INTERACTION OF A TWO LEVEL ATOM WITH A SQUEEZED LIGHT FROM PARAMETRIC OSCILLATORS

By

Kassahun Daniel

A PROJECT SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE IN PHYSICS (QUANTUM OPTICS)
AT
ADDIS ABABA UNIVERSITY
ADDIS ABABA, ETHIOPIA
AUGUST 2017

ADDIS ABABA UNIVERSITY COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCE

DEPARTMENT OF

PHYSICS

The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Programs for acceptance a project work entitled "**INTERACTION OF A TWO LEVEL ATOM WITH A SQUEEZED LIGHT FROM PARAMETRIC OSCILLATORS** " by **Kassahun Daniel** in partial fulfillment of the requirements for the degree of **Master of Science in Physics (Quantum Optics)**.

Dated: August 2017

Advisor: ______________________
Dr. Deribe Hirpo

Examiners: ______________________
Dr. Fesseha Kassahun

____
Dr. Teshome Senbeta

Chairman: ______________________
Dr. Belayneh Mesfin

ADDIS ABABA UNIVERSITY

Date: **August 2017**

Author: **Kassahun Daniel**

Title: **INTERACTION OF A TWO LEVEL ATOM WITH A SQUEEZED LIGHT FROM PARAMETRIC OSCILLATORS**

Department: **Physics**

Degree: **M.Sc.** Convocation: **August** Year: **2017**

Permission is herewith granted to Addis Ababa University to circulate and to have copied for non-commercial purposes, at its discretion, the above title upon the request of individuals or institutions.

Signature of Author

Table of Contents

List of Figures

Abstract

In this project, we have studied the quantum properties of a light produced by a degenerate parametric oscillator that contains a two-level atom. Employing the master equation we have obtained the quantum Langevin equation and the equation of time evolution for the expectation values of the cavity mode and atomic operators. With the aid of the solutions of these equations, we have calculated the correlation properties of noise operators. Using the obtained correlation properties of noise operators and the large time approximation scheme, we determined the mean and variance of photon number, the power spectrum, second order correlation function and quadrature variance.

We have found that the variance of the photon number is greater than the mean photon number, indicating that the light produced by a two level atom has super-Poissonian photon statistics. On the other hand, the photons in the fluorescent light are antibunching. In addition, the power spectrum of the fluorescent light from a two-level atom driven by a coherent light, turns out to be a single peak. It is found that the width of the spectrum increases with ε/κ. Finally, we found that the cavity mode is in a squeezed state and the squeezing occurs in the minus quadrature.

Acknowledgements

First of all, I would like to thank the almighty, God, for letting me to accomplish this stage. Secondly I would like to express my heart felt gratitude and respect goes to my advisor and instructor Dr. Deribe Hirpo for his guidance, continuous support and contribution of valuable suggestions and friendly approach throughout this work. My heart felt gratitude also goes to Dr. Cherinet Amente for his assistance in providing Latex and installing them.

Next I express my sincere thanks to the Department of Physics of Addis Ababa University and to all my Instructors for facilitating conditions for my education and financial support, and also my appreciation goes to Ministry of Education (MOE) for giving me this chance.

Finally, I am also express my gratitude and appreciation to my father Ato Daniel Teklu and my helpmate Emebet Regasa who encourage me a supplying stationeries and moral support.

Chapter 1

Introduction

Nonlinear optics is the study of the behavior of light in nonlinear medium (i,e the medium in which the dielectric polarization responds nonlinearly). When light interacts with nonlinear crystal it induces polarization. The amount of polarization of a material depends linearly or nonlinearly on the applied electric field. Propagation of electromagnetic field in a nonlinear medium is affected by the dielectric polarization induced by the field. When electromagnetic radiation is incident on a nonlinear medium, a response with greater or less than the driving frequency can appear. The frequencies of the response in such a system are related to the driving frequency by an integer multiple and these are called harmonic responses [1]. But under certain conditions, a response with frequency less than the driving frequency can appear. The response with frequency less than the driving frequency is called sub-harmonics and the response with frequency greater than the driving frequency is called supper harmonics. Some examples of nonlinear interactions are; second harmonic generation, sum frequency generation and sub-harmonic generation. Sub harmonic generation [2-6] as well as second harmonic generation [2, 4, 5] is a typical process of leading to the production of squeezing light. In second harmonic generation a photon of frequency

ω interacts with nonlinear material and is up converted into photon with twice the frequency of the initial photon. In sum frequency generation two photons of different frequencies are combined to provide a photon of frequency equal to the sum of the two frequencies. In this project work we consider a degenerate parametric oscillator whose cavity contains a two-level atom. In an optical parametric amplifier, the high frequency is called the pump, the lower frequency of primary interest is called the signal, and the remaining frequency is called the idler [1]. The interaction of the signal light, produced by the parametric amplifier, with the two-level atom leads to the generation of fluorescent light. Thus the cavity mode in this case consists of the signal light and the fluorescent light emitted by the two-level atom. A light mode confined in a cavity, usually formed by two mirrors, is called cavity mode [2]. Also we analyze the quantum statistical properties of the fluorescent and the signal light applying the quantum Langevin equations and large time approximation scheme. This analysis can also be done using the master equation. We also derive the equations of evolution for the expectation values of atomic and cavity mode operators along with the obtained quantum Langevin equation. Applying the resulting equations, we calculate the mean and variance of the photon number, the power spectrum for the cavity mode and for the fluorescent light. We also determine the second order correlation function for the fluorescent light. Finally we calculate quadrature variance for the cavity mode and fluorescence light.

Chapter 2

Cavity Mode Dynamics

2.1 The Master Equation

We consider a single two-level atom inside a degenerate parametric oscillator coupled to a vacuum reservoir. We represent the upper and lower levels of the atom by $|a\rangle$ and $|b\rangle$, respectively. We assume the atom to be at resonance with the cavity mode. In a

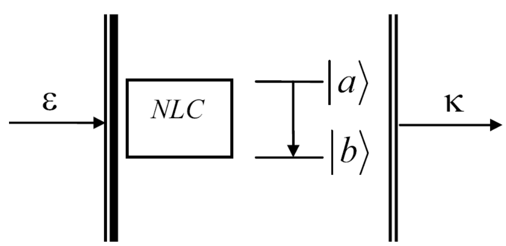

Figure 2.1: A single two level atom inside a parametric oscillator

degenerate parametric oscillator, a pump photon of frequency 2ω is down converted into a pair of highly correlated signal photons each of frequency ω [2, 3, 6, 10]. It so turns out that the signal light is in a squeezed state. With the pump mode treated

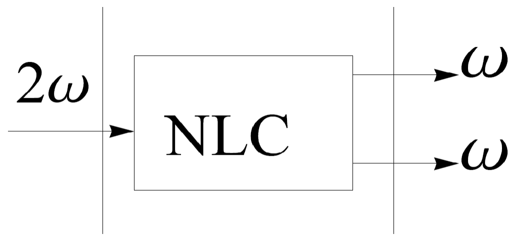

Figure 2.2: Degenerate parametric oscillator

classically, the parametric interaction can be described by the Hamiltonian

$$\hat{H}_1 = \frac{i\varepsilon}{2}(\hat{a}^{\dagger 2} - \hat{a}^2). \tag{2.1.1}$$

The interaction of the cavity mode with the two-level atom is also describable by the Hamiltonian

$$\hat{H}_2 = ig(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-). \tag{2.1.2}$$

where $\hat{a}^\dagger$ and $\hat{a}$ are respectively the creation and annihilation operators for cavity mode, ε is a real constant proportional to the amplitude of the pump mode, g is the atom-cavity mode coupling constant, and $\hat{\sigma}_+ =\mid a\rangle\langle b \mid$, $\hat{\sigma}_- =\mid b\rangle\langle a \mid$ are atomic operators satisfying the commutation relations [2]

$$[\hat{\sigma}_\pm, \hat{\sigma}_\mp] = \pm\hat{\sigma}_z, \tag{2.1.3}$$

$$[\hat{\sigma}_\pm, \hat{\sigma}_z] = \mp 2\hat{\sigma}_\pm, \tag{2.1.4}$$

$$[\hat{\sigma}_z, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm, \tag{2.1.5}$$

in which

$$\hat{\sigma}_z =\mid a\rangle\langle a \mid + \mid b\rangle\langle b \mid . \tag{2.1.6}$$

On account of Eqs. (2.1.1) and (2.1.2), the Hamiltonian describing the parametric interaction and the interaction of the cavity mode with the two-level atom has the form

$$\hat{H} = \frac{i\varepsilon}{2}(\hat{a}^{\dagger 2} - \hat{a}^2) + ig(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-). \tag{2.1.7}$$

Thus the master equation for a two-level atom and parametric oscillator in a cavity coupled to a vacuum reservoir is obtained by substituting the expression (2.1.7) into the relation [2]

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}, \hat{\rho}] + \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}), \tag{2.1.8}$$

one obtains

$$\begin{aligned} \frac{d\hat{\rho}}{dt} &= \frac{\varepsilon}{2}(\hat{a}^{\dagger 2}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger 2} - \hat{a}^2\hat{\rho} + \hat{\rho}\hat{a}^2) \\ &\quad + g(\hat{\sigma}_+\hat{a}\hat{\rho} - \hat{\rho}\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{\sigma}_-) \\ &\quad + \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}). \end{aligned} \tag{2.1.9}$$

This represents the master equation for a two level atom inside subharmonic generator coupled to vacuum reservoir and κ is a cavity damping constant.

2.2 Time evolution of atomic expectation values

In this section we seek to obtain the equation of evolution for cavity mode and atomic operators. The time evolution of the expectation value of an operator $\hat{A}$ can be written as [2]

$$\frac{d}{dt}\left\langle \hat{A}(t) \right\rangle = Tr\left(\frac{d\hat{\rho}}{dt}\hat{A}\right). \tag{2.2.1}$$

Employing the master equation along with Eq. (2.2.1) we can write

$$\begin{aligned}\frac{d}{dt}\langle\hat{\sigma}_-\rangle &= Tr\left(\frac{d\hat{\rho}}{dt}\hat{\sigma}_-\right)\\ &= \frac{\varepsilon}{2}Tr(\hat{a}^{\dagger 2}\hat{\rho}\hat{\sigma}_- - \hat{\rho}\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{a}^2\hat{\rho}\hat{\sigma}_- + \hat{\rho}\hat{a}^2\hat{\sigma}_-)\\ &\quad +gTr(\hat{\sigma}_+\hat{a}\hat{\rho}\hat{\sigma}_- - \hat{\rho}\hat{\sigma}_+\hat{a}\hat{\sigma}_- - \hat{a}^\dagger\hat{\sigma}_-\hat{\rho}\hat{\sigma}_- + \hat{\rho}\hat{a}^\dagger\hat{\sigma}_-^2)\\ &\quad +\frac{\kappa}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{\sigma}_- - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{\sigma}_- - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{\sigma}_-).\end{aligned} \tag{2.2.2}$$

Now using the cyclic property of trace operation together with the identities of commutation relation [2, 4]

$$[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}, \tag{2.2.3}$$

we see that

$$\begin{aligned}\frac{d}{dt}\langle\hat{\sigma}_-\rangle &= \frac{\varepsilon}{2}Tr(\hat{\rho}\hat{\sigma}_-\hat{a}^{\dagger 2} - \hat{\rho}\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\rho}\hat{\sigma}_-\hat{a}^2 + \hat{\rho}\hat{a}^2\hat{\sigma}_-)\\ &\quad +gTr(\hat{\rho}\hat{\sigma}_-\hat{\sigma}_+\hat{a} - \hat{\rho}\hat{\sigma}_+\hat{a}\hat{\sigma}_- - \hat{\rho}\hat{\sigma}_-\hat{a}^\dagger\hat{\sigma}_- + \hat{\rho}\hat{a}^\dagger\hat{\sigma}_-^2)\\ &\quad +\frac{\kappa}{2}Tr(2\hat{\rho}\hat{a}^\dagger\hat{\sigma}_-\hat{a} - \hat{\rho}\hat{\sigma}_-\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{\sigma}_-)\\ &= \frac{\varepsilon}{2}Tr\left(\hat{\rho}\hat{a}^\dagger[\hat{\sigma}_-,\hat{a}^\dagger] + \hat{\rho}[\hat{\sigma}_-,\hat{a}^\dagger]\hat{a}^\dagger + \hat{\rho}\hat{a}[\hat{a},\hat{\sigma}_-] + \hat{\rho}[\hat{a},\hat{\sigma}_-]\hat{a}\right)\\ &\quad +gTr\left(\hat{\rho}\hat{\sigma}_+[\hat{\sigma}_-,\hat{a}] + \hat{\rho}[\hat{\sigma}_-,\hat{\sigma}_+]\hat{a} + \hat{\rho}\hat{a}[\hat{\sigma}_-,\hat{\sigma}_-] + \hat{\rho}[\hat{a},\hat{\sigma}_-]\hat{\sigma}_-\right)\\ &\quad +\frac{\kappa}{2}Tr\left(\rho[\hat{a}^\dagger,\hat{\sigma}_-]\hat{a} + \hat{\rho}\hat{a}^\dagger[\hat{\sigma}_-,\hat{a}]\right).\end{aligned} \tag{2.2.4}$$

Employing Eq. (2.1.3) and assuming that the atomic and cavity mode operators commute

$$[\hat{\sigma}_\pm,\hat{a}^\dagger] = 0, \tag{2.2.5}$$

$$[\hat{\sigma}_\pm,\hat{a}] = 0, \tag{2.2.6}$$

$$[\hat{a}^\dagger,\hat{\sigma}_\pm] = 0, \tag{2.2.7}$$

$$[\hat{a}, \hat{\sigma}_{\pm}] = 0, \tag{2.2.8}$$

we readily find that

$$\frac{d}{dt} \langle \hat{\sigma}_{-} \rangle = -g \langle \hat{\sigma}_z \hat{a} \rangle . \tag{2.2.9}$$

Moreover, employing Eqs. (2.2.1) along with (2.1.9), we see that

$$\begin{aligned} \frac{d}{dt} \langle \hat{\sigma}_z \rangle &= \frac{\varepsilon}{2} Tr(\hat{a}^{\dagger 2} \hat{\rho} \hat{\sigma}_z - \hat{\rho} \hat{a}^{\dagger 2} \hat{\sigma}_z - \hat{a}^2 \hat{\rho} \hat{\sigma}_z + \hat{\rho} \hat{a}^2 \hat{\sigma}_z) \\ &+ g Tr(\hat{\sigma}_+ \hat{a} \hat{\rho} \hat{\sigma}_z - \hat{\rho} \hat{\sigma}_+ \hat{a} \hat{\sigma}_z - \hat{a}^\dagger \hat{\sigma}_- \hat{\rho} \hat{\sigma}_z + \hat{\rho} \hat{a}^\dagger \hat{\sigma}_- \hat{\sigma}_z) \\ &+ \frac{\kappa}{2} Tr(2 \hat{a} \hat{\rho} \hat{a}^\dagger \hat{\sigma}_z - \hat{a}^\dagger \hat{a} \hat{\rho} \hat{\sigma}_z - \hat{\rho} \hat{a}^\dagger \hat{a} \hat{\sigma}_z). \end{aligned} \tag{2.2.10}$$

Applying the cyclic properties of the trace operation together with the identities [4]

$$[\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}, \tag{2.2.11}$$

we then see that

$$\begin{aligned} \frac{d}{dt} \langle \hat{\sigma}_z \rangle &= \frac{\varepsilon}{2} Tr(\hat{\rho} \hat{\sigma}_z \hat{a}^{\dagger 2} - \hat{\rho} \hat{a}^{\dagger 2} \hat{\sigma}_z - \hat{\rho} \hat{\sigma}_z \hat{a}^2 + \hat{\rho} \hat{a}^2 \hat{\sigma}_z) \\ &+ g Tr(\hat{\rho} \hat{\sigma}_z \hat{\sigma}_+ \hat{a} - \hat{\rho} \hat{\sigma}_+ \hat{a} \hat{\sigma}_z - \hat{\rho} \hat{\sigma}_z \hat{a}^\dagger \hat{\sigma}_- + \hat{\rho} \hat{a}^\dagger \hat{\sigma}_- \hat{\sigma}_z) \\ &+ \frac{\kappa}{2} Tr(2 \hat{\rho} \hat{a}^\dagger \hat{\sigma}_z \hat{a} - \hat{\rho} \hat{\sigma}_z \hat{a}^\dagger \hat{a} - \hat{\rho} \hat{a}^\dagger \hat{a} \hat{\sigma}_z) \\ &= \frac{\varepsilon}{2} Tr \left(\hat{\rho} \hat{a}^\dagger [\hat{\sigma}_z, \hat{a}^\dagger] + \hat{\rho} [\hat{\sigma}_z, \hat{a}^\dagger] \hat{a}^\dagger - \hat{\rho} \hat{a} [\hat{\sigma}_z, \hat{a}] - \hat{\rho} [\hat{\sigma}_z, \hat{a}] \hat{a} \right) \\ &+ g Tr \left(\hat{\rho} \hat{\sigma}_+ [\hat{\sigma}_z, \hat{a}] + \hat{\rho} [\hat{\sigma}_z, \hat{\sigma}_+] \hat{a} - \hat{\rho} \hat{a}^\dagger [\hat{\sigma}_z, \hat{\sigma}_-] - \hat{\rho} [\hat{\sigma}_z, \hat{a}^\dagger] \hat{\sigma}_- \right) \\ &+ \frac{\kappa}{2} Tr \left(\hat{\rho} \hat{\sigma}_z [\hat{a}^\dagger, \hat{a}] + \hat{\rho} [\hat{a}^\dagger, \hat{\sigma}_z] \hat{a} - \hat{\rho} \hat{a}^\dagger [\hat{\sigma}_z, \hat{a}] - \hat{\rho} [\hat{a}^\dagger, \hat{a}] \hat{\sigma}_z \right) . \end{aligned} \tag{2.2.12}$$

With the aid of Eq. (2.1.5), and assuming that the atomic and cavity mode operators commute as

$$[\hat{\sigma}_z, \hat{a}] = 0, \tag{2.2.13}$$

$$[\hat{\sigma}_z, \hat{a}^\dagger] = 0, \tag{2.2.14}$$

$$[\hat{a}^\dagger, \hat{\sigma}_z] = 0, \tag{2.2.15}$$

one can easily write Eq. (2.2.14) as

$$\frac{d}{dt}\langle\hat{\sigma}_z\rangle = 2g\left(\langle\hat{\sigma}_+\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_-\rangle\right). \tag{2.2.16}$$

2.3 The quantum Langevin equation

The dynamics of cavity mode coupled to a reservoir can also be described using the quantum Langevin equation. We now seek to obtain the quantum Langevin equation for the cavity mode applying the master equation. With the aid of the relation described by Eq. (2.2.1), the expectation value of the annihilation operator for the cavity mode evolves in time as

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = Tr\left(\frac{d\hat{\rho}}{dt}\hat{a}(t)\right) \tag{2.3.1}$$

Employing the master equation expressed in Eq. (2.1.9), we can write the above relation as

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\rangle &= \frac{\varepsilon}{2}Tr(\hat{a}^{\dagger 2}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^{\dagger 2}\hat{a} - \hat{a}^2\hat{\rho}\hat{a} + \hat{\rho}\hat{a}^3)\\ &+gTr(\hat{\sigma}_+\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{\sigma}_+\hat{a}^2 - \hat{a}^\dagger\hat{\sigma}_-\hat{\rho}\hat{a} + \hat{\rho}\hat{a}^\dagger\hat{\sigma}_-\hat{a})\\ &+\frac{\kappa}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2).\end{aligned} \tag{2.3.2}$$

Applying the cyclic property of trace operation, and taking Eqs. (2.2.3) and (2.2.11) into account one can write Eq. (2.3.2) as

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\rangle &= \frac{\varepsilon}{2}Tr\left(\hat{\rho}\hat{a}^\dagger[\hat{a},\hat{a}^\dagger] + \hat{\rho}[\hat{a},\hat{a}^\dagger]\hat{a}^\dagger\right)\\ &+gTr\left(\hat{\rho}\hat{\sigma}_+[\hat{a},\hat{a}] + \hat{\rho}[\hat{a},\hat{\sigma}_+]\hat{a} - \hat{\rho}\hat{a}^\dagger[\hat{a},\hat{\sigma}_-] - \hat{\rho}[\hat{a},\hat{a}^\dagger]\hat{\sigma}_-\right)\\ &+\frac{\kappa}{2}Tr\left(\hat{\rho}\hat{a}^\dagger[\hat{a},\hat{a}] + \hat{\rho}[\hat{a}^\dagger,\hat{a}]\hat{a}\right).\end{aligned} \tag{2.3.3}$$

With the aid of Eqs. (2.2.8) and the fact that

$$[\hat{a},\hat{a}^\dagger]=1, \tag{2.3.4}$$

and

$$[\hat{a},\hat{a}]=0, \tag{2.3.5}$$

one readily obtains

$$\frac{d}{dt}\langle\hat{a}(t)\rangle=-\frac{\kappa}{2}\langle\hat{a}(t)\rangle+\varepsilon\langle\hat{a}^\dagger(t)\rangle-g\langle\hat{\sigma}_-(t)\rangle. \tag{2.3.6}$$

We now seek to obtain the evolution of expectation values of $\hat{a}^2(t)$, $\hat{a}^\dagger(t)\hat{a}(t)$ and $\hat{a}(t)\hat{a}^\dagger(t)$. With the aid of Eq. (2.2.1) along with the master equation, together with the cyclic property of trace operation one obtains

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^2(t)\rangle &= \frac{\varepsilon}{2}Tr(\hat{\rho}\hat{a}^2\hat{a}^{\dagger 2}-\hat{\rho}\hat{a}^{\dagger 2}\hat{a}^2)+gTr(\hat{\rho}\hat{a}^2\hat{\sigma}_+\hat{a}-\hat{\rho}\hat{\sigma}_+\hat{a}^3-\hat{\rho}\hat{a}^2\hat{a}^\dagger\hat{\sigma_-}+\hat{\rho}\hat{a}^\dagger\hat{\sigma_-}\hat{a}^2)\\ &+\frac{\kappa}{2}Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^3-\hat{\rho}\hat{a}^2\hat{a}^\dagger\hat{a}).\end{aligned} \tag{2.3.7}$$

Applying the property of commutation relation [3]

$$[\hat{A}\hat{B},\hat{C}\hat{D}]=\hat{A}[\hat{B},\hat{C}\hat{D}]+[\hat{A},\hat{C}\hat{D}]\hat{B}, \tag{2.3.8}$$

along with Eqs. (2.2.6), (2.2.8), (2.2.11), (2.3.4) and (2.3.5), one can write Eq. (2.3.7) as

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^2(t)\rangle &= -\kappa\langle\hat{a}^2(t)\rangle+\varepsilon\left(\langle\hat{a}(t)\hat{a}^\dagger(t)\rangle+\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle\right)\\ &-g\left(\langle\hat{a}(t)\hat{\sigma}_-(t)+\langle\hat{\sigma}_-(t)\hat{a}(t)\rangle\right).\end{aligned} \tag{2.3.9}$$

Moreover, on account of Eq. (2.2.1) along with Eq. (2.1.9), we see that

$$\begin{aligned}\frac{d}{dt}\left\langle\hat{a}^\dagger(t)\hat{a}(t)\right\rangle &= \frac{\varepsilon}{2}Tr(\hat{a}^{\dagger 2}\hat{\rho}\hat{a}^\dagger\hat{a}-\hat{\rho}\hat{a}^{\dagger 2}\hat{a}^\dagger\hat{a}-\hat{a}^2\hat{\rho}\hat{a}^\dagger\hat{a}+\hat{\rho}\hat{a}^2\hat{a}^\dagger\hat{a})\\ &+gTr(\hat{\sigma}_+\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}-\hat{\rho}\hat{\sigma}_+\hat{a}\hat{a}^\dagger\hat{a}-\hat{a}^\dagger\hat{\sigma}_-\hat{\rho}\hat{a}^\dagger\hat{a}+\hat{\rho}\hat{a}^\dagger\hat{\sigma}_-\hat{a}^\dagger\hat{a})\\ &+\frac{\kappa}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^{\dagger 2}\hat{a}-\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}-\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}),\end{aligned} \tag{2.3.10}$$

Employing the cyclic property of trace operation, we have

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle &= \frac{\varepsilon}{2}Tr(\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^{\dagger 2} - \hat{\rho}\hat{a}^{\dagger 3}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^3 + \hat{\rho}\hat{a}^2\hat{a}^\dagger\hat{a}) \\ &\quad + gTr(\hat{\rho}\hat{a}^\dagger\hat{a}\hat{\sigma}_+\hat{a} - \hat{\rho}\hat{\sigma}_+\hat{a}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\sigma}_- + \hat{\rho}\hat{a}^\dagger\hat{\sigma}_-\hat{a}^\dagger\hat{a}) \\ &\quad + \frac{\kappa}{2}Tr(2\hat{\rho}\hat{a}^{\dagger 2}\hat{a}^2 - 2\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}) \\ &= \frac{\varepsilon}{2}Tr\left(\hat{\rho}[\hat{a}^\dagger\hat{a}, \hat{a}^{\dagger 2}] - \hat{\rho}[\hat{a}^\dagger\hat{a}, \hat{a}^2]\right) \\ &\quad + gTr\left(\hat{\rho}[\hat{a}^\dagger\hat{a}, \hat{\sigma}_+\hat{a}] - \hat{\rho}[\hat{a}^\dagger\hat{a}, \hat{a}^\dagger\hat{\sigma}_-]\right) \\ &\quad + \kappa Tr\left(\hat{\rho}\hat{a}^\dagger[\hat{\rho}\hat{a}^\dagger, \hat{a}]\hat{a}\right). \end{aligned} \tag{2.3.11}$$

With the aid of Eq. (2.3.8) along with (2.2.11), (2.2.7), (2.2.8), (2.3.4), (2.3.5) and using the commutation relation

$$[\hat{a}^\dagger, \hat{a}^\dagger] = 0, \tag{2.3.12}$$

Eq. (2.3.11) can be expressed in the form

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle &= -\kappa\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle + \varepsilon(\langle\hat{a}^{\dagger 2}(t)\rangle + \langle\hat{a}^2(t)\rangle) \\ &\quad - g(\langle\hat{a}^\dagger(t)\hat{\sigma}_-(t)\rangle + \langle\hat{\sigma}_+(t)\hat{a}(t)\rangle). \end{aligned} \tag{2.3.13}$$

Furthermore, following in a similar manner one readily obtains

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\hat{a}^\dagger(t)\rangle &= -\kappa\langle\hat{a}(t)\hat{a}^\dagger(t)\rangle + \kappa + \varepsilon(\langle\hat{a}^{\dagger 2}(t)\rangle + \langle\hat{a}^2(t)\rangle) \\ &\quad - g(\langle\hat{a}(t)\hat{\sigma}_+(t)\rangle + \langle\hat{\sigma}_-(t)\hat{a}^\dagger(t)\rangle). \end{aligned} \tag{2.3.14}$$

2.4 Correlation properties of noise operator

On the basis of Eq. (2.3.6) one can write

$$\frac{d}{dt}\hat{a}(t) = -\frac{\kappa}{2}\hat{a}(t) + \varepsilon\hat{a}^\dagger(t) - g\hat{\sigma}_-(t) + \hat{F}(t). \tag{2.4.1}$$

This expression is called quantum Langevin equation of the atomic operator $\hat{a}$. Where $\hat{F}(t)$ is a noise operator associated with the vacuum reservoir and whose correlation properties remain to be determined. We note that Eq. (2.3.6) and the expectation value of the expression (2.4.1) will have the same form if

$$\langle \hat{F}(t) \rangle = 0. \tag{2.4.2}$$

Using Eq. (2.4.1) and its complex conjugate along with the relation [2]

$$\frac{d}{dt}\left\langle \hat{a}^{\dagger}(t)\hat{a}(t)\right\rangle = \left\langle \frac{d\hat{a}^{\dagger}(t)}{dt}\hat{a}(t)\right\rangle + \left\langle \hat{a}^{\dagger}(t)\frac{d}{dt}\hat{a}(t)\right\rangle, \tag{2.4.3}$$

we find

$$\begin{aligned}\frac{d}{dt}\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle &= -\kappa\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle + \varepsilon(\langle \hat{a}^{\dagger 2}(t)\rangle + \langle \hat{a}^{2}(t)\rangle) \\ &\quad -g(\langle \hat{\sigma_{+}}(t)\hat{a}(t)\rangle + \langle \hat{a}^{\dagger}(t)\hat{\sigma}_{-}(t)\rangle) \\ &\quad +\langle \hat{F}^{\dagger}(t)\hat{a}(t)\rangle + \langle \hat{a}^{\dagger}(t)\hat{F}(t)\rangle. \end{aligned} \tag{2.4.4}$$

Comparison of Eqs. (2.3.13) and (2.4.4) indicates that

$$\langle \hat{F}^{\dagger}(t)\hat{a}(t)\rangle + \langle \hat{a}^{\dagger}(t)\hat{F}(t)\rangle = 0. \tag{2.4.5}$$

A formal solution of Eq. (2.4.1) can be written as

$$\hat{a}(t) = \hat{a}(0)e^{-\kappa t/2} + \int_{0}^{t} e^{-\kappa(t-t')/2}[\varepsilon\hat{a}^{\dagger}(t') - g\hat{\sigma}_{-}(t') + \hat{F}(t')]dt'. \tag{2.4.6}$$

Multiplying this by $\hat{F}^{\dagger}(t)$ from the left side and taking the expectation value, we get

$$\begin{aligned}\langle \hat{F}^{\dagger}(t)\hat{a}(t)\rangle &= \langle \hat{F}^{\dagger}(t)\hat{a}(0)\rangle e^{-\kappa t/2} + \int_{0}^{t} e^{-\kappa(t-t')/2}[\varepsilon\langle \hat{F}^{\dagger}(t)\hat{a}^{\dagger}(t')\rangle \\ &\quad -g\langle \hat{F}^{\dagger}(t)\hat{\sigma}_{-}(t')\rangle + \langle \hat{F}^{\dagger}(t)\hat{F}(t')\rangle]dt'. \end{aligned} \tag{2.4.7}$$

Since a noise operator at a certain time should not affect a cavity mode and atomic operators at an earlier time, one can write

$$\langle \hat{F}^{\dagger}(t)\hat{a}(0)\rangle = \langle \hat{F}^{\dagger}(t)\rangle\langle \hat{a}(0)\rangle = 0. \tag{2.4.8}$$

$$\langle \hat{F}^{\dagger}(t)\hat{a}^{\dagger}(t')\rangle = \langle \hat{F}^{\dagger}(t)\rangle\langle \hat{a}^{\dagger}(t')\rangle = 0. \tag{2.4.9}$$

and

$$\langle \hat{F}^{\dagger}(t)\hat{\sigma}_{-}(t')\rangle = \langle \hat{F}^{\dagger}(t)\rangle\langle \hat{\sigma}_{-}(t')\rangle = 0. \tag{2.4.10}$$

In view of Eqs. (2.4.8), (2.4.9) and (2.4.10), we see that Eq. (2.4.7) reduces to

$$\langle \hat{F}^{\dagger}(t)\hat{a}(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle \hat{F}^{\dagger}(t)\hat{F}(t')\rangle dt'. \tag{2.4.11}$$

Taking the complex conjugate of Eq. (2.4.6) and multiplying it by $\hat{F}(t)$ from the right and taking the expectation value of the resulting expression together with the assertion that a noise force at a certain time should not affect a cavity mode and atomic operators at an earlier time, one can get

$$\langle \hat{a}^{\dagger}(t)\hat{F}(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle \hat{F}^{\dagger}(t')\hat{F}(t)\rangle dt'. \tag{2.4.12}$$

Thus taking into account Eq. (2.4.5) along with (2.4.11) and (2.4.12), we see that

$$\int_0^t e^{-\kappa(t-t')/2}\langle \hat{F}^{\dagger}(t)\hat{F}(t')\rangle dt' + \int_0^t e^{-\kappa(t-t')/2}\langle \hat{F}^{\dagger}(t')\hat{F}(t)\rangle dt' = 0. \tag{2.4.13}$$

and assuming that

$$\langle \hat{F}^{\dagger}(t)\hat{F}(t')\rangle = \langle \hat{F}^{\dagger}(t')\hat{F}(t)\rangle, \tag{2.4.14}$$

we have

$$\int_0^t e^{-\kappa(t-t')/2}\langle \hat{F}^{\dagger}(t)\hat{F}(t')\rangle dt' = 0. \tag{2.4.15}$$

On the basis of the relation [2]

$$\int_0^t e^{-a(t-t')/2}\langle\hat{F}(t)g(t')\rangle dt' = D, \tag{2.4.16}$$

we assert that

$$\langle\hat{F}(t)g(t')\rangle = 2D\delta(t-t'). \tag{2.4.17}$$

where a is constant and D is constant or some function of time t. We then see that

$$\langle\hat{F}^\dagger(t')\hat{F}(t)\rangle = \langle\hat{F}^\dagger(t)\hat{F}(t')\rangle = 0. \tag{2.4.18}$$

Moreover, employing Eq. (2.4.1) and its complex conjugate along with the relation

$$\frac{d}{dt}\left\langle\hat{a}(t)\hat{a}^\dagger(t)\right\rangle = \left\langle\frac{d\hat{a}(t)}{dt}\hat{a}^\dagger(t)\right\rangle + \left\langle\hat{a}(t)\frac{d\hat{a}^\dagger(t)}{dt}\right\rangle, \tag{2.4.19}$$

one can easily establish that

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\hat{a}^\dagger(t)\rangle &= -\kappa\langle\hat{a}(t)\hat{a}^\dagger(t)\rangle + \varepsilon(\langle\hat{a}^{\dagger 2}(t)\rangle + \langle\hat{a}^2(t)\rangle)\\ &\quad -g(\langle\hat{a}(t)\hat{\sigma}_+(t)\rangle + \langle\hat{\sigma}_-(t)\hat{a}^\dagger(t)\rangle)\\ &\quad +\langle\hat{F}(t)\hat{a}^\dagger(t)\rangle + \langle\hat{a}(t)\hat{F}^\dagger(t)\rangle. \end{aligned} \tag{2.4.20}$$

Comparison of Eqs. (2.3.14) and (2.4.20) indicates that

$$\langle\hat{F}(t)\hat{a}^\dagger(t)\rangle + \langle\hat{a}(t)\hat{F}^\dagger(t)\rangle = \kappa. \tag{2.4.21}$$

Multiplying the complex conjugate of Eq. (2.4.6) from the left by $\hat{F}(t)$ and taking the expectation value of the resulting expression together with the assertion that a noise operator at a certain time should not affect the system variables at earlier time, one obtains

$$\langle\hat{F}(t)\hat{a}^\dagger(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle\hat{F}(t)\hat{F}^\dagger(t')\rangle dt'. \tag{2.4.22}$$

similarly one easily gets

$$\langle \hat{a}(t)\hat{F}^\dagger(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle \hat{F}(t')\hat{F}^\dagger(t)\rangle dt'. \tag{2.4.23}$$

On account of Eqs. (2.4.22) and (2.4.23), we then see that Eq. (2.4.21) can be written as

$$\int_0^t e^{-\kappa(t-t')/2}\langle \hat{F}(t)\hat{F}^\dagger(t')\rangle dt' + \int_0^t e^{-\kappa(t-t')/2}\langle \hat{F}(t')\hat{F}^\dagger(t)\rangle dt' = \kappa \tag{2.4.24}$$

assuming that,

$$\langle \hat{F}(t)\hat{F}^\dagger(t')\rangle = \langle \hat{F}(t')\hat{F}^\dagger(t)\rangle, \tag{2.4.25}$$

we have

$$\int_0^t e^{-\kappa(t-t')/2}\langle \hat{F}(t)\hat{F}^\dagger(t')\rangle dt' = \frac{\kappa}{2}. \tag{2.4.26}$$

On the basis of Eqs. (2.4.16) and (2.4.17), we then see that

$$\langle \hat{F}(t)\hat{F}^\dagger(t')\rangle = \langle \hat{F}(t')\hat{F}^\dagger(t)\rangle = \kappa\delta(t-t'). \tag{2.4.27}$$

Furthermore, employing the relation

$$\frac{d}{dt}\left\langle \hat{a}^2(t)\right\rangle = \left\langle \frac{d\hat{a}(t)}{dt}\hat{a}(t)\right\rangle + \left\langle \hat{a}(t)\frac{d\hat{a}(t)}{dt}\right\rangle, \tag{2.4.28}$$

along with Eq. (2.4.1), we obtain

$$\begin{aligned}\frac{d}{dt}\langle \hat{a}^2(t)\rangle &= -\kappa\langle \hat{a}^2(t)\rangle + \varepsilon(\langle \hat{a}(t)\hat{a}^\dagger(t)\rangle + \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle)\\ &+g(\langle \hat{a}(t)\hat{\sigma}_-(t)\rangle + \langle \hat{\sigma}_-(t)\hat{a}(t)\rangle)\\ &+\langle \hat{F}(t)\hat{a}(t) + \hat{a}(t)\hat{F}(t)\rangle. \end{aligned} \tag{2.4.29}$$

Comparison of Eqs. (2.3.9) and (2.4.29) shows that

$$\langle \hat{F}(t)\hat{a}(t)\rangle + \langle \hat{a}(t)\hat{F}(t)\rangle = 0. \tag{2.4.30}$$

In view of Eq. (2.4.6) and together with the assertion that the noise forces at a certain time should not affect the system variables at earlier time, we have

$$\langle \hat{F}(t)\hat{a}(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle \hat{F}(t)\hat{F}(t')\rangle dt'. \tag{2.4.31}$$

and

$$\langle \hat{a}(t)\hat{F}(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle \hat{F}(t')\hat{F}(t)\rangle dt'. \tag{2.4.32}$$

Upon combining Eqs. (2.4.30), (2.4.31) and (2.4.32), we have

$$\int_0^t e^{-\kappa(t-t')/2}\langle \hat{F}(t)\hat{F}(t')\rangle dt' + \int_0^t e^{-\kappa(t-t')/2}\langle \hat{F}(t')\hat{F}(t)\rangle dt' = 0. \tag{2.4.33}$$

assuming that

$$\langle \hat{F}(t)\hat{F}(t')\rangle = \langle \hat{F}(t')\hat{F}(t)\rangle. \tag{2.4.34}$$

we see that

$$2\int_0^t e^{-\kappa(t-t')/2}\langle \hat{F}(t)\hat{F}(t')\rangle dt' = 0. \tag{2.4.35}$$

Now on the basis of this result, we assert that

$$\langle \hat{F}(t)\hat{F}(t')\rangle = \langle \hat{F}(t')\hat{F}(t)\rangle = 0. \tag{2.4.36}$$

Following a similar procedure, one can verify that

$$\langle \hat{F}^\dagger(t)\hat{F}^\dagger(t')\rangle = \langle \hat{F}^\dagger(t')\hat{F}^\dagger(t)\rangle = 0. \tag{2.4.37}$$

We would like to point out that the Eqs. (2.4.2), (2.4.18), (2.4.27), (2.4.36) and (2.4.37) describe the correlation properties of the noise operators $\hat{F}(t)$ and $\hat{F}^\dagger(t)$. Since Eqs. (2.2.9), (2.2.16) and (2.3.6) are nonlinear and coupled differential equations, it is not possible to obtain their exact solutions. We then seek to obtain the solutions of these equations applying the large-time approximation scheme. In the large time approximation scheme, the cavity damping constant is much greater than the cavity atomic decay rate. In this approximation scheme, the cavity mode variables decay faster than the atomic variables. We can then set the time derivatives of the cavity mode variables equal to zero while keeping the zero-order atomic and cavity mode variables at time t [2,3]. In view of this, from Eq. (2.4.1) we see that

$$\hat{a}(t) = \frac{2}{\kappa}\left(\varepsilon\hat{a}^{\dagger}(t) - g\hat{\sigma}_{-}(t) + \hat{F}(t)\right). \tag{2.4.38}$$

and

$$\hat{a}^{\dagger}(t) = \frac{2}{\kappa}\left(\varepsilon\hat{a}(t) - g\hat{\sigma}_{+}(t) + \hat{F}^{\dagger}(t)\right). \tag{2.4.39}$$

Substituting Eq. (2.4.39) into (2.4.38), we have

$$\begin{aligned}
\hat{a}(t) &= \frac{4\varepsilon}{\kappa^2}(\varepsilon\hat{a} - g\hat{\sigma}_{+} + \hat{F}^{\dagger}(t)) + \frac{2}{\kappa}(\hat{F}(t) - g\hat{\sigma}_{-}) \\
&= \frac{4\varepsilon^2}{\kappa^2}\hat{a} - \frac{4\varepsilon}{\kappa^2}(g\hat{\sigma}_{+} - \hat{F}^{\dagger}(t)) + \frac{2}{\kappa}(\hat{F}(t) - g\hat{\sigma}_{-}) \\
&= -\frac{4\varepsilon/\kappa^2}{1 - 4\varepsilon^2/\kappa^2}(g\hat{\sigma}_{+} - \hat{F}^{\dagger}(t)) + \frac{2/\kappa}{1 - 4\varepsilon^2/\kappa^2}(\hat{F}(t) - g\hat{\sigma}_{-}) \\
&= \frac{-2\kappa g}{\kappa^2 - 4\varepsilon^2}\hat{\sigma}_{-} - \frac{4g\varepsilon}{\kappa^2 - 4\varepsilon^2}\hat{\sigma}_{+} + \frac{4}{\kappa^2 - 4\varepsilon^2}[\frac{\kappa}{2}\hat{F}(t) + \varepsilon\hat{F}^{\dagger}(t)]. \qquad (2.4.40)
\end{aligned}$$

Introducing Eq. (2.4.38) into (2.4.39) one can easily obtain

$$\hat{a}^{\dagger}(t) = \frac{-2\kappa g}{\kappa^2 - 4\varepsilon^2}\hat{\sigma}_{+} - \frac{4g\varepsilon}{\kappa^2 - 4\varepsilon^2}\hat{\sigma}_{-} + \frac{4}{\kappa^2 - 4\varepsilon^2}[\frac{\kappa}{2}\hat{F}^{\dagger}(t) + \varepsilon\hat{F}(t)]. \tag{2.4.41}$$

Upon substituting Eq.(2.4.40) into (2.2.9), and the fact that

$$\hat{\sigma}_z\hat{\sigma}_{\pm} = \pm\hat{\sigma}_{\pm}, \tag{2.4.42}$$

one readily obtains

$$\begin{aligned}\frac{d}{dt}\langle\hat{\sigma}_-\rangle &= \frac{2\kappa g^2}{\kappa^2-4\varepsilon^2}\langle\hat{\sigma}_z\hat{\sigma}_-\rangle + \frac{4g^2\varepsilon/\kappa}{\kappa^2-4\varepsilon^2}\langle\hat{\sigma}_z\hat{\sigma}_+\rangle \\ &\quad -\frac{4g}{\kappa^2-4\varepsilon^2}(\frac{\kappa}{2}\langle\hat{\sigma}_z(t)\hat{F}(t)\rangle + \varepsilon\langle\hat{\sigma}_z(t)\hat{F}^\dagger(t)\rangle) \\ &= \frac{-2\kappa g^2}{\kappa^2-4\varepsilon^2}\langle\hat{\sigma}_-\rangle + \frac{4g^2\varepsilon}{\kappa^2-4\varepsilon^2}\langle\hat{\sigma}_+\rangle \\ &\quad -\frac{4g}{\kappa^2-4\varepsilon^2}(\frac{\kappa}{2}\langle\hat{\sigma}_z(t)\hat{F}(t)\rangle + \varepsilon\langle\hat{\sigma}_z(t)\hat{F}^\dagger(t)\rangle) \\ &= -\frac{2g^2/\kappa}{1-4\varepsilon^2/\kappa^2}\langle\hat{\sigma}_-\rangle + \frac{4g^2\varepsilon/\kappa^2}{1-4\varepsilon^2/\kappa^2}\langle\hat{\sigma}_+\rangle \\ &\quad -\frac{4g}{\kappa^2-4\varepsilon^2}(\frac{\kappa}{2}\langle\hat{\sigma}_z(t)\hat{F}(t)\rangle + \varepsilon\langle\hat{\sigma}_z(t)\hat{F}^\dagger(t)\rangle). \end{aligned} \tag{2.4.43}$$

Introducing Eqs. (2.4.40) and (2.4.41) into (2.2.16) along with the fact that

$$\hat{\sigma}_+^2 = \hat{\sigma}_-^2 = 0, \tag{2.4.44}$$

we obtain

$$\begin{aligned}\frac{d}{dt}\langle\hat{\sigma}_z\rangle &= -\frac{4g^2\kappa}{\kappa^2-4\varepsilon^2}\langle\hat{\sigma}_+\hat{\sigma}_-\rangle + \frac{8g}{\kappa^2-4\varepsilon^2}[\frac{\kappa}{2}\langle\hat{\sigma_+}(t)\hat{F}(t)\rangle + \varepsilon\langle\hat{\sigma}_+(t)\hat{F}^\dagger(t)] \\ &\quad -\frac{4g^2\kappa}{\kappa^2-4\varepsilon^2}\langle\hat{\sigma}_+\hat{\sigma}_-\rangle + \frac{8g}{\kappa^2-4\varepsilon^2}[\frac{\kappa}{2}\langle\hat{F}^\dagger(t)\hat{\sigma}_-(t)\rangle + \varepsilon\langle\hat{F}(t)\hat{\sigma}_-(t)] \\ &= \frac{-8g^2/\kappa}{1-4\varepsilon^2/\kappa^2}\langle\hat{\sigma}_+\hat{\sigma}_-\rangle + \frac{8g}{\kappa^2-4\varepsilon^2}[\frac{\kappa}{2}(\langle\hat{F}^\dagger(t)\hat{\sigma}_-(t)\rangle + \langle\hat{\sigma}_+(t)\hat{F}(t)\rangle) \\ &\quad +\varepsilon(\langle\hat{F}(t)\hat{\sigma}_-(t)\rangle + \langle\hat{\sigma}_+(t)\hat{F}^\dagger(t)\rangle)]. \end{aligned} \tag{2.4.45}$$

one can easily write Eqs. (2.4.43) and (2.4.45) as

$$\frac{d}{dt}\langle\hat{\sigma}_-\rangle = -\frac{\eta}{2}\langle\hat{\sigma}_-\rangle + \frac{\eta\varepsilon}{\kappa}\langle\hat{\sigma}_+\rangle - \frac{4g}{\kappa^2-4\varepsilon^2}\left(\frac{\kappa}{2}\langle\hat{\sigma}_z(t)\hat{F}(t)\rangle + \varepsilon\langle\hat{\sigma}_z(t)\hat{F}^\dagger(t)\rangle\right). \tag{2.4.46}$$

and

$$\begin{aligned}\frac{d}{dt}\langle\hat{\sigma}_z\rangle &= -2\eta\langle\hat{\sigma}_+\hat{\sigma}_-\rangle + \frac{8g}{\kappa^2-4\varepsilon^2}[\frac{\kappa}{2}(\langle\hat{F}^\dagger(t)\hat{\sigma}_-(t)\rangle \\ &\quad +\langle\hat{\sigma}_+(t)\hat{F}(t)\rangle) + \varepsilon(\langle\hat{F}(t)\hat{\sigma}_-(t)\rangle + \langle\hat{\sigma}_+(t)\hat{F}^\dagger(t)\rangle)]. \end{aligned} \tag{2.4.47}$$

where

$$\eta = \frac{4g^2/\kappa}{1 - 4\varepsilon^2/\kappa^2} \tag{2.4.48}$$

We next proceed to find the expectation value of the products involving a noise operator and atomic operator that appear in Eq. (2.4.46) and (2.4.47). Thus a formal solution of Eq. (2.4.46) can be written as

$$\begin{aligned}\langle\hat{\sigma}_-(t)\rangle &= \langle\hat{\sigma}_-(0)\rangle e^{-\eta t/2} + \int_0^t e^{-\eta(t-t')/2}[\eta\frac{\varepsilon}{\kappa}\langle\hat{\sigma}_+(t')\rangle \\ &-\frac{4g}{\kappa^2 - 4\varepsilon^2}(\frac{\kappa}{2}\langle\hat{\sigma}_z(t')\hat{F}(t')\rangle + \varepsilon\langle\hat{\sigma}_z(t')\hat{F}^\dagger(t')\rangle)]dt'. \end{aligned} \tag{2.4.49}$$

We note that Eq. (2.4.46) is a well behaved solution provided that $\eta > 0$ is positive. Thus will be the case if, $\varepsilon/\kappa < 1/2$ [3]. The quantum regression theorem states that it is possible under certain conditions to evaluate a two time correlation function employing the explicit form of a one time correlation function, obtained with the aid of some equation of evolution such as the master equation [2]. In this theorem if

$$\langle\hat{A}(t+\tau)\rangle = G(\tau)\langle\hat{A}(t)\rangle \tag{2.4.50}$$

holds, then the relation

$$\langle\hat{A}(t+\tau)\hat{B}(t)\rangle = G(\tau)\langle\hat{A}(t)\hat{B}(t)\rangle \tag{2.4.51}$$

follows [2]. Then using this theorem to Eq. (2.4.49), we obtain

$$\begin{aligned}\langle\hat{F}(t)\hat{\sigma}_-(t)\rangle &= \langle\hat{F}(t)\hat{\sigma}_-(0)\rangle e^{-\eta t/2} + \int_0^t e^{-\eta(t-t')/2}[\eta(\frac{\varepsilon}{2})\langle\hat{F}(t)\hat{\sigma}_+(t')\rangle \\ &-\frac{4g}{\kappa^2 - 4\varepsilon^2} \times (\frac{\kappa}{2}\langle\hat{F}(t)\hat{\sigma}_z(t')\hat{F}(t')\rangle \\ &+\varepsilon\langle\hat{F}(t)\hat{\sigma}_z(t')\hat{F}^\dagger(t')\rangle)]dt'. \end{aligned} \tag{2.4.52}$$

It is not possible to evaluate the integral that appears in Eq. (2.4.52) as the explicit form of $\hat{\sigma}_z(t')$ is unknown yet. In order to proceed further, we need to adopt a certain approximation scheme [2, 3]. To this end, ignoring the non-commutativity of the atomic and noise operators, we see that

$$\langle \hat{F}(t)\hat{\sigma}_z(t')\hat{F}(t')\rangle = \langle \hat{\sigma}_z(t')\hat{F}(t)\hat{F}(t')\rangle. \tag{2.4.53}$$

and

$$\langle \hat{F}(t)\hat{\sigma}_z(t')\hat{F}^\dagger(t')\rangle = \langle \hat{\sigma}_z(t')\hat{F}(t)\hat{F}^\dagger(t')\rangle. \tag{2.4.54}$$

Moreover, up on neglecting the correlation between $\hat{\sigma}_z(t')$, $\hat{F}(t)\hat{F}(t')$ and $\hat{F}(t)\hat{F}^\dagger(t')$ assumed to be considerably small, we can write the approximately valid relation

$$\langle \hat{\sigma}_z(t')\hat{F}(t)\hat{F}(t')\rangle = \langle \hat{\sigma}_z(t')\rangle\langle \hat{F}(t)\hat{F}(t')\rangle, \tag{2.4.55}$$

and

$$\langle \hat{\sigma}_z(t')\hat{F}(t)\hat{F}^\dagger(t')\rangle = \langle \hat{\sigma}_z(t')\rangle\langle \hat{F}(t)\hat{F}^\dagger(t')\rangle. \tag{2.4.56}$$

The approximation described by Eqs. (2.4.55) and (2.4.56) are referred to as the operator decoupling approximation [2, 3]. Taking Eqs. (2.4.55) and (2.4.56) into account and the fact that a noise operator $\hat{F}(t)$ at a certain time t does not affect the atomic variables at earlier time,

$$\langle \hat{F}(t)\hat{\sigma}_-(0)\rangle = \langle \hat{F}(t)\rangle\langle \hat{\sigma}_-(0)\rangle = 0. \tag{2.4.57}$$

Then Eq. (2.4.52) can be put in the form

$$\begin{aligned}\langle \hat{F}(t)\hat{\sigma}_-(t)\rangle &= -\frac{4g}{\kappa^2-4\varepsilon^2}\int_0^t e^{-\eta(t-t')/2}[\frac{\kappa}{2}\langle \hat{\sigma}_z(t')\rangle\langle \hat{F}(t)\hat{F}(t')\rangle \\ &\quad +\varepsilon\langle \hat{\sigma}_z(t')\rangle\langle \hat{F}(t)\hat{F}^\dagger(t')\rangle]dt'.\end{aligned} \tag{2.4.58}$$

Therefore, with the aid of Eqs. (2.4.27) and (2.4.36) we see that,

$$\begin{aligned}\langle \hat{F}(t)\hat{\sigma}_-(t)\rangle &= -\frac{4g\varepsilon\kappa}{\kappa^2-4\varepsilon^2}\int_0^t \langle \hat{\sigma}_z(t')\rangle e^{-\eta(t-t')/2}\delta(t-t')dt' \\ &= \frac{-2g\varepsilon/\kappa}{1-4\varepsilon^2/\kappa^2}\langle \hat{\sigma}_z(t)\rangle.\end{aligned} \tag{2.4.59}$$

Applying the quantum regression theorem to the complex conjugate of Eq. (2.4.49), one obtain

$$\begin{aligned}\langle\hat{\sigma}_+(t)\hat{F}^\dagger(t)\rangle &= \langle\hat{\sigma}_+(0)\hat{F}^\dagger(t)\rangle e^{-\eta t/2} + \int_0^t e^{-\eta(t-t')/2}[\eta(\frac{\varepsilon}{\kappa})\langle\hat{\sigma}_-(t')\hat{F}^\dagger(t)\rangle \\ &\quad -\frac{4g}{\kappa^2-4\varepsilon^2}(\frac{\kappa}{2}\langle\hat{\sigma}_z(t')\hat{F}^\dagger(t')\hat{F}^\dagger(t)\rangle \\ &\quad +\varepsilon\langle\hat{\sigma}_z(t')\hat{F}(t')\hat{F}^\dagger(t))]dt'. \qquad (2.4.60)\end{aligned}$$

On account of operator decoupling approximation, we can write

$$\langle\hat{\sigma_z}(t')\hat{F}^\dagger(t')\hat{F}^\dagger(t)\rangle = \langle\hat{\sigma_z}(t')\rangle\langle\hat{F}^\dagger(t')\hat{F}^\dagger(t)\rangle \qquad (2.4.61)$$

and

$$\langle\hat{\sigma_z}(t')\hat{F}(t')\hat{F}^\dagger(t)\rangle = \langle\hat{\sigma_z}(t')\rangle\langle\hat{F}(t')\hat{F}^\dagger(t)\rangle. \qquad (2.4.62)$$

Upon substituting Eq. (2.4.61) and (2.4.62) into Eq. (2.4.60), and the noise operator at a certain time should not affect the atomic variables at earlier time, we see that

$$\begin{aligned}\langle\hat{\sigma}_+(t)\hat{F}^\dagger(t)\rangle &= \frac{-4g}{\kappa^2-4\varepsilon^2}\int_0^t e^{-\eta(t-t')/2}[\frac{\kappa}{2}\langle\hat{\sigma}_z(t')\rangle\langle\hat{F}^\dagger(t')\hat{F}^\dagger(t)\rangle \\ &\quad +\varepsilon\langle\hat{\sigma}_z(t')\rangle\langle\hat{F}(t')\hat{F}^\dagger(t)\rangle]dt'. \qquad (2.4.63)\end{aligned}$$

Hence in view of Eqs. (2.4.27) and (2.4.37) and performing the integration we immediately notice that

$$\langle\hat{\sigma}_+(t)\hat{F}^\dagger(t)\rangle = \frac{-2g\varepsilon/\kappa}{1-4\varepsilon^2/\kappa^2}\langle\hat{\sigma}_z(t)\rangle. \qquad (2.4.64)$$

one can also be readily established that

$$\langle\hat{\sigma}_+(t)\hat{F}(t)\rangle = \langle\hat{F}^\dagger(t)\hat{\sigma}_-(t)\rangle = 0, \qquad (2.4.65)$$

and

$$\langle\hat{\sigma}_-(t)\hat{F}(t)\rangle = \langle\hat{F}^\dagger(t)\hat{\sigma}_+(t)\rangle = 0. \qquad (2.4.66)$$

Again using the quantum regression theorem to Eq. (2.4.49), and the assertion that the noise forces at a certain time should not affect a light mode variable at an earlier time and taking the operator decoupling approximation, we have

$$\begin{aligned}\langle\hat{\sigma}_-(t)\hat{F}^\dagger(t)\rangle &= \frac{-4g}{\kappa^2-4\varepsilon^2}\int_0^t e^{-\eta(t-t')/2}[\frac{\kappa}{2}\hat{\sigma}_z(t')\rangle\langle\hat{F}(t')\hat{F}^\dagger(t)\rangle \\ &\quad +\varepsilon\langle\sigma_z(t')\rangle\langle\hat{F}^\dagger(t')\hat{F}^\dagger(t)\rangle]dt'. \qquad (2.4.67)\end{aligned}$$

On account of Eqs. (2.4.27) and (2.4.37), we see that

$$\begin{aligned}\langle\hat{\sigma}_-(t)\hat{F}^\dagger(t)\rangle &= \frac{-4g}{\kappa^2-4\varepsilon^2}\langle\hat{\sigma}_z(t)\rangle(\frac{\kappa^2}{4}) \\ &= \frac{-g}{1-4\varepsilon^2/\kappa^2}\langle\hat{\sigma}_z(t)\rangle. \qquad (2.4.68)\end{aligned}$$

We also notice that

$$\langle\hat{F}(t)\hat{\sigma}_+(t)\rangle = \frac{-g}{1-4\varepsilon^2/\kappa^2}\langle\hat{\sigma}_z(t)\rangle. \qquad (2.4.69)$$

The formal solution of Eq. (2.4.47) can be written as

$$\begin{aligned}\langle\hat{\sigma}_z(t)\rangle &= \langle\hat{\sigma}_+(0)\hat{\sigma}_-(0)\rangle e^{-2\eta t} + \frac{8g}{\kappa^2-4\varepsilon^2}\int_0^t e^{-2\eta(t-t')}[\frac{\kappa}{2}(\langle\hat{F}^\dagger(t')\hat{\sigma}_-(t')\rangle + \langle\hat{\sigma}_+(t')\hat{F}(t')\rangle) \\ &\quad +\varepsilon(\langle\hat{F}(t')\hat{\sigma}_-(t')\rangle + \langle\hat{\sigma}_+(t')\hat{F}^\dagger(t')\rangle)]dt', \qquad (2.4.70)\end{aligned}$$

so that applying the quantum regression theorem to the above equation and taking into account the decoupling approximation, we obtain

$$\begin{aligned}\langle\hat{\sigma}_z(t)\hat{F}(t)\rangle &= \langle\hat{\sigma}_+(0)\hat{\sigma}_-(0)\hat{F}(t)\rangle e^{-2\eta t} \\ &\quad + \frac{8g}{\kappa^2-4\varepsilon^2}\int_0^t e^{-2\eta(t-t')}[\frac{\kappa}{2}(\langle\hat{\sigma}_-(t')\rangle\langle\hat{F}^\dagger(t')\hat{F}(t)\rangle \\ &\quad +\langle\hat{\sigma}_+(t')\rangle\langle\hat{F}(t')\hat{F}(t)\rangle) + \varepsilon(\langle\hat{\sigma}_-(t')\rangle\langle\hat{F}(t')\hat{F}(t)\rangle \\ &\quad +\langle\hat{\sigma}_+(t')\rangle\langle\hat{F}^\dagger(t')\hat{F}(t)\rangle)]dt'. \qquad (2.4.71)\end{aligned}$$

Since the noise operator at a certain time does not affect the atomic variables at an earlier time, then Eq. (2.4.71) can be written as

$$\begin{aligned}\langle\hat{\sigma}_z(t)\hat{F}(t)\rangle &= \frac{8g}{\kappa^2-4\varepsilon^2}\int_0^t e^{-2\eta(t-t')}[\langle\hat{\sigma}_-(t')\rangle\frac{\kappa}{2}(\langle\hat{F}^\dagger(t')\hat{F}(t)\rangle+\varepsilon\langle\hat{F}(t')\hat{F}(t)\rangle) \\ &\quad +\langle\hat{\sigma}_+(t')\rangle(\frac{\kappa}{2}\langle\hat{F}(t')\hat{F}(t)\rangle+\varepsilon\langle\hat{F}^\dagger(t')\hat{F}(t)\rangle)]. \qquad (2.4.72)\end{aligned}$$

With the aid of Eq.(2.4.18) and (2.4.36), we obtain

$$\langle\hat{\sigma}_z(t)\hat{F}(t)\rangle = 0. \qquad (2.4.73)$$

Again applying the quantum regression theorem to Eq. (2.4.70) and in view of the assertion that the noise operator at a certain time does not affect the atomic variables at an earlier time, we see that

$$\begin{aligned}\langle\hat{\sigma}_z(t)\hat{F}^\dagger(t)\rangle &= \frac{8g}{\kappa^2-4\varepsilon^2}\int_0^t e^{-2\eta(t-t')}[\langle\hat{\sigma}_-(t')\rangle\frac{\kappa}{2}(\langle\hat{F}^\dagger(t')\hat{F}^\dagger(t)\rangle+\varepsilon\langle\hat{F}(t')\hat{F}^\dagger(t)\rangle) \\ &\quad +\langle\hat{\sigma}_+(t')\rangle(\frac{\kappa}{2}\langle\hat{F}(t')\hat{F}^\dagger(t)\rangle+\varepsilon\langle\hat{F}^\dagger(t')\hat{F}^\dagger(t)\rangle)], \qquad (2.4.74)\end{aligned}$$

On account of Eqs. (2.4.27) and (2.4.37) and performing the integration one can write Eq. (2.4.74) as

$$\langle\hat{\sigma}_z(t)\hat{F}^\dagger(t)\rangle = \frac{4g/\kappa}{1-4\varepsilon^2/\kappa^2}\left(\frac{\kappa}{2}\langle\hat{\sigma}_+\rangle+\varepsilon\langle\hat{\sigma}_-\rangle\right). \qquad (2.4.75)$$

Now upon substituting Eqs. (2.4.73) and (2.4.75) into (2.4.43), we obtain

$$\begin{aligned}\frac{d}{dt}\langle\hat{\sigma}_-\rangle &= \frac{-2g^2/\kappa}{1-4\varepsilon^2/\kappa^2}\langle\hat{\sigma}_-\rangle+\frac{4g^2\varepsilon/\kappa^2}{1-4\varepsilon^2/\kappa^2}\langle\hat{\sigma}_+\rangle-\frac{4g}{\kappa^2-4\varepsilon^2}\times\left[\frac{4g\varepsilon/\kappa}{1-4\varepsilon^2/\kappa^2}(\frac{\kappa}{2}\langle\hat{\sigma}_+\rangle+\varepsilon\langle\hat{\sigma}_-\rangle)\right] \\ &= \frac{-2g^2/\kappa-8g^2\varepsilon^2/\kappa^3}{(1-4\varepsilon^2/\kappa^2)^2}\langle\hat{\sigma}_-\rangle-\frac{4g^2\varepsilon/\kappa^2-16g^2\varepsilon^3/\kappa^4}{(1-4\varepsilon^2/\kappa^2)^2}\langle\hat{\sigma}_+\rangle \\ &= \frac{-1}{2}\left[\frac{(4g^2/\kappa)(1+4\varepsilon^2/\kappa^2)}{(1-4\varepsilon^2/\kappa^2)^2}\right]\langle\hat{\sigma}_-\rangle-\frac{\varepsilon}{\kappa}\left[\frac{4g^2/\kappa(1+4\varepsilon^2/\kappa^2)}{(1-4\varepsilon^2/\kappa^2)^2}\right]\langle\hat{\sigma}_+\rangle \\ &= -\frac{\Gamma}{2}\langle\hat{\sigma}_-\rangle-\frac{\varepsilon\Gamma}{\kappa}\langle\hat{\sigma}_+\rangle. \qquad (2.4.76)\end{aligned}$$

where

$$\Gamma = \frac{(4g^2/\kappa)(1+4\varepsilon^2/\kappa^2)}{(1-4\varepsilon^2/\kappa^2)^2}. \tag{2.4.77}$$

is the cavity atomic decay rate. Introducing Eqs. (2.4.59), (2.4.64) and (2.4.65) into (2.4.45) along with the relation [2, 3]

$$\langle\hat{\sigma}_+\hat{\sigma}_-\rangle = \frac{\langle\hat{\sigma}_z\rangle + 1}{2}, \tag{2.4.78}$$

We obtains

$$\begin{aligned} \frac{d}{dt}\langle\hat{\sigma}_z\rangle &= \frac{-8g^2/\kappa}{1-4\varepsilon^2/\kappa^2}\left(\frac{\langle\hat{\sigma}_z\rangle+1}{2}\right) + \frac{8g/\kappa^2}{1-4\varepsilon^2/\kappa^2}\left[\frac{-4g\varepsilon^2/\kappa}{1-4\varepsilon^2/\kappa^2}\langle\hat{\sigma}_z\rangle\right] \\ &= \frac{-4g^2/\kappa - 16\varepsilon^2 g^2/\kappa^3}{(1-4\varepsilon^2/\kappa^2)^2}\langle\hat{\sigma}_z\rangle - \frac{4g^2/\kappa}{1-4\varepsilon^2/\kappa^2} \\ &= \frac{(-4g^2/\kappa)(1+4\varepsilon^2/\kappa^2)}{(1-4\varepsilon^2/\kappa^2)^2}\langle\hat{\sigma}_z\rangle - \frac{4g^2/\kappa}{1-4\varepsilon^2/\kappa^2}. \end{aligned} \tag{2.4.79}$$

With the aid of Eqs. (2.4.48) and (2.4.77), one can write Eq. (2.4.79) as

$$\frac{d}{dt}\langle\hat{\sigma}_z\rangle = -\Gamma\langle\hat{\sigma}_z\rangle - \eta. \tag{2.4.80}$$

In view of the fact that $\langle\hat{\sigma}_-(t)\rangle^* = \langle\hat{\sigma}_+(t)\rangle$ and $\langle\hat{\sigma}_z(t)\rangle^*=\langle\hat{\sigma}_z(t)\rangle$, one can write Eq. (2.4.76) as

$$\frac{d}{dt}\langle\hat{\sigma}_+\rangle = \frac{-\Gamma}{2}\langle\hat{\sigma}_+\rangle - \frac{\varepsilon}{\kappa}\Gamma\langle\hat{\sigma}_-\rangle, \tag{2.4.81}$$

where

$$\gamma_c = \frac{4g^2}{\kappa}. \tag{2.4.82}$$

The parametric defined by Eq. (2.4.82) is called the stimulated emission decay constant. Thus, we can write the atomic decay rate expressed in Eq. (2.4.77) as

$$\Gamma = \frac{\gamma_c(1+\frac{4\varepsilon^2}{\kappa^2})}{(1-\frac{4\varepsilon^2}{\kappa^2})^2}. \tag{2.4.83}$$

It can be easily seen that the presence of the parametric amplifier enhance the cavity atomic decay rate.

Chapter 3

Photon Statistics

The photon statistics of a light generated by a two-level atom is described by the mean and variance of the photon number. It would be crucial to classify the photon statistics of a light modes based on the relation between the mean and variance of the photon number. Thus the photon statistics of a light mode for which $(\triangle n)^2 = \overline{n}$ is referred to as Poissonian and the photon statistics of a light mode for which $(\triangle n)^2 > \overline{n}$ is called Supper-Poissonian. Otherwise the photon statistics is said to be Sub-Poissonian [2]. To this end, we calculate the mean and variance of the photon number, the power spectrum and the second order correlation function.

3.1 Mean photon number

In this section employing Eq. (2.3.13), we proceed to calculate the mean photon number for a cavity mode and fluorescent light. First we find the second order cavity mode variables in Eq. (2.3.13). Now multiplying Eq. (2.4.40) from the right by $\hat{\sigma}_-(t)$

and taking the expectation value, we get

$$\begin{aligned}\langle\hat{a}(t)\hat{\sigma}_-(t)\rangle &= \frac{-2\kappa g}{\kappa^2-4\varepsilon^2}\langle\hat{\sigma}_-(t)\hat{\sigma}_-(t)\rangle - \frac{4g\varepsilon}{\kappa^2-4\varepsilon^2}\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle \\ &+\frac{4}{\kappa^2-4\varepsilon^2}\left(\frac{\kappa}{2}\langle\hat{F}(t)\hat{\sigma}_-(t)\rangle + \varepsilon\langle\hat{F}^\dagger(t)\hat{\sigma}_-(t)\rangle\right). \end{aligned} \tag{3.1.1}$$

On account of Eqs. (2.4.44), (2.4.59) and (2.4.65) then Eq. (3.1.1) can be written as

$$\begin{aligned}\langle\hat{a}(t)\hat{\sigma}_-(t)\rangle &= \frac{-4g\varepsilon/\kappa^2}{1-4\varepsilon^2/\kappa^2}\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle + \frac{4/\kappa^2}{1-4\varepsilon^2/\kappa^2}\left(\frac{\kappa}{2}(\frac{-2g\varepsilon/\kappa}{1-4\varepsilon^2/\kappa^2})\langle\hat{\sigma}_z(t)\rangle\right) \\ &= \frac{-4g\varepsilon/\kappa^2}{1-4\varepsilon^2/\kappa^2}\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle - \frac{4g\varepsilon/\kappa^2}{(1-4\varepsilon^2/\kappa^2)^2}\langle\hat{\sigma}_z(t)\rangle. \end{aligned} \tag{3.1.2}$$

Again multiplying Eq. (2.4.40) from the left by $\hat{\sigma}_-(t)$ and taking the expectation value together, we obtain

$$\begin{aligned}\langle\hat{\sigma}_-(t)\hat{a}(t)\rangle &= \frac{-2\kappa g}{\kappa^2-4\varepsilon^2}\langle\hat{\sigma}_-(t)\hat{\sigma}_-(t)\rangle - \frac{4g\varepsilon}{\kappa^2-4\varepsilon^2}\langle\hat{\sigma}_-(t)\hat{\sigma}_+(t)\rangle \\ &+\frac{4}{\kappa^2-4\varepsilon^2}\left(\frac{\kappa}{2}\langle\hat{\sigma}_-(t)\hat{F}(t)\rangle + \varepsilon\langle\hat{\sigma}_-(t)\hat{F}^\dagger(t)\rangle\right), \end{aligned} \tag{3.1.3}$$

Employing Eqs. (2.4.44), (2.4.66) and (2.4.68) one can write Eq. (3.1.3) as

$$\langle\hat{\sigma}_-(t)\hat{a}(t)\rangle = \frac{-4g\varepsilon/\kappa^2}{1-4\varepsilon^2/\kappa^2}\langle\hat{\sigma}_-(t)\hat{\sigma}_+(t)\rangle - \frac{4g\varepsilon/\kappa^2}{(1-4\varepsilon^2/\kappa^2)^2}\langle\hat{\sigma}_z(t)\rangle. \tag{3.1.4}$$

Following a similar procedure one obtain

$$\langle\hat{\sigma}_+(t)\hat{a}(t)\rangle = \frac{-2g/\kappa}{1-4\varepsilon^2/\kappa^2}\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle - \frac{8g\varepsilon^2/\kappa^3}{(1-4\varepsilon^2/\kappa^2)^2}\langle\hat{\sigma}_z(t)\rangle, \tag{3.1.5}$$

and

$$\langle\hat{a}^\dagger(t)\hat{\sigma}_-(t)\rangle = \frac{-2g/\kappa}{1-4\varepsilon^2/\kappa^2}\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle - \frac{8g\varepsilon^2/\kappa^3}{(1-4\varepsilon^2/\kappa^2)^2}\langle\hat{\sigma}_z(t)\rangle. \tag{3.1.6}$$

Upon substituting Eqs. (3.1.2) and (3.1.4) into (2.3.9), there follows

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^2(t)\rangle &= -\kappa\langle\hat{a}^2(t)\rangle + 2\varepsilon\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle + \varepsilon \\ &+\frac{4g^2\varepsilon/\kappa^2}{1-4\varepsilon^2/\kappa^2}(\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle + \langle\hat{\sigma}_-(t)\hat{\sigma}_+(t)\rangle) \\ &+\frac{8g^2\varepsilon/\kappa^2}{(1-4\varepsilon^2/\kappa^2)^2}\langle\hat{\sigma}_z(t)\rangle. \end{aligned} \tag{3.1.7}$$

Employing the relation

$$\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle = \rho_{aa}, \tag{3.1.8}$$

$$\langle\hat{\sigma}_-(t)\hat{\sigma}_+(t)\rangle = \rho_{bb}, \tag{3.1.9}$$

and

$$\rho_{aa} + \rho_{bb} = 1. \tag{3.1.10}$$

One can easily write Eq. (3.1.7) as

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^2(t)\rangle &= -\kappa\langle\hat{a}^2(t)\rangle + 2\varepsilon\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle + \varepsilon + \frac{4g^2\varepsilon/\kappa^2}{1-4\varepsilon^2/\kappa^2} \\ &\quad + \frac{8g^2\varepsilon/\kappa^2}{(1-4\varepsilon^2/\kappa^2)^2}\langle\hat{\sigma}_z(t)\rangle.\end{aligned} \tag{3.1.11}$$

On account of Eq. (2.4.82) we see that the expression (3.1.11) can be written as

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^2(t)\rangle &= -\kappa\langle\hat{a}^2(t)\rangle + 2\varepsilon\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle + \varepsilon + \frac{\gamma_c\varepsilon/\kappa}{1-4\varepsilon^2/\kappa^2} \\ &\quad + \frac{2\gamma_c\varepsilon/\kappa}{(1-4\varepsilon^2/\kappa^2)^2}\langle\hat{\sigma}_z(t)\rangle.\end{aligned} \tag{3.1.12}$$

At a steady state this can be written as

$$\langle\hat{a}^2(t)\rangle_{ss} = \frac{2\varepsilon}{\kappa}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle_{ss} + \frac{\varepsilon}{\kappa} + \frac{\gamma_c\varepsilon/\kappa^2}{1-4\varepsilon^2/\kappa^2} + \frac{2\gamma_c\varepsilon/\kappa^3}{1-4\varepsilon^2/\kappa^2}\langle\hat{\sigma}_z(t)\rangle_{ss}. \tag{3.1.13}$$

one can also easily obtain that

$$\langle\hat{a}^{\dagger 2}(t)\rangle_{ss} = \frac{2\varepsilon}{\kappa}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle_{ss} + \frac{\varepsilon}{\kappa} + \frac{\gamma_c\varepsilon/\kappa^2}{1-4\varepsilon^2/\kappa^2} + \frac{2\gamma_c\varepsilon/\kappa^3}{1-4\varepsilon^2/\kappa^2}\langle\hat{\sigma}_z(t)\rangle_{ss}. \tag{3.1.14}$$

where 'ss' is steady state and it can be defined as a solution after all transients have died out [6]. But the steady state expectation value of $\hat{\sigma}_z$ can be obtained from Eq. (2.4.80). Taking the formal solution of this expression and up on performing the integration we see that

$$\begin{aligned}\langle\hat{\sigma}_z(t)\rangle &= \langle\hat{\sigma}_z(0)\rangle e^{-\Gamma t} - \eta e^{-\Gamma t}\int_0^t e^{\Gamma t'}dt' \\ &= \langle\hat{\sigma}_z(0)\rangle e^{-\Gamma t} - \frac{\eta}{\Gamma}e^{-\Gamma t}(e^{\Gamma t}-1) \\ &= \langle\hat{\sigma}_z(0)\rangle e^{-\Gamma t} - \frac{\eta}{\Gamma}(1-e^{-\Gamma t}).\end{aligned} \tag{3.1.15}$$

One can write the the above equation at a steady state as

$$\langle\hat{\sigma}_z(t)\rangle_{ss} = -\frac{\eta}{\Gamma}. \tag{3.1.16}$$

It can also be shown that

$$(\langle\hat{\sigma}_z(t)\rangle_{ss}+1)/2 = \frac{\Gamma-\eta}{2\Gamma}. \tag{3.1.17}$$

In view of the relation [2, 3]

$$\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle_{ss} = \frac{(\langle\hat{\sigma}_z(t)\rangle_{ss}+1)}{2}, \tag{3.1.18}$$

we see that

$$\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle_{ss} = \frac{\Gamma-\eta}{2\Gamma}. \tag{3.1.19}$$

In which from Eq. (2.4.48) and (2.4.77) follows that

$$\frac{\eta}{\Gamma} = \frac{1-4\varepsilon^2/\kappa^2}{1+4\varepsilon^2/\kappa^2}, \tag{3.1.20}$$

and

$$\frac{\Gamma-\eta}{2\Gamma} = \frac{4\varepsilon^2/\kappa^2}{1+4\varepsilon^2/\kappa^2}. \tag{3.1.21}$$

Now substituting Eqs. (3.1.15) along with (3.1.20) into Eq. (3.1.13), we see that

$$\langle\hat{a}^2(t)\rangle_{ss} = \frac{2\varepsilon}{\kappa}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle_{ss} + \frac{\varepsilon}{\kappa} + \frac{\gamma_c\varepsilon/\kappa^2}{1-4\varepsilon^2/\kappa^2} - \frac{2\gamma_c\varepsilon/\kappa^2}{(1+4\varepsilon^2/\kappa^2)(1-4\varepsilon^2/\kappa^2)}. \tag{3.1.22}$$

In the same manner we have

$$\langle\hat{a}^{\dagger 2}(t)\rangle_{ss} = \frac{2}{\kappa}\varepsilon\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle_{ss} + \frac{\varepsilon}{\kappa} + \frac{\gamma_c\varepsilon/\kappa^2}{1-4\varepsilon^2/\kappa^2} - \frac{2\gamma_c\varepsilon/\kappa^2}{(1+4\varepsilon^2/\kappa^2)(1-4\varepsilon^2/\kappa^2)}. \quad (3.1.23)$$

Furthermore, substituting Eqs. (3.1.5), (3.1.6) and (2.4.82) into Eq. (2.3.13) we obtain

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle &= -\kappa\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle + \varepsilon(\langle\hat{a}^{\dagger 2}(t)\rangle + \langle\hat{a}^2(t)\rangle) \\ &\quad + \frac{\gamma_c/\kappa}{1-4\varepsilon^2/\kappa^2}\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle + \frac{4\gamma_c\varepsilon^2/\kappa^3}{(1-4\varepsilon^2/\kappa^2)^2}\langle\hat{\sigma}_z(t)\rangle. \quad (3.1.24)\end{aligned}$$

At a steady state one can write Eq. (3.1.24) as

$$\begin{aligned}\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle_{ss} &= \frac{\varepsilon}{\kappa}(\langle\hat{a}^{\dagger 2}(t)\rangle_{ss} + \langle\hat{a}^2(t)\rangle_{ss}) + \frac{\gamma_c/\kappa}{1-4\varepsilon^2/\kappa^2}\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle_{ss} \\ &\quad + \frac{4\gamma_c\varepsilon^2/\kappa^3}{(1-4\varepsilon^2/\kappa^2)^2}\langle\hat{\sigma}_z\rangle_{ss}. \quad (3.1.25)\end{aligned}$$

With aid of Eqs. (3.1.16) along with (3.1.20), and (3.1.19) along with (3.1.21) into Eq. (3.1.25) we see that

$$\begin{aligned}\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle_{ss} &= \frac{\varepsilon}{\kappa}(\langle\hat{a}^{\dagger 2}(t)\rangle_{ss} + \langle\hat{a}^2(t)\rangle_{ss}) + \frac{\gamma_c/\kappa}{1-4\varepsilon^2/\kappa^2}(\frac{4\varepsilon^2/\kappa^2}{1+4\varepsilon^2/\kappa^2}) \\ &\quad - \frac{4\gamma_c\varepsilon^2/\kappa^3}{(1-4\varepsilon^2/\kappa^2)^2}(\frac{1-4\varepsilon^2/\kappa^2}{1+4\varepsilon^2/\kappa^2}) \\ &= \frac{\varepsilon}{\kappa}(\langle\hat{a}^{\dagger 2}(t)\rangle_{ss} + \langle\hat{a}^2(t)\rangle_{ss}) + \frac{4\gamma_c\varepsilon^2/\kappa^3}{(1-4\varepsilon^2/\kappa^2)(1+4\varepsilon^2/\kappa^2)} \\ &\quad - \frac{4\gamma_c\varepsilon^2/\kappa^3}{(1-4\varepsilon^2/\kappa^2)}1+4\varepsilon^2/\kappa^2 \\ &= \frac{\varepsilon}{\kappa}(\langle\hat{a}^{\dagger 2}(t)\rangle_{ss} + \langle\hat{a}^2(t)\rangle_{ss}). \quad (3.1.26)\end{aligned}$$

Finally, inserting Eqs. (3.1.22) and (3.1.23) into Eq. (3.1.26), the mean photon number of cavity mode is found at a steady state to be

$$\begin{aligned}\langle\hat{a}^\dagger\hat{a}\rangle_{ss}=\bar{n} &= \frac{4\varepsilon^2}{\kappa^2}\langle\hat{a}^\dagger\hat{a}\rangle_{ss}+\frac{2\varepsilon^2}{\kappa^2}+\frac{2\gamma_c\varepsilon^2/\kappa^2}{1-4\varepsilon^2/\kappa^2}-\frac{4\gamma_c\varepsilon^2/\kappa^3}{(1-4\varepsilon^2/\kappa^2)^2(1+4\varepsilon^2/\kappa^2)}\\ &= \frac{2\varepsilon^2/\kappa^2}{1-4\varepsilon^2/\kappa^2}-\frac{4\gamma_c\varepsilon^2/\kappa^3}{(1-4\varepsilon^2/\kappa^2)^2(1+4\varepsilon^2/\kappa^2)}\\ &\quad+\frac{(\gamma_c/2\kappa)(4\varepsilon^2/\kappa^2)}{(1-4\varepsilon^2/\kappa^2)^2}. \end{aligned} \tag{3.1.27}$$

From Eq.(3.1.27), we see that the first term represents the mean photon number of the signal light in the absence of the two-level atom ($\gamma_c = 0$), the second term corresponds to the mean number of absorbed signal photons and the last term represents the mean number of photons emitted by the two-level atom. Upon adding the last two terms in Eq. (3.1.27) we see that

$$\begin{aligned}\bar{n} &= \frac{2\varepsilon^2/\kappa^2}{1-4\varepsilon^2/\kappa^2}-\frac{2\gamma_c\varepsilon^2/\kappa^3+8\gamma_c\varepsilon^4/\kappa^5}{(1-4\varepsilon^2/\kappa^2)^2(1+4\varepsilon^2/\kappa^2)}\\ &= \frac{2\varepsilon^2/\kappa^2}{1-4\varepsilon^2/\kappa^2}-\frac{2\gamma_c\varepsilon^2/\kappa^3}{(1-4\varepsilon^2/\kappa^2)(1+4\varepsilon^2/\kappa^2)}. \end{aligned} \tag{3.1.28}$$

Since the sum of the last two terms in Eq. (3.1.27) is negative, we conclude that the mean number of photons absorbed by the two-level atom is greater than the mean number of photons emitted by the atom. From Eq. (3.1.27) we note that the mean photon number of the light emitted by the two-level atom, also called fluorescent light, is

$$\bar{n}_f = \frac{2\gamma_c\varepsilon^2/\kappa^3}{(1-4\varepsilon^2/\kappa^2)^2}. \tag{3.1.29}$$

3.2 Variance of the photon number

The variance of the photon number for the cavity light can be expressed as

$$\begin{aligned}(\Delta n)^2 &= \langle \hat{n}^2\rangle - \langle \hat{n}\rangle^2 \\ &= \langle (\hat{a}^\dagger \hat{a})^2\rangle - \langle \hat{a}^\dagger \hat{a}\rangle^2 \\ &= \langle (\hat{a}^\dagger \hat{a})(\hat{a}^\dagger \hat{a})\rangle - \langle \hat{a}^\dagger \hat{a}\rangle^2 \\ &= \langle \hat{a}^\dagger \hat{a}\rangle\langle \hat{a}^\dagger \hat{a}\rangle + \langle \hat{a}^\dagger \hat{a}^\dagger\rangle\langle \hat{a}\hat{a}\rangle + \langle \hat{a}^\dagger \hat{a}\rangle\langle \hat{a}\hat{a}^\dagger\rangle - \langle \hat{a}^\dagger \hat{a}\rangle^2 \\ &= \langle \hat{a}^\dagger \hat{a}\rangle(1 + \langle \hat{a}^\dagger \hat{a}\rangle) + \langle \hat{a}^{\dagger 2}\rangle\langle \hat{a}^2\rangle \\ &= \bar{n}(1+\bar{n}) + \langle \hat{a}^{\dagger 2}\rangle\langle \hat{a}^2\rangle. \end{aligned} \tag{3.2.1}$$

From Eq. (3.1.22) and (3.1.23) we note that

$$\langle \hat{a}^{\dagger 2}\rangle = \langle \hat{a}^2\rangle. \tag{3.2.2}$$

Therefore, Eq. (3.2.1) can also be written as

$$(\Delta n)^2 = \bar{n}(1+\bar{n}) + \langle \hat{a}^2\rangle^2. \tag{3.2.3}$$

From (3.2.3) we observe that the variance of the photon number is greater than the mean photon number and hence the cavity light exhibits supper-Poissonian photon statistics.

3.3 Power spectrum

3.3.1 Power spectrum of the fluorescent light

The power spectrum of the fluorescent light can be expressed as [2, 3]

$$S'(\omega) = \frac{1}{\pi} Re \int_0^\infty \langle \hat{\sigma}_+(t)\hat{\sigma}_-(t+\tau)\rangle_{ss} e^{i\omega\tau} d\tau. \tag{3.3.1}$$

Now we seek to obtain the steady state expectation value $\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t+\tau)\rangle$ expressed in Eq. (3.3.1). Introducing new variables defined by

$$z_\pm = \langle\hat{\sigma}_-(t)\rangle \pm \langle\hat{\sigma}_+(t)\rangle, \tag{3.3.2}$$

from Eq. (2.4.76) and (2.4.81) one obtains

$$\begin{aligned}\frac{d}{dt}(\langle\hat{\sigma}_-(t)\rangle + \langle\hat{\sigma}_+(t)\rangle) &= -\frac{\Gamma}{2}\langle\hat{\sigma}_-(t)\rangle - \frac{\varepsilon\Gamma}{\kappa}\langle\hat{\sigma}_+(t)\rangle - \frac{\Gamma}{2}\langle\hat{\sigma}_+(t)\rangle - \frac{\varepsilon\Gamma}{\kappa}\langle\hat{\sigma}_-(t)\rangle \\ &= -\frac{\Gamma}{2}(\langle\hat{\sigma}_-(t)\rangle + \langle\hat{\sigma}_+(t)\rangle) - \frac{\varepsilon\Gamma}{\kappa}(\langle\hat{\sigma}_-(t)\rangle + \langle\hat{\sigma}_+(t)\rangle) \\ &= -\Gamma(\frac{1}{2} + \frac{\varepsilon}{\kappa})(\langle\hat{\sigma}_-(t)\rangle + \langle\hat{\sigma}_+(t)\rangle) \end{aligned} \tag{3.3.3}$$

In view of Eq. (3.3.2), we see that the expression of Eq. (3.3.3) can be written as

$$\begin{aligned}\frac{d}{dt}z_+ &= -z_+\Gamma(\frac{1}{2} + \frac{\varepsilon}{\kappa}) \\ &= -\lambda_+ z_+, \end{aligned} \tag{3.3.4}$$

In the same procedure one can easily write

$$\frac{d}{dt}z_- = -\lambda_- z_-. \tag{3.3.5}$$

Combining Eqs. (3.3.4) and (3.3.5) we see that

$$\frac{d}{dt}z_\pm = -\lambda_\pm z_\pm, \tag{3.3.6}$$

where

$$\lambda_\pm = \Gamma(\frac{1}{2} \pm \frac{\varepsilon}{\kappa}). \tag{3.3.7}$$

The formal solution of Eq. (3.3.6) can be written as

$$z_\pm(t+\tau) = z_\pm(t)e^{-\lambda_\pm\tau}. \tag{3.3.8}$$

From this expression we see that

$$z_+(t+\tau) + z_-(t+\tau) = z_+(t)e^{-\lambda_+\tau} + z_-(t)e^{-\lambda_-\tau} \tag{3.3.9}$$

In view of Eq. (3.3.2) one can write Eq. (3.3.9) as

$$\begin{aligned} \langle\hat{\sigma}_-(t+\tau)\rangle &= \frac{1}{2}(\langle\hat{\sigma}_-(t)\rangle + \langle\hat{\sigma}_+(t)\rangle)e^{-\lambda_+\tau} + \frac{1}{2}(\langle\hat{\sigma}_-(t)\rangle + \langle\hat{\sigma}_+(t)\rangle)e^{-\lambda_-\tau} \\ &= \frac{1}{2}\langle\hat{\sigma}_-(t)\rangle(e^{-\lambda_+\tau} + e^{-\lambda_-\tau}) + \frac{1}{2}\langle\hat{\sigma}_+(t)\rangle(e^{-\lambda_+\tau} + e^{-\lambda_-\tau}). \end{aligned} \tag{3.3.10}$$

Applying the quantum regression theorem to Eq. (3.3.10) and taking into account Eq. (2.4.44), one obtains

$$\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t+\tau)\rangle = \frac{1}{2}\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle(e^{-\lambda_+\tau} + e^{-\lambda_-\tau}). \tag{3.3.11}$$

Now substituting Eq. (3.1.19) into (3.3.11), we obtain

$$\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t+\tau)\rangle_{ss} = \frac{\Gamma-\eta}{4\Gamma}(e^{-\lambda_+\tau} + e^{-\lambda_-\tau}). \tag{3.3.12}$$

On account of Eq. (3.3.12) the power spectrum specified by Eq. (3.3.1) takes the form

$$\begin{aligned} S'(\omega) &= \left(\frac{\Gamma-\eta}{4\pi\Gamma}\right) Re\int_0^\infty (e^{-(\lambda_+-i\omega)\tau} + e^{-(\lambda_--i\omega)\tau})d\tau \\ &= \left(\frac{\Gamma-\eta}{4\pi\Gamma}\right)\left[Re\int_0^\infty e^{-(\lambda_+-i\omega)\tau}d\tau + Re\int_0^\infty e^{-(\lambda_--i\omega)\tau}d\tau\right] \\ &= \frac{\Gamma-\eta}{4\pi\Gamma}\left(\frac{\lambda_-}{{\lambda_-}^2+\omega^2} + \frac{\lambda_+}{{\lambda_+}^2+\omega^2}\right). \end{aligned} \tag{3.3.13}$$

The normalized power spectrum can be written as

$$S(\omega) = NS'(\omega). \tag{3.3.14}$$

where N is normalization constant, which can be determined from the relation

$$\int_0^\infty NS'(\omega)d\omega = 1. \tag{3.3.15}$$

With the aid of Eq. (3.3.13) one can write Eq. (3.3.15) as

$$\frac{N}{\pi}\left(\frac{\Gamma-\eta}{4\Gamma}\right)\left[\int_0^{\infty}\frac{\lambda_-}{\omega^2+{\lambda_-}^2}d\omega+\int_0^{\infty}\frac{\lambda_+}{\omega^2+{\lambda_+}^2}d\omega\right]=1, \tag{3.3.16}$$

Carrying out the integration using the relation [2]

$$\int_0^{\infty}\frac{d\omega}{(\omega-\omega_o)^2+\lambda^2}=\frac{\pi}{\lambda}, \tag{3.3.17}$$

we find that

$$\begin{aligned}\frac{N}{\pi}\left(\frac{\Gamma-\eta}{4\Gamma}\right)\left(\frac{\pi\lambda_-}{\lambda_-}+\frac{\pi\lambda_+}{\lambda_+}\right)&=1\\ N&=\frac{2\Gamma}{\Gamma-\eta}.\end{aligned} \tag{3.3.18}$$

Therefore, with the aid of Eqs. (3.3.13) and (3.3.18), the normalized power spectrum of Eq. (3.3.14) is expressible as

$$\begin{aligned}S(\omega)&=\left(\frac{2\Gamma}{\Gamma-\eta}\right)\left(\frac{\Gamma-\eta}{4\pi\Gamma}\right)\left(\frac{\lambda_+}{{\lambda_+}^2+\omega^2}+\frac{\lambda_-}{{\lambda_-}^2+\omega^2}\right)\\ &=\frac{1}{2\pi}\left(\frac{\lambda_+}{{\lambda_+}^2+\omega^2}+\frac{\lambda_-}{{\lambda_-}^2+\omega^2}\right).\end{aligned} \tag{3.3.19}$$

Now in view of Eq. (3.3.7), we see that

$$S(\omega)=\frac{\Gamma\left(\frac{1}{2}+\frac{\varepsilon}{\kappa}\right)/2\pi}{\Gamma^2\left(\frac{1}{2}+\frac{\varepsilon}{\kappa}\right)^2+\omega^2}+\frac{\Gamma\left(\frac{1}{2}-\frac{\varepsilon}{\kappa}\right)/2\pi}{\Gamma^2\left(\frac{1}{2}-\frac{\varepsilon}{\kappa}\right)^2+\omega^2}. \tag{3.3.20}$$

We observe that the expression of Eq. (3.3.20) indicates that the power spectrum of the fluorescent light is the sum of two Lorentzians centered at zero frequency and having half width of $\Gamma\left(\frac{1}{2}+\frac{\varepsilon}{\kappa}\right)$ and $\Gamma\left(\frac{1}{2}-\frac{\varepsilon}{\kappa}\right)$. Thus Fig. 3.1, shows that the power spectrum of the fluorescent light is a single peak centered at $\omega=0$. We have found that the half width at half maximum of the power spectrum increases from 0.00165 to 0.0187 as ε/κ increases from 0.25 to 0.35 for the solid line. The power spectrum in this case turns out to be a single peak.

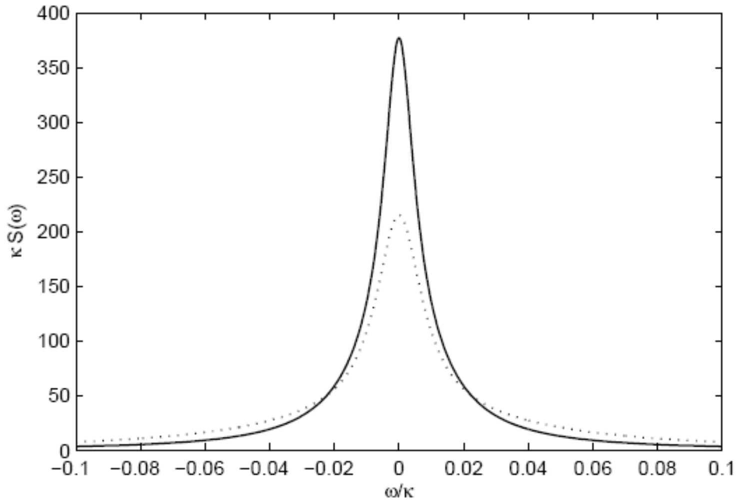

Figure 3.1: Plot of the power spectrum of the fluorescent light [Eq. (3.3.20)] versus ω/κ for $\gamma_c = 0.01$, for $\varepsilon/\kappa = 0.25$ (solid curve) and for $\varepsilon/\kappa = 0.35$(dotted curve)

3.3.2 Power spectrum of the cavity mode

The power spectrum of the cavity mode can be expressed as [3, 7]

$$S'(\omega) = \frac{1}{\pi} Re \int_0^\infty \langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss} e^{i\omega\tau} d\tau. \tag{3.3.21}$$

Now we seek to find the steady state expectations value $\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle$ in Eq. (3.3.21). Taking Eq. (2.3.5) along with its complex conjugate one can write

$$\begin{aligned} \frac{d}{dt}(\langle\hat{a}\rangle + \langle\hat{a}^\dagger\rangle) &= -\frac{\kappa}{2}\left(\langle\hat{a}\rangle + \langle\hat{a}^\dagger\rangle\right) + \varepsilon(\langle\hat{a}\rangle + \langle\hat{a}^\dagger\rangle) - g(\langle\hat{\sigma}_-\rangle + \langle\hat{\sigma}_+\rangle) \\ &= -\kappa(\frac{1}{2} - \frac{\varepsilon}{\kappa})(\langle\hat{a}\rangle + \langle\hat{a}^\dagger\rangle) - g(\langle\hat{\sigma}_-\rangle + \langle\hat{\sigma}_+\rangle) \end{aligned} \tag{3.3.22}$$

In view of Eq. (3.3.2) we see that

$$\frac{d}{dt}(\langle\hat{a}\rangle + \langle\hat{a}^\dagger\rangle) = -\kappa(\frac{1}{2} - \frac{\varepsilon}{\kappa})(\langle\hat{a}\rangle + \langle\hat{a}^\dagger\rangle) - gz_+. \tag{3.3.23}$$

In a similar manner one can also obtain

$$\frac{d}{dt}(\langle\hat{a}\rangle - \langle\hat{a}^\dagger\rangle) = -\kappa(\frac{1}{2} + \frac{\varepsilon}{\kappa})(\langle\hat{a}\rangle - \langle\hat{a}^\dagger\rangle) - gz_-. \tag{3.3.24}$$

We can also put (3.3.23) and (3.3.24) as

$$\frac{d}{dt}\alpha_+ = -\mu_-\alpha_+ - gz_+, \tag{3.3.25}$$

and

$$\frac{d}{dt}\alpha_- = -\mu_+\alpha_- - gz_-, \tag{3.3.26}$$

in which

$$\mu_\mp = \kappa(\frac{1}{2} \mp \frac{\varepsilon}{\kappa}), \tag{3.3.27}$$

and

$$\alpha_\pm = \langle\hat{a}\rangle \pm \langle\hat{a}^\dagger\rangle. \tag{3.3.28}$$

From Eq. (3.3.25) and (3.3.26), we have

$$\frac{d}{dt}\alpha_\pm = -\mu_\mp\alpha_\pm - gz_\pm, \tag{3.3.29}$$

A formal solution of Eq. (3.3.29) can be written as

$$\alpha_\pm(t+\tau) = \alpha_\pm(t)e^{-\mu_\mp\tau} - ge^{-\mu_\mp\tau}\int_0^\tau e^{\mu_\mp\tau'}z_\pm(t+\tau')d\tau'. \tag{3.3.30}$$

In view of Eq. (3.3.8), we see that

$$\begin{aligned}\alpha_\pm(t+\tau) &= \alpha_\pm(t)e^{-\mu_\mp\tau} - ge^{-\mu_\mp\tau}\int_0^\tau e^{\mu_\mp\tau'}z_\pm(t)e^{-\lambda_\pm\tau'}d\tau' \\ &= \alpha_\pm(t)e^{-\mu_\mp\tau} - gz_\pm(t)e^{-\mu_\mp\tau}\int_0^\tau e^{-(\lambda_\pm-\mu_\mp)\tau'}, d\tau' \end{aligned} \tag{3.3.31}$$

Upon performing the integration, we obtain

$$\begin{aligned}\alpha_\pm(t+\tau) &= \alpha_\pm(t)e^{-\mu_\mp\tau} + \frac{gz_\pm(t)e^{-\mu_\mp\tau}}{\lambda_\pm - \mu_\mp}[e^{-(\lambda_\pm-\mu_\mp)\tau} - 1] \\ &= \alpha_\pm(t)e^{-\mu_\mp\tau} + \frac{gz_\pm(t)}{\lambda_\pm - \mu_\mp}(e^{-\lambda_\pm\tau} - e^{-\mu_\mp\tau}), \end{aligned} \tag{3.3.32}$$

On account of Eqs. (3.3.2) and (3.3.28), the above expression can be written as

$$
\begin{aligned}
\langle \hat{a}(t+\tau)\rangle &= \frac{1}{2}(\langle \hat{a}(t)\rangle \pm \langle \hat{a}^\dagger(t)\rangle)e^{-\mu_\mp \tau} + \frac{g(\langle \hat{\sigma}_-(t)\rangle \pm \langle \hat{\sigma}_+(t)\rangle)}{2(\lambda_\pm - \mu_\mp)}(e^{-\lambda_\pm \tau} - e^{-\mu_\mp \tau}) \\
&= \frac{1}{2}(\langle \hat{a}(t)\rangle + \langle \hat{a}^\dagger(t)\rangle)e^{-\mu_- \tau} + \frac{1}{2}(\langle \hat{a}(t)\rangle - \langle \hat{a}^\dagger(t)\rangle)e^{-\mu_+ \tau} \\
&\quad + \frac{g(\langle \hat{\sigma}_-(t)\rangle + \langle \hat{\sigma}_+(t)\rangle)}{2(\lambda_+ - \mu_-)}(e^{-\lambda_+ \tau} - e^{-\mu_- \tau}) \\
&\quad + \frac{g(\langle \hat{\sigma}_-(t)\rangle - \langle \hat{\sigma}_+(t)\rangle)}{2(\lambda_- - \mu_+)}(e^{-\lambda_- \tau} - e^{-\mu_+ \tau}). \qquad (3.3.33)
\end{aligned}
$$

Applying the quantum regression theorem to Eq. (3.3.33) we see that

$$
\begin{aligned}
\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss} &= \frac{1}{2}\left[\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle_{ss} + \langle \hat{a}^{\dagger 2}(t)\rangle_{ss}\right]e^{-\mu_- \tau} + \frac{1}{2}\left[\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle_{ss} - \langle a^{\dagger 2}(t)\rangle_{ss}\right]e^{-\mu_+ \tau} \\
&\quad + \frac{g\left[\langle \hat{a}^\dagger(t)\hat{\sigma}_-(t)\rangle_{ss} + \langle \hat{a}^\dagger(t)\hat{\sigma}_+(t)\rangle_{ss}\right]}{2(\lambda_+ - \mu_-)}\left[e^{-\lambda_+ \tau} - e^{-\mu_- \tau}\right] \\
&\quad + \frac{g\left[\langle \hat{a}^\dagger(t)\hat{\sigma}_-(t)\rangle_{ss} - \langle \hat{a}^\dagger(t)\hat{\sigma}_+(t)\rangle_{ss}\right]}{2(\lambda_- - \mu_+)}\left[e^{-\lambda_- \tau} - e^{-\mu_+ \tau}\right] \\
&= \frac{1}{2}\left[\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle_{ss} + \langle \hat{a}^{\dagger 2}(t)\rangle_{ss} - \frac{g(\langle \hat{a}^\dagger(t)\hat{\sigma}_-(t)\rangle_{ss} + \langle \hat{a}^\dagger(t)\hat{\sigma}_+(t)\rangle_{ss})}{\lambda_+ - \mu_-}\right]e^{-\mu_- \tau} \\
&\quad + \frac{1}{2}\left[\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle_{ss} - \langle \hat{a}^{\dagger 2}(t)\rangle_{ss} - \frac{g(\langle \hat{a}^\dagger(t)\hat{\sigma}_-(t)\rangle_{ss} - \langle \hat{a}^\dagger(t)\hat{\sigma}_+(t)\rangle_{ss})}{\lambda_- - \mu_+}\right]e^{-\mu_+ \tau} \\
&\quad + \frac{g\left[\langle \hat{a}^\dagger(t)\hat{\sigma}_-(t)\rangle_{ss} + \langle \hat{a}^\dagger(t)\hat{\sigma}_+(t)\rangle_{ss}\right]}{2(\lambda_+ - \mu_-)}e^{-\lambda_+ \tau} \\
&\quad + \frac{g\left[\langle \hat{a}^\dagger(t)\hat{\sigma}_-(t)\rangle_{ss} - \langle \hat{a}^\dagger(t)\hat{\sigma}_+(t)\rangle_{ss}\right]}{2(\lambda_- - \mu_+)}e^{-\lambda_- \tau} \\
&= N_1 e^{-\mu_- \tau} + N_2 e^{-\mu_+ \tau} + N_3 e^{-\lambda_+ \tau} + N_4 e^{-\lambda_- \tau}, \qquad (3.3.34)
\end{aligned}
$$

where,

$$N_1 = \frac{1}{2}\left[(\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle_{ss} + \langle \hat{a}^{\dagger 2}(t)\rangle_{ss}) - \frac{g(\langle \hat{a}^\dagger(t)\hat{\sigma}_-(t)\rangle_{ss} + \langle \hat{a}^\dagger(t)\hat{\sigma}_+(t)\rangle_{ss})}{\lambda_+ - \mu_-}\right], \qquad (3.3.35)$$

$$N_2 = \frac{1}{2}\left[(\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle_{ss} - \langle \hat{a}^{\dagger 2}(t)\rangle_{ss}) - \frac{g(\langle \hat{a}^\dagger(t)\hat{\sigma}_-(t)\rangle_{ss} - \langle \hat{a}^\dagger(t)\hat{\sigma}_+(t)\rangle_{ss})}{\lambda_- - \mu_+}\right], \qquad (3.3.36)$$

$$N_3 = \frac{g\left[\langle \hat{a}^\dagger(t)\hat{\sigma}_-(t)\rangle_{ss} + \langle \hat{a}^\dagger(t)\hat{\sigma}_+(t)\rangle_{ss}\right]}{2(\lambda_+ - \mu_-)}, \qquad (3.3.37)$$

and

$$N_4 = \frac{g\left[\langle\hat{a}^\dagger(t)\hat{\sigma}_-(t)\rangle_{ss} - \langle\hat{a}^\dagger(t)\hat{\sigma}_+(t)\rangle_{ss}\right]}{2(\lambda_- - \mu_+)}. \tag{3.3.38}$$

In view of Eq. (3.1.6) along with (3.1.16) and (3.1.19) at a steady state as

$$\begin{aligned}\langle\hat{a}^\dagger(t)\hat{\sigma}_-(t)\rangle_{ss} &= \frac{-2g/\kappa}{1-4\varepsilon^2/\kappa^2}(\frac{\Gamma-\eta}{2\Gamma}) + \frac{8g\varepsilon^2/\kappa^3}{(1-4\varepsilon^2/\kappa^2)^2}(\frac{\eta}{\Gamma})\\ &= \frac{-2g/\kappa}{1-4\varepsilon^2/\kappa^2}(\frac{4\varepsilon^2/\kappa^2}{1+4\varepsilon^2/\kappa^2}) + \frac{8g\varepsilon^2/\kappa^3}{(1-4\varepsilon^2/\kappa^2)^2}(\frac{1-4\varepsilon^2/\kappa^2}{1+4\varepsilon^2/\kappa^2})\\ &= 0.\end{aligned} \tag{3.3.39}$$

Similarly one can verify that

$$\langle\hat{a}^\dagger(t)\hat{\sigma}_-(t)\rangle_{ss} = 0. \tag{3.3.40}$$

Therefore, employing Eqs. (3.3.39) and (3.3.40), we can rewrite Eqs. (3.3.35)-(3.3.38) as follows

$$N_1 = \frac{1}{2}[\bar{n} + \langle\hat{a}^{\dagger 2}(t)\rangle_{ss}], \tag{3.3.41}$$

$$N_2 = \frac{1}{2}[\bar{n} - \langle\hat{a}^{\dagger 2}(t)\rangle_{ss}], \tag{3.3.42}$$

and

$$N_3 = N_4 = 0. \tag{3.3.43}$$

Now using Eq. (3.3.41) - (3.3.43) into (3.3.34), we obtain

$$\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss} = \frac{1}{2}[\bar{n} + \langle\hat{a}^{\dagger 2}(t)\rangle_{ss}]e^{-\mu_-\tau} + \frac{1}{2}[\bar{n} - \langle\hat{a}^{\dagger 2}(t)\rangle_{ss}]e^{-\mu_+\tau}. \tag{3.3.44}$$

Inserting Eq. (3.3.44) into Eq. (3.3.21) and performing the integration, we see that

$$S'(\omega) = \frac{1}{2\pi}[\bar{n} + \langle\hat{a}^{\dagger 2}(t)\rangle_{ss}]\frac{\mu_-}{\mu_-^2+\omega^2} + \frac{1}{2\pi}[\bar{n} - \langle\hat{a}^{\dagger 2}(t)\rangle_{ss}]\frac{\mu_+}{\mu_+^2+\omega^2}. \tag{3.3.45}$$

Upon using this result into Eq. (3.3.15) and performing the integration along with Eq. (3.3.17), the normalized constant N can be put in the form

$$N = \frac{1}{\bar{n}}. \tag{3.3.46}$$

On account of Eqs. (3.3.45) and (3.3.46) into (3.3.14), the normalized power spectrum can be written as

$$S(\omega) = \frac{1}{\bar{n}}\left[\frac{\bar{n}}{2\pi}(\frac{\mu_-}{\mu_-^2+\omega^2} + \frac{\mu_+}{\mu_+^2+\omega^2}) + \frac{\langle\hat{a}^{\dagger 2}(t)\rangle_{ss}}{2\pi}(\frac{\mu_-}{\mu_-^2+\omega^2} - \frac{\mu_+}{\mu_+^2+\omega^2})\right], \quad (3.3.47)$$

In view of Eq. (3.1.26), we have

$$S(\omega) = \frac{\mu_-/2\pi}{\mu_-^2+\omega^2} + \frac{\mu_+/2\pi}{\mu_+^2+\omega^2} + \frac{\langle\hat{a}^{\dagger 2}(t)\rangle_{ss}}{\frac{2\varepsilon}{\kappa}\langle\hat{a}^{\dagger 2}(t)\rangle_{ss}}(\frac{\mu_-/2\pi}{\mu_-^2+\omega^2} - \frac{\mu_+/2\pi}{\mu_+^2+\omega^2}), \quad (3.3.48)$$

Finally, inserting Eq. (3.3.27) into (3.3.48), we then see that

$$\begin{aligned} S(\omega) &= \frac{\kappa(\frac{1}{2}+\frac{\varepsilon}{\kappa})/2\pi}{\kappa^2(\frac{1}{2}+\frac{\varepsilon}{\kappa})^2+\omega^2} + \frac{\kappa(\frac{1}{2}-\frac{\varepsilon}{\kappa})/2\pi}{\kappa^2(\frac{1}{2}-\frac{\varepsilon}{\kappa})^2+\omega^2} \\ &\quad + \frac{\kappa(\frac{\kappa}{2\varepsilon}-1)/4\pi}{\kappa^2(\frac{1}{2}-\frac{\varepsilon}{\kappa})^2+\omega^2} - \frac{\kappa(\frac{\kappa}{2\varepsilon}+1)/4\pi}{\kappa^2(\frac{1}{2}+\frac{\varepsilon}{\kappa})^2+\omega^2}. \quad (3.3.49) \end{aligned}$$

is the power spectrum of the cavity mode. Since the expression for the spectrum does not contain γ_c, the presence of the two-level atom does not affect the width of this spectrum. In fig 3.2, we plot the power spectrum of the cavity mode versus ω/κ for different values of ε/κ. These plots show that the width of the power spectrum increases with ε/κ increases. When the value of ε/κ increases from 0.25 to 0.35, the half width increases from 0.0072 to 0.0108.

3.4 Second order correlation function

The second order correlation function for the light emitted by a two level atom in a cavity in terms of the atomic operators is expressible by [2]

$$g^{(2)}(\tau) = \frac{\langle\hat{\sigma}_+(t)\hat{\sigma}_+(t+\tau)\hat{\sigma}_-(t+\tau)\hat{\sigma}_-(t)\rangle}{\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle^2}. \quad (3.4.1)$$

Now, we seek to obtain the expression $\langle\hat{\sigma}_+(t+\tau)\hat{\sigma}_-(t+\tau)\rangle$. We note that

$$\langle\hat{\sigma}_+(t+\tau)\hat{\sigma}_-(t+\tau)\rangle = \frac{(\langle\hat{\sigma}_z(t+\tau)\rangle+1)}{2}, \quad (3.4.2)$$

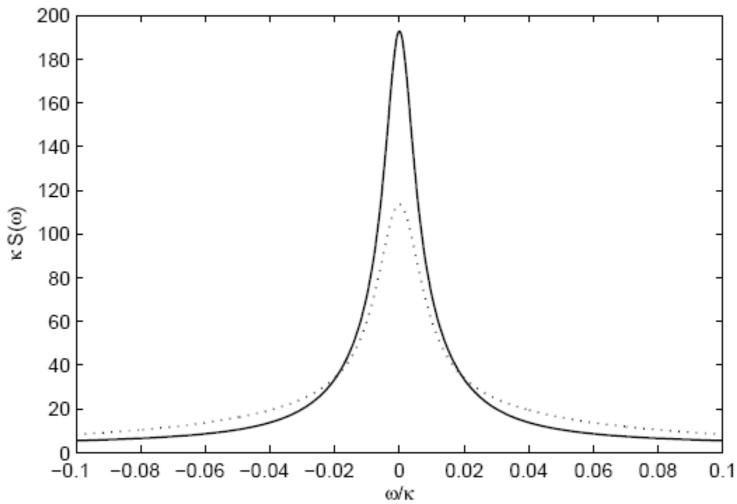

Figure 3.2: Plot of the power spectrum of the cavity mode [Eq. (3.3.49)] versus ω/κ for $\varepsilon/\kappa = 0.25$ (solid curve) and for $\varepsilon/\kappa = 0.35$ (dotted curve)

Upon substituting Eq. (3.1.15) into Eq. (3.4.2), one obtain

$$\begin{aligned}
\langle\hat{\sigma}_+(t+\tau)\hat{\sigma}_-(t+\tau)\rangle &= \frac{\langle\hat{\sigma}_z(t)\rangle e^{-\Gamma\tau} - \frac{\eta}{\Gamma}(1-e^{-\Gamma\tau}) + 1}{2} \\
&= \frac{1}{2}\langle\hat{\sigma}_z(t)\rangle e^{-\Gamma\tau} - \frac{\eta}{2\Gamma} + \frac{\eta}{2\Gamma}e^{-\Gamma\tau} + \frac{1}{2} \\
&= \frac{1}{2}\langle\hat{\sigma}_z(t)\rangle e^{-\Gamma\tau} + \frac{1}{2}e^{-\Gamma\tau} + \frac{1}{2} - \frac{1}{2}e^{-\Gamma\tau} - \frac{\eta}{2\Gamma} + \frac{\eta}{2\Gamma}e^{-\Gamma\tau} \\
&= \frac{1}{2}(\langle\hat{\sigma}_z(t)\rangle + 1)e^{-\Gamma\tau} + \frac{1}{2}(1-e^{-\Gamma\tau}) - \frac{\eta}{2\Gamma}(1-e^{-\Gamma\tau}) \\
&= \frac{1}{2}(\langle\hat{\sigma}_z(t)\rangle + 1)e^{-\Gamma\tau} + \frac{\Gamma-\eta}{2\Gamma}(1-e^{-\Gamma\tau}), \qquad (3.4.3)
\end{aligned}$$

In view of Eq. (2.4.78), one can write the above expression as

$$\langle\hat{\sigma}_+(t+\tau)\hat{\sigma}_-(t+\tau)\rangle = \langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle e^{-\Gamma\tau} + \frac{\Gamma-\eta}{2\Gamma}(1-e^{-\Gamma\tau}). \qquad (3.4.4)$$

Applying the quantum regression theorem to Eq. (3.4.4), one can write as

$$\begin{aligned}
\langle\hat{\sigma}_+(t)\hat{\sigma}_+(t+\tau)\hat{\sigma}_-(t+\tau)\hat{\sigma}_-(t)\rangle &= \langle\hat{\sigma}_+^2(t)\hat{\sigma}_-^2(t)\rangle e^{-\Gamma\tau} + \frac{\Gamma-\eta}{2\Gamma}\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle(1-e^{-\Gamma\tau}) \\
&= \frac{\Gamma-\eta}{2\Gamma}\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle(1-e^{-\Gamma\tau}). \qquad (3.4.5)
\end{aligned}$$

Combination of Eq. (3.4.1) and (3.4.5), yields

$$\begin{aligned} g^{(2)}(\tau) &= \frac{\frac{\Gamma-\eta}{2\Gamma}\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle(1-e^{-\Gamma\tau})}{\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle^2} \\ &= \frac{\frac{\Gamma-\eta}{2\Gamma}(1-e^{\Gamma\tau})}{\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle}. \end{aligned} \tag{3.4.6}$$

Thus in view of Eq. (3.1.19), the steady state second order correlation function takes the form

$$g^{(2)}(\tau) = 1 - e^{-\Gamma\tau}. \tag{3.4.7}$$

We observe that

$$g^{(2)}(0) = 0 \tag{3.4.8}$$

For $\tau > 0$,

$$g^{(2)}(\tau) > 0. \tag{3.4.9}$$

Therefore, for $\tau > 0$,

$$g^{(2)}(\tau) > g^{(2)}(0). \tag{3.4.10}$$

This shows that the Fluorescent light thus exhibits the phenomenon of photon antibunching. This is due to the fact that a two-level atom cannot emit two or more photons simultaneously. After each emission the atom returns to the lower level and it must absorb a photon before another emission can take place. That is the photons have a tendency to arrive at a detector separately rather than in pair. Fig. 3.3, indicates that for relatively small values of τ the second-order correlation function is less than unity which reflects the nonclassical feature of antibunching. We also observe that as ε/κ increases $g^{(2)}(t)$ approaches unity at a faster rate. It is also interesting to consider the dynamics of the two level atom. Thus upon replacing τ by t and t by 0 in Eq. (3.4.4), we see that

$$\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle = \langle\hat{\sigma}_+(0)\hat{\sigma}_-(0)\rangle e^{-\Gamma t} + \frac{\Gamma-\eta}{2\Gamma}(1-e^{\Gamma t}). \tag{3.4.11}$$

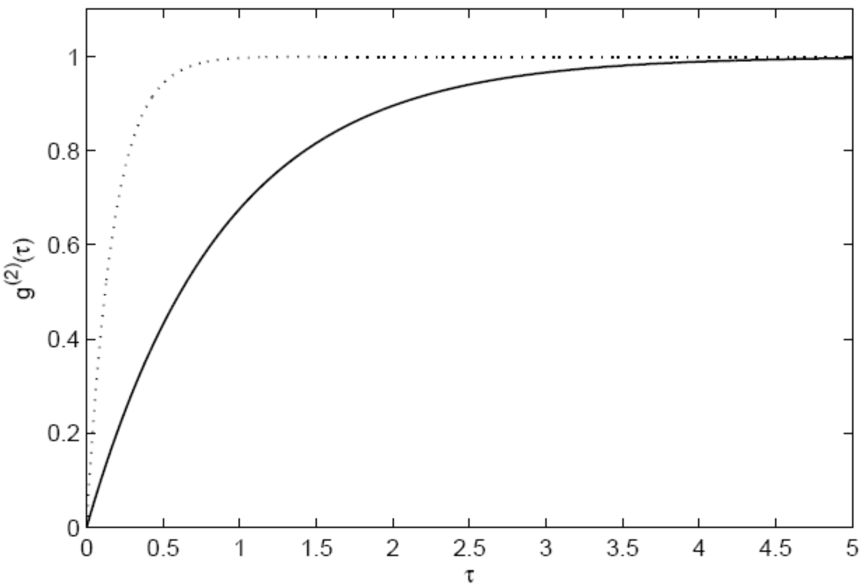

Figure 3.3: Plot of the second order correlation function [Eq. (3.4.7)] versus τ for $\gamma_c/\kappa = 0.01$, for $\varepsilon/\kappa = 0.25$ (solid curve) and for $\varepsilon/\kappa = 0.35$ (dotted curve)

Using the relation

$$\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle = \rho_{aa}(t) = \frac{\hat{\sigma}_z(t)+1}{2}, \tag{3.4.12}$$

together with Eq. (3.1.21) the probability for the two level atom in the upper level is found to be

$$\rho_{aa}(t) = \rho_{aa}(0)e^{-\Gamma t} + \frac{4\varepsilon^2/\kappa^2}{1+4\varepsilon^2/\kappa^2}(1-e^{-\Gamma t}). \tag{3.4.13}$$

If the atom is initially in the upper level, then $\rho_{aa}(0) = 1$. Hence Eq. (3.4.13) takes for this case the form

$$\begin{aligned}\rho_{aa}(t) &= e^{-\Gamma t} + \frac{4\varepsilon^2/\kappa^2}{1+4\varepsilon^2/\kappa^2}(1-e^{-\Gamma t}) \\ &= \frac{e^{-\Gamma t}}{1+4\varepsilon^2/\kappa^2} + \frac{4\varepsilon^2/\kappa^2}{1+4\varepsilon^2/\kappa^2}.\end{aligned} \tag{3.4.14}$$

At a steady state solution one can write Eq. (3.4.14) as

$$\rho_{aa} = \frac{4\varepsilon^2/\kappa^2}{1 + 4\varepsilon^2/\kappa^2}. \tag{3.4.15}$$

We see from Fig.3.4, that the probability for the atom to be in the upper level decays

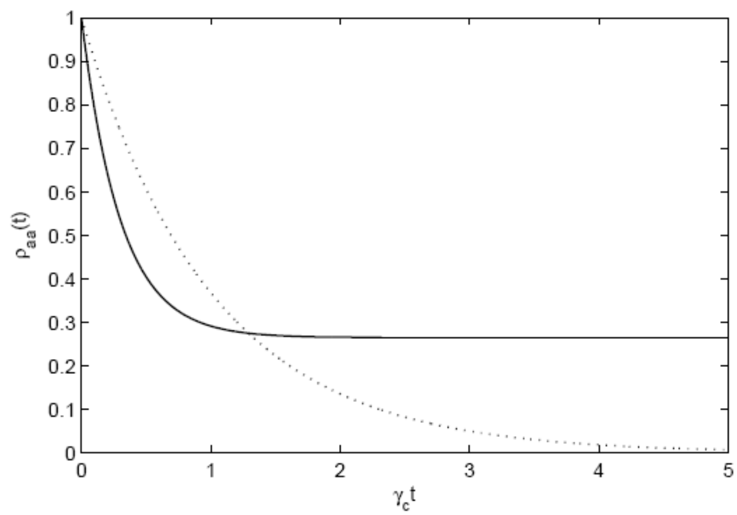

Figure 3.4: Plot of [Eq. (3.4.14)] versus $\gamma_c t$ in the presence of parametric amplifier with $\varepsilon/\kappa = 0.3$ (solid curve) and in the absence of the parametric amplifier, i.e for $\varepsilon = 0$ (dotted curve)

exponentially in the absence of the parametric amplifier and approaches to zero at steady state. However, in the presence of the parametric amplifier the steady state probability for the atom to be in the upper level is different from zero. This is because there are photons in the cavity that can be absorbed by the atom.

Chapter 4

The Quadrature Squeezing

In this section we seek to study the squeezing properties of light produced by degenerate sub-harmonic generation. To this end, we evaluate the quadrature variance for the cavity mode and for the fluorescent light.

4.1 Quadrature variance for the cavity mode

The squeezing properties of a single mode light are described by two quadrature operators defined by [2,7]

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a} \tag{4.1.1}$$

and

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}) \tag{4.1.2}$$

The operators are Hermitian and satisfy the commutation relation

$$[\hat{a}_+, \hat{a}_-] = 2i. \tag{4.1.3}$$

On the basis of Eq. (4.1.3) the uncertainty relation for $\triangle a_+$ and $\triangle a_-$ is

$$\begin{aligned} \Delta a_+ \Delta a_- &\geq \frac{1}{2} |\langle [\hat{a}_+, \hat{a}_-] \rangle| \\ &\geq 1. \end{aligned} \tag{4.1.4}$$

The operators $\hat{a}_+$ and $\hat{a}_-$ represents physical quantities called the plus and minus quadrature respectively. A single mode light is said to be in a squeezed state if either $\Delta a_+ < 1$ or $\Delta a_- < 1$. Such that $\Delta a_+ \Delta a_- \geq 1$. The plus quadrature variance for a light beam in terms of the plus quadrature operator can be expressed as

$$(\Delta a_+)^2 = \langle \hat{a}_+^2 \rangle - \langle \hat{a}_+ \rangle^2, \tag{4.1.5}$$

Hence on account of Eq. (4.1.1) one can verify Eq. (4.1.5) as

$$\begin{aligned} (\Delta a_+)^2 &= \langle (\hat{a}^\dagger + \hat{a})(\hat{a}^\dagger + \hat{a}) \rangle - \langle (\hat{a}^\dagger + \hat{a}) \rangle^2 \\ &= \langle \hat{a}^{\dagger 2} \rangle + \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a} \hat{a}^\dagger \rangle + \langle \hat{a}^2 \rangle \\ &\quad - \left(\langle \hat{a}^\dagger \rangle^2 + \langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle + \langle \hat{a} \rangle \langle \hat{a}^\dagger \rangle + \langle \hat{a} \rangle^2 \right), \end{aligned} \tag{4.1.6}$$

In view of Eq. (3.3.10), we can write

$$\langle \hat{\sigma}_-(t) \rangle = \frac{1}{2} \langle \hat{\sigma}_-(0) \rangle (e^{-\lambda_+ t} + e^{-\lambda_- t}) + \frac{1}{2} \langle \hat{\sigma}_+(0) \rangle (e^{-\lambda_+ t} + e^{-\lambda_- t}), \tag{4.1.7}$$

we then see that

$$\langle \hat{\sigma}_-(0) \rangle = 0. \tag{4.1.8}$$

Similarly one obtain

$$\langle \hat{\sigma}_+(0) \rangle = 0. \tag{4.1.9}$$

It then follows that

$$\langle \hat{\sigma}_-(t) \rangle = \langle \hat{\sigma}_+(t) \rangle = 0. \tag{4.1.10}$$

Taking these results into account, we notice from Eqs. (2.4.40) and (2.4.41) that

$$\langle \hat{a}(t) \rangle = 0, \tag{4.1.11}$$

and

$$\langle \hat{a}^\dagger(t) \rangle = 0. \tag{4.1.12}$$

Thus one can write Eq. (4.1.6) as

$$(\Delta a_+)^2 = \langle \hat{a}^{\dagger 2}\rangle + \langle \hat{a}^\dagger \hat{a}\rangle + \langle \hat{a}\hat{a}^\dagger\rangle + \langle \hat{a}^2\rangle. \tag{4.1.13}$$

We can write the expectation value of operator $\hat{A} = \hat{a}\hat{a}^\dagger$ in the normal order as $\hat{a}^\dagger\hat{a}+1$. So that Eq. (4.1.13) can be written as

$$(\Delta a_+)^2 = 1 + 2\langle \hat{a}^\dagger \hat{a}\rangle + (\langle \hat{a}^{\dagger 2}\rangle + \langle \hat{a}^2\rangle). \tag{4.1.14}$$

One can also establish in a similar manner that the variance of the minus quadrature operator is expressible as

$$(\Delta a_-)^2 = 1 + 2\langle \hat{a}^\dagger \hat{a}\rangle - (\langle \hat{a}^{\dagger 2}\rangle + \langle \hat{a}^2\rangle). \tag{4.1.15}$$

On the basis of Eqs. (4.1.14) and (4.1.15), the plus and minus quadrature variances can be expressed as

$$(\Delta a_\pm)^2 = 1 + 2\langle \hat{a}^\dagger \hat{a}\rangle \pm (\langle \hat{a}^{\dagger 2}\rangle + \langle \hat{a}^2\rangle. \tag{4.1.16}$$

Taking Eq. (3.1.26) into account the plus and minus quadrature variance of the cavity mode can be written as

$$\begin{aligned}(\Delta a_\pm)^2 &= 1 + 2\langle \hat{a}^\dagger \hat{a}\rangle \pm \frac{\kappa}{\varepsilon}\langle \hat{a}^\dagger \hat{a}\rangle \\ &= 1 \pm \frac{\kappa}{\varepsilon}(1 \pm \frac{2\varepsilon}{\kappa})\langle \hat{a}^\dagger \hat{a}\rangle. \end{aligned} \tag{4.1.17}$$

Therefore, on substituting Eq. (3.1.28) into Eq. (4.1.17) leads to

$$\begin{aligned}(\Delta a_+)^2 &= 1 + \frac{\kappa}{\varepsilon}(1 + \frac{2\varepsilon}{\kappa})[\frac{2\varepsilon^2/\kappa^2}{(1-4\varepsilon^2/\kappa^2)} - \frac{2\gamma_c\varepsilon^2/\kappa^3}{(1-4\varepsilon^2/\kappa^2)(1+4\varepsilon^2/\kappa^2)}] \\ &= 1 + \frac{\kappa}{\varepsilon}(1 + \frac{2\varepsilon}{\kappa})[\frac{2\varepsilon^2/\kappa^2(1+4\varepsilon^2/\kappa^2) - 2\gamma_c\varepsilon^2/\kappa^3}{(1+2\varepsilon/\kappa)(1-2\varepsilon/\kappa)(1+4\varepsilon^2/\kappa^2)}] \\ &= 1 + \frac{2\varepsilon/\kappa + 8\varepsilon^3/\kappa^3 - 2\gamma_c\varepsilon/\kappa^2}{} \\ &= 1 + \frac{2\varepsilon/\kappa(1-\gamma_c/\kappa) + 8\varepsilon^3/\kappa^3}{(1-2\varepsilon/\kappa)(1+4\varepsilon^2/\kappa^2)}. \end{aligned} \tag{4.1.18}$$

In a similar manner one can also obtain the minus quadrature variance for the cavity mode as

$$(\Delta a_-)^2 = 1 - \frac{2\varepsilon/\kappa(1-\gamma_c/\kappa) + 8\varepsilon^3/\kappa^3}{(1+2\varepsilon/\kappa)(1+4\varepsilon^2/\kappa^2)}. \tag{4.1.19}$$

From Eqs. (4.1.18) and (4.1.19) we see that $(\Delta a_+)^2 > 1$ and $(\Delta a_-^2) < 1$. Therefore we immediately notice that the cavity mode is in a squeezed state and the squeezing occurs in the minus quadrature. In Fig.4.1, we plot Eq.(4.1.19) versus ε/κ. This plot also shows that the cavity mode is in a squeezed state and the degree of squeezing increases as ε/κ decreases.

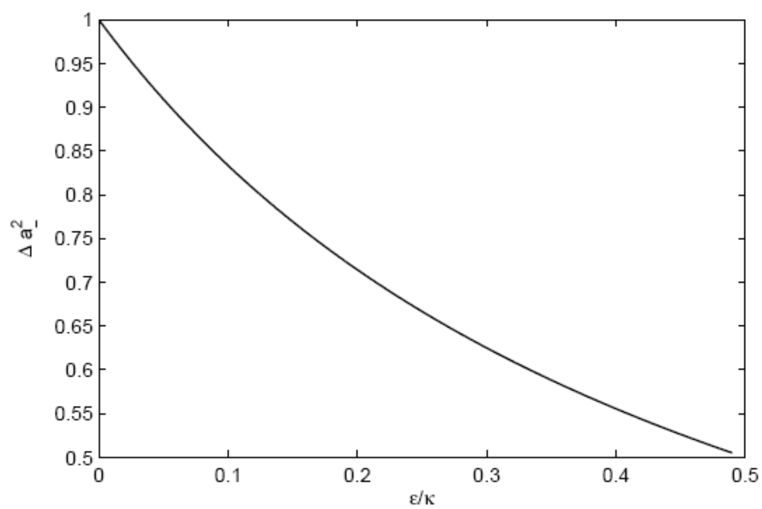

Figure 4.1: Plot of the quadrature variance of the cavity mode [Eq. (4.1.19)] versus ε/κ for $\gamma_c/\kappa = 0.01$

4.2 Quadrature variance for the fluorescent light

Employing Eq. (3.1.29) in (4.1.17) the plus quadrature variance of the fluorescent light can be expressed as

$$\begin{aligned}(\Delta a_+)^2_f &= 1 + \frac{2\gamma_c\varepsilon/\kappa^2(1+2\varepsilon/\kappa)}{(1+2\varepsilon/\kappa)(1-2\varepsilon/\kappa)(1-4\varepsilon^2/\kappa^2)} \\ &= 1 + \frac{2\gamma_c\varepsilon/\kappa^2}{(1-2\varepsilon/\kappa)(1-4\varepsilon^2/\kappa^2)}. \qquad (4.2.1)\end{aligned}$$

similarly the minus quadrature variance of the fluorescent light is

$$(\Delta a_-)^2_f = 1 - \frac{2\gamma_c\varepsilon/\kappa^2}{(1+2\varepsilon/\kappa)(1-4\varepsilon^2/\kappa^2)}. \qquad (4.2.2)$$

From the expression (4.2.1) and (4.2.2) we note that the fluorescent light is in a squeezed state and the squeezing occurs in the minus quadrature. Fig.4.2, indicates

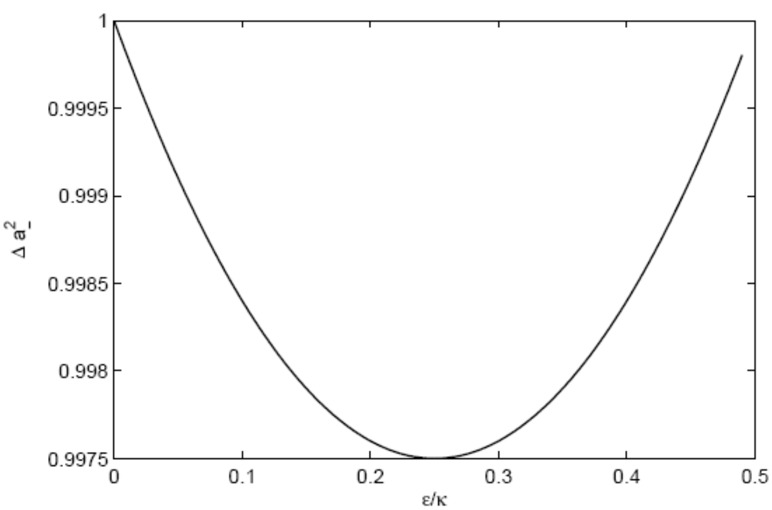

Figure 4.2: Plot of the quadrature variance of the fluorescent light [Eq. (4.2.2)] versus ε/κ for $\gamma_c/\kappa = 0.01$.

that the degree of squeezing of the fluorescent light is very small.

Chapter 5

Conclusion

In this project, we have considered degenerate parametric oscillator whose contains a two-level atom. Employing the master equation for the system under consideration, we have obtained the quantum Langevin equation, the equation of evolution for the expectation values of the cavity mode and atomic operators. Applying the large-time approximation scheme and the correlation properties of noise operators, we have determined the mean and variance of photon number. We have found that the photon statistics of the cavity light is supper-Poissonian. Moreover, we have obtained the normalized power spectrum for the fluorescent light and for the cavity mode. We have obtained the power spectrum in this case turns out to be a single peak at $\omega = 0$. It is found that the width of the spectrum increases with ε/κ. In addition, we have determined the second-order correlation function for the fluorescent light emitted by a two level atom and we have found that the photons in the fluorescent light are antibunching. Finally, we have evaluated the quadrature variance for the cavity mode and fluorescent light produced by two-level atom inside a parametric oscillator coupled to a vacuum reservoir. We observed that the cavity mode is in a squeezed state and the squeezing occurs in the minus quadrature.

Bibliography

[1] Geoffery New, *Introduction to Nonlinear Optics*, (Cambridge University Press, New York, 2011).

[2] Fesseha Kassahun, *Fundamental of Quantum Optics*, (Lulu,North Carolina, 2008).

[3] Eyob Alebachew and K. Fesseha, *arXiv:quant-ph/0610123v2* (2006).

[4] S.M. Barnett, P.M. Radmore, *Methods in Theoretical Quantum Optics* (Clarendon Press, Oxford, 1997).

[5] M.O. Scully, M.S. Zubairy, *Quantum Optics*, (Cambridge University Press, Cambridge, 1997).

[6] D.F. Walls, G.J. Milburn, *Quantum Optics*, (Springer, Berlin, 1995).

[7] Fesseha Kassahun, *arXiv:1209.4723v1* [*quanta-phys*], (2012).

[8] Fesseha Kassahun, *arXiv:1105.1438v3* [*quant-phys*] (2012).

[9] Belay Wedajo, *MSc Thesis*, Addis Ababa University, 2016.

[10] Miguel Orszag, *Quantum Optics* (Springer, Berlin, 1997).

[11] Gilbert Grynberg, Alian Aspect and Cloude Fabre, *Nonlinear optics*, (Cambridge University Press, New York, 2010).

[12] Fesseha Kassahun, *arXiv:1508.00342v1* [*quant-ph*] (2015).

Decclaration

This project is a review of previos work and that all the sources of material used for the project have been correctly acknowledged.

Name: Kassahun Daniel

Signature of Author

This project has been submitted for examination with my approval as Uninersity advisor

Name: Dr. Deribe Hirpo

Signature of Advisor

Place and time of submission:

Physics Departement
Addis Ababa University
August, 2017