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SPECTRAL ANALYSIS OF SINGULARITIES OF DISTRIBUTIONS

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A Project submitted in partial fulfillment of the requirement of the degree of master of science in mathematics

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The undersigned hereby certify that they have read and recommend to the department of mathematics for acceptance of this project entitled **"Spectral analysis of singularities of distributions"** by **Nebiyou Zebene** in partial fulfillment of the requirements for the degree of Master of Science in Differential Equation.

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Abstract

This project report entitled by ″*Spectral analysis of singularities of distributions*″ explored how singularities appeared in the theory of distribution. The main objective of the study is to detect singularities of distributions from both smoothness and analytic point of view. And those singularities that we are talking about can be detected by using the conic set which tells us the direction where the singularities are coming from and using the singular support of a given function which describes the location of singularities. In the study we used a new set called the wave front set which explored both the location and direction of singularities.

Introduction

In differential calculus we usually face with the unpleasant fact that not every function is differentiable. The main purpose of distribution theory is to remedy this flaw. In the theory of distribution we use functions which are infinitely continuously differentiable in $\Re^n$ with a compact support which are called test functions in order to solve problems in differentiability of functions. Many researchers, mathematicians and different authors play their own role for the discovery and development of all theories related to distribution. But there is one limitation on the theory of distribution which is we do not have a general definition of multiplication and composition of distributions.

In this paper we discuss the Fourier analysis of singularities of distributions. When we say singularities it is both from smoothness and from analytic point of view. Here if we consider smooth compactly supported function f, then the Fourier transform of f denoted by F[f] decays faster than any negative power of the dual variable ξ; that is for every number N there exists a constant C_N such that

$$|F[f](\xi)| \leq C_N(1 + |\xi|)^{-N} \tag{1}$$

On the other hand, if the Fourier transform of a distribution with compact support satisfies the estimate (1) then this distribution is actually induced by a smooth function. Therefore,the estimate (1) can be viewed as a characteristic property for smoothness.

In section 3.1 we will see if a distribution f is not smooth we can use the set of directions where F[f] is not rapidly decreasing to describe which are the high frequency components of f causing the singularities. From this analysis for a distribution, f defined on $X \subset \Re^n$ on a C^∞ manifold X we define a new set called the wave front set of f which is denoted $WF(f)$. This set will give us more precise description of singularities; it tells us not only at what points a singularity occur, but it also indicates the direction in the dual space from which the singularities are coming.

In section 3.2 we will review about multiplication of distributions. Particularly about cases in which we can multiply distributions. Then we will use the wave front set to define multiplication of distributions extendedly.
For the first time some remarkable result about smoothness and analyticity of solutions of elliptic differential operators were obtained by Bernstein at the beginning of the twentieth century.

In section 3.3 we prove the simplest facts on the wave front set of solutions of linear partial differential equations in order to identify the region in which solutions of differential equation are not smooth. in particular the wave front set is included in the union of the characteristic set and the wave front set of the right-hand side. It was proved by Petkowsky that all classical solutions of a partial differential equation with constant coefficients are analytic if and only if the equation is elliptic. He also gave a proof of the analyticity of the solutions of (non-linear) analytic elliptic differential equations, that is by extending the results of Bernstein. And hence in section 3.3 we will also study about a corresponding characterization of differential equations with constant coefficients having only continuous solutions, such equations are called hypoelliptic. Moreover in this section we will see conditions for hypoellepticity.

Since singularities need to be seen from an analytic point of view, we shall define a modified wave front set of a distribution f using the notation $WF_A(f)$ in which f is analytic in the complement, which is going to be seen in section 3.4. In section 3.5 and section 3.6 we deal on analogue of section 3.2 and section 3.3 for a new set denoted by C^L and which is defined in section 3.4.

Chapter 1

Preliminary

1.1 Notations

- $C(\Omega)$ the space of continuous functions , where $\Omega \subset \Re^n$
- C^∞ the space of infinitely continuously differentiable functions , where $\Omega \subset \Re^n$
- $D(\Omega) = C_0^\infty(\Omega)$ the space of functions in $C^\infty(\Omega)$ compactly supported in Ω. The elements of $C_0^\infty(\Omega)$ are called test functions.
- S - The set of test functions of the class $C^\infty(\Re^n)$ which decreases as $|x| \to \infty$, together with all their derivatives, faster than any power of $|x|^{-1}$
- D' - The space of generalized functions.
- S' - The space of generalized functions of slow growth (Tempered distribution.)
- $\mathcal{E}'$ - The space of distributions with compact support.
- F[f] the Fourier transform of a function f.

1.2 Basic definitions

Definition 1. *Let Ω be an open domain in $\Re^n$. A distribution (Generalized function) f in Ω is a linear functional on C_0^∞,such that for every compact set $K \subset \Omega$ there are constants C and k such that*

$$|(f,\varphi)| = |f(\varphi)| \leq c \sum_{|\alpha| \leq k} sup|\partial^\alpha \varphi| \tag{1.1}$$

for all $\varphi \in C_0^\infty$ with supp$\varphi \subset K$. The set of all distributions in X is denoted by $D'(X)$. In which f is a linear form on C_0^∞ means f is a function from $C_0^\infty(X)$ to C such that

$$f(a\varphi + b\eta) = af(\varphi) + bf(\eta), \quad a,b \in \mathbb{C}, \quad \varphi, \eta \in C_0^\infty(X).$$

When we say that a distribution f is a continuous functional over D we are saying that if

$$\varphi_k \to \varphi \qquad as \;\; k \to \infty \quad in \;\; D,$$

then

$$f(\varphi_k) \to f(\varphi), \quad as \;\; k \to \infty$$

If the same integer k can be used in equation(2.1) for every K we say that f is of order $\leq k$*, and we denote the set of such distributions by* $D'^k(X)$*. Their union*

$$D'_F(X) = \bigcup D'^k(X)$$

is the space of distributions of finite order.

The simple example of distribution is the functional generated by the function $f(x)$ locally integrable in $\Re^n$:

$$f(\varphi) = \int f(x)\varphi(x)dx, \qquad \varphi \in D$$

such distributions are called **regular** distributions. The remaining are called **singular** distributions.
If $Y \subset X \subset \Re^n$ and $f \in D'(X)$, we can restrict f to a distribution f_Y in Y by setting

$$f_Y(\varphi) = f(\varphi), \varphi \in C_0^\infty(Y).$$

Definition 2. *The operation of the Fourier transform over S is denoted by* $F[\varphi](\xi)$ *is defined by*

$$F[\varphi](\xi) = \int \varphi(x)e^{i(\xi,x)}dx$$

, where $\varphi \in S$

Definition 3. *Let f(x) be an absolutely integrable function over* $\Re^n$*. Then its Fourier transform*

$$F[f](\xi) = \int f(x)e^{i(\xi,x)}dx$$

is a continuous function in $\Re^n$ *and, consequently, defines a generalized function belonging to* S'*,*

$$(F[f], \varphi) = \int F[f](\xi)\varphi(\xi)d\xi, \qquad \varphi \in S$$

Definition 4. *A set* $V \subset \Re^n \backslash \{0\}$ *is called a conic set if, together with any point* ξ*, it contains all the points* $t\xi$ *where* $t > 0$*.*

A conic set is completely determined by its intersection with the unit sphere $S^{N-1} \in \Re^n$. By a conic neighborhood of a point $\xi \in \Re^n$ we mean an open conic set that contains ξ.

Definition 5. *A canonical neighborhood of a point* $\xi \in \Re^n \backslash \{0\}$ *is a set* $V \subset \Re^n$ *such that* V *contains the ball*

$$B(\xi, \varepsilon) = \{\eta \in \Re^n : |\eta - \xi| < \varepsilon\} \quad for\ some\ \ \varepsilon > 0$$

and for any x *in* V *and any* $t > 0$, tx *belongs to* V.

Definition 6. *A smooth function* g *is said to be fast decreasing on a canonical neighborhood* V *if, for any integer* N, *there is a constant* C_N *such that*

$$|g(\xi)| \leq C_N(1 + |\xi|)^{-N} \quad for\ all\ \ \xi \in V.$$

Definition 7. *For a distribution* $f \in D'(\Re^n)$ *a point*

$$(x, \xi) \in \Re^n \times (\Re^n \backslash \{0\})$$

is called a regular directed point of f *if and only if there exist:*
(i) a function $\varphi \in D(\Re^n)$ *with* $\varphi(x) = 1$ *and*
(ii) a closed conical neighborhood $V \subset \Re^n$ *of* ξ, *such that* $F[\varphi f]$ *is fast decreasing on* V.

Definition 8. *If* $f \in D'(X)$ *then the support of* f, *denoted by* $suppf$, *is the set of points in* X *having no open neighborhood to which the restriction of* f *is* 0.

We say that the point x *does not belong to the support of the distribution* f *if and only if there is an open neighborhood* U *of* x *such that* f *is zero on* U.

Definition 9. *If* $f \in D'(X)$ *then the singular support of* f, *denoted by* $singsuppf$, *is the set of points in* X *having no open neighborhood to which the restriction of* f *is a* C^∞ *function.*

The last phrase of definition 9 means that, if there is a smooth function $g \in C^\infty$ such that

$$(f, \varphi) = (g, \varphi) = \int g(x)\varphi(x)dx$$

for all test functions φ supported on X.

Definition 10. *Let* L *be an operator with constant coefficients* $a_\alpha(x) = a_\alpha$*:*

$$L(D) = \sum_{|\alpha|=0}^{m} a_\alpha D^\alpha,$$

the generalized function $q \in D'$ *which satisfies equation*

$$L(D)q = \delta(x)$$

in $\Re^n$ *is said to be the fundamental solution of the differential operator L(D).*

Definition 11. *An n-dimensional manifold is a topological space in which each point has a neighborhood homeomorphic to some open set in* $\Re^n$.

It is a smooth manifold if all partial derivatives exist and are continuous.

Chapter 2

Spectral analysis of singularities of distributions

2.1 The wave front set

Let $f \in \mathcal{E}'$ be a distribution in $\Re^n$ with compact support. Its Fourier transform is a smooth function. We define the conic set $\Sigma(f) \subset \Re^n \backslash \{0\}$ by saying that $\xi \notin \Sigma(f)$ if there exists a conic neighborhood V of ξ such that the estimate (1) holds in V for all N.
If X is an open set in $\Re^n$ and $f \in D'(X)$, we set for $x \in X$

$$\Sigma_x(f) = \bigcap_{\varphi} \Sigma(\varphi f); \qquad \varphi \in C_0^\infty(X), \qquad \varphi(x) \neq 0.$$

It follows immediately from the definition that $\Sigma(f)$ is a closed conic set. Notice that a distribution f is induced by a smooth function if and only if $\Sigma(f) = \emptyset$. From the definition of $singsuppf$ and the cone $\Sigma(f)$ we observe that $singsuppf$ describes only the location of the singularities and the cone $\Sigma(f)$ describes only the direction of the high frequencies causing the singularities. We can combine the two types of information by using the following lemma.

Lemma 1. *Let $g \in \mathcal{E}'(\Re^n)$ and $\varphi \in C_0^\infty(\Re^n)$, Then $\Sigma(\varphi g) \subset \Sigma(g)$.*

Proof. The Fourier transform of $f = \varphi g$ is the convolution

$$\frac{1}{(2\pi)^n} \int F[\varphi](\eta) F[g](\xi - \eta) d\eta$$

that is

$$\begin{aligned} F[f](\xi) &= F[\varphi g](\xi) \\ &= \frac{1}{(2\pi)^n} F[\varphi] * F[g] \\ &= \frac{1}{(2\pi)^n} \int F[\varphi](\eta) F[g](\xi - \eta) d\eta \end{aligned}$$

where $F[\varphi] \in S$. For some $M \geq 0$ we have

$$|F[g](\xi)| \leq C(1+|\xi|)^M.$$

Let $0 < c < 1$ and split the integral into the part where $|\eta| < c|\xi|$ and $|\eta| \geq c|\xi|$. In the second case $|\xi - \eta| \leq (1+\frac{1}{c})|\eta|$. Hence

$$(2\pi)^n|F[f](\xi)| \leq \sup_{|\eta-\xi|<c|\xi|} |F[g](\eta)|||F[\varphi]||_{L^1} + C\int_{|\eta|>c|\xi|} |F[\varphi](\eta)|(1+\frac{1}{c})^M(1+|\eta|)^M d\eta. \quad (2.1)$$

If Γ is an open cone where (1.1) is valid and Γ_1 is a closed cone $\subset \Gamma \cup \{0\}$ we can choose c so that $\eta \in \Gamma$ if $\xi \in \Gamma_1$ and

$$|\xi - \eta| < c|\xi|,$$

for this is obviously possible when $|\xi| = 1$. since

$$|\eta| \geq (1-c)|\xi|$$

it follow from (3.1) and (1.1) that $F[f]$ is rapidly decreasing in Γ_1. In fact, we have for $N \geq 0$

$$sup(1+|\xi|)^N|F[f](\xi)| \leq \frac{1}{(1-c)^N} sup|F[g](\eta)|(1+|\eta|)^N||F[\varphi]||_{L^1}$$

$$+ C(1+\frac{1}{c})^{N+M}\int |F[\varphi](\eta)|(1+|\eta|)^{N+M}d\eta.$$

□

proposition 1. *Let Γ be a conic neighborhood of $\Sigma_x(f)$, $f \in D'(\Omega)$. Then there exist a neighborhood U of x such that $\Sigma(\phi f) \in \Gamma$ for every function $\phi(x) \in C_0^\infty(U)$.*

Theorem 1. *A compactly supported distribution $f \in \mathcal{E}'(\mathcal{X})$ is smooth if and only if $F[f](\xi)$ is fast decreasing on $\Re^n$.*

This theorem is physically reasonable because, if φ is a smooth function, then $\varphi(x)e^{i\xi.x}$ oscillates widely when ξ is large, so that the average of this expression is very small. Theorem (1) implies that any singularity of a distribution can be detected by an absence of fast decrease in some direction: a point x is in the singular support if and only if there is a direction ξ where the Fourier transform is not fast decreasing.
However, if $x \in singsuppf$, there can be directions ξ such that (x,ξ) is regular directed. This brings us finally to the definition of the wavefront set.

Definition 12. *Let $f \in D'(X)$ and X be an open set in $\Re^n$. Then the closed subset of $X \times (\Re^n \backslash \{0\})$ defined by*

$$WF(f) = \{(x,\xi) \in X \times (\Re^n\backslash\{0\});\ \xi \in \Sigma_x(f)\}$$

is called the ***wave front set*** *of f. The projection in X is singsuppf.*

That is the set of points $(x, \xi) \in \Re^n \times (\Re^n \backslash \{0\})$ which are not regular directed for the function f.
In other words, for each point of the singular support of f, the wave front set of f is composed of the directions where the Fourier transform of φf is not fast decreasing, for φ a sufficiently small support. The set $WF(f)$ is conic in the sense that it is invariant under multiplication of the second variable by positive scalars.

Example 1. *The simplest example is $\delta(x)$ in $D'(\Re^n)$, for which*

$$WF(\delta) = \{(0, \xi) : \ \xi \in \Re^n, \ \xi \neq 0\}.$$

Moreover the powers of δ is not defined.

Proof. The singular support of $\delta(x)$ *is* $\{0\}$ and
for $\varphi \in D(\Re^n)$, $F[\varphi\delta](\xi) = \varphi(0)$ is not decreasing if $\varphi(0) \neq 0$. This proves that

$$WF(\delta) = \{(0, \xi) : \ \xi \in \Re^n, \ \xi \neq 0\}.$$

To show that the product is not allowed, consider any point $(0, \xi)$ of $WF(\delta)$, then $(x, -\xi)$ is also a point in $WF(\delta)$ and if so consequently the Hörmander condition is not satisfied (which is given in sec 3.2.3). □

Example 2. *Let X_k in $\Re^n$ be the k-dimensional co-ordinate plane $x_{k+1} = ... = x_n = 0$. By x' we denote the collection $(x_1, ..., x_k)$ and x'' is the collection of remaining co-ordinates, so $x = (x', x'')$.*
For a function $f(x') \in D'(X_k)$, we will compute the wave front set of the distribution $f(x')\delta(x'')$. This distribution acts on a test function in the following way

$$(f(x')\delta(x''), \varphi) = \int f(x')\varphi(x', 0)dx'.$$

The support of this distribution is

$$\{x = (x', 0); \ x' \in suppf\}.$$

Choose a point $x_0 = (x'_0, 0)$ from this set.
Let ϕ be a compactly supported smooth function such that $\phi(x_0) \neq 0$. The Fourier transform of the distribution $\phi f(x')\delta(x'')$ equals $\int f(x')\phi(x', 0)e^{-x'\xi'}dx'$.
That is

$$F[\phi f \delta](\xi) = \int f(x')\phi(x', 0)e^{-x'\xi'}dx'.$$

Where $\xi = (\xi', \xi'')$
Let $\Gamma_k = \{(\xi', \xi'') \neq 0; \ \xi' = 0\}$. On the whole cone Γ_k, the function $\mathcal{F}(\xi) = \int f(x')\phi(x', 0)dx'$ is constant. For every neighborhood of x_0, one can find a function ϕ supported in the neighborhood such that the integral of $f(x')\phi(x', 0)$ does not vanish. So, by proposition 1,

$\Gamma_k \subset \Sigma_{x_0}(f\delta(x''))$. On the other hand, if $\xi_0 \in \Gamma_k$ then $|\xi''| \le C|\xi'|$ for every point $\xi = (\xi', \xi'')$ from a certain conic neighborhood Γ of ξ_0. Therefore for every N

$$|\mathcal{F}(\xi)| \le C_N(1+|\xi'|)^{-N} \le C'_N(1+|\xi|)^{-N}$$

when $\xi \in \Gamma$, and $\xi_0 \notin \Sigma_{x_0}(f\delta(x''))$. We conclude that

$$WF(f(x')\delta(x'')) = \{(x', x''; \xi', \xi'') : \ x' \in suppf, \ \xi' = 0\}$$

proposition 2. *If $f \in \mathcal{E}'(\Re^\backslash)$, then the projection of WF(f) on the second variable is $\Sigma(f)$.*

Proof. The projection W is contained in $\Sigma(f)$ by the definition of WF(f). It is closed since the intersection with the unit sphere is the projection of a compact set in $\Re^n \times S^{n-1}$. If V is a conic neighborhood of W then every $x \in \Re^n$ has a neighborhood U_x, such that

$$\Sigma(\varphi f) \subset V, \quad if \ \ \varphi \in C_0^\infty(U_x).$$

We can cover supp f by a finite number of such neighborhood U_x, and choose $\varphi_j \in C_0^\infty$ with $\Sigma\varphi_j = 1$ near supp f. But then it follows that

$$\Sigma(f) = \Sigma(\sum \varphi_j f) \subset \bigcup \Sigma(\varphi_j f) \subset V.$$

□

from Proposition (2) we have seen that WF(f) contains all information in $singsuppf$ and in $\Sigma(f)$.

Theorem 2. *If X is an open set in $\Re^n$ and S a closed conic subset of $X \times (\Re^n \backslash \{0\})$ then one can find $f \in D'(X)$ with WF(f)=S.*

Proof. It is sufficient to prove the statement when $X = \Re^n$ for otherwise we can apply this case to the closure of S in $\Re^n \times (\Re^n \backslash \{0\})$.
choose a sequence $(x_k, \theta_k) \in S$ with $|\theta| = 1$ so that every $(x, \theta) \in S$ with $|\theta| = 1$ is the limit of a subsequence.
Let $\varphi \in C_0^\infty$ and $F[\varphi](0) = 1$. Then

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \varphi(\xi(x - x_k)) e^{ik^3(x, \theta_k)} \tag{2.2}$$

is a continuous function in $\Re^n$, and we shall prove that $WF(f) = S$.
First we prove that $WF(f) \subset S$. If $(x_0, \xi_0) \notin S$ we can choose an open neighborhood U of x_0 and an open conic neighborhood V of ξ_0 such that

$$(U \times V) \cap S = \emptyset. \tag{2.3}$$

write $f = f_1 + f_2$, where f_1 is the sum of terms in equation(3.2) with $x_k \in U$ and f_2 the sum of terms with $x_k \in U$. Then $f_1 \in C^\infty$ in a neighborhood U_1 of x_0 because all but a finite number of terms vanish in U_1 if $U_1 \subset U$. Now

$$F[f_2](\xi) = \sum_{x_k \in U} \frac{1}{k^{2+n}} F[\varphi](\frac{\xi - k^3\theta_k}{k}) e^{i(x_k, k^3\theta_k - \xi)}. \tag{2.4}$$

Here $\theta_k \in V$ because of equation(3.3). If V_1 is another conic neighborhood of ξ_0 and $\overline{V}_1 \subset V \cup \{0\}$ then

$$|\xi - \eta| \geq c(|\xi| + |\eta|)$$

when $\xi \in V_1$ and $\eta \notin V$, for some $c > 0$. since this is true when $|\xi| + |\eta| = 1$. Thus

$$|\xi - k^3\theta| \geq c(|\xi| + k^3) \geq c|\xi|^{\frac{2}{3}} k, \qquad \xi \in V_1$$

and since $F[\varphi] \in S$ it follows that $F[f_2]$ is rapidly decreasing in V_1. Thus (x_0, ξ_0) is not in $WF(f)$.

Now let $(x_0, \xi_0) \in S$. Choose $\chi \in C_0^\infty$ equal to 1 near x_0. To prove that $(x_0, \xi_0) \in WF(f)$ we must show that $F[\chi f]$ cannot decrease rapidly in a conic neighborhood of ξ_0. To do so we first observe that

$$\chi(x)\varphi(k(x - x_k)) = \varphi_k(k(x - x_k))$$

where $\chi(\frac{x}{k} + x_k)\varphi(x)$ belongs to a bounded set in S. The Fourier transform of χf is a sum of the form of equation(3.4) with φ replaced by φ_k is close to x_0 and k is large then $\varphi_k = \varphi$ and we obtain for any N

$$|F[\chi f](k^3\theta_k)| \geq \frac{1}{k^{n+2}} - C_N \sum_{j \neq k} \frac{1}{j^{n+2}} \frac{1}{(\frac{|k^3\theta_k - j^3\theta_j|}{j})^N}.$$

Here

$$|k^3\theta_k - j^3\theta_j| \geq |k^3 - j^3| \geq k^2 + kj + j^2 \geq kj \qquad if k \neq j$$

so the sum is $O(\frac{1}{k^N})$. If we choose $N > n + 2$ we obtain for large k that

$$|F[\chi f](k^3\theta_3)| \geq \frac{1}{2k^{n+2}}$$

if x_k is close to x_0. Since $(x_0, \frac{\xi_0}{|\xi_0|})$ is a limit point of the sequence (x_k, θ_k) it follows that $F[\chi f]$ cannot decrease rapidly in a conic neighborhood of ξ_0.

□

We shall now determine the wave front set for some classes of distributions which occur very frequently.

Theorem 3. *Let V be a linear subspace of $\Re^n$ and $f = f_0 dS$, where $f_0 \in C^\infty(V)$ and dS is the Euclidean surface measure. Then*

$$WF(f) = suppf \times (V^\perp \backslash \{0\}).$$

Proof. If $\chi \in C_0^\infty$ then

$$F[\chi f](\xi) = \int_V e^{-i(x,\xi)}\chi(x)f_0(x)dS(x).$$

If we write $\xi = \xi' + \xi''$ where $\xi' \in V$ *and* $\xi'' \in V^\perp$, then this is a rapidly decreasing function of ξ' which does not vanish on any open set unless $\chi f = 0$.
Hence $F[\chi f]$ does not decrease rapidly in any open cone meeting $V^\perp$ unless $\chi f = 0$, but there is rapid decrease in every cone where $|\xi| \leq C|\xi'|$. This proves the assertion. □

Theorem 4. *Let X be an open set in $\Re^n$, Γ an open convex cone in $\Re^n$, and set for some* $\gamma > 0$

$$Z = \{z \in \mathbb{C}^n;\ Rez \in X,\ Imz \in \Gamma,\ |Imz| < \gamma\}.$$

If f is an analytic function in Z such that

$$|f(z)| \leq C|Imz|^{-N}, \qquad z \in Z,$$

then

***i**, $f(.+iy)$ has a limit $f_0 \in D'^{(N+1)}(X)$ as $y \to 0$ where$y \in \Gamma$.*
***ii**, If $f_0 = 0$ then $f = 0$*
***iii**, $WF(f_0) \subset X \times (\Gamma^o \backslash 0)$, where Γ^o is dual cone of Γ.*

Now we are interested to prove only (iii), since (i) and (ii) are already proved in [1].

Proof. If $\varphi \in C_0^\infty(X)$ the representation of (f_0, φ) given by

$$(f_0, \varphi) = \int \Phi(x,y)f(x+iy)dx$$

$$+ (N+1)\int\int_{0<t<1} f(x+ity) \sum_{|\alpha|=N+1} \partial^\alpha\varphi(x)(iy)^\alpha/\alpha!t^N dxdt.$$

is valid with N replaced by any integer $v \geq N$ provided that N is also replaced by v in

$$\Phi(x,y) = \sum_{|\alpha|\leq N} \partial^\alpha\varphi(x)(iy)^\alpha/\alpha!.$$

Hence

$$F[\varphi f_0](\xi) = (f_0 e^{-i(.,\xi)}, \varphi) = \int \Phi(x,y)f(x+iy)e^{-i(x+iy,\xi)}dx$$

$$+ (v+1)\int\int_{0<t<1} f(x+ity)e^{-i(x+iy,\xi)} \sum_{|\alpha|=v+1} \partial^\alpha\varphi(x)(iy)^\alpha/\alpha!t^v dxdt.$$

When $\langle y,\xi\rangle < 0$ it follows that

$$|F[\varphi f_0](\xi)| \leq C_{\varphi,v}(e^{(y,\xi)} + \int_0^\infty e^{t(y,\xi)} t^{v-N} dt)$$
$$= C_{\varphi,v}(e^{y,\xi} + (v-N)!(-y,\xi)^{N-v-1}).$$

The right-hand side in $O(|\xi|^{N-v-1})$ in a conic neighborhood of any point in the half space $\langle y,\xi\rangle < 0$. Hence

$$\Sigma(\varphi f_0) \subset \{\xi;\ (y,\xi) \geq 0\}$$

for every $y \in \Gamma$ with $|y| < \gamma$, so $\Sigma(\varphi f_0) \subset \Gamma^0$. □

The hypotheses in the theorem can be weakened in various ways. In particular it is sufficient to assume f analytic for $z \in X_1 + i\Gamma_1$ and $|Imz|$ small when $X_1 \Subset X$ and $\overline{\Gamma_1} \subset \Gamma \cup \{0\}$. We could also have added to f_0 *a* C^∞ term since this does not affect $WF(f_0)$. A converse result is then valid.
We shall also prove that theorem(4) remains valid when singularities are defined as point of non-analyticity.
To prepare for a discussion of the wave front set for homogeneous distributions we shall now prove a modification of Lemma (1), where the modified form of Lemma(1) is stated as follows

Lemma 2. *If $g \in S'$ then $WF(g) \subset \Re^n \times F$ where F is the limit cone of $suppF[g]$ at ∞, consisting of all limits of sequences $t_j x_j$ with $x_j \in suppF[g]$ and $0 < t_j \to 0$.*

Proof. F is obviously closed. For every closed cone Γ with $\Gamma \cap F = \{0\}$ we can choose $\epsilon > 0$ and C so that

$$|\xi - \eta| \geq \epsilon|\xi| \ \ if \ \ \xi \in \Gamma, \quad \eta \in suppF[g] \ \ and \ \ |\xi| > C.$$

In fact, we could otherwise choose $\xi_j \in \Gamma$ and $\eta_j \in suppF[g]$ so that $|\xi_j - \eta_j| < \frac{|\xi_j|}{j}$ and $|\xi_j| > j$. The sequence $\frac{\eta_j}{|\xi_j|}$ will then have a limit point $\theta \in \Gamma \cap F$ with $|\theta| = 1$ which is a contradiction. If $\varphi \in C_0^\infty(\Re^n)$ then the Fourier transform of $f = \varphi g$ is $\frac{1}{(2\pi)^n}F[\varphi] * F[g]$. that is

$$F[f] = F[\varphi g] = \frac{1}{(2\pi)^n}F[\varphi] * F[g].$$

Choose $\psi \in C^\infty(\Re^n)$ so that $\psi(\xi) = 1$ when $|\xi| > 1$ and $\psi(\xi) = 0$ when $|\xi| < \frac{1}{2}$. Then $\phi_R(\xi) = F[\varphi](\xi)\psi(\xi/R)$ is equal to $F[\varphi](\xi)$ when $|\xi| \geq R$, hence

$$(2\pi)^n F[f](\xi) = F[g_\eta](\phi_R(\xi - \eta))$$

if $\xi \in \Gamma$ and $R \leq \epsilon|\xi|$ and $|\xi| > C$.
Since $F[g] \in S'$ it follows that for some N, C', C'' we have when $\xi \in \Gamma$, $|\xi| > C$, $R \leq \epsilon|\xi|$,

$$|F[f](\xi)| \leq C' \sum_{|\alpha+\beta| \leq N} sup|\eta^\alpha D_\eta^\beta \phi_R(\xi - \eta)|$$

$$\leq C''(1+|\xi|)^N \sum_{|\alpha+\beta\leq N} sup_{|\eta|>R/2}|\eta^\alpha D^\beta F[\varphi](\eta|.$$

If we choose $R = \epsilon|\xi|$ the right-hand side is rapidly decreasing since $F[\varphi] \in S'$. □

Theorem 5. *If $f \in D'(\Re^n)$ is homogeneous in $\Re^n \backslash \{0\}$ then*

$$(x,\xi) \in WF(f) \Leftrightarrow (\xi,-x) \in WF(F[f]),\ if\ \xi \neq 0,\ x \neq 0, \tag{2.5}$$

$$x \in suppf \Leftrightarrow (0,-x) \in WF(F[f]),\ if\ x \neq 0 \tag{2.6}$$

$$\xi \in suppF[f] \Leftrightarrow (0,\xi) \in WF(f),\ if\ \xi \neq 0. \tag{2.7}$$

Proof. Assume firs that f is homogeneous in $\Re^n$. To prove (3.5) it is sufficient to show that if $x_0 \neq 0,\ \xi_0 \neq 0$ then

$$(x_0,\xi_0) \notin WF(f) \quad \Rightarrow \quad (\xi_0,-x_0) \notin WF(F[f]), \qquad (3.5)'$$

for $F[f]$ is also homogeneous and $(3.5)'$ applied to $F[f]$ gives the reversed implication since

$$F[F[f]] = (2\pi)^n F^{-1}[f].$$

Choose $\chi \in C_0^\infty(\Re^n)$ equal to 1 in a neighborhood of ξ_0 and $\phi \in C_0^\infty(\Re^n)$ equal to 1 in a neighborhood of x_0 so small that

$$(supp\phi \times supp\chi) \cap WF(f) = \emptyset. \qquad (3.5)''$$

we have to estimate the Fourier transform of $g = \chi F[f]$ in a conic neighborhood of $-x_0$. Let $\phi(x) = 1$ when $|x - x_0| < 2r$ and consider $F[g](-tx)\ when\ |x - x_0| < r$ and t is large.
If f is homogeneous of degree a in $\Re^n$ then

$$\begin{aligned} F[g](-tx) &= F[\chi] * F^{-1}[f](-tx) \\ &= (f, F[\chi](-tx + .)) \\ &= t^{a+n}(f, F[\chi](t(. - x))). \end{aligned}$$

Set $\phi f = f_0$ and $(1-\phi)f = f_1$.
Then $\Sigma(f_0) \cap supp\chi = \emptyset$ by proposition(2) and equation$(3.5)''$ Hence

$$(f_0, F[\chi](t(. - x))) = \int F[f_0](\xi)\chi(\xi/t)e^{i(x,\xi)}d\xi/t^n$$

is rapidly decreasing as $t \to \infty,\ for\ t^N F[f_0](t\xi)\chi(\xi)$ is bounded for every N.
Moreover,

$$(f_1, F[\chi](t(. - x))) = (f, (1-\phi)F[\chi](t(. - x)))$$

is also rapidly decreasing, for

$$y \to t^N(1-\phi(y))F[\chi](t(y-x))$$

is bounded in S for any N. In fact:
$|x-x_0| < r$ by hypothesis and $|y-x_0| > 2r$ in $supp(1-\phi(y))$, hence $t \le t\frac{|y-x|}{r}$ and $|y| \le |y-x| + |x_0| + r$. Since $F[\chi] \in S$ this completes the proof of $(3.5)'$
To prove (3.6) we first observe that since

$$F[F[f]] = (2\pi)^n F^{-1}[f]$$

it follows from Lemma 2 with $g = F[f]$ that

$$x \in supp f \Rightarrow (0,-x) \notin WF(F[f]).$$

Assume now that $(0,-x_0) \notin WF(F[f])$
Choose $\chi \in C_0^\infty$ equal to 1 at 0 so that the Fourier transform of $\chi F[f]$ is rapidly decreasing in a conic neighborhood Γ of $-x_0$. Adding to f a term with support at 0 does not affect (3.6) so we may assume that f is homogeneous of degree a in $\Re^n$ unless $a = -n-k$ equation(3.2.24)[1] is valid for an integer $k \ge 0$. Hence the Fourier transform of $\chi F[f]$ at tx is

$$F[\chi] * F^{-1}[f](tx) = (f, F[\chi](.+tx))$$
$$t^a(f, \varphi_t(.+x)) + \log t \sum_{|\alpha|=k} c_\alpha(\partial^\alpha F[\chi])(tx)/\alpha!$$

where $\varphi_t(x) = t^n F[\chi](tx)$ and the sum should be omitted unless $k = -n-a$ is an integer ≥ 0. When $x \in \Gamma$ the left-hand side tends rapidly to 0 as $t \to \infty$, and so does the sum. Thus

$$(f, \varphi_t(.+x)) = F^{-1}[f] * \varphi_t(x) \to 0 \qquad in \Gamma \ as \ t \to \infty.$$

The convolution converges to

$$(2\pi)^n F^{-1}[f] \ in \ S'(\Re^n).$$

Hence $F^{-1}[f] = 0 \ in \ \Gamma$ so $x_0 \notin supp f$ and (3.6) is proved.
If f and $F[f]$ are homogeneous in $\Re^n$ then (3.7) follows if (3.6) is applied to $F[f]$. If f is not homogeneous then $f(t.) - t^a f$ is a distribution $\neq 0$ supported by 0 for some $t > 0$. Hence $(0,\xi)$ is in $WF(f)$ for every $\xi \neq 0$, and $\xi \in supp F[f]$ since $F[f] = U + V$ where

$$U(\xi) = U_0(\xi) - Q(\xi)\log|\xi|.$$

where U_0 is a bounded function in a neighborhood of 0, and homogeneous of degree k and $C^\infty \ in \ \Re^n \backslash \{0\}$ and Q is the homogeneous polynomial of degree k. And V is a polynomial. □

Theorem 6. *For the distribution*

$$A = \int e^{i\varphi(.,\theta)} a(.,\theta) d\theta$$

defined in theorem(7.8.2)[1] we have

$$WF(A) \subset \{(x, \varphi'_x(x,\theta)) : (x,\theta) \in F, \varphi'_\theta(x,\theta) = 0\}. \tag{2.8}$$

Before the proof we observe that $\varphi'_\theta(x,\theta)=0$ implies $\varphi(x,\theta)=0$ since φ is homogeneous of degree 1 with respect to θ. By hypothesis $Im\varphi \geq 0$ so it follows that $Im\varphi'_x(x,\theta)=0$ Thus $\varphi'_x(x,\theta)$ is real in (3.8).

Proof. Let $\phi \in C_0^\infty(X)$. Then the definition of A means that

$$F[\phi A](\xi)=\int\int e^{i(\varphi(x,\theta)-(x,\xi))}\phi(x)a(x,\theta)dxd\theta$$

as an oscillatory integral. We want to show that this is rapidly decreasing in any closed cone $V \in \Re^n$ which does not intersect

$$\{\varphi'_x(x,\theta);\ (x,\theta)\in F,\ x\in supp\phi,\ \varphi'_\theta(x,\theta)=0\}.$$

Then we have for some $c>0$

$$|\xi-\varphi'_x(x,\theta)|+|\theta||\varphi'_\theta(x,\theta)|\geq c(|\xi|+|\theta|) \qquad (3.8)'$$

$$if(x,\theta)\in F,\quad x\in supp\phi,\quad \xi\in V.$$

To prove $(3.8)'$ we first observe that $\varphi'_x(x,\theta)$ and $|\theta|\varphi'_\theta(x,\theta)$ are continuous in F with the value 0 when $\theta=0$.
By homogeneity it suffices to prove $(3.8)'$ when $|\xi|+|\theta|=1$. By the compactness we only have to show that the left-hand side is never 0 when $(x,\theta)\in F,\ X\in supp\phi,\ \xi\in V$. If $\theta=0$ we have $|\xi-\varphi'_x(x,\theta)|=1$, and when $\theta\neq 0,\ \varphi'_\theta(x,\theta)=0$ we have $\xi\neq\varphi'_x(x,\theta)$ since $\xi\in V$, which proves $(3.8)'$.
Expressing the oscillatory integral by means of the partition of unity in θ (see proof of Theorem 7.8.2)[1] we have

$$F[\phi A](\xi)=\sum_0^\infty\int\int e^{i(\varphi(x,\theta)-(x,\xi))}\phi(x)\chi_v(\theta)a(x,\theta)dxd\theta.$$

Where each terms are in S. With $R=2^{(v-1)}$ the terms with $v\neq 0$ can be written as follows

$$R^N\int\int e^{i(R\varphi(x,\theta)-(x,\xi))}\phi(x)\chi_1(\theta)a(x,R\theta)dxd\theta. \qquad (3.8)''$$

If

$$\Phi(x,\theta)=\frac{(R\varphi(x,\theta)-(x,\xi))}{(R+|\xi|)}\ and\ \xi\in V$$

then by $(3.8)'$ we will have

$$|\Phi'_x|+|\Phi'_\theta|\geq\frac{c(R|\theta|+|\xi|)}{(R+|\xi|)}\geq c$$

in the support of $\phi(x)\chi_1(\theta)a(x,R\theta)$. With $\gamma=\max(1-\rho,\delta)<1$ we have

$$|D_\theta^\alpha D_x^\beta\phi(x)\chi_1(\theta)a(x,R\theta)|\leq C_{\alpha\beta}R^{m+\gamma(|\alpha|+|\beta|)}.$$

By Theorem 7.7.1 see [1] it follows that $(3.8)''$ is estimated for large k by

$$C_k R^{m+N+k\gamma} \frac{1}{(R+|\xi|)^k} \leq C_k R^{-1} |\xi|^{m+N+1+(\gamma-1)k} \quad if\ \xi \in V.$$

Since

$$\sum_1^{\infty} 2^{1-v} = 2$$

we conclude that $F[\chi A](\xi)$ is rapidly decreasing in V. □

2.2 A review of operations with distribution

It is well known that distributions can generally not be multiplied . The first reason is the definition of distributions as objects which generalize the functions but for which the value at some point has no sense in general. But, we may ask under which circumstances it is possible to extend the product of ordinary functions to distributions. In most cases this is just impossible.
For instance we cannot make sense of the square of δ:
a simple way to convince ourself of that is to study the family of functions $\chi_\varepsilon : \Re \to \Re$ for $\varepsilon > 0$ defined by

$$\chi_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon}, & \text{if } |x| \leq \frac{\varepsilon}{2} \\ 0, & \text{otherwise.} \end{cases}$$

for any $\varphi \in D(\Re)$ we have

$$\int_\Re \chi_\varepsilon(x)\varphi(x)dx = \frac{1}{\varepsilon}\int_{\frac{-\varepsilon}{2}}^{\frac{\varepsilon}{2}} \varphi(x)dx$$

$$= \frac{1}{\varepsilon}(\varepsilon\varphi(0) = \varphi(0)$$

and

$$\lim_{\varepsilon \to 0} \chi_\varepsilon = \delta.$$

However, the square of χ_ε does not converge to a distribution:

$$\int_\Re \chi_\varepsilon^2(x)\varphi(x)dx = \frac{1}{\varepsilon^2}\int_{\frac{-\varepsilon}{2}}^{\frac{\varepsilon}{2}} \varphi(x)dx$$

$$= \frac{1}{\varepsilon^2}(\varepsilon\varphi(0)$$

diverges for $\varepsilon \to 0$.
In some other cases it is possible to define a product, but we loose some good properties. Consider the example of the Heaviside step function H, which is defined by

$$H(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Its associated distribution, denoted by θ, is

$$(\theta, \varphi) = \int_{-\infty}^{\infty} H(x)\varphi(x)dx = \int_0^{\infty} \varphi(x)dx.$$

The function H can be obviously be multiplied with itself and $H^n = H$ for any integer $n > 0$. As we shall see, it is possible to define a product of distributions such that $\theta^n = \theta$

as a distribution. But then, we loose the compatibility of the product with the Leibniz rule because, by taking the derivative of both sides we would obtain

$$n\theta^{n-1}\theta' = \theta'.$$

The identity $\theta' = \delta$ and $\theta^{n-1} = \theta$ would give us $n\theta\delta = \delta$ for all integer $n > 1$. Since the left-hand side depends linearly on n and the right-hand side does not and is not equal to zero, we reach a contradiction.

Moreover singularities made it impossible to give a general definition of multiplication of distributions and composition with maps. We shall now show that the definition of both operations can be extended when one takes into account the more refined description of the singularities given by the wave front set. We shall always define such operations by continuous extension from the smooth case, so the first point to discuss is the topology in the space of distributions with a given bound for the wave front set.

Here before using the wave front set in order to define the product of distributions we need to see all the cases in which we can multiply distributions. The cases are listed as follows.

2.2.1 Cases in which we can multiply distributions

1. **A distribution times a smooth function** The product of distributions is well defined when one of the two distributions is a smooth function. Indeed, consider a distribution $f \in D'(\Re^n)$ and a smooth function $g \in C^\infty(\Re^n)$. Then, for all test function $\varphi \in D(\Re^n)$ we can define the product of f and g by $(fg, \varphi) = (f, g\varphi)$.

2. **Distributions with disjoints singular supports** We can also define the product of two distributions when the singularities of the distributions are disjoint.

   Theorem 7. *If f and g are two distributions in $D'(\Re^n)$ such that*

   $$singsuppf \bigcap singsuppg = \emptyset$$

   , then the product fg is well defined.

   Proof. We first notice that, if $\varphi \in D(\Re^n)$ is supported outside the singular support of g, then $g\varphi$ is smooth and we can define the product by $(fg, \varphi) = (f, g\varphi)$. Similarly, $(fg, \varphi) = (g, f\varphi)$ if φ is supported outside the singular support of f. This definition of fg extends to all test functions φ by using a smooth function χ which is equal to zero on a neighborhood of the singular support of f and equal to one on a neighborhood of the singular support of g. Then

   $$(fg, \varphi) = (g, f\chi\varphi) + (f, g(1-\chi)\varphi)$$

   This product is associative and commutative. □

3. **The singular oscillations of the distributions are transversal**

Consider the two distributions $f = \delta \otimes 1$ and $g = 1 \otimes \delta$ in $D'(\Re^2)$, that is $\forall \varphi \in D(\Re^2)$,

$$(f, \varphi) = \int_{\Re} \varphi(0, y)dy$$

and

$$(g, \varphi) = \int_{\Re} \varphi(x, 0)dx.$$

Then we can define their product by $fg = (\delta \otimes 1)(1 \otimes \delta) = \delta \otimes \delta = \delta^2$, that is $(fg, \varphi) = \varphi(0, 0)$, since

$$(fg, \varphi) = \int \int f(x)g(y)\varphi(x, y)dxdy$$

$$= \int f(x)(\int g(y)\varphi(x, y)dy)dx$$

$$= \int f(x)\varphi(x, 0)dx$$

$$= \varphi(0, 0)$$

by the Fubini theorem for distributions. Here f and g are singular on the lines $x = 0$ and $y = 0$ respectively, which have a non empty intersection $\{(0, 0)\}$.

2.2.2 The product of distributions by using Fourier transform

The Fourier transform of a product of distributions (when it is defined) is the convolution of the Fourier transforms of these distributions :

$$F[fg] = F[f] * F[g]$$

if it exists. Therefore, we can define the product of two distributions f and g as the inverse Fourier transform of $F[f] * F[g]$. However, this definition, which requires the Fourier transforms of f and g to be defined and their convolution product to make sense, can be improved. Indeed it does not take into account the fact that the product of two distributions is local, that is its definition on the neighborhood of a point depends only on the restriction of the distributions on that neighborhood. Therefore, we can localize the distributions by multiplying them with a test function:
If $f \in D'(X)$ and $\varphi \in D(X)$, where X an open set in $\Re^n$ then $f\varphi$ is a distribution with

compact support in X and we can extend it to a distribution defined on $\Re^n$ by setting it to equal to zero outside X. Let us still denote by $f\varphi$ this compactly supported distribution on $\Re^n$. It has a Fourier transform $F[f\varphi](\xi)$ which is an entire analytic function of ξ by the PaleyWienerSchwartz Theorem.
Accordingly if we consider a function φ which is of rapid decrease and a tempered distribution f, then the Fourier transform is defined by

$$(F[f], \varphi) = (f, F[\varphi]).$$

The inverse Fourier transform is

$$F^{-1}[f](x) = \frac{1}{(2\pi)^n} F[f](-x), \qquad f \in S'$$

where n is the dimension of spacetime.We can now give a definition of the product of two distributions.

Definition 13. *Let f and g be in $D'(\Re^n)$. We say that $h \in D'(\Re^n)$ is the product of f and g if and only if, for each $x \in \Re^n$, there exists some $\varphi \in D(\Re^n)$, with $\varphi = 1$ near x, so that for each $\xi \in \Re^n$ the integral*

$$F[\varphi^2 h](\xi) = (F[\varphi f] * F[\varphi g])(\xi) = \frac{1}{(2\pi)^n} \int F[\varphi f](\eta) F[\varphi g](\xi - \eta) d\eta \tag{2.9}$$

is absolutely convergent.
When it exists, this product has many desirable properties: it is unique,commutative, associative (when all intermediate products are defined) and it coincides with the product of Theorem 7 when the singular supports of f and g are disjoint.
Let us consider some examples.

Example 3. *If $f = g = \delta$, the product is not defined.(where δ is a Dirac delta function)*

Proof. For any test function φ satisfying the hypothesis of definition (13),

$$\begin{aligned} \varphi\delta(x) &= \varphi(0)\delta(x) \\ &= \delta(x) \end{aligned}$$

and

$$F[\varphi\delta](\xi) = 1,$$

so that

$$\int F[\varphi\delta](\eta) F[\varphi\delta](\xi - \eta) d\eta = \int d\eta$$

which is not absolutely convergent and hence the product is not defined. □

Example 4. *If $f = g = \theta$, the product is well defined.(where θ is the Heaviside function)*

Proof. For any $\varphi \in D(\Re)$, $F[\varphi\theta](\xi) = \int_0^\infty e^{i\xi x}\varphi(x)dx$ satisfies the uniform bounded $|F[\varphi\theta](\xi)| \leq ||\varphi||_{L^1} := \int_\Re |\varphi(x)|dx$.
Moreover an integration by part gives us also

$$F[\varphi\theta](\xi) = \frac{i}{\xi}(\varphi(0) + u(\xi))$$

with $u(x) := \int_0^\infty e^{i\xi x}\varphi'(x)dx$ thus we have the uniform bound $|F[\varphi\theta](\xi)| \leq \frac{1}{|\xi|}(|\varphi(0)|+||\varphi'||_{L^1}$.
Hence, for any $\xi \in \Re$,

$$|f[\varphi\theta](\xi)| \leq C(1 + |\xi|)^{-1}$$

for $C = ||\varphi'||_{L^1} + ||\varphi||_{L^1} + |\varphi(0)|$ and the integral defining $(F[\varphi f] * F[\varphi g])(\xi)$ is absolutely convergent because

$$\int_\Re |F[\varphi\theta]\eta F[\varphi\theta](\xi - \eta)|d\eta \leq \int_\Re \frac{C^2}{(|\xi - \eta| + 1)(|\eta| + 1)}d\eta \leq \overline{C} \int_\Re \frac{1}{(|\eta| + 1)^2}d\eta$$

where

$$\overline{C} = C^2 \sup_\eta \frac{(|\eta| + 1)}{(|\xi - \eta| + 1)}$$

is finite. □

In the previous examples, we saw that the calculation of the product of two distributions by using the Fourier transform looks rather tricky. In particular, it seems that we have to know the Fourier transform of the product of each distribution with an arbitrary function. Moreover even when we are able to define it, the product of distribution does not always satisfy the Leibniz rule $\partial(fg) = (\partial f)g + f(\partial g)$.
For instance the product of θ makes sense (see Example 4) but does not respect the Leibniz rule.

In general, for the convolution integral to be well defined, we just need that the product $F[\varphi f](\eta)F[\varphi g](\xi-\eta)$ decreases fast enough for large η for the integral over η to be absolutely convergent. Note also that, for any distribution f and for any smooth function φ with compact support, since φf is a distribution with compact support,its Fourier transform $F[\varphi f]$ grows at most polynomially at infinity, that is there exists some natural number N and some constant $C > 0$ such that $|F[\varphi f](\xi)| \leq C(1 + |\xi|)^N$ everywhere. Hence it is enough that one of the two factors in the product $F[\varphi f](\eta)F[\varphi g](\xi-\eta)$ is fast decreasing at infinity to ensure that the product is fast decreasing.

2.2.3 Product of distributions using the wave front set

As we have seen in the definition, for each point of the singular support of f, the wavefront set of it is composed of the directions where the Fourier transform of φf is not fast decreasing, for φ a sufficiently small support.Now we see how the definition of wave front set can be used to determine the product of two distributions f and g.Broadly speaking, if a point x belongs to the singular support of f and g, then the product of f and g exists at x if, for all directions η, either $F[\varphi f](\eta)$ or $F[\varphi g](\xi - \eta)$ is rapidly decreasing. In particular, if (x, η) belongs to WF(f),then $(x, -\eta)$ must not belong to WF(g). This is called Hormander's condition.

Theorem 8. *(Product Theorem)*

Let f and g be distributions in $D'(X)$. Assume that there is no point (x, ξ) in WF(f) such that $(x, -\xi)$ belongs to WF(g), then the product fg can be defined. Moreover, if so, then

$$WF(fg) \subset S_+ \bigcup S_f \bigcup S_g, \tag{2.10}$$

where $S_+ = \{(x, \xi + \eta) : (x, \xi) \in WF(f) and (x, \eta) \in WF(g)\}$,
$S_f = \{(x, \xi) : (x, \xi) \in WF(f) and x \in supp g\}$ *and*
$S_g = \{(x, \xi) : (x, \xi) \in WF(g) and x \in supp f\}$.

Now from Hormanders condition and theorem (8) we have the following remarks:

- Theorem (8) is absolutely fundamental for the theory of renormalization in curved spacetimes. With this simple criterion, we can prove that a product of distributions exists even if we cannot calculate their Fourier transforms and we do not know the explicit form of the distributions.
- When Hormanders condition holds, then the product of distributions satisfies the Leibniz rule for derivatives, because derivatives do not extend the wavefront set
- Note that if f and g satisfy Hormanders condition, then their product exists in the sense of Definition 13. The converse is not true in general. However, if the product of distributions is extended beyond Hormanders condition, then it is generally not compatible with the Leibniz rule, as shown by the example of the Heaviside distribution.
- Hormanders condition of the Product Theorem can be rephrased by saying that S_+ does not meet the zero section (of the cotangent bundle over X), that is $S_+ \bigcap (X \times \{0\}) = \emptyset$.

Theorem 9. *Let $X \subset \Re^n$ and $Y \subset \Re^m$ be open sets and let $K \in D'(X \times Y)$. Denote the corresponding linear transformation from $C_0^\infty(Y)$ to $D'(X)$ by κ. Then we have*

$$WF(\kappa\varphi) \subset \{(x, \xi);\ (x, y, \xi, 0) \in WF(K) \quad for\ some\ y \in supp\varphi\}, \qquad \varphi \in C_0^\infty(Y).$$

Proof. Let $x_0 \in X$, choose $\chi \in C_0^\infty(X)$ with $\chi(x_0) = 1$ and

$$K_1 = (\chi \otimes \varphi)K \in \mathcal{E}'(X \times Y).$$

The Fourier transform of $\chi\kappa\varphi$ *is* $F[K_1](\xi, 0)$. Now proposition 2 gives

$$\Sigma(K_1) \subset \{(\xi, \eta);\ (x, y, \xi, \eta) \in WF(K)\ for\ some\ x \in supp\chi,\ y \in supp\varphi\}.$$

Hence it follows that

$$\Sigma(\chi\kappa\varphi) \subset \{\xi;\ (x, y, \xi, 0) \in WF(K)\ for\ some\ x \in supp\chi\ and\ y \in supp\varphi\}.$$

When $supp\chi \to \{x_0\}$ the theorem follow. □

The proof shows that κ maps $C_0^\infty(M)$ continuously into $D'_\Gamma(X)$ if M is a compact subset of Y and

$$\Gamma = \{(x,\xi);\ (x,y,\xi,0) \in WF(K)\ for\ some\ y \in M\}.$$

For the union of all such set we shall use the notation

$$WF(K)_X = \{(x,\xi);\ (x,y,\xi,0) \in WF(K)\ for\ some\ y \in Y\}.$$

It is not necessarily a closed set. If it is empty then κ is a continuous map from $C_0^\infty(Y)$ *to* $C^\infty(X)$.

Example 5. *Let δ_1 and δ_2 be distributions in $D'(\Re^2)$ defined by*

$$(\delta_1, \varphi) = \int \varphi(0,y)dy$$

and

$$(\delta_2, \varphi) = \int \varphi(x,0)dx.$$

Then

$$WF(\delta_1) = \{(0,y;\lambda,0):\ y \in \Re,\ \lambda \neq 0\}$$

and

$$WF(\delta_2) = \{(x,0;0,\mu):\ x \in \Re,\ \mu \neq 0\}.$$

Thus $\delta_1\delta_2$ exist and

$$WF(\delta_1\delta_2) \subset \{(0,0;\lambda,\mu),\ \lambda \neq 0,\ \mu \neq 0\} \cup \{(0,0;\lambda,0),\ \lambda \neq 0\} \cup \{(0,0;0,\mu),\ \mu \neq 0\}$$

where we used

$$supp\delta_1 = \{(0,y):\ y \in \Re\}$$

and

$$supp\delta_2 = \{(x,0):\ x \in \Re\}.$$

Note that the estimate of the wavefront set of $\delta_1\delta_2$ would be much worse if the support of δ_1 and δ_2 had not been taken into account in S_{δ_1} and S_{δ_1} of the Product Theorem. In that case the inclusion is an equality because

$$WF(\delta_1\delta_2) = \{(0,0;\ \lambda,\mu),\ (\lambda,\mu) \neq (0,0)\}.$$

Proof. Let $y \in \Re$, we need to calculate $WF(\delta_1)$ *at* $(0,y)$. Consider a test function $\varphi(x_1,x_2)$ which is equal to one around $(0,y)$. Then

$$\begin{aligned} F[\varphi\delta](\xi) &= \int \varphi(x_1,x_2)\delta(x_1)e^{i(\xi_1x_1+\xi_2x_2)}dx_1dx_2 \\ &= \int \varphi(0,x_2)e^{i\xi_2x_2}dx_2. \end{aligned}$$

Take $\xi = (\xi_1, \xi_2)$ and observe the decay of $F[\varphi\delta_1](\lambda\xi)$. If $\xi_2 \neq 0$ this is a fast decreasing function of λ because $\varphi(0, x_2)$ is a smooth compactly supported function of x_2. On the other hand if $\xi_2 = 0$, then we have

$$F[\varphi\delta_1](\xi_1, 0) = \int \varphi(0, x_2) dx_2,$$

which is independent of ξ_1, so that $F[\varphi\delta_1](\lambda\xi_1, 0)$ is not fast decreasing. This proves that $WF(\delta_1)$ has the given form. A similar proof yields for $WF(\delta_2)$.
The rest follows from the fact that $\delta_1\delta_2$ is the two dimensional delta function. □

2.3 The wave front set of solutions of partial differential equations

A differential operator with C^∞ coefficients of order m in an open set $X \subset \Re^n$ is of the form

$$P = P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha. \tag{2.11}$$

A distribution $E \in D'(\Re^n)$ is called a fundamental solution of P if

$$PE = \delta.$$

By means of the fundamental solution E(x) of the operator P(D) it is possible to construct a solution of the equation

$$P(D)f = g(x) \tag{2.12}$$

with an arbitrary right-hand side g. More accurately, the following theorem is valid.

Theorem 10. *Let $g \in D'$ be a generalized function such that the convolution $E * g$ exist in D'. Then the solution of equation(3.12) exist in D' and is given by the formula*

$$f = E * g \tag{2.13}$$

This solution is unique in the class of generalized function belonging to D' for which a convolution with E exists.

Proof. By the definition of differentiation of convolution we have

$$D^\alpha(E * g) = D^\alpha E * g$$

and

$$P(D)E = \delta(x) \quad ... \quad since\ E\ is\ a\ fundamental\ solution$$

then we obtain

$$\begin{aligned} P(D)f = P(D)(E * g) &= \sum_{|\alpha|=0}^{m} a_\alpha D^\alpha (E * g) \\ &= (\sum_{|\alpha|}^{m} a_\alpha D^\alpha E) * g \\ &= P(D)E * g \\ &= \delta * g \\ &= g \end{aligned}$$

Therefore the formula $f = E * g$ gives the solution of equation (3.12).
We shall prove the uniqueness of the solution of equation (3.12) in the class of the generalized

functions belong to D' for which a convolution with E exist in D'. For this it is sufficient to establish that the corresponding homogeneous equation $P(D)f = 0$ has only a zero solution in this class. and it follows as

$$\begin{aligned} f = f * \delta &= f * P(D)E \\ &= P(D)f * E \\ &= 0 * E \\ &= 0 \end{aligned}$$

□

The importance of fundamental solutions is due to the following two consequence of

$$P(f_1 * f_2) = P(f_1) * f_2 = f_1 * (Pf_2) :$$

$$E * (Pf) = f, \quad f \in \mathcal{E}'(\Re^n), \tag{$*$}$$

$$P(E * g) = g, \quad g \in \mathcal{E}'(\Re^n). \tag{$**$}$$

Thus convolution with E is both a left and a right inverse of P. From $(**)$ it follows that the equation $Pf = g$ has a solution for every $g \in \mathcal{E}'(\Re^n)$, and $(*)$ make it possible to obtain information on the singularities of f from those of Pf.
The principal part P_m is defined by

$$P_m(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha. \tag{2.14}$$

Theorem 11. *If P has a fundamental solution with singsupp$E = \{0\}$ and X is any open set in $\Re^n$, then*

$$singsuppf = singsuppPf, \qquad f \in D'(X).$$

Proof. It is always true that $singsuppPf \subset singsuppf$.
If $f \in \mathcal{E}'$ we obtain from $(*)$ and Theorem 4.2.5 see [1] that

$$singsuppf = singsupp(E * Pf) \subset singsuppPf$$

so the assertion is valid when $f \ in \ \mathcal{E}'$. If $\varphi \in C_0^\infty(X)$ is equal to 1 in an open subset Y, it follows that

$$\begin{aligned} Y \cap singsuppPf &= Y \cap singsuppP(\varphi f) \\ &= Y \cap singsupp\varphi f \\ &= Y \cap singsuppf \end{aligned}$$

$$\Rightarrow singsuppf = singsuppPf$$

□

Theorem 12. *If P is a differential operator of order m with C^∞ coefficients on a manifold X, then*

$$WF(f) \subset CharP \cup WF(Pf), f \in D'(X), \tag{2.15}$$

where the characteristic set Char P is defined by

$$CharP = \{(x,\xi) \in T^*(X)\backslash\{0\},\ P_m(x,\xi) = 0\}. \tag{2.16}$$

Corollary 1. *If P is elliptic, that is $P_m(x,\xi) \neq 0$ in $T^*(X)\backslash\{0\}$, then*

$$WF(f) = WF(Pf), \qquad f \in D'(X).$$

Hence

$$singsuppf = singsuppPf, \qquad f \in D'(X).$$

Theorem 13. *Let B be a real non-singular quadratic form in $\Re^n$, let X be an open set in $\Re^n$ and $f \in D'(X)$ a solution of the equation $B(D)f = \varphi$. If*

$$(x,\xi) \in WF(f)\backslash WF(\varphi) \quad then \quad B(\xi) = 0$$

and

$$I \times \{\xi\} \subset WF(f)$$

if $I \subset X$ is a line segment containing x with direction $B'(\xi)$ such that $I \times \{\xi\}$ does not meet $WF(\varphi)$.
Thus singularities of f with frequency ξ propagate with fixed frequency in the direction $B'(\xi)$ in X until they meet the singularities of φ.

Proof. That $B(\xi) = 0$ follows from theorem 12 choose $\phi \in C_0^\infty(X)$ so that $\phi(x) = 1$ and $L \cap supp\phi \subset I$ if L is the line through I. Then $g = \phi f \in \mathcal{E}'$ and

$$B(D)g = \phi B(D)f + h = \phi\varphi + h$$

where $supph \subset suppd\phi$. since

$$(L \times \xi) \cap WF(B(D)g) = (L \times \{\xi\}) \cap WF(h)$$

it follows that there are points $z_{\pm \in L}$ on either side of x such that $(z_\pm, \xi) \in WF(h)$, hence

$$z_\pm \in L \cap suppd\phi \quad and \quad (z_\pm, \xi) \in WF(f).$$

If y_+ and y_- are arbitrary points in the interior of I on different sides of x we can choose ϕ so that $L \cap suppd\phi$ is as close to $\{y_+, y_-\}$ as we wish. Hence $(y_\pm, \xi) \in WF(f)$. □

In a moment we shall prove that Theorem(13) is valid for much more general differential operators with constant coefficients although we do not have quite so explicit fundamental solutions to work with then. However, we give first an example of a solution of the wave equation in $\Re^4$ which indicates that Theorem(13) gives all conditions which WF(f) must satisfy when $B(D)f = 0$.

Example 6. *There exist a solution $f \in D'(\Re^4)$ of the wave equation*

$$\Box f = (c^{-2}\frac{\partial^2}{\partial t^2} - \triangle_x)f = 0$$

such that for a given y with $|y| = 1$

$$WF(f) = \{(t, cty; sc, -sy), t \in \Re, s \neq 0\}.$$

To construct f we change notation and let E_+, E_- be the advanced and retarded fundamental solutions. These are proportion to $\delta(c^2t^2 - |x|^2)$ when $t \gtrless 0$ so for the solution $E_0 = E_+ - E_-$ of $\Box E_0 = 0$ we have by example(8.2.5)[1], Theorem(8.2.4)[1] and Theorem 12

$$WF(E_0) \subset \{(t, ctx, sc, -sx); t \in \Re, s \in \Re\backslash\{0\}, x \in \Re^3, |x| = 1\}.$$

Let φ be a positive C_0^∞ density on the line L through 0 with direction $(1, cy)$ and set $f = E_0 * \varphi$. Example (8.2.5)[1] gives

$$WF(\varphi) \subset \{(t, cty; \tau, \xi); t \in \Re, \tau + c\langle y, \xi\rangle = 0\}.$$

Now the tangent plane $\tau + c\langle y, \xi\rangle = 0$ of the characteristic cone

$$\tau^2 - c^2|\xi|^2 = 0$$

meets the cone only when (τ, ξ) is proportional to $(c, -y)$ so (8.2.16)[1] gives

$$WF(f) \subset \{(t, cty; sc, -sy); t \in \Re, s \neq 0\}. \tag{2.17}$$

by $(*)$ we have $\{(t, x); ct = \langle x, y\rangle\} \cap suppf \subset L$. If t is so large that $f = \varphi * E_+$ in a neighborhood of $(t, cty) \in L$ it follows from (6.2.7)[1] that the total mass of the measure f at distance $\leq \delta$ from (t, cty) is at least $C\delta^2$ for some $C > 0$. Hence $(t, cty) \in singsuppf$. Since f is real valued the wave front set is symmetric with respect to the origin in the frequency variable. Hence there is equality in (3.17) for a missing point would make the left-hand side empty by Theorem(13). We shall now extend theorem(13) to a general differential operators with constant real coefficients and non-singular characteristic set:

Definition 14. *A differential operator P(D) with constant coefficients in $\Re^n$ is said to be real principal type if the principal symbol P_m is real and*

$$P_m^{'}(\xi) \neq 0 \qquad when\ \xi \in \Re^n\backslash\{0\}. \tag{2.18}$$

For a differential operator P of real principal type we set

$$g(\xi) = P_m'(\xi)|\xi|^{1-m}.$$

This vector field is homogeneous of degree 0 with respect to ξ. In the following lemma we give a lower bound for P in the direction $ig(\xi)\ from\ \xi$.

Lemma 3. *There exist positive constants* $t,\ C_1,\ C_2,\ C_3$ *such that*

$$ImP(\xi + itg(\xi) + iV) \geq C_1(1+|\xi|)^{m-1} + \langle P'_m(\xi), V\rangle - C_2(|V|+1)|V|(|\xi|+|v|)^{m-2}$$

$$if\ \xi \in \Re^n,\quad |\xi| \geq C_3,\quad V \in \Re^n.$$

Theorem 14. *If P(D) is of real principal type then one can find* $E_\pm \in D'(\Re^n)$ *such that* $P(D)E_\pm = \delta + \omega_\pm$ *and*

$$WF(E_\pm) \subset \{(tP'_m(\xi), \xi);\ t \gtrless 0,\ P_m(\xi) = 0,\ \xi \neq 0\} \cup T_0^*\backslash\{0\}. \qquad (2.19)$$

For the proof of this Theorem see [1]

Theorem 15. *Let P(D) be a real principal type. If* $f \in D'(X)$, $P(D)f = \varphi$ *and* $(x, \xi) \in WF(f)\backslash WF(\varphi)$, *then* $P_m(\xi) = 0$ *and*

$$I \times \{\xi\} \subset WF(f)$$

if $I \subset X$ *a line segment containing* x *with direction* $P'_m(\xi)$ *such that* $I \times \{\xi\}$ *does not meet* $WF(\varphi)$.

Finally we shall give a general version of example(6).

Theorem 16. *Let P(D) be of real principal type,* $0 \neq \xi \in \Re^n$ *and* $P_m(\xi) = 0$. *Then one can find* $f \in C^m(\Re^n)$ *such that* $P(D)f \in C^\infty(\Re^n)$ *and*

$$WF(f) = \{(tP'_m(\xi), s\xi);\ t \in \Re,\ s > 0\}. \qquad (2.20)$$

Proof. Set $L = \Re P'_m(\xi)$ and let $\mathcal{F}$ be the set of all $f \in C^m(\Re^n)$ with $Pf \in C^\infty(\Re^n)$, $f \in C^\infty(\complement L)$ and $WF(f) \subset \Re^n \times (\Re_+\xi)$. The theorem states that there is an element $f \in \mathcal{F}$ which is not in C^∞ for all $f \in \mathcal{F}$ implies

$$WF(f) \subset \Re p'_m \times \Re_+\xi$$

and by Theorem 15 $f \in C^\infty$ if the inclusion is strict. Now $\mathcal{F}$ is a *Frechet* space with the seminorms

i $\sup_k |D^\alpha f|,\ |\alpha| \leq m$, K is compact subset of $\Re^n$,

ii $\sup_k |D^\alpha f|,\ \alpha$ arbitrary, K a compact subset of $\complement L$,

iii $\sup -k|D^\alpha P(D)f|,\ \alpha$ arbitrary, K a compact subset of $\Re^n$,

iv $\sup_{\complement\Gamma_N} |\eta|^N |F[\varphi f](\eta)|,\quad N = 1, 2, ...,\qquad \varphi \in C_0^\infty(\Re^n)$.

Here Γ_N is a sequence of conic neighborhood of ξ in $\Re^n$ shrinking to $\Re_+\xi$. We need only use a countable number of compact sets K and functions φ since the semi-norm (iv) can be estimated by the corresponding ones with φ replaced by a function ψ which is 1 in $supp\varphi$. If $\mathcal{F} \subset C^{m+1}$ then the closed graph theorem shows that the inclusion $\mathcal{F} \hookrightarrow C^{m+1}$ is continuous. Thus one can find $N,\ \varphi \in C_0^\infty(\Re^n),\ K_1 \Subset \Re^n \ and\ K_2 \Subset \complement L$ so that

$$\sum_{|\alpha|=m+1} |D^\alpha f(0)| \leq C\{\sum_{|\alpha|\leq m} \sup_{K_1} |D^\alpha f| + \sum_{|\alpha|\leq N} \sup_{K_2} |D^\alpha f|$$

$$+ \sum_{|\alpha|\leq N} \sup_{K_1} |D^\alpha P(D) f| + \sup_{\complement\Gamma_N} (1+|\eta|)^N |F[\varphi f](\eta)|, \qquad f \in \mathcal{F}. \qquad (*1)$$

To show that $(*1)$ is not valid we need to construct approximate solution of the equation $Pf = 0$ concentrated close to L, thus away from K_2. To make the last term small the Fourier transform of f should be concentrated close to the direction ξ. I t is therefore natural to set for $t > 0$

$$f_t = e^{it(x,\xi)} g_t(x).$$

Then

$$\begin{aligned} P(D) f_t(x) &= e^{it(x,\xi)} P(D + t\xi) g_t(x) \\ &= t^{m-1} e^{it(x,\xi)} (\sum_1^n P_m^{(j)}(\xi) D_j g_t + P_{m-1}(\xi) g_t + ...) \end{aligned}$$

where terms indicated by dots contain a negative power of t, and $P_m^j = \partial_j P_m$. A formal solution

$$g_t = g_0 + t^{-1} g_1 + ...$$

may be found by solving the first order equation

$$L g_0 = \sum_1^n P_m^j(\xi) D_j g_0 + P_{m-1}(\xi) g_0 = 0$$

and then successively equations

$$L g_j = \Phi_j$$

where Φ_j is determined by $g_0, ..., g_{j-1}$. The support of g_0 is a cylinder with the axis in the direction $P'_m(\xi)$; we can choose g_0 with $g_0(0) = 1$ and support close to L by prescribing such values on a plane Σ orthogonal to $P'_m(\xi)$. If the other functions g_j are determined by the boundary condition $g_j = 0$ on Σ, it is clear that $supp g_j \subset supp g_0 \ \ for\ j \neq 0$. For

$$g_t = \sum_{j<M} g_j t^{-j}$$

the third sum on the right-hand side of $(*1)$ is $O(t^{m-1-M+N})$. The last term is rapidly decreasing when $t \to \infty$ since

$$F[\varphi f_t](\eta) = \sum_{j<M} t^{-j}(F[\varphi g_j])(\eta - t\xi)$$

and $t + |\eta| \leq C|\eta - t\xi|$ when $\eta \in \Gamma_N$. The first sum on the right-hand side of $(*1)$ is $O(t^m)$, the second sum is 0 for an appropriate choice of g_0, but the left-hand side grows as t^{m+1} since $\xi \neq 0$. If we take $M = N$ this is contradiction. □

Definition 15. *Let $X \subset \Re^n$ be an open set $f \in D'(X)$ and K be a finite set in $\mathbb{N}_0^n$. A differential operator*

$$P(D) = \sum_{\alpha \in K} a_\alpha D^\alpha$$

is called hypoelliptic if $P(D)f \in C^\infty(X)$ implies $f \in C^\infty(X)$.

Theorem 17. *The following conditions on $P(D)$ are equivalent:*

(i), For every open set $X \subset \Re^n$ and $f \in D'(X)$ we have

$$WF(f) = WF(P(D)f).$$

(ii), For every open set $X \subset \Re^n$ and $f \in D'(X)$ we have

$$singsupp f = singsupp P(D)f.$$

(iii), If X is an open set in $\Re^n$, $f \in D'$ and $P(D)f = 0$, then $f \in C^\infty(X)$.

(iv), $\frac{P^{(\alpha)}(\xi)}{P(\xi)} \to 0 \qquad as \ \ \xi \to \infty \ \ in \ \ \Re^n, \qquad \alpha \neq 0$

(v), $P(D)$ has a fundamental solution E with $singsupp E = \{0\}$.

For the proof see [3].

Hypoelliptic differential operators can be also defined as follows

Definition 16. *The differential operator $P(D)$ (and the polynomial $P(\xi)$) is called hypoelliptic if the equivalent conditions in theorem 17 are fulfilled.*

2.4 The wave front set with respect to C^L

Let L_k be an increasing sequence of positive numbers such that $L_0 = 1$ and

$$k \leq L_k, L_{K+1} \leq CL_k \tag{2.21}$$

for some constant C. If $X \subset \Re^n$ is an open set we shall denote by $C^L(X)$ the set of all $f \in C^\infty(X)$ such that for every compact set $K \subset X$ there is a constant C_k with

$$|D^\alpha f(x)| \leq C_k (C_k L_{|\alpha|})^{|\alpha|}, \qquad x \in X$$

for a multi-index α. When $L_k = k+1$ this means that $C^L(X)$ is the set of real analytic functions in X. The class C^L with $L_k = (k+1)^a$, $a > 1$, is called the Gevrey class of order a. For any distribution $f \in D'(X)$ we define $singsupp_L f$ to be the smallest closed subset of X such that f is in C^L in the complement.
When C^L is the real analytic class we use the notation $singsupp_A f$.
The purpose of this section is to show how one can make a spectral analysis of this set parallel to section (3.1) and (3.2). A new difficulty occurs when

$$\sum_{k=1}^{\infty} \frac{1}{L_k} = \infty \tag{2.22}$$

for then the class C^L is quasi-analytic by the Denjoy-Carleman theorem so one can not choose cutoff functions in C^L. (Multiplication by C^∞ functions not in C^L may of course increase $singsupp_L f$). However this difficulty can be circumvented by choosing test functions with adequate bounds for derivatives up to a certain order only.

proposition 3. *Let $x_0 \in X \subset \Re^n$ and $f \in D'(X)$. Then $f \in C^L$ in a neighborhood of x_0 if and only if for some neighborhood U of x_0 there is a bounded sequence $f_N \in \mathcal{E}'(\mathcal{X})$ which is equal to f in U and satisfies*

$$|F[f_N](\xi)| \leq C(\frac{CL_N}{|\xi|})^N, N = 1, 2, ... \tag{2.23}$$

for some constant C.

For the proof see [1]

Definition 17. *If $X \subset \Re^n$ and $f \in D'(X)$ we denote by $WF_L(f)$ the complement in $X \times (\Re^n \backslash 0)$ of the set of (x_0, ξ_0) such that there is a neighborhood $U \subset X$ of x_0, a conic neighborhood Γ of ξ_0 and a bounded sequence $f_N \in \mathcal{E}'(\mathcal{X})$ which is equal to f in U and satisfies (3.23) when $\xi \in \Gamma$. When C^L is the analytic class we use the notation $WF_A(f)$.*

By definition $WF_L(f)$ is a closed subset of $X \times (\Re^n \backslash \{0\})$. The following lemma shows that f_N can always be chosen as products of f and suitable cutoff functions, obtained by regularizing those in the proof of proposition 3.

Lemma 4. *Let $f \in D'(X)$ and let K be a compact subset of X, F a closed cone $\subset \Re^n$ such that $WF_L(f) \cap (K \times F) = \emptyset$. If $\chi_N \in C_0^\infty(K)$ and for all α*

$$|D^{\alpha+\beta}\chi_N| \le C_\alpha (C_\alpha L_N)^{|\beta|}, \qquad |\beta| \le N = 1, 2, ... \tag{2.24}$$

it follows that $\chi_N f$ is bounded in $\mathcal{E}'^{\mathcal{M}}$ if f is of order M in a neighborhood of K, and we have

$$|F[\chi_N f](\xi)| \le C(\frac{CL_N}{|\xi|})^N, \qquad N = 1, 2, ..., \xi \in F. \tag{2.25}$$

For the proof see [1]

Theorem 18. *The projection of $WF_L(f)$ in X is equal to $singsupp_L f$ if $f \in D'(X)$.*

Proof. If $f \in C^L$ in a neighborhood of x_0 it follows from Proposition 3 that $(x_0, \xi_0) \notin WF_L(f)$, $\xi_0 \in \Re^n \backslash \{0\}$.
Assume that $(x_0, \xi_0) \notin WF_L(f)$ *for all* $\xi_0 \in \Re^n \backslash \{0\}$. Then we can choose a compact neighborhood K of x_0 so that

$$WF_L(f) \cap (K \times \Re^n) = \emptyset.$$

By lemma 4 there is a sequence $\chi_N \in C_0^\infty(K)$ which is equal to 1 in a neighborhood U of x_0 such that $\chi_N f$ is bounded in $\mathcal{E}'$ and satisfies (3.23). Hence $x_0 \notin singsupp_L f$ by proposition 3.
The condition (3.24) is satisfied by any fixed function in C^L with support in K. If C^L is non-quasianalytic we can therefore simplify Definition 16 to the existence of a fixed distribution g which is equal to f in a neighborhood of x_0 and has a Fourier transform satisfying (3.23) in a conic neighborhood of ξ_0 and this is parallel to Definition 12. □

Theorem 19. *For all f and L we have $WF(f) \subset WF_L(f) \subset WF_A(f)$; moreover, if $L_j' \le L_j''$ then $WF_{L''}(f) \subset WF_{L'}(f)$.*

Theorem 20. *If $f \in S'(\Re^n)$ and $F = K * f$, where K is defined by*

$$K(z) = \frac{1}{(2\pi)^n} \int e^{i(z,\xi)}/I(\xi) d\xi, \quad z \in \Omega,$$

then F is analytic in

$$\Omega = \{z;\ |Imz| < 1\}$$

and for some C,a,b

$$|F(z)| \le C(1+|z|)^a (1-|Imz|)^{-b}, \quad z \in \Omega. \tag{01}$$

The boundary values $F(.+i\omega)$ are continuous functions of $\omega \in S^{n-1}$ with values in $S'(\Re^n)$, and

$$(f, \varphi) = \int (F(.+i\omega), \varphi) d\omega, \quad \varphi \in S. \tag{02}$$

Conversely, if F is given satisfying (01) then (02) defines a distribution $f \in S'$ with $F = K * f$. We have for any L

$$(\Re^n \times S^{n-1}) \cap WF_L(f) = \{(x, \omega);\ |\omega| = 1, \quad F \textit{ is not in } C^L \textit{ at } x - i\omega\},$$

and an analogous description is valid for $WF(f)$.
That F is not in C^L at $x - i\omega$ means of course that for some neighborhood V of $x - i\omega$ and some constant C we have

$$|\partial_z^\alpha F(z)| \leq C^{1+|\alpha|} L_{|\alpha|}^{|\alpha|} \qquad if\ z \in V \quad and \quad |Imz| < 1.$$

For the real analytic class this means that F can be continued analytically to a full neighborhood of $x - i\omega$.
Moreover
$WF_L(af) \subset WF_L(f)$ if $a \in C^L(X)$ and $f \in D'(X)$.

For the proof see [1].

Theorem 21. *If $X \subset \Re^n$ is open and S is a closed conic set in $X \times (\Re^n \backslash \{0\})$ then one can find $f \in D'(X)$ with $WF(f) = WF_L(f) = S$ for every L.*

Proof. It is sufficient to prove the statement when $X = \Re^n$, and we only have to verify for the chosen f that

$$WF(f) = WF_A(f) = S.$$

Let (x_k, θ_k) be a sequence without repetition which is dense in $\{(x, \theta) \in S;\ |\theta| = 1\}$. with K defined by

$$K(z) = \frac{1}{(2\pi)^n} \int e^{i(z,\xi)}/I(\xi) d\xi, \quad z \in \Omega.$$

Here

$$\Omega\{z \in \mathbb{C}^n;\ |Imz| < 1\} \ \ and \ \ I(\xi) = \int_{|\omega|=1} e^{-(\omega,\xi)} d\omega$$

we set

$$F(z) = \sum_1^\infty 3^{-k} K(\frac{(z - x_k - i\theta_k)}{2}), \quad |Imz| < 1.$$

Since

$$|K(\frac{(z - x_k - i\theta_k)}{2})| \leq K(\frac{i(Imz - \theta_k)}{2}) \leq C(1 - |Imz|)^{-n}$$

it is clear that F is analytic function satisfying (01). Noting that

$$\sum_{N+1}^\infty 3^{-k} = \frac{3^{-N}}{2}$$

and if $|\theta| = 1, \quad t > 0$ then it follows

$$\begin{aligned} |t\theta_k + \theta| &\leq |t\theta_k| + |\theta| \\ &= |t||\theta_k| + 1 \\ &= t + 1 \end{aligned}$$

$$\Rightarrow |t\theta_k + \theta| \leq t + 1$$

thus we obtain

$$|F(x_k - it\theta_k)| > \frac{3^{-k}}{2}|K(\frac{-i(t+1)\theta_k}{2})|$$

$$- |\sum_{j<k} 3^{-j} K((x_k - x_j - \frac{i(t\theta_k + \theta_j))}{2})| \to \infty,$$

Hence F is not even bounded in ω near any point in

$$S' = \{(x, -\theta); (x, \theta) \in S \ \ \textit{and} \ \ |\theta| = 1\}.$$

On the other hand, it is clear that F is analytic near any point in $(\Re^n \times S^{n-1})\backslash S'$. Then by Theorem 20 this completes the proof of the theorem. □

Theorem 22. *Let $f \in D'(X)$, $X \subset \Re^n$ and assume that $WF_L(f) \subset X \times \Gamma^0$ where Γ^0 is the dual of an open convex cone Γ. If $X_1 \Subset X$ and Γ_1 is an open convex cone with closure $\subset \Gamma \cup \{0\}$, then one can find a function F analytic in $\{x + iy;\ x \in X_1,\ y \in \Gamma_1,\ |y| < \gamma\}$, such that*

$$|F(x + iy)| < C|y|^{-N}, \quad y \in \Gamma_1, \quad |y| < \gamma, \quad x \in X_1,$$

and the limit of $F(. + iy)$ when $y \to 0$ in Γ_1 differs from f by an element in $C^L(X_1)$.

Proof. Set $g = \chi f$ where $\chi \in C_0^\infty(X)$ is equal to 1 in X_1. If $V = K * g$ is defined as in Theorem (8.4.11) see [1] we have $V \in C^L$ at every point in $X_1 + i(S^{n=1} \cap \complement(-\Gamma^0))$. choose $M \subset S^{n-1} \subset M$ *and* $\overline{M}$ in the interior of Γ_1^0. Then $g = g_1 + g_2$ where

$$g_1 = \int_{-\omega \notin M} V(. + i\omega) d\omega$$

belongs to C^L *in* X_1 *and* g_2 is the boundary value of the analytic function

$$F(z) = \int_{-\omega \in M} V(z + i\omega) d\omega, \quad Imz \in \Gamma_1, \quad |Imz| < \gamma.$$

then to complete the proof use Lemma 8.4.12 see [1]. □

Corollary 2. *If $f \in D'(X)$ where X is an interval on $\Re$ and if $x_0 \in X$ is a boundary point of suppf, then $(x_0, \pm 1) \in WF_A(f)$.*

Proof. Assume for example that $(x_0, -1) \notin WF_A(f)$. Then we can find F analytic in

$$\Omega = \{z;\ Imz > 0,\ |z - x_0| < r\}$$

with boundary value f. There is an interval

$$I \subset (x - 0 - r, x_0 + r)$$

where $f = 0$. Then by Theorem 3.1.12[1] and Theorem 4.4.1[1] F can be extended analytically across I so that $F = 0$ below I. Thus the uniqueness of analytic continuation gives $F = 0$, hence f=0 in $(x_0 - r, x_0 + r)$. This contradicts that x_0 is a boundary point of $suppf$. □

Note that the corollary can be phrased as a uniqueness theorem:If we know that $WF_A(f)$ does not contain $T_x^*(\Re)\backslash\{0\}$ for any $x \in X$ then f must vanish identically if f vanishes in an open set.

Lemma 5. *If $f \in S'$ then $WF_A(f) \subset \Re^n \times F$ where F is the limit cone of $suppF[f]$ at ∞, consisting of all limits of sequences $t_j x_j$ with $x_j \in suppF[g]$ and $0 < t_j \to 0$.*

Proof. The Fourier transform of $f * K$ *is* $\frac{F[f](\xi)}{I(\xi)}$. where

$$K(z) = \frac{1}{(2\pi)^n} \int e^{i(z,\xi)}/I(\xi) d\xi, \quad z \in \Omega.$$

If Γ is an open cone with $\overline{\Gamma} \cap F = \{0\}$ we can choose a closed cone F' with $F\backslash\{0\}$ in its interior and $\Gamma \cap F' = \{0\}$. Then we have for some $c < 1$

$$\langle y, \xi \rangle \leq c|y||\xi| \quad if\ y \in \Gamma, \quad \xi \in F'.$$

Hence Lemma 8.4.9 [1] shows that $\frac{e^{-\langle y,\xi \rangle}}{I(\xi)}$ and all its ξ derivatives are bounded if $y \in -\Gamma,\ |y| < \frac{2}{(1+c)},\ and\ \xi \in F'$. Since $suppF[f]$ is contained in the union of F' and a compact set, it follows that

$$\frac{F[f](\xi)e^{-\langle y,\xi \rangle}}{I(\xi)} \ is\ in\ S'\ when\ y \in -\Gamma,\ |y| < \frac{2}{(1+c)}.$$

By theorem 7.4.2[1] it follows that $f * K$ has an analytic continuation to $\{z;\ Imz \in -\Gamma,\ |Imz| < \frac{2}{(1+c)}\}$. Hence $WF_A(f) \subset \complement\Gamma$ by Theorem 20 it follows that $WF_A(f) \subset \Re^n \times F$. □

Theorem 23. *If $f \in S'(\Re^n)$ is homogeneous in $\Re^n\backslash\{0\}$ then*

$$(x, \xi) \in WF_L(f) \Leftrightarrow (\xi, -x) \in WF_L(F[f]) \quad if \quad \xi \neq 0, \quad x \neq 0, \tag{2.26}$$

$$x \in suppf \Leftrightarrow (0, -x) \in WF_L(F[f]), \quad x \neq 0, \tag{2.27}$$

$$\xi \in suppF[f] \Leftrightarrow (0, \xi) \in WF_L(f), \quad \xi \neq 0. \tag{2.28}$$

For the proof of Theorem 23 see [1].

2.5 Rule of computation for WF_L

We shall now see analogues for WF_L of the results on WF in section 3.2.

Theorem 24. *If f, $g \in D'(X)$ and $(x,\xi) \in WF_L(f)$ implies $(x,-\xi) \notin WF_L(g)$ then the product fg is defined and*

$$WF_L(fg) \subset \{(x,\xi+\eta);\ (x,\xi) \in WF_L(f) \ or\ \xi = 0,\ (x,\eta) \in WF_L(g)\ or\ \eta = 0\}.$$

The way of proving this Theorem is similar to that of Theorem 9. Thus by begin with a special case which fit the notations in Theorem 20 the proof of Theorem 24 can be performed.

Theorem 25. *Let $f \in \mathcal{E}'(\Re^\backslash)$, split the coordinates in $\Re^n$ into two groups $x' = (x_1, ..., x_{n'})$ and $x'' = (x_{n'+1}, ..., x_n)$, and set*

$$f_1(x') = \int f(x', x'')dx''$$

Then

$$WF_L(f_1) \subset \{(x', \xi^{'});\ (x', x'', \xi^{'}, 0) \in WF_L(f)\ \ for\ some\ x''\}.$$

Proof. By using the notation in Theorem 20 we have

$$\langle f, \varphi \otimes \phi\rangle = \int_{|\omega|=1} \langle F(. + i\omega), \varphi \otimes \phi\rangle d\omega,$$

$$\varphi \in C_0^\infty(\Re^{n'}),\quad \phi \in C_0^\infty(\Re^{n-n'}).$$

Take $\phi(x'') = \chi(\delta x'')$ *where* $\chi = 1$ in the unit ball, and let $\delta \to 0$. Since F is exponentially decreasing at infinity it follows then that

$$\begin{aligned}\langle f_1, \varphi\rangle &= \int_{|\omega|=1} \langle F(. + i\omega),\ \varphi \otimes 1\rangle d\omega \\ &= \int_{|\omega|=1} \langle F_1(. + i\omega'),\ \varphi\rangle d\omega\end{aligned}$$

where

$$F_1(z') = \int F(z', x'')dx'' = \int F(z', x'' + iy'')dx'',\quad |Imz'|^2 + |y''|^2 < 1,$$

is an analytic function when $|Imz'| < 1$ which is bounded by $C(1 - |Imz'|)^{-N}$.
If $|\omega_0'| = 1$ and $(x', x'', \omega_0') \notin WF_L(f)$ for every $x'' \in \Re^{n-n'}$ then $F_1 \in C^L$ *at* $x' - i\omega_0'$. Hence Lemma 8.4.12 [1] implies that $(x', \omega_0') \notin WF_L(f_1)$ □

Theorem 26. *Let $X \subset \Re^n,\ Y \subset \Re^m$ be open sets and $k \in D'(X \times Y)$ be a distribution such that the projection suppk $\to X$ is proper. If $f \in C^L(Y)$ then*

$$WF_L(\kappa f) \subset \{(x,\xi);\ (x,y,\xi,0) \in WF_L(k)\ \ for\ some\ y \in suppf\}.$$

Here κ is the linear operator with kernel K.

Proof. Replacing K by $K(1 \otimes f)$ we may assume that $f = 1$. Without changing K over a given compact subset of X we may replace K by a distribution of compact support, and then the statement is identical to Theorem 25. □

Theorem 27. *If $f \in \mathcal{E}'(\mathcal{Y})$ and $WF_L(f) \cap WF_L^{'}(k)_Y = \emptyset$ then*

$$WF_L(\kappa f) \subset WF_L(k)_X \cup (WF_L^{'}(k) \circ WF_L(f)).$$

For the proof of this Theorem see [1].

Theorem 28. *Let $f \in D'(X)$, $X \subset \Re^n$, and assume that φ is a real valued real analytic function in X and x^0 a point in $suppf$ such that*

$$d\varphi(x^0) \neq 0, \quad \varphi(x) \leq \varphi(x^0) \quad if \;\; x \in suppf. \tag{2.29}$$

Then it follows that

$$(x^0, \pm d\varphi(x^0)) \in WF_A(f). \tag{2.30}$$

Proof. Replacing φ by $\varphi(x) - |x - x^0|^2$ we may assume that

$$\varphi(x) < \varphi(x^0) \quad if \;\; x^0 \neq x \in suppf.$$

Since $d\varphi(x^0) \neq 0$ we may take φ as a coordinate locally, so we may assume that $\varphi(x) = x_n$ and that $x^0 = 0$.
Choose a neighborhood Y of $0 \in \Re^{n-1}$ so that $Y \times \{0\} \Subset X$. Since $suppf \cap (Y \times \{0\}) = \{0\}$ we can choose an open interval $I \subset \Re$ with $0 \in I$ so that

$$Y \times I \Subset X \quad and \quad (\partial Y \times I) \cap suppf = \emptyset.$$

If $a(x')$ is an entire analytic function of $x' = (x_1, ..., x_{n-1})$ then Theorem 26 (with $X \times Y,\ x\ and\ y\ replaced\ by\ I \times Y,\ x_n\ and x'$) gives that

$$F_a(x_n) = \int_Y f(x)a(x')dx'$$

is well defined as a distribution in I and that

$$WF_A(F_a) \subset \{(x_n, \xi_n);\ (x', x_n, 0, \xi_n) \in WF_A(f)\ for\ some\ x' \in Y\}.$$

Here (x', x_n) must be close to 0 if x_n is small. If we say $(0, e_n) \notin WF_A(f),\ e_n = (0, ..., 0, 1)$, then we can choose I so that $(x, e_n) \notin WF_A(f)$ if $x \in Y \times I$. Hence $(x_n, 1) \notin WF_A(F_a)\ if\ x_n \in I$, so Corollary 2 gives that $F_a = 0$ in I, because $F_a = 0$ when $x_n > 0$. Thus if f_1 is f restricted to $Y \times I$,

$$\langle f_1, a \otimes \varphi \rangle = 0$$

for all real analytic a and all $\varphi \in C_0^\infty(I)$. Since a is free to vary in a dense subset of $C^\infty(\Re^{n-1})$ it follows from theorem 5.1.1 [1] that $f = 0$ in $Y \times I$. This contradiction proves (3.30). □

Definition 18. *If F is a closed subset of a C^2 manifold X then the exterior normal set $N_e(F) \subset T^*(X)\backslash\{0\}$ is defined as the set of all (x^0, ξ^0) such that $x^0 \in F$ and there is a real valued function $\varphi \in C^2(X)$ with $d\varphi(x^0) = \xi^0 \neq 0$ and*

$$\varphi(x) \leq \varphi(x^0) \qquad \textit{when } x \in F. \tag{2.31}$$

proposition 4. *For every closed subset F of the C^2 manifold X the projection of $N_e(F)$ in X is dense in ∂F. If $x^0 \in F$, $\varphi \in C^1(X)$, $d\varphi(x^0) = \xi^0 \neq 0$ and $\varphi(x) \leq \varphi(x^0)$ when $x \in F$, then $(x^0, \xi^0) \in \overline{N_e(F)}$. If $X \subset \Re^n$ and Y is a convex open set $\subset X\backslash F$, $x^0 \in F \cap \partial Y$, we have $(x^0, \xi^0) \in \overline{N_e(F)}$ for some ξ^0 with $\langle x - x^0, \xi^0 \rangle > 0$, $x \in Y$.*

For the proof of this Proposition see [1].

Theorem 29. *For every $f \in D'(X)$ we have*

$$\bar{N}(suppf) \subset WF_A(f). \tag{2.32}$$

The importance of this theorem will be enhanced in section 3.6, that is if f satisfies a differential equation $P(x, D)f = 0$ with analytic coefficients, then $WF_A(f)$ is contained in the characteristic set of P. Thus the principal symbol $p(x, \xi)$ vanishes on $WF_A(f)$, so it must vanish on $\overline{N}(suppf)$ by (3.32).

2.6 WF_L for solution of partial differential equations

Theorem 30. *If $P(x,D)$ is a differential operator of order m with real analytic coefficient in X, then*

$$WF_L(f) \subset Char P \cup WF_L(Pf), \qquad f \in D'(X), \tag{2.33}$$

In order to prove this theorem let us see the following Lemmas

Lemma 6. *There is a constant C' such that if*

$$j = j_1 + ... + j_k \;\; and \;\; j + |\beta| \le 2N,$$

then

$$|D^\beta R_{j_1} ... R_{j_k} \chi_{2N}| \le C'^{N+1} N^{j+|\beta|} |\xi|^{-j}, \qquad \xi \in V \tag{$*^0$}$$

where $R = R_1 + R_2 + ... + R_m$ and $R_j|\xi|^j$ is a differential operator of order less than or equal to j with analytic coefficient which are homogeneous of degree 0 with respect to ξ when $\xi \in V$ and $x \in K$.

Lemma 7. *Let K be a compact set in $\Re^n$ and K' a neighborhood of K in $\mathbb{C}$. If $a_1, ..., a_{j-1}$ are analytic and $|a_1| < 1, ..., |a_{j-1}| < 1$ in K', $j \le N$, we have*

$$|D_{i_1} a_1 D_{i_2} ... a_{j-1} D_{i_j} \chi_N| \le C'^{N+1} N^j. \tag{2.34}$$

Proof of Theorem 30. We need to show that if (x_0, ξ_0) does not belong to the right-hand side of (3.33) and $\xi \neq 0$, then $(x_0, \xi_0) \notin WF_L(f)$. The hypothesis means that we can choose a compact neighborhood K of x_0 and a closed conic neighborhood V of ξ_0 in $\Re^n \backslash \{0\}$ such that

$$P_m(x,\xi) \neq 0 \quad in \;\; K \times V$$

and

$$(K \times V) \cap WF_L(Pf) = \emptyset.$$

Now by using Theorem 1.4.2 [1] we now choose a sequence $\chi_N \in C_0^\infty(K)$ equal to 1 in a fixed neighborhood U of x_0 such that for every α

$$|D^{\alpha+\beta} \chi_N| \le C_\alpha (C_\alpha N)^{|N|}, \qquad |\beta| \le N.$$

Then the sequence $f_N = \chi_{2N} f$ is bounded in $\mathcal{E}'$ and equal to f in U. Now to complete the proof it is sufficient showing that (3.23) is valid in V when $|\xi| > N$, for (3.23) follows from the boundedness of f_N when $|\xi| \le N \le L_N$. To estimate $F[f_N](\xi)$ in V we must solve the equation

$${}^t P g(x) = \chi_{2N}(x) e^{-i}(x,\xi) \tag{$*'$}$$

approximately. Writing $g = e^{-i(x,\xi)}h/P_m(x,\xi)$ and noting that the principal symbol of ${}^tP$ *is* $P_m(x,\xi)$, we obtain instead of $(*')$ an equation of the form

$$h - Rh = \chi_{2N}, \qquad R = R_1 + R_2 + ... + R_m$$

where $R = R_1 + R_2 + ... + R_m$ and $R_j|\xi|^j$ is a differential operator of order less than or equal to j with analytic coefficient which are homogeneous of degree 0 with respect to ξ when $\xi \in V$ *and* $x \in K.$ and

$$h = \sum R^k \varphi$$

Then a solution is given by

$$\sum_0^\infty R^k \chi_{2N}.$$

However, we must not introduce derivatives of very high order so we set

$$h_N = \sum_{j_1+...+j_k \leq N-m} R_{j_1}...R_{j_k}\chi_{2N}.$$

Then after some calculation we obtain

$$h_N - Rh_N = \chi_{2N} - \sum_{j_1+...+j_k > N-m \geq j_2+...+j_k} R_{j_1}...R_{j_k}\chi_{2N} = \chi_{2N} - e_N.$$

This means

$${}^tP(x,D)(e^{-i(x,\xi)}h_N(x,\xi)/P_m(x,\xi)) = e^{-(x,\xi)}(\chi_{2N}(x) - e_N(x,\xi)).$$

with integral denoting action of distribution we obtain

$$\int f(x)\chi_{2N}(x)e^{-i(x,\xi)}dx = \int f(x)e_N(x,\xi)e^{-i(x,\xi)}dx$$

$$+ \int \varphi(x)e^{-i(x,\xi)}h_N(x,\xi)/P_m(x,\xi)dx. \quad (*'')$$

Here $\varphi = P(x,D)f$. Then by Lemma 6 and Lemma 7 if M is the order of f in a neighborhood of K, we can estimate the first term on the right-hand side of $(*'')$ for large N and $|\xi| > N,\ \xi \in V$, by

$$C \sum_{|\alpha| \leq M} (1 + |\xi|)^{M-|\alpha|} \sup_x |D^\alpha e_N(x,\xi)|.$$

The number of terms in e_N can not exceed 2^N, and each term can be estimated by means of $(*^0)$, which gives the bound

$$C_1|\xi|^{M+m-N}C'^{N+1}N^{N+M}2^N.$$

If N is replaced by $N + m + M$ this is an estimate of the desired form $(*')$ even for the analytic class. To estimate the last term in $(*'')$ we observe that $(*^0)$ gives

$$|D^\beta h_N| \leq C_1^{N+1} N^{|\beta|}, \quad |\beta| \leq N, \quad \xi \in V, \quad |\xi| > N.$$

We have a similar bound for $h_N|\xi|^m/P_m(x,\xi)$. The proof is therefore completed by the following lemma.

Lemma 8. *Let $f \in D'(X)$, let K be a compact subset of X and V a closed cone $\subset \Re^n \backslash \{0\}$ such that*

$$WF_L(f) \cap (K \times V) = \emptyset.$$

If $h_N \in C_0^\infty(K)$ and (8.6.10)[1] is fulfilled, then

$$|F[h_N f](\xi)| \leq C_2^{N+1} (\frac{L_{N-M-n}}{|\xi|})^{N-M-n} \quad if\ \xi \in V,\ |\xi| > N,\ N > M + n. \tag{2.35}$$

Here M is the order of f in a neighborhood of K.

Theorem 31. *If $f \in D'(X)$ is a solution of a differential equation $P(x,D)f = 0$ with analytic coefficients, then the principal symbol $P_m(x,\xi)$ must vanish on $N(suppf)$. Thus $f = 0$ in a neighborhood of a non-characteristic C^1 surface if this is true on one side.*

Theorem 32. *Let $P(x,D)$ be a differential operator with analytic coefficients and let $\mathcal{S}$ be the smallest subset of $C^\infty(T^* \backslash \{0\})$ which contains all C^∞ functions vanishing on $CharP$ and is closed under Poisson brackets. If $f \in D'(X)$ and $P(x,D)f = 0$ it follows then that all functions in $\mathcal{S}$ must vanish on $N(suppf)$.*

In particular, if the functions in $\mathcal{S}$ have no common zeros then we conclude that f vanishes identically if X is connected and f vanishes in an open set. If f vanishes on one side of a C^1 surface with normal ξ at x, then f vanishes in a neighborhood of x unless all functions in $\mathcal{S}$ vanish at (x,ξ). This is an improvement of the classical uniqueness theorem of Holmgren as the following example shows:

Example 7. *If $P(x,\xi) = \xi_1^2 + x_1^2\xi_2^2 + ... + x_{n-1}^2\xi_n^2$ then $\xi_1, x_1\xi_2, ..., x_{n-1}\xi_n$ vanish on $CharP$. Taking Poisson brackets we obtain*

$$\{\xi_1, x_1\xi_2\} = \xi_2, \quad \{\xi_2, x_2\xi_3\} = \xi_3, \qquad , \{\xi_{n-1}, x_{n-1}\xi_n\} = \xi_n$$

so the functions in $\mathcal{S}$ have no common zeros.

Example 8. *If $P(x,\xi) = x_2^2\xi_1^2 + \xi_2^2 + \xi_3^2$ then $\mathcal{S}$ contains ξ_1, ξ_2, x_2 and since $\{\xi_2, x_2\} = 1 \in \mathcal{S}$ there are no common zeros. However, the solutions of $P(x,D)f = 0$ need not be analytic. In fact;*

$$f_\tau(x) = \exp(\tau x_3 + i x_1 \tau^2 - x_2^2\tau^2/2)$$

is a solution for every τ. Hence

$$f(x) = \int_0^\infty f_\tau(x)e^{-\tau}d\tau$$

is a C^∞ solution when $|x_3| < 1$, but f is not real analytic since

$$D_1^k f(0) = \int_0^\infty \tau^{2k}e^{-\tau}d\tau = (2k)!.$$

For differential operators with constant coefficients forming Poisson brackets is of no use, for the Poisson bracket of any two functions of ξ is 0.

Theorem 33. *Let the plane $\langle x, N\rangle = 0$, $N \in \Re^n$, be characteristics with respect to the differential operator P(D), that is, $P_m(N) = 0$. Then there exist a solution f of the equation $P(D)f = 0$ such that $f \in C^\infty(\Re^n)$ and supp$f = \{x : \langle x, N\rangle \leq 0\}$.*

For the proof of this Theorem see [1]

Theorem 34. *Let X_1 and X_2 be open convex sets in $\Re^n$ such that $X_1 \subset X_2$, and let P(D)) be a differential operator with constant coefficients. Then the following conditions are equivalent:*
(i) Every $f \in D'(X_2)$ satisfying the equation $P(D)f = 0$ in X_2 and vanishing in X_1 must also vanish in X_2.
(ii) Every hyperplane which is characteristic with respect to P and intersects X_2 also intersects X_1.

For the proof of this Theorem see [1]

Corollary 3. *If the support of a solution $f \in D'(\Re^n)$ of the solution $P(D)f = 0$ is contained in a half space with non-characteristic boundary, then $f = 0$.*

Proof. Every characteristic plane intersect the half space. □

Theorem 35. *Let P(D) be a real principal type. If $f \in D'(X)$, $P(D)f = \varphi$ and $(x, \xi) \in WF_L(f)\backslash WF_L(\varphi)$, then $P_m(\xi) = 0$ and*

$$I \times \{\xi\} \subset WF_L(f)$$

if $I \subset X$ a line segment containing x with direction $P'_m(\xi)$ such that

$$(I \times \{\xi\}) \cap WF_L(\varphi) = \emptyset.$$

For the proof see [1]

If $\mu \in S'(\Re^n)$ we can define a characteristic set as follows.
First we let Γ be the set of all $\xi_0 \in \Re^n\backslash\{0\}$ such that there is a complex conic neighborhood V of ξ_0 and an analytic function ϕ in

$$V_C = \{\xi \in V,\ |\xi| > C\}$$

for some C such that

$$\phi F[\mu] = 1 \quad in\ V_C \cap \Re^n$$

and

$$|\phi(\xi)| \le C_1 |\xi|^N, \qquad \xi \in V_C, \tag{2.36}$$

for some N and C_1. We shall denote by $Char\mu$ the complement of Γ in $\Re^n \backslash \{0\}$.

Theorem 36. *If $\mu \in S'(\Re^n)$ and $f \in \mathcal{E}'(\Re^\backslash)$, then*

$$WF_A(f) \subset WF_A(\mu * f) \cup (\Re^n \times Char\mu). \tag{2.37}$$

Proof. We shall use the interpretation of WF_A in Theorem 20.With the notation in that theorem we must show that $f * K(z)$ is analytic at $x_0 - i\xi_0$ if $\xi_0 \notin Char\mu$, $|\xi_0| = 1$ and $(x_0, \xi_0) \notin WF_A(\varphi)$, $\varphi = \mu * f$. Choose V and ϕ as above so that (3.37) is valid and $\phi F[\mu] = 1$ *in* $V_C \cap \Re^n$. Let W' *and* W'' be closed conic neighborhoods of ξ_0 in $\Re^n \backslash \{0\}$ such that W'' is contained in the interior of W' and $W' \subset V$.
Choose $\chi \in C^\infty$ with $0 \le \chi \le 1$ equal to 1 in a neighborhood of W''_{3C} and $supp\chi \subset W'_{2C}$ so that χ is homogeneous of degree 0 when $|\xi| > 3C$. Then the Fourier transform of $f * K(. + iy)$, $|y| < 1$, can be decomposed as follows

$$F[f]e^{-(y,\xi)}/I(\xi) = F[f](1 - \chi(\xi))e^{-(y,\xi)}/I(\xi) + F[\varphi]\phi(\xi)\chi(\xi)e^{-(y,\xi)}/I(\xi).$$

If we introduce the Fourier transforms

$$K_1(z) = \frac{1}{(2\pi)^n} \int (1 - \chi(\xi))e^{i(z,\xi)}/I(\xi)d\xi,$$

$$K_2(z) = \frac{1}{(2\pi)^n} \int \chi(\xi)\phi(\xi)e^{i(z,\xi)}/I(\xi)d\xi$$

which is rapidly decreasing when $Rez \to \infty$, $|Imz| < 1$, it follows that

$$K * f(z) = K_1 * f(z) + K_2 * \varphi(z), \qquad |Imz| < 1.$$

It is clear that K_1 remains analytic when $|Imz + \xi_0|$ is sufficiently small, so $K_1 * f(z)$ is analytic at $x_0 - i\xi_0$. To study the properties of K_2 we shall follow the proof of lemma 8.4.10 [1] although we must now work in all variable and apply Stokes' formula. Let $\chi_1(\xi)$ *be a* C^∞ function with support in W''_{3C} which is 1 in W'''_{4C} for another conic neighborhood W''' of ξ_0 and is homogeneous of degree o for $|\xi| > 4C$. We want to move the integration to the cycle $(x = Rez)$

$$\xi \to \xi + i\delta\chi_1(\xi)|\xi|x(1 + |x|^2)^{-\frac{1}{2}}, \qquad \xi \in \Re^n,$$

where $0 < \delta \le 1$ is chosen so small that we do not leave V_C when $\xi \in supp\chi_1$. To estimate the integrand we shall use Lemma 8.4.9 [1] and the inequality

$$Re(i\langle x + iy, \xi + i\eta\rangle - \langle \xi + i\eta, \xi + i\eta\rangle^{\frac{1}{2}}) \le -\langle x, \eta\rangle - \langle y, \xi\rangle - (|\xi|^2 - |\eta|^2)^{\frac{1}{2}}$$

valid when $|\eta| < |\xi|$. (This follows from the fact that $Rew^2 \leq (Rew)^2$.) When $\eta = \rho|\xi|x(1+|x|^2)^{-\frac{1}{2}}, \;\; 0 \leq \rho \leq 1$, we obtain the estimate

$$|\xi|(-\rho|x|^2/(1+|x|^2)^{\frac{1}{2}} - (1-\rho^2|x|^2/(1+|x|^2))^{\frac{1}{2}}) - \langle y, \xi\rangle.$$

The parenthesis is a convex function of ρ which is -1 for $\rho = 0$ and $-(1+|x|^2)^{\frac{1}{2}}$ when $\rho = 1$. Hence

$$Re(i\langle x+iy, \xi+i\eta\rangle - \langle \xi+i\eta, \xi+i\eta\rangle^{\frac{1}{2}}) \leq -|\xi|(1-\rho+\rho(1+|x|^2)^{\frac{1}{2}}) - \langle y, \xi\rangle.$$

Using Stokes' formula as just indicated it follows that for some $\delta > 0$ there is an analytic continuation of K_2 to

$$\{z;\; |Imz| < 1-\delta+\delta(1+|Rez|^2)^{\frac{1}{2}}, \;\; |Imz+\xi_0| < \delta\}$$

where the second restriction as in the discussion of K_1 comes from the set where the integration contour has not been deformed. An integration by part shows that K_2 is rapidly decreasing at infinity in this set. The properties of K_2 shows that the boundary value $K_2 * \varphi(.-i\xi_0)$ is equal to the convolution of φ and the boundary values $K_2(.-i\xi_0)$ which are analytic except at 0. Write $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1 \in \mathcal{E}'$ and φ_2 vanishes when $|x - x_0| < r$, say. Then

$$K_2 * \varphi_2(z) = \varphi_2(K_2(.-z))$$

is analytic when z is so close to $x_0 - i\xi_0$ that $K(.-z)$ is uniformly in S when $|x - x_0| \geq r$. By Theorem 8.4.8 [1] we have $WF_A(K_2(.-i\xi_0)) \subset \{(0, t\xi_0), \; t > 0\}$. And from Lemma 8.4.12 [1] if $(x_0, \xi_0) \notin WF_A(\varphi_1)$ it follows by the analogue of (8.2.16) [1] for WF_A that

$$x_0 \notin singsupp_A \varphi_1 * K_2(.-i\xi_0).$$

Hence $K * f$ is analytic at $x_0 - i\xi_0$ which completes the proof. □

Conclusion

From our study we observe that the smoothness of a function is directly dependent on the estimate (1). Here if the estimate (1) is satisfied that is well. Because the function will be smooth. But if the estimate (1) is not satisfied that is if the Fourier transform is not fast decay we can detect the point and the direction in which the Fourier transform is not fast decay by using the wave front set. Moreover in order to say the product of two distributions f and g exist at a point x, for all directions ξ, we have seen that the Hörmander's condition need to be fulfilled, that is if (x, ξ) belong to $WF(f)$, then $(x, -\xi)$ must not belong to $WF(g)$.

For a differential operator P, if a fundamental solution E with $singsuppE = \{0\}$ is obtained in $X \subset \Re^n$, then we arrived on a fact that

$$singsuppf = singsuppPf \qquad for f \in D'(X)$$

particularly if P is elliptic the equality

$$WF(f) = WF(Pf), \qquad for f \in D'(X)$$

holds true. Moving forward in our study led us to define an extended set of functions C^L and the wave front set WF_L defined corresponding to C^L. Specifically we have seen how the analytic wave front set of a distribution f denoted by WF_A is defined and used when C^L is the analytic class.

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