SPIN-SUM TO DIRAC SOLUTIONS IN A VERY STRONG MAGNETIC FIELD

By

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SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE IN PHYSICS
AT
ADDIS ABABA UNIVERSITY
ADDIS ABABA, ETHIOPIA
JUNE 2009

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ADDIS ABABA UNIVERSITY

Date: **June 2009**

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Title: ***Spin-sum to Dirac Solutions in a Very Strong Magnetic Field***

Department: **Physics**

Degree: **M.Sc.** Convocation: **June** Year: **2009**

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Abstract

Neutron stars are the collapsed cores of some massive stars. These compact objects (NS) have incredibly high densities and the strong magnetic fields (Pulsars have magnetic field $\geq 10^{12}G$,Magnetars have magnetic field $\sim 10^{15}G$)[1]. Phenomena like this trigger people to look for the solutions of Dirac equation in strong magnetic fields. In this thesis we discuss the properties of the solutions of the Dirac equation in presence of a uniform background magnetic field. The nature of the solutions, their ortho-normality properties, dependence of these solutions on the choice of the vector potential giving rise to the magnetic field and explicit calculation the spin-sum of the solutions are areas of emphasis.

Acknowledgements

If I use any of my body parts or spirit just to write a phrase to express my deepest gratitude I should certainly write '***Thank You God*** !!!' first . ***Thank You God***!!!
I am utterly grateful to my dearest adviser Dr.Legesse Wetro , whose constructive assistance and encouragement paved the way for the betterment of my Thesis.
My Special thank goes to the best family over the whole wide world,My family, I love you all. Merki you are one of them .
I am also very thankful to those friends of mine who really like me. I do not feel like stoping with out expressing my gratitude to Addis Ababa university, Physics Department. I owe you one .

Addis Ababa , Ethiopia *Bililign Tsige*

DEDICATED TO

MY BELOVED FAMILY

Introduction

There are three different final stages that a star will enter based on its initial mass. A star will either become a white dwarf, a neutron star or a black hole. If a star starts out with right around the size of the Sun (and up to $1.5M_{\odot}$) it will wind up being a white dwarf; a star with up to $2.5M_{\odot}$ is destined to become a neutron star, and any star larger than $2.5M_{\odot}$ is going to end up as black hole . One of these compact objects of great physical interest is the neutron star which has incredibly high $density\ of\ \sim 10^{14}g/c.c$, $mass \cong 1.4M_{\odot}$ and $radius\ \sim 10km$. Pulsars have more wonders like very high $spin\ rate \sim 36,000rpm$ and very strong $magneticfield > 10^{12}G$. The cores of the NS can sustain a magnetic field to the order of $10^{12}G$ or more[2]. Therefore the processes of elementary particle decay and scattering cross-sections in the presence of a back ground magnetic field are very important since they shade information on the internal structures of such stars. The realistic back ground fields could be very complicated in their structure but for certain regions such as poles,we treat them as uniform. While making the fields uniform we are taking the advantage of the fact that under this assumption the Dirac equation is exactly solvable. Once the Dirac equation is exactly solved then we can use those solutions and calculate the elementary particle decay and the scattering cross-section. Finding the exact solutions of the Dirac equation in background magnetic fields and explicit calculation of the spin- sum of this solutions are the main objectives of this thesis. Hence first we begin with the introduction of the Dirac equation then we will find the solutions of this equation in free space. Discussion on the nature of this free space solutions that is the spin-sum and the ortho-normality of the solutions is also included. We next introduce the background field into the calculations and solve the Dirac equation exactly. The ortho-normality property of the spinors will be checked and the spin-sum correction to the solution will be included. Since the spinor solutions in the presence of the magnetic field depend on the choice of the vector potential, all the results we obtain using these solutions are not gauge invariant.

Chapter 1

The Dirac Equation

1.1 Introduction

Although the general principles of the non-relativistic quantum mechanics are valid and true under all circumstances, even when the system happens to be relativistic, there are shortcomings of the theory:

(I)The non-relativistic(NR) quantum mechanics theory is not consistent with special theory of relativity and hence does not treat space and time on equal footing

(II) The explanation given to the spin of electron does not come from the very nature of the theory but rather put by hand

(III) Cannot explain fine structures in atomic spectrum of various atoms

(IV) It cannot explain the existence of anti-particle

(V) It is a single particle theory and hence does not support many particle interactions

The non-relativistic Schrödinger equation is :

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + V(x)\right)\psi(x,t) \tag{1.1.1}$$

where V(x) is the potential energy and and m is the mass of the particle. Hence the Schrödinger equation , which has NR hamiltonian $H = \frac{p^2}{2m} + V(x)$, displays unsymmetrical footing in the temporal and spatial components. On the quest for finding relativistically consistent quantum mechanical equations there have been few attempts to replace the Schrödinger equation. The first attempt was due to Schrödinger(1926),Gordon(1926) and Klein(1927) which is now known to us as Klein-Gordon equation[3]. The Klein-Gordon equation is :

$$-\hbar^2 \frac{\partial^2 \psi(x,t)}{\partial t^2} = \left(-\hbar^2 c^2 \nabla^2 + m^2 c^4\right) \psi(x,t) \tag{1.1.2}$$

where c is the speed of light. The above scalar wave equation displays symmetrical footing in both spatial and temporal components. Though it is relativistically consistent, the Klein-Gordon equation has been initially abandoned for the reasons that it failed to explain the negative energy solutions and it led to negative probability density :

$$\rho = \frac{i\hbar}{2mc^2} \left(\psi^\star \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^\star}{\partial t}\right) \tag{1.1.3}$$

Then latter by reinterpreting Klein-Gordon equation as a basis for quantum field theory (many particle theory), Pauli and Weisskopf showed that this could describe spin zero particles (π mesons)[3,4]. A successful relativistic wave equation for a particle such as an electron was first obtained by the British physicist P. A. M. Dirac in 1928[4]. In the same way as the Schrödinger equation cannot be derived from classical mechanics because it is essentially new physics, any relativistic equation can only be guessed at by a process of induction, and its truth or otherwise must be established by testing its predictions against experiment. Paul A. M.Dirac is known to us through the Dirac equation for spin-1/2 particles. But his main interest was in foundational problems. First, Dirac was never satisfied with the probabilistic formulation of quantum mechanics. Second, if we tentatively accept the present form of quantum mechanics, he was insisting that it has to be consistent with special relativity. He wrote several important papers on this subject. During World War II, he was looking into constructing representations of the Lorentz group using harmonic oscillator wave functions. The Lorentz group is the language of special relativity, and the present form of quantum mechanics starts with harmonic oscillators. Presumably, therefore, he was interested in making quantum mechanics Lorentz-covariant by constructing representations of the Lorentz group using harmonic wave functions [5].This newly evolved equation is:

$$i\hbar\frac{\partial\psi}{\partial t} = (-i\hbar c\alpha.\nabla + \beta mc^2)\psi \tag{1.1.4}$$

where α, β are matrices that cannot simply be numbers, if they were, the equation would not even be form invariant (having the same coefficients)with respect to spatial rotations. P.A.M Dirac is successful because

(I) The Dirac equation is consistent with special theory of relativity

(II) The equation is linear in time derivatives, predicts positive probability density:

$$\rho = \psi^\star\psi \tag{1.1.5}$$

(III) The theory gave successful interpretation of the negative energy solutions which leads to considerable triumph of relativistic quantum mechanics by forecasting and describing anti-particles. The energy eigenvalue solution of the Dirac equation in free space is

$$E = \pm\sqrt{p^2c^2 + m^2c^4}$$

The above equation implies that an electron can be both in positive and negative energy states. An electron is the most stable elementary particles which is observed naturally to exist in the positive energy state. We can apply some perturbation to the electron in the positive energy state and make it fall dawn to the negative energy state. Its stability will be in doubt if it falls to the negative energy states. Hence Dirac postulated :

i) The negative energy states are completely filled in accordance with the Pauli exclusion principle by negative sea of electrons, which completely rule out the possibility of transition of electrons from the positive to the negative energy state. This postulate guaranty the stability of the positive energy state.

ii) Completely filled negative sea of electrons have no observable effect. However deviations from the norm are observable. That is, a hole manifests itself as particle with positive charge, the positron that is, the anti-particle of electron. In general we can have pair creation or the reverse process the pair annihilation. Since these processes conserve energy but not momentum we do need a nucleus involved during the reaction.

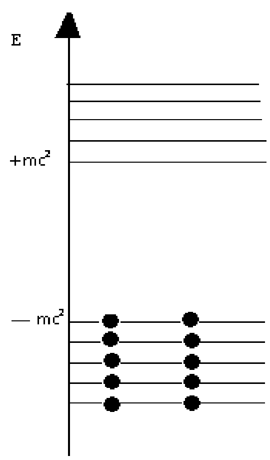

Figure 1.1: *The negative energy state with $E < mc^2$ are occupied by electrons and form the 'Dirac sea' and their effects are unobservable where as real i.e.observable electrons in general exist in state of positive energy*

1.2 The Solutions of Dirac Equation in Free Space and Their Properties

On the basis of the special theory of relativity the energy relation for a particle of rest mass m is

$$E^2 = \boldsymbol{p}^2 c^2 + m^2 c^2 \tag{1.2.1}$$

where c is the speed of light and $\boldsymbol{p}$ is the three momentum . The Schrödinger form of Dirac equation reads

$$i\frac{\partial \psi}{\partial t} = \mathcal{H}\psi \tag{1.2.2}$$

where $\mathcal{H}$ is the free Hamiltonian of an electron with mass m given by

$$\mathcal{H} = -i\alpha.\nabla + \beta m \tag{1.2.3}$$

Throughout this work including the above two equations we will make use of the natural units with $c = \hbar = 1$

N.B In this paper when ever there is a bold typeface vector then it is a 3 vector else a 4 vector.The Greek letters like μ, ν=0,1,2,3 while i,j=1,2,3

The dot product of any two arbitrary 4 vectors $A = (A_o, \boldsymbol{A})andB = (B_o, \boldsymbol{B})$ is a scalar and Lorentz covariant given as $A.B = A_\mu B^\mu = A^\mu B_\mu = g_{\mu\nu} A^\mu B^\nu, \quad \mu = 0, 1, 2, 3$

Since α^i, β in Eq.(1.2.3)are 4x4 matrices then the eigenvector of(1.2.2) will be a column matrix. The hamiltonian $\mathcal{H}$ is linear in momentum operator and rest energy. In the Dirac-Pauli 2x2 representation

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{1.2.4}$$

where σ_i are the three Pauli matrices i=1,2,3 and **1** is a 2x2 unit matrix.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1.2.5)$$

From Eq.(1.2.2), Eq.(1.2.3) we derive the standard and relativistically consistent form of the free Dirac equation as :

$$[i\not{\partial} - m\mathbf{1}]\psi = 0 \quad (1.2.6)$$

$$\Rrightarrow \ [-\not{p} + m\mathbf{1}]\psi = 0$$

where $\not{p} = i\not{\partial} = i\gamma^\mu \partial_\mu$ and in 2×2 matrix notation

$$\gamma^o = \beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \gamma^i = \beta\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

Multiplying the standard form of the Dirac equation $[-\not{p} + m\mathbf{1}]\psi$ with $(\not{p} + m\mathbf{1})$ we will get

$$(-\not{p} + m\mathbf{1})(-\not{p} + m\mathbf{1})\psi = 0$$

$$(-\not{p}\not{p} + m^2)\psi = 0 \quad \text{but since}$$

$$\not{p}\not{p} = \gamma_\mu\gamma_\nu p^\mu p^\nu \quad \text{and} \quad \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$$

$$\not{p}\not{p} = \frac{1}{2}\{\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu\}p^\mu p^\nu$$

where $p^\mu = i\frac{\partial}{\partial x_\mu} \quad p_\mu = i\frac{\partial}{\partial x^\mu}$ and the Minkowiski metric tensor

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (1.2.7)$$

The p's commute for the simple reason that ordinary differentiations commute. Hence

$$\gamma_\mu\gamma_\nu p^\mu p^\nu = \gamma_\nu\gamma_\mu p^\mu p^\nu$$

$$\Rightarrow \not{p}\not{p} = g_{\mu\nu}p^{\mu}p^{\nu} = p_{\mu}p^{\mu} \quad ,$$

so that

$$(\Box^2 + m^2)\psi = 0 \tag{1.2.8}$$

where $\Box^2 = \frac{\partial^2}{c^2\partial t^2} - \nabla^2$

The above equation has exactly the form as the Klein-Gordon equation. Thus the Dirac equation will have free spinors given by

$$\psi(x) \sim u_s(\boldsymbol{p})e^{-ip.x} \tag{1.2.9}$$

for positive energy solutions $E_p = +\sqrt{\boldsymbol{p}^2 + m^2}$

and

$$\psi(x) \sim v_s(\boldsymbol{p})e^{ip.x} \tag{1.2.10}$$

for negative energy solutions $E_p = -\sqrt{\boldsymbol{p}^2 + m^2}$.

In the above solutions of the Dirac equation (Eq. (1.2.9) and Eq.(1.2.10))

$p.x = p_o x_o - \boldsymbol{p}.\boldsymbol{x}$ and u_s, v_s are the spin wave functions , which are four component column matrix and s is the spin index which can have two values $s = +, -$. Each spinors given above has two components each, indicating a degree of freedom which we need to define. The degree of freedom describe the Spin $\frac{1}{2}$ attribute. This attribute is the so called helicity. Defined as the component of particle spin along direction of motion or projection of spin on to the direction of momentum[6]. Mathematical representation of this quantity is

$\Sigma.\hat{\mathbf{p}} \equiv \frac{p_x\Sigma_x + p_y\Sigma_y + p_z\Sigma_z}{|\mathbf{p}|}$ where the spin operator

$\Sigma \equiv (\Sigma^1, \Sigma^2, \Sigma^3) \equiv (\sigma^{23}, \sigma^{31}, \sigma^{12})$ and the σ^{ij}s are defined as :

$$\sigma^{ij} = \frac{i}{2}\left[\gamma^i, \gamma^j\right] \tag{1.2.11}$$

Using the definition of the helicity, the commutation relation between helicity and the free Hamiltonian can be obtained :

$$[\mathcal{H}, \Sigma\cdot\mathbf{p}] = [-i\alpha\cdot\nabla + \beta m, \Sigma\cdot\mathbf{p}]$$

Since β is a diagonal matrix hence $[\beta, \Sigma] = 0$

$$\begin{aligned}
\Rightarrow [\mathcal{H}, \Sigma\cdot\mathbf{p}] &= [-i\alpha\cdot\nabla, \Sigma\cdot\mathbf{p}] \\
&= (\alpha\cdot\mathbf{p})(\Sigma\cdot\mathbf{p}) - (\Sigma\cdot\mathbf{p})(\alpha\cdot\mathbf{p}) \\
&= \begin{pmatrix} 0 & \sigma\cdot\mathbf{p} \\ \sigma\cdot\mathbf{p} & 0 \end{pmatrix}\begin{pmatrix} \sigma\cdot\mathbf{p} & 0 \\ 0 & \sigma\cdot\mathbf{p} \end{pmatrix} - \begin{pmatrix} \sigma\cdot\mathbf{p} & 0 \\ 0 & \sigma\cdot\mathbf{p} \end{pmatrix}\begin{pmatrix} 0 & \sigma\cdot\mathbf{p} \\ \sigma\cdot\mathbf{p} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & (\sigma\cdot\mathbf{p})^2 \\ (\sigma\cdot\mathbf{p})^2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & (\sigma\cdot\mathbf{p})^2 \\ (\sigma\cdot\mathbf{p})^2 & 0 \end{pmatrix} = 0
\end{aligned}$$

Hence

$$[\mathcal{H}, \Sigma\cdot\mathbf{p}] = 0 \tag{1.2.12}$$

Similarly we can obtain

$$[\mathbf{p}, \Sigma\cdot\mathbf{p}] = 0 \tag{1.2.13}$$

In general two observables A, B will have simultaneous eigenstate $|a, b>$ only if they are compatible, that is $[A, B] = 0$ then $A|a, b> = a|a, b>$, $B|a, b> = b|a, b>$
In place of A and B let's use Σ_z and $\mathcal{H}$respectively. Then

$$[\Sigma_z, \mathcal{H}] = [\Sigma_z, -i\alpha\cdot\nabla + \beta m] = [\Sigma_z, \alpha\cdot\mathbf{p} + \beta m] = [\Sigma_z, \alpha\cdot\mathbf{p}]$$

$$\text{Hence,} \qquad [\Sigma_z, \mathcal{H}] \neq 0 \tag{1.2.14}$$

Bonding the idea of simultaneous eigenstate with Eq.(1.2.14) one can infer that the spinors which are the eigenstates of the hamiltonian $\mathcal{H}$ will not be eigenstate of Σ_z unless $p_x, p_y = 0$. The Heisenberg's equation of motion or equation of motion for an operator A in Heisenberg picture is given as

$$\frac{dA}{dt} = \frac{1}{i\hbar}[A, H]$$

which reveals that the helicity is a constant of time or a conserved quantity.

The stationary sate solutions for the free Dirac spinors can very easily be obtained form Eq.(1.2.2) as :

$$(i\partial\!\!\!/ - m\mathbf{1})\psi = (i\gamma^o\partial_o - m\mathbf{1})\psi = 0$$

$$i\begin{pmatrix} \frac{\partial}{\partial t} & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial t} & 0 & 0 \\ 0 & 0 & -\frac{\partial}{\partial t} & 0 \\ 0 & 0 & 0 & -\frac{\partial}{\partial t} \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = m\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

which provides the four independent solutions or the spinors given in Eq.(1.2.9)and Eq.(1.2.10) for stationary state $U_r^s(0)$:

$$U_1^+(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, U_2^-(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, V_1^+(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, V_2^-(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.2.15}$$

The next objective is to find non-stationary solutions for the the free dirac equation using the Lorentz transformation, that is by transforming to a coordinate system which moves with velocity of -V relative to the rest system, the free wave function of the electron with velocity $+V$ are constructed from the spinors in Eq.(1.2.15) of the electron at rest[6]. The lorentz transformation is:

$$p'_\nu = \Lambda^\nu_\mu p^\mu = (\delta^\nu_\mu + \varepsilon^\nu_\mu)p^\mu$$

where Λ_μ^ν is the transformation matrix, ε_μ^ν is the infinitesimal transformation matrix and

$$\begin{aligned} \delta_\mu^\nu &= 1, \mu = \nu \\ &= 0 \ \text{ otherwise} \end{aligned}$$

The corresponding infinitesimal lorentz boost along z -axis will be $\varepsilon_\mu^\nu = \triangle\omega_z I_\mu^\nu$ where ω is the infinitesimal rotation angle and
$\sinh(w) = \gamma\mathbf{B}, \cosh(w) = \gamma, \mathbf{B} = V, \gamma = \frac{1}{\sqrt{1-\mathbf{B}^2}}$,

$$I_\mu^\nu = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

so that the boosted spinors(along Z direction) will be

$$\psi' = S\psi \tag{1.2.16}$$

where the 4×4 matrix S is given as :

$$\begin{aligned} S &= lim \left(1 - \frac{i(-\omega)}{4N}\sigma_{\alpha\beta}I^{\alpha\beta}\right)^N \\ & \quad N \to \infty \\ &= e^{\frac{i\omega}{4}\sigma_{\alpha\beta}I^{\alpha\beta}} = e^{\frac{i\omega}{4}[\sigma_{03}I^{03}+\sigma_{30}I^{30}]} = e^{\frac{i\omega}{4}[\sigma_{03}-\sigma_{30}]} \end{aligned}$$

From Eq.(1.2.11) $\quad \sigma_{\alpha\beta} = \frac{i}{2}[\gamma_\alpha, \gamma_\beta]$

Hence,

$$[\sigma_{03} - \sigma_{30}] = -2i \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$$

$$
\begin{aligned}
S &= e^{\left[\frac{i\omega}{4}(-2i)\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}\right]} \\
&= 1 + (\frac{\omega}{2})\frac{1}{1!}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} + (\frac{\omega}{2})^2\frac{1}{2!}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} + (\frac{\omega}{2})^3\frac{1}{3!}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} + - - - - - -
\end{aligned}
$$

The above 2×2 notation provides a 4×4 matrix

$$
S = \begin{pmatrix} \cosh(\frac{\omega}{2}) & 0 & \sinh(\frac{\omega}{2}) & 0 \\ 0 & \cosh(\frac{\omega}{2}) & 0 & -\sinh(\frac{\omega}{2}) \\ \sinh(\frac{\omega}{2}) & 0 & \cosh(\frac{\omega}{2}) & 0 \\ 0 & -\sinh(\frac{\omega}{2}) & 0 & \cosh(\frac{\omega}{2}) \end{pmatrix}
$$

While using the obtained value of the above S matrix in collaboration with Eq.(1.2.15) and Eq.(1.2.16) gives the boosted spinors $U^s_r(\mathbf{p}), \mathbf{p} = p_z\hat{z}$

$$
\text{for} \quad \cosh(\frac{w}{2}) = \sqrt{\frac{(E+m)}{2m}}, \ \sinh(\frac{w}{2}) = \frac{\mathbf{p}}{\sqrt{2m(E+m)}} \quad as:
$$

$$
\begin{aligned}
U_1^+(\mathbf{p}) &= \begin{pmatrix} \sqrt{\frac{(E+m)}{2m}} \\ 0 \\ \frac{\mathbf{p}}{\sqrt{2m(E+m)}} \\ 0 \end{pmatrix}, U_2^-(\mathbf{p}) = \begin{pmatrix} 0 \\ \sqrt{\frac{(E+m)}{2m}} \\ 0 \\ \frac{-\mathbf{p}}{\sqrt{2m(E+m)}} \end{pmatrix} \\
V_1^+(\mathbf{p}) &= \begin{pmatrix} \frac{\mathbf{p}}{\sqrt{2m(E+m)}} \\ 0 \\ \sqrt{\frac{(E+m)}{2m}} \\ 0 \end{pmatrix}, V_2^-(\mathbf{p}) = \begin{pmatrix} 0 \\ \frac{-\mathbf{p}}{\sqrt{2m(E+m)}} \\ 0 \\ \sqrt{\frac{(E+m)}{2m}} \end{pmatrix} \qquad (1.2.17)
\end{aligned}
$$

1.2.1 The Ortho-normality of the Free Solutions

From Eq.(1.2.17)

$$
\overline{U}_1(\boldsymbol{p})U_2(\boldsymbol{p}) = U_1^\dagger(\boldsymbol{p})\gamma^o U_2(\boldsymbol{p}) = \overline{V}_1(\boldsymbol{p})U_2(\boldsymbol{p}) = \overline{V}_1(\boldsymbol{p})V_2(\boldsymbol{p}) = 0
$$

and

$$\overline{U}_1(\boldsymbol{p})U_1(\boldsymbol{p}) = \overline{U}_2(\boldsymbol{p})U_2(\boldsymbol{p}) = 1, \overline{V}_1(\boldsymbol{p})V_1(\boldsymbol{p}) = \overline{V}_2(\boldsymbol{p})V_2(\boldsymbol{p}) = -1$$

this could be summarized as:

$$\overline{U}_r(\boldsymbol{p})U_{r'}(\boldsymbol{p}) = \varepsilon_r \delta_{rr'} \begin{cases} \varepsilon_r = 1, & \text{for r ,r'=1,2 (E>0)} \\ \varepsilon_r = -1, & \text{for r,r'=1,2 (E<0)} \end{cases}$$

and

$$U_r^\dagger(\boldsymbol{p})U_{r'}(\boldsymbol{p}) = \frac{E}{m}\delta_{rr'} = \gamma\delta_{rr'} = \begin{cases} \gamma & \text{for r=r'} \\ 0 & \text{otherwise} \end{cases}$$

where $\delta_{rr'} = 1$ for $r = r'$,and is zero other wise,$\gamma = \frac{1}{\sqrt{1-\mathrm{v}^2}}$

1.2.2 The Spin-sum of the Free Dirac Solutions

In this subsection we will find the spin-sum $\sum_s U^s(\mathbf{P})\overline{U}^s(\mathbf{p})$ for the free dirac solutions where s is the spin degrees of freedom. Hence for the positive energy solutions:

$$\begin{aligned} P_U(\boldsymbol{p}) &= \sum_s U^s(\boldsymbol{P})\overline{U}^s(\boldsymbol{p}) \\ &= U^+(\boldsymbol{P})\overline{U}^+(\boldsymbol{p}) + U^-(\boldsymbol{P})\overline{U}^-(\boldsymbol{p}) \end{aligned}$$

therefore

$$\begin{aligned} P_U(\boldsymbol{p}) &= \begin{pmatrix} \sqrt{\frac{(E+m)}{2m}} \\ 0 \\ \frac{\mathbf{p}}{\sqrt{2m(E+m)}} \\ 0 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{(E+m)}{2m}} & 0 & \frac{-\mathbf{p}}{\sqrt{2m(E+m)}} & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ \sqrt{\frac{(E+m)}{2m}} \\ 0 \\ \frac{-\mathbf{p}}{\sqrt{2m(E+m)}} \end{pmatrix} \begin{pmatrix} 0 & \sqrt{\frac{(E+m)}{2m}} & 0 & \frac{\mathbf{p}}{\sqrt{2m(E+m)}} \end{pmatrix} \end{aligned}$$

or

$$P_U(\boldsymbol{p}) \quad = \quad \frac{1}{2m}\begin{pmatrix} E+m & 0 & -\mathbf{p} & 0 \\ 0 & E+m & 0 & \mathbf{p} \\ \mathbf{p} & 0 & -E+m & 0 \\ 0 & -\mathbf{p} & 0 & -E+m \end{pmatrix}$$

In 2×2 matrix notation the above matrix may be rewritten as

$$P_U(\boldsymbol{p}) = \frac{1}{2m}\left[E\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} + \mathbf{p}\begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix} + m\begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}\right]$$

Hence in 4×4 matrix notation

$$P_U(\boldsymbol{p}) = \frac{1}{2m}\left[E\gamma_o + P\gamma_3 + m\right]$$

where $\mathbf{p} = \mathbf{p}^3, E = \mathbf{p}^o = \mathbf{p}_o$

$$P_U(\boldsymbol{p}) = \frac{1}{2m}\left[\gamma_o\mathbf{p}^o + \gamma_3\mathbf{p}^3 + m\right]$$

$$P_U(\boldsymbol{p}) \quad = \quad \frac{(\gamma_\mu p^\mu + m)}{2m} = \frac{(\not{p} + m)}{2m}. \tag{1.2.18}$$

Similarly for the negative energy solutions the spin-sum is:

$$P_V(\boldsymbol{p}) = V^+(\boldsymbol{P})\overline{V}^+(\boldsymbol{p}) + V^-(\boldsymbol{P})\overline{V}^-(\boldsymbol{p})$$

$$P_V(\boldsymbol{p}) \quad = \quad \begin{pmatrix} \frac{\mathbf{p}}{\sqrt{2m(E+m)}} \\ 0 \\ \sqrt{\frac{(E+m)}{2m}} \\ 0 \end{pmatrix}\begin{pmatrix} \frac{\mathbf{p}}{\sqrt{2m(E+m)}} & 0 & -\sqrt{\frac{(E+m)}{2m}} & 0 \end{pmatrix}$$

$$+ \quad \begin{pmatrix} 0 \\ \frac{-\mathbf{p}}{\sqrt{2m(E+m)}} \\ 0 \\ \sqrt{\frac{(E+m)}{2m}} \end{pmatrix}\begin{pmatrix} 0 & \frac{-\mathbf{p}}{\sqrt{2m(E+m)}} & 0 & -\sqrt{\frac{(E+m)}{2m}} \end{pmatrix}$$

or

$$P_V(\boldsymbol{p}) \quad = \quad \frac{1}{2m}\begin{pmatrix} E-m & 0 & -\mathbf{p} & 0 \\ 0 & E-m & 0 & \mathbf{p} \\ \mathbf{p} & 0 & -(E+m) & 0 \\ 0 & -\mathbf{p} & 0 & -(E+m) \end{pmatrix}$$

In 2×2 matrix notation the above matrix gives:

$$P_V(\boldsymbol{p}) = \frac{1}{2m}\left[E\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} + \mathbf{p}\begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix} - m\begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}\right]$$

Hence in 4×4 matrix notation

$$P_V(\boldsymbol{p}) = \frac{1}{2m}\left[E\gamma_o + P\gamma_3 - m\right]$$

where $\mathbf{p} = \mathbf{p}^3, E = \mathbf{p}^o = \mathbf{p}_o$

$$P_V(\boldsymbol{p}) = \frac{1}{2m}\left[\gamma_o\mathbf{p}^o + \gamma_3\mathbf{p}^3 - m\right]$$

$$P_V(\boldsymbol{p}) \quad = \quad \frac{(\gamma_\mu p^\mu - m)}{2m} = \frac{(\not{p} - m)}{2m}. \tag{1.2.19}$$

At the end we come up with the result :

$$P_U(\boldsymbol{p}) \quad = \quad -P_V(-p) \tag{1.2.20}$$

Chapter 2

Charged Fermions in the Presence of Magnetic Field

2.1 Introduction

The interaction of charged fermions in a uniform background magnetic field has many fascinating features. Understanding this features is quite mandatory and indispensable as it will give the right picture about different high energy processes. One of this features is the change in the dispersion relation of an electron. Since we make use of a uniform background magnetic field we will have a convenient and one of the few exact solutions of the Dirac equation in the presence of external field[7,8,9]. In this chapter we will seek for the proper ways of introducing gauge conditions which provide the background field and will make use of one of these conditions to find the exact solutions to the Dirac equation in uniform background magnetic field. Not only this but we will also discuss the fundamental properties of these solutions.

2.2 The Gauge Condition

When we have charged fermions in a classical uniform magnetic field there would be certain gauge conditions

$$A_o \rightarrow A_o,$$
$$\boldsymbol{A} \rightarrow \boldsymbol{A} + \nabla\Lambda(\mathbf{x}) \tag{2.2.1}$$

and the momentum changes as:

$$\boldsymbol{p} \rightarrow \boldsymbol{p} + eQ\nabla\Lambda(\mathbf{x}), \tag{2.2.2}$$

Where the four vector $A^\mu = (A_o, \mathbf{A})$, Q is the sign of the fermions charge,it can take values $Q = \pm 1$. Λ is a scalar function of $\mathbf{x}$ and e is the proton charge. From Eq.(2.2.2)we can tell that in the presence of a classical uniform magnetic field the momentum the canonical momentum $\boldsymbol{p}$, does not remain gauge invariant as in case of the the free space . Because

$$\Pi_o = p_o - eQA_o \quad and \tag{2.2.3}$$
$$\Pi_i = p_i - eQA_i = p_i + eQ\nabla\Lambda(\boldsymbol{x}) - eQ(A_i + \nabla\Lambda(\boldsymbol{x})) \quad i = 1,2,3 \tag{2.2.4}$$

remain gauge invariant under the gauge transformation Eq.(2.2.1)and Eq.(2.2.2). We can say

$$\Pi_\mu = p_\mu - eQA_\mu \tag{2.2.5}$$

is also gauge invariant

The commutation relation of the kinematic momentum components are

$$[\Pi^\mu, \Pi^\nu] = [\Pi^\mu \Pi^\nu] - [\Pi^\nu \Pi^\mu] \tag{2.2.6}$$

and defining the antisymmetric field tensor

$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

We will have for

Case I: $\mu = i, \nu = j$

$[\prod^i, \prod^j] = -eQ\{[A^i, \boldsymbol{p}^j] + [\boldsymbol{p}^i, A^j]\}$

$[\prod^i, \prod^j] = -ieQ\{\frac{dA^j}{dx_i} - \frac{dA^i}{dx_j}\} = -ieQ\{\partial^i A^j - \partial^j A^i\} = -ieQF^{ij}$

For

Case II $\mu = 0, \nu = 0$

$[\prod^o, \prod^o] = -ieQF^{oo}$

For

Case III $\mu = 0, \nu = i$

$[\prod^o, \prod^i] = -eQ\{[A^o, \boldsymbol{p}^i] + [\boldsymbol{p}^o, A^i]\}$

$[\prod^o, \prod^i] = -ieQ\{\frac{dA^i}{dx_o} - \frac{dA^o}{dx_i}\} = -ieQ\{\partial^o A^i - \partial^i A^o\} = -ieQF^{oi}$

For

Case IV $\mu = i, \nu = 0$

$[\prod^i, \prod^o] = -eQ\{[A^i, \boldsymbol{p}^o] + [\boldsymbol{p}^i, A^o]\}$

$[\prod^i, \prod^o] = -ieQ\{\frac{dA^o}{dx_i} - \frac{dA^i}{dx_o}\} = -ieQ\{\partial^i A^o - \partial^o A^i\} = -ieQF^{io}(see\ Append\ B)$

Combining all the above four cases,we derive.

$$[\Pi^\mu, \Pi^\nu] = -ieQF^{\mu\nu} \tag{2.2.7}$$

Considering the magnetic field to be along the z-axis then we cannot simply choose the solutions of the Dirac equation in such a way that $\Pi^x = \Pi^y = 0$, as in the case of free fermions because the gauge fields giving rise to the magnetic field will never allow us to do so. We like to Keep the gauge fields along the z-direction .Under this gauge the components of the vector potential become

$$A_o = A_y = A_z = 0, \qquad\qquad A_x = -yB_z \tag{2.2.8}$$

2.3 Solution of the Dirac Equation in the Presence of a Uniform Background Magnetic Field

It is already stated that the free Dirac equation is given by

$$[i\partial\!\!\!/ - m\mathbf{1}]\psi = 0 \quad where \;\; \partial\!\!\!/ = \gamma^\mu \partial_\mu \tag{2.3.1}$$

But whenever there is a uniform background magnetic field around a fermion with mass m and charge eQ, the generalized momentum is given by

$$p^\mu \rightarrow p^\mu - eQA^\mu \tag{2.3.2}$$

and

$$p^o \quad \rightarrow \quad p^o \tag{2.3.3}$$

$$\boldsymbol{p} \quad \rightarrow \quad \boldsymbol{p} - eQ\mathbf{A} \tag{2.3.4}$$

Hence upon the interaction given by the above equations the standard form of Dirac equation Eq(2.3.1) will have a form $[i\gamma^o\partial_o + \gamma^i\Pi_i - m]\psi = 0$ giving

$$i\frac{\partial\psi}{\partial t} = \mathcal{H}_B\psi \tag{2.3.5}$$

where $\mathcal{H}_B$ the Dirac Hamiltonian in the presence of a magnetic field

$$\mathcal{H}_B = \alpha \cdot \Pi + \beta m \tag{2.3.6}$$

The eigen states of Eq.(2.3.5),as discussed in section 1,could be given as

$$\psi = e^{-iEt}\begin{pmatrix} \phi \\ \chi \end{pmatrix} \tag{2.3.7}$$

Hence Eq.(2.3.5) may be expanded to

$$i\frac{\partial}{\partial t}\{e^{-iEt}\begin{pmatrix}\phi\\ \chi\end{pmatrix}\} = \alpha\cdot(-i\nabla - eQ\boldsymbol{A})\{e^{-iEt}\begin{pmatrix}\phi\\ \chi\end{pmatrix}\} + m\beta\begin{pmatrix}\phi\\ \chi\end{pmatrix}\{e^{-iEt}\} \quad (2.3.8)$$

$$Ee^{-iEt}\begin{pmatrix}\phi\\ \chi\end{pmatrix} = \begin{pmatrix}0 & \sigma\cdot(-i\nabla - eQ\boldsymbol{A})\\ \sigma\cdot(-i\nabla - eQ\boldsymbol{A}) & 0\end{pmatrix} e^{-iEt}\begin{pmatrix}\phi\\ \chi\end{pmatrix} + me^{-iEt}\begin{pmatrix}\phi\\ -\chi\end{pmatrix} \quad (2.3.9)$$

In general for any arbitrary vector T in three dimensions

$$\sigma\cdot T = \begin{pmatrix}T_z & T_x - iT_y\\ T_x + iT_y & -T_z\end{pmatrix} \quad (2.3.10)$$

where σ is as given in Eq.(1.2.5)

Using the above equations we can rewrite Eq.(2.3.5) as

$$(E-m)\phi = \sigma\cdot(-i\nabla - eQ\mathbf{A})\chi, \quad (2.3.11)$$

$$(E+m)\chi = \sigma\cdot(-i\nabla - eQ\mathbf{A})\phi \quad (2.3.12)$$

Just to get rid of χ we multiply from the left side of Eq.(2.3.11) by $(E+m)$ and use Eq.(2.3.12) to obtain

$$(E^2 - m^2)\phi = [\sigma\cdot(-i\nabla - eQ\mathbf{A})]^2\,\phi \quad (2.3.13)$$

$$(\sigma\cdot\mathbf{a})(\sigma\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b} + i\sigma\cdot(\mathbf{a}\times\mathbf{b}) \quad (2.3.14)$$

(*see Append.A*) The right hand side of Eq.(2.3.13) becomes

$$\begin{aligned}[\sigma\cdot(-i\nabla - eQ\mathbf{A})]^2\phi &= [-i\nabla - eQ\mathbf{A}][-i\nabla - eQ\mathbf{A}]\phi\\ &+ i\sigma\cdot[(-i\nabla - eQ\mathbf{A})\times(-i\nabla - eQ\mathbf{A})]\phi\\ &= [-\nabla^2 + ieQ(\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla) + e^2Q^2\mathbf{A}^2]\phi\\ &+ i\sigma\cdot[ieQ(\mathbf{A}\times\nabla + \nabla\times\mathbf{A})]\phi\end{aligned}$$

Thus, based on the choice we made for the vector potential ,when applied to Eq.(2.3.13):

$$(E^2 - m^2)\phi = \left[-\nabla^2 + (eQ\mathcal{B})^2 y^2 - eQ\mathcal{B}(2iy\frac{\partial}{\partial x} + \sigma_3) \right] \phi \tag{2.3.15}$$

Since the co-ordinates x and z do not appear in the above equation but through their derivatives, its solutions can be written as

$$\phi \equiv e^{i\boldsymbol{p}\cdot\mathbf{X}_{\not{y}}} f(y) \tag{2.3.16}$$

In the above expression the notation $\mathbf{X}$ stands for spatial coordinates and is different form x, which is one of the components of $\mathbf{X}$. $\mathbf{X}_{\not{y}}$ represent the 3 vector $\mathbf{X}$ with out its y-component. This means

$$\boldsymbol{p} \cdot \ \mathbf{X}_{\not{y}} = p_x x + p_z z$$

The expression $f(y)$ on the right side of Eq.(2.3.16) is a 2-component matrix which depends on the y-coordinate and some momentum component. From the expressions given in Eqs. (2.3.15)and (2.3.16) we notice that $f(y)$ is an eigenstate of the σ_3 with eigenvalues $s = \pm 1$, so that

$\sigma_3 f_s = s f_s$. Let

$$f_+(y) = \begin{pmatrix} F_+^1(y) \\ F_+^2(y) \end{pmatrix}, \quad f_-(y) = \begin{pmatrix} F_-^1(y) \\ F_-^2(y) \end{pmatrix} \tag{2.3.17}$$

Using the eigenvalue equation and assumptions for $f_+(y), f_-(y)$ we end up with more refined equations :

$$\begin{aligned} F_+^1(y) &= F_+^1(y) \\ -F_+^2(y) &= F_+^2(y) \\ F_-^1(y) &= -F_-^1(y) \\ -F_-^2(y) &= -F_-^2(y) \end{aligned} \tag{2.3.18}$$

From the above set of equation we derive two independent solutions written in general as:

$$f_+(y) = \begin{pmatrix} F_+(y) \\ 0 \end{pmatrix}, \quad f_-(y) = \begin{pmatrix} 0 \\ F_-(y) \end{pmatrix} \tag{2.3.19}$$

We now insert Eq.(2.3.19) first in Eq.(2.3.16) then the result to Eq.(2.3.15)

For $\underline{s = +1}$

We get for LHS

$$(E^2 - m^2)e^{i\boldsymbol{p}\cdot\mathbf{X}_{\not y}} \begin{pmatrix} F_+(y) \\ 0 \end{pmatrix} = \begin{pmatrix} (E^2 - m^2)F_+(y) \\ 0 \end{pmatrix} e^{i\boldsymbol{p}\cdot\mathbf{X}_{\not y}} \tag{2.3.20}$$

Now,since $\quad -\nabla^2\phi = \{p_x^2 + p_z^2\} \begin{pmatrix} F_+(y) \\ 0 \end{pmatrix} e^{i\boldsymbol{p}\cdot\mathbf{X}_{\not y}} - \begin{pmatrix} \frac{d^2F_+(y)}{dy^2} \\ 0 \end{pmatrix} e^{i\boldsymbol{p}\cdot\mathbf{X}_{\not y}}$

and

$\{-eQ\mathcal{B}(2iy\frac{\partial}{\partial x} + \sigma_3) + (eQ\mathcal{B})^2y^2\}\phi = \{eQ\mathcal{B}(2yp_x) - eQ\mathcal{B} + (eQ\mathcal{B})^2y^2\} \begin{pmatrix} F_+(y) \\ 0 \end{pmatrix} e^{i\boldsymbol{p}\cdot\mathbf{X}_{\not y}}$

The RHS of Eq.(2.3.15) will become

$$\begin{aligned} &= \{p_x^2 + p_z^2\} \begin{pmatrix} F_+(y) \\ 0 \end{pmatrix} e^{i\boldsymbol{p}\cdot\mathbf{X}_{\not y}} - \begin{pmatrix} \frac{d^2F_+(y)}{dy^2} \\ 0 \end{pmatrix} e^{i\boldsymbol{p}\cdot\mathbf{X}_{\not y}} \\ &+ \{eQ\mathcal{B}(2yp_x) - eQ\mathcal{B} + (eQ\mathcal{B})^2y^2\} \begin{pmatrix} F_+(y) \\ 0 \end{pmatrix} e^{i\boldsymbol{p}\cdot\mathbf{X}_{\not y}} \end{aligned}$$

Equating the LHS with RHS for $s = +1$, we then get

$\frac{d^2F_+}{dy^2} - [\ eQ\mathcal{B}y + p_x]\ ^2F_+ + [\ E^2 - m^2 - p_z^2 + eQ\mathcal{B}]\ F_+ = 0,$

Similarly for $\underline{s = -1}$we get

$\frac{d^2F_-}{dy^2} - \{eQ\mathcal{B}y + p_x\}^2F_- + \{E^2 - m^2 - p_z^2 - eQ\mathcal{B}\}F_- = 0$

In general for a given $s = \pm 1$ the differential equation satisfied by F_s obtained from Eq.(2.3.15) is

$$\frac{d^2F_s}{dy^2} - \{eQ\mathcal{B}y + p_x\}^2F_s + \{E^2 - m^2 - p_z^2 + eQ\mathcal{B}s\}F_s = 0 \tag{2.3.21}$$

We will use the dimensionless variable ξ (*see Append.D*)

$$\xi = \sqrt{e|Q|\mathcal{B}}\left(y + \frac{p_x}{eQ\mathcal{B}}\right) \tag{2.3.22}$$

Which transforms Eq.(2.3.21) to the form

$$\left[\frac{d^2}{d\xi^2} - \xi^2 + a_s\right] F_s = 0 \tag{2.3.23}$$

where

$$a_s = \frac{E^2 - m^2 - p_z^2 + eQ\mathcal{B}s}{e|Q|\mathcal{B}} \tag{2.3.24}$$

At the end we derive, special form of Hermite's equation Eq.(2.3.23) (*see Append.C*), in which its solutions exist provided that $a_s = 2\nu + 1$ for non-zero integer ν. Eq. (2.3.24) can then be rearranged to give us energy eigenvalues

$$E^2 = m^2 + p_z^2 + (2\nu + 1)e|Q|\mathcal{B} - eQ\mathcal{B}s \tag{2.3.25}$$

And the celebrated solutions of F_s, for the particular form of a_s given above are (*see Append.C*)

$$I_\nu(\xi) \equiv N_\nu e^{-\xi^2/2} H_\nu(\xi) \tag{2.3.26}$$

where H_ν are Hermite polynomials of order ν, and N_ν are the corresponding normalization constants given by (*see Append.C*)

$$N_\nu = \left(\frac{\sqrt{e|Q|\mathcal{B}}}{\nu! 2^\nu \sqrt{\pi}}\right)^{1/2} \tag{2.3.27}$$

From the above two equations one can build the so called completeness relation as:

$$\sum_\nu I_\nu(\xi) I_\nu(\xi_\star) = \sqrt{e|Q|\mathcal{B}}\delta(\xi - \xi_\star) = \delta(y - y_\star),$$

where $\xi_\star$ is obtained from ξ just by replacing y by another position coordinate $y_\star$. So far Q was arbitrary. We now specialize to the case of electrons, for which Q=

-1. If we have charged particles in a magnetic field then their cyclotron orbit will be quantized, such that this charged particles can only occupy orbits with discrete energy values, called Landau levels[10]. The relativistic form of this Landau energy levels is given by

$$E_n^2 = m^2 + p_z^2 + 2ne\mathcal{B} \tag{2.3.28}$$

Comparing Eq.(2.3.28) with Eq.(2.3.25) we notice that the solutions are twofold degenerate for:

$$s = +1, \nu = n - 1 \quad \textit{and} \quad s = -1, \nu = n \tag{2.3.29}$$

But for n=0

$(2\nu + 1)e|Q|\mathcal{B} = eQ\mathcal{B}, \quad Q = -|Q|$

$\Rightarrow \nu = -\frac{1}{2}(1+s)$ and because ν and n are non negative integers, $s = -1$. Therefore unlike the other n states , $n = 0$ is a non-degenerate state.Taking the square root of the right hand side the energy eigenvalue solutions given in the Eq.(2.3.28), can have both the positive E_n and negative $-E_n$ values, where the subscript n represents the n-*th* Landau level .

Positive Energy Solutions Using Eq.(2.3.19) for the positive energy

$$f_+^{(n)}(y) = \begin{pmatrix} I_{n-1}(\xi) \\ 0 \end{pmatrix}, \quad f_-^{(n)}(y) = \begin{pmatrix} 0 \\ I_n(\xi) \end{pmatrix} \tag{2.3.30}$$

For $n = 0$, f_+ does not exit. We will consistently incorporate this fact by defining

$$I_{-1}(y) = 0 \tag{2.3.31}$$

in addition to the definition of I_n given in Eq.(2.3.26) for non negative integer n . It is customary to denote the column spinor with out the temporal and spatial exponential dependence, and hence the positive energy solutions of the Dirac equation can be written as

$$\psi = e^{-ip\cdot X_{\not{y}}} U_s(y, n, \boldsymbol{P}_{\not{y}}) \tag{2.3.32}$$

where $p \cdot X_{\not{y}} = p^{\mu}(X_{\not{y}})_{\mu}$. Comparing Eq.s.(2.3.7) (2.3.16)and (2.3.32) we notice that upper components of the spinors $U_s(y, n, \boldsymbol{P}_{\not{y}})$are as given in

$$U_+(y, n, \boldsymbol{P}_{\not{y}}) = \begin{pmatrix} I_{n-1}(\xi) \\ 0 \\ \chi_+ \end{pmatrix} \quad , U_-(y, n, \boldsymbol{P}_{\not{y}}) = \begin{pmatrix} 0 \\ I_n(\xi) \\ \chi_- \end{pmatrix}$$

The lower components of the spinors denoted by χ can be solved using combinations of Eq.s.(2.3.12) and (2.3.16) . *When s=+1* the gauge provides us

$$(E+m) \begin{pmatrix} \chi_+^1 \\ \chi_+^2 \end{pmatrix} e^{i\boldsymbol{p}\cdot \mathbf{X}_{\not{y}}} = \begin{pmatrix} -i\nabla_3 & (-i\nabla_1 - eQA_1 - \nabla_2) \\ (-i\nabla_1 - eQA_1 + \nabla_2) & i\nabla_3 \end{pmatrix} e^{i(p_x x + p_z z)} \begin{pmatrix} I_{n-1}(\xi) \\ 0 \end{pmatrix} \tag{2.3.33}$$

so that

$\chi_+^1 = \frac{p_z}{E_n + m} I_{n-1}(\xi)$

$(E+m)x_+^2 = eQ\mathcal{B}\left(\frac{p_x}{eQ\mathcal{B}} + y\right) I_{n-1}(\xi) + \frac{\partial I_{n-1}(\xi)}{\partial y} \qquad , Q = -|Q| = -1$

$(E+m)\chi_+^2 = -\sqrt{e\mathcal{B}}\left[\xi I_{n-1}(\xi) - \frac{\partial I_{n-1}(\xi)}{\partial \xi}\right]$

using relation

$$I_n(\xi) = \frac{1}{(2n)^{1/2}}\left[\xi I_{n-1}(\xi) - \frac{I_{n-1}(\xi)}{\partial \xi}\right] (\textit{See Append.C}) \tag{2.3.34}$$

This gives us

$\chi_+^2 = -\frac{\sqrt{2ne\mathcal{B}}}{(E_n+m)} I_n(\xi)$

Thus, $\quad U_+(y, n, \boldsymbol{p}_{\not{y}}) = \begin{pmatrix} I_{n-1}(\xi) \\ 0 \\ \frac{p_z}{E_n+m} I_{n-1}(\xi) \\ -\frac{\sqrt{2ne\mathcal{B}}}{(E_n+m)} I_n(\xi) \end{pmatrix}$

Similarly when s=-1 We get

$$U_{-}(y,n,\boldsymbol{p}_{\mathrm{y}}) = \begin{pmatrix} 0 \\ I_n(\xi) \\ -\frac{\sqrt{2ne\mathcal{B}}}{(E_n+m)} I_{n-1}(\xi) \\ -\frac{p_z}{E_n+m} I_n(\xi) \end{pmatrix}$$

2.3.1 The Negative Energy Solutions

By negative energy we mean we have energy eigenvalue as $E = -E_n$. To find the negative energy solutions first we start from the Dirac equation Eq.(2.3.5) but now

$$\psi = e^{-i(-|E|)t} \begin{pmatrix} \phi \\ \chi \end{pmatrix} \tag{2.3.35}$$

Combining Eqs.(2.3.5) and (2.3.35) gives

$$-Ee^{iEt} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} 0 & \sigma \cdot (-i\nabla - eQ\boldsymbol{A}) \\ \alpha \cdot (-i\nabla - eQ\boldsymbol{A}) & 0 \end{pmatrix} e^{iEt} \begin{pmatrix} \phi \\ \chi \end{pmatrix} + me^{iEt} \begin{pmatrix} \phi \\ -\chi \end{pmatrix} \tag{2.3.36}$$

using the above equation we can rewrite Eq.(2.3.5) for negative energy as

$$(E+m)\phi = \sigma \cdot (i\nabla + eQ\mathbf{A})\chi, \tag{2.3.37}$$

$$(E-m)\chi = \sigma \cdot (i\nabla + eQ\mathbf{A})\phi \tag{2.3.38}$$

As we can see from Eqs.(2.3.37) and (2.3.38) for the negative energy spinors it is easier to start with the two lower components first and then find the upper components. Hence the next strategy is to get rid of ϕ and obtain ;

$$(E^2 - m^2)\chi = [\sigma \cdot (i\nabla + eQ\mathbf{A})]^2 \chi \tag{2.3.39}$$

Taking the advantage of the similarities between Eqs.(2.3.13) and (2.3.39) (*See Append.C*)we write

$$(E^2 - m^2)\chi = \left[-\nabla^2 + (eQ\mathcal{B})^2 y^2 - eQ\mathcal{B}(2iy\frac{\partial}{\partial x} + \sigma_3)\right]\chi \tag{2.3.40}$$

Since the co-ordinates x and z do not appear in the above equation but through their derivatives, the solutions can be written as

$$\chi \equiv e^{-i\boldsymbol{p}\cdot\mathbf{X}_{y}} g(y) \tag{2.3.41}$$

where $g(y)$ is a 2-components column matrix which depends not only on the y-coordinate but also some momentum components. Similar to the case of positive energy, here $g(y)$ is also eigenstate of the σ_3 which has eigenvalues $s = \pm 1$. Therefore $g(y)$ will have two refined independent solutions written in general as

$$g_+(y) = \begin{pmatrix} G_+(y) \\ 0 \end{pmatrix}, \quad g_-(y) = \begin{pmatrix} 0 \\ G_-(y) \end{pmatrix} \tag{2.3.42}$$

Using Eq.(2.3.42) first in Eq.(2.3.41), then the result in Eq.(2.3.40) results:
$\frac{d^2G_+}{dy^2} - [\ eQ\mathcal{B}y - p_x]\ ^2 G_+ + [\ E^2 - m^2 - p_z^2 + eQ\mathcal{B}]\ G_+ = 0$ for $s = +1$ and
$\frac{d^2G_-}{dy^2} - [\ eQ\mathcal{B}y - p_x]\ ^2 G_- + [\ E^2 - m^2 - p_z^2 - eQ\mathcal{B}]\ G_- = 0$ for $s = -1$.In general for a given $s = \pm 1$

$$\frac{d^2G_s}{dy^2} - [\ eQ\mathcal{B}y - p_x]\ ^2 G_s + [\ E^2 - m^2 - p_z^2 + eQ\mathcal{B}s]\ G_s = 0 \tag{2.3.43}$$

Defining a dimensionless variable

$$\tilde{\xi} \equiv \sqrt{e|Q|\mathcal{B}}\left(y - \frac{p_x}{eQ\mathcal{B}}\right) \tag{2.3.44}$$

equation (2.3.43) will be transformed to special form of Hermites equation

$$\left[\frac{d^2}{d\tilde{\xi}^2} - \tilde{\xi}^2 + a_s\right] G_s = 0 \tag{2.3.45}$$

where

$$a_s = \frac{E^2 - m^2 - p_z^2 + eQ\mathcal{B}s}{e|Q|\mathcal{B}} \tag{2.3.46}$$

As in the case of the positive energy, for the negative energy the solutions exist provided that $a_s = 2\nu + 1$ for $\nu = 0, 1, 2, 3 \cdots$. This provides the energy eigenvalues

$$E^2 = m^2 + p_z^2 + (2\nu + 1)e|Q|\mathcal{B} - eQ\mathcal{B}s \tag{2.3.47}$$

and solutions for G_s as

$$I_\nu(\tilde{\xi}) \equiv N_\nu e^{-(\tilde{\xi})^2/2} H_\nu(\tilde{\xi}) \tag{2.3.48}$$

where H_ν are Hermite polynomials of order ν , and N_ν are the corresponding normalization constants given by

$$N_\nu = \left(\frac{\sqrt{e|Q|\mathcal{B}}}{\nu! 2^\nu \sqrt{\pi}} \right)^{1/2} \tag{2.3.49}$$

(*See Append.C*) Since for electron we have $Q = -|Q|$, we can compare Eq.(2.3.47) with relativistic form of Landau energy levels given by Eq.(2.3.28) to obtain a two fold degenerate general solutions for:

$$s = 1, \nu = n - 1 \;\; \textit{and for} \;\; s = -1, \nu = n \tag{2.3.50}$$

For $n = 0$, we get $\nu = -\frac{1}{2}(1 + s)$ and since ν can not be negative it implies that $s = -1$, or $n = 0$ is a non-degenerate state. Hence we can write the negative energy solutions using Eq.(2.3.42) as

$$g_+(y) = \begin{pmatrix} I_{n-1}(\xi) \\ 0 \end{pmatrix}, \quad g_-(y) = \begin{pmatrix} 0 \\ I_n(\xi) \end{pmatrix} \tag{2.3.51}$$

When $n = 0$, the solution g_+ does not exit (i.e $I_{-1}(y) = 0.$) The lower component of the spinor is determined through Eqs.(2.3.51) and (2.3.41). The upper components denoted by ϕ earlier can be solved using Eq.(2.3.37). The negative energy solutions of the Dirac equations can be written as

$$\psi = e^{ip.X_{\not{y}}} V_s(y, n, \boldsymbol{P}_{\not{y}}) \tag{2.3.52}$$

where $p \cdot X_{\not{y}} = p^{\mu}(X_{\not{y}})_{\mu}$ and $V_s(y, n, \boldsymbol{P}_{\not{y}})$ are the spinors .

Combining (2.3.35) ,(2.3.41),(2.3.51) together with Eq.(2.3.52) gives

$$V_{+}(y, n, \boldsymbol{P}_{\not{y}}) = \begin{pmatrix} \phi_{+} \\ I_{n-1}(\tilde{\xi}) \\ 0 \end{pmatrix} \quad , V_{-}(y, n, \boldsymbol{P}_{\not{y}}) = \begin{pmatrix} \phi_{-} \\ 0 \\ I_{n}(\tilde{\xi}) \end{pmatrix} \tag{2.3.53}$$

where ϕ is a 2 component matrix.

<u>Now we determine ϕ_{+}</u> Let

$$\phi_{+} = e^{-i\boldsymbol{p}\cdot\mathbf{X}_{\not{y}}} \begin{pmatrix} \phi_{+}^{1} \\ \phi_{+}^{2} \end{pmatrix} \tag{2.3.54}$$

Then from Eq.(2.3.37)

$$(E+m) \begin{pmatrix} \phi_{+}^{1} \\ \phi_{+}^{2} \end{pmatrix} e^{-i(\boldsymbol{p}\cdot\mathbf{X}_{\not{y}})} = \begin{pmatrix} i\nabla_3 & (i\nabla_1 + eQA_1 + \nabla_2) \\ (i\nabla_1 + eQA_1 - \nabla_2) & -i\nabla_3 \end{pmatrix} e^{-i(p_x x + p_z z)} \begin{pmatrix} I_{n-1}(\tilde{\xi}) \\ 0 \end{pmatrix}$$

so that

$$(E+m)\phi_{+}^{1} = p_z I_{n-1}(\tilde{\xi})$$

Then,

$$\phi_{+}^{1} = \tfrac{p_z}{E_n + m} I_{n-1}(\tilde{\xi})$$

and $\qquad (E+m)\phi_{+}^{2} = -eQ\mathcal{B}\left(-\tfrac{p_x}{eQ\mathcal{B}} + y\right) I_{n-1}(\tilde{\xi}) - \tfrac{\partial I_{n-1}(\tilde{\xi})}{\partial y} \qquad , Q = -|Q|$

$(E+m)\phi_{+}^{2} = \sqrt{e\mathcal{B}}\left[\tilde{\xi} I_{n-1}(\tilde{\xi}) - \tfrac{\partial I_{n-1}(\tilde{\xi})}{\partial \tilde{\xi}}\right]$ using ,

$$I_n(\tilde{\xi}) = \frac{1}{(2n)^{1/2}} \left[\tilde{\xi} I_{n-1}(\tilde{\xi}) - \frac{I_{n-1}(\tilde{\xi})}{\partial \tilde{\xi}}\right] \tag{2.3.55}$$

Thus, $\qquad \phi_+^2 = \frac{\sqrt{2ne\mathcal{B}}}{(E_n+m)} I_n(\tilde{\xi})$

$$\Rightarrow V_+(y,n,\boldsymbol{p}_{\not{y}}) = \begin{pmatrix} \frac{p_z}{E_n+m} I_{n-1}(\tilde{\xi}) \\ \frac{\sqrt{2ne\mathcal{B}}}{(E_n+m)} I_n(\tilde{\xi}) \\ I_{n-1}(\tilde{\xi}) \\ 0 \end{pmatrix}$$

<u>Let us next determine ϕ_-</u> $\quad$ Let $\phi_- = e^{-i\boldsymbol{p}\cdot\mathbf{X}_{\not{y}}} \begin{pmatrix} \phi_-^1 \\ \phi_-^2 \end{pmatrix}$, then from Eq.(2.3.37)

$$(E+m) \begin{pmatrix} \phi_-^1 \\ \phi_-^2 \end{pmatrix} e^{-i(\boldsymbol{p}\cdot\mathbf{X}_{\not{y}})} = \begin{pmatrix} i\nabla_3 & (i\nabla_1 + eQA_1 + \nabla_2) \\ (i\nabla_1 + eQA_1 - \nabla_2) & -i\nabla_3 \end{pmatrix} e^{-i(p_x x + p_z z)} \begin{pmatrix} 0 \\ I_n(\tilde{\xi}) \end{pmatrix}$$

so that

$$(E+m)\phi_-^2 = -p_z I_n(\tilde{\xi}) \tag{2.3.56}$$

From this follows

$$\phi_-^2 = -\frac{p_z}{E_n+m} I_n(\tilde{\xi})$$

In the same way

$(E+m)\phi_-^1 = -eQ\mathcal{B}\left(-\frac{p_x}{eQ\mathcal{B}} + y\right) I_n(\tilde{\xi}) + \frac{\partial I_n(\tilde{\xi})}{\partial y} \qquad , Q = -|Q|$

$$(E+m)\phi_-^1 = \sqrt{e\mathcal{B}}\left[\tilde{\xi} I_n(\tilde{\xi}) + \frac{\partial I_n(\tilde{\xi})}{\partial \tilde{\xi}}\right] \tag{2.3.57}$$

but because , $I_{n-1}(\tilde{\xi}) = \frac{1}{(2n)^{1/2}}\left[\tilde{\xi} I_n(\tilde{\xi}) + \frac{I_n(\tilde{\xi})}{\partial \xi}\right]$

Hence, $\qquad \phi_-^1 = \frac{\sqrt{2ne\mathcal{B}}}{E_n+m} I_{n-1}(\tilde{\xi})$

So that $\qquad V_-(y,n,\boldsymbol{p}_{\not{y}}) = \begin{pmatrix} \frac{\sqrt{2ne\mathcal{B}}}{E_n+m} I_{n-1}(\tilde{\xi}) \\ -\frac{p_z}{E_n+m} I_n(\tilde{\xi}) \\ 0 \\ I_n(\tilde{\xi}) \end{pmatrix}$

Finally collecting all spinors together, the positive energy spinors , U_s are

$$U_+(y,n,\boldsymbol{p}_{\not{y}}) = \begin{pmatrix} I_{n-1}(\xi) \\ \\ 0 \\ \\ \frac{p_z}{E_n+m} I_{n-1}(\xi) \\ \\ -\frac{\sqrt{2ne\mathcal{B}}}{(E_n+m)} I_n(\xi) \end{pmatrix} \quad U_-(y,n,\boldsymbol{p}_{\not{y}}) = \begin{pmatrix} 0 \\ \\ I_n(\xi) \\ \\ -\frac{\sqrt{2ne\mathcal{B}}}{(E_n+m)} I_{n-1}(\xi) \\ \\ -\frac{p_z}{E_n+m} I_n(\xi) \end{pmatrix} \tag{2.3.58}$$

and the negative energy spinors ,V_s are

$$V_+(y,n,\boldsymbol{p}_{\not{y}}) = \begin{pmatrix} \frac{p_z}{E_n+m} I_{n-1}(\tilde{\xi}) \\ \\ \frac{\sqrt{2ne\mathcal{B}}}{(E_n+m)} I_n(\tilde{\xi}) \\ \\ I_{n-1}(\tilde{\xi}) \\ \\ 0 \end{pmatrix} \quad , V_-(y,n,\boldsymbol{p}_{\not{y}}) = \begin{pmatrix} \frac{\sqrt{2ne\mathcal{B}}}{E_n+m} I_{n-1}(\tilde{\xi}) \\ \\ -\frac{p_z}{E_n+m} I_n(\tilde{\xi}) \\ \\ 0 \\ \\ I_n(\tilde{\xi}) \end{pmatrix} \tag{2.3.59}$$

Remark

In general the above solutions are eigenstates of Π_x and Π_z but not of Π_y. Which can be discovered from Eq.(2.2.7)which gives us $[\Pi^x, \Pi^y] = -ieQF^{12}$. It implies that Π_x and Π_y do not commute (i.e we cannot have simultaneous eigenstates of both).

The dimensionless variable $\tilde{\xi}$ is obtained from ξ upon changing the sign of the p_x-term .

2.4 The Wave Functions in the Zero Field Limit

The pairs of solutions in Eq.(2.3.58) and Eq.(2.3.59) that we obtained from the Dirac equations in the presence of magnetic field are exact solutions and not perturbative excitations around the free Dirac equation solutions. When $\mathcal{B} \rightarrow 0$ the term to the right hand side of Eq.(2.3.24)or Eq.(2.3.46) will blow up. Consequently it is not possible to put $\mathcal{B} \rightarrow 0$ in the final solutions Eq.(2.3.58), Eq.(2.3.59) and retrieve the free Dirac solutions. Mathematically in the zero field limit, the quantization condition given in Eq.(2.3.24) fails and in that limit the solutions of Eq.(2.3.23)or Eq.(2.3.45) become indeterminate. Just because the solution becomes indeterminate does not mean there is no physical explanation behind this. Physically we can say the solutions in Eq.(2.3.58) and Eq.(2.3.59) are specific to a gauge condition giving rise to a magnetic field along the z- direction. And what we best can do is gauge transform this solutions and obtain an equivalent solutions in background magnetic field. The choice of the background gauge does not permit us to obtain the free solutions in any limit as the free solutions belong to another gauge orbit, namely the pure gauge solutions.

2.5 The n=0 Solutions

Based on our prior knowledge of the n=0 sate, the n=0 solution is non-degenerate. The n=0 's non-degenerate state has only one available solution for for both the positive and negative energies. And they are the s=-1 state in Eq.(2.3.58) and s=-1 state in Eq.(2.3.59), for the positive and negative energies respectively. For this non-degenerate state ,$(I_{-1}(\xi) = 0)$, and

$$\Sigma_z U_-(y,n,\boldsymbol{p}_y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} U_-(y,n,\boldsymbol{p}_y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ I_n(\xi) \\ -\frac{\sqrt{2ne\mathcal{B}}}{(E_n+m)} I_{n-1}(\xi) \\ -\frac{p_z}{E_n+m} I_n(\xi) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ I_n(\xi) \\ 0 \\ -\frac{p_z}{E_n+m} I_n(\xi) \end{pmatrix}$$

$$= -\begin{pmatrix} 0 \\ I_n(\xi) \\ 0 \\ -\frac{p_z}{E_n+m} I_n(\xi) \end{pmatrix} = -(U_-(y,n,\boldsymbol{p}_y))$$

We conclude that $$\Sigma_z U_-(y,0,\boldsymbol{p}_y) = -(U_-(y,0,\boldsymbol{p}_y)) \qquad (2.5.1)$$

Following similar procedure for the negative energy spinor one obtains

$$\Sigma_z V_-(y,0,\boldsymbol{p}_y) = -(V_-(y,0,\boldsymbol{p}_y)) \qquad (2.5.2)$$

The collaborated usage of the above two consecutive equations leads to a physical generalization that only in the $n = 0$ state the wave functions are eigenstates of Σ_z, and the other higher Landau states solutions do not have any definite Σ_z eigenvalue such as in the $n = 0$ case(-1). The astronomical world has so many wonders like the neutron star cores which can sustain a strong magnetic field to the order of 10^{12}G or more. In actual calculations when the strength of the magnetic field is high we require to work with $n = 0$ solutions. Why?Let's roughly estimate the magnitude of the magnetic field suitable for the $n = 0$ approximation.Suppose that the typical energy of electron in a system is E and the magnitude of the magnetic field is $\mathcal{B}$

from experimental observation. In calculations if it happens that $2ne\mathcal{B} > E^2 - m^2$, then the dispersion relation $2ne\mathcal{B} = E^2 - m^2 - p_z^2$ provides for any positive n , p_z^2 will be negative, which is not possible. Consequently when $2e\mathcal{B}$ is greater than the square of the typical electron energy of the system minus the rest mass square of the electron then we have only the $n = 0$ level contributing to the energy levels and only those corresponding wave functions must be used in calculating the other details of the system. As an example if the typical electron energy of system is of the order of $1Mev$ then for magnetic field greater than 10^{14}Gauss we must have the $n = 0$ level contribution in the energy. For lower magnitude of the magnetic field the other Landau levels will start to contribute in the electron energy. In general for a fixed energy of an electron and for very low magnetic field we will have as many possible Landau levels. One of the great astronomical physical interest is the calculation of scattering cross-sections and decay rates in the presence of magnetic field. In these calculations a benchmark value of the background magnetic field is used:

$$\mathcal{B}_e = \frac{m^2}{e} = 4.41 \times 10^{13} Gauss \tag{2.5.3}$$

where m is mass of electron and e is the charge of a proton. $\mathcal{B}_e$ is some times called the 'critical field' although nothing critical happens at this field strength. The only condition which defines the critical field is that when the magnitude of a magnetic field reaches the value $\mathcal{B}_e$, the electron cyclotron frequency equals its rest mass in natural units. The electron cyclotron frequency $\omega_c = \frac{e\mathcal{B}}{m}$. When ever the magnitude of the magnetic field is above the critical field, there will be a considerable electronic wave functions modification. Generally in a typical calculations with the presence of magnetic fields the relative strength of the field is measured in terms of the critical field. If the field strength is much higher than $\mathcal{B}_e$ then for electrons with an energy of a few MeV only the lowest Landau level contribution is expected.

2.6 Ortho-normality of the Solutions

Using the positive energy spinors given in Eq.(2.3.45)

$$U_+^\dagger(y,n,\boldsymbol{p}_y)U_-(y,n,\boldsymbol{p}_y) = \begin{pmatrix} I_{n-1}(\xi) & 0 & \frac{p_z}{E_n+m}I_{n-1}(\xi) & \frac{-\sqrt{2ne\mathcal{B}}}{E_n+m}I_n(\xi) \end{pmatrix} \begin{pmatrix} 0 \\ I_n(\xi) \\ \frac{-\sqrt{2ne\mathcal{B}}}{E_n+m}I_{n-1}(\xi) \\ -\frac{p_z}{E_n+m}I_n(\xi) \end{pmatrix}$$

$$U_+^\dagger(y,n,\boldsymbol{p}_y)U_-(y,n,\boldsymbol{p}_y) = p_z\frac{\sqrt{2ne\mathcal{B}}}{(E_n+m)^2}(I_n^2(\xi) - I_{n-1}^2(\xi)) \qquad (2.6.1)$$

From the above equation one can infer that in general the two positive energy spinors are not orthogonal to each other. The same is true of the negative energy spinors. If the magnitude of the magnetic field $\mathcal{B}$ is small as compared to the critical field then we expect many Landau levels, as has been indicated in the previous section. And if we keep decreasing this field at certain limit $n \sim n-1$. This limit is denoted by $n \to \infty$ and it is in this limit that the spinors in Eq.(2.6.1) become orthogonal. In most cases the maximum Landau level can be calculated from the maximum attainable energy of the electron and the field strength. To obtain the maximum attainable energy of the electron, observational data or experimental limits are used. Once we know the maximum electron energy E_{max} then the highest Landau level is given by:

$$n_{max} = \frac{E_{max}^2 - m^2}{2e\mathcal{B}}$$

Hence if $\mathcal{B} \sim 10^{10} Gauss$ and $E_{max} \sim 1Mev$ then $n_{max} \sim 10^3$. Decreasing the field strength further increases n_{max} and the possible number of Landau levels becomes very high.In such a condition we say $n \to \infty$. Taking the $n \to \infty$ limit in to account we can reconstruct the product of the positive spinors and reform the ortho-normality as:

$$\begin{aligned} U_+^\dagger(y,n,\boldsymbol{p}_{\not{y}})U_-(y,n,\boldsymbol{p}_{\not{y}})\mid_{n\to\infty} &= \begin{pmatrix} I_n(\xi) & 0 & \frac{p_z}{E_n+m}I_n(\xi) & \frac{-\sqrt{2ne\mathcal{B}}}{E_n+m}I_n(\xi) \end{pmatrix} \begin{pmatrix} 0 \\ I_n(\xi) \\ \frac{-\sqrt{2ne\mathcal{B}}}{E_n+m}I_n(\xi) \\ -\frac{p_z}{E_n+m}I_n(\xi) \end{pmatrix} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} U_+^\dagger(y,n,\boldsymbol{p}_{\not{y}})U_+(y,n,\boldsymbol{p}_{\not{y}})\mid_{n\to\infty} &= \begin{pmatrix} I_n(\xi) & 0 & \frac{p_z}{E_n+m}I_n(\xi) & \frac{-\sqrt{2ne\mathcal{B}}}{E_n+m}I_n(\xi) \end{pmatrix} \begin{pmatrix} I_n(\xi) \\ 0 \\ \frac{p_z}{E_n+m}I_n(\xi) \\ \frac{-\sqrt{2ne\mathcal{B}}}{E_n+m}I_n(\xi) \end{pmatrix} \\ &= I_n^2\left[\frac{(E_n+m)^2+p_z^2+2ne\mathcal{B}}{(E_n+m)^2}\right] = \frac{2E_n}{E_n+m}I_n^2 \end{aligned}$$

When combined the above two equations provide :

$$U_s^\dagger(y,n,\boldsymbol{p}_{\not{y}})U_{s'}(y,n,\boldsymbol{p}_{\not{y}})\mid_{n\to\infty} = \frac{2E_n}{E_n+m}I_n^2\delta_{s,s'} \tag{2.6.2}$$

where $\delta_{s,s'}$ is 1 when $s = s'$ and zero other wise. Incidentally the above relation holds for the $n = 0$ state. The lowest Landau state is trivially orthogonal as one of the spinors do not exist.In general performing similar calculations done for the positive spinors provide,

$$\begin{aligned} U_s^\dagger(y,n,\boldsymbol{p}_{\not{y}})V_{s'}(y,n,-\boldsymbol{p}_{\not{y}})\mid_{n\to\infty} &= 0 && (2.6.3) \\ V_s^\dagger(y,n,\boldsymbol{p}_{\not{y}})V_{s'}(y,n,\boldsymbol{p}_{\not{y}})\mid_{n\to\infty} &= \frac{2E_n}{E_n+m}I_n^2\delta_{s,s'} && (2.6.4) \end{aligned}$$

As a coincidence the above relations also hold true for the $n = 0$ state.

Based on the above relations in general the solutions of the Dirac equation are not ortho-normal. But for special cases where the magnetic field strength is very small we will have huge number of Landau levels and for very high Landau levels the solutions tend to be ortho-normal. As a matter of fact this ortho-normality holds always true for the other extreme $n = 0$.

Chapter 3

Spin-sum of the Dirac Solutions in a Uniform Back-ground Magnetic Field

3.1 Introduction

The calculations of scattering cross-sections and decay rates of elementary particles in compact objects like the neutron star are some of the few methods in which valid information about this stars is obtained. And these crucial calculations make use of the spin-sum where we sum over all the spin degrees of freedom using the corresponding spinor solutions. Hence in this chapter we explicitly calculate the spin-sum in back-ground magnetic field using the spinor solutions obtained in the previous chapter.

3.2 The Spin-sum

The spin sum can be written in compact form as

$$P_U(y, y_\star, n, \boldsymbol{p}_{\not{y}}) = \sum_s U_s(y, n, \boldsymbol{p}_{\not{y}}) \overline{U}_s(y_\star, n, \boldsymbol{p}_{\not{y}})$$

for the positive energy solutions(positive spinors) and

$$P_V(y, y_\star, n, \boldsymbol{p}_{\not{y}}) = \sum_s V_s(y, n, \boldsymbol{p}_{\not{y}})\overline{V}_s(y_\star, n, \boldsymbol{p}_{\not{y}})$$

for the negative energy solutions(negative spinors). The two spinors in each sums in general can have two different position coordinates hence their spatial dependence is shown to be explicitly different. Depending on the very nature of the solutions given in Eq.(2.3.58), the spin-sum for the positive spinors can be rewritten as

$$P_U(y, y_\star, n, \boldsymbol{p}_{\not{y}}) = \sum_s U_s(y, n, \boldsymbol{p}_{\not{y}})\overline{U}_s(y_\star, n, \boldsymbol{p}_{\not{y}}) = \frac{1}{E_n + m}\sum_{i,j=n-1}^{n} I_i(\xi)I_j(\xi_\star)T_{i,j} \tag{3.2.1}$$

where the $T_{i,j}s$ are 4×4 matrices. In order to obtain the $T_{i,j}s$ we need to have the matrix multiplication of the spinors to the left hand side of Eq.(3.2.1). Performing the summation over all the possible spin degrees of freedom provides:

$$P_U(y, y_\star, n, \boldsymbol{p}_{\not{y}}) = \sum_s U_s(y, n, \boldsymbol{p}_{\not{y}})\overline{U}_s(y_\star, n, \boldsymbol{p}_{\not{y}}) = U_+(y, n, \boldsymbol{p}_{\not{y}})\overline{U}_+(y_\star, n, \boldsymbol{p}_{\not{y}}) + U_-(y, n, \boldsymbol{p}_{\not{y}})\overline{U}_-(y_\star, n, \boldsymbol{p}_{\not{y}}) \tag{3.2.2}$$

But $\overline{U}_+(y_\star, n, \boldsymbol{p}_{\not{y}}) = U_+^\dagger(y_\star, n, \boldsymbol{p}_{\not{y}})\gamma^o$ where $\dagger$ depicts the adjoint and $\gamma^o = \beta$

$$\Longrightarrow \overline{U}_+(y_\star, n, \boldsymbol{p}_{\not{y}}) = \begin{pmatrix} I_{n-1}(\xi_\star) & 0 & \frac{p_z}{E_n+m}I_{n-1}(\xi_\star) & -\frac{\sqrt{2ne\mathcal{B}}}{E_n+m}I_n(\xi_\star) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\Longrightarrow \overline{U}_+(y_\star, n, \boldsymbol{p}_{\not{y}}) = \begin{bmatrix} I_{n-1}(\xi_\star) & 0 & -\frac{p_z}{E_n+m}I_{n-1}(\xi_\star) & \frac{\sqrt{2ne\mathcal{B}}}{E_n+m}I_n(\xi_\star) \end{bmatrix} \text{ so that}$$

$$U_+(y, n, \boldsymbol{p}_{\not{y}})\overline{U}_+(y_\star, n, \boldsymbol{p}_{\not{y}}) = \begin{pmatrix} I_{n-1}(\xi) \\ 0 \\ \frac{p_z}{E_n+m}I_{n-1}(\xi) \\ -\frac{\sqrt{2ne\mathcal{B}}}{(E_n+m)}I_n(\xi) \end{pmatrix} \begin{bmatrix} I_{n-1}(\xi_\star) & 0 & -\frac{p_z}{E_n+m}I_{n-1}(\xi_\star) & \frac{\sqrt{2ne\mathcal{B}}}{E_n+m}I_n(\xi_\star) \end{bmatrix}$$

or

$$U_+(y, n, \boldsymbol{p}_{\not{y}})\overline{U}_+(y_\star, n, \boldsymbol{p}_{\not{y}}) =$$

$$\begin{pmatrix} I_{n-1}(\xi)I_{n-1}(\xi_\star) & 0 & -\frac{p_z}{E_n+m}I_{n-1}(\xi)I_{n-1}(\xi_\star) & \frac{\sqrt{2ne\mathcal{B}}}{E_n+m}I_{n-1}(\xi)I_n(\xi_\star) \\ 0 & 0 & 0 & 0 \\ \frac{p_z}{E_n+m}I_{n-1}(\xi)I_{n-1}(\xi_\star) & 0 & -\frac{(p_z)^2}{(E_n+m)^2}I_{n-1}(\xi)I_{n-1}(\xi_\star) & p_z\frac{\sqrt{2ne\mathcal{B}}}{(E_n+m)^2}I_{n-1}(\xi)I_n(\xi_\star) \\ -\frac{\sqrt{2ne\mathcal{B}}}{E_n+m}I_n(\xi)I_{n-1}(\xi_\star) & 0 & p_z\frac{\sqrt{2ne\mathcal{B}}}{(E_n+m)^2}I_n(\xi)I_{n-1}(\xi_\star) & -\frac{2ne\mathcal{B}}{(E_n+m)^2}I_n(\xi)I_n(\xi_\star) \end{pmatrix}$$

Following similar procedures, for s=-1:

$\overline{U}_-(y_\star, n, \boldsymbol{p}_{\not{y}}) = U^\dagger_-(y_\star, n, \boldsymbol{p}_{\not{y}})\gamma^o \Longrightarrow \overline{U}_-(y_\star, n, \boldsymbol{p}_{\not{y}}) = \begin{bmatrix} 0 & I_n(\xi_\star) & \frac{\sqrt{2ne\mathcal{B}}}{E_n+m}I_{n-1}(\xi_\star) & \frac{p_z}{E_n+m}I_n(\xi_\star) \end{bmatrix}$

Thus

$$U_-(y, n, \boldsymbol{p}_{\not{y}})\overline{U}_-(y_\star, n, \boldsymbol{p}_{\not{y}}) = \begin{pmatrix} 0 \\ I_n(\xi) \\ -\frac{\sqrt{2ne\mathcal{B}}}{(E_n+m)}I_{n-1}(\xi) \\ -\frac{p_z}{E_n+m}I_n(\xi) \end{pmatrix} \begin{bmatrix} 0 & I_n(\xi_\star) & \frac{\sqrt{2ne\mathcal{B}}}{E_n+m}I_{n-1}(\xi_\star) & \frac{p_z}{E_n+m}I_n(\xi_\star) \end{bmatrix}$$

or

$$U_-(y, n, \boldsymbol{p}_{\not{y}})\overline{U}_-(y_\star, n, \boldsymbol{p}_{\not{y}}) =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I_n(\xi)I_n(\xi_\star) & \frac{\sqrt{2ne\mathcal{B}}}{E_n+m}I_n(\xi)I_{n-1}(\xi_\star) & \frac{p_z}{E_n+m}I_n(\xi)I_n(\xi_\star) \\ 0 & -\frac{\sqrt{2ne\mathcal{B}}}{E_n+m}I_{n-1}(\xi)I_n(\xi_\star) & -\frac{2ne\mathcal{B}}{(E_n+m)^2}I_{n-1}(\xi)I_{n-1}(\xi_\star) & -p_z\frac{\sqrt{2ne\mathcal{B}}}{(E_n+m)^2}I_{n-1}(\xi)I_n(\xi_\star) \\ 0 & -\frac{p_z}{E_n+m}I_n(\xi)I_n(\xi_\star) & -p_z\frac{\sqrt{2ne\mathcal{B}}}{(E_n+m)^2}I_n(\xi)I_{n-1}(\xi_\star) & -\frac{(p_z)^2}{(E_n+m)^2}I_n(\xi)I_n(\xi_\star) \end{pmatrix}$$

Thus

$$P_U(y,y_\star,n,\boldsymbol{p}_{\not y}) = U_+(y,n,\boldsymbol{p}_{\not y})\overline{U}_+(y_\star,n,\boldsymbol{p}_{\not y}) + U_-(y,n,\boldsymbol{p}_{\not y})\overline{U}_-(y_\star,n,\boldsymbol{p}_{\not y}) =$$

$$\begin{bmatrix} I_{n-1}(\xi)I_{n-1}(\xi_\star) & 0 & -\frac{p_z}{E_n+m}I_{n-1}(\xi)I_{n-1}(\xi_\star) & \frac{\sqrt{2ne\mathcal{B}}}{E_n+m}I_{n-1}(\xi)I_n(\xi_\star) \\ 0 & I_n(\xi)I_n(\xi_\star) & \frac{\sqrt{2ne\mathcal{B}}}{E_n+m}I_n(\xi)I_{n-1}(\xi_\star) & \frac{p_z}{E_n+m}I_n(\xi)I_n(\xi_\star) \\ \frac{p_z}{(E_n+m)}I_{n-1}(\xi)I_{n-1}(\xi_\star) & -\frac{\sqrt{2ne\mathcal{B}}}{E_n+m}I_{n-1}(\xi)I_n(\xi_\star) & \frac{-(2ne\mathcal{B}+p_z^2)}{(E_n+m)^2}I_{n-1}(\xi)I_{n-1}(\xi_\star) & 0 \\ -\frac{\sqrt{2ne\mathcal{B}}}{E_n+m}I_n(\xi)I_{n-1}(\xi_\star) & -\frac{p_z}{(E_n+m)}I_n(\xi)I_n(\xi_\star) & 0 & -\frac{(p_z^2+2ne\mathcal{B})}{(E_n+m)^2}I_n(\xi)I_n(\xi_\star) \end{bmatrix} \tag{3.2.3}$$

One can rewrite Eq.(3.2.1) once more as

$$P_U(y,y_\star,n,\boldsymbol{p}_{\not y}) =$$

$$\frac{1}{E_n+m}[I_n(\xi)I_n(\xi_\star)T_{n,n}+I_{n-1}(\xi)I_n(\xi_\star)T_{n-1,n}+I_{n-1}(\xi)I_{n-1}(\xi_\star)T_{n-1,n-1}+I_n(\xi)I_{n-1}(\xi_\star)T_{n,n-1}] \tag{3.2.4}$$

A straight forward comparison between Eqs.(3.2.3) and (3.2.4) and use of the dispersion relation Eq.(2.3.28) gives

$$T_{n,n} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (E_n+m) & 0 & p_z \\ 0 & 0 & 0 & 0 \\ 0 & -p_z & 0 & -(E_n-m) \end{pmatrix} \tag{3.2.5}$$

In 2×2 matrix notation the above matrix can be written as,

$$T_{n,n} = E_n\begin{pmatrix} \frac{1}{2}(\mathbf{1}-\sigma_3) & 0 \\ 0 & -\frac{1}{2}(\mathbf{1}-\sigma_3) \end{pmatrix} + p_z\begin{pmatrix} 0 & \frac{1}{2}(\mathbf{1}-\sigma_3) \\ -\frac{1}{2}(\mathbf{1}-\sigma_3) & 0 \end{pmatrix}$$

$$+m\begin{pmatrix} \frac{1}{2}(\mathbf{1}-\sigma_3) & 0 \\ 0 & \frac{1}{2}(\mathbf{1}-\sigma_3) \end{pmatrix} \tag{3.2.6}$$

Expanding Eq.(3.2.6)in 2×2 matrix notation

$$T_{n,n} = \frac{1}{2}E_n \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} + \frac{1}{2}E_n \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} + \frac{p_z}{2} \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} + \frac{p_z}{2} \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix} + \frac{m}{2} \begin{pmatrix} (\mathbf{1} - \sigma_3) & 0 \\ 0 & (\mathbf{1} - \sigma_3) \end{pmatrix} \quad (3.2.7)$$

In 4×4 matrix notation

$$T_{n,n} = \frac{E_n}{2}\gamma^o + \frac{E_n}{2}\gamma^5\gamma^3 + \frac{p_z}{2}\gamma^o\gamma^5 - \frac{p_z\gamma^3}{2} + \frac{m}{2}(1-\sigma_z) \quad \text{where} \qquad E_n = p^o = p_o \quad (3.2.8)$$

Using the properties of the gamma matrices the above 4×4 matrix becomes

$$T_{n,n} = \frac{1}{2}\left[p^o\gamma_0 + p^3(-\gamma^3) - p^o\gamma^3\gamma^5 + p^3\gamma^0\gamma^5 + m(1 - \sigma_z)\right]$$

$$\text{or} \qquad = \frac{1}{2}\left[m(1 - \sigma_z) + \not{p}_\parallel + \tilde{\not{p}}_\parallel\gamma_5\right] \quad (3.2.9)$$

where

$$\sigma_z = i\gamma^1\gamma^2, \quad \gamma^5 = i\gamma^o\gamma^1\gamma^2\gamma^3 \quad \text{and} \quad \not{p}_\parallel = p^o\gamma_0 + p^3\gamma_3 \ , \ \tilde{\not{p}}_\parallel = p^o\gamma_3 + p^3\gamma_0 \quad (3.2.10)$$

Again,Comparing Eqs.(3.2.3) and (3.2.4)once more provides

$$T_{n-1,n-1} = \begin{pmatrix} (E_n + m) & 0 & -p_z & 0 \\ 0 & 0 & 0 & 0 \\ p_z & 0 & -(E_n - m) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.2.11)$$

In 2×2 matrix notation the above matrix can be written as,

$$T_{n-1,n-1} = E_n \begin{pmatrix} \frac{1}{2}(\mathbf{1} + \sigma_3) & 0 \\ 0 & -\frac{1}{2}(\mathbf{1} + \sigma_3) \end{pmatrix} + p_z \begin{pmatrix} 0 & -\frac{1}{2}(\mathbf{1} + \sigma_3) \\ \frac{1}{2}(\mathbf{1} + \sigma_3) & 0 \end{pmatrix} + m \begin{pmatrix} \frac{1}{2}(\mathbf{1} + \sigma_3) & 0 \\ 0 & \frac{1}{2}(\mathbf{1} + \sigma_3) \end{pmatrix} \quad (3.2.12)$$

A further expansion Eq.(3.2.12), in 2×2 matrix notation provides

$$T_{n-1,n-1} = \tfrac{E_n}{2}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} + \tfrac{E_n}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} + \tfrac{p_z}{2}\begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} + \tfrac{p_z}{2}\begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$$

$$+\frac{m}{2}\begin{pmatrix} (\mathbf{1}+\sigma_3) & 0 \\ 0 & (\mathbf{1}+\sigma_3) \end{pmatrix}$$

In 4×4 matrix notation the above equation will be

$$T_{n-1,n-1} = \frac{1}{2}\left[p^o\gamma_0 + p^3(-\gamma^3) - p^o\gamma^5\gamma^3 - p^3\gamma^0\gamma^5 + m(1+\sigma_z)\right]$$

$$\text{or} \qquad = \frac{1}{2}\left[m(1+\sigma_z) + \not{p}_{\parallel} - \tilde{\not{p}}_{\parallel}\gamma_5\right] \tag{3.2.13}$$

Having similar comparison and analysis for rest of $T_{i,j}$ matrices

$$T_{n-1,n} = \sqrt{2ne\mathcal{B}}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.2.14}$$

In 2×2 matrix notation the above matrix looks like

$$T_{n-1,n} = \frac{\sqrt{2ne\mathcal{B}}}{2}\left[\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\right] \tag{3.2.15}$$

When converted back to the 4×4 matrix notation the above equation becomes ,

$$T_{n-1,n} = -\frac{1}{2}\sqrt{2ne\mathcal{B}}(\gamma_1 + i\gamma_2) \tag{3.2.16}$$

Similarly $T_{n,n-1}$ will be given as

$$T_{n,n-1} = \sqrt{2ne\mathcal{B}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{3.2.17}$$

The 2×2 matrix notation of the $T_{n,n-1}$ matrix is

$$T_{n,n-1} = \frac{\sqrt{2ne\mathcal{B}}}{2}\left[\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\right] \tag{3.2.18}$$

Which when converted back to the 4×4 notations the above 2×2 matrix becomes ,

$$T_{n,n-1} = -\frac{1}{2}\sqrt{2ne\mathcal{B}}(\gamma_1 - i\gamma_2) \tag{3.2.19}$$

So now we feed all the $T_{i,j}$ values from Eqs.(3.2.9),(3.2.13),(3.2.16),(3.2.19) to the spin-sum machine given by Eq.(3.2.1) or Eq.(3.2.4) so as to get the output :

$$\begin{aligned} P_U(y, y_\star, n, \boldsymbol{p}_{\not{y}}) &= \sum_s U_s(y, n, \boldsymbol{p}_{\not{y}})\overline{U}_s(y_\star, n, \boldsymbol{p}_{\not{y}}) \\ &= \frac{1}{2(E_n + m)} \times \{[m(1 + \sigma_z) + \not{p}_\parallel - \tilde{\not{p}}_\parallel\gamma_5]I_{n-1}(\xi)I_{n-1}(\xi_\star) \\ &+ [m(1 - \sigma_z) + \not{p}_\parallel + \tilde{\not{p}}_\parallel\gamma_5]I_n(\xi)I_n(\xi_\star) \\ &- \sqrt{2ne\mathcal{B}}[\gamma_1 - i\gamma_2]I_n(\xi)I_{n-1}(\xi_\star) \\ &- \sqrt{2ne\mathcal{B}}[\gamma_1 + i\gamma_2]I_{n-1}(\xi)I_n(\xi_\star)\} \end{aligned} \tag{3.2.20}$$

3.2.1 Calculation of the Spin-sum for the Negative Energy Solutions

Beside their vital difference there are certain similarities between the forms of the positive and the negative spinors which enable us to write the spin-sum for the negative spinors as

$$P_V(y, y_\star, n, \boldsymbol{p}_{\not{y}}) = \sum_s V_s(y, n, \boldsymbol{p}_{\not{y}})\overline{V}_s(y_\star, n, \boldsymbol{p}_{\not{y}}) = \frac{1}{E_n + m}\sum_{i,j=n-1}^{n} I_i(\tilde{\xi})I_j(\tilde{\xi}_\star)\tilde{T}_{i,j} \tag{3.2.21}$$

where $\tilde{T}_{i,j}s$ are the 4×4 matrices. Next we perform summation over all the possible spin degrees of freedom using left hand side of Eq.(3.2.21) to obtain the exact form of $\tilde{T}_{i,j}s$. Therefore:

$$\begin{aligned} P_V(y,y_\star,n,\boldsymbol{p}_{\not{y}}) &= \sum_s V_s(y,n,\boldsymbol{p}_{\not{y}})\overline{V}_s(y_\star,n,\boldsymbol{p}_{\not{y}}) \\ &= V_+(y,n,\boldsymbol{p}_{\not{y}})\overline{V}_+(y_\star,n,\boldsymbol{p}_{\not{y}}) + V_-(y,n,\boldsymbol{p}_{\not{y}})\overline{V}_-(y_\star,n,\boldsymbol{p}_{\not{y}}) \end{aligned}$$

But $\overline{V}_+(y_\star,n,\boldsymbol{p}_{\not{y}}) = V_+^\dagger(y_\star,n,\boldsymbol{p}_{\not{y}})\gamma^o$ where † shows the adjoint.

Thus, $\overline{V}_+(y_\star,n,\boldsymbol{p}_{\not{y}}) = \begin{bmatrix} \frac{p_z}{E_n+m}I_{n-1}(\tilde{\xi}_\star) & \frac{\sqrt{2ne\mathcal{B}}}{E_n+m}I_n(\tilde{\xi}_\star) & -I_{n-1}(\tilde{\xi}_\star) & 0 \end{bmatrix}$

so that

$$V_+(y,n,\boldsymbol{p}_{\not{y}})\overline{V}_+(y_\star,n,\boldsymbol{p}_{\not{y}}) = \begin{pmatrix} \frac{p_z}{E_n+m}I_{n-1}(\tilde{\xi}) \\ \frac{\sqrt{2ne\mathcal{B}}}{(E_n+m)}I_n(\tilde{\xi}) \\ I_{n-1}(\tilde{\xi}) \\ 0 \end{pmatrix} \begin{bmatrix} \frac{p_z}{E_n+m}I_{n-1}(\tilde{\xi}_\star) & \frac{\sqrt{2ne\mathcal{B}}}{E_n+m}I_n(\tilde{\xi}_\star) & -I_{n-1}(\tilde{\xi}_\star) & 0 \end{bmatrix}$$

Hence,

$$V_+(y,n,\boldsymbol{p}_{\not{y}})\overline{V}_+(y_\star,n,\boldsymbol{p}_{\not{y}}) =$$

$$\begin{pmatrix} \frac{p_z^2}{(E_n+m)^2}I_{n-1}(\tilde{\xi})I_{n-1}(\tilde{\xi}_\star) & p_z\frac{\sqrt{2ne\mathcal{B}}}{(E_n+m)^2}I_{n-1}(\tilde{\xi})I_n(\tilde{\xi}_\star) & -\frac{p_z}{E_n+m}I_{n-1}(\tilde{\xi})I_{n-1}(\tilde{\xi}_\star) & 0 \\ p_z\frac{\sqrt{2ne\mathcal{B}}}{(E_n+m)^2}I_n(\tilde{\xi})I_{n-1}(\tilde{\xi}_\star) & \frac{2ne\mathcal{B}}{(E_n+m)^2}I_n(\tilde{\xi})I_n(\tilde{\xi}_\star) & -\frac{\sqrt{2ne\mathcal{B}}}{(E_n+m)}I_n(\tilde{\xi})I_{n-1}(\tilde{\xi}_\star) & 0 \\ \frac{p_z}{E_n+m}I_{n-1}(\tilde{\xi})I_{n-1}(\tilde{\xi}_\star) & \frac{\sqrt{2ne\mathcal{B}}}{E_n+m}I_{n-1}(\tilde{\xi})I_n(\tilde{\xi}_\star) & -I_{n-1}(\tilde{\xi})I_{n-1}(\tilde{\xi}_\star) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Following similar procedures, for s=-1 :

$\overline{V}_-(y_\star,n,\boldsymbol{p}_{\not{y}}) = V_-^\dagger(y_\star,n,\boldsymbol{p}_{\not{y}})\gamma^o \Longrightarrow \overline{V}_-(y_\star,n,\boldsymbol{p}_{\not{y}}) = \begin{bmatrix} \frac{\sqrt{2ne\mathcal{B}}}{E_n+m}I_{n-1}(\tilde{\xi}_\star) & -\frac{p_z}{E_n+m}I_n(\tilde{\xi}_\star) & 0 & -I_n(\tilde{\xi}_\star) \end{bmatrix}$

so that

$$V_-(y,n,\boldsymbol{p}_{\not y})\overline{V}_-(y_\star,n,\boldsymbol{p}_{\not y}) = \begin{pmatrix} \frac{\sqrt{2ne\mathcal{B}}}{E_n+m} I_{n-1}(\tilde{\xi}) \\ -\frac{p_z}{E_n+m} I_n(\tilde{\xi}) \\ 0 \\ I_n(\tilde{\xi}) \end{pmatrix} \times \begin{bmatrix} \frac{\sqrt{2ne\mathcal{B}}}{E_n+m} I_{n-1}(\tilde{\xi}_\star) & -\frac{p_z}{E_n+m} I_n(\tilde{\xi}_\star) & 0 & -I_n(\tilde{\xi}_\star) \end{bmatrix}$$

Thus,

$$V_-(y,n,\boldsymbol{p}_{\not y})\overline{V}_-(y_\star,n,\boldsymbol{p}_{\not y}) =$$

$$\begin{pmatrix} \frac{2ne\mathcal{B}}{(E_n+m)^2} I_{n-1}(\tilde{\xi}) I_{n-1}(\tilde{\xi}_\star) & \frac{-p_z\sqrt{2ne\mathcal{B}}}{(E_n+m)^2} I_{n-1}(\tilde{\xi}) I_n(\tilde{\xi}_\star) & 0 & -\frac{\sqrt{2ne\mathcal{B}}}{(E_n+m)} I_{n-1}(\tilde{\xi}) I_n(\tilde{\xi}_\star) \\ -p_z \frac{\sqrt{2ne\mathcal{B}}}{(E_n+m)^2} I_n(\tilde{\xi}) I_{n-1}(\tilde{\xi}_\star) & \frac{(p_z)^2}{(E_n+m)^2} I_n(\tilde{\xi}) I_n(\tilde{\xi}_\star) & 0 & \frac{p_z}{(E_n+m)} I_n(\tilde{\xi}) I_n(\tilde{\xi}_\star) \\ 0 & 0 & 0 & 0 \\ \frac{\sqrt{2ne\mathcal{B}}}{E_n+m} I_n(\tilde{\xi}) I_{n-1}(\tilde{\xi}_\star) & -\frac{p_z}{E_n+m} I_n(\tilde{\xi}) I_n(\tilde{\xi}_\star) & 0 & -I_n(\tilde{\xi}) I_n(\tilde{\xi}_\star) \end{pmatrix}$$

Consequently, $P_V(y,y_\star,n,\boldsymbol{p}_{\not y}) = V_+(y,n,\boldsymbol{p}_{\not y})\overline{V}_+(y_\star,n,\boldsymbol{p}_{\not y}) + V_-(y,n,\boldsymbol{p}_{\not y})\overline{V}_-(y_\star,n,\boldsymbol{p}_{\not y}) =$

$$\begin{bmatrix} \frac{(p_z^2+2ne\mathcal{B})}{(E_n+m)^2} I_{n-1}(\tilde{\xi}) I_{n-1}(\tilde{\xi}_\star) & 0 & \frac{-p_z}{E_n+m} I_{n-1}(\tilde{\xi}) I_{n-1}(\tilde{\xi}_\star) & \frac{-\sqrt{2ne\mathcal{B}}}{E_n+m} I_{n-1}(\tilde{\xi}) I_n(\tilde{\xi}_\star) \\ 0 & \frac{(p_z^2+2ne\mathcal{B})}{(E_n+m)^2} I_n(\tilde{\xi}) I_n(\tilde{\xi}_\star) & -\frac{\sqrt{2ne\mathcal{B}}}{E_n+m} I_n(\tilde{\xi}) I_{n-1}(\tilde{\xi}_\star) & \frac{p_z}{E_n+m} I_n(\tilde{\xi}) I_n(\tilde{\xi}_\star) \\ \frac{p_z}{(E_n+m)} I_{n-1}(\tilde{\xi}) I_{n-1}(\tilde{\xi}_\star) & \frac{\sqrt{2ne\mathcal{B}}}{E_n+m} I_{n-1}(\tilde{\xi}) I_n(\tilde{\xi}_\star) & -I_{n-1}(\tilde{\xi}) I_{n-1}(\tilde{\xi}_\star) & 0 \\ \frac{\sqrt{2ne\mathcal{B}}}{E_n+m} I_n(\tilde{\xi}) I_{n-1}(\tilde{\xi}_\star) & -\frac{p_z}{(E_n+m)} I_n(\tilde{\xi}) I_n(\tilde{\xi}_\star) & 0 & -I_n(\tilde{\xi}) I_n(\tilde{\xi}_\star) \end{bmatrix} \tag{3.2.22}$$

A complete expansion of the term given at the right hand side of Eq.(3.2.21)provides

$P_V(y, y_\star, n, \boldsymbol{p}_y) =$

$$\frac{1}{E_n + m}[I_n(\tilde{\xi})I_n(\tilde{\xi_\star})\tilde{T}_{n,n}+I_{n-1}(\tilde{\xi})I_n(\tilde{\xi_\star})\tilde{T}_{n-1,n}+I_{n-1}(\tilde{\xi})I_{n-1}(\tilde{\xi_\star})\tilde{T}_{n-1,n-1}+I_n(\tilde{\xi})I_{n-1}(\tilde{\xi_\star})\tilde{T}_{n,n-1}] \tag{3.2.23}$$

Comparison between the above two consecutive equations furnishes the $\tilde{T}_{i,j}$ values as:

$$\tilde{T}_{n,n} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (E_n - m) & 0 & p_z \\ 0 & 0 & 0 & 0 \\ 0 & -p_z & 0 & -(E_n + m) \end{pmatrix} \tag{3.2.24}$$

Rewriting the above matrix in 2×2 matrix notation we get

$$\tilde{T}_{n,n} = E_n \begin{pmatrix} \frac{1}{2}(\mathbf{1} - \sigma_3) & 0 \\ 0 & -\frac{1}{2}(\mathbf{1} - \sigma_3) \end{pmatrix} + p_z \begin{pmatrix} 0 & \frac{1}{2}(\mathbf{1} - \sigma_3) \\ -\frac{1}{2}(\mathbf{1} - \sigma_3) & 0 \end{pmatrix}$$

$$-m \begin{pmatrix} \frac{1}{2}(\mathbf{1} - \sigma_3) & 0 \\ 0 & \frac{1}{2}(\mathbf{1} - \sigma_3) \end{pmatrix} \tag{3.2.25}$$

More detailed form of the above 2×2 notation will be exhibited as

$$\tilde{T}_{n,n} = \frac{1}{2}E_n \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} + \frac{1}{2}E_n \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} + \frac{p_z}{2} \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} + \frac{p_z}{2} \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$$

$$-\frac{m}{2} \begin{pmatrix} (\mathbf{1} - \sigma_3) & 0 \\ 0 & (\mathbf{1} - \sigma_3) \end{pmatrix}$$

Therefore with the use of *Append.D* and properties of the gamma matrices, in 4×4 notation we have

$\tilde{T}_{n,n} = \frac{E_n}{2}\gamma^o + \frac{E_n}{2}\gamma^5\gamma^3 + \frac{p_z}{2}\gamma^o\gamma^5 - \frac{p_z\gamma^3}{2} - \frac{m}{2}(1 - \sigma_z)$ where $E_n = p^o = p_o$

$$\tilde{T}_{n,n} = \frac{1}{2}\left[p^o\gamma_0 + p^3(-\gamma^3) - p^o\gamma^3\gamma^5 + p^3\gamma^0\gamma^5 - m(1 - \sigma_z)\right]$$

$$\tilde{T}_{n,n} = \frac{1}{2}\left[-m(1 - \sigma_z) + \not{p}_{\parallel} + \tilde{\not{p}}_{\parallel}\gamma_5\right] \tag{3.2.26}$$

The other possible $\tilde{T}_{i,j}$ matrix is $\tilde{T}_{n-1,n-1}$ which can be written as ,

$$\tilde{T}_{n-1,n-1} = \begin{pmatrix} (E_n - m) & 0 & -p_z & 0 \\ 0 & 0 & 0 & 0 \\ p_z & 0 & -(E_n + m) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.2.27}$$

The 2×2 representation of the above matrix becomes,

$$\tilde{T}_{n-1,n-1} = E_n \begin{pmatrix} \frac{1}{2}(\mathbf{1} + \sigma_3) & 0 \\ 0 & -\frac{1}{2}(\mathbf{1} + \sigma_3) \end{pmatrix} + p_z \begin{pmatrix} 0 & -\frac{1}{2}(\mathbf{1} + \sigma_3) \\ \frac{1}{2}(\mathbf{1} + \sigma_3) & 0 \end{pmatrix}$$

$$-m \begin{pmatrix} \frac{1}{2}(\mathbf{1} + \sigma_3) & 0 \\ 0 & \frac{1}{2}(\mathbf{1} + \sigma_3) \end{pmatrix} \tag{3.2.28}$$

$$\text{Or } \tilde{T}_{n-1,n-1} = \frac{E_n}{2} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} + \frac{E_n}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} + \frac{p_z}{2} \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} + \frac{p_z}{2} \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$$

$$-\frac{m}{2} \begin{pmatrix} (\mathbf{1} + \sigma_3) & 0 \\ 0 & (\mathbf{1} + \sigma_3) \end{pmatrix}$$

Hence the corresponding 4×4 representation the above matrix equation becomes

$$\tilde{T}_{n-1,n-1} = \frac{1}{2} \left[p^o\gamma_0 + p^3(-\gamma^3) - p^o\gamma^5\gamma^3 - p^3\gamma^0\gamma^5 - m(1 + \sigma_z) \right]$$

$$\tilde{T}_{n-1,n-1} = \frac{1}{2} \left[-m(1 + \sigma_z) + \not{p}_\parallel - \tilde{\not{p}}_\parallel \gamma_5 \right] \tag{3.2.29}$$

To complete the expanded summation given by Eq.(3.2.23) we still need the other two $\tilde{T}_{i,j}$ matrices, $\tilde{T}_{n,n-1}$ and $\tilde{T}_{n-1,n}$ where

$$\tilde{T}_{n-1,n} = \sqrt{2ne\mathcal{B}} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.2.30}$$

In 2×2 notation the above matrix becomes

$$\tilde{T}_{n-1,n} = \frac{\sqrt{2ne\mathcal{B}}}{2}\left[\begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\right] \tag{3.2.31}$$

From the 2×2 notation given above one can recover 4×4 notation as,

$$\tilde{T}_{n-1,n} = \frac{1}{2}\sqrt{2ne\mathcal{B}}(\gamma_1 + i\gamma_2) \tag{3.2.32}$$

Similarly for $\tilde{T}_{n,n-1}$,

$$\tilde{T}_{n,n-1} = \sqrt{2ne\mathcal{B}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{3.2.33}$$

The 2×2 notation of the $\tilde{T}_{n,n-1}$ matrix is

$$\tilde{T}_{n,n-1} = \frac{\sqrt{2ne\mathcal{B}}}{2}\left[\begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix} - i\begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\right] \tag{3.2.34}$$

which when converted back to the 4×4 notation becomes ,

$$\tilde{T}_{n,n-1} = \frac{1}{2}\sqrt{2ne\mathcal{B}}(\gamma_1 - i\gamma_2) \tag{3.2.35}$$

Right now the summation machine given by Eq.(3.2.21) or(3.2.23) is ready hence we supply the input from Eq.(3.2.26),Eq(3.2.29),Eq(3.2.32),Eq(3.2.35) so as to get the output:

$$\begin{aligned}
P_V(y, y_\star, n, \boldsymbol{p}_{\not y}) &= \sum_s V_s(y, n, \boldsymbol{p}_{\not y})\overline{V}_s(y_\star, n, \boldsymbol{p}_{\not y}) \\
&= \frac{1}{2(E_n + m)} \times \{[-m(1+\sigma_z) + \not p_\parallel - \tilde{\not p}_\parallel\gamma_5]I_{n-1}(\tilde{\xi})I_{n-1}(\tilde{\xi}_\star) \\
&+ [-m(1-\sigma_z) + \not p_\parallel + \tilde{\not p}_\parallel\gamma_5]I_n(\tilde{\xi})I_n(\tilde{\xi}_\star) \\
&+ \sqrt{2ne\mathcal{B}}[\gamma_1 - i\gamma_2]I_n(\tilde{\xi})I_{n-1}(\tilde{\xi}_\star) \\
&+ \sqrt{2ne\mathcal{B}}[\gamma_1 + i\gamma_2]I_{n-1}(\tilde{\xi})I_n(\tilde{\xi}_\star)\}
\end{aligned} \tag{3.2.36}$$

Some of the most amazing features of the results that we acquire from equations (3.2.20) and (3.2.36) are first

$$P_U(y, y_\star, n, \boldsymbol{p}_y) = -P_V(y, y_\star, n, -p_y) \tag{3.2.37}$$

which is similar to the result in free space Eq.(1.2.20).

Second, this spin-sum is quite indispensable (corner stone)in calculations of scattering cross-sections and decay-rates involving charged fermions in back-ground magnetic field.

Chapter 4

Discussion and Conclusion

4.1 Gauge Dependence

At the beginning of the second chapter we gave short summary on gauge condition and there we have chosen gauge configuration Eq.(2.2.8)which provides the background magnetic field along the z-coordinate axis. This equation is used to solve the Dirac equation in the presence of background magnetic field . Therefore most of the results that we obtain throughout this thesis are specific to the choice of the background gauge field . The spinor solutions are dependent on the gauge choice and hence are not gauge invariant. On the contrary physical observables (Quantities) like the dispersion relation, the decay-rates and scattering cross-section are gauge invariant. If I made other gauge choice instead of Eq.(2.2.8) then the spinor solutions, the Spin-sum Eq.(3.2.20), Eq.(3.2.36) and the ortho-normality in section 2.6 would be different and hence are gauge dependent.But in the calculations of decay-rates and scattering cross-sections we always perform integration and consequently the end results will not depend upon which gauge we started with. As we have already stated the energy of the electron given in Eq.(2.3.25)with $Q = -1$ is not gauge dependent quantity, any gauge we choose we will get the same dispersion relation of the electrons. One of the many unique features of the $n = 0$ state is that this state is not gauge choice dependent as it depends directly on the form of the dispersion relation.

The above discussion highlights the fact that, most of the quantities calculated in this thesis using the exact solutions in the presence of the uniform magnetic field relay heavily on the choice of the vector potential. That is, the solutions of the Dirac equation in the presence of a uniform magnetic field along the z-direction obtained using various vector potential will be different but related by smooth gauge transformations. Second the free Dirac solutions can be gauge rotated where the gauge fields are pure gauge configuration. But since there is no connection between the gauge configurations giving rise to a magnetic field along the z- coordinate axis and pure gauge fields, we can not retrieve the free Dirac solutions as a limit of the exact solutions in a magnetic field.

4.2 Conclusion

In this thesis we solved the Dirac equation for both the free field and in the presence of a uniform background magnetic field specified by a particular vector potential. The dispersion relation of the electron is seen to change from its form in the vacuum and we see the emergence of Landau levels designating the quantized nature of the transverse motion of the electrons. The Dirac solutions in the presence of background field are dependent on the Landau levels and the energy of electron is seen to be degenerate except the lowest Landau level energy i.e $n = 0$. It is seen that there is no way to get back the free Dirac solutions from the exact solutions in presence of the magnetic field by letting the field strength goes zero in the solutions, a fact which is related to the gauge invariance of the system. Unlike the free Dirac solutions, the Dirac solutions in the presence of the back ground magnetic field are not in general ortho-normal. When we have a magnetic field of much smaller magnitude than the critical magnetic field then the Landau level number will be very high, making the solutions ortho-normal. The spin-sum of this solutions are derived explicitly using the exact solutions of the Dirac equation in a magnetic field. Though the calculations are exact and important in astrophysical applications we need to be a bit careful about the gauge dependence of the the result. As most of the quantities calculated in this thesis depend on the choice of the vector potential giving rise to the magnetic field so the gauge invariance of the calculations become less transparent. In the penultimate section we discuss about the gauge invariance of the calculations in presence of a magnetic field and show that although the spin-sums may not be gauge invariant but physical quantities like scattering-cross sections and decay rates of elementary particles in presence of a magnetic field can be gauge invariant. The main application of the overall work is in the calculations of decay-rates and scattering cross-sections of charged particles in the presence of a magnetic field.Hopefully the astrophysics research group will use the results of this thesis.

Appendices. Supplements

4.3 A . Identities

From the properties of the Pauli matrices ($\sigma_i \quad , i = 1, 2, 3$)we have

$\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ and $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$ as anti-commutation and commutation relation of the three Pauli matrices respectively. where

$$\delta_{ij} = \begin{cases} 1, & \text{for i=j} \\ 0, & \text{otherwise} \end{cases}$$

and ϵ_{ijk} is Levi-Civita symbol defined by

$$\epsilon_{ijk} = \begin{cases} +1, & \text{even permutaions of i,j,k} \\ -1, & \text{odd permutations} \\ 0, & \text{otherwise} \end{cases}$$

as a result, for any two 3 vectors **a** and **b**

$$\begin{aligned} (\sigma \cdot \mathbf{a})(\sigma \cdot \ \mathbf{b}) &= \sum_j \sigma_j \mathbf{a}_j \sum_k \sigma_k \mathbf{b}_k \\ &= \sum_j \sum_k \sigma_j \sigma_k \mathbf{a}_j \mathbf{b}_k \end{aligned}$$

From the commutation and the anti commutation relation we obtain the identity relation

$$\begin{aligned} (\sigma \cdot \mathbf{a})(\sigma \cdot \mathbf{b}) &= \sum_j \sum_k (\delta_{jk} \mathbf{a}_j \mathbf{b}_k + i\epsilon_{ijk}\sigma_i \mathbf{a}_j \mathbf{b}_k) \\ (\sigma \cdot \mathbf{a})(\sigma \cdot \ \mathbf{b}) &= (\mathbf{a} \cdot \ \mathbf{b}) + i\sigma \cdot (\mathbf{a} \times \mathbf{b}) \end{aligned}$$

4.4 The Normalization Constants and Relations Between Different I's

let's multiply two generating functions

$$e^{-t^2+2t\xi}e^{-s^2+2s\xi} = \sum_{m,\nu=0}^{\infty} H_m(\xi)H_\nu(\xi)\frac{s^m}{m!}\frac{t^\nu}{\nu!}$$

so that

$$\begin{aligned}\sum_{\nu=0}^{\infty}\frac{(st)^\nu}{\nu!\nu!}\int_{-\infty}^{\infty} e^{-\xi^2}\left[H_\nu(\xi)\right]^2 d\xi &= \int_{-\infty}^{\infty} e^{-t^2+2t\xi-s^2+2s\xi-\xi^2}d\xi \\ &= \int_{-\infty}^{\infty} e^{-(\xi-s-t)^2}e^{2st}d\xi \\ &= \pi^{1/2}e^{2st} = \pi^{1/2}\sum_{\nu=0}^{\infty}\frac{2^\nu(st)^\nu}{\nu!}\end{aligned}$$

Equating the the left and side with the right hand side

$$\int_{-\infty}^{\infty} e^{-\xi^2}\left[H_\nu(\xi)\right]^2 d\xi = 2^\nu\pi^{1/2}\nu! \tag{4.4.1}$$

The probability of finding a particle over the entire space is 1. i.e

$$\int_{-\infty}^{\infty} I_\nu^\star(y)I_\nu(y)dy = 1$$

$$\begin{aligned}\Rrightarrow |N_\nu|^2\int_{-\infty}^{\infty} e^{-\xi^2}\left[H_\nu(\xi)\right]^2(\frac{dy}{d\xi})d\xi &= 1 \\ \frac{|N_\nu|^2}{\sqrt{e|Q|\mathcal{B}}}\left(2^\nu\pi^{1/2}\nu!\right) &= 1\end{aligned}$$

$$N_\nu = \left(\frac{\sqrt{e|Q|\mathcal{B}}}{2^\nu\pi^{1/2}\nu!}\right)^{1/2} \tag{4.4.2}$$

From equation Eq.(2.3.19) $I_\nu(\xi) = N_\nu e^{-\xi^2/2} H_\nu(\xi)$ and from equation(4.5.8)

$$\begin{aligned} N_\nu &= \left(\frac{\sqrt{e|Q|\mathcal{B}}}{2^\nu \pi^{1/2} \nu!}\right)^{1/2} \\ N_\nu &= \frac{1}{(2\nu)^{1/2}} \left(\frac{\sqrt{e|Q|\mathcal{B}}}{2^{\nu-1}\pi^{1/2}(\nu-1)!}\right)^{1/2} = \frac{N_{\nu-1}}{(2\nu)^{1/2}} \end{aligned}$$

hence,

$$I_\nu(\xi) = \left(\frac{\sqrt{e|Q|\mathcal{B}}}{2^\nu \pi^{1/2} \nu!}\right)^{1/2} e^{-\xi^2/2} H_\nu(\xi)$$

Using the first recurrent relation Eq.(4.5.2)

$$H_\nu(\xi) = 2\xi H_{\nu-1}(\xi) - \frac{\partial H_{n-1}(\xi)}{\partial \xi}$$

$$\begin{aligned} \Rrightarrow I_\nu(\xi) &= 2\xi H_{\nu-1}(\xi) N_\nu e^{-\xi^2/2} - \frac{\partial H_{n-1}(\xi)}{\partial \xi} N_\nu e^{-\xi^2/2} \\ &= \frac{2\xi}{(2\nu)^{1/2}} I_{\nu-1}(\xi) - \frac{1}{(2\nu)^{1/2}} \frac{\partial}{\partial \xi} \left(I_{\nu-1}(\xi)\right) - \xi \frac{I_{\nu-1}(\xi)}{(2\nu)^{1/2}} \end{aligned}$$

hence

$$I_\nu(\xi) = \frac{1}{(2\nu)^{1/2}} \left[\xi I_{\nu-1}(\xi) - \frac{\partial I_{\nu-1}(\xi)}{\partial \xi}\right] \tag{4.4.3}$$

To find the other relation between $I_{\nu-1}(\xi)$ and $I_\nu(\xi)$ a similar procedure can be maintained. From Eq.(4.5.9), using some definition and recurrence relation :

$$I_{\nu+1}(\xi) = \frac{1}{[2(\nu+1)]^{1/2}} \left[\xi I_\nu(\xi) - \frac{\partial I_\nu(\xi)}{\partial \xi}\right]$$

and also

$$\begin{aligned} I_{\nu+1}(\xi) &= N_{\nu+1} e^{-\xi^2/2} H_{\nu+1}(\xi) \\ &= N_{\nu+1} e^{-\xi^2/2} \left[2\xi H_\nu(\xi) - 2\nu H_{\nu-1}(\xi)\right] \\ &= \frac{2\xi}{[2(\nu+1)]^{1/2}} \left[I_\nu(\xi)\right] - \frac{2\nu}{[2(\nu)]^{1/2}[2(\nu+1)]^{1/2}} \left[I_{\nu-1}(\xi)\right] \end{aligned}$$

Comparing the above two consecutive equations provides a relation

$$I_{\nu-1}(\xi) \;=\; \frac{1}{(2\nu)^{1/2}}\left[\xi I_\nu(\xi) + \frac{\partial I_\nu(\xi)}{\partial \xi}\right] \tag{4.4.4}$$

4.5 D . The Gamma Matrices and the Dimensionless Variable

By definition $\gamma_o = \gamma^o = \beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$ where $\mathbf{1}$ is a 2×2 unit matrix.And also by definition

$\gamma^k = \beta\alpha^k = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}\begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$ where $k = 1, 2, 3$ and σ^k are the Pauli matrices

$\gamma_k = g_{kj}\gamma^j = -\gamma^k$ in general. The other 4×4 gamma matrix is

$\gamma^5 = \gamma_5 = i\gamma^o\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$ similarly

$\sigma_z = i\gamma^1\gamma^2 = i\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} -\sigma_1\sigma_2 & 0 \\ 0 & -\sigma_1\sigma_2 \end{pmatrix} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$

where σ_z is a 4×4 matrix. Multiplications of $\gamma^5\gamma^3$ and $\gamma^o\gamma^5$ in 2×2 notation are very straight forward:

$\gamma^5\gamma^3 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$,

$\gamma^o\gamma^5 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$

Lastly the anti commutation relation of the gamma matrices has the following form

$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ where $\nu = \mu = 0, 1, 2, 3$[12].

Next we will have a short summary on dimension of the variable ξ. In natural units

$\hbar = c = 1$, because energy $E = \hbar\omega$ the dimension of energy will be inverse of the dimension of time. Considering the relation between energy and mass provides a similar dimension for both . Beside the above two relations the dimension of length will be the same as the dimension of time for the simple reason that $c = 1$ in natural units[4]. Merging all the three ideas [Length=L]=[Time=T]=[Energy=E]$^{-}$=[Mass=M]$^{-}$ from Eq.(2.3.15)

$\xi = \sqrt{e|Q|\mathcal{B}}\left(y + \frac{p_x}{eQ\mathcal{B}}\right)$

But if V=velocity ,Q=charge, B=magnetic field then

dim(Force=F)=$[Q][V][B]$

$\Rightarrow$ dim(momentume=P)=$[T][Q][V][B]$=$[T][Q][B]$

$\Rightarrow$ dim$\left(\frac{P}{QB}\right)$=$\frac{[T][Q][B]}{[Q][B]} = [T] = [L]$

dim(QB)=[Q][B]=$\frac{P}{T} = \frac{[M]}{[T]} = \frac{1}{[T]^2}$ therefore our variable ξ *is dimension less* in the natural unit.

N.B The dirac delta function has the following important property:

$$\delta\left(g(x)\right) = \sum_a \frac{\delta(x-a)}{|g'(a)|}$$

$$where \quad g(a)=0$$

$$g'(a) \neq 0$$

4.6 Lorentz Transformation and Ortho-normality of free space solutions

For a lorentz boost along z-axis the transformation of the 4- vector $x^\mu = (x_o, \mathbf{x})$is of the form[13]

$$z' = \gamma(z - \mathbf{B}x_o) \quad , \quad x'_o = \gamma(x_o - \mathbf{B}z)$$

$$y' = y, x' = x$$

The above transformation can be rewritten as

$$x^{\mu'} = \Lambda^{\mu}_{\nu} x^{\nu}$$

$$\begin{pmatrix} x'_o \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\mathbf{B} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\mathbf{B} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_o \\ x \\ y \\ z \end{pmatrix}$$

where $\gamma = \frac{1}{\sqrt{1-\mathbf{B}^2}} \mathbf{B} = V$ in natural units

Using standard parameterizations

$$\sinh\omega = \gamma\mathbf{B} \qquad \cosh\omega = \gamma$$

For infinitesimal rotation through angle ω

$$\begin{pmatrix} x'_o \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\omega & 0 & 0 & -\sinh\omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\omega & 0 & 0 & \cosh\omega \end{pmatrix} \begin{pmatrix} x_o \\ x \\ y \\ z \end{pmatrix}$$

For a small ω

$$\begin{pmatrix} x'_o \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -\omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\omega & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_o \\ x \\ y \\ z \end{pmatrix} \tag{4.6.1}$$

The lorentz transformation of the 4- vector p^{μ} is

$$p^{\mu'} = \Lambda^{\mu}_{\nu} p^{\nu}$$

or

$$p'^{\mu} = (\delta^{\mu}_{\nu} + \varepsilon^{\mu}_{\nu}) p^{\nu}$$

where

$$\varepsilon^{\mu}_{\nu} = \Delta\omega I^{\mu}_{\nu}$$

Hence,from Eq.(4.7.1) for boost along z-axis

$$I^{\mu}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{4.6.2}$$

Since

$$\cosh(\omega) = \gamma$$

$$\cosh(\frac{\omega}{2}) = \cosh^2(\frac{\omega}{2}) + \sinh^2(\frac{\omega}{2}) = 2\cosh^2(\frac{\omega}{2}) - 1$$

Hence,

$$\cosh(\frac{\omega}{2}) = \sqrt{\frac{(1+\gamma)}{2}} = \sqrt{\frac{E+m}{2m}}$$

Similarly

$$\sinh(\frac{\omega}{2}) = \left(\frac{\gamma\mathbf{B}}{2}\right)\left(\frac{1}{\cosh(\frac{\omega}{2})}\right) = \frac{P}{\sqrt{2m(E+m)}}$$

The product of the positive free space spinors is

$$\begin{aligned} \overline{U}_1(\mathbf{P})U_2(\mathbf{P}) &= U_1^{\dagger}\gamma^{o}(\mathbf{p})U_2(\mathbf{p}) \\ &= \begin{pmatrix} \sqrt{\frac{E+m}{2m}} & 0 & -\frac{P}{\sqrt{2m(E+m)}} & 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ \sqrt{\frac{E+m}{2m}} \\ 0 \\ -\frac{P}{\sqrt{2m(E+m)}} \end{pmatrix} \\ &= 0 \end{aligned}$$

similarly

$$\begin{aligned}\overline{V}_2(\mathbf{P})V_2(\mathbf{P}) &= V_1^{\dagger}\gamma^{o}(\mathbf{p})V_2(\mathbf{p}) \\ &= \begin{pmatrix} 0 & -\frac{P}{\sqrt{2m(E+m)}} & 0 & -\sqrt{\frac{E+m}{2m}} \end{pmatrix} \times \begin{pmatrix} 0 \\ -\frac{P}{\sqrt{2m(E+m)}} \\ 0 \\ \sqrt{\frac{E+m}{2m}} \end{pmatrix} \\ &= -1\end{aligned}$$

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Declaration

I hereby declare that this thesis is my original work and has not been presented for a degree in any other university. All sources of material used for the thesis have been duly acknowledged.

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June, 2009