Addis Ababa University

College of Natural and Computational Sciences

Department of Mathematics

Thesis on

Fourier analysis and the Dirichlet problem

By: - Shiferaw Yeneabat

Advisor: - Hunduma Legesse (PhD)

A thesis submitted to Addis Ababa University Natural and Computational Sciences in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

September, 2017

Addis Ababa, Ethiopia

Addis Ababa University

College of Natural and Computational Sciences

Department of Mathematics

Approved by the Board of Examiners

September, 2017

Advisor: - ______________________

Dr. Hunduma Legesse

Examiners:- ______________________

Dr. Taddesse Abdi

____

Dr. Tesfa Biset

Acknowledgement

First, I would like to express my deepest gratitude to God for giving me patience.

Next, I would like to express my heartfelt gratitude and thanks to my

Advisor Dr. HundumaLegesse, for helpful discussion, comments and providing the necessary materials in the preparation of thisthesis.

Finally, I would like to extend gratefulness to my families and all who encouraged me to complete my thesis and their heart full help while writing the paper.

Abstract

In this thesis, using the properties of convergence of Fourier series and some other properties of trigonometric polynomials, in particular that they are sums of holomorphic and anti-holomorphic functions, we are able to solve the Dirichlet problem on the Disc. Then, applying the result for the unit Disc along with a couple of Möbius transformations, we are able to solve the Dirichlet problem on the upper half plane.

Tables of content

Content **page**

Introduction

Fourier analysis began as an attempt to approximate periodic function with infinite summation of trigonometric polynomials. For certain function, these sums, known as Fourier series converge exactly to the original function.

The Dirichlet problem is named after LejuneDirichlet who proposed a solution by variation method which becomes Dirichlet's principle.

The Dirichlet problem for Laplace equation has the form

$$\Delta u = 0 \quad in\, \Omega$$

$$u = f \quad on\ \ \partial\Omega$$

To solve Dirichlet problem on the disc and on the upper half plane we need the concept of harmonic function, Boundary value problem, maximum principle, Fourier series and Poisson Kernel function. In this Thesis we want to solve the Dirichlet problem on the unit disc, and then applying the result for the unit disc along with a couple of Möbius transformation, we are able to solve the Dirichlet problem on the upper half plane.

This thesis consists of two chapters. In the first chapter, preliminaries are presented briefly and,chapter two focuses on the Dirichlet problem.

CHAPTER 1

Preliminaries

1.1 Harmonic Function

Definition1.1. The equation $\frac{\partial^2 u}{\partial {x_1}^2} + \cdots + \frac{\partial^2 u}{\partial {x_n}^2} = 0$ is called

Laplace's equation. The operator

$\Delta = \frac{\partial^2}{\partial {x_1}^2} + \cdots + \frac{\partial^2}{\partial {x_n}^2}$is called the Laplacian. In terms of this operator Laplace's equation becomes simply $\Delta u = 0$, Smooth function $u(X), X\epsilon\mathbb{R}^n$ that satisfy Laplace's equation are called harmonic functions. Laplace's equation is one of the most important partial differential equation of mathematical physics. We will be concerned with harmonic functions of two variables, that is, solutions of

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\text{Or } \Delta u = u_{xx} + u_{yy} = 0$$

We say that a function $u(x, y)$, is harmonic if all its first and second order partial derivatives exist and are continuous and satisfy Laplace's equation. In the case of functions of two variables, there is an intimate connection between analytic functions and harmonic functions.

Theorem1.1. If f= u + i v is analytic, and the functions u and v have continuous second –order partial derivatives then u and v are harmonic.

Proof: The harmonicity of u and v is a simple consequence of the Cauchy-Reiman equations

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}and\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$using these we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial \partial v}{\partial x \partial y} = \frac{\partial \partial v}{\partial y \partial x} = \frac{-\partial^2 u}{\partial y^2}$$

Which show that u is harmonic. The verification that v is harmonic is the same. If u is harmonic on a domain D, and v is a harmonic function such that u + iv is analytic we say that v is a harmonic conjugate of u. The harmonic conjugate V is unique, up to adding a constant. Indeed, if V_0 is another harmonic conjugate for u, so that $u + iv_0$ is also analytic, then the difference$i(v - v_0)$ is also analytic, and $v - v_0$ is a real–valued analytic function, hence constant on D.

1.2 Boundary Value Problem

A Boundary value problem is a problem of finding a function which satisfies a given partial differential equation and particular boundary condition. There are three types of boundary conditions that are usually associated with Laplace equation. These are

1. Dirichlet Boundary value problem (DBVP)
2. Neumann Boundary value problem (NBVP)
3. Robin's Boundary value problem (RBVP)

1. Dirichlet Boundaryvalue problem (DBVP): The boundary condition (BC) is of Dirichlet type if the solution u(x,y) to Laplace equation in the domain Ω is specified on the boundary $\partial\Omega$ i.e $u(x, y) = \mathrm{f(x, y)}$on Ω where $f(x, y)$ is a given function. The Laplace equation together with the Dirichletboundary condition is called DBVP (Dirichlet problem). The Dirichlet problem for Laplace equation has the form Δu=0 in Ω and $u = f \; on \; \partial\Omega$.

2. Neumann Boundary value problem: The boundary condition (BC) is Neumann type if the directional derivatives $\frac{\partial u}{\partial n}$ along the outward normal to the boundary is specified on ∂Ω. i.e.$\frac{\partial u}{\partial n}(x, y) = g(x, y) \; for \; (x, y) \in \partial\Omega$.

In physical terms, the normal component of the solution gradient is known on the boundary.

The Laplace equation together with Neumann BC is called Neumann Boundary value problem (Neumann problem) which is expressed as $\Delta u = 0 \ in \ \Omega , \frac{\partial u}{\partial n}(x, y) = g(x, y) \, for \ (x, y) \in \partial\Omega$

3. Robin's Boundary value problem: The boundary condition (BC) Robin's type (mixed type) if Dirichlet BC specified on part of the boundary $\partial\Omega$ and Neumann BC is specified on the remaining part of the boundary $\partial\Omega$

Example:$\frac{\partial u}{\partial n}$+ c(u-g)=0, where c is constant and g is a given function that can vary over the boundary

The Laplace equation together with Robin's (mixed) BC is called Robin's Boundary value problem

1.3. Maximum Principle

Before we prove the uniqueness for the interior Dirichlet problem for Laplace equation, we first prove maximum principle.

Theorem1.2. (Maximum principle)

Let Ω be a connected bounded open set in $\mathbb{R}^2$.

Let u satisties $u_{xx} + u_{yy}$=0 in Ω and continuous on $\bar{\Omega} = \Omega \cup \partial\Omega$. Then u attains its maximum on the boundary $\partial\Omega$ unless u is constant.

Proof: - let M= $\{u(x, y) : (x, y) \in \partial\Omega\}$

Suppose that the maximum of $u \ on \ \bar{\Omega}$is not attained on $\partial\Omega$.

Let $p = (x_{0,}y_0) \in \Omega, \quad (x_0, y_0) \notin \partial\Omega$ such that u attained its maximum at p

Let $M_0 = u\ (x_0, y_0), then\ M_0 > M$

Define $v\ (x, y) = u(x, y) + \frac{M_0 - M}{4R^2}\{(x - x_0)^2 + (y - y_0)^2\}, (x, y) \in \Omega$ and R is radius of the circle with center (x_0, y_0) and containing Ω

Notice that $v(x_0, y_0) = \ u(x_0, y_0) = M_0 \ on \ \partial\Omega$.

Then, we have $v(x, y) \leq M + \frac{M_0 - M}{4} < M_0$

Now u attains its maximum in Ω

Since, $v(x_{0,}y_0) = M_0\ and\ v(x, y) < M_0\ on\ \partial\Omega$ say at Q (x, y)

So, $v_{xx} + v_{yy} \leq 0\ at\ Q$ … (1)

But on Ω, $v_{xx} + v_{yy} = 0 + \frac{M_0 - M}{R^2} > 0$

$\Rightarrow v_{xx} + v_{yy} > 0$ … (2)

But (1) and (2) lead to contradiction. So our supposition was wrong thus u attains its maximum on the boundary ∂Ω.

The maximum principle is useful, for instance for demonstrating convergence of a sequence of harmonic functions. To show that a sequence of harmonic functions converges in a disc, it suffices to obtain good estimates automatically persist in the interior.

Theorem1.3.(continuity theorem) The solution of the Dirichlet problem depends continuously on the boundary data

Proof: - let u_1 and u_2 be the solution of

$$\Delta u_1 = 0, in\ \Omega$$

$$U_1 = f_1 on\ \partial\Omega$$

And $\Delta u_2 = 0, in\ \Omega$

$u_2 = f_2$ on ∂Ω

If $v = u_1 - u_2$ then v satisfies $\Delta v = 0$ in Ω

$v = f_1 - f_2$ on ∂Ω

Bymaximum and a minimum principle f_1-$f_2$attains the maximum and minimum of v on ∂Ω

Thus, if $|f_1 - f_2| < \varepsilon$,then $-\varepsilon < v_{min} \leq v_{max} < \varepsilon$ on ∂Ω

Thus, at any interior point in Ω we have

$-\varepsilon < v_{min} \leq v \leq v_{max} < \varepsilon$ therefore $|v| < \varepsilon$ in Ω

Hence, $|u_1 - u_2| < \varepsilon$

Theorem1.4. let $\{u_n\}$ be a sequence of functions harmonic in Ω and continuous in Ω.

Let $f_i$be the values of $u_i$on Ω . If a sequence $u_n$converges uniformly on ∂Ω ,then it converges uniformly in Ω.

Proof : by hypothesis $\{f_n\}$ converge uniformly on ∂Ω.

Thus, for $\varepsilon > 0$,there exists an integer N such that everywhere on ∂Ω ,

$|f_n-f_m| < \varepsilon$,for n , m >N

It follows from the continuity theoremthat for all n,m >N

$|u_n-u_m| < \varepsilon$ in Ω and hence the theorem is proved

1.4 Fourier series

In calculus we learn that any sufficiently nice function can be approximated by an infinite sum of polynomials called a Taylor series. So if we consider only periodic functions, it does not seem too unreasonable to assume that we could approximate them with infinite sums of sines and cosines (if this apparent guess seems too non-rigorous, rest assured that these summation can be shown to arise quite naturally.

The existence and convergence of these summations is the basis for Fourier analysis. First we will determine what form these summations should take and then we will explore some basic properties of their convergence.

Basic Properties of Fourier series

We begin our rigorous study of Fourier series. We set the stage by introducing the main objects in the subject and then formulate some basic problems which we have already touched upon earlier. Our first result disposes of the question of uniqueness. Are two functions with the same Fourier coefficients necessarily equal? Indeed a simple argument shows that if both functions are continuous, then in fact they must agree. Next we take closer look at the partial sums of a Fourier series. Using the formula for the Fourier coefficients (which involves an integration) we make the key observation that these sums can be written conveniently as integral

$\frac{1}{2\pi}\int D_N(x-y)f(y)dy\ where\ \{D_N\}$is a family function called the Dirichlet kernel. The above expression is the convolution of with the function $D_{N.}$ Convolution will play a critical role in our analysis. In general, given a family of function $\{G_n\}$, we are led to investigate the limiting properties as n tends to infinity of the convolutions $\frac{1}{2}\int G_n(x-y)f(y)dy.$ we find that if the family $\{G_n\}$, satisfies the three important of good kernels, then convolutions above tend to $f(x) as\ n \to \infty$ (at least when f is continuous). In this sense, the family $\{G_n\}$, is an "approximation to the identity". Unfortunately, the Dirichlet kernel D_N do not belong to the category of good kernel, which indicates that the question of convergence of Fourier series is subtle. Instead of pursuing at this stage the problem of convergence, we consider various other methods of summing the Fourier series of a function. The first method, which involves average of partial sums, leads to convolution with good kernels and yields an important theorem of Fejer. From this we deduce that a continuous function on the circle can be approximated uniformly by trigonometric polynomials. Second we may also sum the Fourier series in the sense of Abel and again encounter, a family of good kernels lead to a solution of the Dirichlet problem for the steady-state heat equation in the disc.

Function on the Circle

There is a natural connection between $2\pi -$ periodic function on $\mathbb{R}$ like the exponentials $e^{ik\theta}$, function on an interval of length 2π, and function on the unit circle. This connections arises as follows.

A point on the unit circle takes the form $e^{i\theta}$, where θ is a real number that is unique up to integer multiples of 2π. If f is a function on the unit circle, then we may define for each real number θ

$f(\theta) = F(e^{i\theta})$and observe that with this definition, the function f is periodic on $\mathbb{R}$ of period 2π, that is $f(\theta + 2\pi) = f(\theta)$ for all θ. The integrality, continuity and other smoothness properties of F are determined by those off. For instance, we say that F is integrable on the circle if f is integrable on every interval of length 2π. Also, F is continuous on the circle if f is continuous on $\mathbb{R}$ which is the same as saying that f is continuous on any interval of length 2π. Moreover, F is continuously differentiable if f has a continuous derivative, and so forth. Since f has period 2π, we may restrict to any interval of length 2π, say $[0, 2\pi]$ or $[-\pi, \pi]$ and still capture the initial function F on the circle. We note that f must take the same value at the end points of the interval since they correspond to the same point on the circle. Conversely, any function on $[0, 2\pi]$ for which$f(0) = f(2\pi)$ can be extended to a periodic function on $\mathbb{R}$ which can then be identified as a function on the circle. In particular a continuous function f on the interval $[0,2\pi]$ gives rise to a continuous function on the circle if and only if $f(\theta) = f(2\pi)$.

In conclusion functions on $\mathbb{R}$ that 2π −periodic are functions on the interval of length 2π that take on the same value at its end-points are two equivalent descriptions of the same mathematical objects, namely, functions on the circle. In this connection, we mention an item of notational usage. When our functions are defined on an interval on the line, we often use x as the independent variable. However, when we consider these as functions on the circle, we usually replace the variable x by θ. As the reader will note, we are not strictly bound by this rule since this practice is mostly a matter of convenience.

Definitions and Examples

We now begin our study of Fourier analysis with the precise definition of the Fourier series of a function. Here, it is important to pin down where our function is originally defined, if f is an integral function given on an interval [0,1] then the k^{th} Fourier coefficient of f is defined by

$$\hat{f}(\text{k}) = \int_0^1 f(x)e^{-2\pi ikx}dx\,, \quad k \in z$$

And the Fourier series of f is given by

$$\sum_{k=-\infty}^{\infty} \hat{f}(\text{k})e^{2\pi ikx}$$

We shall sometimes write a_k for the Fourier coefficient of f and use notation

$$f(x) \sim \sum_{k=-\infty}^{\infty} a_k e^{2\pi ikx}$$

To indicate that the series on the right hand side is the Fourier series of f.

Example: If f is an integrable function on the interval$[-\pi, \pi]$, then the k^{th} Fourier coefficient of f is

$$\hat{f}(\text{k}) = a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}d\theta \quad and$$

The Fourier series of f is

$$f(\theta) \sim \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}$$

Here we use θ is a variable since we think of it as an angle ranging from π *to* π. Also if f is defined on $[0,2\pi]$, that the formula are the same as above,

except that we integrate from 0 to 2π in the definition of the Fourier coefficients. We may also consider the Fourier coefficients and Fourier series for a function defined on the circle. By our previous discussion, we may think of a function on the circle as a function of f on $\mathbb{R}$ which is $2\pi-$ periodic. We may restrict the function f to any interval of length 2π, for instance$[0,2\pi]$ or$[-\pi, \pi]$, and compute its Fourier coefficients. Fortunately, f is periodic and the resulting integrals are independent of the chosen interval.

Thus, the Fourier coefficients of a function on the circle are well defined. Finally, we shall sometimes consider a function g given on [0,1], then

$$\hat{g}(k) = a_k = \int_0^1 g(x)e^{-2\pi ikx}dx \text{ and}$$

$$g(x) \sim \sum_{k=-\infty}^{\infty} a_k e^{2\pi ikx}$$

Here we use x for a variable ranging from 0 to 1 of course, if f is initially given on $[0,2\pi]$, then $g(x) = f(2\pi x)$ is defined on [0,1] and a change of variables shows that the k^{th}Fourier coefficient of f equals the k^{th} Fourier coefficient of g.

Fourier series are part of a large family called the trigonometric series which, by definition, are expressions of the form

$$\sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx}$$

If a trigonometric series involves only finitely many non-zero terms that is, $c_k = 0\ for\ all\ large\ |k|$ it is called trigonometric polynomial, its degree is the largest value of $|k|$ for which $c_k \neq 0$.

The N^{th} partial sum of the Fourier series of f for N a positive integer, is a particular example of a trigonometric polynomial. It is given by

$$S_N(f)(x) = \sum_{k=-N}^{N} \hat{f}\,(k)e^{2\pi ikx}$$

Convolutions

The notion of convolution of two functions plays a fundamental role in Fourier analysis; it appears naturally in the context of Fourier series but also serves more generally in the analysis of functions in other settings.

Example: Given two-2π periodic integrable functions f and g on $\mathbb{R}$, we define their convolutions $f * g$ on $[-\pi, \pi]$ by

$(f * g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)dy$ --------- (1)

The above integral makes sense for each x. since the product of two integrable functions is again integrable. Also, since the functions are periodic, we can change variables to see that

$(f * g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)g(y)dy$ ------- (2)

Here, convolutions correspond to "weighted averages" For instance , if g=1 in(1) then $f * g$ is constant and equal to $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)dy$, which we may interpret as the average value of f on the circle . Also,the convolution

(f $*$ g)(x) Plays a role similar to, and in some sense replace,the point wise product f(x)g(x) of the two functions f and g.

In the context of basic properties of Fourier series, our interest in convolutions originates from the fact that the partial sums of the Fourier series of f can be expressed as follows:

$$S_N(\mathrm{f})(\mathrm{x}) = \sum_{\mathrm{n}=-\mathrm{N}}^{\mathrm{N}} \hat{\mathrm{f}}(\mathrm{n})e^{inx}$$

$$\sum_{n=-N}^{N} (\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}dy)e^{inx}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)(\sum_{N=-N}^{n} e^{in(x-y)}\, dy$$

$$=(f * D_N)(x)$$

Where D_N is the N^{th} Dirichlet kernel given by

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}$$

so we observe that the problem of understanding S_N (f) reduces to the understanding of the convolution $f * D_N$

We begin by gathering some of the main properties of convolutions.

Proposition1.1. suppose that f, g and h are 2π- periodic integrable functions.

Then

I. $F * (g + h) = (f * g) + (g * h)$
II. $cf * g = c(f * g) = f * (cg)\ for\ any\ c \epsilon\ \mathbb{C}$
III. $f * g = g * f$
IV. $(f * g) * h = f * (g * h)$
V. $f * g\ \ is\ continuous$

The first four points describe the algebraic properties of convolution; Linearity, commutability and associativity.
Property (v) exhibit an important principle. The convolution of $f\ * g$ is more regular than f or ghere, $f\ * g$ is continuous while f and g are Riemann integrable

Proof: properties (i) and (ii) follow at once from the linearity of the integration.

The other properties are easily deduced if we assume also that f and g are continuous.

In this case, we may freely interchange the order of integration.

To prove (iii), one first notes that if F is continuous and 2π- periodic, then

$\int_{-\pi}^{\pi} F(y)dy = \int_{-\pi}^{\pi} F(x-y)dy$for any x $\in \mathbb{R}$

The verification of this identity consists of a change of variables y ↔ -y, followed by a translation y ↔ $y - x$. Then, one takes F(y) =f(y) g(x-y).

Suppose we have a Fourier series

We begin by assuming that we have a 2π periodic function, f, and a summation of trigonometric functions of varying frequencies that converges it. Thus we have

$$f(\theta) = \sum_{k=0}^{\infty} a_k \sin(k\theta) + \ b_k \cos(k\theta). However\ by\ Euler's\ identlty\ we\ also\ have$$

$$e^{ik\theta} = \cos(k\theta) + isin\ (k\theta)$$

So with the correct complex $C_k s, we\ have$

$$f(\theta) = \sum_{k=0}^{\infty} a_k \sin(k\theta) + b_k \cos(k\theta) = \sum_{k=-\infty}^{\infty} C_k e^{ik\theta}$$

Assuming that this sum converges uniformly, we now must determine the actual values for the $C_k s.$ Now let us make slight change to consider functions that are periodic on the interval [0,1] this prevents us dividing out constants later

So we are now solving for the $C_k s$ in

$$f(x) = \sum_{k=-\infty}^{\infty} C_{k_e} 2\pi ikx, \qquad then\ we\ have$$

$$f(x) = \sum_{k=-\infty}^{\infty} C_k e^{2\pi ikx}$$

Now multiply each side by $e^{-2\pi imx}$and integrate from 0 to 1.So,

$$\int_0^1 f(x) e^{-2\pi imx} \quad dx = \int_0^1 \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx} \ e^{-2\pi imx} dx$$

$$\int_0^1 f(x)e^{-2\pi imx} \quad dx = \sum_{k=-\infty}^{\infty} c_k \int_0^1 e^{2\pi ikx}e^{-2\pi imx}dx$$

Then, we note that the $e^{2\pi ikx}$ are orthogonal with respect to the inner product$\langle g, h\rangle$ =$\int_0^1 g(x)h(x)dx$. In other words, it can easily be verified that

$$\int_0^1 e^{2\pi ikx}e^{-2\pi imx} \quad dx = \begin{cases} 0, (k \neq m) \\ 1, (k = m) \end{cases}$$

Thus, we determine that $C_m = \int_0^1 f(x)e^{-2\pi imx}dx$

Now that we have explored what these summations should look like when everything works out. Now it is time to make some definitions out of our discoveries. First it would be helpful to have a standard name and notation for those $c_k s$ that are just found

Definition1.2. (Fourier coefficients)

Let $\hat{f}(k) = \int_0^1 f(x)e^{2\pi ikx}dx$ be the k^{th} Fourier coefficients of f

Now we should formally define what a Fourier series actually is

Definition1.3. (Fourier series). The Fourier series of f is the trigonometric polynomial with the Fourier coefficients defined above:

$$\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{2\pi ikx}$$

It is important to keep that these definitions describe how to determine the Fourier series for a given function but they do not imply anything about the convergence of the series. In fact even when the Fourier series does converge, it does not always converge to the function that it is based on.

When does a Fourier series converge? As it turns out, a certain degree of required for a Fourier series to converge to the function that it is based on. There are several conditions that are sufficient for the convergence of a

Fourier series (assuming we have decided what sort of convergent we are talking about.) For pointwise convergence,differentiability or local bounded variations are both sufficient.

For our purposes, it is only necessary for us to know that the Fourier series of a nice enough function does converge to the function. Thus it makes sense to use them for the Dirichlet problem on the disc

1.5. The Poisson Integral Formula for the unit Disc

We wish to extend a given continuous complex –valued function $h(e^{i\theta})$ on the unit circle continuously to the closed unit disc$\{|z| \leq 1\}$. So, as to be harmonic on the interior of the disc. Any such extension is unique, since the difference of two such extensions is zero on the boundary, hence zero on the entire disc, by the maximum principle. Our strategy is to derive a formula for the extension in the case that $h(e^{i\theta})$ is a trigonometric polynomial and then to show that the formula provides an extension even when $h(e^{i\theta})$ is only continuous.

We start with the trigonometric monomial $e^{ik\theta}$a harmonic extension is given explicitly by $r^{|k|}e^{ik\theta}$,if $k \geq 0$ this extension is $r^k e^{ik\theta} = z^k$ which is analytic.

If $k < 0$ this extension is $r^{-k}e^{ik\theta} = z^{-(-k)}$, which is conjugate analytic hence harmonic. Proceeding by linearity, we see that the trigonometric polynomial

$$h(e^{i\theta}) = \sum_{K=-N}^{N} a_k e^{ik\theta}$$

has the (unique) harmonic extension.

$$\hat{h}(re^{i\theta}) = \sum_{K=-N}^{N} a_k r^{|k|} e^{ik\theta}$$

We capture the coefficient a_m by multiplying $h(e^{i\theta})$ by $e^{-im\theta}$ and integrating. The orthogonality relations for complex exponentials yield

$$a_m = \int_{-\pi}^{\pi} h(e^{i\theta})e^{-im\theta}\frac{d\theta}{2\pi}$$

Substituting this expression in to the formula for $\tilde{h}(re^{i\theta})$, we obtain

$$\tilde{h}(re^{i\theta}) = \sum_{k=-\infty}^{\infty}\left(\int_{-\pi}^{\pi} h(re^{i\varphi})e^{-ik\varphi}\frac{d\varphi}{2\pi}\right)r^{|k|}e^{ik\theta}$$

$$= \int_{-\pi}^{\pi} h(re^{i\varphi})\left[\sum_{k=-\infty}^{\infty} r^{|k|}\, e^{-ik\varphi}e^{ik\varphi}\right]\frac{d\varphi}{2\pi}$$

In order to simplify this expression, we introduce the Poisson kernel function defined by

$$p_r(\theta) = \sum_{k=-\infty}^{\infty} r^{|k|}\, e^{ik\theta}$$

For each fixed $\rho < 1$ this series converges uniformly for $r \le \rho \; and \; \pi \le \theta \le \pi$ by the Weierstrass M- test, since then$\left|r^{|k|}e^{ik\theta}\right| \le \rho^{|k|}$. Interms the Poisson kernel the formula for the harmonic extension $h(re^{i\theta})$ becomes

$$\tilde{h}(re^{i\theta}) = \int_{-\pi}^{\pi} h(e^{i\varphi})p_r(\theta-\varphi)\frac{d\varphi}{2\pi}$$

If we make a change of variable $\varphi \to \theta - \varphi$ and use the 2π −periodicity of $p_r(\theta)$, we obtain an alternative form

$$\tilde{h}(re^{i\theta}) = \int_{-\pi}^{\pi} h(e^{i(\vartheta-\varphi)})p_r(\varphi)\frac{d\varphi}{2\pi}$$

Now we look at the Poisson kernel function $p_r(\theta)$ more closely. Setting $z = re^{i\theta}$ and setting $j = -k \;\; where \;\; k < 0, we \; obtain$

$$p_r(\theta) = 1 + \sum_{k=1}^{\infty} z^k + \sum_{J=1}^{\infty} z^{-j}$$

Summing these two geometric series, we obtain

$$p_r(\theta) = 1 + \frac{z}{1 - z} + \frac{\bar{z}}{1 - \bar{z}}, \qquad z = re^{i\theta}$$

Putting this over the common denominator

$$|1 - z|^2 = (1 - z)(1 - \bar{z}) = 1 + r^2 - 2r\cos\theta$$

We obtain

$p_r(\theta) = \frac{1-|z|^2}{|1-z|^2} = \frac{1-r^2}{1+r^2-2r\cos\theta}$, z= $re^{i\theta}$

Since the Poisson kernel is 2π −periodic, we focus on its behavior on the period interval $-\pi \le \theta \le \pi$.

CHAPTER2

The Dirichlet Problem

2.1 The Dirichlet Problem on the Disc

Given a connected open set Ω and a function f defined on the boundary $\partial\Omega$ ofΩ, the solution to the Dirchlet problem is a function of u, such that

$$\Delta u = 0, x \in \Omega$$
$$u = f\ , x \in \partial\Omega$$

Therefore, for the open disc we are looking for a function that is harmonic on the interior of the disc and periodic on the circle. Next we will show that the solution will be unique and then we will explicitly solve the Dirichlet problem on the disc.

Theorem 2.1.1.(Uniqueness Theorem)

The solution u(if exists) to the Dirichlet problem

$\Delta u = 0\ in\ \Omega$ and $u = f, 0n\ \ \partial\Omega$ is unique

Proof: let $u_1(x,y)\ and\ u_2\,(x,y)$ be two solution of the Dirichlet problem. Then $u_1 and\ u_2\ satisty$

$$\Delta u_1 = 0, \Delta u_2 = 0, in\ \Omega$$

$$\text{And } u_1 = f, u_2 = f\ on\ \partial\Omega$$

Since $u_1\ and\ \ u_2\ are\ harmonic\ in\ \Omega\ (u_1 - \ u_2)$ is also harmonic in Ω

But $u_1 - \ \ u_2$ =0 on $\partial\Omega$

The maximum – minimum principle gives$u_1 - \ \ u_2 = 0\ \ at\ all\ interior$

$points\ of\ \Omega. then\ we\ have$

$$u_1 = \ u_2$$

Thus, the solution is unique

Example:let D=(x,y)∈ $\mathbb{R}^2$: $x^2 + y^2<1$ =(r,θ):$0 \leq \theta < 2\pi$

Consider the Dirichlet problem

$$\Delta u = 0 \; in \; \Omega$$

$u(1,\theta) = \cos\theta - \sin\theta,\; 0 \leq \theta < 2\pi$

Consider u(x,y)= x-y. Obviously $\Delta u = 0$

On the boundary x=$\cos\theta$ $and\; y = \sin\theta$.

And hence u (x,y)= x-y=$\cos\theta - \sin\theta \;\; on\;\; \partial\Omega$

Thus, u(x,y) =x-y is a solution of the Dirichlet problem .

By uniqueness theorem u(x,y)=x-y is the unique solution of the given problem .

2.1.1 Solving the Dirichlet Problem on the Unit Disc

When solving the Dirichlet problem on the unit disc, we first observe that we are looking for a harmonic function that approximate the function f, from the interior from the disc. Notice that f is periodic it would be sufficient to find a harmonic function that is equivalent to the Fourier series of f on the boundary of the disc.

Definition2.1.let D⊆ $\mathbb{C}$ $and\; f: D \to \mathbb{C}$,

Then f is said to be holomorphic at $z_o \in D$ $if\; and\; only\; if\;\; f\; is\; differetiable\; in\; some$

neighbourhood of $z_o$including at z_o.

That is, there exist$\delta > 0$ such that f is differentiable on B(z_o,δ)

First we note that the Fourier series of f is equivalent to the $\lim r \to 1^- \; of$

$$u(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k + \sum_{k=-\infty}^{-1} \hat{f}(k) \bar{z}^{|k|} \; where \;\; z = re^{2\pi i\theta}$$

In order for this expression to be useful in this situation we need to know that this function is harmonic for this case, we will use the fact that every holomorphic function are harmonic. Therefore, u is the sum of a holomorphic function

$$\sum_{k=0}^{\infty} \hat{f}(k)z^k, and\ \ anti-holomorphic\ function \sum_{k=-\infty}^{-1} \hat{f}(k)\bar{z}^{|k|}$$

In the unit disc, we know that u is harmonic. With this in mind, we have essentially solved the Dirichlet problem on the unit disc. First let us combine the two sums in the definition of u,and then we have

$$u(re^{2\pi i\theta}) = \sum_{k=-\infty}^{\infty} \hat{f}(k)z^{|k|}e^{2\pi ik}$$

Then, if this converges,we have

$$\sum_{k=-\infty}^{\infty} \hat{f}(k)z^{\overset{|k|}{r}} e^{2\pi ik\theta} = \int_{-1/2}^{1/2} f(t) \sum_{k=-\infty}^{\infty} r^{|k|}e^{2\pi ik(\theta-t)}\, dx$$

Now as is often done in Fourier analysis, we try to find a closed form for this convolution operator. The notion of convolution of two functions plays a fundamental role in Fourier analysis it appears naturally in the context of Fourier series but also serves more generally in the analysis of functions in other settings so here is how it is done

$$\sum_{k=-\infty}^{\infty} r^{|k|}e^{2\pi ikt} = \sum_{k=0}^{\infty} r^{|k|}e^{2\pi ikt} + \sum_{k=1}^{\infty} r^{|k|}e^{-2\pi ikt}$$

Noting that each of these is now a geometric series, we have

$$\sum_{k=-\infty}^{\infty} r^{|k|}e^{2\pi ikt} = \frac{1}{1-re^{2\pi it}} + \frac{re^{-2\pi it}}{1-re^{-2\pi it}}$$

$$= \frac{1-re^{-2\pi it} + re^{-2\pi it}(1-re^{2\pi it})}{(1-re^{2\pi it})(1-re^{-2\pi it})}$$

$$= \frac{1-re^{-2\pi it}+re^{-2\pi it}-r^2}{1-re^{2\pi it}-re^{-2\pi it}+r^2}$$

$$= \frac{1-r^2}{1-\ r(e^{2\pi it}-\ e^{-2\pi it})+r^2}$$

$$= \frac{1-r^2}{1-2\ rcos(2\pi t)+r^2}$$

So we finally have

$$\sum_{k=-\infty}^{\infty} r^{|k|}e^{2\pi ik(t)} = \frac{1-r^2}{1-2\ rcos(2\pi t)+r^2}$$

This convolution operator is known as the Poisson kernel for the unit disc and it is denoted by p_r(t)=$\frac{1-r^2}{1-2\ rcos(2\pi t)+r^2}$

Thus,the solution to the Dirichlet problem on the unit disc is given by

$$u\left(re^{2\pi i\theta}\right) = \int_{-1/2}^{1/2} f(t)p_r(\theta - t)dt$$

2.2 TheDirichlet Problem on the Upper half Plane and Möbius Transformation

AMöbiustransformation is a function which changes the complex plane in a particular manner. Fundamentally, the change can be considered as an arrangement of rotation, inversions, dilations and translations on the complex projective area. This category of changes formally contains the function of the form $m(z) = \frac{az+b}{cz+d}$, where a, b, c, and d are complex constants and $z \neq \frac{-d}{c}$

Generalized circles (circles on the projective space) can be clearly indicated by mapping them on generalized circles under Möbius transformation. Then our goal is finding a transformation which maps the unit circle to the x-axis (ensuring that the internal part of the circle is equivalent to the upper half of the plane). So, by employing this transformation, a Dirichlet, problem on the upper half plane can be changed to one on the disc, solved it on the disc, and lastly changed it back to the upper half plane for the ultimate solution.

Which transformations should be used?

The primary step is changing a function on the upper half plane to a function on the disc. Then a transformation, m, is required which takes points in the disc to points on the upper half plane in a particular way.

The idea is that we compose $f: u \to D \; with \; m: D \; \to U \; to \; obtain \; a \; function$

$f\big(m(z)\big): D \to \mathbb{C} \; where$

D is the unit disc and U is the upper half plane. Then, so as to find a M$\ddot{o}$bius transformations which uses the unit disc to the upper half plane we employ our knowledge of the general form ofM$\ddot{o}$bius transformation as well as facts about its action on particular points

Since there are four unknowns it makes sense to use four points to define our transformation.

Transferring the points

$$z = \; -i, 1, i, -1 \; on \; the \; unit \; circle \; to \; the \; point \; z = \; -1, 0, 1, \infty$$

On the x-axis with the transformation $m(z) = \frac{az+b}{cz+d}$ $, we \; obtain$

$$a + b = 0 \; , ai + b = ci + d, -ai + b = ci - d \; and - c + d = 0,$$

So this method of equation can be solved to find out that $a = -b = -ic = -id$

It is important to analyze that the ratio in the transformation could cross out any change in the extent of the constants, any value can be selected for a and decided the other indeterminacies from that

$$for \; a = 1, we \; get \; a = 1, b = -1, c = i \; , d = i \; \; so \; m(z) = \frac{z - i}{iz + i}$$

It is also necessary to change this later to take the upper half plane back tothe disc. So, by transferring the points $z = -1{,}0{,}1, \infty \; on \; the \; x - \; axis \; to$

$$the \; points \; z = -i, 1, i, -1 \; on \; the \; unit \; circle \; with \; the \; transformation$$

$$m^{-1}(z) = \frac{dz - b}{-cz + a}\ , we\ get\ m^{-1}(z) = \frac{iz + 1}{-iz + 1}$$

2.3 Transforming the Dirichlet problem

Using transformation, we can solve the Dirichlet problem on the upper half plane. So we seek a function

$$u(z) such\ that \begin{cases} \Delta u = 0, z \in U \\ u = f, z \in \partial U \end{cases}$$

Composing the function f with our transformation, m, we have $f\ (m(z))$. This is now a boundary condition on the unit disc. Next we use our solution for the Dirichlet problem on the unit disc, to find

$$u'\left(re^{2\pi i\theta}\right) = \int_{-1/2}^{1/2} f(m(e^{2\pi it}))\ p_r(\theta - t)dt$$

So we have solved the Dirichlet problem for $f\big(m(z)\big)$ on the unit circle and we just have to use the inverse transformation we found earlier to transform it back to the upper half plane composing our function

$u' : D \to \mathbb{C}\ with\ m^{-1} : U \to D,$ We should obtain our function $u : U \to \mathbb{C}$. now we need to determine the function. After composing the functions, we have the form of the function $u(z) = u'(m^{-1}(z) = \int_?^? f(m(m^{-1}(t)p_?(?)dt$.

In order to complete this solution, we now need to determine how the bounds of the integrate and the Poison kernel were affected by composing with our inverse transformation. First we examine the bounds. The basic observation here is that the bounds on the solution to the Dirichlet problem on the disc span the entire circle, so the bounds on the solution of Dirichlet problem on the upper half plane should span the entire circle that the x-axis makes up on the projective space

Therefore, we have an integral from -∞ to ∞

Now we must consider how the Poisson kernel is affected by the composition. The problem here is that the Poisson kernel given as a function of the angle for each radius rather than as a function of the complex number itself. So we need to find a way of expressing the Poisson kernel as a function of z. Toachieve this we use the following.

$$p_r(t) = \frac{1 - r^2}{1 - 2r\cos(2\pi t) + r^2} = p(z) = \frac{1 - z\bar{z}}{1 - z - \bar{z} + z\bar{z}}$$

$$where\ z = re^{2\pi i\theta}$$

Now that we have the Poisson kernel as an actual function of z, we can just compose it with m^{-1}.So, going through a very messy calculation we obtain

$p(m^{-1})(x + iy) = \frac{y}{x^2 + y^2}$.

This function is called the Poisson kernel for the upper half plane and it is denoted by $p_y(x) = \frac{y}{x^2 + y^2}$

Thus we finally have a solution to the Dirichlet problem on the upper half plane. Filling in the question marks from our previous step we have

$$u(x + iy) = \int_{-\infty}^{\infty} f(t)p_y(x - t)dt = p_y * f(x)\ is\ our\ solution$$

Now since it would be nice to have our solution on the real plane rather than the complex plane we make one last (almost insignificant) change to

$$u(x, y) = \int_{-\infty}^{\infty} f(t)p_y(x - t)dt = p_y * f(x)$$

Conclusion

On this Thesis we have shown that,

- The solution to the Dirichlet problem on the unit disc is

$$u\left(re^{2\pi i\theta}\right) = \int_{-1/2}^{1/2} f(t)p_r(\theta - t)dt$$

- In composing the function with the inverse transformation the bounds on the solution to the Dirichet problem on the disc span the entire circle , so the bounds on the solution of Dirichlet problem on the upper half plane should span the entire circle that the x –axis makes up on the projective space.
- The solution of the Dirichlet problem on the upper half plane is

$$u(x,y) = \int_{-\infty}^{\infty} f(t)p_y(x - t)dt = p_y * f(x)$$

Bibliography

1**. Elias M. Stein andRami Shakarchi**. Fourier analysis, An introduction.PrincetonUniversity press (2003)

2. **Schaum's Outline series**. Theory and problems with application to Boundary Value problem. Mc Graw-Hill (1974)

3. **Theodore W.Gamelin**. Complex analysis. Springer- verlag New York, Inc (2001)

4. **Walter Rudin**. Real and complex Analysis Mc. Graw –Hill, Inc, (1966)