A FUZZY PROGRAMMING APPROACH TO BI-LEVEL LINEAR PROGRAMMING PROBLEMS

A PROJECT REPORT SUBMITTED TO THE DEPARTMENT OF MATHEMATICS; ADDIS ABABA UNIVERSITY AS A PARTIAL FULFILLMENT TO MSC. DEGREE IN MATHEMATICS.

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January, 2013

Addis Ababa, Ethiopia
Acknowledgements
ABSTRACT

Bi-level programming is characterized as a mathematical programming to solve decentralized problems with two decision makers in a hierarchical organization. They become more important for contemporary decentralized organization where each unit seeks to optimize its own objective. In this report we have considered a bi-level linear programming and applied fuzzy mathematical programming (FMP) approach to obtain the solution of the system. We have suggested FMP method for the minimization of the objectives in terms of the linear membership functions. FMP is a supervised search procedure (supervised by the upper decision maker (DM)). The upper level decision maker provides the preferred values of decision variables under his control (to enable the lower level DM to search for his optimum in a wider feasible space) and the bounds of his objective function (to direct the lower level DM to search for his solutions in the right direction).
Acknowledgements

I would like to thank the one Alpha and Omega-God, the whole secret of my life. Many thanks to my lovely wife W/ro. Bilisumma Waktole for her large contribution in the progress of my life.

I would also like to express my heartfelt acknowledgements to my advisor Dr. Semu Mitiku Kassa for his valuable and crucial advices, for his cooperation and continuous follow up in commenting and correcting me from the very beginning up to the end and for his careful reading of the paper and helpful suggestions.
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CHAPTER-0

INTRODUCTION

Decision making problems in decentralized organizations are often modeled as stackelberg games, and they are formulated as bi-level mathematical programming problems. A bi-level problem with a single decision maker at the upper level and two or more decision makers at the lower level is referred to as a decentralized bi-level programming problem. Real-world applications under non-cooperative situations are formulated by bi-level mathematical programming problems and their effectiveness is demonstrated.

The use of fuzzy set theory for decision problems with several conflicting objectives was first introduced by Zimmermann. Thereafter, various versions of fuzzy programming (FP) have been investigated and widely circulated in literature. The use of the concept of tolerance membership function of fuzzy set theory to bi-linear programming problems (BLPPs) for satisfactory decisions was first introduced by Lai in 1996 [3]. Shih and Lee further extended Lai’s concept by introducing the compensatory fuzzy operator for solving BLPPs [5]. Sinha studied alternative BLP techniques based on fuzzy mathematical programming (FMP).

The basic concept of these fuzzy mathematical programming (FMP) approaches is the same as fuzzy goal programming (FGP) approach which implies that the lower level DMs optimizes, his/her objective function, taking a goal or preference of the higher level DMs in to consideration. In the decision process, considering the membership functions of the fuzzy goals for the decision variables of the higher level DM, the lower level DM solves a FMP problem with a constraint on an overall satisfactory degree of the higher level DMs. If the proposed solution is not satisfactory, to the higher level DMs, the solution search is continued by redefining the elicited membership functions until a satisfactory solution is reached [5]. The main difficulty that arises with the FMP approach of Sinha is that there is possibility of rejecting the solution again and again by the higher level DMs and re-evaluation of the problem is repeatedly needed to reach the satisfactory decision, where the objectives of the DMs are over conflicting [5].
Taking into account vagueness of judgments of the decision makers, we will present interactive fuzzy programming for bi-level linear programming problems. In the interactive method, after determining the fuzzy goals of the decision makers at both levels, a satisfactory solution is derived by updating some reference points with respect to the satisfactory level. In the real world, we often encounter situations where there are two or more decision makers in an organization with a hierarchical structure, and they make decisions in turn or at the same time so as to optimize their objective functions. In particular, consider a case where there are two decision makers; one of the decision makers first makes a decision. Such a situation is formulated as a bi-level programming problem. Although a large number of algorithms for obtaining Stackelberg solutions have been developed, it is also known that solving the mathematical programming problems for obtaining Stackelberg solution is NP-hard (Shimizu, Ishizuka and Bard, 1997). From such difficulties, a new solution concept which is easy to compute and reflects structure of bi-level programming problems had been expected. Lai (1996) and Shih, Lai and Lee (1996) proposed a solution method, which is different from the concept of Stackelberg solutions, for bi-level linear programming problems with cooperative relationship between decision makers. Sakawa, Nishizaki and Uemura (1998) present interactive fuzzy programming for bi-level linear programming problems. In order to overcome the problem in the methods of Lai (1996) and Shih, Lai and Lee (1996), after eliminating the fuzzy goals for decision variables, they formulate the bi-level linear programming problem.

In their interactive method, after determining the fuzzy goals of the decision makers at all the levels, a satisfactory solution is derived efficiently by updating the satisfactory degree of the decision maker at the upper level with considerations of overall satisfactory balance among all the levels. By eliminating the fuzzy goals for the decision variables to avoid such problems in the method of Lai (1996) and Shih, Lai and Lee (1996), Sakawa, Nishizaki and Uemura (1998) develop interactive fuzzy programming for bi-level linear programming problems. Moreover, from the viewpoint of experts’ imprecise or fuzzy understanding of the nature of parameters in a problem formulation process, they extend it to
Interactive fuzzy programming for bi-level linear programming problems with fuzzy parameters (Sakawa, Nishizaki and Uemera, 2000a). Interactive fuzzy programming can also be extended so as to manage decentralized bi-level linear programming problems by taking into consideration individual satisfactory balance between the upper level DM and each of the lower level DMs as well as overall satisfactory balance between the two levels (Sakawa and Nishizaki, 2002a). Moreover, by using some decomposition methods which take advantage of the structural features of the decentralized bi-level problems, efficient methods for computing satisfactory solutions are also developed (Kato, Sakawa and Nishizaki, 2002, Sakawa, Kato and Nishizaki, 2001). Bi-level programming is characterized as mathematical programming to solve decentralized planning problems. We have considered a bi-level linear programming problem (BLPP) and applied fuzzy mathematical programming (FMP) potentially for agriculture, bio-fuel production, economic systems, governmental policy, network flow design, transportation design and etc.

In this project, we discuss a procedure for solving bi-level linear programming problems through linear fuzzy mathematical programming (FMP) approach. In order to reach the optimal solution of bi-level linear programming problems, using fuzzy programming approach, the report contains three chapters. In chapter I, we discuss the basic concept of fuzzy set, membership function, binary operation on fuzzy numbers and linear programming using fuzzy approach, in chapter II, the basic concept of bi-level programming, characteristics and mathematical formulation of bi-level linear programming problems will be presented, in chapter III, the procedure for solving bi-level linear programming problems and FMP solution approach) are discussed! Moreover, the selection of compromise solution to FMP models and comparison of optimal solution with other FP approach is also included.
CHAPTER 1

FUZZY PROGRAMMING

1.1. Fuzzy set theory

Fuzzy set theory has been developed to solve problems where the descriptions of activities and observations are imprecise, vague, or uncertain. The term “fuzzy” refers to a situation where there are no well-defined boundaries of the set of activities or observations to which the descriptions apply. For example, one can easily assign a person 180cm tall to the “class of tall men”. But it would be difficult to justify the inclusion or exclusion of a 173cm tall person to that class, because the term “tall” does not constitute a well-defined boundary. This notion of fuzziness exists almost everywhere in our daily life, such as a “class of red flowers,” a “class of good shooters,” a “class of comfortable speeds for travelling,” “a number close to 10,” etc. These classes of objects cannot be well represented by classical set theory. In classical set theory, an object is either in a set or not in a set. An object cannot partially belong to a set. In fuzzy set theory, we extend the image set of the characteristic function from the binary set $B=\{0,1\}$ which contains only two alternatives, to the unit interval $U=[0,1]$ which has an infinite number of alternatives. We even give the characteristic function a new name, the membership function, and a new symbol $\mu$, instead of $\chi$. The vagueness of language, and its mathematical representation and processing, is one of the major areas of study in fuzzy set theory.

1.1. Definition of fuzzy and crisp sets

Definition 1.1. Let $X$ be a set of points (objects) called universal or referential set. An ordinary (crisp) subset $A$ is characterized by a membership function $\chi_A$ as mapping from the element of $X$ to the element of the set $\{0,1\}$, defined by

$$\chi_A(x)=\begin{cases} 
1, & \text{if } x \in A \\
0, & \text{if } x \notin A 
\end{cases} \tag{1}$$
Where \( \{0, 1\} \) is called a valuation set. However, in the fuzzy set t, the membership function will have not only 0 and 1 but also any number in between i.e.

If the valuation set is allowed to be the real interval \([0, 1]\), A is called a fuzzy set.

**Definition 1.2.** Let X be any referential set. A fuzzy subset A of X is a collection of objects with graded membership. A fuzzy subset A of a set X is specified by its membership function \( \mu_A : X \to [0,1] \), assigning to each \( x \in X \) the degree or grade to which \( x \) belongs to A.

**Definition 1.3.** A fuzzy set A in X is a set of ordered pairs \( A = \{(x, \mu_A(x)) : x \in X \} \) where \( \mu_A : X \to [0,1] \).

Example:-let \( X = \{a, b, c\} \) and define the fuzzy set A as follows:

\[
\mu_A(a) = 1.0, \mu_A(b) = 0.7, \mu_A(c) = 0.4
\]

\( A = \{(a, 1.0), (b, 0.7), (c, 0.4)\} \)

Note:-The statement, \( \mu_A(b) = 0.7 \) is interpreted as saying that the membership grade of ‘b’ in the fuzzy set A is seven-tenths. i.e. the degree or grade to which b belongs to A is 0.7.

Finally, we might present the values in a table:

<table>
<thead>
<tr>
<th>x</th>
<th>( \mu_A(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1.0</td>
</tr>
<tr>
<td>b</td>
<td>0.7</td>
</tr>
<tr>
<td>c</td>
<td>0.4</td>
</tr>
</tbody>
</table>

Remark

i. The value zero is to represent a complete non-membership, the value one is used to represent complete membership, and values in between are used to represent intermediate degrees of membership.

ii. A fuzzy set is an extension or the generalization of a crisp set. Crisp sets allow only full membership or no membership at all, whereas fuzzy sets allow also partial degree of membership.
iii. A membership function is just a function. Its domain is some universal set $X$ and its range is the unit interval $U=[0,1]$ ,i.e. $\mu_A: X \rightarrow [0,1]$ ,where the domain $X$ is the standard (crisp) set.

iv. The only difference between a traditional set and a fuzzy set is the image of their membership functions.

V. The nearer the value of $\mu_A(x)$ to 1, the higher the grade of membership of $x$ in $A$.

**Fuzzy set operations**

We will define intersection of fuzzy sets to represent ‘and’, union of fuzzy sets to represent ‘or’ and complement of fuzzy sets ‘not’.

All the definitions for complements, unions, and intersections are given in terms of membership functions. Thus to define a fuzzy complement $A^c$ we define its membership function in terms of the membership function of the fuzzy set $A$.

1. **Subsets:** A fuzzy set $A$ is a subset of a fuzzy set $B$ if $\mu_A(x) \leq \mu_B(x)$ , $\forall x \in X$

Example:- if we have the fuzzy sets $A$ and $B$,

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\mu_A(x)$</th>
<th>$\mu_B(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>1.0</td>
<td>0.9</td>
</tr>
<tr>
<td>$b$</td>
<td>0.7</td>
<td>0.0</td>
</tr>
<tr>
<td>$c$</td>
<td>0.4</td>
<td>0.4</td>
</tr>
</tbody>
</table>

This is easier to see if we use a table to show both fuzzy sets;

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\mu_B(x)$</th>
<th>$\mu_A(x)$</th>
<th>$\mu_B(x) \leq \mu_A(x)$ then $B(x) \leq A(x)$ if and only if the final comparison column is all 1.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0.9</td>
<td>1.0</td>
<td>1</td>
</tr>
<tr>
<td>$b$</td>
<td>0.0</td>
<td>0.7</td>
<td>1</td>
</tr>
<tr>
<td>$c$</td>
<td>0.4</td>
<td>0.4</td>
<td>1</td>
</tr>
</tbody>
</table>
Fuzzy power set: Suppose that X is a crisp universal set, let the class of all fuzzy sets defined upon X be denoted by \( p(X) \) and called the fuzzy power set.

Note: it is impossible to give a simple example of a fuzzy power set since it is infinite in size. Since we cannot list all the real numbers between 0 and 1, we cannot list all the fuzzy subsets of a crisp set \( X=\{a, b, c\} \).

2. Intersection: The intersection of fuzzy sets A and B is the fuzzy set \( A \cap B \) with a membership grade for every \( x \in X \) given by \( \mu_{A\cap B}(x) = \min \{ \mu_A(x), \mu_B(x) \} \). The minimum operator is represented by the symbol “\( \land \)” so that \( \min\{a, b\} \) can also be written \( a \land b \) and the intersection membership function is often written as \( \mu_{A\cap B}(x) = \mu_A(x) \land \mu_B(x) \).

3. Union: The union of fuzzy sets A and B is the fuzzy set \( AUB \) with a membership grade for every \( x \in X \) given by \( \mu_{AUB}(x) = \max \{ \mu_A(x), \mu_B(x) \} \) for all \( x \) in \( X \). The maximum operator is represented by the symbol “\( \lor \)” so that \( \max\{a, b\} \) can be written \( a \lor b \) and the membership function for union can be written as: \( \mu_{AUB}(x) = \mu_A(x) \lor \mu_B(x) \).

4. Complement: The standard complement operator introduced by Zadeh is \( A^c(x) = 1 - A(x) \). This formula represents one of the most important differences between fuzzy set theory and standard set theory. In set theory it is always true that a set and its complement have nothing in common. In fuzzy set theory a set and its complement can be identical.

i. Set difference: The difference of two fuzzy sets A and B is the fuzzy set \( A-B \) with membership function: \( (\mu_A-\mu_B)(x) = \max\{\mu_A(x) - \mu_B(x), 0\} \)

ii. Algebraic sum: the sum of fuzzy sets A and B is the fuzzy set \( A \oplus B \) with a membership grade for every \( x \in X \) given by \( (\mu_A \oplus \mu_B)(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x) \) for all \( x \) in \( X \).

iii. Algebraic product: the product of fuzzy sets A and B is the fuzzy set \( A \cdot B \) with a membership grade for every \( x \in X \) given by \( (\mu_A \cdot \mu_B)(x) = \mu_A(x) \cdot \mu_B(x) \) for all \( x \) in \( X \).
Example:-consider X={Addis, Dave, Beki, Nafyad}. Suppose A is the fuzzy subset of “good looking students” and B is the fuzzy subset of “intelligent students”. Then

<table>
<thead>
<tr>
<th></th>
<th>Nafyad</th>
<th>Addis</th>
<th>Dave</th>
<th>Beki</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_A$</td>
<td>0.2</td>
<td>0.3</td>
<td>0.6</td>
<td>0.8</td>
</tr>
<tr>
<td>$\mu_B$</td>
<td>0.7</td>
<td>0.4</td>
<td>0.1</td>
<td>0.5</td>
</tr>
<tr>
<td>$\mu_A^c$</td>
<td>0.8</td>
<td>0.7</td>
<td>0.4</td>
<td>0.2</td>
</tr>
<tr>
<td>$\mu_{A \cap B}$</td>
<td>0.2</td>
<td>0.3</td>
<td>0.1</td>
<td>0.5</td>
</tr>
<tr>
<td>$\mu_{A \cup B}$</td>
<td>0.7</td>
<td>0.4</td>
<td>0.6</td>
<td>0.8</td>
</tr>
<tr>
<td>$\mu_{A \cap B}$</td>
<td>0.14</td>
<td>0.12</td>
<td>0.06</td>
<td>0.4</td>
</tr>
<tr>
<td>$\mu_{\bar{A} \cap \bar{B}}$</td>
<td>0.76</td>
<td>0.58</td>
<td>0.64</td>
<td>0.9</td>
</tr>
<tr>
<td>$\mu_B^c$</td>
<td>0.3</td>
<td>0.6</td>
<td>0.9</td>
<td>0.5</td>
</tr>
<tr>
<td>$\mu_{A \cup B}$</td>
<td>0.2</td>
<td>0.3</td>
<td>0.6</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Membership grade operations:--

Support and $\alpha$-cut

Sometimes, we might only need objects of a fuzzy set instead of its membership function, that is, to transfer a fuzzy set in to a crisp set. In order to do so, we need two concepts, support and $\alpha$-cut.

It is often necessary to consider those elements in a fuzzy set which have non-zero membership grades. These elements are called the support of that fuzzy set.

**Definition:** Given a fuzzy set $A$, its support $S(A)$ is an ordinary crisp subset on $X$ defined as $S(A) = \{x: \mu_A(x) > 0 \text{ and } x \in X\}$.

**Definition:** Given a fuzzy set $A$, its $\alpha$-cut $A^\alpha$ is defined as $A^\alpha = \{x: \mu_A(x) \geq \alpha \text{ and } x \in X\}$ where $\alpha$ is called the confidence level. That is, for every $\alpha \in [0,1]$, a given fuzzy set $A$ yields a crisp set $A^\alpha$ which contains those elements of the universe $X$ who have membership grade in $A$ of at least $\alpha$: $A^\alpha = \{x \in X: \mu_A(x) \geq \alpha\}$.
A membership function $\mu_A$ is termed as:

(i). **normal** if there exists an $x \in X$ such that $\mu_A(x)=1$, i.e. $A \neq \emptyset$ and $A$ is subnormal if it is not normal. A fuzzy set $A$ is empty if and only if $\mu_A(x) = 0$, $\forall x \in X$.

(ii). **convex**: if and only if for every pair of points $x, y \in X$ the membership function of $A$ satisfies:

$$\mu_A(\lambda x + (1-\lambda)y) \geq \min\{ \mu_A(x), \mu_A(y) \}$$

where $\lambda \in [0,1]$. Finally, the support, or strong $\alpha$-cut at zero must be bounded, that is, $A^{\alpha} = [a, b]$ with $a \leq b$ and neither ‘a’ nor ‘b’ is permitted to be infinite. i.e. a fuzzy set $A$ is bounded if and only if the sets $A^{\alpha} = \{x / \mu_A(x) \geq \alpha \}$ are bounded for all $\alpha > 0$, i.e. for every $\alpha > 0$, there exists a finite $R(\alpha)$ such that $\|x\| \leq R(\alpha)$ for all $x$ in $A^{\alpha}$.

**1.2. Fuzzy Number**

**1.2.1. Definition of fuzzy numbers**

The term fuzzy number is used to handle imprecise numerical quantities, such as “close to 10,” “about 60,” “several,””much greater than”, etc. A general definition of a fuzzy number is given by Dubois and Prade[1]: any fuzzy subset $M = \{(x, \mu_M(x))\}$ where $x$ takes its number from the real line $R$ and, $\mu_M(x) \in [0,1]$.

**Definition 1.2.1.** A fuzzy number $A$ is a normal and convex subset of real numbers whose membership function is piecewise continuous.

Note: A fuzzy number is a special type of fuzzy set.

**Definition 1.2.2.** A fuzzy quantity is a fuzzy subset of the set of real numbers. The family of all fuzzy quantities are usually denoted by $F(R)$.

**Remarks:**

1. $S(A) = \{x \in R : \mu_A(x) > 0 \}$ is the support of $A$.
2. $A^\alpha = \{x: \mu_A(x) \geq \alpha \text{ and } x \in R\}$ is the $\alpha$-cut of $A$
3. $\mu_A(x) = 1$, $x \in R$ if and only if a fuzzy set $A$ is normal
4. If $A$ is not a fuzzy number then there exists $\alpha \in [0,1]$ such that $A^\alpha$ is not a convex subset of $R$. 
1.2.3. Types of Fuzzy Numbers

1. Triangular fuzzy number (TRFN): A fuzzy number A=(a_1,a,a_2) for a_1 \leq a \leq a_2 is called triangular fuzzy number if its membership function \( \mu_A \) is given by

\[
\mu_A(x) = \begin{cases} 
0, & \text{if } x < a_1, x > a_2 \\
\frac{x - a_1}{a - a_1}, & \text{if } a_1 \leq x \leq a \\
\frac{a_2 - x}{a_2 - a}, & \text{if } a < x \leq a_2
\end{cases}
\]

Notation: The TRFN A is denoted by the triple A=(a_1,a,a_2) and has a shape of triangle. Moreover, the \( \alpha \)-cut of a TRFN A=(a_1,a,a_2) is the closed interval given by \( A^\alpha = [(a-a_1)\alpha + a_1, (a-a_2)\alpha + a_2] \).

Definition :- For any two triangular fuzzy numbers A=(a, b, c) and B=(p, q ,r), A \leq B if and only if

i. a \leq p   
ii. a - b \leq p - q   
iii. a +c \leq p + r

2. Trapezoid fuzzy number (TRDFN): A fuzzy number A is called trapezoid fuzzy number if its membership function is given by

\[
\mu_A(x) = \begin{cases} 
0, & \text{if } x < a_1, x > a_2 \\
\frac{x - a_1}{\alpha - a_1}, & \text{if } a_1 \leq x \leq a' \\
1, & \text{if } a' \leq x \leq a'' \\
\frac{a_2 - x}{a_2 - a}, & \text{if } a'' < x \leq a_2
\end{cases}
\]

Notation: The TRDFN is denoted by the quadruplet A=(a_1,a',a'',a_2) and has the shape of trapezoid. The \( \alpha \)-cut of a TRDFN

\[
A=(a_1,a',a'',a_2) \text{ is a closed interval given by } A^\alpha = [a_1(\alpha), a_2(\alpha)][(a'-a_1)\alpha + a_1, (a''-a_2)\alpha + a_2].
\]
3. L-R fuzzy number: A fuzzy number $A$ is called L-R fuzzy number if its membership function is given by

$$
\mu_A(x) = \begin{cases} 
L\left(\frac{x-a}{\alpha}\right); & \text{if } (a-\alpha) \leq x < a, \alpha > 0 \\
1, & \text{if } a \leq x \leq b \\
R\left(\frac{x-b}{\beta}\right); & \text{if } b < x \leq (b+\beta), \beta > 0 \\
0; & \text{otherwise}
\end{cases}
$$

Where $L(.)$ and $R(.)$ are piecewise continuous functions such that $L(.)$ is increasing and $R(.)$ decreasing. $L$ is called the left reference function and $R$ is called the right referential. $\alpha$ and $\beta$ are the left and the right spreads respectively.

1.2.4. Binary operation.

**Definition 1.3.3.1.** Let $\circ$ be a binary operation on $\mathbb{R}$, and $A$ and $B$ be two fuzzy numbers, then $\circ$ induces a binary operation on $F(\mathbb{R})$ i.e. $\circ : F(\mathbb{R}) \times F(\mathbb{R}) \rightarrow F(\mathbb{R})$ is given by

$$
\mu_C(x) = \max\{\min(\mu_A(a),\mu_B(b))\}.
$$

**Arithmetic operation on fuzzy numbers using $\alpha$-cut method.**

In this section, we consider arithmetic operation, addition, subtraction, multiplication, and division on fuzzy numbers using $\alpha$-cut method. Since a fuzzy number is a convex fuzzy set defined on some subsets of the real numbers, the $\alpha$-cuts of $A$ and $B$ are closed intervals. Let $A$ and $B$ be fuzzy numbers $\alpha \in [0,1]$ , and $A^\alpha$, $B^\alpha$ be $\alpha$-cuts of $A$ and $B$ respectively.

Let $A^\alpha = [a_1, a_2]$ and $B^\alpha = [b_1, b_2]$ where $a_1 = \min\{x/\mu_A(x) \geq \alpha\}$ and $a_2 = \max\{x/\mu_A(x) \geq \alpha\}$ for $A^\alpha = [a_1, a_2]$ =\{x/\mu_A(x) \geq \alpha\}$
and \(b_1 = \min\{x/\mu_B(x) \geq \alpha\}\) and \(b_2 = \max\{x/\mu_B(x) \geq \alpha\}\) for \(B^\alpha = \{x/\mu_B(x) \geq \alpha\}\).

Remark: The notation \([a_1,a_2]\) comes from the branch of mathematics called interval analysis. Therefore, the set of pairs \((\alpha, A^\alpha)\) for every \(\alpha \in [0,1]\), completely characterize a fuzzy number and consist of a level of presumption \(\alpha\) and of an interval of confidence \(A^\alpha = [a_1,a_2]\). Since \(A\) is a fuzzy number, at each presumption level \(\alpha\), the interval of confidence is simply an interval on the real number line (if the domain is the integers or natural numbers instead of intervals one gets consecutive sequences but this makes little difference to the algebra involved). In interval analysis for two intervals \([a_1,a_2]\) and \([b_1,b_2]\) we have:

**Definition (Addition(+) and subtraction(-)):** if \(x \in [a_1,a_2], y \in [b_1,b_2]\) then \(x+y \in [a_1+b_1, a_2+b_2]\) and \(x-y \in [a_1-b_2, a_2-b_1]\). Therefore, if \(A^\alpha = [a_1,a_2], B^\alpha = [b_1,b_2]\) then:

\[A(+)B = [a_1,a_2](+) [b_1,b_2] = [a_1+b_1, a_2+b_2]\]

Similarly, the subtraction of \(A\) and \(B\), denoted by \(A(-)B\) is defined as

\[A(-)B = [a_1,a_2](-) [b_1,b_2] = [a_1-b_2, a_2-b_1]\]

**Definition 1.3.3.2.** (Multiplication (-)) :-The multiplication of \(A\) and \(B\), denoted by \(A(.)B\) is defined as

\[A(.)B = [a_1,a_2](.) [b_1,b_2] = [\min(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2), \max(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2)].\]

In case these intervals are in \(R^+\), the non-negative real line, the multiplication formula gets simplified to

\[A(.)B = [a_1 b_1, a_2 b_2].\]

**Definition 1.2.4.1.** (scalar multiplication and inverse):-Let \(A = [a_1,a_2]\) be a closed interval in \(R^+\) and \(k \in R^+\) identifying the scalar \(k\) as the closed interval \([k_1,k_2]\), the scalar multiplication \(k.A\) is defined as

\[k.A = [k_1, k_2](.) [a_1, a_2] = [k_1 a_1, k_2 a_2].\]
For $A = [a_1, a_2]$ in $\mathbb{R}^+$ if $x \in [a_1, a_2]$ and $0 \notin [a_1, a_2]$ then $\frac{1}{x} \in \left[ \frac{1}{a_2}, \frac{1}{a_1} \right]$. Therefore, the inverse of $A$, denoted by $A^{-1}$, is defined to be $A^{-1} = [a_1, a_2]^{-1} = \left[ \frac{1}{a_2}, \frac{1}{a_1} \right]$, provided that $0 \notin [a_1, a_2]$.

**Definition 1.2.4.2. (Division (:) )** :- The division of two closed intervals $A = [a_1, a_2]$ and $B = [b_1, b_2]$ of $\mathbb{R}$, denoted by $A(:)B$, is defined as the multiplication of $[a_1, a_2]$ and $\left[ \frac{1}{b_2}, \frac{1}{b_1} \right]$

$$A(:)B = [a_1, a_2](:)[b_1, b_2] = [a_1, a_2] \cdot \left[ \frac{1}{b_2}, \frac{1}{b_1} \right] = \left[ \min \left( \frac{a_1}{b_2}, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_2}{b_1} \right), \max \left( \frac{a_1}{b_2}, \frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_2}{b_1} \right) \right]$$

**Definition 1.2.4.3. (max(\(\vee\)) and min(\(\wedge\)) operations)**. Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ be two closed intervals in $\mathbb{R}$. Then the max(\(\vee\)) and min(\(\wedge\)) operations on $A$ and $B$ are defined as :

$$A(\vee)B = [a_1, a_2](\vee)[b_1, b_2] = [a_1 \vee b_1, a_2 \vee b_2]$$

$$A(\wedge)B = [a_1, a_2](\wedge)[b_1, b_2] = [a_1 \wedge b_1, a_2 \wedge b_2]$$

Let $A=[a,b,c]$ be a triangular fuzzy number. Then from its membership function we can find $A^\alpha$ as follows:

We first set $\alpha \in [0,1]$ to both left and right reference functions of $A$. That is, $\alpha = \frac{x-a}{b-a}$ and $\alpha = \frac{c-x}{c-b}$. Expressing $x$ in terms of $\alpha$, we have $x = (b-a)\alpha + a$ and $x = c - (c-b)\alpha$

which gives the $\alpha$-cut of $A$ is $A^\alpha = [x,x] = [(b-a)\alpha + a, c - (c-b)\alpha]$.

1. Addition of triangular fuzzy numbers: let $X=(a,b,c)$ and $Y=(p,q,r)$ be two triangular fuzzy numbers. Then $X^\alpha = [(b-a)\alpha + a, c - (c-b)\alpha]$ and $Y^\alpha = [(q-p)\alpha + p, r - (r-q)\alpha]$ are the $\alpha$-cuts of fuzzy numbers $X$ and $Y$ respectively.
Therefore, 

\[ X^\alpha + Y^\alpha = [a + p + (b-a+q-p)\alpha, c+r-(c-b+r-q)\alpha]. \]

To find the membership function \( \mu_{x+y}(x) \), we equate to \( x \) both the first and second components which gives

\[ x = a + p + (b-a+q-p)\alpha \text{ and } x = c+r-(c-b+r-q)\alpha. \]

Now, expressing \( \alpha \) in terms of \( x \), we get \( \alpha \) together with the domain of \( x \),

\[
\alpha = \frac{x-(a+p)}{(b+q)-(a+p)}, \quad \text{if } (a+p) \leq x \leq (b+q) \text{ and }
\]

\[
\alpha = \frac{(c+r)-x}{(c+r)-(b+q)}, \quad \text{if } (b+q) < x \leq (c+r) \text{ which gives }
\]

\[
\mu_{x+y}(x) = \begin{cases} 
\alpha = \frac{x-(a+p)}{(b+q)-(a+p)}, & \text{if } (a+p) \leq x \leq (b+q) \\
\alpha = \frac{(c+r)-x}{(c+r)-(b+q)}, & \text{if } (b+q) < x \leq (c+r)
\end{cases}
\]

and use the same procedure for the rest three operations.

**1.3. fuzzy linear programming**

Crisp linear programming is one of the most important operational research techniques. It is a problem of maximizing or minimizing a crisp objective function subject to crisp constraints (crisp linear-inequalities and/or equations). It has been applied to solve many real world problems but it fails to deal with imprecise data, that is, in many practical situations it may not be possible for the decision maker to specify the objective and/or the constraint in crisp manner rather he/she may have put them in “fuzzy sense”. So many researchers succeeded in capturing such vague and imprecise information by fuzzy programming problem(FLPP.). In this case, the type of the problem he/she put in the fuzziness should be specified, that means, there is no general or unique definition of fuzzy
linear problems. The fuzziness may appear in a linear programming problem in several ways such as the inequality may be fuzzy (P₁-FLP), the objective function may be fuzzy (P₂-FLP), or the parameters c, A, b may be fuzzy (P₃-FLP) and so on.

Definition 1: If an imprecise aspiration level is assigned to the objective function, then this fuzzy objective is termed as fuzzy goal. It is characterized by its associated membership function by defining the tolerance limits for achievement of its aspired level.

We consider the general model of a linear programming

$$\max c^T x$$

subject to

$$A_i x \leq b_i (i=1,2,...,m)$$

$$x \geq 0,$$

Where $A_i$ is an $n$-vector, $C$ is an $n$-column vector and $x \in \mathbb{R}^n$.

To a standard linear programming problem (1.3.1) above, taking into account the imprecision or fuzziness of a decision maker’s judgment, Zimmermann considers the following linear programming problem with a fuzzy goal (objective function) and fuzzy constraints.

$$C^T x \leq Z_o$$

$$A_i x \leq b_i, (i=1,2,...,m)$$

$$x \geq 0,$$

Where the symbol $\leq$ denotes a relaxed or fuzzy version of the ordinary inequality $\leq$. From the decision maker’s preference, the fuzzy goal (1.3.1a) and the fuzzy constraints (3.1b) mean that the objective function $C^T X$ should be
“essentially smaller than or equal to” a certain level Zo, and that the values of the constraints AX should be “essentially smaller than or equal to” b, respectively. Assuming that the fuzzy goal and the fuzzy constraints are equally important, he employed the following unified formulation.

\[ Bx \leq b' \]
\[ x \geq 0, \]

where \( B = \begin{bmatrix} C \\ A \end{bmatrix} \) and \( b' = \begin{bmatrix} Z_0 \\ b_0 \end{bmatrix} \)

2. Fuzzy decision:- is the fuzzy set of alternatives resulting from the intersection of the fuzzy constraints and fuzzy objective functions. Fuzzy objective functions and fuzzy constraints are characterized by their membership functions.

1.3.1. Solution techniques of solving some fuzzy linear programming problems.

The solution techniques for fuzzy linear programming problems follow the following procedure. We consider the following linear programming problem with fuzzy goal and fuzzy constraints (the coefficients of the constraints are fuzzy numbers).

\[
\begin{align*}
\max & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \sum_{j=1}^{m} \tilde{a}_{ij} x_i \leq \tilde{b}_i, \ 1 \leq i \leq m \\
& x_i \geq 0, \text{ where } \tilde{a}_i \text{ and } \tilde{b}_i \text{ are fuzzy numbers with the following linear membership functions:}
\end{align*}
\]

\[
\mu_{x_i} = \begin{cases} 
1 & \text{if } x \leq a_i \\
\frac{a_i + d_i - x}{d_i} & \text{if } a_i < x < a_i + d_i \\
0 & \text{if } x \geq a_i + d_i
\end{cases}
\]
and $x \in \mathbb{R}$, $d_{ij}>0$ is the maximum tolerance for the corresponding constraint coefficients and $p_i$ is the maximum tolerance for the $i^{th}$ constraint. For defuzzification of the problem, we first fuzzify the objective function. This is done by calculating the lower and upper bounds of the optimal values. These optimal values $z_l$ and $z_u$ can be defined by solving the following standard linear programming problems, for which we assume that both of them have finite optimal values.

Let $z_l=\min(z_1,z_2)$ and $z_u=\max(z_1,z_2)$ . The objective function takes values between $z_l$ and $z_u$ while the constraint coefficients take values between $a_{ij}$ and $a_{ij}+d_{ij}$ and the right-hand side numbers take values between $b_i$ and $b_i+p_i$. Then, the fuzzy set optimal values, $G$, which is a subset of $\mathbb{R}^n$ is defined by:

$$
\mu_{\mathcal{J}}(x) = \begin{cases} 
0 & \text{if } \sum_{j=1}^{n} c_j x_j \leq z_l \\
\frac{\sum_{j=1}^{n} c_j x_j - z_l}{z_u - z_l} & \text{if } z_l < \sum_{j=1}^{n} c_j x_j < z_u \\
1 & \text{if } \sum_{j=1}^{n} c_j x_j \geq z_u
\end{cases}
$$
The fuzzy set of the $i^{th}$ constraint, $C_i$, which is a subset of $\mathbb{R}^n$ is defined by:

$$
\mu_i(x) = \begin{cases} 
0 & \text{if } b \leq \sum_{j=1}^{n} a_j x_j, \\
\frac{b - \sum_{j=1}^{n} a_j x_j}{\sum_{j=1}^{n} d_j x_j + p} & \text{if } \sum_{j=1}^{n} a_j x_j < b < \sum_{j=1}^{n} (a_j + d_j) x_j + p, \\
1 & \text{if } b \geq \sum_{j=1}^{n} (a_j + d_j) x_j + p.
\end{cases}
$$

using the above membership functions $\mu_i(x)$ and $\mu_G(x)$ and following Bellmann and Zadeh approach, we construct the membership function $\mu_D(x)$ as follows.

$$
\mu_D(x) = \min_i(\mu_G(x), \mu_i(x))
$$

where $\mu_D(.)$ is the membership function of the fuzzy decision set. The min. section is selected as the aggregation operator. Then the optimal decision $x^*$ is the solution of

$$
x^* = \arg\max_x \min_i\{\mu_G(x), \mu_i(x)\}
$$

Then, problem (1) is reduced to the following crisp problem by introducing the auxiliary variable $\lambda$ which indicates the common degree of satisfaction of both the fuzzy constraints and objective function.

$$
\begin{align*}
\max \lambda \\
\text{s.t. } & \mu_G(x) \geq \lambda \\
& \mu_i(x) \geq \lambda \\
& x \geq 0, 0 \leq \lambda \leq 1, 1 \leq i \leq m
\end{align*}
$$

This problem is equivalent to the following non-convex optimization problem
\[
\max \lambda \\
\lambda(z_i - z_j) - \sum_{j=1}^{n} c_i x_j + z_i \leq 0 \\
\sum_{j=1}^{n} (a_i + \lambda d_i)x_j + \lambda p_i - b_i \leq 0 \\
x \geq 0, 0 \leq \lambda \leq 1, 1 \leq i \leq m
\]

which contains the cross product terms \( \lambda x_j \) that makes non-convex. Therefore, the solution of this problem requires the special approach such as fuzzy decisive method adopted for solving general non-convex optimization problems.

Here solving the above linear programming problem gives us an optimum \( \lambda^* \in [0,1] \). Then the solution of the problem is any \( x \geq 0 \) satisfying the problem constraint with \( \lambda = \lambda^* \).

Example:- Solve the optimization problem:

\[
\max x_1 + x_2 \\
\text{s.t. } 1 \times x_1 + 2 \times x_2 \leq 3 \quad \ldots \ldots \ldots \quad \ast \\
2 \times x_1 + 3 \times x_2 \leq 4 \\
x_1, x_2 \geq 0
\]

Solution: the problem takes fuzzy parameters as:

\( 1 = L(1,1), 2 = L(2,1), 2 = L(2,2), 3 = L(3,2), b_1 = 3 = L(3,2) \) and \( b_2 = 4 = L(4,3) \) as used by shaocheng [11]. That is

\[
(a_i) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad (d_i) = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \implies (a_i + d_i) = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}
\]

To solve this problem, first, we must solve the following two sub problems,
\[ z_1 = \max x_1 + x_2 \quad \text{and} \quad z_2 = \max x_1 + x_2 \]
\[ \text{s.t. } 2x_1 + 3x_2 \leq 3 \quad \text{s.t. } x_1 + 2x_2 \leq 5 \]
\[ 4x_1 + 5x_2 \leq 4 \quad 2x_1 + 3x_2 \leq 7 \]
\[ x_1, x_2 \geq 0 \quad x_1, x_2 \geq 0 \]

The optimal solutions of these sub problems are
\[ x_1 = 1 \quad x_1 = 3.5 \]
\[ x_2 = 0 \quad \text{and} \quad x_2 = 0 \]
respectively.
\[ z_1 = 1 \quad z_2 = 3.5 \]

By using these optimal values, the problem * can be reduced to the following equivalent non-linear programming problem

\[
\max \lambda \\
\text{s.t.} \quad \frac{x_1 + x_2 - 1}{3.5 - 1} \geq \lambda \\
\frac{3 - x_1 - 2x_2}{x_1 + x_2} \geq \lambda \\
\frac{4 - 2x_1 - 3x_2}{2x_1 + 2x_2} \geq \lambda \\
x_1, x_2 \geq 0, \quad 0 \leq \lambda \leq 1,
\]

That is
\[
(1 + \lambda)x_1 + (2 + \lambda)x_2 \leq 3 - 2\lambda \\
(2 + 2\lambda)x_1 + (3 + 2\lambda)x_2 \leq 4 - 3\lambda \\
x_1, x_2 \geq 0, \quad 0 \leq \lambda \leq 1
\]

Now applying the fuzzy decisive set method for the problem ** above, we obtain the optimal value of \( \lambda \) at the twenty fifth iteration of the decisive set method, i.e. \( \lambda^* = 0.183215916 \). Hence, \( x_1^* = 1.45804, x_2^* = 7.8 \times 10^{-10} \approx 0 \) and \( \lambda^* = 0.183215916 \) are optimal solutions to **. This means that, the vector \((x_1^*, x_2^*)\) is a solution to the problem * which has the best membership grade \( \lambda \)
CHAPTER -2

BI-LEVEL PROGRAMMING

2. Basic definitions:

2.1. Decision making: is a process of choosing an action (solution) from a set of possible actions to optimize a given objective.

2.1.2. Decision making under multi objectives: In most real situation a decision maker needs to choose an action to optimize more than one objective simultaneously. Most of these objectives are usually conflicting. For example, a manufacturer wants to increase his profit and at the same time want to produce a product with better quality. Mathematically a multi objective optimization with k objectives, for a natural number K>1, can be given as:

$$\max F(x)=(f_1(x), f_2(x),... f_k(x))$$

S.t. $x \in s \subseteq IR^n$

2.1.3. Hierarchical decision making: An optimization problem which has other optimization problems in the constraint set and has a decision maker for each objective function controlling part of the variables is called multi-level optimization problem. If there are only two nested objective functions then it is called a bi-level optimization problem. The decision maker at the first level, with objective function $f_1$, is called the leader and the other decision makers are called the followers. A solution is supposed to fulfill all the feasibility conditions and optimize each objectives it is uncommon to find a solution which makes all the decision makers happy. Hence to choose an action the preference of the decision makers for all the levels or objectives play a big role.
2.2. Bi-level programming (BLP): is a mathematical programming problem that solves decentralized planning problems with two decision makers (DMs) in a two level or hierarchical organization. It has been studied extensively since the 1980s.

It often represents an adequate tool for modeling non-cooperative hierarchical decision process, where one player optimizes over a subset of decision variables, while taking in to account the independent reaction of the other player to his or course of action. In the real world, we often encounter situations where there are two or more decision makers in an organization with a hierarchical structure, and they make decisions in turn or at the same time so as to optimize their objective functions. In particular, consider a case where there are two decision makers; one of the decision makers first makes a decision, and then the other who knows the decision of the opponent makes a decision. Such a situation is formulated as a bi-level programming problem. We call the decision maker who first makes a decision the leader, and the other decision maker the follower. For bi-level programming problems, the leader first specifies (decides) a decision and then the follower determines a decision so as to optimize the objective function of the follower with full knowledge of the decision of the leader. According to this rule, the leader also makes a decision so as to optimize the objective function of self. This decision making process is extremely practical to such decentralized systems as agriculture, government policy, economic systems, finance, warfare, transportation, network designs, and is especially for conflict resolution.

Bi-level programming is particularly appropriate for problems with the following characteristics:
• Interaction: Interactive decision-making units within a predominantly hierarchical structure.

• Hierarchy: Execution of decision is sequential, from upper to lower level.

• Full information: Each DM is fully informed about all prior choices when it is his turn to move.

• Nonzero sum: The loss for the cost of one level is unequal to the gain for the cost of the other level. External effect on a DM’s problem can be reflected in both the objective function and the set of feasible decision space.

• Each DM controls only a subset of the decision variables in an organization

2.2.1. Mathematical formulation of a bi-level linear programming problem (BLPP):

For the bi-level programming problems, the leader first specifies a decision and then the follower determines a decision so as to optimize the objective function of self with full knowledge of the decision of the leader. According to this rule, the leader also makes a decision so as to optimize the objective function of self. The solution defined as the above mentioned procedure is a stackelberg solution.

A bi-level LPP for obtaining the stackelberg solution is formulated as:

\[
\begin{align*}
\max_{x_1} z_1(x_1, x_2) &= c_1 x_1 + d_1 x_2 \quad \text{leader} \\
\text{subject to} \quad \begin{array}{c}
Ax_1 + Bx_2 \leq b \\
A_2 x_1 + B_2 x_2 \leq b_2 \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\max_{x_2} z_2(x_1, x_2) &= c_2 x_1 + d_2 x_2 \quad \text{follower} \\
\end{align*}
\]

where \(c_i, i=1,2\) are \(n_1\)-dimensional row coefficient vector, \(d_i, i=1,2\) are \(n_2\)-dimensional row coefficient vector, \(A\) is an \(m \times n_1\) coefficient matrix, \(B\) is an \(m \times n_2\) coefficient matrix.
coefficient matrix, b-is an m-dimensional column constant vector. In the bi-level linear programming problem above, 

\[ z_1(x_1,x_2) \] and \[ z_2(x_1,x_2) \] represent the objective functions of the leader and the follower, respectively, and \( x_1 \) and \( x_2 \) represent the decision variables of the leader and the follower respectively. Each decision maker knows the objective function of self and the constraints. The leader first makes a decision, and then the follower makes a decision so as to maximize the objective function with full knowledge of the decision of the leader. Namely, after the leader chooses \( x_1 \), he solves the following linear programming problem:

\[
\max_{x_2} z_2(x_1,x_2) = c_2x_1 + d_2x \\
\text{s.t. } Bx_2 \leq b - Ax_1 \quad (**) \\
\quad x_2 \geq 0,
\]

and chooses an optimal solution \( x_2(x_1) \) to the problem above as a rational response. Assuming that the follower chooses the rational response, the leader also makes a decision such that the objective function \( z_1(x_1,x_2(x_1)) \) is maximized.

2.3. BLP problem description

The linear bi-level programming problem is similar to standard linear programming, except that the constraint region is modified to include a linear objective function constrained to be optimal with respect to one set of variables. The linear BLPP characterized by two planners at different hierarchical levels each independently controlling only a set of decision variables, and with different conflicting objectives. The lower-level executes its policies after and in view of, the decision of the higher level, and the higher level optimizes its objective independently which is usually affected by the reactions of the lower level. Let
the control over all real-valued decision variables in the vector $x=(x_1, x_2, \ldots, x_N)$ be partitioned between two planners, hereafter known as level-one (the superior or top planner) and level-two (the inferior or bottom planner). Assume that the level-one has control over the vector $x_1=(x_1^1, x_1^2, \ldots, x_1^{N_1})$, the first $N_1$ components of the vector $x$, and that the level-two has control over the vector $x_2=(x_2^1, x_2^2, \ldots, x_2^{N_2})$ the remaining $N_2$ components. Further, assume that $f_1, f_2: \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \to \mathbb{R}$ linear. Then, the linear BLPP can be formulated as:

$$\max f_1(x_1, x_2) \text{ where } x_2 \text{ solves} \tag{1}$$

$$\max f_2(x_1, x_2) \tag{2}$$

s.t $(x_1, x_2) \in S$-----------------------------(2),

where $S \subseteq \mathbb{R}^{N_1+N_2}$ is the feasible choices of $(x_1, x_2)$, and is closed and bounded. For any fixed choice of $x_1$, level-two will choose a value of $x_2$ to maximize the objective function $f_2(x_1, x_2)$. Hence, for each feasible value of $x_1$, level-two will react with a corresponding value of $x_2$. This induces a functional reaction ship between the decisions of level-one and the reactions of level-two. Say, $x_2=W(x_1)$.

We will assume that the reaction function, $W(\cdot)$, is completely known by level-one.

Definition:-The set $Wf_2(S)$ given by $Wf_2(S)=\{(x_1^*, x_2^*) \in S: f_2(x_1^*, x_2^*)=\max f_2(x_1, x_2)\}$ is the set of rational reactions of $f_2$ over $S$. Hence level-one is really restricted to choosing a point in the set of rational reactions of $f_2$ over $S$. So, if level-one wishes to maximize its objective function, $f_1(x_1, x_2)$, by controlling only the vector $x_1$, it must solve the following mathematical programming problem:

$$\max f_1(x_1, x_2) \tag{3}$$

s.t $(x_1, x_2) \in Wf_2(S)$-----------------------------(3)
For convenience of notation and terminology, we will refer to \( S^1 = Wf_2(S) \) as the level-one feasible region or in general, the feasible region, and \( S^2 = S \) as the level-two feasible region.

The following are the basic concepts of the bi-level linear programming problem

of (*)

(i). The feasible region of the bi-level linear programming problem:

\[
S = \{(x_1, x_2) / Ax_1 + Bx_2 \leq b\}
\]

(ii). The decision space (feasible set) of the follower after \( x_1 \) is specified by the leader:

\[
S(x_1) = \{x_2 \geq 0 / Bx_2 < b - Ax_1, x_1 > 0\}
\]

(iii). The decision space of the leader:

\[
S_X = \{x_1 > 0 / \text{there is an } x_2 \text{ such that } Ax_1 + Bx_2 < b, x_2 > 0\}
\]

(iv). The set of rational responses of the follower for \( x_1 \) specified by the leader

\[
R(x_1) = \left\{ \begin{array}{l}
\quad x_1 = \text{arg max } z(x_1, x_2) \\
\quad x_2 \in S(x_1)
\end{array} \right\}
\]

(v). Inducible region:

\[
IR = \{(x_1, x_2) / (x_1, x_2) \in S, x_2 \in R(x_1)\}
\]

(vi). Stackleberg solution:

\[
\{(x_1, x_2) / (x_1, x_2) \in \text{arg max } z(x_1, x_2), (x_1, x_2) \in IR\}
\]

Computational methods for obtaining Stackelberg solution to bi-level linear programming problems are classified roughly in to three categories. These are;

1. The vertex enumeration approach (Bialas and Karwan, 1994): This takes advantage of the property that there exists a stackelberg solution in a set of extreme points of the feasible region. The solution search procedure of the
method starts from the first best point namely an optimal solution to the upper level problem which is the first best solution, is computed, and then it is verified whether the first best solution is also an optimal solution to the lower level problem. If the first best point is not the stackelberg solution, the procedure continues to examine the second best solution to the problem of the upper level, and so forth.

2. The Kuhn-Tucker approach: In this approach, the leader’s problem with constraints involving the optimality conditions of the follower’s problem is solved.

3. The penalty function approach: In this approach, a penalty term is appended to the objective function of the leader so as to satisfy the optimality of the follower’s problem.

4. Fuzzy approach:-that will be discussed in detail under the next chapter
CHAPTER 3

FUZZY APPROACH TO BI-LEVEL LINEAR PROGRAMMING PROBLEMS

3.1. fuzzy bi-level linear programming

As discussed under chapter two, a bi-level linear programming problem is formulated as:

\[
\text{max } f_1(x_1, x_2) = C_{11}x_1 + C_{12}x_2 \\
\text{subject to: } A_1x_1 + A_2x_2 \leq b \\
x_1, x_2 \geq 0
\]

Where \( x_i, i=1,2; \) is an \( n_i \)-dimensional decision variable column vector

\( C_{i1}, i=1,2 ; \) is an \( n_1 \)-dimensional constant column vector.

\( C_{i2}, i=1,2 ; \) is an \( n_2 \)-dimensional constant column vector

\( b \) is an \( m \)-dimensional constant column vector, and

\( A_i, i=1,2 ; \) is an \( m \times n_i \) coefficient matrix. For the sake of simplicity, we use the following notations:

\( X=(x_1,x_2) \in \mathbb{R}^{n_1+n_2}, C_i=(C_{i1},C_{i2}), i=1,2 \) and \( A=[A_1,A_2] \), and

Let DM_1 denotes the decision maker at the upper level and DM_2 denotes the decision maker at the lower level. In the bi-level linear programming problem (3.1.1) above, \( f_1(x_1, x_2) \) and \( f_2(x_1, x_2) \) represent the objective functions of DM_1 and DM_2 respectively; and \( x_1 \) and \( x_2 \) represent the decision variables of DM_1 and DM_2 respectively.
Instead of searching through vertices as the $k^{th}$ best algorithm, or the transformation approach based on Kuhn-Tucker conditions, we here introduce a supervised search procedure (supervised by DM$_1$) which will generate (non-dominated) satisfactory solution for a bi-level programming problem. In this solution search, DM$_1$ specifies (decides) a fuzzy goal and a minimal satisfactory level for his objective function and decision vector and evaluates a solution proposed by DM$_2$, and DM$_2$ solves an optimization problem, referring to the fuzzy goal and the minimal satisfactory level of DM$_1$. The DM$_2$ then presents his/her solution to the DM$_1$. If the DM$_1$ agrees to the proposed solution, a solution is reached and it is called a satisfactory solution here. If he/she rejects this proposal, then DM$_1$ will need to re-evaluate and change former goals and decisions as well as their corresponding leeway or tolerances until a satisfactory solution is reached. It is natural that decision makers have fuzzy goals for their objective functions and their decision variables when they take fuzziness of human judgments into consideration. For each of the objective functions $f_i(x)$, $i=1,2$, assume that the decision makers have fuzzy goals such as “the objective function $f_i(x)$ should be substantially less than or equal to some value $q_i$“ and the range of the decision on $x_i$, $i=1,2$, should be “around $x_i^*$ with its negative and positive –side tolerances $p_i^-$ and $p_i^+$, respectively.

We obtain optimal solution of each DM$_1$ and DM$_2$ calculated in isolation. If the individual optimal solution $x_i^0$, $i=1,2$; are the same then a satisfactory solution of the system has been attained. But this rarely happens due to conflicting objective functions of the two DMs. The decision-making process then begins at the first level. Thus, the first-level DM provides his preferred ranges for $f_1$ and decision
vector $x_1$ to the second level DM. This information can be modeled by fuzzy set theory using membership functions [3]

### 3.2. Fuzzy programming formulation of BLPPs

To formulate the fuzzy programming model of a BLPP, the objective functions $f_i$ ($i=1,2$) and the decision vectors $x_i$ ($i=1,2$) would be transformed into fuzzy goals by means of assigning an aspiration level (the optimal solutions of both of the DMs calculated in isolation can be taken as the aspiration levels of their associated fuzzy goals) to each of them. Then, they are to be characterized by the associated membership functions by defining tolerance limits for achievement of the aspired levels of the corresponding fuzzy goals.

### 3.3. Fuzzy programming approach for bi-level LPPs

In the decision making context, each DM is interested in maximizing his or her own objective function, the optimal solution of each DM when calculated in isolation would be considered as the best solution and the associated objective value can be considered as the aspiration level of the corresponding fuzzy goal because both the DMs are interested of maximizing their own objective functions over the same feasible region defined by the system of constraints. Let $x_i^B$ be the best (optimal) solution of the $i$th level DM. It is quite natural that objective values which are equal to or larger than $f_i^B = f_i(x_i^B) = \max f_i(x), i=1,2., x \in S$ should be absolutely satisfactory to the $i$th level DM. If the individual best (optimal) solution $x_i^B, i=1,2$ are the same, then a satisfactory optimal solution of the system is reached. However, this rarely happens due to the conflicting nature of the objectives. To obtain a satisfactory solution, higher level DM should give some
tolerance (relaxation) and the relaxation of decision of the higher level DM depends on the needs, desires and practical situations in the decision making situation. Then the fuzzy goals take the form $f_i(x) \leq f_i(x_i^B)$, $i=1,2$, $x_i \geq x_i^B$.

To build membership functions, goals and tolerance should be determined first. However, they could hardly be determined without meaningful supporting data. Using the individual best solutions, we find the values of all the objective functions at each best solution and construct a payoff matrix

$$
\begin{bmatrix}
  f_i(x) & f_i(x) \\
  f_i(x_1^0) & f_i(x_1^0) \\
  f_i(x_2^0) & f_i(x_2^0)
\end{bmatrix}
$$

The maximum value of each column ($f_i(x_i^0)$) gives upper tolerance limit or aspired level of achievement for the $i^{th}$ objective function where $f_i^u = f_i(x_i^0) = \max f_i(x_i^0)$, $i=1,2$.

The minimum value of each column gives lower tolerance limit or lowest acceptable level of achievement for the $i^{th}$ objective function where $f_i^L = \min f_i(x_i^0)$, $i=1,2$. For the maximization-type objective function, the upper tolerance limit $f_t^u$, $t=1,2$, are kept constant at their respective optimal values calculated in isolation but the lower tolerance limit $f_t^L$ are changed. The idea being that $f_i(x) \rightarrow f_t^u$, then the fuzzy objective goals take the form $f_i(x) \leq f_i(x_t^u)$, $i=1,2$. And the fuzzy goal for the control vector $x_i$ is obtained as $x_i \geq x_t^u$. Now, in the decision situation, it is assumed that all DMs that are up to $i^{th}$ motivation to cooperate each other to make a balance of decision powers, and they agree to give a possible relaxation of their individual optimal decision. The $i^{th}$ level DM must adjust his/her goal by
assuming the lowest acceptable level of achievement $f_i^L$ based on indefiniteness of the decentralized organization. Thus, all values of $f_i(x) \geq f_i^u$ are absolutely acceptable (desired) to objective function $f_i(x)$ satisfactory to the $i^{th}$ level DM. All values of $f_i(x)$ with $f_i(x) \leq f_i^L$ are absolutely unacceptable (undesired) to the objective function $f_i(x)$ for $i=1,2$. Based on this interval of tolerance, we can establish the following linear membership functions for the defined fuzzy goals as fig. 3.1.2. below

By identifying the membership functions $\mu_1(f_1(x))$ and $\mu_2(f_2(x))$ for the objective functions $f_1(x)$ and $f_2(x)$, and following the principle of the fuzzy decision by Bellman and Zadeh, the original bi-level linear programming problem (3.1.1) can be interpreted as the membership function maxmin problem defined by:

$$\max \min \{ \mu_i(f_i(x)) \}$$

$$\text{s.t. } A_1x_1 + A_2x_2 \leq b,$$

$$x_1, x_2 \geq 0$$
Then the linear membership functions for decision vector $x_1$ can be formulated as:

\[
\mu_i(x_i) = \begin{cases} 
  x_i - (x_i^o - e_i^-) / e_i^- ; & \text{if } x_i^o - e_i^- \leq x_i \leq x_i^o \\
  (x_i^o + e_i^+) - x_i / e_i^+ ; & \text{if } x_i^o \leq x_i \leq (x_i^o + e_i^+) \\
  0 ; & \text{otherwise}
\end{cases} \tag{3.1.4}
\]

Where $x_1^o$ is the optimal solution of first level DM

$e_i^-$ the negative tolerance value on $x_i$

$e_i^+$ the positive tolerance value on $x_i$.

To derive an overall satisfactory solution to the membership function maximization problem (3.1.3), we first find the maximizing decision of the fuzzy decision proposed by Bellman and Zadeh (1970). Namely, the following problem is solved for obtaining a solution which maximizes the smaller degree of satisfaction between those of the two decision makers:

\[
\begin{align*}
\max & \min \{ \mu_1(f_1(x)), \mu_2(f_2(x)), \mu_{x_1}(x_1) \} \\
\text{s.t.} & \quad A_1x_1 + A_2x_2 \leq b \tag{3.1.5}
\end{align*}
\]

By introducing an auxiliary variable $\lambda$, this problem can be transformed into the following equivalent problem:

\[
\begin{align*}
\max & \quad \lambda \\
\text{s.t.} & \quad \mu_1(f_1(x)) \geq \lambda \\
& \quad \mu_2(f_2(x)) \geq \lambda \\
& \quad \mu_{x_1}(x_1) \geq \lambda \tag{3.1.6} \\
& \quad A_1x_1 + A_2x_2 \leq b \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]
Solving problem (3.1.6), we can obtain a solution which maximizes the smaller satisfactory degree between those of both decision makers. It should be noted that if the membership functions $\mu_i(f_i(x))$, $i=1,2$ are linear membership functions such as (3.1.2), problem (3.1.6) becomes a linear programming problem. Let $x^*$ denotes an optimal solution to problem (3.1.6). Then we define the satisfactory degree of both decision makers under the constraints as

$$\lambda^* = \min\{\mu_1(f_1(x^*), \mu_2(f_2(x^*))\} \ldots$$

If DM$_1$ is satisfied with the optimal solution $x^*$, it follows that the optimal solution $x^*$ becomes a satisfactory solution; however, DM$_1$ is not always satisfied with the solution $x^*$. It is quite natural to assume that DM$_1$ specifies (decides) the minimal satisfactory level $\delta \in [0,1]$ for his membership function $\mu_1(f_1(x))$ subjectively. Consequently, DM$_2$ optimizes his objective under the new constraints as the following problem:

$$\max \mu_2(f_2(x))$$

$$\mu_1(f_1(x)) \geq \delta \ldots$$

$$A_1x_1 + A_2x_2 \leq b$$

$$x_1, x_2 \geq 0$$

If an optimal solution to problem (3.1.7) exists, it follows that DM$_1$ obtains a satisfactory solution having a satisfactory degree larger than or equal to the minimal satisfactory level specified (decided) by DM$_1$’s own self. However, the larger the minimal satisfactory level is assessed, the smaller DM$_2$’s satisfactory degree becomes. Consequently, a relative difference between the satisfactory degrees of DM$_1$ and DM$_2$ becomes larger than it is feared that overall satisfactory
balance between both levels cannot be maintained. To take account of overall satisfactory balance between both levels, DM_1 needs to compromise (agree) with DM_2 on DM_1’s own minimal satisfactory level. To do so, the following ratio of the satisfactory degree of DM_2 to that of DM_1 is defined as:

\[
\Delta = \frac{\mu(f_2(x^*))}{\mu(f_1(x^*))} \tag{3.1.8}
\]

which is originally introduced by Lai(1996).

Let \( \Delta_l \) and \( \Delta^u \) denote the lower bound and the upper bound of \( \Delta \) specified by DM_1. If \( \Delta > \Delta^u \), i.e. \( \mu(f_2(x^*)) > \Delta^u \mu(f_1(x^*)) \), then DM_1 updates (improves) the minimal satisfactory level \( \delta \) by increasing \( \delta \). Then DM_1 obtains a larger satisfactory degree and DM_2 accepts a smaller satisfactory degree. Conversely, if \( \Delta < \Delta_l \) i.e. \( \mu_2(f_2(x^*)) < \Delta_l \mu_1(f_1(x^*)) \), then DM_1 updates the minimal satisfactory level \( \delta \) by decreasing \( \delta \), and DM_1 accepts a smaller satisfactory degree and DM_2 obtains a larger satisfactory degree.

At an iteration \( l \), let \( \mu(f_1(x^l)) \), \( \mu_2(f_2(x^l)) \), \( \lambda^l \), and \( \Delta^l = \frac{\mu_2(f_2(x^l))}{\mu_1(f_1(x^l))} \) denote DM_1’s and DM_2’s satisfactory degrees, a satisfactory degree of both levels and the ratio of satisfactory degrees between both DMs, respectively, and let a corresponding solution be \( x^l \) at the iteration \( l \). The iterated interactive process terminates if the following two conditions are satisfied and DM_1 concludes the solution as a satisfactory solution.

**Termination conditions of the interactive processes for bi-level linear programming problems.**
i. DM\(_1^{rs}\) satisfactory degree is larger than or equal to the minimal satisfactory level \(\delta\) specified by DM\(_1\), i.e. \(\mu_1(f_1(x^\ell)) \geq \delta\).

ii. The ratio \(\Delta^\ell\) of satisfactory degrees lies in the closed interval between the lower and upper bounds specified by DM\(_1\), i.e. \(\Delta^\ell \in [\Delta_{\text{min}}, \Delta_{\text{max}}]\).

Condition (i) is DM\(_1^{rs}\)'s required condition for solutions, and Condition (ii) is provided in order to keep overall satisfactory balance between both levels. Unless the conditions are satisfied simultaneously, DM\(_1\) needs to update the minimal satisfactory level \(\delta\).

Procedure for updating the minimal satisfactory level \(\delta\).

1. If condition (i) is not satisfied, then DM\(_1\) decreases the minimal satisfactory level by \(\delta\).
2. If the ratio \(\Delta^\ell\) exceeds its upper bound, then DM\(_1\) increases the minimal satisfactory level \(\delta\). Conversely, if the ratio \(\Delta^\ell\) is below its lower bound, then DM\(_1\) decreases the minimal satisfactory level \(\delta\).

3.4. Algorithm of interactive fuzzy programming for BLPPs.

Step 1: Find the solution of the first level and second level independently with the same feasible set given.

Step 2: Do these solutions coincide?
- If yes, an optimal solution is reached.
- If No, go to step 3.

Step 3: Define a fuzzy goal, construct a pay off matrix, and then find upper tolerance limit \(f_t^u\) and lower tolerance limit \(f_t^l\).
Step 4: Build membership functions for maximization objective functions $\mu_i(f_i(x))$ and decision vector $x_1$ using equation (3.1.2) and (3.1.4) respectively.

Step 5: Set $\ell=1$ and solve the auxiliary problems (3.1.6). If DM$_1$ is satisfied with the optimal solution, the solution becomes a satisfactory solution $x^*$. Otherwise, ask DM$_1$ to specify (decide) the minimal satisfactory level $\delta$ together with the lower and the upper bounds $[\Delta \text{ min}, \Delta \text{ max}]$ of the ratio of satisfactory degrees $\Delta^\ell$ with the satisfactory degree $\lambda^*$ of both decision makers and the related information about the solution in mind.

Step 6: Solve problem (3.1.7), in which the satisfactory degree of DM$_1$ is maximized under the condition that the satisfactory degree of DM$_1$ is larger than or equal to the minimal satisfactory level $\delta$, and then an optimal solution $x^\ell$ to problem (3.1.7) is proposed to DM$_1$ together with $\lambda^\ell$, $\mu_1(f_1(x^\ell))$, $\mu_2(f_2(x^\ell))$ and $\Delta^\ell$.

Step 7: If the solution $x^\ell$ satisfies the termination conditions and DM$_1$ accepts it, then the procedure stops, and the solution $x^\ell$ is determined to be a satisfactory solution.

Step 8: Ask DM$_1$ to revise the minimal satisfactory level $\delta$ in accordance with the procedure for updating minimal satisfactory level. Return to step 7.

**Example: Solve (Linear BLPP)**

$$\max f_1(x) = 5x_1 + 6x_2 + 4x_3 + 2x_4$$

where $x_3, x_4$ solves

$$\max f_2(x) = 8x_1 + 9x_2 + 2x_3 + 4x_4$$

s.t. $3x_1 + 2x_2 + x_3 + 3x_4 \leq 40$
\[
\begin{align*}
&x_1 + 2x_2 + x_3 + 2x_4 \leq 30 \\
&2x_1 + 4x_2 + x_3 + 2x_4 \leq 35 \\
&x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

**Solution**

Step 1: Find the solution of the top-level and lower-level independently with the same feasible set. i.e.

\[
\begin{align*}
\text{max } f_1(x) &= 5x_1 + 6x_2 + 4x_3 + 2x_4 \\
\text{s.t. } &3x_1 + 2x_2 + x_3 + 3x_4 \leq 40 \\
&x_1 + 2x_2 + x_3 + 2x_4 \leq 30 \\ &2x_1 + 4x_2 + x_3 + 2x_4 \leq 35 \\
&x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

(1)

Then we find the optimal solution

\[
\begin{align*}
f_1 &= 125 \text{ at } x_1^o = (5, 0, 25, 0) \\
f_2 &= 118.125 \text{ at } x_2^o = (11.25, 3.125, 0, 0)
\end{align*}
\]

But this is not a satisfactory solution (since \(x_1^o \neq x_2^o\))

Step 2: Define fuzzy goals, construct the payoff matrix and we need to find the upper and lower tolerance limit.

- objective function as: \(f_1 \leq 125, \ f_2 \leq 118.125\)
- decision variables as: \(x_1 \geq 5, \ x_2 \geq 0\)
- payoff matrix =

\[
\begin{bmatrix}
\text{f}_1(x_1^o) & f_2(x_1^o) \\
x_1^o & 125 & 90 \\
x_2^o & 75 & 118.125
\end{bmatrix}
\]

- upper tolerance limits are: \(f_1^u = 125, \ f_2^u = 118.125\)
- lower tolerance limits are: \(f_1^l = 75 \text{ and } f_2^l = 90\)

Step 3: Build membership functions for:
Objective functions as

\[
\mu f_1(f_1(x)) = \begin{cases} 
1 & : f_1(x) \geq 125 \\
\frac{f_1(x) - 75}{125 - 75} & : 75 \leq f_1(x) \leq 125 \\
0 & : f_1(x) \leq 75 
\end{cases}
\]

\[
\mu f_2(f_2(x)) = \begin{cases} 
1 & : f_2(x) \geq 118.125, \\
\frac{f_2(x) - 90}{118.125 - 90} & : 90 \leq f_2(x) \leq 119.125 \\
0 & : f_2(x) \leq 90 
\end{cases}
\]

Decision variable function as

Let the upper level DM specifies (decides) \( x_1 = 5 \) with 2.5 (negative) and 2.5 (positive) tolerance and \( x_2 = 0 \) with 0 (negative) and 3 (positive) tolerance values.

\[
\mu x_1(x_1) = \begin{cases} 
x_i - (5 - 2.5) & : 2.5 \leq x_i \leq 5 \\
\frac{(5 + 2.5) - x_i}{2.5} & : 5 < x_i \leq 7.5 \\
0 & : \text{otherwise} 
\end{cases}
\]

\[
\mu x_2(x_2) = \begin{cases} 
x_i : \text{if } x_i \leq 3 \\
\frac{3 - x_i}{3} & : 0 \leq x_i \leq 3 \\
0 & : \text{otherwise} 
\end{cases}
\]

Solve 4: solve the auxiliary problem

\[
\max \lambda \\
\text{s.t. } \mu f_1(f_1(x)) \geq \lambda \\
\mu f_2(f_2(x)) \geq \lambda \\
\mu x_1(x_1) \geq \lambda \\
3x_1 + 2x_2 + x_3 + 3x_4 \leq 40 \\
3x_1 + 2x_2 + x_3 + 2x_4 \leq 30 \\
2x_1 + 4x_2 + x_3 + 2x_4 \leq 35
\]
\[ X_1, x_2, x_3, x_4 \geq 0, \lambda \in [0,1] \]

The result of the first iteration including an optimal solution to the problem is:

\[ x_1^1 = 6.41, \quad x_2^1 = 1.95, \quad x_3^1 = 10.52, \quad x_4^1 = 1.42, \quad \text{and} \quad \lambda = 0.316 \]

\[ f_1(x) = 88.67, \quad f_2(x) = 95.55, \quad \mu_1(f_1(x)) = 0.2734 \]

suppose that DM_1 is not satisfied with the solution obtained in iteration 1, and then let him specify (decide) the minimal satisfactory level at \( \delta = 0.3 \) and the bounds of the ratio at the interval \([\Delta_{\text{min}}, \Delta_{\text{max}}] = [0.3, 0.4]\), taking account of the result of the first iteration. Then, the problem with the minimal satisfactory level is written as:

\[
\text{max} \quad \mu f_2(f_2(x)) \\
\text{s.t.} \quad \mu f_1(f_1(x)) \geq 0.3
\]

\[ x \in S \]

Applying simplex algorithm, the result of the second iteration including an optimal solution to problem (3) is

\[ x_1^2 = 6.71, \quad x_2^2 = 2.05, \quad x_3^2 = 10.52, \quad x_4^2 = 1.42 \quad \text{and} \quad \lambda^2 = 0.316; \]

\[ f_1^2 = 90.77, \quad f_2^2 = 98.85, \quad \mu_{f_1}(f_1(x)) = 0.3154, \quad \text{and} \quad \Delta^2 = 0.3165 \]

Therefore, this solution satisfies the termination conditions.
Title:

FUZZY PROGRAMMING APPROACH TO BI-LEVEL LINEAR PROGRAMMING PROBLEMS

Introduction

The use of fuzzy set theory for decision problems with several conflicting objectives was first introduced by Zimmermann. Thereafter, various versions of fuzzy programming (FP) have been investigated and widely circulated in literature. The use of the concept of tolerance membership function of fuzzy set theory to bi-linear programming problems (BLPPs) for satisfactory decisions was first introduced by Lai in 1996 [3]. Shih and Lee further extended Lai’s concept by introducing the compensatory fuzzy operator for solving BLPPs [5]. Sinha studied alternative BLP techniques based on fuzzy mathematical programming (FMP).

Fuzzy set theory

Definition: Let X be any referential set. A fuzzy subset A of X is a collection of objects with graded membership and characterized by its membership function $\mu_A : X \rightarrow [0,1]$.

Example: let $X = \{a, b, c\}$ and define the fuzzy set A as follows:

$\mu_A(a) = 1.0$, $\mu_A(b) = 0.7$, $\mu_A(c) = 0.4$

Fuzzy set operations

Union: The union of fuzzy sets A and B is the fuzzy set $A \cup B$ with a membership grade for every $x \in X$ given by $\mu_{A \cup B}(x) = \max \{\mu_A(x), \mu_B(x)\}$ for all $x$ in $X$.

Intersection: The intersection of fuzzy sets A and B is the fuzzy set $A \cap B$ with a membership grade for every $x \in X$ given by $\mu_{A \cap B}(x) = \min \{\mu_A(x), \mu_B(x)\}$.

A fuzzy set A is termed as:

(i). **normal** if there exists an $x \in X$ such that $\mu_A(x) = 1$ and A is subnormal if it is not normal.

(ii). **convex**: A is convex if and only if for every pair of points $x, y \in X$ the membership function of A satisfies:

$\mu_A(\lambda x + (1-\lambda)y) \geq \min(\mu_A(x), \mu_A(y))$ where $\lambda \in [0,1]$. 
Fuzzy Number

Definition: A fuzzy number $A$ is a normal and convex subset of real numbers whose membership function is piecewise continuous.

Types of Fuzzy Numbers

A fuzzy number $A=(a_1,a,a_2)$ for $a_1 \leq a \leq a_2$ is called triangular fuzzy number if its membership function $\mu_A$ is given by

$$\mu_A(x) = \begin{cases} 
0, & \text{if } x < a_1 \text{ or } x > a_2 \\
\frac{x-a_1}{a-a_1}, & \text{if } a_1 \leq x \leq a \\
\frac{a_2-x}{a_2-a}, & \text{if } a < x \leq a_2
\end{cases}$$

fuzzy linear programming

The fuzziness may appear in a linear programming problem in several ways such as the inequality may be fuzzy ($p_1$–FLP), the objective function may be fuzzy ($p_2$–FLP), or the parameters $c$, $A$, $b$ may be fuzzy ($p_3$–FLP) and so on.

Definition 1: If an imprecise aspiration level is assigned to the objective function, then this fuzzy objective is termed as fuzzy goal. It is characterized by its associated membership function by defining the tolerance limits for achievement of its aspired level.

We consider the general model of a linear programming

$$\max c^T x$$

$$\text{s.t. } A_i x \leq b_i (i=1,2,...,m)$$

$$\text{------------------------------------------(1.3.1)}$$

Where $A_i$ is an $n$-vector $C$ is an $n$-column vector and $x \in \mathbb{R}^n$. 
To a standard linear programming problem (1.3.1) above, taking in to account the imprecision or fuzziness of a decision maker’s judgment, Zimmermann considers the following linear programming problem with a fuzzy goal (objective function) and fuzzy constraints.

\[ C^T x \leq Z_0 \] (1.3.1a)

\[ A_i x \leq b_i, \quad (i=1,2,\ldots,m) \] (1.3.1.b)

\( x \geq 0 \), Where the symbol \( \leq \) denotes a relaxed or fuzzy version of the ordinary inequality \( \leq \). From the decision maker’s preference, the fuzzy goal (1.3.1a) and the fuzzy constraints (3.1b) mean that the objective function \( C^T X \) should be “essentially smaller than or equal to” a certain level \( Z_0 \), and that the values of the constraints \( A X \) should be “essentially smaller than or equal to” \( b \), respectively.

Fuzzy decision: is the fuzzy set of alternatives resulting from the intersection of the fuzzy constraints and fuzzy objective functions. Fuzzy objective functions and fuzzy constraints are characterized by their membership functions.

1.3.1. Solution techniques of solving fuzzy linear programming problems. The solution techniques for fuzzy linear programming problems follow the following procedure. We consider the following linear programming problem with fuzzy goal and fuzzy constraints,

\[
\max \sum_{j=1}^{n} c_j x_j \quad \text{------------------------------------------(1)}
\]

\[
\text{s.t.} \quad \sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq \tilde{b}_i, \quad 1 \leq i \leq m
\]

\( x_i \geq 0 \), where \( \tilde{a}_{ij} \) and \( \tilde{b}_i \) are fuzzy numbers with the following linear membership functions:

\[
\mu_{a_i} = \begin{cases} 
1 & \text{if} \quad x \leq a_i \\
\frac{a_i + d_i - x}{d_i} & \text{if} \quad a_i < x < a_i + d_i \\
0 & \text{if} \quad x \geq a_i + d_i
\end{cases}
\]
\[
\mu_i = \begin{cases} 
1 & \text{if } x \leq b_i \\
\frac{b_i + p_i - x}{p_i} & \text{if } b_i < x < b_i + p_i \\
0 & \text{if } x \geq b_i + p_i
\end{cases}
\]

and \( x \in \mathbb{R} \), \( d_{ij} > 0 \) is the maximum tolerance for the corresponding constraint coefficients and \( p_i \) is the maximum tolerance for the \( i \)th constraint. For defuzzification of the problem, we first fuzzify the objective function. This is done by calculating the lower and upper bounds of the optimal values. These optimal values \( z_l \) and \( z_u \) can be defined by solving the following standard linear programming problems, for which we assume that both of them have finite optimal values.

\[
\begin{align*}
\text{max } & \sum_{j=1}^{n} c_i x_i \\
\text{s.t.} & \sum_{j=1}^{n} \left( a_{ij} + d_{ij} \right) x_i \leq b_i & \text{and } \sum_{j=1}^{n} a_{ij} x_i \leq b_i + p_i \quad \forall i \\
& x \geq 0 , 1 \leq i \leq m \\
& x_i \geq 0 , 1 \leq i \leq m
\end{align*}
\]

Let \( z_l = \min(z_1, z_2) \) and \( z_u = \max(z_1, z_2) \). The objective function takes values between \( z_l \) and \( z_u \), while the constraint coefficients take values between \( a_{ij} \) and \( a_{ij} + d_{ij} \) and the right-hand side numbers take values between \( b_i \) and \( b_i + p_i \). Then, the fuzzy set optimal values, \( G \), which is a subset of \( \mathbb{R}^n \) is defined by:

\[
\mu_G(x) = \begin{cases} 
0 & \text{if } \sum_{j=1}^{n} c_i x_i \leq z_l \\
\sum_{j=1}^{n} c_i x_i - z_l & \text{if } z_l < \sum_{j=1}^{n} c_i x_i < z_u \\
\frac{z_u - z_l}{z_u - z_l} & \text{if } \sum_{j=1}^{n} c_i x_i \geq z_u
\end{cases}
\]

The fuzzy set of the \( i \)th constraint, \( C_i \), which is a subset of \( \mathbb{R}^n \) is defined by:
using the above membership functions $\mu_c(x)$ and $\mu_G(x)$ and following Bellmann and Zadeh approach, we construct the membership function $\mu_D(x)$ as follows.

$$
\mu_D(x) = \min_i(\mu_G(x), \mu_c(x))
$$

where $\mu_D(.)$ is the membership function of the fuzzy decision set. The min. section is selected as the aggregation operator. Then the optimal decision $x^*$ is the solution of

$$
\mu_D(x^*) = \max \mu_D(x) = \max \min_i(\mu_G(x), \mu_c(x))
$$

Then by using the method of defuzzification, problem (1) is reduced to the following crisp problem by introducing the auxiliary variable $\lambda$ which indicates the common degree of satisfaction of both the fuzzy constraints and objective function.

$$
\max \lambda
$$

s.t. $\mu_G(x) \geq \lambda$

$\mu_c(x) \geq \lambda$

$x \geq 0$, $0 \leq \lambda \leq 1$, $1 \leq i \leq m$

This problem is equivalent to the following non-convex optimization problem

$$
\max \lambda
$$

$$
\lambda(z_i - \bar{z}) - \sum_{j=1}^{n} c_j x_j + z_i \leq 0
$$

$$
\sum_{j=1}^{n} (a_{ij} + \lambda d_{ij}) x_j + \lambda p_i - b_i \leq 0
$$

$x \geq 0$, $0 \leq \lambda \leq 1$, $1 \leq i \leq m$
Therefore, the solution of this problem requires the special approach adopted for solving general non-convex optimization problems

Here solving the above linear programming problem gives us an optimum $\lambda^* \in [0,1]$. Then the solution of the problem is any $x \geq 0$ satisfying the problem constraint with $\lambda = \lambda^*$.

**BI-LEVEL PROGRAMMING**

Bi-level programming (BLP): is a mathematical programming problem that solves decentralized planning problems with two decision makers (DMs) in a two level or hierarchical organization.

**Mathematical formulation of a bi-level linear programming problem (BLPP):**

For the bi-level programming problems, the leader first specifies a decision and then the follower determines a decision so as to optimize the objective function of self with full knowledge of the decision of the leader. According to this rule, the leader also makes a decision so as to optimize the objective function of self. The solution defined as the above mentioned procedure is a stackelberg solution.

A bi-level LPP for obtaining the stackelberg solution is formulated as:

$$\max z_1(x_1, x_2) = c_1 x_1 + d_1 x_2$$

$x_1$

Where $x_2$ solves

$$\max z_2(x_1, x_2) = c_2 x_1 + d_2 x_2$$

$x_2$

S.t. $Ax_1 + Bx_2 \leq b$

where $c_i$, $i=1,2$ are $n_1$-dimensional row coefficient vector, $d_i$, $i=1,2$ are $n_2$-dimensional row coefficient vector, $A$ is an $mxn_1$ coefficient matrix, $B$ is an $mxn_2$ coefficient matrix, $b$-is an $m$-dimensional column constant vector. In the bi-level linear programming problem above, $z_1(x_1, x_2)$ and $z_2(x_1, x_2)$ represent the objective functions of the leader and the follower, respectively, and $x_1$ and $x_2$ represent the decision variables of the
leader and the follower respectively. Each decision maker knows the objective function of self and the constraints. The leader first makes a decision, and then the follower makes a decision so as to maximize the objective function with full knowledge of the decision of the leader. Namely, after the leader chooses $x_1$, he solves the following linear programming problem:

$$\max z_2(x_1, x_2) = c_2 x_1 + d_2 x_2$$

subject to:

$$B x_2 \leq b - A x_1$$

$$x_2 > 0,$$

and chooses an optimal solution $x_2(x_1)$ to the problem above as a rational response.

Definition:- The set $W_{z_2}(S)$ given by

$$W_{z_2}(S) = \{(x_1^*, x_2^*) \in S : z_2(x_1^*, x_2^*) = \max z_2(x_1, x_2)\}$$

is the set of rational reactions of $f_2$ over $S$.

Assuming that the follower chooses the rational response, the leader also makes a decision such that the objective function $z_1(x_1, x_2(x_1))$ is maximized as

$$\max z_1(x_1, x_2)$$

subject to:

$$(x_1, x_2) \in W_{z_2}(S)$$

FUZZY APPROACH TO BI-LEVEL LINEAR PROGRAMMING PROBLEMS

Fuzzy programming formulation of BLPPs

To formulate the fuzzy programming model of a BLPP, the objective functions $f_i$ (i=1,2) and the decision vector $x_i$ (i=1,2) would be transformed into fuzzy goals by means of assigning an aspiration level (the optimal solutions of both of the DMs calculated in isolation can be taken as the aspiration levels of their associated fuzzy goals) to each of them. Then, they are to be characterized by the associated membership functions by defining tolerance limits for achievement of the aspired levels of the corresponding fuzzy goals.

Fuzzy programming approach for bi-level LPPs

In the decision making context, each DM is interested in maximizing his or her own objective function, the optimal solution of each DM when calculated in isolation would
be considered as the best solution and the associated objective value can be considered as the aspiration level of the corresponding fuzzy goal. To build membership functions, goals and tolerance should be determined first. Using the individual best solutions, we find the values of all the objective functions at each best solution and construct a payoff matrix:

\[
\begin{bmatrix}
  f_i(x) & f_j(x) \\
  x_i^0 & f_i(x_i^0) \\
  x_j^0 & f_j(x_j^0)
\end{bmatrix}
\]

The maximum value of each column \((f_i(x_i^0))\) gives upper tolerance limit or aspired level of achievement for the \(i^{th}\) objective function where \(f_i^u = f_i(x_i^0) = \max f_i(x_i^0), i=1,2\).

The minimum value of each column gives lower tolerance limit or lowest acceptable level of achievement for the \(i^{th}\) objective function where \(f_i^L = \min f_i(x_i^0), i=1,2\). For the maximization-type objective function, the upper tolerance limit \(f_i^u\), \(t=1,2\), are kept constant at their respective optimal values calculated in isolation but the lower tolerance limit \(f_i^L\) are changed. The idea being that \(f_i(x) \rightarrow f_i^u\), then the fuzzy objective goals take the form \(f_i(x) \leq f_i(x_i^u), i=1,2\). And the fuzzy goal for the control vector \(x_i\) is obtained as \(x_i \cong x_i^u\). Thus, all values of \(f_i(x) \geq f_i^u\) are absolutely acceptable(desired) to objective function \(f_i(x)\) satisfactory to the \(i^{th}\) level DM. All values of \(f_i(x)\) with \(f_i(x) \leq f_i^L\) are absolutely unacceptable(undesired) to the objective function \(f_i(x)\) for \(i=1,2\).

\[
\mu(f_i(x)) = \begin{cases} 
1; & \text{if} \ f_i(x) \geq f_i^u \\
\frac{f_i(x) - f_i^L}{f_i^u - f_i^L}; & \text{if} \ f_i^L \leq f_i(x) \leq f_i^u, i=1,2, \ldots \ldots (3.1.2) \\
0; & \text{if} \ f_i(x) \leq f_i^L 
\end{cases}
\]

By identifying the membership functions \(\mu_1(f_1(x))\) and \(\mu_2(f_2(x))\) for the objective functions \(f_1(x)\) and \(f_2(x)\), the original bi-level linear programming problem (3.2) can be interpreted as the membership function maxmin problem defined by:

\[
\max \min \{ \mu_i(f_i(x)) \} \\
i=1,2.
\]

s.t. \(A_1x_1 + A_2x_2 \leq b,\)

\(x_1, x_2 \geq 0\)
Then the linear membership functions for decision vector $x_1$ can be formulated as:

$$\mu_i(x) = \begin{cases} 
\frac{x_i - (x_i^o - e_i^-)}{e_i^-}; & \text{if } x_i^o - e_i^- \leq x_i \leq x_i^o \\
\frac{(x_i^o + e_i^+)_i - x_i}{e_i^+}; & \text{if } x_i^o \leq x_i \leq (x_i^o + e_i^+) \\
0; & \text{Otherwise}
\end{cases}$$

Where $x_1^o$ is the optimal solution of first level DM

- $e_i^-$ the negative tolerance value on $x_i$
- $e_i^+$ the positive tolerance value on $x_i$.

To derive an overall satisfactory solution to the membership function maximization problem (3.1.3), the following problem is solved for obtaining a solution which maximizes the smaller degree of satisfaction between those of the two decision makers:

$$\max \min \{\mu_1(f_1(x)), \mu_2(f_2(x)), \mu_{x_1}(x_1)\}$$

s.t. $A_1x_1 + A_2x_2 \leq b$ .........................................................(3.1.5)

$x_1, x_2 \geq 0$

By introducing an auxiliary variable $\lambda$, this problem can be transformed into the following equivalent problem:

$$\max \lambda$$

s.t. $\mu_1(f_1(x)) \geq \lambda$

$\mu_2(f_2(x)) \geq \lambda$

$\mu_{x_1}(x_1) \geq \lambda$.................................................................(3.1.6)

$A_1x_1 + A_2x_2 \leq b$

$x_1, x_2 \geq 0$

Solving problem (3.1.6), we can obtain a solution which maximizes the smaller satisfactory degree between those of both decision makers. Let $x^*$ denotes an optimal solution to problem (3.1.6). Then we define the satisfactory degree of both decision makers under the constraints as

$$\lambda^* = \min\{\mu_1(f_1(x^*)), \mu_2(f_2(x^*))\}$$.........................................................(3.1.6.1)

If DM$_1$ is satisfied with the optimal solution $x^*$, it follows that the optimal solution $x^*$ becomes a satisfactory solution; however; DM$_1$ is not always satisfied with the solution $x^*$. Then it is quite natural to assume that DM$_1$ specifies (decides) the minimal satisfactory level $\delta \in [0,1]$ for his membership function $\mu_1(f_1(x))$ subjectively.
Consequently, if $DM_1$ is not satisfied with the solution $x^*$ to problem (3.1.6), then $DM_2$ optimizes his objective under the new constraints as the following problem:

$$\max \mu_2(f_2(x)) \quad \mu_1(f_1(x)) \geq \delta$$

$$A_1x_1 + A_2x_2 \leq b$$

$$x_1, x_2 \geq 0$$

If an optimal solution to problem (3.1.7) exists, it follows that $DM_1$ obtains a satisfactory solution having a satisfactory degree larger than or equal to the minimal satisfactory level specified (decided) by $DM_1$'s own self. However, the larger the minimal satisfactory level is assessed, the smaller $DM_2$'s satisfactory degree becomes. Consequently, a relative difference between the satisfactory degrees of $DM_1$ and $DM_2$ becomes larger than it is feared that overall satisfactory balance between both levels cannot be maintained. To take account of overall satisfactory balance between both levels, $DM_1$ needs to compromise (agree) with $DM_2$ on $DM_1$'s own minimal satisfactory level. To do so, the following ratio of the satisfactory degree of $DM_2$ to that of $DM_1$ is defined as:

$$\Delta = \frac{\mu_2(f_2(x^*))}{\mu_1(f_1(x^*))}$$

which is originally introduced by Lai (1996).

Let $\Delta^L$ and $\Delta^U$ denote the lower bound and the upper bound of $\Delta$ specified by $DM_1$. If $\Delta > \Delta^L$, then $DM_1$ updates (improves) the minimal satisfactory level $\delta$ by increasing $\delta$. Then $DM_1$ obtains a larger satisfactory degree and $DM_2$ accepts a smaller satisfactory degree. Conversely, if $\Delta < \Delta^L$, then $DM_1$ updates the minimal satisfactory level $\delta$ by decreasing $\delta$, and $DM_1$ accepts a smaller satisfactory degree and $DM_2$ obtains a larger satisfactory degree.

At an iteration $l$, let $\mu_2(f_2(x^{l'}))$, $\mu_1(f_1(x^{l'}))$, $\lambda'$, and $\Delta' = \frac{\mu_2(f_2(x^{l'}))}{\mu_1(f_1(x^{l'}))}$ denote $DM_1$'s and $DM_2$'s satisfactory degrees, a satisfactory degree of both levels and the ratio of satisfactory degrees between both $DM$s, respectively, and let a corresponding solution be $x^{l'}$ at the iteration $l$. The iterated interactive process terminates if the following two conditions are satisfied and $DM_1$ concludes the solution as a satisfactory solution.
Termination conditions of the interactive processes for bi-level linear programming problems.

i. DM₁’’s satisfactory degree is larger than or equal to the minimal satisfactory level \( \delta \) specified by DM₁, i.e. \( \mu_1(f_1(x^\ell)) \geq \delta \).

ii. The ratio \( \Delta^\ell \) of satisfactory degrees lies in the closed interval between the lower and upper bounds specified by DM₁, i.e. \( \Delta^\ell \in [\Delta_{\text{min}}, \Delta_{\text{max}}] \).

1. If condition (i) is not satisfied, then DM₁ decreases the minimal satisfactory level by \( \delta \).
2. If the ratio \( \Delta^\ell \) exceeds its upper bound, then DM₁ increases the minimal satisfactory level \( \delta \). Conversely, if the ratio \( \Delta^\ell \) is below its lower bound, then DM₁ decreases the minimal satisfactory level \( \delta \).

3.4. Algorithm of interactive fuzzy programming for BLPPs.

Step 1: Find the solution of the first level and second level independently with the same feasible set given.
Step 2: Do these solutions coincide?
   - If yes, an optimal solution is reached.
   - If No, go to step 3.
Step 3: Define a fuzzy goal, construct a pay off matrix, and then find upper tolerance limit \( f_{i,L} \) and lower tolerance limit \( f_{i,U} \).
Step 4: Build membership functions for maximization objective functions \( \mu_{f_i}(f_i(x)) \) and decision vector \( x_1 \) using equation (3.1.2) and (3.1.4) respectively.
Step 5: Set \( \ell = 1 \) and solve the auxiliary problems (3.1.6). If DM₁ is satisfied with the optimal solution, the solution becomes a satisfactory solution \( x^* \). Otherwise, ask DM₁ to specify(decide) the minimal satisfactory level \( \delta \) together with the lower and the upper bounds \([\Delta_{\text{min}}, \Delta_{\text{max}}]\) of the ratio of satisfactory degrees \( \Delta^\ell \) with the satisfactory degree \( \lambda^* \) of both decision makers and the related information about the solution in mind.
Step 6: Solve problem (3.1.7), in which the satisfactory degree of DM₁ is maximized under the condition that the satisfactory degree of DM₁ is larger than or equal to the minimal satisfactory level \( \delta \), and then an optimal solution \( x^\ell \) to problem (3.1.7) is proposed to DM₁ together with \( \lambda^\ell, \mu_1(f_1(x^\ell)), \mu_2(f_2(x^\ell)) \) and \( \Delta^\ell \).
Step 7: If the solution \( x^\ell \) satisfies the termination conditions and DM₁ accepts it, then the procedure stops, and the solution \( x^\ell \) is determined to be a satisfactory solution.
Step 8: Ask DM₁ to revise the minimal satisfactory level \( \delta \) in accordance with the procedure for updating minimal satisfactory level. Return to step 7.
Title:

FUZZY PROGRAMMING APPROACH TO BI-LEVEL LINEAR PROGRAMMING PROBLEMS

Introduction

Fuzzy set theory

Fuzzy set: Let X be any referential set. A fuzzy subset A of X is a collection of objects with graded membership and characterized by its membership function \( \mu_A : X \rightarrow [0,1] \).

Fuzzy set operations

Union: The union of fuzzy sets A and B is the fuzzy set \( A \cup B \) with a membership grade for every \( x \in X \) given by \( \mu_{A \cup B}(x) = \max \{ \mu_A(x), \mu_B(x) \} \) for all \( x \) in \( X \).

Intersection: The intersection of fuzzy sets A and B is the fuzzy set \( A \cap B \) with a membership grade for every \( x \in X \) given by \( \mu_{A \cap B}(x) = \min \{ \mu_A(x), \mu_B(x) \} \).

A fuzzy set A is termed as:

(i). normal if there exists an \( x \in X \) such that \( \mu_A(x) = 1 \) and A is subnormal if it is not normal.

(ii). convex: A is convex if and only if for every pair of points \( x, y \in X \) the membership function of A satisfies:

\[ \mu_A(\lambda x + (1-\lambda)y) \geq \min\{ \mu_A(x), \mu_A(y) \} \text{ where } \lambda \in [0,1]. \]

Fuzzy Number: A fuzzy number A is a normal and convex subset of real numbers whose membership function is piecewise continuous.

Fuzzy linear programming

We consider the general model of a linear programming

\[
\begin{align*}
\text{max} & \quad c^T x \\
\text{s.t.} & \quad A_i x \leq b_i (i=1,2,\ldots,m) \\
& \quad x \in \mathbb{R}^n.
\end{align*}
\]

Where \( A_i \) is an \( n \)-vector C is an \( n \)-column vector and \( x \in \mathbb{R}^n \).
To a standard linear programming problem (1.3.1) above, taking into account the imprecision or fuzziness of a decision maker’s judgment, Zimmermann considers the following linear programming problem with a fuzzy goal (objective function) and fuzzy constraints.

\[ C^T x \leq Z_0 \] \hspace{1cm} (1.3.1a)

\[ A_i x \leq b_i, (i=1,2,...,m) \] \hspace{1cm} (1.3.1b)

\[ x \geq 0, \text{ Where the symbol } \leq \text{ denotes a relaxed or fuzzy version of the ordinary inequality } \leq \]

**BI-LEVEL PROGRAMMING**

Bi-level programming (BLP): is a mathematical programming problem that solves decentralized planning problems with two decision makers (DMs) in a two level or hierarchical organization.

**Mathematical formulation of a bi-level linear programming problem (BLPP):** A BLPP for obtaining the Stackelberg solution is formulated as:

\[
\begin{align*}
\text{max } z_1(x_1,x_2) &= c_1 x_1 + d_1 x_2 \\
\text{Where } x_2 \text{ solves:} &\hspace{1cm} (\ast) \\
\text{max } z_2(x_1,x_2) &= c_2 x_1 + d_2 x_2 \\
\text{S.t. } A x_1 + B x_2 &\leq b
\end{align*}
\]

where \( c_i, i=1,2 \) are \( n_1 \)-dimensional row coefficient vector, \( d_i, i=1,2 \) are \( n_2 \)-dimensional row coefficient vector, \( A \) is an \( m \times n_1 \) coefficient matrix, \( B \) is an \( m \times n_2 \) coefficient matrix, \( b \) is an \( m \)-dimensional column constant vector. In the bi-level linear programming problem above, \( z_1(x_1,x_2) \) and \( z_2(x_1,x_2) \) represent the objective functions of the leader and the follower, respectively, and \( x_1 \) and \( x_2 \) represent the decision variables of the leader and the follower respectively.

**Challenges while applying computational methods for solving BLPPs.**

Unlike the \( K^{th} \) best and Kuhn-Tucker algorithms where DM2’s objectives are only used to restrict DM1’s feasibilities, i.e., the solutions are dominated, which may not be practical for solving real-world problems, both decision variables and objective functions are considered in fuzzy approach, and thus the proposed statsfactory solution should be
more practical and reasonable. Since the solution search is based on the change of membership function instead of vertex enumeration, even a large-scale problem can be solved with little computation. In this case, in a decentralized organization, a non-dominated solution may be more meaningful and good enough than classical solutions.

**FUZZY APPROACH TO BI-LEVEL LINEAR PROGRAMMING PROBLEMS**

**Fuzzy programming approach for bi-level LPPs**

In the decision making context, each DM is interested in maximizing his or her own objective function, the optimal solution of each DM when calculated in isolation would be considered as the best solution and the associated objective value can be considered as the aspiration level of the corresponding fuzzy goal.

To build membership functions, goals and tolerance should be determined first. However, they could hardly be determined without meaningful supporting data. Using the individual best solutions, we find the values of all the objective functions at each best solution and construct a payoff matrix

\[
\begin{bmatrix}
    f_1(x) & f_2(x) \\
    f_1(x_0) & f_2(x_0) \\
    f_1(x_0) & f_2(x_0)
\end{bmatrix}
\]

where \( f_i^u = f_i(x_i^0) = \max f_i(x_i^0), \) and \( f_i^l = \min f_i(x_i^0), \) \( i=1,2. \) For the maximization-type objective function, the upper tolerance limit \( f_i^u, \) \( t=1,2, \) are kept constant at their respective optimal values calculated in isolation but the lower tolerance limit \( f_i^l \) are changed. The idea being that \( f_i(x) \rightarrow f_i^u, \) then the fuzzy objective goals take the form \( f_i(x) \geq f_i(x_i^u), \) \( i=1,2. \) And the fuzzy goal for the control vector \( x_i \) is obtained as \( x_i \equiv x_i^u. \) The Objective function \( f_i(x) \) for \( i=1,2. \) is defined as

\[
\mu(f_i(x)) = \begin{cases} 
1; & \text{if } f_i(x) \geq f_i^u \\
\frac{f_i(x) - f_i^l}{f_i^u - f_i^l}; & \text{if } f_i^l \leq f_i(x) \leq f_i^u, \ t=1,2, \\
0; & \text{if } f_i(x) \leq f_i^l.
\end{cases}
\]

Now the original bi-level linear programming problem (3.1) can be interpreted as the membership function maxmin problem defined by:

\[
\max \min \{ \mu_i(f_i(x)) \} \quad i=1,2. \quad \text{-------------------------------------3.1.3}
\]

\[
s.t. \ A_1x_1 + A_2x_2 \leq b,
\]
Then the linear membership functions for decision vector $x_1$ can be formulated as:

$$
\mu_i(x_i) = \begin{cases} 
\frac{x_i - (x_i^o - e_i^-)}{e_i^-}, & \text{if } x_i^o - e_i^- \leq x_i \leq x_i^o \\
\frac{(x_i^o + e_i^+) - x_i}{e_i^+}, & \text{if } x_i \leq x_i^o \leq (x_i^o + e_i^+) \\
0; & \text{otherwise}
\end{cases}
$$

Where $x_1^o$ is the optimal solution of first level DM

$e_1^-$ the negative tolerance value on $x_1$

$e_1^+$ the positive tolerance value on $x_1$.

To derive an overall satisfactory solution to the membership function maximization problem (3.1.3), the following problem is solved which maximizes the smaller degree of satisfaction between those of the two decision makers:

$$
\max \min \{\mu_1(f_1(x)), \mu_2(f_2(x)), \mu_{x_1}(x_1)\}
$$

s.t. $A_1x_1 + A_2x_2 \leq b$…………………..(3.1.5)

$x_1, x_2 \geq 0$

By introducing an auxiliary variable $\lambda$, this problem can be transformed into the following equivalent problem:

$$
\max \lambda 
$$

s.t. $\mu_1(f_1(x)) \geq \lambda$

$\mu_2(f_2(x)) \geq \lambda$

$\mu_{x_1}(x_1) \geq \lambda$……………………………..(3.1.6)

$A_1x_1 + A_2x_2 \leq b$

$x_1, x_2 \geq 0$

Solving problem (3.1.6), we can obtain a solution which maximizes the smaller satisfactory degree between those of both decision makers. Let $x^*$ denotes an optimal
solution to problem (3.1.6). Then we define the satisfactory degree of both decision
decisions) the minimal satisfying degree of both decision
makers under the constraints as

$$\lambda^* = \min\{\mu_1(f_1(x^*), \mu_2(f_2(x^*))\}$$  (3.1.6.1)

If DM_1 is satisfied with the optimal solution x^*, it follows that the optimal solution x^* becomes a satisfactory solution. However, DM_1 is not always satisfied with the solution x^*. Then it is quite natural to assume that DM_1 specifies (decides) the minimal satisfactory level \(\delta \in [0,1]\) for his membership function \(\mu_1(f_1(x))\) subjectively. Consequently, if DM_1 is not satisfied with the solution x^* to problem (3.1.6), then DM_2 optimizes his objective under the new constraints as the following problem:

$$\max \mu_2(f_2(x))$$

$$\mu_1(f_1(x)) \geq \delta$$  (3.1.7)

$$A_1x_1 + A_2x_2 \leq b$$

$$X_1, x_2 \geq 0$$

If an optimal solution to problem (3.1.7) exists, it follows that, DM_1 needs to compromise (agree) with DM_2 on DM_1's own minimal satisfactory level. To do so, the following ratio of the satisfactory degree of DM_2 to that of DM_1 is defined as:

$$\Delta = \frac{\mu_2(f_2(x^*))}{\mu_1(f_1(x^*))}$$  (3.1.8)

which is originally introduced by Lai(1996).

Let \(\Delta^L\) and \(\Delta^U\) denote the lower bound and the upper bound of \(\Delta\) specified by DM_1. If \(\Delta > \Delta^U\) then DM_1 updates (improves) the minimal satisfactory level \(\delta\) by increasing \(\delta\). Then DM_1 obtains a larger satisfactory degree and DM_2 accepts a smaller satisfactory degree. Conversely, if \(\Delta < \Delta^L\), then DM_1 updates the minimal satisfactory level \(\delta\) by decreasing \(\delta\), and DM_1 accepts a smaller satisfactory degree and DM_2 obtains a larger satisfactory degree.

At an iteration \(l\), let \(\mu_1(f_1(x^l)), \mu_1(f_2(x^l))\), \(\lambda',\) and \(\Delta' = \frac{\mu_2(f_2(x^l))}{\mu_1(f_1(x^l))}\) denote DM_1's and DM_2's satisfactory degrees, a satisfactory degree of both levels and the ratio of satisfactory degrees between both DMs, respectively, and let a corresponding solution
be $x^i$ at the iteration $l$. The iterated interactive process terminates if the following two conditions are satisfied and DM$_1$ concludes the solution as a satisfactory solution.

- $\mu_1(f_1(x^i)) \geq \delta$ which is the DM$_1$'s required condition for solutions.
- $\Delta^i \in [\Delta \text{ min}, \Delta \text{ max}]$ which is provided in order to keep overall satisfactory balance between both levels.

3.4. Algorithm of interactive fuzzy programming for BLPPs.

Step 9: Find the solution of the first level and second level independently with the same feasible set given.

Step 10: Do these solutions coincide?
- If yes, an optimal solution is reached.
- If No, go to step 3.

Step 11: Define a fuzzy goal, construct a payoff matrix, and then find upper tolerance limit $f_i^u$ and lower tolerance limit $f_i^L$.

Step 12: Build membership functions for maximization objective functions $\mu_{fi}(f_i(x))$ and decision vector $x_1$ using equation (3.1.2) and (3.1.4) respectively.

Step 13: set $\ell=1$ and solve the auxiliary problems (3.1.6). If DM$_1$ is satisfied with the optimal solution, the solution becomes a satisfactory solution $x^*$. Otherwise, ask DM$_1$ to specify (decide) the minimal satisfactory level $\delta$ together with the lower and the upper bounds $[\Delta \text{ min}, \Delta \text{ max}]$ of the ratio of satisfactory degrees $\Delta^i$ with the satisfactory degree $\lambda^*$ of both decision makers and the related information about the solution in mind.

Step 14: Solve problem (3.1.7), in which the satisfactory degree of DM$_1$ is maximized under the condition that the satisfactory degree of DM$_1$ is larger than or equal to the minimal satisfactory level $\delta$, and then an optimal solution $x^\ell$ to problem (3.1.7) is proposed to DM$_1$ together with $\lambda^\ell$, $\mu_1(f_1(x^\ell))$, $\mu_2(f_2(x^\ell))$ and $\Delta^\ell$.

Step 15: If the solution $x^\ell$ satisfies the termination conditions and DM$_1$ accepts it, then the procedure stops, and the solution $x^\ell$ is determined to be a satisfactory solution.

Step 16: Ask DM$_1$ to revise the minimal satisfactory level $\delta$ in accordance with the procedure for updating minimal satisfactory level. Return to step 7.
REFERENCES


**Conclusion**

The FMP approach is simple to implement, interactive and applicable to BLPP. The satisfactory solution obtained is realistic. We can take any membership function other than linear. The results will hold good, however, the problem will become a non linear programming problem. We observe that even though the decision-making process is from higher to lower level, the lower level becomes the most important. This is because the decision vector under the control of the lower level DM is not given any tolerance limits. Hence this decision vector either remains unchanged or closest to its valued obtained in isolation. But at higher level, the decision vectors are given some tolerance and hence they are free to move within the tolerance limits. The tolerance levels can also be considered as variables and if the DMs cooperate then the entire system as a whole can be optimized. We can easily apply the same approach to non linear BLPPs.