$ $\forall x \in B_{2r}$, since $|\langle\zeta\rangle| \leq |\zeta|$, we have for $\delta < \frac{c}{36}$,

$$\begin{aligned}
&\Re\Big(-\langle\zeta\rangle\Big(-2itZ_{x'}(x')v.(Z(x)-Z(x')) - (tZ_{x'}(x')v).(tZ_{x'}(x')v)\Big)\Big) \\
&\leq |\zeta|\Big(3t|v||Z(x)-Z(x')| + \frac{9}{4}|v|^2t^2\Big) \\
&\leq |\zeta||v|t(3|Z(x)-Z(x')| + \frac{9}{4}|v|) \\
&\leq |\zeta||v|t(\frac{3c}{16} + \frac{9}{4}\,\frac{c}{36}) \text{ since } v\in\Gamma_\delta, \delta < \frac{c}{36} \\
&= \frac{1}{4}tc|v||\zeta| \\
&\leq \frac{1}{4}tc_1'c|v||\zeta| = \frac{1}{4}c't|v||\zeta|
\end{aligned}$$

Therefore, $\Re Q_2(x,x',\zeta,t) \leq -(1-\kappa)|\zeta||Z(x)-Z(x')|^2 - tc''|v||\zeta|,\ c'' = \frac{c'}{2}$. Thus we get a similar estimate for $\Re Q_1$.

Consider I_0:

Now $\Re Q_1 \leq -(1-\kappa)|\zeta||Z(x)-Z(x')|^2 - c''|v||\zeta|$. Choosing $|v|$ small, for $\xi\in\Gamma_1$, $\zeta =^t Z_{x'}^{-1}(x')\xi$, $|\zeta|\geq 1$ we have

$$\begin{aligned}
|I_0| &\leq e^{-c''|v||\zeta|} \sup_{x'\in\overline{B_{2r}}} |(f\tilde{g})(Z(x')+iZ_{x'}(x')v)| \int_{B_{2r}} |\Delta| e^{-(1-\kappa)|\zeta||Z(x)-Z(x')|^2} dZ(x') \\
&\leq de^{-c''|v||\zeta|} \int_{B_{2r}} |\Delta| e^{-(1-\kappa)|\zeta||Z(x)-Z(x')|^2} dZ(x') \\
&\leq d'e^{-b|\zeta|} , b>0 \\
&\leq d'e^{-M(b|\zeta|)} \text{ since } M(t)\leq t
\end{aligned}$$

Since $\frac{I_0}{e^{-M(c'|\zeta|)}}$ is bounded on $\overline{B_{2r}}\times\{\zeta : |\zeta|\leq 1\}$, there exist $a_0, b_0 > 0$ such that

$$|I_0| \leq a_0 e^{-M(b_0|\zeta|)} \text{ for } |x'| < 2r, \zeta =^t Z_{x'}^{-1}(x')\xi\ , \xi\in\Gamma_1 \qquad (4.2.4)$$

Consider

$$I_1^\lambda = \lim_{\lambda \to 0^+} \sum_{j=1}^{m} \int_{D_\lambda} e^{Q_2} \tilde{g}(Z(x') + iZ_{x'}(x')tv) \frac{\partial f}{\partial \overline{\tilde{z}}_j}(Z(x') + iZ_{x'}(x')tv)$$
$$\Delta(Z(x) - Z(x') - iZ_{x'}(x')tv, \zeta) d\overline{\tilde{z}_j} \wedge d\tilde{z} :$$

For $\zeta \in \{{}^t Z_{x'}^{-1}(x')\Gamma_1\}, |\zeta| \geq 1,$, with $C' = \sup_{(x',t) \in \overline{B_{2r}} \times [0,t]} |\tilde{g}(Z(x') + iZ_{x'}(x')tv)|$, we have

$$\begin{aligned}
&\left| e^{Q_2} \tilde{g}(Z(x') + iZ_{x'}(x')tv) \frac{\partial f}{\partial \overline{\tilde{z}}_j}(Z(x') + iZ_{x'}(x')tv) \right| \\
&\leq C' e^{\Re Q_2} A e^{M^*(Bt|v|)} \\
&\leq A' e^{-(1-\kappa)|\zeta||z-z'|^2} e^{-bt|\zeta|} e^{-M^*(Bt|v|)} \text{ (taking } v \text{ small)} \\
&= A' e^{-(1-\kappa)|\zeta||z-z'|^2} e^{-[M^*(Bt|v|) + Bt|v| \frac{b}{B}|\zeta|]} \\
&\leq A' e^{-\inf_s M*(s) + s\frac{b}{B}|\zeta|} e^{-(1-\kappa)|\zeta||z-z'|^2} \\
&\leq A' e^{-M(B'|\zeta|)} e^{-(1-\kappa)|\zeta||z-z'|^2} \text{ since } M(t) \leq \inf_s \{M^*(s) + st\} \ \forall t \geq 0
\end{aligned}$$

Thus

$$\begin{aligned}
&\lim_{\lambda \to 0^+} |I_1^\lambda| \\
&\leq A' e^{-M(B'|\zeta|)} \int_0^1 \int_{B_{2r}} |\Delta| e^{-(1-\kappa)|\zeta||z-z'|^2} d\overline{\tilde{z}_j} \wedge d\tilde{z} \text{ for } x' \in B_{2r}, \zeta \in \{{}^t Z_{x'}^{-1}(x')\Gamma_1\}, |\zeta| \geq 1
\end{aligned}$$

Therefore,

$$\lim_{\lambda \to 0^+} |I_1^\lambda| \leq a_1 e^{-M(b_1|\zeta|)} \text{ for } x' \in B_{2r}, \zeta \in \{{}^t Z_{x'}^{-1}(x')\Gamma_1\}, |\zeta| \geq 1$$

for some $a_1, b_1 > 0$ independent of λ. But since $\frac{|I_1^\lambda|}{e^{-M(b_1|\zeta|)}}$ is bounded on $\overline{B_{2r}} \times \{\zeta : |\zeta| \leq 1\}$, there exist $A_1, B_1 > 0$ such that

$$\lim_{\lambda \to 0^+} |I_1^\lambda| \leq A_1 e^{-M(b_1|\zeta|)} \tag{4.2.5}$$

Consider

$$I_2^\lambda = \lim_{\lambda \to 0^+} \sum_{j=1}^{m} \int_{D_\lambda} e^{Q_2} f(Z(x') + iZ_{x'}(x')tv) \frac{\partial \tilde{g}}{\partial \bar{\tilde{z}}_j}(Z(x') + iZ_{x'}(x')tv)$$
$$\Delta(Z(x) - Z(x') - iZ_{x'}(x')tv, \zeta) d\overline{\tilde{z}_j} \wedge d\tilde{z} :$$

Since $\frac{\partial \tilde{g}}{\partial \bar{\tilde{z}}_j} \equiv 0$ for $|x'| \le r$, the integral over $|x'| \le r$ is zero. Then for $|x| < \frac{r}{2}$ and $|x'| \ge r$ and taking $|v|$ small we have

$$\Re Q_2 \le -(1-\kappa)|\zeta||Z(x) - Z(x')|^2 - c''|\zeta|$$

for $\zeta \in \{{}^t Z_{x'}^{-1}(x')\Gamma_1\}, |\zeta| \ge 1$. Since f increases M^*- exponentially, for all $\varrho > 0$ there exist a constant $d > 0$ such that

$$|f(Z(x') + iZ_{x'}(x')tv)| \le d e^{\varrho M^* \frac{t|v|}{\varrho}}.$$

Also since $\tilde{g}$ is almost holomorphic, there exist $C > 0$ such that

$$|\frac{\partial \tilde{g}}{\partial \bar{\tilde{z}}_j}| \le C^{N+1} \frac{M_N}{N!} |tv|^N \ \forall j = 1, \ldots, m, \forall N = 1, 2 \ldots.$$

Equivalently,

$$|\frac{\partial \tilde{g}}{\partial \bar{\tilde{z}}_j}| \le C e^{-M^*(Ct|v|)}.$$

Taking $C > 1$ and $\varrho = \frac{1}{C}$, we obtain

$$|f(Z(x') + iZ_{x'}(x')tv)\frac{\partial \tilde{g}}{\partial \bar{\tilde{z}}_j}| \le dCe^{\varrho M^*(\frac{t|v|}{\varrho}) - \frac{1}{C}M^*(Ct|v|)} \le \frac{d}{\varrho} e^{-\varrho M^*(\frac{t|v|}{\varrho}) + \varrho M^*(\frac{t|v|}{\varrho})} = C'$$

Thus we can get $A_2, B_2 > 0$ independent of t such that

$$\lim_{\lambda \to 0^+} |I_2^\lambda| \le A_2 e^{-M(B_2|\zeta|)} \ \forall \zeta \in \{{}^t Z_{x'}^{-1}(x')\Gamma_1\}, x' \in B_{\frac{r}{2}} \tag{4.2.6}$$

Combining (4.2.4), (4.2.5) and (4.2.6) we can find constants $a, b > 0$ such that

$$|\mathcal{F}_u(z,\zeta)| \le a e^{-M(b|\zeta|)}$$

for (z,ζ) in a conic neighborhood of $(0,\xi^0)$ in $\mathbb{R}T'_{\mathcal{X}}$.
For the converse let $u \in \mathcal{E}^{M'}(\Omega)$, Ω open neighborhood of 0 in $\mathcal{X}$. let B_0 be a ball centered at zero such that $Z(B_0) = \Omega$. Suppose that there exist $a, b > 0$ such that

$$|\mathcal{F}_u(z,\zeta)| \le a e^{-M(b|\zeta|)}$$

for (z,ζ) in a conic neighborhood of $(0,\xi^0)$, that is, on a set of the form $\{(z,\zeta) : z \in Z(V), \{\zeta \in^t Z_x^{-1}(x)\Gamma\}\}$, where V is a neighborhood of 0 ($V \subset B_0$) and Γ is a conic neighborhood of ξ^0. Applying the inversion formula (4.2.3) we have

$$u_{\tilde{K}}(z) = \frac{1}{(2\pi^3)^{\frac{m}{2}}} \lim_{\epsilon\to 0^+} \iint_{\mathbb{R}T'_{\mathcal{X}_{|\tilde{K}}}} e^{i\zeta.(z-z')-\langle\zeta\rangle[z-z']^2-\epsilon\langle\zeta\rangle^2} \mathcal{F}_u(z',\zeta)\langle\zeta\rangle^{\frac{m}{2}} dz' d\zeta,$$

where $\tilde{K} = Z(K)$; K is a compact neighborhood of 0 in U. Extend the map Z as a map $\tilde{Z} : \mathbb{R}^m \to \mathbb{C}^m$ and let $\tilde{Z}(\mathbb{R}^m) = \tilde{\Omega}$. Then the properties of wellposedness remains valid in $\tilde{\Omega}$. Choose $d > 0$ such that $\Omega' = \{Z(x') : |Z(x')| \le d\} \subset Z(V)$. Then we can write $u^\epsilon_{\tilde{K}}(z)$ as

$$u^\epsilon_{\tilde{K}}(z) = u^\epsilon_{\tilde{K},0}(z) + u^\epsilon_{\tilde{K},1}(z),$$

where

$$u^\epsilon_{\tilde{K},0}(z) = \frac{1}{(2\pi^3)^{\frac{m}{2}}} \int\int_{\Omega'} e^{i\zeta.(z-z')-\langle\zeta\rangle[z-z']^2-\epsilon\langle\zeta\rangle^2} \mathcal{F}_u(z',\zeta)\langle\zeta\rangle^{\frac{m}{2}} dz' d\zeta$$

and

$$u^\epsilon_{\tilde{K},1}(z) = \frac{1}{(2\pi^3)^{\frac{m}{2}}} \int\int_{\tilde{\Omega}\setminus\Omega'} e^{i\zeta.(z-z')-\langle\zeta\rangle[z-z']^2-\epsilon\langle\zeta\rangle^2} \mathcal{F}_u(z',\zeta)\langle\zeta\rangle^{\frac{m}{2}} dz' d\zeta$$

so that $u(z) = u_{\tilde{K},0}(z) + u_{\tilde{K},1}(z)$.

Consider $u_{\tilde{K},0}(z)$: Let $\mathcal{C}_j$, $1 \leq j \leq N$ be open, acute cones such that

$$\mathbb{R}^m = \cup_{j=1}^N \overline{\mathcal{C}_j}$$

and $\overline{\mathcal{C}_j} \cap \overline{\mathcal{C}_k}$ has measure zero when $j \neq k$. We may assume that $\xi^0 \in \mathcal{C}_1$ and $\xi^0 \notin \overline{\mathcal{C}_j}$ for $j \geq 2$. This implies that we can get acute, open cones Γ^j , $2 \leq j \leq N$ and a constant $C > 0$ such that

$$\xi^0.\Gamma^j < 0 \text{ and } y.\xi \geq C|y||\xi| \ \forall y \in \Gamma^j,\ \forall \xi \in \mathcal{C}_j.$$

For each $j = 2, \ldots, N,\ z = Z_\#(x + iy) \in Z_\#(V + i\Gamma^j_\delta)$ define

$$f^\epsilon_j(z)) = \frac{1}{(2\pi^3)^{\frac{m}{2}}} \int_{\xi\in\mathcal{C}_j} \int_{\Omega'} e^{i\zeta.(z-z')-\langle\zeta\rangle[z-z']^2-\epsilon\langle\zeta\rangle^2} \mathcal{F}_u(z',\zeta)\langle\zeta\rangle^{\frac{m}{2}} dz' d\zeta.$$

Then for $2 \leq j \leq N$, each f^ϵ_j is entire and as $\epsilon \to 0^+$, the f^ϵ_j converges uniformly on compact subsets of $Z_\#(\mathbb{R}^m + i\Gamma^j)$ to

$$f_j(z) = \frac{1}{(2\pi^3)^{\frac{m}{2}}} \int_{\xi\in\mathcal{C}_j} \int_{\Omega'} e^{i\zeta.(z-z')-\langle\zeta\rangle[z-z']^2} \mathcal{F}_u(z',\zeta)\langle\zeta\rangle^{\frac{m}{2}} dz' d\zeta,$$

which is also holomorphic on $Z_\#(\mathbb{R}^m + i\Gamma^j_\delta)$. Moreover, f_j increases M^*-exponentially on $\mathbb{R}^m + i\Gamma^j_\delta$ for some $0 \leq \delta < 1$. Indeed,

$$\begin{aligned}
&|f_j(Z(x) + iZ_x(x)y)| = \\
&|\frac{1}{(2\pi^3)^{\frac{m}{2}}} \int_{\xi\in\mathcal{C}_j} \int_{\Omega'} e^{i\zeta.(Z(x)+iZ_x(x)y-Z(x'))-\langle\zeta\rangle[Z(x)+iZ_x(x)y-Z(x')]^2} \mathcal{F}_u(z',\zeta)\langle\zeta\rangle^{\frac{m}{2}} dZ(x') d\zeta| \\
&\leq \frac{1}{(2\pi^3)^{\frac{m}{2}}} \int_{\xi\in\mathcal{C}_j} \int_{\Omega'} |e^{i\zeta.(Z(x)+iZ_x(x)y-Z(x'))-\langle\zeta\rangle[Z(x)+iZ_x(x)y-Z(x')]^2} \mathcal{F}_u(z(x'),\zeta)\langle\zeta\rangle^{\frac{m}{2}} ||dZ(x')||d\zeta| \\
&\leq C \int_{\xi\in\mathcal{C}_j} \int_{\Omega'} e^{-c''|y||\zeta|} e^{M(L|\zeta|)} |\zeta|^{\frac{m}{2}} e^{-(1-\kappa)|Z(x)-Z(x')|^2} |dZ(x')||d\zeta| \\
&\leq C' \int_{\xi\in\mathcal{C}_j} e^{-c''|y||\zeta|} e^{M(L|\zeta|)} |\zeta|^{\frac{m}{2}} |d\zeta|
\end{aligned}$$

Thus

$$|f_j(Z(x)+iZ_x(x)y)| \leq C' \int_{\mathcal{C}_j} e^{[M(|L\zeta|)-c|y||\zeta|}|\zeta|^{\frac{m}{2}}|d\zeta|.$$

Now since for each $\varrho > 0$ there exist $c_\varrho > 0$ such that (see Lemma 1.1.10)

$$M(L|\zeta|) = 2M(L|\zeta|) - M(L|\zeta|) \leq \varrho M(c_\varrho L|\zeta|) - M(L|\zeta|),$$

choosing $L = L_\varrho \doteq \frac{c}{c_\varrho}$, we have

$$\begin{aligned}
|f_j(Z(x)+iZ_x(x)y)| &\leq C' \int_{\mathcal{C}_j} e^{\varrho M(c_\varrho L|\zeta|)-c|y||\zeta|}e^{-M(L|\zeta|)}|\zeta|^{\frac{m}{2}}|d\zeta| \\
&= C' \int_{\mathcal{C}_j} e^{\varrho M(c_\varrho L_\varrho|\zeta|)-L_\varrho c_\varrho|y||\zeta|}e^{-M(L|\zeta|)}|\zeta|^{\frac{m}{2}}|d\zeta| \\
&= C' \int_{\mathcal{C}_j} e^{\varrho[M(c_\varrho L_\varrho|\zeta|)-\frac{L_\varrho c_\varrho}{\varrho}|y||\zeta|]}e^{-M(L|\zeta|)}|\zeta|^{\frac{m}{2}}|d\zeta| \\
&\leq C' e^{\varrho \sup_{r>0}[M(r)-r\frac{|y|}{\varrho}]} \int_{\mathcal{C}_j} e^{-M(L|\zeta|)}|\zeta|^{\frac{m}{2}}|d\zeta| \\
&= C' e^{\varrho\omega^*(\frac{|y|}{\varrho})} \int_{\mathcal{C}_j} e^{-M(L|\zeta|)}|\zeta|^{\frac{m}{2}}|d\zeta| \\
&\leq C_1 e^{\varrho M^*(\frac{|y|}{\varrho})}.
\end{aligned}$$

Then each f_j $(j \geq 2)$ has boundary value $bf_j \in D'(Z(V))$.
Let

$$g_1^\epsilon(Z(x)) = \frac{1}{(2\pi^3)^{\frac{m}{2}}} \int_{\xi\in\mathcal{C}_1}\int_{\Omega'} e^{i\zeta.(Z(x)-Z(x'))-\langle\zeta\rangle[z-z']^2-\epsilon\langle\zeta\rangle^2}\mathcal{F}_u(z',\zeta)\langle\zeta\rangle^{\frac{m}{2}}dz'd\zeta.$$

Then by the decay of the FBI transform g_1^ϵ are smooth for all $\epsilon > 0$ and converges uniformly on $\mathbb{R}^m$ to the function

$$g_1(Z(x)) = \frac{1}{(2\pi^3)^{\frac{m}{2}}} \int_{\xi\in\mathcal{C}_1}\int_{\Omega'} e^{i\zeta.(Z(x)-Z(x'))-\langle\zeta\rangle[Z(x)-Z(x')]^2}\mathcal{F}_u(Z(x'),\zeta)\langle\zeta\rangle^{\frac{m}{2}}dZ(x')d\zeta.$$

Clearly $g_1(Z(x))$ is smooth on $\mathbb{R}^m$. We will show that $g_1(Z(x))$ is in $\mathcal{E}^M$ for x near the origin.

Let $\{\mathcal{M}_1, \dots, \mathcal{M}_m\}$ be the vector fields that satisfy

$$\mathcal{M}_j Z_k = \delta_j^k, \ 1 \le j, k \le m.$$

Then $g_1(Z(x))$ will be in $\mathcal{E}^M$ near 0 if there exist $C > 0$ such that for all α,

$$|\mathcal{M}^\alpha g_1(Z(x))| \le C^{|\alpha|+1} M_{|\alpha|} \tag{4.2.7}$$

for x near 0, where $\mathcal{M}^\alpha = \mathcal{M}_1^{\alpha_1} \dots \mathcal{M}_m^{\alpha_m}$. We will therefore establish estimate (4.2.7). Observe that since

$$\mathbb{C}^m \ni z \mapsto e^{i\zeta.(z-Z(x'))-\langle\zeta\rangle[z-Z(x')]^2}$$

is an entire function, for any α,

$$\mathcal{M}^\alpha g_1(Z(x)) = \frac{1}{(2\pi^3)^{\frac{m}{2}}} \int_{\xi\in\mathcal{C}_1} \int_{\Omega'} \partial_z^\alpha \left\{ e^{i\zeta.(z-Z(x'))-\langle\zeta\rangle[z-Z(x')]^2} \right\}|_{z=Z(x)} \mathcal{F}_u(Z(x'),\zeta)\langle\zeta\rangle^{\frac{m}{2}} dZ(x')d\zeta.$$

We begin by estimating the term $\partial_z^\alpha \left\{ e^{i\zeta.(z-Z(x'))-\langle\zeta\rangle[z-Z(x')]^2} \right\}$ for z and x' bounded. Clearly

$$\partial_z^\alpha \left\{ e^{i\zeta.(z-Z(x'))-\langle\zeta\rangle[z-Z(x')]^2} \right\} = \prod_{j=1}^m \partial_{z_j}^{\alpha_j} F(h_j(z_j)),$$

where $F(w) = e^w$ for $w \in \mathbb{C}$ and for each $1 \le j \le m$,

$$h_j(z_j) = i\zeta_j(z_j - Z_j(x')) - \langle\zeta\rangle[z_j - Z_j(x')]^2.$$

By the formula of Faá di Bruno,

$$\partial_{z_j}^{\alpha_j} F(h_j(z_j)) = \sum_{S_j} \frac{\alpha_j!}{\alpha_j^1! \dots \alpha_j^m!} F^{(\alpha_j^1+\dots+\alpha_j^m)}(h_j(z_j)) \prod_{k=1}^m \left(\frac{h_j^{(k)}(z_j)}{k!} \right)^{\alpha_j^k}$$

where the sum is taken over the set

$$S_j = \{(\alpha_j^1, \ldots, \alpha_j^m) : \sum_{k=1}^{m} k\alpha_j^k = \alpha_j\}$$

and each α_j^k is a nonnegative integer.

Note that $h_j^{(1)}(z_j) = i\zeta_j - 2\langle\zeta\rangle(z_j - Z_j(x')), h_j^{(2)}(z_j) = -2\langle\zeta\rangle$, and $h_j^{(k)}(z_j) = 0$ for $k \geq 3$. It follows that

$$\partial_{z_j}^{\alpha_j} F(h_j(z_j)) = \sum_{S_j} \frac{\alpha_j!}{\alpha_j^1!\alpha_j^2!}(-1)^{\alpha_j^2}\langle\zeta\rangle^{\alpha_j^2}\Big(i\zeta_j - 2\langle\zeta\rangle(z_j - Z_j(x'))\Big)^{\alpha_j^1} e^{h_j(z_j)}$$

and

$$S_j = \{(\alpha_j^1, \alpha_j^2) : \alpha_j^1 + 2\alpha_j^2 = \alpha_j\}.$$

Hence

$$\partial_z^\alpha \Big\{e^{i\zeta.(z-Z(x'))-\langle\zeta\rangle[z-Z(x')]^2}\Big\} = e^{h(z)} \prod_{j=1}^{m} \Bigg(\sum_{S_j} \frac{\alpha_j!}{\alpha_j^1!\alpha_j^2!}(-1)^{\alpha_j^2}\langle\zeta\rangle^{\alpha_j^2}\Big(i\zeta_j - 2\langle\zeta\rangle(z_j - Z_j(x'))\Big)^{\alpha_j^1}\Bigg) \tag{4.2.8}$$

where $h(z) = i\zeta.(z - Z(x')) - \langle\zeta\rangle[z - Z(x')]^2$. Since $z_j = Z_j(x)$ and $Z_j(x')$ will lie in a bounded set, there exist $c > 0$ such that

$$\Big|i\zeta_j - 2\langle\zeta\rangle(z_j - Z_j(x'))\Big|^{\alpha_j^1} \leq c^{\alpha_j^1}|\zeta|^{\alpha_j^1} \leq c^{|\alpha|}|\zeta|^{\alpha_j^1}.$$

Likewise, the factor $e^{h(z)}$ is bounded for $z = Z(x)$. Therefore, the term

$$\Big|\partial_z^\alpha \Big\{e^{i\zeta.(z-Z(x'))-\langle\zeta\rangle[z-Z(x')]^2}\Big\}|_{z=Z(x)}\Big|$$

is bounded by the sum of a finite number of terms of the form (after possibly increasing c)

$$c^{|\alpha|+1}\Big(\frac{\alpha_1!}{\alpha_1^1!\alpha_1^2!} \cdots \frac{\alpha_m!}{\alpha_m^1!\alpha_m^2!}\Big)|\zeta|^{\alpha_1^1+\ldots+\alpha_m^1+\alpha_1^2+\ldots+\alpha_m^2} \tag{4.2.9}$$

We thus see that $|\mathcal{M}^\alpha g_1(Z(x))|$ is bounded by a finite sum of the form

$$c^{|\alpha|+1}\left(\frac{\alpha_1!}{\alpha_1^1!\alpha_1^2!}\cdots\frac{\alpha_m!}{\alpha_m^1!\alpha_m^2!}\right)\int_{\mathcal{C}_1}\int_{\Omega'}|\zeta|^{\alpha_1^1+\ldots+\alpha_m^1+\alpha_1^2+\ldots+\alpha_m^2+\frac{m}{2}}|\mathcal{F}_u(Z(x'),\zeta)||d\zeta||dZ(x')|.$$

By the hypothesis on the decay of $\mathcal{F}_u(Z(x'),\zeta)$, $|\mathcal{M}^\alpha g_1(Z(x))|$ is bounded by a finite sum of the form

$$c^{|\alpha|+1}\left(\frac{\alpha_1!}{\alpha_1^1!\alpha_1^2!}\cdots\frac{\alpha_m!}{\alpha_m^1!\alpha_m^2!}\right)\int_{\mathcal{C}_1}\int_{\Omega'}|\zeta|^{\alpha_1^1+\ldots+\alpha_m^1+\alpha_1^2+\ldots+\alpha_m^2+\frac{m}{2}}e^{-M(b|\zeta|)}|d\zeta||dZ(x')|$$

for some $c>0$. Since $|\zeta|$ is comparable to $|\xi|$, $\xi\in\mathbb{R}^m$ and Ω' is a bounded set, the latter expression is bounded by

$$c^{|\alpha|+1}\left(\frac{\alpha_1!}{\alpha_1^1!\alpha_1^2!}\cdots\frac{\alpha_m!}{\alpha_m^1!\alpha_m^2!}\right)\int_{\mathbb{R}^m}|\xi|^{\alpha_1^1+\ldots+\alpha_m^1+\alpha_1^2+\ldots+\alpha_m^2+\frac{m}{2}}e^{-M(b|\xi|)}d\xi \qquad (4.2.10)$$

and hence for $|\xi|\geq 1$, it is bounded by

$$c^{|\alpha|+1}\left(\frac{\alpha_1!}{\alpha_1^1!\alpha_1^2!}\cdots\frac{\alpha_m!}{\alpha_m^1!\alpha_m^2!}\right)\int_{\mathbb{R}^m}|\xi|^{\alpha_1^1+\ldots+\alpha_m^1+\alpha_1^2+\ldots+\alpha_m^2+m}e^{-M(b|\xi|)}d\xi \qquad (4.2.11)$$

We next estimate the integral (4.2.11). Using Lemma 1.1.10 (f) with $L=b$, $t=|\xi|$, $k=r=\alpha_1^1+\ldots+\alpha_m^1+\alpha_1^2+\ldots+\alpha_m^2+m$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^m}|\xi|^{\alpha_1^1+\ldots+\alpha_m^1+\alpha_1^2+\ldots+\alpha_m^2+m}e^{-M(b|\xi|)}d\xi\\
&\leq\int_{\mathbb{R}^m}\sqrt{A}\left(\frac{H}{b}\right)^{\alpha_1^1+\ldots+\alpha_m^1+\alpha_1^2+\ldots+\alpha_m^2+m}M_{\alpha_1^1+\ldots+\alpha_m^1+\alpha_1^2+\ldots+\alpha_m^2+m}e^{\frac{1}{2}M(b|\xi|)}e^{-M(b|\xi|)}d\xi\\
&\leq\sqrt{A}\left(\frac{H}{b}\right)^{\alpha_1^1+\ldots+\alpha_m^1+\alpha_1^2+\ldots+\alpha_m^2+m}M_{\alpha_1^1+\ldots+\alpha_m^1+\alpha_1^2+\ldots+\alpha_m^2+m}\int_{\mathbb{R}^m}e^{-\frac{1}{2}M(b|\xi|)}\,d\xi\\
&\leq C^{\alpha_1^1+\ldots+\alpha_m^1+\alpha_1^2+\ldots+\alpha_m^2+m}M_{\alpha_1^1+\ldots+\alpha_m^1+\alpha_1^2+\ldots+\alpha_m^2+m}
\end{aligned}$$

Observe that for each $1 \leq j \leq m$,

$$\begin{aligned}
\frac{\alpha_j!}{\alpha_j^1!\alpha_j^2!} &= \frac{[(\alpha_j^1+\alpha_j^2)+\alpha_j^2]!}{\alpha_j^1!\alpha_j^2!} \text{ since } \alpha_j = \alpha_j^1 + 2\alpha_j^2 \\
&\leq \frac{2^{\alpha_j}(\alpha_j^1+\alpha_j^2)!}{\alpha_j^1!} \\
&\leq \frac{2^{2\alpha_j}\alpha_j^1!\alpha_j^2!}{\alpha_j^1!} = 2^{2\alpha_j}\alpha_j^2!
\end{aligned}$$

Therefore, using the in equality $\alpha_j^2! \leq M_{\alpha_j^2}$, (4.2.11) is bounded by

$$\begin{aligned}
&c^{|\alpha|+1}4^{|\alpha|}\alpha_1^2!\dots\alpha_m^2!C^{\alpha_1^1+\dots+\alpha_m^1+\alpha_1^2+\dots+\alpha_m^2+m}M_{\alpha_1^1+\dots+\alpha_m^1+\alpha_1^2+\dots+\alpha_m^2+m} \\
&\leq c^{|\alpha|+1}4^{|\alpha|}M_{\alpha_1^2}\dots M_{\alpha_m^2}C^{\alpha_1^1+\dots+\alpha_m^1+\alpha_1^2+\dots+\alpha_m^2+m}M_{\alpha_1^1+\dots+\alpha_m^1+\alpha_1^2+\dots+\alpha_m^2+m} \\
&\leq C^{|\alpha|+1}4^{|\alpha|}M_{\alpha_1^1\dots\alpha_m^1+2(\alpha_1^2+\dots+\alpha_m^2)+m} \\
&\leq C^{|\alpha|+1}4^{|\alpha|}M_{|\alpha|+m} \\
&\leq C^{|\alpha|+1}M_{|\alpha|} \qquad (4.2.12)
\end{aligned}$$

Finally, we need to estimate the number of terms that arise from the product in (4.2.8). For n a positive integer, let

$$S_n = \{(n_1, n_2) : n_1 + 2n_2 = n\}$$

where n_1, n_2 are nonnegative integers. If $(n_1, n_2) \in S_n$, then $n_2 \leq \frac{n}{2}$ and for each such n_2, there is at most one n_1 such that $(n_1, n_2) \in S_n$. Hence S_n has no more than $\frac{n}{2}$ elements. It follows that the product in (4.2.8) leads to no more than

$$\begin{aligned}
(\frac{\alpha_1}{2})\dots(\frac{\alpha_m}{2}) \leq |\alpha|^m &\leq m!e^{|\alpha|} \\
&\leq c^{|\alpha|+1} \qquad (4.2.13)
\end{aligned}$$

elements where c is independent of α.

From (4.2.12) and (4.2.13), we conclude that there is $C > 0$ independent of α

such that

$$\left|\mathcal{M}^\alpha g_1(Z(x))\right| \leq C^{|\alpha|+1} M_{|\alpha|}$$

and hence $g_1(Z(x))$ is in $\mathcal{E}^M$ near 0.

Thus if V is an open set, then there is an almost analytic extension f_1 of g_1 such that $f_{1|V} = g_1$ and

$$\left|\frac{\partial f_1}{\partial \bar{z}_k}(z)\right| \leq C^{N+1} \frac{M_N}{N!}\bigg(dist(z, Z(V))\bigg)^N, \forall k = 1, \ldots, m$$

for some $C > 0$, $z \in Z_\#(V + i\Gamma^1_\delta)$.

In the sense of distributions, for each $j \geq 2$,

$$\lim_{\Gamma^j \ni y \to 0} f_j(Z_\#(x+iy)) = \lim_{\epsilon \to 0^+} f_j^\epsilon(Z(x))$$

and

$$\lim_{\Gamma^1 \ni y \to 0} f_1(Z_\#(x+iy)) = \lim_{\epsilon \to 0^+} g_1^\epsilon(Z(x)).$$

Thus

$$u_{\tilde{K},0} = \lim_{\epsilon \to 0^+} \sum_{j=1}^N f_j^\epsilon(Z(x)) = \lim_{y \to 0} \sum_{j=1}^N f_j(Z_\#(x+iy)).$$

It remains to show that that $u^\epsilon_{\tilde{K},1}(z)$ converges to a holomorphic function. However, by wellposedness, for $z' \in \tilde{\Omega} \setminus \Omega'$ and $z \in Z(V)$, we have

$$\Re\bigg(i\zeta.(z-z') - \langle\zeta\rangle[z-z']^2\bigg) \leq -(1-\kappa)|\zeta||z-z'|^2 \leq -c_0 < 0$$

and hence we may let ϵ go to zero in the integral for $u^\epsilon_{\tilde{K},1}(z)$. Thus we obtain

$$u_{\tilde{K},1}(z) = \frac{1}{(2\pi^3)^{\frac{m}{2}}} \int\int_{\tilde{\Omega}\setminus\Omega'} e^{i\zeta.(z-z') - \langle\zeta\rangle[z-z']^2} \mathcal{F}_u(z',\zeta)\langle\zeta\rangle^{\frac{m}{2}} dz' d\zeta,$$

which defines a holomorphic function. □

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