DEPARTEMENT OF MATHEMATICS

Upper and Lower Solutions for BVPs on the Half-line with Derivative Depending Nonlinearity

Prepared by: Ibrahim Yimer

Advisor: Tadesse Abdi(Ph.D)

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The undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled **Upper and Lower Solutions for BVPs on the Half-line with Derivative Depending Nonlinearity** by Ibrahim Yimer in partial fulfillment of the requirements for the degree of master of Science.

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Advisor: ____________________
Dr.Tadesse Abdi

Examining committee: ____________________

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Abstract

This paper concerns the existence of solutions of a second order non linear boundary value problem with a derivative depending non linearity and posed on the positive half line. The derivative operator is time dependent. Upon a priori estimate and under a suitable growth condition, the Schauder's fixed point theorem combined with the method of upper and lower solutions on unbounded domains are used to prove existence of solutions. A uniquness theorem is also viewed and some examples are used to illustrates the obtained results.

Notations

$\mathbb{R}$	The set of all real numbers.
$\mathbb{N}$	The set of all natural numbers.
$\|t\|$	Euclidean norm of $t = (t_1, t_2, ..., t_n) \in \mathbb{R}^n,$ i.e. $\|t\| = (\sum_{i=1}^{n} t_i^2)^{1/2}$.
$\bar{B}(y_0, a)$	Closed ball of radius a about the point y_0.
$C[0, \infty)$	The set of continuous function on $[0, \infty)$.
$C^1[0, \infty)$	The set of 1-times continuously differentiable function on $[0, \infty)$.

Chapter 1

Introduction

The method of upper and lower solutions provides an effective tool to produce existence theorems for frist and second order initial and boundary value problems. In this paper we are concerned with the existence of solution to the following boundary value problem,

$$x^{''}(t) - k^2(t)x(t) + q(t)f(t, x(t), x^{'}(t)) = 0, t > 0 \tag{1.1}$$
$$x(0) = 0, x(+\infty) = 0$$

where $q \in C(0, \infty) \bigcap L^1(0, \infty)$ while the nonlinearity $f : I \times R \times R \longrightarrow R$ and the coefficient $k : I \longrightarrow (0, \infty)$ are continuous. Here $I = (0, +\infty)$ refers to the positive half line.Since boundary value problems on the infinite intervals arise in many application from physics, chemistry and biology, there has been so much work devoted to the investigation of positive solutions for such BVPs in the last couple of years.superlinear or sublinear nonlinearity are considered. when nonlinearity is not necessarily positive,the Schauder fixed point theorem is combined with the method of upper and lower solutions on unbounded domains to prove existence of solutions. In this case ,the Nagumo condition is assumed in the nonlinearity. A uniqueness result is also given under a monotonicity condition.
The positivity of solutions is motivated by the fact that the unknown x may refer to density,a temperature or the concentration of the product. For instance the linear differential equation $-x^{''} + cx^{'} + \lambda x = 0(c, \lambda > 0)$ which can be rewritten in reduced form as:$-x^{''} + k^2x = 0$ stems from combustion theory.
Indeed, if we set

$$x(t) = u(t)z(t)$$

Then

$$x^{'} = u^{'}z + uz^{'}$$
$$\Rightarrow x^{''} = u^{''}z + 2u^{'}z^{'} + uz^{''}$$

Thus,

$$\begin{aligned} -x^{''} + cx^{'} + \lambda x &= -(u^{''}z + 2u^{'}z^{'} + uz^{''}) + c(u^{'}z + uz^{'}) + \lambda uz \\ &= -u^{''}z + (-2z^{'} + cz)u^{'} + (-z^{''} + cz^{'} + \lambda z)u \end{aligned}$$

Setting $-2z^{'} + cz = 0$,we have

$$\Rightarrow z^{'} = \frac{c}{2}z$$
$$\Rightarrow z(t) = Ae^{\frac{ct}{2}},$$

where A is constant.
Consequently,

$$\begin{aligned} -z^{''} + cz^{'} + \lambda z &= -(\frac{cz}{2})^{'} + \frac{c^2}{2}z + \lambda z \\ &= -\frac{c}{2}z^{'} + \frac{c^2}{2}z + \lambda z \\ &= -\frac{c^2}{4}z + \frac{c^2}{2}z + \lambda z \\ &= (\lambda + \frac{c^2}{4})z \end{aligned}$$

And hence,

$$\begin{aligned} -x^{''} + cx^{'} + \lambda x &= -u^{''}z + (-z^{''} + cz^{'} + \lambda z)u \\ &= -u^{''}z + (\lambda + \frac{c^2}{4})zu \\ &= [-u^{''} + (\lambda + \frac{c^2}{4})u]z = 0 \end{aligned}$$

$$\Rightarrow -u^{''} + (\lambda + \frac{c^2}{4})u = 0$$

$$\Rightarrow -u^{''} + k^2u = 0. where, k^2 = \lambda + \frac{c^2}{4}$$

Therefore, $-x^{''} + cx^{'} + \lambda x = 0$ was reduced to $-x^{''} + k^2x = 0$.

In order to construct a Green's function of the corresponding linear problem:

$$x^{''}(t) - k^2x(t) = 0, t > 0 \tag{1.2}$$
$$x(0) = 0, \lim_{t\to\infty} x(t) = 0$$

A Green's function is constructed out of two independent solutions y_1 and y_2 of the homogenuous equation.Then the Green's function $G(x, s)$ is

$$G(x, s) = \begin{cases} \frac{y_1(s)y_2(x)}{w(y_1,y_2)(s)}, if x \geq s \\ \frac{y_1(x)y_2(s)}{w(y_1,y_2)(s)}, if x \leq s \end{cases}$$

where $w(y_1, y_2)(s)$ is wronskian of y_1 and y_2.
If k is costant, then the Green's function $G(t, s)$ of problem (1.2) can be explicitly expressed by:

$$G(t, s) = \begin{cases} \frac{(e^{kt} - e^{-kt})e^{-ks}}{2k}, if t \geq s \\ \frac{(e^{ks} - e^{-ks})e^{-kt}}{2k}, if t \leq s \end{cases}$$

In the general case when the constant k is replaced by a bounded function $k = k(t)$, Green's functions of the associated problem (1.2) cannot be explicitly expressed by elementary functions. This makes our approach more difficult.

Chapter 2

Basic Notions and Mathematical prelude

2.1 Definitions

Definition 2.1.1. *[6] A Banach space is a vector space with a norm such that every cauchy sequence converges to a limit in the space. We also say the space is complete.*

Example 2.1.1. *[6] The familiar vector space R^n with the norms defined for $x = (x_1, x_2, ..., x_n)^T$,*

$$\|x\|_1 = |x_1| + |x_2| + ... + |x_n|$$

$$\|x\|_2 = (|x_1|^2 + |x_2|^2 + ... + |x_n|^2)^{\frac{1}{2}}$$

$$\|x\|_\infty = \max |x_i|$$

are all Banach spaces.

Definition 2.1.2. *[6] Let $T : X \to Y$ be a mapping between two Banach spaces. Then 1)T is called continuous if for each $x \in X$ and each $\varepsilon > 0$ there exists a $\delta = \delta(x, \varepsilon) > 0$ such that for every $y \in X$*

$$\|y - x\|_x < \delta \Rightarrow \|T(y) - T(x)\|_y < \varepsilon.$$

2)A bounded linear operator $T : X \to Y$ is called completely continuous if, for every weakly convergent sequence (x_n) from X,the sequence $(T(x_n))$ is norm convergent in Y.
3)If X a reflexive Banach space, then every completely continuous operator $T : X \to Y$ is compact.

Definition 2.1.3. *[3] we say that*
a)A function α is a lower solution of problem (1.1) if $\alpha \in C^1[0, \infty) \bigcap C^2(0, \infty)$ and

$$\alpha^{''}(t) - k^2(t)\alpha(t) + q(t)f(t, \alpha(t), \alpha^{'}(t)) \geq 0$$

$$\alpha(0) \leq 0, \alpha(+\infty) \leq 0.$$

b)A function β is an upper solution of problem (1.1) if $\beta \in C^1[0, \infty) \bigcap C^2(0, \infty)$ and

$$\beta^{''}(t) - k^2(t)\beta(t) + q(t)f(t, \beta(t), \beta^{'}(t)) \leq 0$$

$$\beta(0) \geq 0, \beta(+\infty) \geq 0.$$

Example 2.1.2. *[4] consider the boundary value problem:*

$$x^{''}(t) - (\sin t + 3)^2 x(t) + e^{-t}(t + x(t)) = 0, t > 0$$

$$x(0) = 0, x(+\infty) = 0$$

Since $\alpha(t) \equiv 0$ *is a lower solution. Because,*

$$\Rightarrow \alpha^{''}(t) - (\sin t + 3)^2 \alpha(t) + e^{-t}(t + \alpha(t)) = te^{-t}$$

$$\Rightarrow te^{-t} = \frac{t}{e^t} \geq 0, \forall t > 0$$

and in addition $\alpha(0) \leq 0, \alpha(+\infty) \leq 0$ *Similarly* $\beta(t) = t$ *is un upper solution. Because,*

$$\Rightarrow \beta^{''}(t) - (\sin t + 3)^2 \beta(t) + e^{-t}(t + \beta(t)) = 2te^{-t} - t(\sin t + 3)^2$$

We know that

$$\Rightarrow 2te^{-t} - t(\sin t + 3)^2 = t[\frac{2}{e^t} - (\sin t + 3)^2] \leq 0, \forall t > 0$$

and in addition $\beta(0) \geq 0, \beta(+\infty) \geq 0$
Therefore, $\alpha(t) \equiv 0$ *and* $\beta(t) = t$ *are a lower solution and an upper solution of a given example respectively.*

2.2 Mathematical tools

The theorem which we state and prove in this section are very important in understanding the whole work of this paper.

Theorem 2.2.1. *(Lebesgue dominated convergence theorem)*
Suppose $f_n : R \to [-\infty, \infty]$ *are (Lebesgue) measurable functions such that the pointwise limit* $f(x) = \lim_{n\to\infty} f_n(x)$ *exists. Assume there is an integral* $g : R \to [0, \infty]$ *with* $|f_n(x)| \leq g(x)$ *for each* $x \in R$*. Then* f *is integrable as is* f_n *for each* n*, and*

$$\lim_{n\to\infty} \int_R f_n d\mu = \int_R \lim_{n\to\infty} f_n d\mu = \int_R f d\mu$$

Proof. Since $|f_n(x)| \leq g(x)$ and g is integrable,$\int_R |f_n| d\mu \leq \int_R g d\mu < \infty$. So f_n is integrable. We know f is measurable (as a pointwise limit of measurable functions) and then, similarly,$|f(x)| = \lim_{n\to\infty} |f_n(x)| \leq g(x)$ implies that f is integrable too.

The proof does not work properly if $g(x) = \infty$ for some x. We know that $g(x) < \infty$ almost everywhere. So we can take $E = \{x \in R : g(x) = \infty\}$ and multiply g and each of the functions f_n and f by $1 - x_E$ to make sure all the functions have finite values. As we are changing them all only on the set E of measure zero, this change does not affect the integrals or the conclusions. We assume then all have finite values.
Let $h_n = g - f_n$, so that $h_n \geq 0$. By Fatous Lemma

$$\liminf_{n\to\infty} \int_R (g - f_n) d\mu \geq \int_R \liminf_{n\to\infty} (g - f_n) d\mu = \int_R (g - f) d\mu$$

and that gives

$$\liminf_{n\to\infty}(\int_R gd\mu - \int_n f_n d\mu) = \int_R gd\mu - \limsup_{n\to\infty}\int_R f_n d\mu \geq \int_R gd\mu - \int_R fd\mu$$

or

$$\limsup_{n\to\infty}\int_R f_n d\mu \leq \int_R fd\mu \tag{2.1}$$

Repeat this Fatous Lemma argument with $g + f_n$ rather than $g - f_n$. We get

$$\liminf_{n\to\infty}\int_R (g+f_n)d\mu \geq \int_R \liminf_{n\to\infty}(g+f_n)d\mu = \int_R (g+f)d\mu$$

and that gives

$$\liminf_{n\to\infty}(\int_R gd\mu + \int_R f_n d\mu) = \int_R gd\mu + \int_R \liminf_{n\to\infty} f_n d\mu \geq \int_R gd\mu + \int_R fd\mu$$

or

$$\int_R \liminf_{n\to\infty} f_n d\mu \geq \int_R fd\mu \tag{2.2}$$

combining (2.1) and (2.2), we get

$$\int_R fd\mu \leq \int_R \liminf_{n\to\infty} f_n d\mu \leq \limsup_{n\to\infty}\int_R f_n d\mu \leq \int_R fd\mu$$

which forces

$$\int_R fd\mu = \int_R \liminf_{n\to\infty} f_n d\mu = \limsup_{n\to\infty}\int_R f_n d\mu$$

and that gives the result because if $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$ (for a sequence $(a_n)_{n=1}^{\infty}$), it implies that $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$. □

Theorem 2.2.2. *(Leray-Schauder fixed point theorem)*
Assume that X is a Banach space and that $T : X \to X$ is a continuous compact mapping. Moreover assume that the set

$$\bigcup_{0\leq\lambda\leq 1}\{x \in X : x = \lambda T(x)\}$$

is bounded. Then T has a fixed points.

Proof. Assume that the mapping T satisfies the hyphothesis in the theorem. Pick a $R > 0$ such that $x = \lambda T(x)$ and $0 \leq \lambda \leq 1$ impilies that $||x|| < R$.
Define the mapping $\widetilde{T} : x \to x$ as follows;

$$\widetilde{T} = \begin{cases} T(x), if\ ||T(x)|| \leq R \\ \frac{R}{||T(x)||}T(x), if\ ||T(x)|| > R \end{cases}$$

This implies that $\widetilde{T} : x \to x$ is a compact operator. To show this take a bounded sequence $(x_n)_{n=1}^{\infty}$ in x. Then there exists a subsequence $(x_{nk})_{k=1}^{\infty}$ such that $||T(x_{nk})|| <$

R for all k or $||T(x_{nk})|| > R$ for all k. In the first case $(T(x_{nk}))_{k=1}^{\infty}$ has convergent subsequence, since $\widetilde{T}(x_{nk}) = T(x_{nk})$ and T is compact mapping. In the second case we get that $(T(x_{nk}))_{k=1}^{\infty}$ has a convergent subsequence, denote it by $(T(x_l))_{l=1}^{\infty}$ for convenience. But then it follows that also $||T(x_l)|| \geq R$ for all l. Hence we obtain $\widetilde{T}(x_l) = \frac{R}{||T(x_l)||}T(x_l)$.
Set $k = \overline{C_0\widetilde{T}(B(0,r))}$.
Here k is convex (it is the convex hull of a set), compact (the convex hull of a compact set is compact and $\widetilde{T}$ is a compact mapping) subset of x such that $\widetilde{T} : k \to k$. Schauder fixed point theorem implies that $\widetilde{T}$ has a fixed point $x_0 \in k$. But x_0 is a fixed point for T if $||T(x_0)|| \leq R$. Assume that $||T(x_0)|| \leq R$. This yields a contradiction since $x_0 = T(x_0) = \lambda T(x_0)$, where $\lambda = \frac{R}{||T(x_0)||} \in (0,1)$, since according to the hyphothesis of the theorem it should follow that $||T(x_0)|| = ||x_0|| < R$. $\square$

Chapter 3

Auxiliary Lemmas And Modified Problem

First, let us state some assumptions:

(H_0) The function $k : I \to [0, \infty)$ is bounded and continuous and $\exists d \in [\underline{k}, \overline{k}], \forall p > 0, \lim_{n\to\infty} e^{-pt} \int_0^t e^{ps}[k^2(s) - d^2]ds$ exists.
Where $\overline{k} := \sup_{t\in[0,\infty)} k(t) > 0$ and $\underline{k} := \inf_{t\in[0,\infty)} k(t) > 0$.

(H_1) There exists $\alpha \leq \beta$ lower and upper solutions of problem (1.1) respectively.

(H_2) $\alpha_0 := \sup_{t\in[o,\infty)}\{|\alpha(t)|\phi_2^{-1}(t)\} < \infty$ and $\beta_0 := \sup_{t\in[o,\infty)}\{|\beta(t)|\phi_2^{-1}(t)\} < \infty$,where $\phi_2^{-1}(t) := \frac{1}{\phi_2(t)}$.

(H_3) There exist continuous functions $\psi : I \to [0, \infty)$ and $h : R \to [1, \infty)$ such that

$$\int_0^\infty \psi(s)q(s)ds < \infty,$$

$$\int_0^\infty \frac{ds}{h(s)} = +\infty$$

and

$$|f(t, x, y)| \leq \psi(t)h(y), \forall(t, x, y) \in D_\alpha^\beta \times R$$

where D_α^β is defined by

$$D_\alpha^\beta := \{(t, x) \in (0, \infty) \times R : \alpha(t) \leq x(t) \leq \beta(t)\}.$$

(H_4) $\alpha_1 := \sup_{t\in R^+} \alpha^{'}(t) < \infty, \beta_1 := \inf_{t\in R^+} \beta^{'}(t) > -\infty$ and for any $y \in R$ and $t \in (0, \infty)$,we have

$$y < \beta^{'}(t) \Rightarrow f(t, \beta(t), y) \leq f(t, \beta(t), \beta^{'}(t))$$

$$y > \alpha^{'}(t) \Rightarrow f(t, \alpha(t), y) \geq f(t, \alpha(t), \alpha^{'}(t))$$

$(H_4)^{'}$ $\alpha_1 := \sup_{t\in R^+} |\alpha^{'}(t)| < \infty$,and $\beta_1 := \inf_{t\in R^+} |\beta^{'}(t)| < \infty$.

3.1 Basic Lemmas

Lemma 3.1.1. *[5] Assume that (H_0) holds. Then the cauchy problem*

$$x^{''}(t) - k^2(t)x(t) = 0, t > 0 \tag{3.1}$$

$$x(0) = 0, x^{'}(0) = 1$$

has a unique solution ϕ_1 defined on $[0, +\infty)$. Moreover ϕ_1 is defined nondecreeasing and unbounded.

Proof. By the existence and uniqueness of solution for intial value problem, we know that problem (3.1) has the unique solution ϕ_1 defined on $[0, \infty)$.
Suppose the contrary that $\phi_1^{'}(t) = 0$ for some $t_0 \in (0, \infty)$. By the boundary condition $x^{'}(0) = 1, t_0 > 0$. We may assume that $\phi_1^{'} > 0$ on $[0, t_0)$. Thus ϕ_1 is strictly increasing on $[0, t_0)$. On the other hand,we have from problem (3.1) that $\phi_1^{''}(t_0) = k^2(t_0)\alpha_1(t_0) > 0$ and accordingly t_0 is a minimum value point. This is a contradiction.

By the above proof, we can infer that $\phi_1^{''}(t) = k^2(t)\phi_1(t) > 0, t \in (0, \infty)$. Therefore,$\phi_1$ is unbounded. □

Lemma 3.1.2. *[5] Assume that (H_0) holds. Then the problem*

$$x^{''}(t) - k^2(t)x(t) = 0, t > 0 \tag{3.2}$$

$$x(0) = 0, \lim_{t\to\infty} x(t) = 0$$

has a unique solution ϕ_2 defined on $[0, +\infty)$, moreover

$$\phi_2(t) > 0, \phi_2^{'}(t) < 0, t \in [0, \infty).$$

Proof. Let us divide the proof into several steps.

i.We show that equation (3.2) has a solution u with $u(t) > 0$ in $[0, \infty)$.
Let us consider the following

$$x^{''}(t) - k^2(t)x(t) = 0, t \in (0, n) \tag{3.3}$$

$$x(0) = 1, x(n) = 0.$$

We claim that for each $n \in N$, problem (3.3) has a positive solution u with

$$u(t) > 0, u^{'}(t) < 0, u^{''}(t) > 0, \forall t \in (0, n) \tag{3.4}$$

In fact, suppose on the contrary that $t_1 \in (0, n)$ is such that$u(t_1) = 0$ and $u(t) > 0$ for $t \in (0, t_1)$.
Then $u^{'}(t_1) < 0$, since the other case $u^{'}(t_1) = 0$ would imply that $u(t) \equiv 0$ for $t \in (0, n)$, which is a contradiction. Notice that $u^{'}(t_1) < 0$ and $u(t_1) = 0$ imply that u is negative and concave down in $(t_1, t_1 + \sigma) \subseteq (t_1, n]$. Using (3.3), it follows that σ may be taken as $n - t_1$ so that $u(n) = 0$. This contradicts the boundary condition $u(n) = 0$. Thus, we get that $u(t) > 0$ in $[0, n)$, and subsequently, $u^{''}(t) > 0, t \in [0, n)$. This together

with the fact $u^{'}(t) < 0, t \in [0, n)$.
Therefore

$$u(t) > 0, u^{'}(t) < 0, u^{''}(t) > 0, \forall t \in (0, n).$$

ii.For each $n \geq 1$.We show that $u_n(t) < u_{n+1}(t), t \in (0, n)$.
Let $w(t) := u_{n+1}(t) - u_n(t), t \in [0, n]$
Then

$$w^{''}(t) - k^2(t)w(t) = 0, t \in (0, n).$$

$$w(0) = 0, w(n) = a$$

where $a := u_{n+1}(n) > 0$.
We claim that

$$w(t) > 0, w^{'}(t) > 0, w^{''}(t) > 0, \forall t \in (0, n).$$

In fact, suppose on the contrary that $t_2 \in (0, n)$ is such that $w(t_2) > 0$ and $w(t) > 0$ for $t \in (t_2, n)$.
Applying the same method used in (i), we may deduce

$$w(t) > 0, w^{'}(t) > 0, w^{''}(t) > 0, \forall t \in (0, n).$$

iii.Define the functions $\widetilde{u}_n : [0, \infty) \to [0, \infty)$ by

$$\widetilde{u}_n(t) = \begin{cases} u_n(t), t \in [0, n] \\ 0, t \in (n, \infty) \end{cases}$$

Then $\widetilde{u}_n \in C[0, \infty)$. Moreover, we have from (i) and (ii) that

$$0 < \widetilde{u}_n(t) < 1, t \in (0, \infty).$$

and

$$\widetilde{u}_1(t) \leq \widetilde{u}_2(t) ... \leq \widetilde{u}_n(t) ..., t \in [0, \infty).$$

Let $u^*(t) := \lim_{t\to\infty} \widetilde{u}_n(t), t \in [0, \infty)$.
Then $u^* \in C[0, \infty) \bigcap C^2[0, \infty)$ and u^* is actually a solution of the problem (3.2).

iv.By the maximum principle we can infer that u^* is a unique solution of the problem (3.2). □

Lemma 3.1.3. *[5] Assume that (H_0) holds. Then*

$$\lim_{t\to\infty} \frac{\phi_2^{'}(t)}{\phi_2(t)} = -d$$

Proof. Since the equation $x^{''} - k(t)x = 0, t \geq t_0 \geq 0$ has a solution $f : [t_0, +\infty) \to R$ satisfying $\lim_{t\to\infty} f(t) = 0$ and $\lim_{t\to\infty} \frac{f^{'}(t)}{f(t)} = -m$ if and only if there exists a continuous function $b : [t_0, \infty) \to R$ such that, $\forall \geq t_0$ and $\forall q > 0, b(t) = k(t) - m^2$ and $\lim_{t\to\infty} e^{-qt} \int_0^t e^{qs} b(s)ds = 0$. Then the problem (3.2) has a solution $x(t), t \in [0, \infty)$ satisfies

$$\lim_{t\to\infty} x(t) = 0, \lim_{t\to\infty} \frac{x^{'}(t)}{x(t)} = -d$$

Then
i.If $x(0) = 0$ and $x^{'}(0) < 0$. Since $\lim_{t\to\infty} x(t) = 0$, hence there exists $\eta > 0$ and $t_1 \in (0, \infty)$, such that

$$x(t_1) = \min\{\frac{x(t)}{t} \in [0, \eta]\} < 0.$$

So $x^{''}(t_1) \geq 0$. On the other hand, $x^{''}(t_1) = k^2(t_1)x(t_1) < 0$. This a contradiction.
ii.if $x(0) = 0$ and $x^{'}(0) > 0$. Then a desired contradiction can be deduce by the same method from (i).
iii.If $x(0) = 0$ and $x^{'}(0) = 0$. Then $x(t) \equiv 0, t \in [0, \infty)$. However, this contradicts $\lim_{t\to\infty} \frac{x^{'}(t)}{x(t)} = -d$.
Hence $x(0) = b \neq 0$, denote $u(t) := \frac{x(t)}{b}$ then $u(0) = 1$, $\lim_{t\to\infty} u(t) = 0, \lim_{t\to\infty} \frac{u^{'}(t)}{u(t)} = -d$. By Lemma (3.1.2), we get that $u(t) \equiv \phi_2(t)$ in $[0, \infty)$.
Therefore,

$$\lim_{t\to\infty} \frac{\phi_2{}'(t)}{\phi_2(t)} = -d.$$

□

Lemma 3.1.4. *[5] Assume that (H_0) holds. Then there exists $M > 0$ such that*

$$\sup_{t\in[0,\infty)} \phi_1(t)\phi_2(t) < M$$

Proof. Assume that (H_0) holds. Then by Liouville formula,

$$\phi_1(t) = c_1\phi_2(t) + c_2\phi_2(t) \int_0^t \frac{1}{\phi_2^2(s)} ds,$$

for some constants c_1 and c_2.
Applying the L'Hopital rule, it follows that

$$\begin{aligned}
\lim_{t\to\infty} \phi_1(t)\phi_2(t) &= c_1 \lim_{t\to\infty} \phi_2^2(t) + c_2 \lim_{t\to\infty} \phi_2^2(t) \int_0^t \frac{1}{\phi_2^2(s)} ds \\
&= c_2 \lim_{t\to\infty} \frac{\phi_2^{-2}(t)}{-2\phi_2^{-3}(t)\phi_2'(t)} \\
&= c_2 \lim_{t\to\infty} \frac{\phi_2(t)}{-2\phi_2'(t)} \\
&= \frac{c_2}{2d}.
\end{aligned}$$

Hence, there exists $M > 0$ such that

$$\sup_{t\in[0,\infty)} \phi_1(t)\phi_2(t) < M.$$

□

Lemma 3.1.5. *[5] Assume that (H_0) holds. Then for any function $y \in L^1[0, \infty)$, the problem*

$$\begin{cases} x^{''}(t) - k^2(t)x(t) + y(t) = 0, & t > 0 \\ x(0) = 0, \ x(+\infty) = 0 \end{cases} \tag{3.5}$$

is equivalent to the integral equation

$$x(t) = \int_0^\infty G(t,s)y(s)ds, \quad t > 0 \tag{3.6}$$

where

$$G(t,s) = \begin{cases} \phi_1(t)\phi_2(s), & s \geq t, \\ \phi_1(s)\phi_2(t), & t \geq s. \end{cases} \tag{3.7}$$

Proof. First we show that the unique solution of problem (3.5) can be represented by equation (3.6).
In fact, we know that the equation

$$x^{''}(t) - k^2(t)x(t) = 0, \quad t \in (0, \infty),$$

has known two linear independent solutions ϕ_1 and ϕ_2,
since $W(\phi_1(0)\phi_2(0)) = \phi_1(0)\phi_2^{'}(0) - \phi_1^{'}(0)\phi_2(0) = -\phi_1(0) = -1 \neq 0$.
Now by the method of variation of constants, we can obtain that the unique solution of the problem (3.5) can be represented by

$$x(t) = \int_0^\infty G(t,s)y(s)ds, \quad where\ G(t,s)\ is\ from\ Eq\ (3.7).$$

Next we check that the function defined by equation (3.6) is a solution of problem (3.5).
Then, we know that

$$\begin{aligned} x(t) &= \int_0^\infty G(t,s)y(s)ds \\ &= \int_0^t \phi_1(s)\phi_2(t)y(s)ds + x(t) + \int_t^\infty \phi_2(s)\phi_1(t)y(s)ds. \end{aligned}$$

This implies that

$$x^{'}(t) = \phi_2^{'}(t)\int_0^t \phi_1(s)y(s)ds + \phi_1^{'}(t)\int_t^\infty \phi_2(s)y(s)ds,$$

$$x^{''}(t) = \phi_2^{''}(t)\int_0^t \phi_1(s)y(s)ds + \phi_2^{'}(t)\phi_1(t)y(t) + \phi_1^{''}(t)\int_t^\infty \phi_2(s)y(s)ds - \phi_1^{'}(t)\phi_2(t)y(t).$$

So that

$$x^{''}(t) - k^2(t)x(t) = W(\phi_1(t),\phi_2(t))y(t) = W(\phi_1(0),\phi_2(0))y(t) = -y(t).$$

It is easy to see that $G(0,s) = 0$ implies $x(0) = 0$.
Applying the facts that $\sup_{t\in[0,\infty)} \phi_1(t)\phi_2(t) < M$ and $y \in L^1[0,\infty)$.
It follows that for any $\epsilon > 0$, there exists $N_1 > 0$ such that

$$\int_t^\infty \phi_1(s)\phi_2(s)|y(s)|ds \leq \int_t^\infty |y(s)|ds < \epsilon/3, \quad \forall t \geq N_1.$$

From $\lim_{t\to\infty}\phi_2(t)=0$, we have that there exists N_2 such that

$$\phi_1(N_1)\phi_2(t)\int_0^\infty |y(s)|ds < \epsilon/3, \quad \forall t > N_2.$$

Let $N := \max\{N_1, N_2\}$. Then for $t > N$, we get

$$\begin{aligned}
|x(t)| &= |\int_0^t \phi_1(s)\phi_2(t)y(s)ds + \int_t^\infty \phi_1(t)\phi_2(s)y(s)ds| \\
&\leq \int_0^{N_1} \phi_1(s)\phi_2(t)|y(s)|ds + \int_{N_1}^t \phi_1(s)\phi_2(t)|y(s)|ds + \int_t^\infty \phi_1(t)\phi_2(s)|y(s)|ds \\
&\leq \phi_1(N_1)\phi_2(t)\int_0^\infty |y(s)ds| + 2m\int_{N_1}^\infty |y(s)|ds \\
&< \epsilon/3 + 2\epsilon/3 = \epsilon.
\end{aligned}$$

Therefore, $\lim_{t\to\infty} x(t) = 0$. Now, we have from (Lemma 3.1.5) that for $y \in L^1[0,\infty)$, the problem

$$x''(t) - h^2x(t) + y(t) = 0, \quad t \in (0,\infty),$$
$$x(0) = 0, \quad \lim_{t\to\infty} x(t) = 0.$$

and

$$x''(t) - H^2x(t) + y(t) = 0, \quad t \in (0,\infty),$$
$$x(0) = 0, \quad \lim_{t\to\infty} x(t) = 0.$$

are equivalent to the integral equation

$$x_1(t) = \int_0^\infty G_1(t,s)y(s)ds,$$

and

$$x_2(t) = \int_0^\infty G_2(t,s)y(s)ds$$

where

$$G_1(t,s) = \begin{cases} \frac{(e^{ht}-e^{-ht})e^{-hs}}{2h}, & s \geq t, \\ \frac{(e^{hs}-e^{-hs})e^{-ht}}{2h}, & t \geq s, \end{cases}$$

and

$$G_2(t,s) = \begin{cases} \frac{(e^{Ht}-e^{-Ht})e^{-Hs}}{2H}, & s \geq t, \\ \frac{(e^{Hs}-e^{-Hs})e^{-Ht}}{2H}, & t \geq s, \end{cases}$$

respectively. □

Lemma 3.1.6. *[5] For any $(t,s) \in [0,\infty) \times [0,\infty)$,*

$$G_2(t,s) \leq G(t,s) \leq G_1(t,s) < 1/2h$$

Where $h = \underline{k}$

Proof. From

$$G_1(t,s) = \begin{cases} \frac{(e^{ht}-e^{-ht})e^{-hs}}{2h}, & s \geq t, \\ \frac{(e^{hs}-e^{-hs})e^{-ht}}{2h}, & t \geq s, \end{cases}$$

We can easily deduce that $G_1(t,s) < 1/2h$ in $[0,\infty) \times [0,\infty)$.
Next, we only show that $G(t,s) \leq G_1(t,s)$. The other case can be treated by the same way.
Suppose on the contrary that there exists $(t_0, s_0) \in (0,\infty) \times (0,\infty)$, such that $G(t_0,s_0) > G_1(t_0,s_0)$.
Let

$$\hat{y} := \begin{cases} 0, & 0 \leq t \leq s_0 - \epsilon, \\ t - s_0 + \epsilon, & s_0 - \epsilon \leq t \leq s_0, \\ s_0 + \epsilon - t, & s_0 \leq t \leq s_0 + \epsilon, \\ 0, & s_0 + \epsilon \leq t < \infty. \end{cases}$$

Then $\hat{y} \in [0,\infty) \cap L^1[0,\infty)$ and $\hat{y}(t) \geq 0$ in $[0,\infty)$. Let $x_1(t), x_2(t)$ be the solution of

$$x^{''}(t) - h^2 x(t) + \hat{y}(t) = 0, \quad t \in (0,\infty),$$

$$x(0) = 0, \quad \lim_{t\to\infty} x(t) = 0.$$

and

$$x^{''}(t) - H^2 x(t) + \hat{y}(t) = 0, \quad t \in (0,\infty),$$

$$x(0) = 0, \quad \lim_{t\to\infty} x(t) = 0.$$

Let $\hat{x}(t)$ be the solution of

$$x^{''}(t) - k^2 x(t) + \hat{y}(t) = 0, \quad t \in (0,\infty),$$

$$x(0) = 0, \quad \lim_{t\to\infty} x(t) = 0.$$

□

Lemma 3.1.7. *[2] For any $t \in (0,\infty)$, we have*

$$0 \leq \phi_1^{'}(t)\phi_2(t) \leq 1 \quad and \quad -1 \leq \phi_1(t)\phi_2^{'}(t) \leq 0.$$

Proof. Since $\{\phi_1(t)\phi_2(t)\}$ is the fundamental system, we have that $\phi_1(t)\phi_2^{'}(t) - \phi_1^{'}(t)\phi_2(t) = -1$ which means that $\phi_1^{'}(t)\phi_2(t) + (-\phi_1(t)\phi_2^{'}(t)) = 1$. Then our claim follows from the sign and the monotonicity of ϕ_1, ϕ_2. From (Lemma 3.1.1) and (Lemma 3.1.2), we have $\phi_1(t) \geq 0, \phi_2(t) \geq 0, \phi_1^{'}(t) \geq 0$ and $\phi_2^{'}(t) < 0$ and $\{\phi_1, \phi_2\}$ are monotonic functions. □

Lemma 3.1.8. *[2] $\forall t \in (0,\infty)$, we have*

$$K_0 := \int_0^\infty k^2(s)\phi_2(s)ds < \infty.$$

Proof. By (Lemma 3.1.3), it is clear that the function $u = \frac{\phi_2^{'}}{\phi_2}$ satisfies the Ricatti equation $u^{'} + u^2 = k^2$ both with the terminal condition $u(+\infty) = -d$. Hence if, by contradiction, $u(0) = -\infty$, then $u^{'}(0) = -\infty$, which is impossible. Thus $-\infty < u(0) < 0$ and as a consequence $-\infty < \phi_2^{'}(0) < 0$ which implies that $0 < k_0 = -\phi_2^{'}(0) < \infty$, as claimed. □

3.2 The Modified Problem

Now consider the Banach space
$x = \{x \in C^1[0,\infty) : \lim_{t\to+\infty} x(t)$ and $\lim_{t\to+\infty} x^{'}(t)$ exists$\}$ equippied with the norm
$||x|| = \max\{\sup_{t\in[0,\infty)} |x(t)|, \sup_{t\in[0,\infty)} |x^{'}(t)|\}$.

Lemma 3.2.1. *[5] Let $M \subseteq X$. Then M is relatively compact in X if the following conditions hold*
(a) M is bounded in X,
(b) the functions belonging to $Y : y(t) = \phi_2^\theta(t)x(t), x \subseteq M$ are locally equicontinuous on $[0,+\infty)$,
(c) the functions belonging to $Y : y(t) = \phi_2^\theta(t)x(t), x \subseteq M$ are equiconvergent.

Proof. Define a cone of X,
$P = \{x \in X : x(t) \geq 0, t \geq (0,\infty)$, and $x(t) \geq \frac{h}{H}q(t)\|x\|\}$ Define operator T:

$$Tx(t) = \int_0^\infty G(t,s)m(s)f(s, x(s) + M\phi_2^\mu(s))ds$$

Let

$$M_1 = \int_T^\infty G(s,s)\phi_2(s)a(s)m(s)ds < \infty. \quad M_2 = \int_T^\infty G(s,s)\phi_2^{1-\mu p}(s)b(s)m(s)ds < \infty.$$

(a) Let $D \subseteq P$ is bounded, Then there exists $M > 0$ such that $\|x\| \leq M, \forall x \in D$. We show that $T(D)$ is bounded in X.
For any $x \in D$, applying $(H_0) - (H_4)$ *and* $(H_4)^{'}$, we have

$$\begin{aligned}
\|Tx(t)\| &= \sup_{t\in R^+} |\int_0^\infty G(t,s)m(s)f(s, x(s) + \phi_2^\mu(s))ds| \\
&\leq \sup_{t\in R^+} |\int_0^\infty \phi_2^\theta(t)G(t,s)m(s)f(s, x(s) + \phi_2^\mu(s))ds| \\
&\leq \frac{H}{h}\int_0^\infty \phi_2(s)G(s,s)m(s)(a(s) + b(s)|x(s)|^p)ds \\
&\leq \frac{H}{h}(M_1 + M_2\|x\|^p) \\
&\leq \frac{H}{h}(M_1 + M^pM_2).
\end{aligned}$$

Which implies that $T(D)$ is bounded in X.

(b) We show that the functions belonging to $\{(Tx(t)\phi_2^\theta(t)) : x \in D\}$ are locally equicontinuous on $[0,\infty]$.

For any $x \in D$ and any $T > 0$, if $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, we have that

$$\begin{aligned}
|\phi_2^\theta(t_1)Tx(t_1) - \phi_2^\theta(t_2)Tx(t_2)| &= |\int_0^\infty (G(t_1,s)\phi_2^\theta(t_1) - G(t_2,s)\phi_2^\theta(t_2))m(s)f(s,x(s)+\phi_2^\mu(s))ds| \\
&\le \int_0^T |G(t_1,s)\phi_2^\theta(t_1) - G(t_2,s)\phi_2^\theta(t_2)|m(s)(a(s)+b(s)|x|^p)ds \\
&+ \int_T^\infty |G(t_1,s)\phi_2^\theta(t_1) - G(t_2,s)\phi_2^\theta(t_2)|m(s)(a(s)+b(s)|x|^p)ds \\
&= \int_0^T |G(t_1,s)\phi_2^\theta(t_1) - G(t_2,s)\phi_2^\theta(t_2)|m(s)(a(s)+b(s)|x|^p)ds \\
&+ |\phi_1(t_1)\phi_2^\theta(t_1) - \phi_1(t_2)\phi_2^\theta(t_2)| \int_T^\infty \phi_2(s)m(s)(a(s)+b(s)|x|^p)ds.
\end{aligned}$$

Then, we know that

$$\int_T^\infty \phi_2(s)a(s)m(s)ds < \infty. \quad \int_T^\infty \phi_2^{1-\mu p}(s)b(s)m(s)ds < \infty.$$

So for any $\epsilon > 0$, there exists $\delta > 0$, such that for any $t_1, t_2 \in [0, T] : t_1 < t_2$ and $|t_1 - t_2| < \delta$,
$|\phi_2^\theta(t_1)Tx(t_1) - \phi_2^\theta(t_2)Tx(t_2)| < \epsilon, \forall x \in D$. Since T is arbitrary, the functions belonging to $\{(Tx(t)\phi_2^\theta(t))\} : x \in D$ are locally equicontinuous on $[0, \infty]$.

(c) We show that the functions belonging to $\{Y : y(t) = \phi_2^\theta(t)x(t), x \subseteq M\}$ are equiconvergent.
Let

$$\sigma := \frac{\theta - 1}{2}.$$

Then $\sigma > 0$. Since $\lim_{t\to\infty} \phi_2(t) = 0$, we have that for all $\epsilon > 0$, there exists $T > 0$, such that

$$|\phi_2(t) - 0| < (\frac{h\epsilon}{H(M_1 + M_2M^p)})^{\frac{1}{\sigma}}, \ \forall t \in (T, \infty).$$

Thus, it follows from the above $\epsilon > 0$, there exists $T > 0$, such that $x \in D$ and $t \ge T$ imply

$$\begin{aligned}
0 &\le \phi_2^\theta(t)Tx(t) \\
&= \int_0^\infty \phi_2^\theta(t)G(t,s)m(s)f(s,x(s)+M\phi_2^\mu(s))ds \\
&= \phi_2^\sigma(t)\int_0^\infty \phi_2^{1+\sigma}(t)G(t,s)m(s)f(s,x(s)+M\phi_2^\mu(s))ds \\
&\le \phi_2^\sigma(t)\frac{H}{h}\int_0^\infty \phi_2(s)G(s,s)m(s)(a(s)+b(s)|x(s)|^p)ds \\
&\le \phi_2^\sigma(t)\frac{H}{h}(\int_0^\infty \phi_2(s)G(s,s)a(s)m(s)ds + \|x\|\int_0^\infty \phi_2(s)^{1-\mu p}G(s,s)b(s)m(s)ds) \\
&\le \phi_2^\sigma(t)\frac{H}{h}(M_1 + M_2\|x\|^p) \\
&\le \phi_2^\sigma(t)\frac{H}{h}(M_1 + M_2M^p) \\
&< \epsilon.
\end{aligned}$$

This is, the functions belonging to $\{Y : y(t) = \phi_2^{\theta}(t)x(t), x \subseteq M\}$ are equiconvergent. $\square$

The method of upper and lower solutions, involves truncation and penalization techniques. For this purpose we introduce the following functions. Choosing $M_0 > \max\{M_1, \|\alpha^{'}\|_{\infty}, \|\beta^{'}\|_{\infty}\|$, we define the truncation map $u : T \times R \times R \to R^2$ as follows:

$$u(t,x,y) = \begin{cases} (\beta(t), \beta^{'}(t)), \ if \ x(t) < \beta(t), \\ (\alpha(t), \alpha^{'}(t)), \ if \ x(t) > \alpha(t), \\ (x, M_0), \quad if \ \beta(t) \leq x(t) \leq \alpha(t), \ y > M_0, \\ (x, -M_0), \quad if \ \beta(t) \leq x(t) \leq \alpha(t), \ y < -M_0, \\ (x,y), \quad otherwise. \end{cases}$$

Given two continuous functions α and β such that $\alpha \leq \beta$, we define the truncated function $\widetilde{f}$ by

$$\widetilde{f}(t,x,y) = \begin{cases} f_R(t, \beta(t), y) + \frac{\beta(t)-x}{1+|\beta(t)-x|}, \beta(t) < x, \\ f_R(t,x,y), \alpha(t) \leq x \leq \beta(t), \\ f_R(t, \alpha(t), y) + \frac{x-\alpha(t)}{1+|\alpha(t)-x|}, x < \alpha(t). \end{cases}$$

Where

$$f_R(t,x,y) = \begin{cases} f(t,x,-R), y < -R, \\ f(t,x,y), |y| \leq R, \\ f(t,x,R), y > R. \end{cases}$$

and the real number R is such that $R > \max\{|\alpha_1|, |\beta_1|\}$.
Finally, consider the modified problem

$$x^{''}(t) - k^2(t)x(t) + q(t)\widetilde{f}(t, x(t), x^{'}(t)) = 0, t \in (0, \infty) \tag{3.8}$$

$$x(0) = 0, x(+\infty) = 0.$$

Chapter 4

Existence and Uniqueness of solution

4.1 A priori Estimates

Proposition 4.1.1. *[2] Assume that either* (H_1) *and* (H_4) *or* (H_1) *and* $(H_4)'$ *hold. Then all possible solutions of the problem* (3.8) *satisfy*

$$\alpha(t) \leq x(t) \leq \beta(t), \forall t \in I$$

Proof: We prove that $x(t) \leq \beta(t), \forall t \in I$. Suppose, on the contrary that $\sup_{t\in[0,\infty]}(x-\beta)(t) > 0$. since $(x-\beta)(+\infty) = -\beta(+\infty) \leq 0$ and $(x-\beta)(0) = -\beta(0) \leq 0$, then there exists $t_0 \in (0,\infty)$ such that $x(t_0) - \beta(t_0) = \sup(x-\beta)(t_0) > 0$: hence $(x^{''} - \beta^{''})(t_0) \leq 0$ and $x^{'}(t_0) - \beta^{'}(t_0) = 0$. Moreover, by the definition of an upper solution, we have the successive estimates:

$$\begin{aligned}(x^{''} - \beta^{''})(t_0) &= k^2(t_0)x(t_0) - q(t_0)\widetilde{f}(t_0, x(t_0), x^{'}(t_0)) - \beta^{''}(t_0)\\ &\geq k^2(t_0)x(t_0) - q(t_0)\tilde{f}(t_0, x(t_0), x^{'}(t_0)) - k^2(t_0)\beta(t_0) + q(t_0)f(t_0, \beta(t_0), \beta^{'}(t_0))\end{aligned}$$

Hence

$$\begin{aligned}(x^{''} - \beta^{''})(t_0) &\geq k^2(t_0)(x-\beta)(t_0) - q(t_0)\tilde{f}(t_0, x(t_0), x^{'}(t_0)) + q(t_0)f(t_0, \beta(t_0), \beta^{'}(t_0))\\ &= k^2(t_0)(x-\beta)(t_0) - q(t_0)f_R(t_0, \beta(t_0), \beta^{'}(t_0)) - q(t_0)\frac{(\beta - x)(t_0)}{1 + |(\beta - x)(t_0)|}\\ &+ q(t_0)f(t_0, \beta(t_0), \beta^{'}(t_0))\\ &> -q(t_0)[f_R(t_0, \beta(t_0), \beta^{'}(t_0)) - f(t_0, \beta(t_0), \beta^{'}(t_0))].\end{aligned}$$

To check that the last right-hand term is nonnegative, we distinguish between two cases:

- (a) In cases (H_4) holds, consider the subcases:

  (a_1) $\beta^{'}(t_0) < -R$ implies that $|\beta_1| > R$ which does not hold true.

  (a_2) If $-R \leq \beta^{'}(t_0) \leq R$, then $f_R(t_0, \beta(t_0), \beta^{'}(t_0)) = f(t_0, \beta(t_0), \beta^{'}(t_0))$.

  (a_3) If $\beta^{'}(t_0) > R$, then $f_R(t_0, \beta(t_0), \beta^{'}(t_0)) = f(t_0, \beta(t_0), R) \leq f(t_0, \beta(t_0), \beta^{'}(t_0))$ follows from the first part of (H_4).

- (b) If $(H_4)'$ hold then $-R \leq \beta'(t_0) \leq R$. Consequently $f_R(t_0, \beta(t_0), \beta'(t_0)) = f(t_0, \beta(t_0), \beta'(t_0))$. Our claim is then proved leading to a contradiction.

Similarly, we can prove that $x(t) \geq \alpha(t), \forall t \in [0, \infty]$
Remark 4.1: [2] Assumption (H_4) is essential in proposition 4.1.1.

4.2 The Truncated Problem

Theorem 4.2.1. *[2] Under assumptions* (H_0), (H_1) *and* (H_3), *the truncated problem* (3.8) *has at least one solution in* x.

Proof:Since solving problem (3.8) amounts to proving existence of a fixed point for T, let us consider the operator $T : X \longrightarrow X$ defined by

$$(Tx)(t) = \int_0^\infty G(t,s)q(s)\tilde{f}(s, x(s), x'(s))ds \tag{4.1}$$

(a) $T : X \to X$ is well defined. Let $x \in X$.From $\int_0^\infty \psi(s)q(s)ds < \infty$ and $|f(t,x,y)| \leq \psi(t)h(y), \forall (t,x,y) \in D_\alpha^\beta \times R$,where $D_\alpha^\beta := \{(t,x) \in (0,\infty) \times R : \alpha(t) \leq x \leq \beta(t)\}$,we get

$$\begin{aligned}
(Tx)(t) &\leq \int_0^\infty G(t,s)q(s)\widetilde{f}(s, x(s), x'(s))ds \\
&\leq \int_0^\infty G(t,s)q(s)(\psi(s)h(x'(s)) + 1)ds \\
&\leq \frac{1}{2\underline{k}} \int_0^\infty q(s)(H_0\psi(s) + 1)ds \\
&< \infty,
\end{aligned}$$

Where $H_0 = H_0(x) = \max_{0 \leq t \leq ||x'||} h(t)$. From the monotonicity of ϕ_1 and ϕ_1' together with Lemma 3.1.7, we obtain that

$$\begin{aligned}
|(Tx)'(t)| &= |\int_0^\infty G(t,s)q(s)\widetilde{f}(s, x(s), x'(s))ds| \\
&= |\int_0^t \phi_1(s)\phi_2'(t)q(s)\widetilde{f}(s, x(s), x'(s))ds + \int_t^\infty \phi_1'(t)\phi_2(s)q(s)\widetilde{f}(s, x(s), x'(s))ds| \\
&\leq \int_0^t |\phi_1(t)\phi_2'(t)|q(s)|\widetilde{f}(s, x(s), x'(s))|ds + \int_t^\infty \phi_1'(s)\phi_2(s)q(s)|\widetilde{f}(s, x(s), x'(s))|ds \\
&\leq \int_0^t q(s)|\widetilde{f}(s, x(s), x'(s))|ds + \int_t^\infty q(s)|\widetilde{f}(s, x(s), x'(s))|ds.
\end{aligned}$$

Hence

$$\begin{aligned}
|(Tx)'(t)| &\leq \int_0^\infty q(s)|\widetilde{f}(s, x(s), x'(s))|ds \\
&\leq \int_0^\infty q(s)(H_0\psi(s) + 1)ds \\
&< \infty.
\end{aligned}$$

Lemma 3.1.5 implies that $\lim_{t\to\infty} Tx(t) = 0$. Moreover

$$\lim_{t\to\infty}(Tx)^{'}(t) = \lim_{t\to\infty}[\int_0^t \phi_1(s)\phi_2^{'}(t)q(s)\widetilde{f}(s,x(s),x^{'}(s))ds + \int_t^\infty \phi_1^{'}(t)\phi_2(s)q(s)\widetilde{f}(s,x(s),x^{'}(s))ds].$$

For $s \le t$, we have

$$\begin{aligned}\lim_{t\to\infty}\phi_1(s)\phi_2^{'}(t) &= \lim_{t\to\infty}\phi_1(s)\phi_2(t)\frac{\phi_2^{'}(t)}{\phi_2(t)}\\ &= \lim_{t\to\infty}G(t,s)\lim_{t\to\infty}\frac{\phi_2^{'}(t)}{\phi_2(t)}\\ &= 0.\end{aligned}$$

Hence for any $\varepsilon > 0$, there exist $N > 0$ such that for $t \ge N$, we have

$$\phi_1(s)\phi_2^{'}(t) \le \frac{\varepsilon}{\int_0^\infty q(s)(H_0\psi(s)+1)ds} := \varepsilon$$

and then

$$\int_0^t \phi_1(s)\phi_2^{'}(t)q(s)\tilde{f}(s,x(s),x^{'}(s))ds \le \int_0^\infty q(s)(H_0\psi(s)+1)ds \le \varepsilon.$$

For $s \ge t$, we have

$$\phi_1^{'}(t)\phi_2(s) \le \phi_1^{'}(t)\phi_2(t) \le 1.$$

Therefore,

$$\lim_{t\to\infty}|\int_t^\infty \phi_1^{'}(t)\phi_2(s)q(s)\tilde{f}(s,x(s),x^{'}(s))ds| \le \lim_{t\to\infty}|\int_t^\infty q(s)(H_0\psi(s)+1)| = 0.$$

It follows that

$$\lim_{t\to\infty}(Tx)^{'}(t) = 0.$$

b) $T : X \longrightarrow X$ is continuous. Let $(x_n)_{n\in N}$ be a sequence converging to some limit x in X; then there exists $\gamma > 0$ such that $\|x\| \le \gamma$ and $\|x_n\| \le \gamma$.
Let $H_\gamma = \max_{0\le t\le\gamma} h(t)$. We have

$$\int_0^\infty q(s)|\tilde{f}(s,x_n(s),x_n^{'}(s)) - \tilde{f}(s,x(s),x^{'}(s)|ds \le 2\int_0^\infty q(s)(H_0\psi(s)+1)ds < \infty \quad (4.2)$$

and

$$\|Tx_n - Tx\| = \max\{\sup_{t\in[0,\infty)} |\int_0^\infty G(t,s)q(s)[\tilde{f}(s,x_n(s),x_n^{'}(s)) - \tilde{f}(s,x(s),x^{'}(s))]ds|,$$

$$\sup_{t\in[0,\infty)} |\int_0^\infty G_t(t,s)q(s)[\tilde{f}(s,x_n(s),x_n^{'}(s)) - \tilde{f}(s,x(s),x^{'}(s))]ds|\}$$

$$\leq \max\{\frac{1}{2\underline{k}}\int_0^\infty q(s)[\tilde{f}(s,x_n(s),x_n^{'}(s)) - \tilde{f}(s,x(s),x^{'}(s))]|ds,$$

$$\sup_{t\in[0,\infty)} [\int_0^t |\phi_1(t)\phi_2^{'}(t)|q(s)|\tilde{f}(s,x_n(s),x_n^{'}(s)) - \tilde{f}(s,x(s),x^{'}(s))|ds$$

$$+ \int_t^\infty \phi_1^{'}(s)\phi_2(s)q(s)|\tilde{f}(s,x_n(s),x_n^{'}(s)) - \tilde{f}(s,x(s),x^{'}(s))|ds]\}$$

$$\leq \max\{\frac{1}{2\underline{k}}\int_0^\infty q(s)|\tilde{f}(s,x_n(s),x_n^{'}(s)) - \tilde{f}(s,x(s),x^{'}(s))|ds,$$

$$\int_0^\infty q(s|\tilde{f}(s,x_n(s),x_n^{'}(s)) - \tilde{f}(s,x(s),x^{'}(s))|ds\}.$$

$$\leq \max\{1,\frac{1}{2\underline{k}}\}\int_0^\infty q(s)|\tilde{f}(s,x_n(s),x_n^{'}(s)) - \tilde{f}(s,x(s),x^{'}(s))|ds.$$

From continuity of f, (4.2) and the Lebesgue dominated convergence theorem, the last term goes to 0 as $n \longrightarrow \infty$.
c) $T : X \longrightarrow X$is compact. Let B be any bounded subset of X and let $x \in B$. Then there exists $\gamma > 0$ such that $\|x\| < \gamma$.
First, notice that as above we have

$$\|Tx\| \leq \max\{1,\frac{1}{2\underline{k}}\}\int_0^\infty q(s)(H_r\psi(s)+1)ds$$
$$< \infty.$$

Now, given $T > 0$ and t_0, $t_1 \in [0,T]$, we have the estimates

$$|(Tx)(t_0) - (Tx)(t_1)| \leq \int_0^\infty |G(t_0,s) - G(t_1,s)|q(s)(H_r\psi(s)+1)ds$$

$$\leq \int_0^T |G(t_0,s) - G(t_1,s)|q(s)(H_r\psi(s)+1)ds$$

$$+ \int_T^\infty |\phi_1(t_0)\phi_2(s) - \phi_1(t_1)\phi_2(s)|q(s)(H_r\psi(s)+1)ds$$

$$\leq \int_0^T |G(t_0,s) - G(t_1,s)|q(s)(H_r\psi(s)+1)ds$$

$$+ |\phi_1(t_0) - \phi_2(t_1)|\int_0^\infty q(s)(H_r\psi(s)+1)ds.$$

By(H_3), the continuity of the Green's function and the Lebsgue dominated convergence theorem, we get

$$\lim_{|t_1-t_0|\to 0}\int_0^T |G(t_0,s) - G(t_1,s)|q(s)(H_r\psi(s)+1)ds = 0.$$

In addition, from (H_3) we have$\int_0^\infty \psi(s)q(s)ds > \infty$ and the continuity of ψ_1 imply that

$$\lim_{|t_1-t_0|\to 0} |\phi_1(t_0)\phi_2(s) - \phi_1(t_1)\phi_2(s)| \int_0^\infty q(s)(H_r\psi(s)+1)ds = 0.$$

Hence the right-hand term goes to 0 as $|t_1 - t_0| \longrightarrow 0$. Moreover, for $t_0 \leq t_1$, the following estimates hold true.

$$\begin{aligned}
|(Tx)^{'}(t_0) - (Tx)^{'}(t_1)| &\leq \int_0^\infty |G_t(t_0,s) - G_t(t_1,s)|q(s)(H_r\psi(s)+1)ds \\
&= \int_0^{t_0} |G_t(t_0,s) - G_t(t_1,s)|q(s)(H_r\psi(s)+1)ds \\
&+ \int_{t_0}^{t_1} |G_t(t_0,s) - G_t(t_1,s)|q(s)(H_r\psi(s)+1)ds \\
&+ \int_{t_1}^{T} |G_t(t_0,s) - G_t(t_1,s)|q(s)(H_r\psi(s)+1)ds \\
&+ \int_{T}^{\infty} |G_t(t_0,s) - G_t(t_1,s)|q(s)(H_r\psi(s)+1)ds \\
&\leq \int_0^{t_0} \phi_1(s)|\phi_2^{'}(t_0) - \phi_2^{'}(t_1)|q(s)(H_r\psi(s)+1)ds \\
&+ \int_{t_0}^{t_1} |\phi_1^{'}(t_0)\phi_2(s) - \phi_{t_1}(s)\phi_2^{'}(t_1)|q(s)(H_r\psi(s)+1)ds \\
&+ \int_{t_1}^{T} \phi_2(s)|\phi_1^{'}(t_0) - \phi_1^{'}(t_1)|q(s)(H_r\psi(s)+1)ds \\
&+ \int_{T}^{\infty} \phi_2(s)|\phi_1^{'}(t_0) - \phi_1^{'}(t_1)|q(s)(H_r\psi(s)+1)ds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
|(Tx)^{'}(t_0) - (Tx)^{'}(t_1)| &\leq \phi_1(t_0)|\phi_2^{'}(t_0) - \phi_2^{'}(t_1)| \int_0^{t_0} q(s)(H_r\psi(s)+1)ds \\
&+ (|\phi_1^{'}(t_1)\phi_2(t_1)| + |\phi_1(t_1)\phi_2^{'}(t_1)|) \int_{t_0}^{t_1} q(s)(H_r\psi(s)+1)ds \\
&+ |\phi_1^{'}(t_0) - \phi_1^{'}(t_1)| \int_{t_1}^{T} q(s)(H_r\psi(s)+1)ds \\
&+ |\phi_1^{'}(t_0) - \phi_1^{'}(t_1)| \int_{T}^{\infty} q(s)(H_r\psi(s)+1)ds,
\end{aligned}$$

each of the four terms above tends to 0 as $|t_1 - t_0|$ tends to 0, proving that Tx is almost equicontinuous.

To prove equiconvergence, we first notice that $\lim_{t\to 0|\to\infty} Tx(t) = 0$. Moreover from $\lim_{t\to 0|\to\infty} \phi_2(t) = 0$ and $\int_0^\infty q(s)(H_r\psi(s)+1)ds < \infty$, for any $\varepsilon > 0$, there exists $N > 0$

such that for $t \geq N$, the following estimates hold true:

$$\begin{aligned}
0 &\leq \sup_{x \in B} |Tx(t) - 0| \\
&\leq \sup_{x \in B} \int_0^\infty G(t,s) q(s) |\tilde{f}(s, x(s), x^{'}(s))| ds \\
&\leq \int_0^t \phi_1(s)\phi_2(t) q(s)(H_r \psi(s) + 1) ds + \int_t^\infty \phi_1(t)\phi_2(s) q(s)(H_r \psi(s) + 1) ds \\
&= \int_0^N \phi_1(s)\phi_2(t) q(s)(H_r \psi(s) + 1) ds + \int_N^t \Phi_s(t)\Phi_2(t) q(s)(H_r \psi(s) + 1) ds \\
&+ \int_t^\infty \phi_1(t)\phi_2(s) q(s)(H_r \psi(s) + 1) ds \\
&\leq \int_0^N \phi_1(s)\phi_2(t) q(s)(H_r \psi(s) + 1) ds + \int_N^\infty \phi_1(t)\phi_2(t) q(s)(H_r \psi(s) + 1) ds \\
&+ \int_N^\infty \phi_1(t)\phi_2(t) q(s)(H_r \psi(s) + 1) ds \\
&\leq \phi_1(N)\phi_2(t) \int_0^N q(s)(H_r \psi(s) + 1) ds + M \int_N^\infty q(s)(H_r \psi(s) + 1) ds \\
&+ M \int_N^\infty q(s)(H_r \psi(s) + 1) ds
\end{aligned}$$

Hence

$$\sup_{x \in B} |Tx(t)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Furthermore, for any $\varepsilon > 0$, there exist $N > 0$ such that for $t \geq N$

$$\begin{aligned}
|\phi_1(s)\phi_2^{'}(t)| &= |G(t,s)(\frac{\phi_2^{'}(t)}{\phi_2(t)} + d - d)| \\
&\leq G(t,s)|\frac{\phi_2^{'}(t)}{\phi_2(t)} + d| + dG(t,s) \\
&\leq \frac{\varepsilon}{2\int_0^\infty (H_r \psi(s) + 1)} ds
\end{aligned}$$

and

$$\begin{aligned}
\int_t^\infty \phi_1^{'}(t)\phi_2(s) q(s)(H_r \psi(s) + 1) ds &\leq \int_t^\infty q(s)(H_r \psi(s) + 1) ds \\
&\leq \frac{\varepsilon}{2}.
\end{aligned}$$

As a consequence, for $t \geq N$, we obtain the estimates

$$\begin{aligned}
\sup_{x\in B}|(Tx)^{'}(t)-\lim_{t\to\infty}(Tx)^{'}(t)| &= \sup_{x\in B}|\int_0^{\infty} G_t(t,s)q(s)|\tilde{f}(s,x(s),x^{'}(s))ds| \\
&\leq \sup_{x\in B}[|\int_0^{t}\phi_1(s)\phi_2^{'}(t)q(s)\tilde{f}(s,x(s),x^{'}(s))ds| \\
&+|\int_t^{\infty}\phi_1^{'}(t)\phi_2(s)q(s)\tilde{f}(s,x(s),x^{'}(s))ds|] \\
&\leq \int_0^{t}|\phi 1(s)\phi_2^{'}(t)|q(s)(H_r\psi(s)+1)ds \\
&+\int_t^{\infty}\phi_1^{'}(t)\phi_2(s)q(s)(H_r\psi(s)+1)ds \\
&\leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}$$

Thus we have proved equiconvergence of T ending the proof that T is completely continuous. Finally, by the Leray-Schauder fixed point theorem, we deduce that T has at least a fixed point x, solution of problem (3.8).

4.3 The Original Problem

Theorem 4.3.1. *[2] Assume that either Assumptions $(H_0)-(H_4)$ or $(H_0)-(H_3)$ and $(H_4)^{'}$ hold. Then problem (1.1) has at least one solution x having the representation*

$$x(t)=\int_0^{\infty} G(t,s)q(t)f(s,x(s),x^{'}(s))ds$$

With

$$\alpha(t)\leq x(t)\leq \beta(t), t\in[0,\infty),$$

Where $G(t,s)$ is the Green's function defined in (3.7).

Proof. From $\int_0^{\infty}\frac{ds}{h(s)}=+\infty$, we can find two real numbers $R > max\{|\alpha_1|,|\beta_1|\}$ and $\eta>0$ such that

$$\int_{\eta}^{R}\frac{ds}{h(s)}\geq k_0\max\{\alpha_0,\beta_0\}+\int_0^{\infty}\psi(s)q(s)ds \tag{4.3}$$

and

$$\eta\geq \max\{\sup_{t\in[\gamma,\infty)}\frac{\beta(t)-\alpha(0)}{t},\sup_{t\in[\gamma,\infty)}\frac{\beta(0)-\alpha(t)}{t}\},$$

for some $\gamma>0$.
Note that $\alpha(t)\leq\alpha_0\phi_2(t)<\alpha_0$ and $\beta(t)\leq\beta_0\phi_2(t)<\beta_0$.
(a) By (theorem 4.2.1), problem (3.8) has at least one solution in x. In addition, (proposition 4.1.1) implies that any solution x of problem (3.8) satisfies the bounds

$$\alpha(t)\leq x(t)\leq\beta(t).$$

Hence $\widetilde{f}(t,x(t)x^{'}(t))=f_R(t,x(t)x^{'}(t)),\forall t\in(0,\infty)$.
(b) It remains to prove that $|x(t)|\leq R$, for every $t\in[0,\infty)$.

Case 1. Assume that $|x^{'}(t)| > \eta, \forall t \in [0, \infty)$ and that $x^{'}(t) > \eta, \forall t \in [0, \infty)$. Then for $t \geq \gamma$, we have

$$\begin{aligned}
\frac{\beta(t) - \alpha(0)}{t} &\geq \frac{x(t) - x(0)}{t} \\
&= \frac{1}{t}\int_0^t x^{'}(s)ds \\
&> \eta \\
&\geq \frac{\beta(t) - \alpha(0)}{t},
\end{aligned}$$

which is a contradiction. Hence there exists $t_0 \in [0, \infty)$ such that $|x^{'}(t0)| \leq \eta$.
Case 2. If $|x^{'}(t)| \leq \eta, \forall t \in [0, \infty)$, then one may take $R = max\{|\alpha_1|, |\beta_1|, \eta\}$.
Case 3. There exists an interval $[t_0, t_1] \subset [0, \infty)$ such that either

$$|x^{'}(t_0)| = \eta, and, x^{'}(t) > \eta, \forall t \in (t_0, t_1]$$

or

$$|x^{'}(t_1)| = \eta, and, x^{'}(t) > \eta, \forall t \in [t_0, t_1).$$

For the sake of brevity, we only consider the first case. Using the fact that $|x(t)| \leq \max(\frac{|\alpha(t)|}{\phi_2(t)}, \frac{|\beta(t)|}{\phi_2(t)})\phi_2(t)$, we get

$$\begin{aligned}
\int_{x^{'}(t_0)}^{x^{'}(t_1)} \frac{ds}{h(s)} &= \int_{t_0}^{t1} \frac{x^{''}(s)}{h(x^{'}(s))}ds \\
&= \int_{t_0}^{t1} \frac{k^2(s)x(s) - q(s)f_R(s, x(s), x^{'}(s))}{h(x^{'}(s))}ds \\
&\leq \int_{t_0}^{t1} \frac{k^2(s)|x(s)| + q(s)\psi(s)h(x^{'}(s))}{h(x^{'}(s))}ds \\
&\leq \int_{t_0}^{t1} \frac{h(x^{'}(s))(k^2(s)|x(s)| + q(s)\psi(s))}{h(x^{'}(s))}ds \\
&\leq \int_{t_0}^{t1} (k^2(s)|x(s)| + q(s)\psi(s))ds \\
&\leq \int_{t_0}^{t1} k^2(s)|x(s)|ds + \int_{t_0}^{t1} q(s)\psi(s)ds \\
&\leq \max\{\sup_{t\in[0,\infty)} (|\beta(t)|\phi_2^{-1}(t)), \sup_{t\in[0,\infty)} (|\alpha(t)|\phi_2^{-1}(t))\} \\
&\quad \int_{t_0}^{t1} k^2(s)\phi_2(s)ds + \int_{t_0}^{t1} q(s)\psi(s)ds.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{x^{'}(t_0)}^{x^{'}(t_1)} \frac{ds}{h(s)} &\leq k_0 \max(\beta_0, \alpha_0) + \int_0^{\infty} q(s)\psi(s)ds \\
&\leq \int_{\eta}^{R} \frac{ds}{h(s)}.
\end{aligned}$$

Then $x'(t_1) \leq R$. Since t_0 and t_1 are arbitrary, we obtain that if $x'(t) \geq \eta$, then $x'(t) \leq R$, $t \in [0, \infty)$ yielding that $f_R(t, x(t), x'(t)) = f(t, x(t), x'(t))$. This means that x is a solution of problem (1.1), which completes the proof of the theorem.

Remark 4.3.1. *[2] The condition $h(s) \geq 1$ in (H_3) is not essential; in fact it is sufficient to suppose $h(s) \geq h_0$ for some $h_0 > 0$. Indeed, in this case $\frac{h(s)}{h_0} \geq 1$ and then we have to write in the above estimates:*

$$\begin{aligned}
\int_{x'(t_0)}^{x'(t_1)} \frac{ds}{h(s)} &\leq \int_{t_0}^{t1} \frac{k^2(s)|x(s)| + q(s)\psi(s)h(x'(s))}{h(x'(s))} ds \\
&\leq \int_{t_0}^{t1} \frac{h(x'(s))(\frac{1}{h_0}k^2(s)|x(s)| + q(s)\psi(s))}{h(x'(s))} ds \\
&\leq \int_{t_0}^{t1} (\frac{1}{h_0}k^2(s)|x(s)| + q(s)\psi(s)) ds \\
&\leq \int_{t_0}^{t1} \frac{1}{h_0}k^2(s)|x(s)| ds + \int_{t_0}^{t1} q(s)\psi(s) ds \\
&\leq \frac{1}{h_0}k_0 \max(\beta_0, \alpha_0) + \int_0^\infty q(s)\psi(s) ds.
\end{aligned}$$

So we have just to modify (4.3) by

$$\int_\eta^R \frac{ds}{h(s)} > \frac{1}{h_0}k_0 \max(\beta_0, \alpha_0) + \int_0^\infty q(s)\psi(s) ds.$$

Our second existence result is

Theorem 4.3.2. *[2] Assume that all conditions of Theorem 4.3.1 are satisfied but (H_2) replaced by*
$(H_2)'$ $k_1 := \int_0^\infty k^2(t) \max(|\alpha(t)|, |\beta(t)|) dt < \infty$. Then problem (1.1) has at least one solution x having the representation

$$x(t) = \int_0^\infty G(t, s)q(t)f(s, x(s), x'(s)) ds$$

and such that $\alpha(t) \leq x(t) \leq \beta(t), t \in [0, \infty)$.

Proof. From $\int_0^\infty \frac{ds}{h(s)} = +\infty$, we can find real numbers $\widetilde{R} > \max\{|\alpha_1|, |\beta_1|\}$ and $\eta > 0$ such that

$$\int_\eta^{\widetilde{R}} \frac{ds}{h(s)} \geq k_1 + \int_0^\infty q(s)\psi(s) ds. \tag{4.4}$$

Then the proof runs parallel to the proof of Theorem 4.2.1 with R replaced by $\widetilde{R}$. However, in Case 3 of the proof of Theorem 4.2.1, we have the following estimates instead:

$$
\begin{aligned}
\int_{x'(t_0)}^{x'(t_1)} \frac{ds}{h(s)} &= \int_{t_0}^{t1} \frac{x''(s)}{h(x'(s))} ds \\
&= \int_{t_0}^{t1} \frac{k^2(s)x(s) - q(s)f_R(s, x(s), x'(s))}{h(x'(s))} ds \\
&\leq \int_{t_0}^{t1} \frac{k^2(s)|x(s)| + q(s)\psi(s)h(x'(s))}{h(x'(s))} ds \\
&\leq \int_{t_0}^{t1} \frac{h(x'(s))(k^2(s)|x(s)| + q(s)\psi(s))}{h(x'(s))} ds \\
&\leq \int_{t_0}^{t1} (k^2(s)|x(s)| + q(s)\psi(s)) ds \\
&\leq \int_{t_0}^{t1} k^2(s)|x(s)| ds + \int_{t_0}^{t1} q(s)\psi(s) ds \\
&\leq k_1 + \int_0^{\infty} q(s)\psi(s) ds \\
&\leq \int_{\eta}^{\widetilde{R}} \frac{ds}{h(s)}.
\end{aligned}
$$

Finally, we complete the proof using (4.4). □

Remark 4.3.2. *[2] Contrarily to* (H_2)*, assumption* $(H_2)'$ *allows the upper and lower solutions to be unbounded.*

4.4 Uniqueness of solution

The following result complements Theorems 4.3.1 and 4.3.2

Theorem 4.4.1. *[2] Assume that* $f = f(t, x, y)$ *is continuously differentiable in* x *and* y *for each* $t \geq 0$ *and satisfies either the conditions of Theorem 4.3.1 or Theorem 4.2.1 together with*
(H_5) $f(t, x, y)$ *is nonincreasing in* x *for each* t *and* y *fixed.*
Then problem (1.1) has a unique solution x *such that*

$$\alpha(t) \leq x(t) \leq \beta(t), \forall t \geq 0.$$

Proof. Suppose there exist two distinct solutions x_1, x_2 of problem (1.1) and let $z := x_1 - x_2$. By the mean value theorem, there exist θ, φ such that

$$f(t, x_2, x_2') = f(t, x_1, x_1') - z\frac{\partial f}{\partial x}(t, \theta, \varphi) - z'\frac{\partial f}{\partial y}(t, \theta, \varphi).$$

Assume that $z(t_1) > 0$ for some t_1 and that z has a positive maximum at some $t_0 < \infty$.

Then, with (H_5), we have

$$\begin{aligned}
0 &\geq z''(t_0) \\
&= k^2(t_0)z(t_0) + q(t_0)[f(t_0, x_2(t_0), x_2'(t_0)) - f(t_0, x_1(t_0), x_1'(t_0))] \\
&= k^2(t_0)z(t_0) - q(t_0)\frac{\partial f}{\partial x}(t_0, \theta, \varphi)z(t_0) - q(t_0)\frac{\partial f}{\partial y}(t_0, \theta, \varphi)z'(t_0) \\
&= z(t_0)[k^2(t_0) - q(t_0)\frac{\partial f}{\partial x}(t_0, \theta, \varphi)] \\
&> 0.
\end{aligned}$$

Leading to a contradiction. Hence $\sup z(t) = \lim_{t\to\infty} z(t)$. But $\lim_{t\to\infty} z(t) = \lim_{t\to\infty}[x_1(t) - x_2(t)] = 0$, which is again a contradiction, ending the proof of the theorem. □

□

4.5 Illustrative Example

Example [2] Consider the boundary value problem

$$x''(t) - k^2(t)x(t) + q(t)[x(t)(x')^\theta(t) + c(t)] = 0, t > 0 \tag{4.5}$$
$$x(0) = 0, x(+\infty) = 0,$$

Where $\theta = \frac{1}{2p+1}$ and p is a positive integer. The positive functions $c = c(t)$ and $q = q(t)$ satisfy $q(t), \phi_2(t)q(t) \in C(0, +\infty) \bigcap L^1(0, \infty)$ and $0 \leq c(t) \leq -\phi_2(t)(\phi_2')^\theta(t)$. The function k verifies (H_0).
Then $\alpha(t) \equiv 0$ and $\beta(t) = \phi_2(t)$ are respectively lower solution and upper solution with $\alpha \leq \beta$. Moreover
$\alpha_0 = \sup_{t\in[0,\infty)}\{|\alpha(t)|\phi_2^{-1}(t)\} = 0$ and $\beta_0 = \sup_{t\in[0,\infty)}\{|\beta(t)|\phi_2^{-1}(t)\} = 1$.
Then (H_1) and (H_2) are satisfied. As for (H_3), one may take $\psi(t) = \phi_2(t)$ and $h(y) = (|y| + 1 + \gamma) \geq 1$ with $\gamma := -(\phi_2')^\theta(0)$ so that, for $0 \leq x \leq \phi_2(t)$, we have

$$\begin{aligned}
|f(t, x, y)| &= |xy^\theta + c(t)| \\
&\leq \phi_2(t)|y|^\theta - \phi_2(t)(\phi_2')^\theta(t) \\
&\leq \phi_2(t)(|y|^\theta - (\phi_2')^\theta(0)) \\
&\leq \phi_2(t)(|y| + 1 + \gamma).
\end{aligned}$$

as well as

$$\int_0^\infty \psi(s)q(s)ds < \infty,$$

and

$$\begin{aligned}
\int_0^\infty \frac{ds}{h(s)} &= \int_0^\infty \frac{ds}{s + 1 + \gamma} \\
&= +\infty.
\end{aligned}$$

Regarding (H_4),we have

$$\begin{cases} \alpha_1 := \sup_{t\in R^+} \alpha^{'}(t) = 0 \\ \beta_1 := \inf_{t\in R^+} \phi_2^{'}(t) = \phi_2^{'}(0) = -\int_0^\infty \phi_2^{''}(s)ds = -\int_0^\infty k^2(s)\phi_2(s)ds = -k_0. \end{cases}$$

In addition,for any $y \in R$ and $t \in (0, \infty)$,we have

$$\begin{cases} y < \beta^{'}(t) \Rightarrow f(t, \beta(t), y) = \phi_2(t)y^\theta + c(t) \le \phi_2(t)(\phi_2)^{'\theta}(t) + c(t) = f(t, \beta(t), \beta^{'}(t)) \\ y > \alpha^{'}(t) \Rightarrow y > 0 \Rightarrow f(t, \alpha(t), y) = c(t) = f(t, \alpha(t), \alpha^{'}(t)). \end{cases}$$

Therefore,theorem 4.3.1 yields that problem (5.1) has atleast one solution x such that

$$0 \le x(t) \le \phi_2(t), \forall t \ge 0.$$

Summary

In this work, we have discussed some existence results and also a uniqueness theorem for problem (1.1). This problem has the particularity that the differential operator is time depending.
The existence of an upper solution and a lower solution to a boundary value problem implies the existence of solutions lying in between the upper and lower solutions. When in each problem,we have developed the upper and lower solution method on infinite intervals of the positive half line together with the Leray-Schauder fixed point theorem to prove the existence of problem (1.1).

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